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FORMULAS FOR REINFORCED CONCRETE BEAMS,

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(Read before the General Section, February 22, 1906.)

The rise of reinforced concrete is, perhaps, the most interesting phenomenon of the past decade in the field of constructive engineering. Since the introduction of Bessemer and open hearth steel into bridge building, twenty years ago, there has been no invention of equal importance to the civil engineer and architect.

Apart from bridges of very long span, there is hardly a branch of heavy construction to-day in which concrete, with or without metallic reinforcement, is not largely employed in place of timber, cast iron, steel or masonry.

This development, as is well known, is of very recent growth. Even ten years ago concrete steel construction was quite in its infancy. To-day only the most conservative engineers rule it out altogether, although there is naturally much difference of opinion as to its proper field. So rapid a rise is in itself a proof that this new material must possess many advantages. Its strong points are indeed not far to seek.

Advantages of Reinforced Concrete.—Reinforced concrete combines the resistance to fire and the low cost of maintenance which we find in the best masonry, with great tensile as well as compressive strength. It has also a comparatively high degree of elasticity, so that it can be deformed considerably without fracture, a property of the utmost importance in a structural material.

Besides this, it is economical in first cost as compared with other fireproof and permanent materials. In buildings concrete steel is used in beams, girders, columns, floors, partitions and even outside walls. It is usually cheaper than steel protected by fireproofing, and not much dearer than timber mill construction. The saving in insurance premiums alone will frequently off-set the extra cost.

To the bridge engineer reinforced concrete has proved of great value. In railroad work it is used very generally for culverts and arches under embankments, and other short spans. These designs are safer and more permanent than timber trestles or open bridges with steel beams and wooden ties. Their maintenance costs almost nothing.

For highway bridges, both of long and short span, the reinforced concrete arch has no equal for economy combined with aesthetic value.

In other structures, such as retaining walls, dams, etc., hitherto built of solid masonry, which depends entirely upon its weight for stability, the tensile strength of reinforced concrete has led to new and more economical designs, in which the stresses are more rationally distributed.

Objections to Reinforced Concrete.—Although steel concrete is now so widely used, and its economy is generally admitted, there is still much discussion as to its permanence and reliability. It may, therefore, be of interest to mention some of the objections commonly made, and to discuss them briefly.

The points to be considered fall under these headings:—

The permanence of steel concrete, more especially the liability of the steel to corrosion in outdoor structures.

The uniformity and reliability of the concrete itself.

The proper designing of reinforced concrete structures and the calculation of stresses.

With regard to the *permanence* of steel concrete, I believe that most American engineers, at least, are now well satisfied that the metal is fully protected if the concrete covering is of sufficient thickness. Some eminent engineers, however, do not think that our experience in this respect is long enough to permit a definite conclusion to be reached. Sir Benjamin Baker, for instance, in a recent discussion before the Institution of Civil Engineers, expressed the opinion that a test extending over twenty years would be required to settle this question. Our experience with reinforced concrete hardly extends over twenty years, but a considerable amount of iron and steel has been in use embedded in cement mortar in various suspension and truss bridges and in buildings for fifty years or more. The subsequent demolition of some of these structures has given many good opportunities for observing the condition of the metal work. These observations have been very reassuring as to the permanence of the embedded iron. The writer has examined several buildings of this kind while they were being torn down, and has been much struck with the uniformly perfect preservation of the iron, even under seemingly adverse conditions.

In modern reinforced construction, the steel is far more carefully protected than in the older bridges and buildings just referred to, and should be even less liable to corrosion. Concrete steel has not been long in general use and, of course, few complete structures have been demolished to make place for new ones. Still, many beams, slabs, etc., have been cut into or entirely taken down, but in the author's experience, very few cases have been observed in which the steel has become seriously corroded.

It is, of course, not impossible that a longer and wider experience may give less favourable results, but this does not seem at all probable. There is no apparent reason why structures that have stood uninjured, say for five years, in exposed positions, should deteriorate later since the strength as well as the impermeability of concrete is not impaired but rather increased by age. It seems, therefore, that even with our present experience we may feel quite confident that the steel reinforcement will be indefinitely preserved, while the durability of Portland cement concrete itself is, of course, hardly open to question.

The second point to be considered is the *uniformity* and *reliability* of the concrete itself.

Concrete is not a new material, but has been used for fifty years or more though, perhaps, not very extensively. Of late years, its properties have been studied more carefully by scientific methods with a view to greater economy and certainty in its composition. More especially, an interesting series of experiments has recently been made in France and the United States on the effect which the relative fineness or coarseness of the aggregate has on the strength of the concrete. Many other points are being studied with equal thoroughness.

There are still many unexplained problems and indefinite room for further study, but even now the strength of concrete of given proportions can be predicted with considerable accuracy, and a safe minumum strength assumed in designing. Individual tests will naturally show variations in strength, though less than those that prevail in the case of timber and not much greater than allowed by the best specifications for structural steel.

It is often asserted, that even when the cement and aggregates are carefully selected and the workmanship good, unexpected weaknesses are likely to occur, and doubtless this sometimes happens. A similar criticism was frequently made in the case of structural steel. Some fifteen or twenty years ago, when steel was first coming into use for bridgework, its unreliable character as compared with wrought iron was much dwelt upon. Some of the most eminent engineers adopted the new material with hesitation, and their early specifications were almost prohibitive. Occasional fractures in steel bars that occurred during manufacture and erection were often cited as proving the unreliable character of structural steel. It is quite true that we now use a softer grade of steel, but even in the earlier bridges, failures in completed structures due to the brittleness of the material were practically unknown.

Fractures in steel plates not easy to explain still occur to some extent, but the reliability of strucural steel is no longer seriously questioned by anyone, and its superiority, even in this respect, over commercial wrought iron is, to-day, generally admitted.

Concrete, plain and reinforced, like other forms of construction, must be made from carefully selected material of proper composition. Its manipulation must be conscientious, but does not require any unusual amount of intelligence. If a certain proportion of cement and aggregates mixed in a certain way gives good results at one time, it can hardly fail to do so at another. Uniform and reliable cements are readily obtained and most localities furnish sufficiently clean sand, gravel or crushed stone to do excellent concrete work. With even ordinarily good management weak or badly set concrete should be a matter of the very rarest occurrence.

