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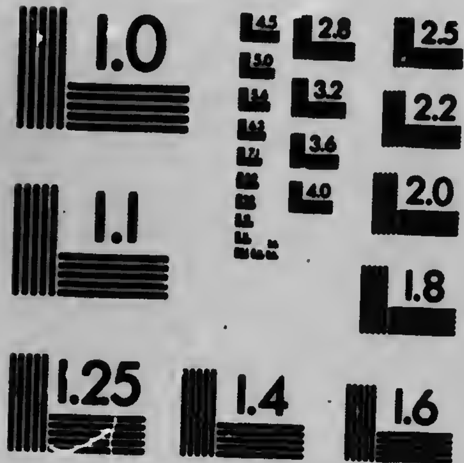

AN

ARITHMETIC

FOR

HIGH SCHOOLS

AND

COLLEGIATE INSTITUTES

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BY

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PREFACE

The preparation of this work on arithmetic was undertaken by me only after much urging from many sides. It would seem that, while much time is given to this subject in our High Schools, the results are not what might be expected. I have thought that if less time were given to the solving of strange and artificial problems and more to the theory of the subject and to the careful working out of normal problems, the subject would have a higher value educationally and practically. Students would be better prepared for dealing with such problems as present themselves later, when they are called upon actually to apply their knowledge. On this account I have given much attention to the theory of the subject, and to methods of computation.

The chapters on commercial applications are somewhat concise, the object being to avoid the confusion too often caused by the presentation of details.

Chapters on Series and Logarithms have been introduced. The difficulties connected with these parts of the subject are not great, certainly not so great as are often found in pointless problems, and in overcoming these difficulties the student will feel that he is making progress. Further, a knowledge of Series makes it possible to treat the subject of Annuities in a direct and natural manner, and allows the student to grasp the full significance of the recurring decimal. It is not necessary to refer to the practical value of a knowledge of Logarithms.

While the responsibility for the plan and the treatment of the subject necessarily rests upon me, I am under obligations to many friends. In particular, I wish to offer my thanks to Dr. Glashan of Ottawa and to Dr. McDonald of the University of California for helpful advice and suggestive criticism. To Dr. Goggin of the Canada Publishing Company I also owe much; to his sound judgment in matters pertaining to education, frequent appeal was made.

Toronto, June 3, 1903. A. T. DeL.

A re-issue of this book has given me the opportunity to make certain corrections and to meet the generally expressed wish that the answers to the examples be given. I may also say that it is proposed to issue, from time to time, at nominal price, sets of graded problems. This will secure freshness and variety in the matter of exercises and, at the same time, furnish additional work for those schools in which it is found advisable to give special attention to the important subject of arithmetic.

A. T. DeL.

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INTRODUCTION

Arithmetic is the science of **number.** The source of number is to be found in the question, **How many?**, asked with respect to a collection of objects admitting a common name. The answer to this question is a number. Originating in this way, number soon offers itself as a measure of **quantity.** For example, if a length, call it one foot, be chosen as **unit,** *i.e.*, a standard with which to compare other lengths, and if a second length contains the unit five times, the **measure** of the latter length is said to be five. The length itself is five feet. The number five is here an abstraction reached through comparing the two lengths in the way indicated; it gives the **ratio** of the latter length to the unit. In the **quantity** five feet there appear two elements, one foot,—the unit,—and the number five,—the measure. Hence a quantity is defined when the unit and the measure are given.

Arithmetic investigates the relations among numbers and the operations on and with them. When physical quantities admit measurement, relations among the numbers which are their measures, reveal facts about the actual quantities, and arithmetic finds practical applications.

In dealing with numbers a first requisite is a **numeration** or a system of names. The reader is already familiar with the system,

one, two, three, . . . ten, . . . hundred, . . .

and with the special part played in it by the number ten. The next need is a **notation** or method of writing numbers. The reader is acquainted also with the symbols or figures,

1, 2, 3, . . . 9,

employed to denote one, two, three, . . . nine, and with the decimal notation in which these nine symbols or digits, combined with the symbol 0 (read zero or nought), serve to denote larger numbers. For example, 4305 means four thousand, three hundred, no tens and five, or four thousand, three hundred and five. Here the 4, say, has an intrinsic value, four, and a value due to its place, the complete value appearing in the expression four thousand.

A numeration and a notation being accepted, the student is ready to make a study of the processes of arithmetic and to apply the results to practical ends. In the treatment of the subject that is to follow it will be assumed that the reader already possesses a knowledge of the elementary processes, so that the chief concern will be the presentation of the theory of arithmetic and the application to problems of its operations and results.

The plan followed may be briefly sketched. In Part I is given a connected elementary treatment of the ordinary theory. Part II deals with applications which afford a certain training and which in the main can be described as of practical use. In Part III appear certain developments and more difficult applications, an acquaintance with which is to be supposed in one who may wish to become a teacher of the subject.

PART I

THE SIMPLER THEORY

CHAPTER I

THE SIMPLE RULES

1. **Addition.** In one bag snppose there are 5 marbles and in another 4; snppose also that we wish to know how many marbles there are in the two bags. By connting it is fonnd that there are 9; we have fonnd a **sum** or made an **addition.** The 5 marbles and the 4 marbles are **addends** and their **sum** is 9 marbles. We may now say that the sum of \$5 and \$4 is \$9, or, passing to nnmbers, that the sum of 5 and 4 is 9. In symbols, this is written

$$5+4=9.$$

The sign of addition $+$ is called *plus*, and the relation $5+4=9$ is read *five plus four equal nine.*

In the same way we may have the sum of several quantities or numbers. For example,

$$\$3+\$9+\$12+\$23=\$47;$$
$$8+4+7+9=28.$$

For the addition of several numbers the following **laws** can be seen to hold in any given case:

(1.) The sum is not affected by any change in the order of the **terms.**

Ex. $5+9+7=5+7+9=9+5+7=21.$

(II.) The terms may be grouped into partial sums without affecting the result.

Ex. $5+7+3+8=(5+7)+(3+8)=12+11=23.$

Of the truth of these laws we are at once satisfied, but it would be out of place here to raise the question of a proof.

The appearance of zero as an addend has no effect on the sum. In symbols,

$$7+0=7=0+7.$$

In the case of the addition of several quantities, the units must be the same, if the total is to be given as one sum.

2. **Multiplication.** Suppose that in an addition all the addends are the same as in $7+7+7+7=28$. Here we say that there are four sevens, or that 7 is taken four times. Hence 4 times 7 make 28 and this fact is written

$$7\times4=28, \text{ or } 7.4=28.$$

We have performed the **multiplication** of 7 by 4; 7 is the **multiplicand,** 4 the **multiplier**, and 28 the **product.** Plainly also $\$7\times4=\$28$. Hence in a multiplication the multiplicand may be either a number or a quantity; the multiplier however is a number, indicating *how many times* the multiplicand is to be taken. If the multiplicand is a number the product is a number; if it is a quantity the product is a quantity measured by the same unit.

We pass easily to the idea of the product of several numbers. For example $3\times7\times5$ is to be understood as meaning that 3 is to be multiplied by 7 and the result multiplied by 5, so that the product is 21×5 or 105.

For the product of several numbers, the following laws are seen to hold in any given case:

(I.) The product is not affected by any change in the order of the **factors.**

Ex. $3\times7\times5=3\times5\times7=7\times3\times5=105.$

(II.) The factors may be collected into groups without affecting the result.

Ex. $2\times5\times7\times3=(2\times5)\times(7\times3)=10\times21=210.$

(III.) The product of the sum or the difference of two numbers and a third number is equal to the sum or the difference of the products of each of the two numbers and the third number.

Ex. $(7-5)\times3=(7\times3)-(5\times3)=21-15=6.$

As in the case of addition, we are at once satisfied of the truth of these statements, but the question of the proof is not raised.

If zero occurs as a multiplicand or as a multiplier the product is zero. In symbols,

$$0\times11=0=11\times0.$$

If a quantity is multiplied by several numbers in succession, the quantity must be retained as the multiplicand; the remaining factors may be taken in any order.

3. **Subtraction.** The addition of 19 and 13 yields the sum 32. This result furnishes also an answer to the two equivalent questions:

(1) *What number added to 19 will give the number 32?*
(2) *If 19 be taken from 32 what number remains?*

The answer is 13, and this fact is written

$$32-19=13.$$

The process of finding such a number is called **subtraction.**

The sign of subtraction – is called *minus*, and the relation 32 – 19 = 13 is read *thirty-two minus nineteen equals thirteen*.

Plainly, then, *subtraction is the inverse of addition*. In the subtraction just made 32 is called the **minuend**, 19 the **subtrahend**, and 13 the **remainder** or the **difference**.

The minuend is not to be less than the subtrahend. In the subtraction, 8 – 8, the remainder is 0, in accordance with the relation 8+0=8.

In the subtraction of quantities, as in their addition, the same unit must be employed in the statement of the quantities.

4. **Division.** The multiplication of 13 and 17 yields the product 221. This result furnishes also answers to the questions:

(*1*) *What must 13 be multiplied by to give the product 221?*

(*2*) *What number multiplied by 17 will give the product 221?*

The answer to (*1*) is 17, to (*2*) is 13. Since 13×17 =17×13, the two questions are seen to be essentially one, which may be stated in general terms thus: *Given the product of two numbers, and one of the numbers, what is the other number?* The process of finding this number is called **division**. Thus *division is the inverse of multiplication*.

The divisions just made are written in symbols thus:

$$221 \div 13 = 17, \text{ or } \tfrac{221}{13} = 17;$$

$$221 \div 17 = 13, \text{ or } \tfrac{221}{17} = 13.$$

In the division $221 \div 13 = 17$, 221 is called the **dividend**, 13 the **divisor**, and 17 the **quotient**, *i.e.*, the number telling *how many times* 13 is contained in 221.

If the original multiplication is concerned with quantity as in

$$\$13 \times 17 = \$221,$$

the two inverse problems implied in questions (*1*) and (*2*) yield *two distinct interpretations*. First. $\$221 \div \$13 = 17$ means that \$13 is contained in \$221 seventeen times. Next, $\$221 \div 17 = \$13$ means that the one-seventeenth part of \$221 is \$13. It is plain that either of these results follows from the other; further, since they both follow at once from the relation, $13 \times 17 = 221$, (among the *numbers* 13, 17, 221), there is at bottom only *one process*.

The two interpretations may be illustrated by the following simple problems:

(1) How many yards of cotton at 9 ct. a yard may be bought for 63 ct.?

Here, for every time 9 ct. is contained in 63 ct. one yard may be bought.

The number of times 9 ct. is contained in 63 ct.

$$= \frac{63 \text{ ct.}}{9 \text{ ct.}} \text{ or } 7;$$

$\therefore$ 7 yd. may be bought.

(2) If 7 yd. of cotton cost 63 ct., find the cost of 1 yd.

Here, 7 yd. cost 63ct.;

1 yd. is the one-seventh part of 7 yd.;

$\therefore$ cost of 1 yd. = one-seventh of the cost of 7 yd.,

$$= \frac{63 \text{ ct.}}{7} = 9 \text{ ct.}$$

Since the product of 0 and any number is 0, it is plain that a division by 0 is without meaning. Hence division by zero is not an admitted operation.

The operations of addition, subtraction and multiplication, as explained, can always be performed; that is, the results are always numbers or quantities of the same kind as those appearing in the operations. That this is not true of division is shown by the following example:

Ex. If 3 yd. of cotton cost 25 ct. find the cost of 1 yd.

The cost of 1 yd. is one-third of 25 ct.

The formal division is:

$$\begin{array}{r} 3)25 \text{ ct.}(8 \text{ ct.} \\ \underline{24 \text{ ct.}} \\ 1 \text{ ct.} \end{array}$$

Hence the cost is 8 ct. and one-third of 1 ct.; the division of the number 1 by 3 can only be indicated and the result of the division of 1 ct. by 3, namely, one-third of 1 ct., we denote by $\frac{1}{3}$ ct. The cost of 1 yd. is then $8\frac{1}{3}$ ct., which is read eight and one-third cents.