As in all other forms of construction, good workmanship and careful inspection are requisite for the best results, but the monolithic character of concrete work is greatly in its favor. The continuity between the different beams and girders and floor slabs in a building adds greatly to their strength and minimizes the risk from local weakness. In this respect, concrete construction is probably superior to most structural steel work, the strength of which depends largely on riveted or bolted connections.

Different Classes of Structures.—Designing in concrete steel involves much more than merely the calculation of stresses and requires careful study and good practical judgment. There is as yet no general agreement as to the best shapes for reinforcing, and there is much variation in arranging them. Nevertheless, considerable experience has been gained, and many actual tests have been made, so that the safe load which a beam, arch or column will carry can to-day be computed with much accuracy.

It is beyond the scope of this paper to treat in detail the different forms of concrete steel construction. They may, however, be divided into a few general classes, so that the principles involved in the calculation of stresses are comparatively few.

The object in reinforcing any concrete construction is, of course, to increase its strength over that of plain concrete. The latter has a high compressive but little tensile and shearing strength. Hence, steel is mainly used where there are considerable tensile and shearing stresses, although in some cases it is also used to strengthen concrete in compression.

Tensile stresses may occur by themselves, as in tie rods and the tension members of skeleton trusses, or else in combination with compressive and shearing stresses. Some concrete trusses have lately been built like lattice girders in which the tension diagonals and bottom chords (and also the top chords), contain steel embedded in concrete, while the compression members are of plain concrete, but this form of construction is very exceptional.

As noted above, steel is also used in purely compression members such as columns in buildings. In this case the metal may be disposed in one of two ways. In the first method it consists of vertical rods, placed as a rule near the outer circumference of the column. The steel is supposed to carry part of the load by compression, the rest being supported directly by the concrete.

In the second method the steel is used far more efficiently in the shape of a helicoidal wrapping surrounding the concrete. The steel here is really used in tension. When the column is loaded, it inevitably shortens longtitudinally and tends to spread laterally. The spiral wrapping resists this lateral expansion, keeps the concrete from splitting and thus greatly increases the supporting power of the column.

By far the greatest use of steel concrete is however in members subject to cross bending, so that there is both tension and compression in the same cross section. This occurs in beams and girders of all kinds, in arches under moving loads, and in many other places.

Beam Formulas.

The remainder of this paper is devoted to a brief account of the methods for calculating the stresses in reinforced concrete beams.

Many different formulas have from time to time been published for this purpose, of which some are based on theoretical considerations while others are merely empirical summaries of a limited number of experiments.

Those of the second class have a certain value for comparing the results of actual tests, but are not adapted for general use.

The theoretical formulas apparently differ greatly in principle, but this is due largely to the use of different symbols for the same quantities and to different algebraic arrangements This multiplication of methods is to be regretted, as it has led many engineers to mistrust the results of a calculation on which the experts appear to differ so widely.

A number of the best known formulas are given below, using the same symbols throughout. It has occurred to the writer that it might be of interest to show how readily they can be derived from a few general principles, although they may differ in arrangement and in the value of the constants used.

The correctness of any theory or formula of the kind depends, of course, entirely on its agreement with actual tests. Many laboratory tests have been made both in Europe and America, and also many individual tests in connection with building operations. As is always the case where tests by different experimenters are compared there are many discrepancies, and a systematic series of experiments on a large scale would be very desirable.

Still, even with our present knowledge, the agreement between the theoretical and actual breaking strength of a beam is sufficiently close for practical purposes, if the grade of concrete and the strength of the steel is known.

The formulas for concrete steel beams are based on the common theory of flexure for homogeneous material with certain modifications.

These modifications result from the composite nature of the beam and the properties of cement concrete as distinguished from steel or timber.

In properly reinforced concrete, the steel and concrete must work in unison, and the arrangement and proportioning must be such as to utilize the strength of both materials to the limit of their respective capacities.

The concrete as far as possible should be used to resist compressive, and the steel tensile, stresses, but as the steel is embedded in the concrete, it is impossible to separate their functions entirely.

Where there is steel only in the lower part of the beam, as is usually the case, all compression will be carried by the concrete, but except under certain extreme conditions, the tensile stresses will be carried in part by the steel and in part by the concrete.

In structural steel beams the modulus of elasticity is practically the same for all grades of steel, and remains constant for all stresses below the elastic limit, hence formulas can readily be obtained for the stresses and deformations under any given load.

The modulus of cement concrete on the other hand depends on the composition of the concrete, the mode of mixing, the age of the concrete, etc.; furthermore, it probably varies with the unit stress acting on the concrete. The neutral axis, which in steel beams is fixed by the geometrical figure of the cross-section, in the case of concrete depends also on the ratio of the modulus of the steel to that of the concrete, and the ratio of the quantity of steel used to that of concrete. The latter ratio is fixed for any given beam, the former varies with the unit stresses according to laws which are little known. Hence, the position of the neutral axis changes with the loading. For this reason, it is not feasible to devise any general formula for stresses and strains under different working loads.

It is, however, possible to determine the maximum loading a beam can carry without rupture and from this to proportion beams with proper factors of safety.

It is true, as stated above, that under working loads the concrete usually carries part of the tensile stresses. If, however, the loading be gradually increased, a point will eventually be reached at which the concrete on the tension side will tear apart, showing fine cracks. The beam will still be far from failure, as the tension in the steel at this stage will usually be well below the elastic limit. If the loading be still further increased the entire tensile stress will now be carried by the steel reinforcement. Failure will not occur until the steel is stressed beyond its elastic limit, or the concrete crushed on the compressive side of the beam. The elongation which the concrete itself can stand without cracking, appears to be somewhat increased by the reinforcement. Still, in a well proportioned beam there will usually be some tearing apart of the concrete on the tension side under a load which does not utilize fully either the tensile strength of the steel or the compressive strength of the concrete. Hence, it will result in more economical and still perfectly safe construction, to assume that at breaking loads all tensile stresses are resisted by the steel reinforcements only. Some engineers believe that no appreciable cracks appear on the concrete until the elastic limit of the steel has been passed and hence they take into account the tensile resistance both of the concrete and of the steel. The tests on which this view is based still await full corroboration.