The result in the example just considered involves the quantity $\frac{1}{3}$ ct. The unit of money here is one cent, and $\frac{1}{3}$, serving to measure the quantity $\frac{1}{3}$ ct., we shall call a number. Such a number will be called a **fractional number**, or simply a **fraction**, to distinguish it from those previously occurring which will be called **integral numbers** or **integers.**

Consider next the example:

Ex. If 3 lb. of sugar are worth 17 ct. find the value of 1 lb.

As before we denote the result by $5\frac{2}{3}$ ct., which is read five and two-thirds cents, and for the same reason $\frac{2}{3}$ is called a fraction.

In a formal division as

$$\begin{array}{r} 3)17(5 \\ \underline{15} \\ 2 \end{array}$$

we may say that the quotient is $5\frac{2}{3}$, or, if we wish to avoid fractions, that the quotient is 5 and the **remainder** is 2.

It is to be noted that the divisions

$$\frac{\$15}{\$3}, \quad \frac{15 \text{ yd.}}{3 \text{ yd.}}, \quad \frac{15 \text{ tens}}{3 \text{ tens}}, \quad \frac{15}{3},$$

all yield the same quotient 5.

5. **Powers.** A product as $7 \times 7 \times 7 \times 7$ is generally written 7^4. This number is called the **fourth power** of 7 and is read **seven to the fourth**. The number 4, indicating how many times the factor occurs, is called the **index** of the power, or the **exponent**. Usually 7^2, 7^3, are read **seven squared, seven cubed** and called **the square of seven, the cube of seven.**

It is easy to shew that

(1) $7^3 \times 7^5 = 7^{5+3} = 7^8$,
(2) $3^{11} \div 3^7 = 3^{11-7} = 3^4$,
(3) $(5^3)^7 = 5^{3 \times 7} = 5^{21}$.

The reasoning employed in proving these relations is general and we may state the following **laws of indices:**

(*I.*) *The product of two powers of the same number is a power of that number whose index is the sum of the indices of those powers.*

(*II.*) *The quotient of the power of a number by a power of that number of lower index is a power of that number whose index is the difference of the indices of those powers.*

(*III.*) *A power of a number, raised to any power, is a power of that number whose index is the product of the indices of those powers.*

6. **Important Theorems.** The following theorems are given formal statement because of the frequent use made

of them. They are implied in the meanings we attach to the simple rules and may be verified in any given case.

(*I.*) *If equals are added to or subtracted from equals the wholes or the remainders are equal.*

(*II.*) *If equals are multiplied or divided by equals the products or quotients are equal.*

(*III.*) *If the multiplier is multiplied or divided by any number the product is multiplied or divided by that number.*

(*IV.*) *If the divisor is multiplied or divided by any number the quotient is divided or multiplied by that number.*

Exercises

1. Shew that the number

137594

may be read, one thousand three hundred and seventy-five hundred and ninety-four.

Give other ways of reading the number.

2. Explain the successive steps in the subtraction

$$\begin{array}{r} 30045 \\ \underline{13789} \\ 16256 \end{array}$$

3. Write out an explanation of the multiplication

$$\begin{array}{r} 5389 \\ \underline{2037} \\ 37723 \\ 16167 \\ \underline{10778} \\ 10977393 \end{array}$$

4. Perform the multiplication

537892×441639

employing only three partial multiplications.

5. Show that

$$47325 = 9999\times4+999\times7+99\times3+9\times2+(4+7+3+2+5).$$

Hence shew that if this number is divided by 9 the remainder is the same as when the sum of its digits is divided by 9.

Is this statement general ? If so, offer reasons for its general acceptance.

If the following numbers are divided by 9 find the remainders in each case :

37259, 493825, 671238,
75416, 275083, 439647.

6. When 53824 is divided by 9 the remainder is 4 and when 49875 is divided by 9 the remainder is 6 ; shew that when the product 53824 × 49875 is divided by 9 the remainder is the same as when the product 4×6 is divided by 9 and is therefore 6.

7. A person finds 48011798 as the product of 8734 and 5497; shew without multiplying that he is in error.

8. 3794324 = 100×37943 + 24. Now 100×37943 being an integral number of hundreds is divisible by 4 ; 24 is also divisible by 4. Thus the original number is divisible by 4.

What conclusion is to be drawn from this ?

In a similar way obtain the following results :

(*a*) A number is divisible by 8 if the number by its last three digits is divisible by 8.

(*b*) A number is divisible by 5 if the number given by its last digit is divisible by 5, *i.e.*, if the last digit is 0 or 5.

(*c*) A number is divisible by 2 if the number given by its last digit is divisible by 2.

(*d*) A number is divisible by 25 if the number given by its last two digits is divisible by 25.

(*e*) A number is divisible by 125 if the number given by its last three digits is divisible by 125.

9. Obtain the quotients in the following cases without dividing :

(1) 37475÷25, (2) 537625÷125, (3) 376÷2, (4) 7395÷5.

10. Obtain the following products by a simple division :

(1) 5937×5, (2) 39587×25, (3) 473967×125.

11. Formulate laws for subtraction and division analogous to those stated for addition and multiplication, indicating any limitations.

12. How many integers between 1 and 300 are exactly divisible by 13?

13. How many integers between 200 and 400 are exactly divisible by 29?

14. What is the least number which (1) added to, (2) subtracted from, 3597 will give a result divisible by 37?

15. Shew that there is no essential difference between simple addition and compound addition.

Compare the other simple and compound rules.

16. What number divided by 1293 will give 37 as quotient and 597 as remainder?

17. Divide 75382 by 63 employing two short divisions, explaining how the remainder is found.

18. Divide 89372154 by 693 employing short divisions only, explaining how to find the remainder.

19. The cost of 7 yd. of cotton at 5ct. a yd. is 5ct.×7 or 35ct. Shew that this may be regarded in such a way as to give for result 7ct.×5 or 35ct.

20. Shew that by repeated divisions by 7 the number 76598 may be expressed in the form,

$$4\times7^5+3\times7^4+6\times7^3+2\times7^2+1\times7+4$$

i.e., $$4.7^5+3.7^4+6.7^3+2.7^2+1.7+4.$$

Here no digit higher than 7 is employed. The given number is said to be

436214

in the *scale of seven*, just as, for example, 3725 in the ordinary scale of 10 means

$$3.10^3+7.10^2+2.10+5.$$

21. Shew that 14326 in the scale of seven is equal to 30124 in the scale of 6.

CHAPTER II

MEASURES AND MULTIPLES

1. Measures or Factors, and Multiples. When one number is contained an integral number of times in another, the former is a **measure** or **factor** of the latter; the latter is a **multiple** of the former. For example, 5 is a factor of 15; it may be employed to measure 15, for 15=3 fives; also 15 is a multiple of 5.

In this chapter the word factor or measure will mean integral factor or measure, and divisible will mean exactly divisible.

A number which has no other factor than itself and unity is a **prime number** or a **prime**. A number which has factors other than itself and unity is a **composite number**.

A composite number can always be represented as a product of prime numbers.

Ex. $15561=5187\times3=1729\times3\times3=247\times7\times3\times3=19\times 13\times7\times3\times3$.

It is plain in any given case that:

A composite number can be expressed as a product of prime factors in only one way.

A formal proof of this theorem is not here offered.

The truth of the following theorem is also readily seen:

A composite number is divisible by every number formed by multiplying together any two or more of its prime factors; unity, the prime factors themselves, and all the numbers that can be thus formed make up all the factors of the number.

In the case of very large numbers a great many trials may be necessary to determine whether or not the number is composite, and the resolution into factors may be a tedious operation.

The multiples of a number are formed by multiplying it by 1, 2, 3, 4, ... ; these are called the first, the second, the third, ... multiples of the given number. Manifestly the number of such multiples is unlimited.

2. **Common Measures.** The numbers 15 and 35 written as products of prime factors are 5×3 and 5×7. Thus 5 is a **common factor** or a **common measure** of 15 and 35. By what has been seen, it is the only common measure, other than 1, of these numbers.

Next take the numbers 60 and 84. Written as products of prime factors these are $2\times2\times3\times5$, $2\times2\times3\times7$. Then all the common measures of 60 and 84 are seen to be 1, 2, 3, 2×2, 3×2, $3\times2\times2$. Of these $3\times2\times2$ or 12 is the **greatest common measure.** It is evident then that:

All the common measures of two numbers are factors of the greatest common measure, and every factor of the greatest common measure is a common measure of the numbers.

It may be that, when the numbers are resolved into prime factors, they are seen to have no common factor other than 1; the numbers are then said to be **prime to each other** or to be **relative primes.**

We thus see that all the common measures, and therefore the greatest common measure of two numbers may be at once found if the numbers are first resolved into prime factors. As the resolution into prime factors may be difficult, we shall explain the method of finding the greatest common measure of two numbers given by Euclid [The Elements, Book vii, Prop. 2.].

Lemma. *Every common factor of two numbers is a factor of the sum or difference of any multiples of these numbers.*

Consider the numbers 30 and 45; they have a common factor 5, for $30=5\times6$ and $45=5\times9$. Then every multiple of 30 or 45 is an integral number of fives; hence also the sum or difference of any multiples of 30 and 45 is an integral number of fives, *i.e.*, has five as a factor. The case in which the difference of the multiples should happen to be zero is included in this statement. Though here particular numbers have been taken the reasoning is general.

Now take the numbers 299 and 943 and examine the operation:

```
299)943(3
    897
     46)299(6
        276
         23)46(2
            46
```

Here 299, the smaller, is divided into 943, the larger, the quotient being 3 and the remainder 46 which is necessarily less than 299. Then 46 is divided into 299, with quotient 6 and remainder 23. Then 23 is found to divide 46 exactly.

Now, by the lemma, every factor common to 299 and 943 is a factor of $943-299\times3$ or 46 and is therefore a common factor of 46 and 299. Also every common factor of 46 and 299 is a factor of $46+299\times3$ or 943, and hence is a common factor of 299 and 943. Thus 46 and 299 have precisely the same common factors as 299 and 943. In like manner it is seen that 23 and 46 have the same common factors as 46 and 299, and therefore as 299 and 943. But 23 is itself the greatest common measure of

28 and 46; it is therefore the greatest common measure of 299 and 943.

As in the case of the lemma, particular numbers have been taken but the reasoning is general.

To find the G. C. M. (denoting thus the greatest common measure) of three numbers, we resolve the numbers into their prime factors and the G. C. M. is at once recognized. Or in the case of numbers not readily factored, we find the G. C. M. of the first two numbers by Euclid's method, and then the G. C. M. of this and the third number; the number thus found is the G. C. M. sought. For every common measure of the first two numbers is a factor of their G.C.M.; the greatest number that will divide exactly this G. C. M. and the third number is then the G. C. M. of the three numbers. The case of more than three numbers will now present no difficulty.

3. **Common Multiples.** Consider the numbers 4 and 9, *i.e.*, 2×2 and 3×3; these numbers are prime to each other. Every multiple of 4 must contain the factors 2×2; every multiple of 9 must contain the factors 3×3. Since in the two sets of factors there is no factor in common, the smallest number that will contain both 4 and 9 is $2\times2\times3\times3$ or 4×9 *i.e.*, 36. All other numbers that will contain both 4 and 9 will have factors additional to $2\times2\times3\times3$, *i.e.*, will be multiples of 36. The **least common multiple** of 4 and 9 is then 36. In like manner we can shew in any given case that:

The least common multiple of two numbers, prime to each other, is their product, and all other common multiples are multiples of their least common multiple.

Take next 20 and 35, *i.e.*, $5\times2\times2$ and 5×7. Here there is one factor, 5, common to 20 and 35 Every multiple of 20 must have among its prime factors both 5 and 2×2; every multiple of 35 must have among its prime factors both 5 and 7. The smallest number fulfilling both requirements is $5\times(2\times2)\times(7)$ or 140. This then is the L. C. M. (denoting thus the least common multiple) of 20 and 35. As in the earlier case, it is seen that all other common multiples of 20 and 35 are multiples of 140, their L.C.M. We may in like manner treat any two numbers whose prime factors may be found.

It has just been shewn that the L.C.M. of 20 and 35, *i.e.*, of $5\times(2\times2)$ and 5×7, is $5\times(2\times2)\times7$, and 5 is their G.C.M. Now $5\times(2\times2)\times7=\frac{5\times(2\times2)\times5\times7}{}=\frac{20\times35}{}$.

It thus appears that:

The L. C. M. of two numbers is the quotient of their product by their G.C.M.