To sum up, then, two different assumptions are made:—

(*a*) That the steel carries all tensile stresses.

(*b*) That the concrete takes part of them.

Most formulas now in use are based on the first assumption, but some engineers with a high reputation as experimenters and practical designers, have adopted the latter hypothesis in calculating stresses.

A second mooted point is the shape of the stress curve, that is, the law according to which stresses in given cross-sections vary from zero at the neutral axis to a maximum at the outer edge of a beam.

Apart from these two points, the difference in the various formulas is largely a matter of algebraic arrangements. There is, of course, much difference of opinion as to the proper values for the working stresses and the elastic moduli, which should be assumed in any given case. The entire question will be more clearly understood by working out a general formula and showing how methods of calculation given by different writers can be derived from it.

The most usual case is a rectangular beam reinforced on the tension side only and loaded vertically. It is assumed that the steel adheres to the concrete and that the two materials act together perfectly. The fact that the co-efficient of expansion of steel and concrete is the same makes this last assumption possible. There are also supposed to be no initial stresses on either material. As in the case of steel beams, sections plane before bending are assumed to remain plane after bending. Experiments appear to prove this to be correct. When the beam is loaded and bends, there will be compression extending down for a certain distance from the top, the lower edge of this compression area being the neutral axis. Below this point the concrete is supposed to open up and all tensile stresses to be carried by the steel.

Figure 1.

The preceding diagram shows the dimensions of the beam and the position of the reinforcement.

Let b and h = Breadth and total depth of beam.

d = The efficient depth, i.e., the depth to the centre of the steel.

y_1 = Distance to neutral axis from top of beam.

y_3 = Distance of centroid of compressive stresses from neutral axis.

λ_1 = Compression of concrete (at top) per unit of length.

λ_2 = Extension of steel per unit of length.

E_c = Modulus of elasticity of concrete.

E_s = Modulus of elasticity of steel.

$e = \dfrac{E_s}{E_c}$

C = Compressive stress per square inch on extreme fibre of concrete at maximum load.

T = Tensile stress per square inch in steel.

P_c = Total compressive stress on cross-section.

S = Area of steel reinforcement.

M_0 = Moment of resistance of cross-section.

If E_c is taken as constant for all stresses, the stress diagram will be triangular, as shown in stress diagram (A) Fig. 1. This is the case for steel, timber or other homogeneous materials. This assumption is also made very generally for concrete steel beams and is contained in the building codes of most cities, and of the Prussian and other governments.

This gives the total compressive stress $P_c = \frac{1}{2} C b y_1$ and the distance from centroid to neutral axis $y_3 = \frac{2}{3} y_1$.

Actual compressive tests in concrete seem to indicate that the rectilinear assumption is not correct and that the line of stress is curved.

It is frequently taken as a parabola with its vertex on the neutral axis of the beam and its axis vertical

$$\text{This gives } P_c = \tfrac{2}{3} C b y_1.$$
$$\text{and } y_3 = \tfrac{3}{8} y_1$$

After making a careful study of many recent tests Capt. John S. Sewall, U.S.A., proposes as a safe compromise, a curve which gives:—

$$P_c = \tfrac{5}{8} C b y_1$$
$$\text{and } y_3 = \tfrac{3}{8} y_1$$

Mr. A. L. Johnson makes $P_c = \frac{5}{8} C b y_1$,

$$y_3 = \tfrac{2}{3} y_1$$

but he also takes into account the tensile tresses in the concrete.

As a *general* expression we can use $P_c = n\,C\,b\,y_1$

and $y_3 = my_1$

In homogeneous beams the moment of resistance and the stresses can be found by simple statics.

The algebraic sum of the axial forces acting on any cross-section and also the moments of these forces about any point in the section must equal zero. These conditions give two equations which solve the problem. In a composite beam, these two statical equations are not enough. We must add a third taken from elasticity.

The three equations thus found are the following:—

$$P_c = ST = n\,C\,b\,y_1 \tag{1}$$

$$M_0 = P_c\,(y_3 + y_2) = ST(y_3 + y_2) = P_c\,y_3 + ST\,y_2 \tag{2}$$

$$\frac{y_1}{y_2} = \frac{\lambda_1}{\lambda_2} = \frac{\frac{C}{E_c}}{\frac{T}{E_s}} = \frac{Ce}{T} \tag{3}$$

As $y = d - y_1$ and $y_3 = my_1$

we have by substitution

$$ST = n\,C\,b\,y_1 \tag{4}$$

$$M_0 = n\,C\,b\,y_1\,(my_1 + d - y_1) \tag{5}$$

$$y_1 = \frac{d}{1 + \frac{T}{Ce}} \tag{6}$$

In the last three equations e depends on the materials only, and n and m on the assumption made for the stress curve, so that these three quantities are constants for any given case. There are seven remaining quantities and only three equations. Two cases must be considered.

First.—To find the strength of a beam of given dimensions and reinforcement.

Here b, d and S are known.

There remain four other quantities, T, C, M_0 and y_1

From (4) and (6) we obtain

$$y_1 = \sqrt{\frac{Sed}{bn} + \left(\frac{Se}{2bn}\right)^2} - \frac{Se}{2bn} \tag{7}$$

which shows that the neutral axis depends merely on the amount of steel reinforcement and the elastic moduli, i.e., it is fixed in any given beam if definite values for the moduli of concrete and steel are assumed.

T and C are interdependent; that is, if a given value is assumed for T, there will be a corresponding value for C and vice versa.

The breaking strength of a concrete beam is generally supposed to correspond to a stress in the steel equal to its elastic limit, as beyond this point its extension is so great as to completely rupture the concrete surrounding it.

If we take T = elastic limit of steel, the maximum moment of resistance will be

$$M_0 = TS\,(m\,y + d - y_1)$$

and the corresponding value for C will be given by equation (1). If the value of C thus obtained is excessive, T must be reduced. Second:—For a given loading and span to design a beam in which the concrete and steel shall not be strained beyond safe limits.