For any two given numbers this may be shewn to be true.

If the numbers are not easily factored, their G.C.M. may be found by Euclid's method.

The L. C. M. of several numbers whose prime factors are known may at once be written down. It is also easy to shew that the L.C.M. of three given numbers may be found by finding first the L.C.M. of two of the numbers and then the L.C.M. of this and the third number.

Exercises

1. Find the G. C. M. of 2509 and 5597 and give a complete statement of the reasons for concluding that the result found is the G. C. M.

2. Shew that to determine whether 227 is a prime or not it is necessary to test for its divisibility by 2, 3, 5, 7, 11, 13 only.

3. Shew by indirect reasoning that any common multiple of two integral numbers is a multiple of their L. C. M.

4. Given three numbers no two of which have a common factor, as 24, 55, 91, shew that their L. C. M. is their product.

5. Find the least number which when divided by 15, 18, 24 will in each case leave a remainder 13, and find other numbers that satisfy this condition.

6. Find the least integer by which 720 can be multiplied to give a product which is the square of some number.

7. Find the least integer by which 1056 can be divided to give a quotient which is the square of some number.

8. Find the least integer by which 75 can be multiplied to give a product which is the cube of some number.

9. Find the least integer by which 2160 can be divided to give a quotient which is the cube of some number.

10. The product of four consecutive integers is 73440; find these integers.

11. Find the G.C.M. of 64753 and 208909 and shew that the work might have been abbreviated thus:

64753	208909
6153	14650
1465	2344
586	879
	293

12. In the preceding example shew that after the first division the remainder may be divided by 10 and the work shewn thus:

64753	208909
6153	1465
293	

CHAPTER III

VULGAR FRACTIONS

1. **Meaning of Fraction.** In the treatment of division attention was directed to the problems:

(1) If 3 yd. of cotton cost 25ct., find the cost of 1 yd.;

(2) If 3 lb. of sugar are worth 17ct., find the value of 1 lb.

The results were found to be $8\frac{1}{3}$ct., $5\frac{2}{3}$ct. We agreed to regard $\frac{1}{3}$ and $\frac{2}{3}$ as numbers because they served to measure quantities, namely, one-third of one cent and one-third of two cents; they were called **fractional numbers** or **fractions** to distinguish them from the numbers previously met, namely, **integral numbers** or **integers.**

The fraction $\frac{1}{3}$ is read one-third: by this is meant one-third of one, *i.e.*, one of the three equal parts that make up the unit. This is the essential property of $\frac{1}{3}$ and it is expressed thus: $\frac{1}{3}\times3=1$, or $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$.

The fraction $\frac{2}{3}$ is read two-thirds: by this is meant one-third of two, *i.e.*, one of the three equal parts that make up two units. This is the essential property of $\frac{2}{3}$ and it is expressed thus: $\frac{2}{3}\times3=2$, or $\frac{2}{3}+\frac{2}{3}+\frac{2}{3}=2$. Now one-third of two means one-third of one together with one-third of one, *i.e.*, two of the three equal parts that make up the unit. Hence $\frac{2}{3}$ has either of the two meanings:

(1) *The quotient of 2 by 3, i.e., one-third of two;*

(2) *Two-thirds, or two of the three equal parts that make up the unit.*

The latter meaning is the one generally given to $\frac{2}{3}$. On this account the integers 2 and 3 used in writing the fraction $\frac{2}{3}$ are called, the former the **numerator**, the latter the **denominator** of the fraction. The equivalence of (*1*) and (*2*) is expressed thus:

$$\frac{1}{3} \text{ of } 2 = \frac{2}{3}.$$

We see then that while a fraction arises through the division of an integer by an integer, and therefore denotes a quotient, it may also be regarded as denoting one or more of the equal parts of the unit.

2. Fundamental Theorem. Consider the fractions

$$\frac{2}{3} \text{ and } \frac{10}{15}$$

noting that $10=2\times5$, $15=3\times5$.

Here $\frac{10}{15}$ means the quotient of 10 by 15, *i.e.*, of 2 fives by 3 fives, which is the quotient of 2 by 3 or the fraction $\frac{2}{3}$. Hence

$$\frac{2}{3}=\frac{10}{15}.$$

Or we may reach this result thus:

3 thirds = 15 fifteenths, (each denoting the unit);
$\therefore$ 1 third $= \frac{1}{3}$ of 15 fifteenths,
$= 5$ fifteenths;
$\therefore$ 2 thirds = 10 fifteenths,
i.e., $\frac{2}{3}=\frac{10}{15}$.

The reasoning is general and we have the theorem:

If the numerator and the denominator of a fraction be each multiplied or divided by the same number the value of the fraction is not changed.

A fraction is in its **lowest terms** when the numerator and the denominator are prime to each other.

3. Addition and Subtraction. To find the value of

$$\frac{2}{7}+\frac{3}{7}$$

we may say that this sum means $\frac{1}{7}$ of 2 together with $\frac{1}{7}$ of 3, or in all $\frac{1}{7}$ of 5, *i.e.*, $\frac{5}{7}$. Or we may say that the meaning is 2 sevenths and 3 sevenths, *i.e.*, 5 sevenths or $\frac{5}{7}$. The method of adding fractions with the same denominator is then manifest.

Next to find the valne of

$$\tfrac{2}{5}+\tfrac{3}{7}.$$

By the fundamental theorem,

$$\tfrac{2}{5}=\tfrac{2\times 7}{5\times 7}=\tfrac{14}{35}\,;\ \tfrac{3}{7}=\tfrac{3\times 5}{7\times 5}=\tfrac{15}{35}.$$
$$\therefore \tfrac{2}{5}+\tfrac{3}{7}=\tfrac{14}{35}+\tfrac{15}{35}=\tfrac{29}{35}.$$

Hence to add fractions with different denominators we express them as fractions with the same denominator. The simplest common denominator is the L. C. M. of the given denominators.

The qnestion of subtraction may be treated in the same way.

4. **Multiplication.** Before touching the question of multiplication of fractions let us seek the value of $\frac{2}{3}$ of $\frac{5}{7}$. The meaning is evident; just as we construct mentally $\frac{2}{3}$ of 1 we may construct mentally $\frac{2}{3}$ of any quantity. It is evident that $\frac{2}{3}$ of any quantity is twice as mnch as $\frac{1}{3}$ of that quantity. Now,

$$\tfrac{5}{7}=\tfrac{5\times 3}{7\times 3}=\tfrac{15}{21}\,;\ \text{(15 twenty-firsts)}$$
$$\therefore \tfrac{1}{3}\text{ of }\tfrac{5}{7}=\tfrac{1}{3}\text{ of }\tfrac{15}{21}=\tfrac{5}{21}\,;$$
$$\therefore \tfrac{2}{3}\text{ of }\tfrac{5}{7}=\tfrac{5}{21}+\tfrac{5}{21}=\tfrac{10}{21}=\tfrac{2\times 5}{3\times 7}.$$

An expression as $\frac{2}{3}$ of $\frac{5}{7}$ is called a **compound fraction**, and the rule for finding its value is at once seen.

Let us now turn to the multiplication of fractions. Take first

$$\tfrac{5}{23}\times 3.$$

The meaning attached to mnltiplication gives

$$\tfrac{5}{23}\times 3=\tfrac{5}{23}+\tfrac{5}{23}+\tfrac{5}{23}=\tfrac{5\times 3}{23}=\tfrac{15}{23}$$

and such a multiplication presents no difficulty.

Next consider

$$\tfrac{5}{7}\times\tfrac{2}{3}.$$

Here $\frac{2}{3}$ appears as a multiplier. Since up to this point the multiplier has always indicated the number of times the multiplicand has been taken (as an addend), the operation proposed does not come within the range of multiplication as thus previously understood. Now we have seen that

$$\tfrac{5}{7}\times 2=\tfrac{10}{7}.$$

In the multiplication proposed the multiplier is $\frac{2}{3}$, *i.e.*, one-third of 2. Therefore, in accordance with our idea of multiplication we say that the product when the multiplier is $\frac{2}{3}$ is one-third of the product with a multiplier 2. Hence

$$\tfrac{5}{7}\times\tfrac{2}{3}=\tfrac{1}{3}\text{ of }\tfrac{10}{7}=\tfrac{5\times 2}{7\times 3}.$$

Thus $\frac{5}{7}\times\frac{2}{3}$ and $\frac{2}{3}$ of $\frac{5}{7}$ have the same value, and a meaning has been assigned to multiplication of fractions. The rule may be stated as follows:

The product of two fractions is a fraction whose numerator is the product of their numerators, and denominator the product of their denominators.

The following examples will help the student to see that in the multiplication thus defined the older meaning has been conserved, and that the rules for working problems demand no new statement

Ex. 1. Find the cost of $\frac{3}{4}$ yd. of cloth at \$$\frac{4}{5}$ a yd.

To retain the language suited to a like problem involving integers only we should say

The cost = the product of \$$\frac{4}{5}$ by $\frac{3}{4}$

= \$$\frac{4}{5}\times\frac{3}{4}$.

But we know that

The cost = $\frac{3}{4}$ of \$$\frac{4}{5}$.

Now the meaning given to multiplication of fractions requires that

[illegible] of [illegible] = [illegible] × [illegible].

Therefore the rule in such problems covers the case in which the numbers are fractional.

Ex. 2. Find the area of a rectangle [illegible] ft. by [illegible] ft.

The rule in like problems involving integers only makes it desirable to say

The area = ([illegible] × [illegible]) square feet.

But if a figure is constructed it is readily seen that

The area = [illegible] of [illegible] of 1 square foot.

Therefore as in the preceding example the rule in such problems covers the case in which the numbers are fractional.

The product of several fractions may now easily be found.

5. Division. We know that

$$\tfrac{2}{3}\times\tfrac{5}{7}=\tfrac{10}{21}.$$

As in the case of integers we shall say

$$\tfrac{10}{21}\div\tfrac{5}{7}=\tfrac{2}{3}.$$

To obtain this result we may then divide 10 by 5 and 21 by 7.

If as in $\tfrac{3}{8}\div\tfrac{5}{11}$, these divisions are not both exact, we may proceed thus:

$$\tfrac{3}{8}\div\tfrac{5}{11}=\tfrac{3\times5\times11}{8\times5\times11}\div\tfrac{5}{11}=\tfrac{3\times11}{8\times5}=\tfrac{3}{8}\times\tfrac{11}{5}.$$

Hence the rule:

The quotient of one fraction by another is equal to the product of that fraction by the reciprocal of the other.

6. Complex Fractions. The fractions dealt with up to this point are the quotients of integers. We may now consider quotients such as $\dfrac{2\frac{2}{3}}{3\frac{1}{4}}$, for

$$\frac{2\frac{2}{3}}{3\frac{1}{4}}=\tfrac{8}{3}\div\tfrac{13}{4}=\tfrac{8}{3}\times\tfrac{4}{13}=\tfrac{32}{39}.$$

These quotients we shall call fractions—**complex fractions**,—*to* distinguish them from those previously treated which will be called **simple fractions.**

The rules and operations devised in the case of simple fractions may be extended to complex fractions.

7. Measures and Multiples. Since

$$\frac{3}{14}\times 3=\frac{9}{14}$$

we say as in the case of the integers that $\frac{3}{14}$ is a measure or factor of $\frac{9}{14}$ and that $\frac{9}{14}$ is a multiple of $\frac{3}{14}$.

If one fraction is a factor of another and both are in their lowest terms it is easily seen that the numerator of the former is a factor of the numerator of the latter while its denominator is a multiple of the denominator of the latter.

Ex. $\frac{3}{14}$ is a factor of $\frac{15}{7}$.

For $\frac{15}{7}\div\frac{3}{14}=\frac{15}{7}\times\frac{14}{3}=5\times 2=10.$

If one fraction is a multiple of another and both are in their lowest terms it is also evident that the numerator of the former is a multiple of the numerator of the latter while its denominator is a factor of the denominator of the latter.

Ex. $\frac{16}{21}$ is a multiple of $\frac{4}{105}$.

For $\frac{16}{21}\div\frac{4}{105}=\frac{16}{21}\times\frac{105}{4}=4\times 5=20.$

We are now in a position to find the G. C. M. of several fractions in their lowest terms.

Take, for example, $\frac{12}{35}$, $\frac{24}{25}$, $\frac{30}{49}$. The numerator of every common measure must be a common factor of 12, 24, 30, and the denominator must be a multiple of 35, 25, 49. The *greatest* common measure will then be a fraction with the *greatest* numerator and the least denominator satisfying these conditions. The numerator is therefore the G.C.M.

of 12, 24, 30 which is 6, and the denominator is the L. C. M. of 35, 25, 49 which is 1225.

$\therefore$ The G.C.M. of $\frac{12}{35}$, $\frac{24}{25}$, $\frac{30}{49}$ is $\frac{6}{1225}$.

Hence: *The G. C. M. of two or more fractions in their lowest terms is the fraction whose numerator is the G. C. M. of their numerators, and denominator the L. C. M. of their denominators.*

Similar reasoning will shew that: *The L. C. M. of two or more fractions in their lowest terms is a fraction whose numerator is the L. C. M. of their numerators and denominator the G. C. M. of their denominators.*

8. **Ratio and Proportion.** When we compare two numbers or two quantities of the same kind with the view of finding how large or how small one is relatively to the other, we are said to seek their **ratio**. Take the two numbers 3 and 12; plainly 12 is 4 times 3 and 3 is $\frac{1}{4}$ of 12. So, of the two quantities \$2 and \$3, the former is $\frac{2}{3}$ of the latter, the latter $\frac{3}{2}$ of the former. Thus the ratio of two numbers or of two like quantities is expressed by the fraction with those numbers, or the measures of those quantities as numerator and denominator.

The two numbers or quantities whose ratio is stated are called the **terms** of the ratio, the former the **antecedent**, the latter the **consequent**. The numerator and the denominator of a fraction are also called its **terms**.

Two ratios are equal when the fractions expressing those ratios are equal. For example, the ratio of 2 to 3, is equal to the ratio of 10 to 15, since $\frac{2}{3}=\frac{10}{15}$. We state this fact thus: 2 is to 3 as 10 is to 15, or in symbols 2:3::10:15.

The four numbers 2, 3, 10, 15 are said to be in **proportion** or to be **proportionals**.

9. Convention. Suppose that it is required to find the value of

$$\tfrac{1}{2}+\tfrac{1}{3}\times\tfrac{7}{10}\div\tfrac{3}{5}-\tfrac{4}{7}\text{ of }\tfrac{5}{8}\div\tfrac{7}{10}.$$

It is plain that, in the absence of some agreement as to which operations are first to be performed, the problem is indefinite. The convention ordinarily made is that first the fractions connected by "of" are to be taken together; next multiplications and divisions are to be performed in the order in which they occur; and finally the additions and subtractions are to be made in order. In the example offered the successive steps are here shewn:

$$\tfrac{1}{2}+\tfrac{1}{3}\times\tfrac{7}{10}\div\tfrac{3}{5}-\tfrac{4}{7}\text{ of }\tfrac{5}{8}\div\tfrac{7}{10}=\tfrac{1}{2}+\tfrac{1}{3}\times\tfrac{7}{10}\div\tfrac{3}{5}-\tfrac{5}{14}\div\tfrac{7}{10}$$
$$=\tfrac{1}{2}+\tfrac{1}{3}\times\tfrac{7}{10}\times\tfrac{5}{3}-\tfrac{5}{14}\times\tfrac{10}{7}=\tfrac{1}{2}+\tfrac{7}{18}-\tfrac{25}{49}=\tfrac{441+343-450}{882}=\tfrac{167}{441}.$$

In practice several steps may frequently be combined.

When expressions are enclosed by brackets they are to be regarded as making up one value. Thus

$$\tfrac{2}{3}\times(\tfrac{3}{5}+\tfrac{2}{7}\times\tfrac{1}{3})\div\{\tfrac{3}{7}\text{ of }(\tfrac{1}{2}+\tfrac{1}{5})\}$$
$$=\tfrac{2}{3}\times(\tfrac{3}{5}+\tfrac{2}{21})\div\{\tfrac{3}{7}\text{ of }\tfrac{7}{10}\}$$
$$=\tfrac{2}{3}\times\tfrac{73}{105}\div\tfrac{3}{10}=\tfrac{2}{3}\times\tfrac{73}{105}\times\tfrac{10}{3}=\tfrac{292}{189}.$$

Exercises

1. Reduce to equivalent fractions with a common numerator

$$\tfrac{5}{11},\ \tfrac{4}{13},\ \tfrac{6}{25},\ \tfrac{12}{17},\ \tfrac{24}{35},$$

and arrange the fractions in order of magnitude.

2. Find the value of

$$\tfrac{4}{5}+\frac{2\frac{1}{3}}{5\frac{1}{4}}\div\tfrac{3}{5}\text{ of }\tfrac{11}{12}-\tfrac{5}{14}\times\tfrac{12}{5}.$$

3. Shew that the G. C. M. of several fractions as

$$\tfrac{2}{15},\ \tfrac{14}{15},\ \tfrac{24}{35},$$

may be found by reducing them to a common denominator.

4. Shew that the L. C. M. of several fractions as

$$\frac{7}{15},\ \frac{4}{15},\ \frac{11}{24},$$

may be found by reducing them to a common denominator.

5. By considering the complementary fractions shew that if the same number is added to each term of a proper fraction (*i.e.*, a fraction less than one) the value of the fraction is increased.

NOTE.—Two fractions whose sum is unity are called complementary.

6. Shew that if the same number is added to each term of an improper fraction, the value of the fraction is diminished.

7. In the case of the division of two fractional quantities, *e.g.*,

$$\$\tfrac{3}{8} \div \$\tfrac{4}{7}$$

point out the difficulty in applying the ordinary rule, and shew how it is overcome.

8. How much must be added to the numerator of $\frac{11}{15}$ to give a fraction equal to $\frac{4}{5}$?

9. How much must be taken from the numerator of $\frac{11}{16}$ to give a fraction equal to $\frac{3}{8}$?

10. Find the number which added to both terms of $\frac{3}{15}$ will give a fraction equal to $\frac{3}{4}$.

11. Find the number which taken from both terms of $\frac{17}{25}$ will give a fraction equal to $\frac{1}{9}$.

12. If four numbers are in proportion shew that:

(*a*) The ratio of the first to the third is equal to the ratio of the second to the fourth.

(*b*) The product of the extremes is equal to the product of the means.

CHAPTER IV

DECIMALS

1. Introduction. In the number 2222, the successive 2's from the right give a scale of values increasing 10-fold, or from the left a scale of values diminishing in like manner. Thus the 2 on the right—2 units—denotes one-tenth of the 2 just before it, one-hundredth of the 2 next to the left and one-thousandth of the 2 on the extreme left. Suppose now a **point** is introduced just after the 2 units, solely to indicate that the place just before it is that of the units, and let 2's be added as in 2222·222. Then the 2 just after the point would be expected to mean one-tenth of the 2 next to the left, *i.e.*, one-tenth of 2 units which is 2 tenths; the next 2 would be expected to mean one-tenth of the 2 tenths, or one-hundredth of the 2 units which is 2 hundredths; and the next 2 to mean 2 thousandths.

We agree so to regard them. Then 35·279 means 35 and 2 tenths, 7 hundredths and 9 thousandths; since 2 tenths equals 200 thousandths, and 7 hundredths equals 70 thousandths, the part ·279 may be read 279 thousandths.

The point is called the **decimal point** and 35·279 or ·279 is called a **decimal fraction** or a **decimal.**

In writing decimals which have no integral part it is well always to write 0 in the units place. The fact that the figures written are named by referring to the units place is emphasized, and the function of the decimal point is less likely to be overlooked.

It is evident that $35{\cdot}279 = 35\frac{279}{1000} = \frac{35279}{1000}$, so that a decimal may be expressed as a vulgar fraction. Hence the simple rules for decimals may be derived from those for vulgar fractions. The student is recommended so to derive them. In the text, however, no explicit reference will be made to the rules for vulgar fractions, the desire being to shew that the rules for decimals follow naturally from the rules for integers.

From what has been said as to the meaning of decimal, it is easy to compare two decimals differing only in the position of the decimal point as 2·783 and 27·83. In the latter the 2, the 7, the 8 and the 3 denote ten times as much as the 2, the 7, the 8 and the 3, respectively, in the former. Therefore 27·83 = 2·783×10. Hence the following:

Fundamental Principle. *If in a decimal fraction the point be moved one place, two places, three places, etc., to the right, the resulting decimals are 10 times, 100 times, 1000 times, etc., as great as the original decimal; while if the point be moved one place, two places, three places, etc., to the left the resulting decimals are one-tenth, one-hundredth, one-thousandth, etc., of the original decimal.*

It is to be noted that, since the value of a number depends only on its significant figures and the places they occupy, zeros may be added to the right of a decimal, jnst as they may be snpplied at the left of an integral nnmber, without changing its valne.

2. **Addition and Subtraction.** The addition or subtraction of decimals presents no difficnlties. As in the case of integers the decimals shonld be arranged so that figures carrying the same names are in columns ; the work then calls for no new considerations.

Ex. 1. Addition.

```
 23·715
  1·2034
   ·0173
138·7
--------
163·6357
```

Ex. 2. Subtraction.

```
13·017
 2·93854
--------
10·07846
```

3. **Multiplication.** If the multiplier is integral, as for example in 27·316×23 the product can be at once found.

Ex.

$$\begin{array}{r} 27{\cdot}316 \\ 23 \\ \hline 81{\cdot}948 \\ 546{\cdot}32 \\ \hline 628{\cdot}268 \end{array}$$

Here 3 times 6 thousandths are 18 thousandths, etc., while 20 times 6 thousandths are 120 thousandths, etc.

If, however, the multiplier is a decimal as in 23·357× 1·23, we have the same difficulty as in the case of vulgar fractions. The multiplier denotes the number of times the multiplicand is to be taken, and here 1·23 cannot signify a number of times. But let us first multiply by 123.

$$\begin{array}{r} 23{\cdot}357 \\ 123 \\ \hline 70{\cdot}071 \\ 467{\cdot}14 \\ 2335{\cdot}7 \\ \hline 2872{\cdot}911 \end{array}$$

Now 1·23 is one-hundredth of 123; if then, as is in entire accord with what we have seen to hold in the case of integers, we say that the product with multiplier 1·23 is to be one-hundredth of what it would be with multiplier 123, we have a meaning for the multiplication proposed. For

$$23{\cdot}357\times123=2872{\cdot}911$$
$$\therefore\ 23{\cdot}357\times1{\cdot}23=28{\cdot}72911.$$

This, then, will be the accepted meaning of multiplication of decimals and as in the case of vulgar fractions examples may be cited to shew that it faithfully interprets previously formed ideas of multiplication.

Ex. 1. Find the cost of 3·5 yd. of cloth at \$1·5 a yd.

Result = \$1·5 × 3·5 = \$5·25.

Ex. 2. Find the area of a rectangle 3·5 ft. by 4·5 ft.

Result = (3·5×4·5) sq. ft. = 15·75 sq. ft.

The rule for the multiplication of two decimals is now readily seen to be as follows:

Multiply the given decimals as if they were integers and mark off in the result as many decimal places as there are in both the given numbers.

The idea of product may now be extended to the case of several factors.

4. Division. As in vulgar fractions, the enlarged idea of multiplication makes it necessary that we accept divisions in which the quotient is not integral. The process of division is illustrated by the following example.

Ex. Find the quotient of 2·72118 by 2·31.

$$\text{The } \textit{quotient } \frac{2{\cdot}72118}{2{\cdot}31} = \text{the } \textit{quotient } \frac{2{\cdot}72118\times100}{2{\cdot}31\times100}$$

$$= \text{the } \textit{quotient } \frac{272{\cdot}118}{231}$$

This last is found as in ordinary division.

```
231)272·118(1·178
    231
     41·1
     23·1
     18·01
     16·17
      1·848
      1·848
```

The rule is then:

Multiply divisor and dividend by a number 10, 100, 1000 etc., sufficient to make the divisor integral and then proceed as in ordinary division.