Here M is given, also T and C, and n and m are also known quantities, while b, d, S and y_1 remain to be determined. As there are only three equations, they are not independent variables. If, however, the ratio $\frac{b}{d}$ be assumed, S may be readily expressed as a function of T, C, and e, and suitable values for b and d obtained.

As stated above, most formulas in general use can be derived from equations (1) to (6) and some of the best known, together with what the author has found on the whole the most convenient method of calculation, are given in the sequel.

THACHER'S FORMULA.

Mr. Edwin Thacher's well known formula* may be readily derived from the equations given above.

His assumptions are the following:

Steel on tension side only.

Steel carries all tension stresses.

Compressive stress curve rectilinear.

And therefore, $n = \frac{1}{2}$, and $m = \frac{2}{3}$,

Hence, $P_c = \frac{1}{2}\,C\,b\,y_1$ and $y_3 = \frac{2}{3} y_1$

and from (2)

$$M_0 = \tfrac{1}{2}\,C\,b\,y_1\,(\tfrac{2}{3}\,y_1) + ST\,y_2$$
$$= \tfrac{1}{3}\,C\,b\,y^2_1 + ST\,y_2$$

and by substituting from (3)

$$C = \frac{Ty_1}{ey_2}$$

$$M_0 = \tfrac{1}{3}\left(\frac{T\,by_1{}^3}{ey_2} + ST\,y_2\right)$$

$$= \frac{T}{3}\left(\frac{by_1{}^3}{ey_2} + 3Sy_2\right)$$

or if M_0 is expressed in foot pounds instead of inch pounds

$$M_0 = \frac{T}{36}\left(\frac{y_1{}^3}{ey_2} + 3\,Sy_2\right)$$

the form given by Mr. Thacher.

* Buel and Hill, "Reinforced Concrete," p. 26.

From (7) by substituting $n = \frac{1}{2}$, the position of the neutral axis is given

$$y_1 = \sqrt{2\frac{Sed}{b} + \left(\frac{Se}{b}\right)^2} - \frac{Se}{b}$$

The amount of steel reinforcement corresponding to a given $\frac{T}{C}$ between the tensile and compressive stresses may be readily derived from (4) and (6) namely:

$$S = \frac{bd}{2\left[\frac{T}{C} + \frac{1}{e}\left(\frac{T}{C}\right)^2\right]}$$

or for a width of beam of one inch we have Thacher's form for the required area of steel

$$a = \frac{S}{b} = \frac{d}{2\left[\frac{T}{C} + \frac{1}{e}\left(\frac{T}{C}\right)^2\right]}$$

The following table is given by Thacher, using the ultimate strength of the steel in calculating the breaking strength.

MATERIAL.	E_s	E_c	C	T	Area of Steel S	Ultimate bending moment in inch lbs.
Steel	30,000,000			64,000		
Concrete 1 : 2 : 4 one month old		1,460,000	2,400		$\frac{bd}{142} = 0.71\%$	$427\ bd^2$
Concrete 1 : 2 : 4 six months old		2,580,000	3,700		$\frac{bd}{100} = 1\%$	$615\ bd^2$
Concrete 1 : 3 : 6 one month old		1,220,000	2,050		$\frac{bd}{165} = 0.61\%$	$367\ bd^2$
Concrete 1 : 3 : 6 six months old		1,860,000	3,100		$\frac{bd}{109} = 0.92\%$	$555\ bd^2$

SEWALL'S FORMULA.*

Capt. John S. Sewall, Corps of Engineers, U.S.A., has made a careful study of concrete beams, and recommends the following method: — He assumes:

* Trans. Am. Soc. C.E, Vol. LIV, Part E, p. 490. Int. Eng. Congress.

Steel in tension side only.

Steel carries all tension.

Stress curve, a compromise between a right line and parabola.

He makes $n = \frac{5}{8}$, and $m = \frac{3}{5}$, so that

$$P_c = \tfrac{5}{8}\, C\, b\, y_1,\quad y_3 = \tfrac{3}{8}\, y_1$$

$$\text{Hence, } M_0 = \tfrac{3}{8}\, C\, b\, y_1^2 + S T\, y_2$$

$$\text{and } y_2 = \frac{T}{eC}\, y_1$$

Capt. Sewall recommends using the elastic limit of the steel and the ultimate strength of the concrete in computing the breaking strength and a factor of $2\frac{1}{2}$ for working loads.

In a later and very interesting paper,* Capt. Sewall takes for the breaking strength of a beam a higher value than the elastic limit in the steel and assumes that when T reaches the elastic limit of steel, the maximum compression of the concrete is equal to eight-tenths of the ultimate resistance of concrete. He then has the following expression for the bending moment.

$$M_0 = 0.292\ C\, b\, y_1^2 + S T\, y_2$$

In view of the uncertainty as to the value of E_c and as to the shape of the stress curve, he proposes using a constant value for the position of the neutral axis, and shows that the variations are practically not very great within the usual range of stresses.

Making the lever arm of the stresses, i.e., the distance from the centroid of the compressive area to the centre of the steel $= 0.85d$, he gets the simple formula: $M_0 = 0.85\ d\, S\, T$.

A. W. Buel's Formula.†

This is a general formula, taking into account beams having steel on the compressive side and also the tensile strength of concrete if desired.

It is based on the moment of inertia of the steel and of the concrete beam, and can readily be derived for our special case where there is no tension carried by the concrete, and no steel on the compressive side of the beam.

A triangular stress diagram is assumed.

Using the notation employed in our general discussion, we have

$$y_1 = -\frac{Se}{b} + \sqrt{2\, Sed + \left(\frac{Se}{b}\right)^2}$$

* Proc. Am. Soc. C.E., Dec. 1905.

†Buel and Hill "Reinforced Concrete" p. 20.

Then the moments of inertia of the steel and concrete about the neutral axis are respectively

$$I_s = S\, y_2{}^2$$
$$\text{and } I_c = \tfrac{1}{3}\, b\, y_1{}^3$$
$$\text{then } T = \frac{M_o}{\frac{I_s}{y_2}} \cdot \frac{1}{2}$$
$$C = \frac{M_o}{\frac{I_c}{y_1}} \cdot \frac{1}{2}$$
$$M_o = \frac{2\ I_s\ T}{y_2} = \frac{2\ I_c\ C}{y_1}$$

which three equations suffice for solving the problem.

Example:—

Take a beam 4″ wide, 16″ deep, with one steel bar ⅞″ diameter, placed 1½″ from the bottom

Assume C = 700 lbs. per square inch.
E_c = 3,600,000
E_s = 29,000,000

$$\text{then } y_1 = -\frac{0.60 \times 8.06}{4} + \sqrt{\frac{16.11}{4}\ 0.6 \times 14.5 + \left(\frac{6 \times 8.06}{4}\right)^2}$$
$$= 4.83 \text{ inches}$$
$$I_s = 0.6\ (967)_2 = 56.07$$
$$I_c = \tfrac{4}{3}\,(4.83)_3 = 150.52$$
$$M_o = 2\,\frac{150.52 \times 700}{4.83} = 43601.9 \text{ inch lbs.}$$

VON EMPERGER'S FORMULA.

Dr. F. von Emperger, in a paper read before the International Engineering Congress at St. Louis,* gave a formula of a somewhat novel form.

He takes all tensile stresses as being carried by the steel, and the tensile stress for a maximum load as equal to the ultimate strength of the metal.

His formula differs from the previous ones, in assuming "that the compressive and tensile stresses are uniformly distributed over their respective sections."

Using the notation previously used by us, his equations become as follows:—

Let $p = \frac{S}{bd}$ = ratio of steel to total section, and the other symbols remain as before.

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* Trans. Am. Soc. C.E. 1905, Vol. LIV Part E, page 535. Int. Eng. Congress.

Then for rectilinear variation of the compressive stress we should have $y_1 = -15\,d\,p + d\sqrt{15\,p\,(15\,p + 2)}$

$$\text{if } e = \frac{E_s}{E_c} = 15$$

This formula is used by Dr. von Emperger as a "close approximation," although the stress is taken as uniformly distributed in his assumption, and not as varying.

The moment of resistance will then be:

$$M_0 = p\,b\,d\,T\,(d - \tfrac{1}{2}\,y_1)$$

$$= \frac{bd^2}{2}\,p\left[2 + 15\,p - \sqrt{15\,p\,(15\,p + 2)}\,\right]\,T$$

$$= \frac{bd^2}{6}\,3\,p\left[2 + 15\,p - \sqrt{15\,p\,(15\,p + 2)}\,\right]\,T$$

Let $W = \frac{bd^2}{6}$ which is the section modulus of resistance of a concrete beam of width b and depth d,

$$\text{and } C_s = 3\,p\left[2 + 15\,p - \sqrt{15\,p\,(15\,p + 2)}\,\right]$$

a constant depending entirely on the percentage of reinforcing metal and T = tensile stress in the steel.

We can then write:

$$M_0 = W\,C_s\,T$$

The author of the formula gives a diagram by which this equation can be readily solved.

WENTWORTH'S FORMULA.

Mr. C. A. Wentworth, in a discussion at the International Engineering Congress, * gave some valuable data and his formula, which has been adopted by the Bureau of Yards and Docks of the United States Navy, is as follows:—

The tension is entirely carried by the steel and the compressive stress diagram is triangular.

If, furthermore, A_s = area of steel per foot of width of beam = $\frac{12S}{b}$, and $r = \frac{T}{C}$,

our previous notation for other quantities being retained, then for a beam 12 inches wide.

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1 Trans. Am. Soc. C.E. 1905, Vol. LIV., p. 599. Int. Eng. Congress.

$$T\,A_s = \frac{12\,C y_1}{2}$$

$$A_s = \frac{6\,C y_1}{T} = \frac{6\,y_1}{r}$$

$$\frac{T}{d - y_1} = \frac{C\,E_s}{y_1\,E_c} = \frac{C\,e}{y_1}$$

$$\frac{y_1}{d - y_1} = \frac{e}{r}$$

$$\frac{y_1}{d} = \frac{e}{r + e}$$

$$M_o = 6\,C\,y_1\,(d - \frac{y_1}{3}) \text{ in inch pounds.}$$

The working stresses to be used are:—

For stone concrete, 500 lbs. per square inch, or 550 if there is a granolithic finish.

For cinder concrete, 400 and 440 lbs. respectively.

For steel, 16,000 lbs. per square inch.

The following table gives the position of neutral axis, area of steel, and moment of resistance for working loads under the above stresses for a beam 12″ wide:

	y_1	A_s	M_o
Stone concrete..	0.273 d	.051	744 d^2
Cinder concrete...	0.333 d	.050	711 d^2
Stone concrete, granolithic top.. ..	0.292 d	.060	870 d^2
Cinder concrete, granolithic top.. ..	0.355 d	.059	826 d^2

Hennebique's Formula.*

As M. Hennebique has done a very large amount of work all over the world, the method used by him is of interest. It is, however, theoretically open to criticism.

He assumes that there is no tension on the concrete and also that the moment of the tensile stresses about the neutral axis equals the moment of the compressive stresses.

He further assumes that the compressive stress is uniform over the compressed area. The last two assumptions are, of course, wrong, but if the average allowable stress is taken low enough, so that the stress on the outer fibre does not become excessive, the practical results are satisfactory.

* "Reinforced Concrete," by Considere (Moisseiff's Translation), p. 33.

Hennebique takes this uniform compression equal to 350 lbs. per square inch, which for a triangular stress diagram would give 700 lbs. as the maximum stress. The working stress in the steel is 14,000 lbs. per square inch.

Hence, by his assumptions:—

$$\frac{M_0}{2} = 350\, by_1\, \frac{y_1}{2} = \frac{350\, by_1^2}{2}$$

$$\text{or } M_0 = 350\, by_1^2$$

$$\text{and } y_1 = 0.053\sqrt{\frac{m}{b}}$$

$$\text{similarly } \tfrac{1}{2} M = ST\, y_2 = ST\, y_1$$

as his assumptions of the quality of the tensile and compressive moments about the neutral axis makes $y_1 = y_2$

From the last equation if $T = 14{,}000$

$$S = \frac{M}{28{,}000 y_1}$$

In the methods so far given, the concrete was not assumed as carrying any tensile stresses. In the next three formulas the concrete is taken as carrying part of the tensile stresses.