5. **Approximations.** Suppose that the result of a computation is 17·358973; for practical purposes it might be sufficient to retain only two decimal places. Noting that the result lies between 17·35 and 17·36 and that it is nearer in value to the latter, we say that 17·36 is the result *correct to two places of decimals*. Sometimes it is said that in such a case 17·35 is the result correct to two places of decimals, the meaning being that it is 17·35 if figures after the second place are not regarded. We shall retain the former meaning and say that 17·36 is an **approximation** to, or an **approximate value** of, 17·358973, correct to two places of decimals.

Since then approximate results are often sufficient, it may be that the work of computation itself may be shortened. That this can be done in the case of such additions as ordinarily occur, and in the case of subtraction, is at once seen.

Ex. 1.

(a)	(b)
2·37859423	2·3786
3·01588	3·0158
7·503698723	7·5037
28·073845	28·0738
40·971967953	40·9719

It is supposed that the sum is required correct to 3 places of decimals. In (a) the complete work is given. In (b) the decimals are written correct to four places of decimals. The sum correct to three places of decimals is 40·972.

Ex. 2.

(a)	(b)
13·70239586	13·7024
5·938249738	5·9382
7·764146122	7·7642

It is supposed that the difference is required to three places of decimals. In (a) the complete work is given. In (b) the decimals are written correct to four places of decimals. The difference required is 7·764.

The following examples exhibit a contracted process for multiplication:

Ex. 1. Find the product 17·3789543×8 correct to three places of decimals.

```
     (a)                (b)              (c)
 17·3789543        17·3789543       17·3789543
          8                 8                8
 ----------        ----------       ----------
139·0316344        139·0316         139·031
```

In (a) the complete work is given. In (b) we work to the fourth place of decimals and have regard to what is "carried" into this place from the multiplication in the next place. In (c) the multiplication is to three places; this is given to shew the need for multiplying to the fourth place. The answer is 139·032.

Ex. 2. Find the product 271·3845×29·378 correct to two places of decimals.

```
      (a)               (b)              (c)
                      × ××××
   271·3845          271·3845         271·3845
     29·378            29·378           8 7392
 ------------        --------         --------
   2·1710760         5427·690         5427·690
  18·996915          2442·460         2442·460
  81·41535             81·415           81·415
2442·4605              18·997           18·997
5427·690                2·171            2·171
 ------------        --------         --------
7972·7338410         7972·733         7972·733
```

In (a) the complete work is given. In (b) we decide as in *Ex.* 1 to work to three decimal places, *i.e.*, to thousandths, and begin the multiplication with the 2 tens of the multiplier. The 2 tens multiplied into the 5 ten-thousandths of the multiplicand give 10 thousandths, a result in the third place, and the multiplication by 2 begins at this place. Having completed the multiplication by 2, which began at 5, we make a mark above the 5. Then we multiply by the 9 units of the multiplier: 9 units multiplied into the 4 thousandths of the multiplicand gives 36 thousandths, a result in the third place. Thus the multiplication begins at 4, the first figure to the right of the marked 5. Having completed the multiplication by 9 which began at 4, attention having been paid to the 4 carried from the product 5×9, we make a mark above the 4. In the same way the multiplication by 3 is seen to begin at 8, the first figure to the left of the one just marked, and so on. In estimating the amount to be "carried" we should, for example, regard 48 as 50, 54 as 50, and 75 now as 70 again as 80 as might be judged best. As one becomes expert in the process, the judgment becomes sharpened in this respect. Thus in the multiplication (b) with multiplier 8, to get the amount to be carried, we have 3×8=24, but we see also that, on account of the next earlier figure, this should be 30.

In (c) there is essentially the same work as in (b), but when the 2 is placed under the figure where multiplication by it is to begin, the remaining figures of the multiplier are written in reverse order so that each is below the figure where multiplication by it is to begin.

The result required is 7972·73.

A contracted process for division will now be indicated.

Ex. Divide correctly to 3 places of decimals 23·62782364 by 3·2759.

Here the quotient will have one figure before the decimal point so that the required result will have four figures.

```
            (a)                                (b)
                                             ×××
3·2759)23·62782364(7·213          3·2759)23·62782364(7·213
       22·9313                           22·9313
       -------                           -------
        69652                              6965
        65518                              6552
        -----                              ----
         41343                              413
         32759                              328
         -----                              ---
          85846                              85

            (c)                                (d)
3·2759)23·62782364(7·213          3·2759)23·62782364(7·213
       22·931                       3127 22·9313
       ------                            -------
        696                                6965
        655                                6552
        ---                                ----
         41                                 413
         33                                 328
         --                                 ---
          8                                  85
```

In (a) is given the ordinary division. To determine the last figure we note that 85846÷32759 is more nearly equal to 3 than to 2; it is not necessary to multiply out by 3. In (b) after the first partial division, instead of bringing down the 2 from the next place in the dividend, we drop the last figure 9 of the divisor, placing a mark above it to indicate that it has been dropped; the reason for this is found in the fact that the figure of the quotient is given by the first one or two figures of the divisor and the dividend, so that, when 9 is marked out, there remains a sufficient number of figures to shew that the next figure of the quotient is 2. The amount to be carried from the part marked out is to be regarded. The work proceeds thus until the result sought is reached. In (c) the process begins by marking out 9, the number of figures remaining in the divisor being sufficient to lead to four figures in the quotient. But as there is doubt as to whether for last figure of the quotient we should take 3 or 2, and as certainty is first of all to be secured, (b) is to be preferred to (c). Thus the division should be begun with a number of figures in the divisor one or even two more than the number required in the quotient. In (d) the work is essentially as in (b) except that, instead of marking out the figures of the divisor, we place the successive figures of the quotient below the last figure of the divisor used in obtaining them.

The result required is 7·213.

6. The Conversion of Vulgar Fractions into Decimals. It has been observed that decimals may always be expressed as vulgar fractions. Thus

$$0{\cdot}375=\frac{375}{1000}=\frac{3\times5\times5\times5}{8\times5\times5\times5}=\frac{3}{8}.$$

$$2{\cdot}25=2\tfrac{25}{100}=2\tfrac{1}{4}.$$

There arises then the question whether, conversely, vulgar fractions may always be expressed as decimals. The vulgar fractions will be supposed given in their lowest terms.

Ex. 1. Express $\frac{3}{4}$ as a decimal.

The given fraction is the quotient of 3 by 4. If the division is performed we obtain the result given below.

```
4)3·0(0·75
  2·8
  ----
  0·20
  0·20
```

$\therefore \frac{3}{4}=0{\cdot}75.$

Ex. 2. Express $\frac{2}{3}$ as a decimal.

Proceed as in *Ex.* 1.

After the first partial division it is seen that the operation will ever present the partial quotient 6 and the remainder 3. Hence $\frac{2}{3}$ does not, in this way at least, yield an ordinary decimal. We may say that

$$\tfrac{2}{3} = 6\tfrac{2}{3} \text{ tenths} = 0{\cdot}6\tfrac{2}{3} = 0{\cdot}66\tfrac{2}{3} = 0{\cdot}666\tfrac{2}{3} = \cdots$$

or that, correctly to four places of decimals,

$$\tfrac{2}{3} = 0{\cdot}6667$$

Ex. 3. Express as a decimal $\frac{7}{12}$.

Proceed as before.

```
12)7·0(0·583
   6·0
   1·00
    ·96
      40
      36
       4
```

As in *Ex.* 2, after the second partial division, the partial quotient 3 and the remainder 4 will recur and we may write

$$\tfrac{7}{12} = 0{\cdot}58\tfrac{4}{12} = 0{\cdot}583\tfrac{4}{12} = 0{\cdot}5833\tfrac{4}{12} = \cdots$$

From these examples it is plain that not all vulgar fractions can, in the way given, be expressed as ordinary decimals. The reason for this will appear.

In *Ex.* 1, the result might have been reached thus:

$$\tfrac{3}{4} = \frac{3}{2\times 2} = \frac{3\times 5\times 5}{(2\times 5)\times(2\times 5)} = \frac{75}{100} = 0{\cdot}75.$$

In like manner

$$\tfrac{7}{25} = \frac{7}{5\times 5} = \frac{7\times 2\times 2}{(5\times 2)\times(5\times 2)} = \frac{28}{100} = 0{\cdot}28$$

and

$$\tfrac{11}{40} = \frac{11}{10\times 2\times 2} = \frac{11\times 5\times 5}{10\times(2\times 5)\times(2\times 5)} = \frac{275}{1000} = 0{\cdot}275.$$

It thus appears that:

(*I*) *In order that a vulgar fraction may be expressed as a decimal it is necessary and sufficient that the denominator contain no prime factors other than 2 and 5.*

Such a fraction may be brought to the form of a decimal by multiplying numerator and denominator by the power of 2 or of 5 necessary to make the denominator a power of 10.

In the case of $\frac{2}{3}$ which cannot, therefore, be expressed as an ordinary decimal we have

$$\tfrac{2}{3}=\frac{6\tfrac{2}{3}}{10};$$

$\therefore\ \frac{2}{3}\times 10=6\frac{2}{3}$, (multiplying each number by 10).

$\therefore\ \frac{2}{3}\times 9=6$, (subtracting $\frac{2}{3}$ from each number).

$\therefore\ \frac{2}{3}=\frac{6}{9}$, (dividing each number by 9),

as is otherwise evident. After the analogy of the notation for the decimal fraction, we agree to denote $\frac{2}{3}$ by $0\cdot\dot{6}$, the dot above the 6 indicating that, in the equivalent vulgar fraction, 6 is to have the denominator 9; we therefore write,

$$\tfrac{2}{3}=0\cdot\dot{6}$$

In like manner,

$$\tfrac{5}{11}=0\cdot45\tfrac{5}{11},\ (=0\cdot4545\tfrac{5}{11}=\ \cdot\ \cdot\ \cdot)$$

$$\therefore\ \tfrac{5}{11}=\frac{45\tfrac{5}{11}}{100}$$

$$\therefore\ \tfrac{5}{11}\times 100=45\tfrac{5}{11}$$

$$\therefore\ \tfrac{5}{11}\times 99=45$$

$$\therefore\ \tfrac{5}{11}=\tfrac{45}{99}.$$

Following the earlier analogy let us then write,

$$\tfrac{5}{11}=0\cdot\dot{4}\dot{5}$$

Similarly,

$$\frac{15}{37} = 0\cdot405\tfrac{15}{37} = \frac{405}{999} = 0\cdot\dot{4}0\dot{5}.$$

We have seen that

$$\frac{2}{3} = 0\cdot6\tfrac{2}{3} = 0\cdot66\tfrac{2}{3} = 0\cdot666\tfrac{2}{3} = \cdots$$

Thus $\frac{2}{3}$ can be represented indifferently by

$$0\cdot\dot{6},\ 0\cdot\dot{6}\dot{6},\ 0\cdot\dot{6}6\dot{6},\ \cdots$$

and since

$$\frac{6}{9} = \frac{66}{99} = \frac{666}{999} = \cdots$$

the notation is consistent.

Further let us agree that $0\cdot0\dot{6}$, $0\cdot00\dot{6}$, $0\cdot0\dot{4}\dot{5}$, are to denote $\frac{6}{90}$, $\frac{6}{900}$, $\frac{45}{990}$, so that the moving of the point one place to the left will mean as in ordinary decimals a division by 10.

Turn now to *Ex.* 8. It has been shewn that

$$\frac{7}{12} = 0\cdot58\tfrac{4}{12} = 0\cdot583\tfrac{4}{12} = 0\cdot5833\tfrac{4}{12} = \cdots$$

From the work it is seen that $\frac{4}{12} = 0\cdot\dot{3}\ (=\frac{1}{3})$.

$$\therefore\ \frac{7}{12} = \frac{58\frac{4}{12}}{100} = \frac{58}{100} + \frac{1}{100} \text{ of } 0\cdot\dot{3}$$

$$= 0\cdot58 + 0\cdot00\dot{3}, \qquad \text{(as agreed)}$$

$$= 0\cdot58\dot{3} \qquad \text{(let us write)}.$$

As a further example it can be shewn that

$$\frac{113}{148} = 0\cdot76\dot{3}5\dot{1}.$$

Since in any division the remainder is less than the divisor, the division indicated by a fraction will either terminate or will present a recurrence of figures in the quotient. Therefore every vulgar fraction can be expressed in one or other of the forms above given.