CONSIDERE'S FORMULA.*

This eminent investigator considers the concrete as carrying part of the tensile stresses, and assumes them uniformly distributed over the concrete below the neutral axis. The compressive stress he takes as varying uniformly between the neutral axis and the extreme fibre.

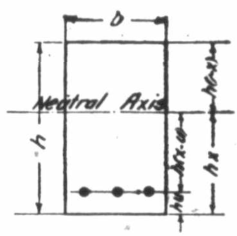

Figure 2.

Referring to Fig. 2, it is seen that he expresses all dimensions in terms of h.

$$\text{Furthermore, let } p = \frac{\text{area of steel}}{\text{cross section of beam}}$$

$$= \frac{S}{bh}$$

and t = max. tensile stress on concrete

* Reinforced Concrete by Considere (Moisseiff's Translation), p. 20.

Then equation (3) in our general solution takes the form

$$\frac{h\,(1-x)}{h\,(x-u)} = \frac{\dfrac{C}{E_c}}{\dfrac{T}{E_s}}$$

$$\text{or } C = \frac{T}{e}\,\frac{(1-x)}{(x-u)}$$

Equating tensile and compressive stresses

$$t\,b\,h\,x + T\,b\,p\,h = \frac{C}{2}\,b\,h\,(1-x)$$

substituting the value of C above

$$t\,x + T\,p = \frac{T}{2\,e}\,\frac{(1-x)^2}{(x-u)}$$

Finally, taking moments about the centroid of compressive forces

$$M_0 = T\,p\,b\,h\,[\tfrac{2}{3}\,h\,(1-x) + h\,(x-u)] + t\,b\,h\,x[\tfrac{2}{3}\,h\,(1-x) + \tfrac{1}{2}\,h\,x]$$

$$= b\,h^2\left(t\,x\,\frac{4+x}{6} + T\,p\,\frac{x-3\,u+2}{3}\right)$$

M. Considère takes $T =$ elastic limit of the steel and t equal to the stress which will extend the concrete 0.015 to 0.020 per cent., i.e., from 171 to 427 lbs. per square inch, according to the grade of concrete used.

A. L. Johnson's Formula.*

This formula has been quite widely used. It assumes part of the tension as carried by the concrete and uniformly distributed. As to the compression, the assumed curve is such that

$$P_c = \tfrac{5}{8}\,C\,b\,y_1 \text{ and } y_3 = \tfrac{3}{8}\,y_1$$

The modulus of concrete is taken as $\frac{3}{5}\,E_c$ for rock concrete, i.e., $\frac{3}{5}$ of the original modulus, and for cinder concrete $= \frac{1}{2}\,E_c$.

The tensile stress acting on the extreme edge of the concrete beam is taken equal to $\frac{8}{10}\,t$, if $t =$ the tensile strength of concrete This last assumption is in accordance with Considère's experiments.

If, further, P_t = total tensile stress carried by concrete,

$$\text{then } P_c = S\,T + P_t$$

$$\text{or, } \tfrac{5}{8}\,C\,by_1 = S\,T + \tfrac{8}{10}\,t\,by_2$$

$$\text{But as } y_2 = \tfrac{2}{3}\,\frac{T}{C}\,\frac{E_c}{E_s}\,y_1\,, \qquad \text{(A)}$$

$$\tfrac{5}{8}\,C\,by_1 = S\,T + \tfrac{8}{15}\,\frac{T\,E_c\,t\,by_1}{C\,E_s}$$

$$S = \frac{15\,C\,by_1 - 64\,t\,by_1\left(\dfrac{T\,E_c}{C\,E_s}\right)}{120\,T} \qquad \text{(B)}$$

$$\text{and } M_0 = S\,T\left(y_2 + \frac{2\,y_1}{3}\right) + \tfrac{8}{10}\,t\,by_2\left(\frac{y_2}{2} + \frac{2\,y_1}{3}\right) \qquad \text{(C)}$$

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* Corrugated Bars for Reinforced Concrete, 1905, p. 115.

Equations (A), (B) and (C) are sufficient for obtaining the strength of a beam.

Mr. Johnson makes $T = 50{,}000$ lbs., the elastic limit of the high carbon steel used by him.

The following Table is computed from the above formula:—

	Average Rock Concrete 1 : 3 : 6	Special Rock Concrete 1 : 2 : 4	Cinder Concrete 1 : 2 : 5
E_s	29,000,000	29,000,000	29,000,000
E_c	3,000,000	2,400,000	750,000
T. (Elastic limit of Steel)	50,000	50,000	50,000
C	2,000	2,400	750
t	200	200	80
y_2	$1.72\ y_1$	$1.15\ y_1$	$0.862\ y_1$
S. (Area of Steel)	$0.0195\ by_1$	$0.0263\ by_1$	$0.00827\ by_1$
M_0	$2750\ by_1^2$	$2620\ by_1^2$	$693\ by_1^2$
If $b = 12''$ and $g = \frac{1}{10}h$			
y_2	$0.331\ h$	$0.418\ h$	$0.483\ h$
S	$0.077\ h = 0.64\%$	$0.132h = 1.1\%$	$0.048\ h = 0.4\%$
M_0	$3620\ h^2$	$5505\ h^2$	$1935\ h^2$

W. K. Hatt's Formula.

In a valuable paper read before the Western Society of Engineers,* Professor Hatt has developed some formulas that may be stated as follows:—

He assumes the tensile stress as partly carried by the concrete, the distribution, both for the tensile and compressive stresses being given by a parabola with axis horizontal and its origin on the outer edge.

* Journal West. Soc. Engr's 1904, p. 223.

Let E_t = modulus of concrete in tension.

$$t = \frac{E_c}{E_t},$$

$$q = \frac{E_s}{E_t}$$

p = ratio of steel area to total cross-section of beam

λ_3 = extension of concrete on extreme element per unit of length.

$$\text{Then } \frac{\lambda_1}{\lambda_3} = \frac{\dfrac{C}{E_c}}{\dfrac{t}{E_t}} = \frac{h\,x}{h\,(1-x)} \text{ or } C = \frac{t\;l\;x}{1-x}$$

$$\text{and } \frac{\lambda_2}{\lambda_3} = \frac{\dfrac{T}{E_s}}{\dfrac{t}{E_t}} = \frac{h\,(u-x)}{h\,(1-x)} \text{ or } T = \frac{t\,q\,(u-x)}{1-x}$$

Equating the tensile and compressive stresses

$$\tfrac{2}{3}\, C\, by_1 = \tfrac{2}{3}\, t\, bh\,(1-x) + p\,b\,h\,T$$

$$\text{or, } \tfrac{2}{3}\, C\, x = \tfrac{2}{3}\, t\,(1-x) + p\,T$$

Inserting values of C and T given above

$$\tfrac{2}{3}\, t\, \frac{l\,x^2}{1-x} = \tfrac{2}{3} t\,(1-x) + p\,t\,q\,\frac{(u-x)}{(1-x)}$$

$$\text{whence, } x = \frac{\dfrac{4+3pq}{2} + \sqrt{4l + \tfrac{9}{4}\,p^2\,q^2 + p\,[6q\,(u\,(l-1)+1)\,]}}{2\,(b-1)}$$

Having solved x, C and T may be computed from the first two equations.

Taking moments about the neutral axis we have

$$M_0 \left(\tfrac{5}{12}\,(1-x)\,2 + \tfrac{5}{12}\,\frac{lx^3}{(1-x)} + pq\left(\frac{u-x}{(1-x)}\right)^2 \right) t\,bh^2$$

or if the expression in brackets is called K

$$M_0 = K\, t\, bh^2$$

Professor Hatt replaces the above complicated form by a closely approximate straight line formula :

If P = percentage of steel = $100p$

then for 1 : 2 : 4 stone concrete

$$K = [\,\tfrac{1}{3} + (\,\tfrac{7}{4}u - 1)\,P]$$

or if $u = \tfrac{7}{8}$,

$$K = (\,0.33 + 0.53\,P\,)$$

which for $t = 240$, gives

$$M_0 = (127\,P + 80)\,bh^2 = (165\,P + 104)\,bd^2$$

For 1 : 5 gravel concrete the corresponding expressions are:

$$M_0 = (180\,P + 80)\,bh^2 = (234\,P + 104)\,bd^2$$

The above expressions include tension in concrete, hence they give the bending moment at the "first crack."

The stresses after the concrete ceases to act on the tension side may readily be obtained by omitting the tensile stresses.

The equation for M_0 then becomes:

$$M_0 = bh^2 \left[\tfrac{5}{12} cx^2 + p\, T\, (a - x) \right]$$

CONDRON'S FORMULA.*

Mr. T. L. Condron has lately made a study of 202 tests of concrete steel beams, which have been made in different laboratories in the United States within the last few years. As the result of this comparison, Mr. Condron suggests the following expression for the bending moments at breaking loads:

For corrugated bars:

$$M_0 = (450\, P + 55)\, bd^2$$

and for plain bars:

$$M_0 = (275\, P + 55)\, bd^2$$

He further suggests a general empirical formula for steel, having an elastic limit of 60,000 lbs. per square inch.

$$M_0 = \frac{T\, bd^2}{60000}\, (500\, P + 50)$$

$$\text{or } M_0 = \frac{T\, bd^2}{120}\, (P + \tfrac{1}{10})$$

$$\text{whence } d = \sqrt{\frac{120\, M_0}{T\, b}\, (P + \tfrac{1}{10})}$$

$$S = \frac{1.2\, M_0}{T\, d} - \frac{bd}{1000}$$

The following table has been computed from the above formula:

T	$P = 0.5\%$		$P = 0.75\%$		$P = 1.0\%$		$P = 1.25\%$	
	M_0	C	M_0	C	M_0	C	M_0	C
60,000	$300 bd$	1667	$425 bd^2$	2110	$550 bd^2$	2500	$675 bd^2$	2880
40,000	200 "	1110	283 "	1406	367 "	1667	450 "	1920
30,000	150 "	833	212 "	1055	275 "	1250	338 "	1440
20,000	100 "	555	142 "	703	183 "	833	225 "	960
18,000	90 "	500	128 "	633	165 "	750	203 "	864
15,000	75 "	417	106 "	528	137.5"	625	169 "	720
12,000	60 "	333	85 "	422	111 "	500	135 "	576
10,000	50 "	278	71 "	352	92 "	417	112 "	480

*Journal West. Soc. Eng'rs, 1904, p. 415.

TALBOT'S FORMULA.

Professor A. N. Talbot, of the University of Illinois, as the result of an elaborate series of experiments, has prepared the following simple formula.*

$$M_0 = ST\,(0.906 - 6.5p\,)\,d$$

in which $p = \dfrac{S}{bd}$.

Taking P = percentage of steel = $100p$
and T = 15,000 lbs. per square inch.
Then for a width of 12″ and working loads

$$\frac{M_0}{d^2} = 1800\,P\,(0.906 - 0.065\,P)$$

This equation gives a flat curve, which agrees closely with the straight line as given by Mr. Condron.

CONVENIENT WORKING METHOD.

The writer has found the following a simple method for computations. The formulas are similar in form to those given in Taylor and Thompson's excellent treatise on concrete.

Assumptions.

All tensions carried by steel.

The compressive stress curve is the one given by Capt. Sewall, so that its area is equal to ⅝ of the maximum fibre stress multiplied by the area in compression and the distance of its centroid from the top is ⅜ of the distance from the top to the neutral axis. The formulas are for working stresses:

Symbols used:—

b = width of beam.
d = distance from top to centre of steel.
T = unit stress in steel.
C = Max. unit compressive stress in concrete.

$$e = \frac{E_s}{E_c} = \frac{\text{modulus of steel.}}{\text{modulus of concrete.}}$$

$$p = \frac{\text{cross-section of steel.}}{\text{cross-section of beam above centre of steel.}}$$

$$x = \frac{\text{distance from top to neutral axis.}}{\text{distance from top to centre of steel.}}$$

M_0 = moment of resistance.