A result as 0·25 is called an ordinary decimal or simply a decimal. Results as $0{\cdot}\dot{3}$, $0{\cdot}7\dot{6}35\dot{1}$ are called **recurring, circulating** or **periodic decimals**; $0{\cdot}\dot{3}$, $0{\cdot}\dot{4}\dot{5}$ are called **pure** recurring decimals, consisting as they do of a recurring part only; $0{\cdot}7\dot{6}35\dot{1}$, $0{\cdot}58\dot{3}$ are called **mixed** recurring decimals as they have one part an ordinary decimal and another a recurring decimal.

By definition, a pure recurring decimal may be at once written as a vulgar fraction. The following example will illustrate how to express a mixed recurring decimal as a fraction.

Ex.

$$0{\cdot}25\dot{3}8\dot{7} = 0{\cdot}25 + 0{\cdot}00\dot{3}8\dot{7}$$

$$= \tfrac{25}{100} + \tfrac{387}{99900}$$

$$= \frac{25\times999+387}{99900} = \frac{25\times1000+387-25}{99900}$$

$$= \frac{25387-25}{99900} = \tfrac{25362}{99900} = \tfrac{4227}{16650}.$$

The following rule may be stated:

To reduce a mixed recurring decimal to a vulgar fraction, write as numerator the difference between the number formed by the figures of the decimal and the number formed by the figures of its non-periodic part, and as denominator as many nines as there are figures in the periodic part, followed by as many zeros as there are figures in the non-periodic part, after the point.

From (*I*) it is evident that recurring decimals must arise from fractions in whose denominators appear factors other than 2 and 5. From an example as

$$\tfrac{5}{11} = 0{\cdot}\dot{4}\dot{5} = \tfrac{45}{99}$$

it is seen that the denominator of a fraction yielding a pure recurring decimal must be a factor of some one of the numbers 9, 99, 999, ... Since neither 2 nor 5 is a factor of any such number it follows that the denominator of such a fraction cannot have either 2 or 5 as a factor.

In the case of a mixed recurring decimal as $0{\cdot}25\dot{3}8\dot{9}$ we can suppose that the last figure of the non-periodic part is always different from the last figure of the period. For if they were the same, as in $0{\cdot}21\dot{3}7\dot{1}$, this decimal would have been written $0{\cdot}2\dot{1}3\dot{7}$. Therefore, in reducing such a decimal to a vulgar fraction, as for example

$$0{\cdot}25\dot{3}8\dot{9}=\frac{25389-25}{99900}=\frac{25364}{999\times10\times10},$$

we shall always find a numerator which, not ending in 0, cannot have both 5 and 2 as factors, and a denominator in which all the factors 2 and 5 appear in the factors 10. Therefore 5 and 2 occur as factors in the denominator each as many times as there are figures in the non-periodic part of the decimal. In the reduction of the vulgar fraction to its lowest terms, either the 2's or the 5's in the denominator must persist. Hence mixed recurring decimals must arise from fractions in whose denominator appear factors not prime to 10 as well as factors prime to 10.

From these considerations we have:

(*II*) *In order that a fraction may be expressed as a pure recurring decimal it is necessary and sufficient that its denominator be prime to 10.*

(*III*) *In order that a fraction may be expressed as a mixed recurring decimal it is necessary and sufficient that its denominator, while containing factors prime to 10, be not prime to 10.*

EXERCISES

1. Find by the contracted method the following products:
   (1) 73·2509×23·5738, to four places of decimals;
   (2) 13·72564×3·275, to three places of decimals;
   (3) 0·137842×0·376589, to five places of decimals;
   (4) 3·6789×5·3827, to three places of decimals;
   (5) 2·13789×3·5269×1·37285, to four places of decimals.

2. Find by the contracted method the following quotients:
   (1) 7·3569407÷2·237859, to four places of decimals;
   (2) 0·3758674÷0·0893765, to four places of decimals;
   (3) 37·2039÷0·87538, to three places of decimals;
   (4) 5·93725÷ 0·837, to four places of decimals;
   (5) 3·698573÷13·57389, to five places of decimals.

3. Express as decimals—simple or recurring—the following vulgar fractions:

(1) $\frac{7}{8}$; (2) $\frac{11}{25}$; (3) $\frac{17}{160}$; (4) $\frac{15}{64}$; (5) $\frac{54}{125}$; (6) $\frac{2}{3}$; (7) $\frac{1}{7}$; (8) $\frac{12}{13}$; (9) $\frac{16}{27}$; (10) $\frac{24}{37}$; (11) $\frac{5}{6}$; (12) $\frac{7}{12}$; (13) $\frac{13}{55}$; (14) $\frac{123}{275}$; (15) $\frac{112}{385}$; (16) $\frac{23}{105}$.

In each case by an examination of the vulgar fraction, account for the precise form of the decimal.

4. By reducing to vulgar fractions find the sum of

$$0{\cdot}\dot{2}\dot{3},\ 0{\cdot}\dot{5}7\dot{4},\ 0{\cdot}\dot{2}35\dot{7},$$

and derive a method of finding the sum without the reduction to vulgar fractions.

5. By reducing to vulgar fractions find the sum of

$$0{\cdot}1\dot{3}\dot{7},\ 0{\cdot}23\dot{5}8\dot{9},\ 0{\cdot}234\dot{5}63\dot{7},$$

and derive a method of finding the sum without the reduction to vulgar fractions.

6. Explain in each case how to find the vulgar fractions equivalent to

(1) $0{\cdot}3\dot{5}\dot{7}$; (2) $0{\cdot}13\dot{5}7\dot{8}$; (3) $0{\cdot}253\dot{7}45\dot{8}$; (4) $13{\cdot}3\dot{7}2\dot{5}$.

7. Find the error made in taking 0·4285 as the equivalent of $\frac{3}{7}$.

8. Find the value, correct to four places of decimals, of

$$\frac{3{\cdot}7259\times 2{\cdot}37365\times 0{\cdot}98723}{2{\cdot}18974\times 1{\cdot}38279}.$$

9. Find, correct to seven places of decimals, the value of

(1) $$\frac{1}{5}+\frac{1}{3.5^3}+\frac{1}{5.5^5}+\frac{1}{7.5^7}+\frac{1}{9.5^9}+\frac{1}{11.5^{11}}+\frac{1}{13.5^{13}}$$

(2) $$\frac{1}{11}+\frac{1}{3.11^3}+\frac{1}{5.11^5}+\frac{1}{7.11^7}+\frac{1}{9.11^9}+\frac{1}{11.11^{11}}+\frac{1}{13.11^{13}}.$$

10. Find the sum

$$\frac{3}{8}+\frac{5}{7}+\frac{7}{10}$$

and express the result as a decimal correct to three places of decimals. Also, expressing each fraction as a decimal to a sufficient number of places, find the sum correct to three places of decimals and compare results.

11. Find the product

[illegible] × [illegible]

and express the result as a decimal correct to three places of decimals. Also, expressing each fraction as a decimal to a sufficient number of places, find the product correct to three places of decimals and compare results.

12. Write down five fractions which lead to finite or simple decimals, five which lead to pure recurring decimals, and five which lead to mixed recurring decimals.

CHAPTER V

INVOLUTION AND EVOLUTION

1. **Involution.** The operation of finding a given power of a given number is called **involution.** For example,

$$11^4 = 11\times11\times11\times11 = 14641.$$

In the case of the power of a fraction, as $(\frac{5}{7})^3$, we have

$$\left(\frac{5}{7}\right)^3 = \frac{5}{7}\times\frac{5}{7}\times\frac{5}{7} = \frac{5^3}{7^3} = \frac{125}{343},$$

so that the power is a fraction whose numerator is that power of the numerator, and whose denominator is that power of the denominator, of the given fraction.

It appears then that involution demands only multiplication and therefore presents no difficulty.

The following forms will be of immediate use:

(1)
$$\begin{aligned} 24^2 &= 24\times20+24\times4,\\ &= 20\times20+4\times20+20\times4+4\times4,\\ &= 20^2+2\times(20\times4)+4^2. \end{aligned}$$

Or, if h and k are any two numbers,

$$\begin{aligned} (h+k)^2 &= (h+k)\times h+(h+k)\times k,\\ &= h^2+2hk+k^2,\\ &= h^2+(2h+k)\times k. \end{aligned}$$

(2)
$$\begin{aligned} 24^3 &= 24^2\times20+24^2\times4,\\ &= (20^2+2\times20\times4+4^2)\times20\\ &\qquad +(20^2+2\times20\times4+4^2)\times4,\\ &= 20^3+3\times(20^2\times4)+3\times(20\times4^2)+4^3. \end{aligned}$$

Or, as in (1),

$$(h+k)^3=(h+k)^2\times h+(h+k)^2\times k,$$
$$=h^3+3h^2k+3hk^2+k^3,$$
$$=h^3+(3h^2+3hk+k^2)\times k.$$

NOTE: The student should give a verbal statement of the results in (1) and (2). It would be well also to illustrate (1) by a diagram.

2. **Square Root.** The product 9×9 or 81 has been called the square of 9. On the other hand 9 is called the **square root** of 81; 81 is said to be a **square number**, or a **perfect square**, or simply a **square.**

The squares of numbers expressed by one figure are expressed by one or two figures. The table for such squares is familiar to the student.

The squares of numbers expressed by two figures are integers expressed by three or four figures, for $10^2=100$, $100^2=10{,}000$, and all integers of two figures are less than 100.

To devise a method for finding the square root of an integer of three or four figures, we shall first construct the square of an integer of two figures and then seek to recover from the square these two figures, *i.e.*, the two parts, of the root.

Ex. $47^2=40^2+2\times40\times7+7^2=1600+560+49=1600+609=2209.$

We first see that 2209 lies between 40^2 and 50^2; the first two figures of 2209 (from the left) suffice to give this fact, *i.e.*, to determine that 4 is the first figure, and therefore that 40 is the first part, of the root. If now k is the other figure of the root, or rather the remaining part of the root,

$$(40+k)^2=2209.$$
$$\therefore\ 40^2+2\times40\times k+k^2=2209;$$
$$\therefore\ 2\times40\times k+k^2=609,$$

where 609 is the remainder when 40^2 is taken from 2209. But k is a number expressed by one figure, so that $2\times40\times k$ must make up the greater part of 609. Hence if we divide 609 by 2×40 we get an indication as to the value of k. Here the indication is 7 and we have only to make sure that $2\times40\times7+7^2=609$, which is seen to be true. In making the verification it is well to note that $2\times40\times7+7^2=7\times(2\times40+7)$. The process thus explained may be presented in concise form thus:

(a)

$$
\begin{array}{lr}
 & 2209(40+7 \\
 & \underline{1600} \\
2\times40=80 & 609 \\
80\times7+7^2=87\times7= & \underline{609}
\end{array}
$$

(b)

$$
\begin{array}{lr}
 & 2209(47 \\
 & \underline{16} \\
87 & 609 \\
 & \underline{609}
\end{array}
$$

In (a) all the work is given. In (b) we carry in mind that 4 here means 40 and therefore that 2×4 means 80, so that, after dividing by 80 and getting the indication 7, we add 7 to 80 by writing it after 8. The process then serves to recover 47 the square root of 2209.

As a further example we find the square root of 841.

$$
\begin{array}{lr}
 & 841(29 \\
 & \underline{4} \\
49 & 441 \\
 & \underline{441}
\end{array}
$$

$\therefore$ The square root of 841 is 29.

Next consider the squares of integers expressed by three figures. It is plain that the squares will be integers expressed by five or six figures, for $100^2=10,000$ and $1000^2=1,000,000$. Before seeking a method for extracting the square root of such numbers, we construct a square.

Ex. $357^2=300^2+2\times300\times57+57^2=127449$,

or, $357^2=350^2+2\times350\times7+7^2=127449$.

From 127449 we wish to find a way of recovering 357. First we see that 127449 lies between 300^2 and 400^2, and the first two figures of the square suffice to determine this fact and therefore to shew that the first part of the root is 300. Taking 300^2 or

90000 from 127449, we have 37449. As in the earlier example and for the same reason, we divide 2×300 into 37449 to get an indication as to the remaining part. The indication is 60, but on trial this is found to be too great. We might now make trial with 59, or with other numbers, and we should in the end come upon 57. But, for the moment, we shall seek only the second figure of the root and we make trial with 50. This is found to be too small; we know then that 127449 lies between 350^2 which is 122500, and, 360^2 which is 129600, and this fact reveals itself in the second two figures of 127449. We now consider 350 as one part of the root, and we take away 350^2 from 127449 to find the remaining part of the root. Now $350^2=300^2+2\times300\times50+50^2$ and we have already taken away 300^2, so that from the remainder we have to take only $2\times300\times50+50^2$ or 32500, which leaves 4949. As before, we divide 2×350 into 4949 to get an indication as to the remaining figure of the root. The indication is 7 and on trial this is found to be correct.

The work may be presented thus:

(a)

	127449(300+50+7
	90000
$300\times2=600$	37449
$600\times50+50^2=$	32500
$350\times2=700$	4949
$700\times7+7^2=$	4949

(b)

	12'74'49(357
	9
65	374
	325
707	4949
	4949

In (a) the complete work is given. In (b) all unnecessary figures are omitted. The number 127449 is marked off into periods of two from the right for reasons that appeared in the explanation.

By a continuation of the process, the square root of any square integer expressed by a greater number of figures may now be found.

Again since

$$1{\cdot}5^2=2{\cdot}25,\ 1{\cdot}23^2=1{\cdot}5129,\ \ldots\ldots$$

it is seen that the square of a decimal has an even number of decimal places. The square root may be found without regard to the decimal point and then the decimal point can be introduced; or the figures may be marked off in periods of two from the decimal point, and the decimal point introduced at the proper time.

Ex.

	1·87′69(1·37
	1
2·3	0·87
	0·69
2·67	0·1869
	0·1869

The square root sought is therefore 1·37.

If an integer or a decimal is not a square, the square root may be found to any degree of approximation.

Ex.

	43· (6·5574
	36·
12·5	7·00
	6·25
13·05	0·7500
	0·6525
13·107	97500
	91749
13·1144	575100
	524576

Here we may say at successive steps that:

The	square	root of	43	is greater	than	6	and	less than	7.
"	"	"	43	"	"	6·5	"	"	6·6
"	"	"	43	"	"	6·55	"	"	6·56
"	"	"	43	"	"	6·557	"	"	6·558
"	"	"	43	"	"	6·5574	"	"	6·5575

and further that the square root of 43 is 6·557 correct to three places of decimals.

It is to be noted that the operation, in the example just treated, will not terminate. For if it did, the square root of 43 would be either an integer, or a fraction which can be supposed in its lowest terms. It is not an integer, nor can it be a fraction; for, if so, the square of a fraction in its lowest terms would be equal to 43, an integer, and this is impossible.

It is plain that the square root of a fraction is the quotient of the square root of the numerator by the square root of the denominator. If the denominator is not a square, as in $\frac{5}{7}$, it is best to say

$$\frac{5}{7}=\frac{5\times7}{7\times7}=\frac{35}{49}.$$

The square root of 35 may now be found to any degree of approximation and then the result divided by 7.

The symbol for square root is $\sqrt{}$; thus the square root of 539 is written $\sqrt{539}$.

3. **Cube Root.** The product $9\times9\times9$ or 729 has been called the cube of 9. On the the other hand 9 is called the **cube root** of 729; 729 is said to be a **cube number**, or a **perfect cube**, or simply a **cube**.

The cubes of integers expressed by one figure are integers expressed by one, two or three figures. The student should make a table of cubes of the first 9 integers.

The cubes of integers expressed by two figures are integers expressed by four, five or six figures, for $10^3=1000$ and $100^3=1,000,000$.

To obtain a method for finding the cube root of an integer expressed by four, or five, or six integers, we shall first construct the cube of an integer of two figures.

Ex. $42^3=(40+2)^3=40^3+3\times40^2\times2+3\times40\times2^2+2^3,$
$=64000+9600+480+8=64000+10088=74088.$

It is now proposed to recover from 74088 its cube root. It is first seen that 74088 lies between 40^3 and 50^3, a fact which reveals itself in the first two figures of 74088. Let h be the remaining part, so that h is a number expressed by one figure. Then must

$$(40\times h)^3=74088.$$
$$\therefore\ 40^3+3\times40^2\times h+3\times40\times h^2+h^3=74088;$$
$$\therefore\ 3\times40^2\times h+3\times40\times h^2+h^3=10088,$$

where 10088 is the remainder when 40^3 is taken from 74088. Now, h being a number expressed by one figure, $3\times40^2\times h$ must make up a large part of 10088. Hence, if we divide 10088 by 3×40^2, *i. e.*, by 4800, we get an indication as to the other part of the root. The indication yielded is 2; testing we find that 2 fulfils all requirements. In making the test it is well to note that

$$3\times(40^2\times2)+3\times(40\times2^2)+2^3=(3\times40^2+3\times40\times2+2^2)\times2.$$

The work may be presented thus:

(a)

	74088(40+2
	64000
$3\times40^2=4800$	10088
$3\times40\times2=$ 240	
$2^2=$ 4	
$5044\times2=$	10088

(b)

	74088(42
	64
4800	10088
240	
4	
5044	10088

In (a) the complete work is given, while in (b) certain unnecessary figures are omitted. Thus the cube root 42 has been recovered.

Next consider the cubes of integers expressed by three figures. It is evident that their cubes will be integers expressed by seven, eight, or nine figures, for $100^3=1,000,000$ and $1000^3=1,000,000,000$. We shall construct the cube of an integer of three figures.

Ex. $451^3=400^3+3\times400^2\times51+3\times400\times51^2+51^3,$
$=64000000+51\times(3\times400^2+3\times400\times51+51^2),$
$=64000000+27733851=91733851.$

It is proposed now to recover the cube root of this last number. It is first seen that the number lies between 400^3 and 500^3, so that 400 is the first part, *i.e.*, 4 is the first figure, of the root. Taking 400^3 from the number we have 27733851. We now divide this by 3×400^2 to get an indication as to the remaining part. The indication is 50 and as in the case of the square we find that the root lies between 450^3 which is 91125000 and 460^3 which is 97336000, and the second figure of the root is revealed in the three figures following 91. Now 450^3—the cube of the part of the root now found—is equal to

$$400^3+3\times400^2\times50+3\times400\times50^2+50^3.$$

Therefore, to subtract 450^3 from the original number, we have only to subtract

$$(3\times400^2+3\times400\times50+50^2)\times50$$

from the remainder 27733851. The remaining figure of the root is found in like manner. The work is given below.

(a)

$$\begin{array}{rl}
3\times400^2 & =480000 \\
3\times400\times50 & =\ \ 60000 \\
50^2 & =\ \ \ \ 2500 \\
\hline
3\times400^2+3\times400\times50+50^2 & =542500 \\
50^2 & =\ \ \ \ 2500 \\
\hline
3\times(400^2+2\times400\times50+50^2) & =607500 \\
(=3\times450^2) & \\
3\times450\times1 & =\ \ \ \ 1350 \\
1^2 & =\ \ \ \ \ \ \ \ 1 \\
\hline
3\times450^2+3\times450\times1+1^2 & =608851
\end{array}
\qquad
\begin{array}{r}
91'733'851(400+50+1 \\
64\ 000\ 000 \\
\hline
27\ 733\ 851 \\
\\
27\ 125\ 000 \\
\hline
608\ 851 \\
\\
608\ 851 \\
\hline
\end{array}$$

(b)

$$\begin{array}{r}
4800 \\
600 \\
25 \\
\hline
5425 \\
25 \\
\hline
607500 \\
1350 \\
1 \\
\hline
608851
\end{array}
\qquad
\begin{array}{r}
91'733'851(451 \\
64 \\
\hline
27733 \\
\\
27125 \\
\hline
608851 \\
\\
608851 \\
\hline
\end{array}$$

In (a) the complete work is given; in (b) unnecessary figures are omitted. The explanation given above furnishes the reason for marking off the figures from the right in periods of three.

By a continuation of the process, the cube root of any integral cube may be found.

As in the case of square root, it may be shewn that:

(1) The cube root of a decimal may be found, care being taken to mark off the periods of three figures from the decimal point.

(2) If a number is not a cube, an approximation to its cube root may be found.

(3) The cube root of a fraction is the quotient of the cube root of its numerator by the cube root of its denominator.

(4) The cube root of a fraction as $\frac{5}{7}$ is best found by regarding the fraction as

$$\frac{5\times7\times7}{7\times7\times7} \text{ or } \frac{245}{343}.$$

(5) The process of finding the cube root of a number, which is not a cube, as 4 or 7, does not terminate, *i.e.*, the cube root, being neither an integer nor a fraction, is an irrational number.

The symbol for cube root is $\sqrt[3]{\ }$; thus the cube root of 597 is written $\sqrt[3]{597}$.

4. Higher Roots. It rarely happens that there is need to extract higher roots, and then these roots are best found by indirect methods. It may, however, be pointed out that the fourth root may be obtained by two successive operations of square root, the sixth root by finding first a square root and then a cube root or first a cube root and then a square root; and so on.

The operation of finding a root is called **evolution.**

5. Irrational Numbers. It has been shewn that in the case of such indicated operations as $\sqrt{3}$, $\sqrt[3]{5}$ it is impossible to find the roots sought, either as integers or as fractions, *i.e.* these roots cannot be expressed as *ratios*. Yet it will be seen that $\sqrt{3}$, $\sqrt[3]{5}$ may be employed to measure quantity. We shall therefore speak of them as numbers, and, in virtue of the fact that they cannot be expressed as *ratios*, we shall call them **irrational numbers.**

The extraction of roots is not the only source of irrational numbers. Special irrational numbers of the kind here met, as $\sqrt{3}$, $\sqrt[3]{7}$, are called **surd numbers** or **surds**.

A complete theory of operations with surd numbers—as their addition, subtraction, multiplication and division, their powers and roots—has been constructed. Its presentation does not fall within the plan of this book. However, a few examples are given to show the way in which surd numbers are treated.

Ex. 1. Find the sum of $\sqrt{3}$ and $\sqrt{5}$.

The simplest complete expression for the sum is:

$$\sqrt{3}+\sqrt{5},$$

but it cannnot be said that the result has been found. In an actual problem, as an approximate value of either $\sqrt{3}$ or $\sqrt{5}$ would be sufficient, so also would an approximate value for their sum. We should then say

$\sqrt{3}+\sqrt{5}=1{\cdot}7321+2{\cdot}2361$, each root correct to 4 decimal places;
$=3{\cdot}968$, the sum correct to three decimal places.

Ex. 2. Find the product of $\sqrt{3}$ and $\sqrt{5}$.

As in the preceding example we may say that the product is

$$\sqrt{3}\times\sqrt{5}$$

and to this result like remarks would apply.

Approximately

$\sqrt{3}\times\sqrt{5}=1{\cdot}7321\times2{\cdot}2361$, each root correct to 4 decimal places;
$=3{\cdot}873$, the product correct to three decimal places.

Under certain assumptions we can say

$$\begin{aligned}(\sqrt{3}\times\sqrt{5})^2&=\sqrt{3}\times\sqrt{5}\times\sqrt{3}\times\sqrt{5}\\&=(\sqrt{3}\times\sqrt{3})\times(\sqrt{5}\times\sqrt{5})\\&=3\times5\\&=15\\ \therefore\ \sqrt{3}\times\sqrt{5}&=\sqrt{15}.\end{aligned}$$

The result is correct, but the student should point the assumptions.

As a verification, $\sqrt{15}=3{\cdot}8730\ldots$

Exercises

1. Find the square roots of the following numbers:

169, 289, 361, 441, 1024, 2916, 6084, 9801.

2. Find the cube roots of the following numbers:

1331, 1728, 4913, 9261, 15625, 24389, 103823, 405224.

3. Find the square roots of the following numbers:

15129, 54756, 92416, 370881, 574564, 801025.

4. Find the cube roots of the following numbers:

1367631, 12812904, 107171875, 401947272.

5. Find the square roots correct to three places of decimals of the following numbers:

7, 13, 29, 73, 127.

6. Find the cube roots correct to two places of decimals of the following numbers:

11, 23, 99, 153, 513.

7. Find the square roots of:

8·41, 28·09, 1·7424, 10·6929, 0·4489.

8. Find the cube roots of:

2·197, 12·167, 2·299968, 0·636056.

9. Find, to the nearest thousandth, the square roots of:

0·7, 0·07, 0·312, 0·0312, 0·00312.