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*Journal West. Soc. Eng'rs, 1905, p. 329.

Then, from our general discussion, we readily deduce the following equations:

$$x = \frac{1}{1 + \frac{T}{Ce}} = \sqrt{\tfrac{4}{5}\,ep\,(\tfrac{4}{5}\,ep + 2)} - \tfrac{4}{5}\,ep \qquad (A)$$

$$\frac{T}{C} = \frac{e\,(1 - x)}{x} \qquad (B)$$

$$p = \frac{5}{8\,\frac{T}{C}\left(1 + \frac{T}{Ce}\right)} \qquad (C)$$

$$M_0 = \tfrac{5}{8}\,C\,bd^2\,(1 - \tfrac{2}{5}\,x) = p\,T\,bd^2\,(1 - \tfrac{2}{5}x) \qquad (D)$$

CASE I.

To design a beam of any assumed quality of steel and concrete to carry a given load.

Here we have given T, C, E_s and E_c

To find p, b, d, for any definite M_0

Example:—

Let $C = 700$ lbs per sq. in. $T = 14{,}000$ lbs. per sq. in.

$E_s = 30{,}000{,}000$, $E_c = 3{,}000{,}000$.

then $\frac{T}{C} = 20$ and $e = 10$.

From (A), $x = \frac{1}{1 + \frac{T}{Ce}} = \frac{1}{1 + \frac{20}{10}} = \frac{1}{3}$

From (C), $p = \frac{5}{8 \times 20 \times (1 + \frac{20}{10})} = 0.0104$ or 1.04%

$M_0 = \frac{5}{8}\,700 \times \frac{1}{3}\,(1 - \frac{2}{5} \times \frac{1}{3}) = 126.7\,bd^2$

which gives bd^2 for any value of M_0 that may be given.

CASE II.

To determine the stresses in a given beam under any given loading.

Here we have given b, d, p, also E and E_c

To determine T and C for any value of M_0

Example:—

Let $b = 6''$, $d = 10''$, $p = 0.0104$

$E_s = 30{,}000{,}000$ $E_c = 3{,}000\,000$, so that $e = 10$

Then from (A):

$$x = \sqrt{\tfrac{4}{5}\,0.104\,(\tfrac{4}{5}\,0.104 + 2)} - \tfrac{4}{5}\,0.104 = \tfrac{1}{3}$$

and from (B):

$$\frac{T}{C} = 10\,\frac{1 - \frac{1}{3}}{\frac{1}{3}} = 20$$

If T is, say, 14,000 lbs. per square inch, C will be 700 lbs. per square inch from (D):

$$M_0 = \tfrac{5}{8}\ 700 \times 6 \times 100\left(1 - \tfrac{2}{5} \times \tfrac{1}{3}\right) = 227500 \text{ inch lbs.}$$

General Considerations.—The above discussion has been confined to the determination of the moment of resistance in a concrete steel beam and the stresses parallel to its axis. The magnitude of the bending moment at different points in the length of the beam (which the moment of resistance must, of course, equal), has not been taken up.

Furthermore, certain other important internal stresses have not been touched upon.

The bending moments at different points under the existing dead and live loads are computed in the same way as for timber or steel beams, but there is almost always, at least, partial continuity over the supports.

On this account, there will be a negative bending moment which may be as large as $\frac{1}{24} Wl$ at the supports for a beam of length l, carrying a total load W uniformly distributed. The tensile stresses thus produced in the upper part of the beam, must be resisted by inserting horizontal steel members near the top over the supports, and extending a proper distance each way. The bending moment at the centre of the span has a maximum value of $\frac{Wl}{8}$ for uniform loads if there is not continuity and a maximum of $\frac{Wl}{24}$ for perfect continuity and full loading on the two adjacent spans.

It is conservative practice to make the moment at the centre equal to $\frac{Wl}{10}$

The internal stresses in concrete and concrete steel beams still require much investigation, but the study of tests already made, and the analogy of iron and wooden beams, have provided some valuable practical information.

In a beam on two supports, the bending moment is at a maximum at the centre and decreases according to the law of loading to the ends. If the depth of the beam is uniform, the stresses parallel with the axis vary according to the same law, becoming zero at the supports. There are, however, other stresses in the beam, analogous to those in the web members of a steel truss or the web plate of a plate girder. These increase from the centre to the supports, and are greatest at any point in a direction making an angle of 45 degrees with the longtitudinal axis of the beam.

There are conjugate tensile and compressive stresses, the former being at right angles to the latter. As the concrete is not capable of withstanding tensile stress, no beams can be considered properly reinforced in which these tensile stresses, as well as those parallel with the axis, are not carried by steel.

Numerous tests of beams not thus reinforced have corroborated the above theory. Such beams almost invariably fail by tension in the concrete near the ends, the cracks being inclined downwards towards the abutment. Although the above reasoning has not been universally adopted, the need of special reinforcing for web stresses near the end is very generally admitted. This has usually consisted of vertical stirrups embedded in the concrete, but not attached to the main reinforcing bar. Web members making an angle of 45 degrees with the horizontal reinforcement, and forming an integral part thereof, have come into use within the past few years. In the opinion of some of the most competent students of this problem, this system of reinforcement has great advantages over any other as to safety, efficiency and an economical utilization of the metal in both the main bars and web members.

There are, naturally, many other debateable questions in connection with concrete steel beams, which cannot be dwelt upon here. It may be worth while to at least mention one of these, namely, the question of the adhesion of the concrete to the embedded steel.

This is essential to the strength of the beam, and can generally be counted upon, although shocks, extreme temperature, etc., may cause a separation of the two materials. For this reason deformed bars have been to some extent introduced in the place of the plain bars previously used. In these bars, by means of slight projections upon the surface, an actual mechanical bond is substituted, which resists the tendency of the concrete to slide along the steel better than the smooth surface of the plain bar.

The same advantage is obtained in the bar with attached web members, as the latter are fixed in the concrete and keep the entire reinforcement from sliding.

The various types of deformed bars, and the reinforcement with integrally attached web members are American contributions to Reinforced Concrete Construction.