10. Find, to the nearest hundredth, the cube roots of:

0·3, 0·05, 0·23, 0·023, 0·0023.

11. By resolving into factors, find the square roots of:

3969, 6084, 27225, 50625;

and the cube roots of:

9261, 42875, 373248, 681472.

12. Find, to the nearest tenth, the fourth root of 7 and the sixth root of 11.

13. Find the square roots correct to two places of decimals of the following fractions, first by extracting the square roots of the numerators and the denominators and performing the divisions, and then by considering the equivalent fractions with square denominators:

[illegible], [illegible], [illegible], [illegible].

14. Find the cube roots correct to two places of decimals, first by extracting the cube roots of the numerators and denominators and performing the divisions, and then by considering the equivalent fractions with cube denominators:

[illegible], [illegible], [illegible], [illegible].

15. Test the following relations by working to three places of decimals in the case of square roots, and to two places of decimals in the case of cube roots:

$\sqrt{5}\times\sqrt{7}=\sqrt{35}$; $\sqrt{11}\times\sqrt{13}=\sqrt{143}$; $\sqrt{95}\div\sqrt{19}=\sqrt{5}$; $\sqrt{75}=5\sqrt{3}$;

$\sqrt[3]{3}\times\sqrt[3]{5}=\sqrt[3]{15}$; $\sqrt[3]{5}\times\sqrt[3]{7}=\sqrt[3]{35}$; $\sqrt[3]{55}\div\sqrt[3]{11}=\sqrt[3]{5}$; $\sqrt[3]{88}=2\sqrt[3]{11}$.

PART II

APPLIED ARITHMETIC

The most useful and perhaps the most interesting applications of arithmetic are to be found in the problems that arise in business transactions and in geometrical measurements. Such problems form the subject matter of Sections I and II. In Section III are treated certain types of problems that do not fall into any definite class, and in Section IV is given a series of sets of problems for solution.

In every problem of applied arithmetic, the student should first make sure that he understands the nature and meaning of the application. Thus in a problem of commercial arithmetic it is necessary to know the meaning of the terms employed, to understand the transactions appearing in it, and to have in mind any convention of business life that may bear upon it; in a problem of mensuration, there is a like need of a knowledge of the implied geometry. Not infrequently the difficulty of a problem is due to a failure on the part of the student to grasp its full meaning rather than to an inability to supply the necessary reasoning.

SECTION I

COMMERCIAL ARITHMETIC

CHAPTER I

PERCENTAGE: SIMPLE APPLICATIONS

1. Definition. The term **per cent.**, *i. e.*, *per centum*, means *on each hundred*, so that, for example, 6 per cent. means 6 on each hundred, and 6 per cent. of any number or quantity means $\frac{6}{100}$ of that number or quantity. The expression per cent. is frequently denoted by the symbol %; for example, 5 per cent. is written 5%. Plainly 100% of any quantity is that quantity and 100% is $\frac{100}{100}$ or 1.

From what has been said it follows that any percentage can be expressed formally as a fraction: *e. g.*, $8\% = \frac{8}{100} = \frac{2}{25}$. Conversely any fraction can be expressed as a percentage. For take the fraction $\frac{2}{5}$. Then

$$\tfrac{2}{5} = \tfrac{2}{5} \text{ of } 1 = \tfrac{2}{5} \text{ of } 100\% = 40\%.$$

We might also have said:

$$\tfrac{2}{5} = \frac{2 \times 20}{5 \times 20} = \frac{40}{100} = 40\%.$$

EXERCISES

1. Express the following percentages as fractions in their lowest terms:

$$25\%,\ 22\tfrac{1}{2}\%,\ 33\tfrac{1}{3}\%,\ 12\tfrac{1}{2}\%,\ 6\%,\ 20\%,$$

$$45\%,\ 87\tfrac{1}{2}\%,\ 40\%,\ 18\%,\ 23\%,\ 19\tfrac{1}{3}\%.$$

2. Express as percentages :

$\frac{1}{4}$, $\frac{3}{8}$, $\frac{5}{16}$, $\frac{5}{8}$, $\frac{7}{12}$, $\frac{17}{25}$, $\frac{111}{200}$, $\frac{1\cdot281}{3\cdot726}$,

0·03, 0·375, $0\cdot03\frac{1}{8}$, 3·15.

When the result is not integral find it also to the nearest hundredth of one per cent.

3. Express the pound as a percentage of the kilogramme to the nearest hundredth of one per cent.

4. Express the yard as a percentage of the metre, to the nearest hundredth of one per cent.

5. A candidate makes 210 marks out of a total of 275 in Grammar, 217 out of a total of 350 in Arithmetic, and 130 out of a total of 200 in History. Find to the nearest unit his percentage in each subject, and on the whole examination.

6. The population of a city in 1890 was 185,000; in 1900 the population was 235,000. Find by what per cent. the population increased.

7. A merchant bought 1000 yards of cloth at $1·20 a yard, and paid duty thereon at $33\frac{1}{3}$ per cent.; other charges amount to $50·00. At how much a yard must it be sold that he may gain 20 per cent.?

8. The result of a weight analysis is

Sulphur	2·555	grammes
Copper	2·5185	"
Iron	2·2265	"
Total	7·3	"

Find the percentage composition.

9. A man held a 25 per cent. interest in an estate and transferred 25 per cent. of his interest to another man for $5250. Find the estimated value of the estate.

10. By selling cloth at $1·75 a yard, a merchant gains $16\frac{2}{3}$ per cent. At what price should it be sold to make the gain $33\frac{1}{3}$ per cent.?

11. An article was sold at a gain of 10 per cent.; had it been sold for $7·20 less the loss would have been 5 per cent. Find the cost of the article.

12. A sold a horse to B, gaining 16⅔ per cent.: B sold it to C for $95·20 thus losing 20 per cent. What did the horse cost A?

13. If a number be increased 15 per cent. of itself and the amount increased 20 per cent, the result is 414; find the number.

14. A town, of which the population in 1870 was 7500, for two successive decades increased its population by a certain percentage. If the population in 1890 was 10800, find the rate of increase for the decade.

15. A barrel of sugar containing 200 pounds cost $10. If 3°/ₒ of it be lost in weighing, what per cent. is gained by selling it at 8 cents a pound?

16. A merchant bought goods to the value of $3750. He lost 12½°/ₒ of them by fire; for what must the remainder be sold to yield a profit of 5°/ₒ on the investment?

17. A tea-chest contains 100 pounds of tea; 60 pounds are sold at a loss of 15°/ₒ. For what per cent. in advance of cost must the remainder be sold in order to make a gain of 10°/ₒ on the whole?

18. A merchant buys cloth which depreciates, and he sells 35 yards for the cost of 25 yards. Find his loss per cent.

19. If a manufacturer reduces the working day from 10½ hours to 10 hours without a reduction of wages, by what per cent. are wages increased?

20. What per cent. is gained by using a yard measure ¾ of an inch too short?

21. A man sold two horses at $180 each, on the one gaining 20 per cent. and on the other losing 20 per cent. Find his gain or loss.

22. A man sold two lots at $1200 each. On one of the lots his gain was 20 per cent. If on the whole he neither gained nor lost, at what per cent. loss was the second lot sold?

23. A merchant marks his goods at an advance of 30 per cent. on the cost price and in selling makes a reduction of 5 per cent. What profit per cent. does he make on his sales?

Find the cost price and the marked price of goods sold for $117·35.

24. A merchant marks his goods at an advance of 25 per cent. on cost. He allows a customer a reduction of 10 per cent. on his bill and still makes a profit of $5·60 on the transaction. What was the amount of the bill?

25. At what advance on cost should a merchant mark his goods so that, giving a reduction of 10 per cent. on the marked price, he may make a profit of 25 per cent.?

The reduction is what fraction of the cost price?

Find the cost price and the selling price of goods marked at $17·25.

26. At what price should cloth which cost 84 cents a yard be marked that it may be sold at a reduction of 10 per cent. from the marked price and still yield a profit of 8 per cent.?

27. Tea costing 50 cents a pound is mixed with an inferior quality at 20 cents a pound in the proportion of 2 pounds of the former to 3 pounds of the latter, and the mixture is sold at 40 cents a pound. Find the gain per cent.

28. In the erection of a bridge, five times as much was paid for building material as for labor. Had 19 per cent. less been expended for material and 15 per cent. more for wages, it would have cost $4680. What was its actual cost?

29. Two pounds of tea and 6 pounds of sugar cost $2·20; if sugar rises 50 per cent. in price and tea 10 per cent. they would cost $2·66. Find the prices of tea and sugar.

30. A man whose yearly salary is $2000, after paying rent and living expenses, has a balance of $900 at the end of the year. Rents advance 20 per cent. and living expenses 25 per cent. and his balance now is $640. Find the sum now paid for rent.

31. There are two numbers whose sum is 10; if one of them is increased 25 per cent. and the other increased $16\frac{2}{3}$ per cent. the sum of the numbers is then 12. Find the numbers.

32. A merchant's total sales of goods amounted to $1165. He sold $33\frac{1}{3}$ per cent. of them at an advance of $12\frac{1}{2}$ per cent. on cost, 60 per cent. at an advance of 25 per cent., and the remainder at a loss of 40 per cent. What did the goods cost him?

33. A merchant buys cloth and sells it so as to gain 20 per cent. Had he bought it at 20 per cent. less, and sold it for 20 cents a yard less, his profit would have been 30 per cent. Find the cost price of the cloth.

2. **Trade Discount.** A certain article is manufactured to be sold by the dealers at $10. The manufacturer in selling to the dealers makes a reduction or allows a **discount** of 20 per cent.; this means a reduction of $2 from the list or catalogue price, and the cost to the dealer is $8.

A merchant who sells his goods on short credit, bills to be paid at the end of each calendar month, finds it an advantage to allow a discount of 2 per cent. on all bills for payment at time of sale. For immediate payment he would then deduct 80 cents from a bill of $40·00 and the cost to the purchaser would be $39·20.

A wholesale dealer finds that an article of nominal price $30·00 in his distributed catalogues—a price on which he has hitherto allowed a discount of 20 per cent. to the retail trade—may now be obtained by him at a lower cost; this makes it possible for him further to reduce his bills by 10 per cent. Instead of recalling his catalogues he may announce to the trade this further discount. The reduction on the catalogue price is then 20 per cent. and 10 per cent. The price after the first discount is $24·00; on this there is a reduction of 10 per cent. or $2·40 and the net price is $21·60.

The foregoing are illustrations of Trade Discount. The rate of discount is usually given as a percentage of the price to be discounted or of the amount of the bill.

Exercises

1. The amount of a bill of goods is $137·50 with 2 per cent. off for cash. Cash payment is made. Find the sum.

2. The amount of a bill is $28·56 with 1¼ per cent. off for cash. Find the sum necessary for cash payment.

3. The cash payment of a bill on which a discount of 1 per cent. for cash has been allowed, is \$313·83; find the amount of the bill.

4. Merchandise to the amount of \$540·25 was purchased on May 3, the terms being 3 mo., or 3 per cent. off 60 da., or 5 per cent. off 30 da. What amount would meet the bill on May 17? On June 27?

5. A bill of goods is \$720·00, discounts 20 per cent., 10 per cent. Find the cost of the goods.

6. Find the difference between discounting a bill of \$1,800 at 20 per cent. and 10 per cent., and discounting at 30 per cent.

7. What single discount is equivalent to the discounts 20 per cent., 10 per cent.?

8. Shew that the discounts 30 per cent. and 20 per cent. are equivalent to the discounts 20 per cent. and 30 per cent.

9. A merchant buys goods listed at \$180·00, 33⅓ per cent. off and 2 per cent. for cash, paying at once. In selling he allows 10 per cent. off the list price; what gain per cent. does he make?

10. A dealer buys goods catalogued at \$300 with 20 per cent., 10 per cent., 5 per cent. off and sells them for \$250; find his gain per cent.

11. What single discount is equivalent to three discounts of 10 per cent.?

12. What second discount, taken with 20 per cent. off, is the same as a discount of 30 per cent.?

13. What discount followed by a 10 per cent. discount, is the same as 40 per cent. off?

14. What discount twice taken is the same as 19 per cent. off? as 30 per cent. off?

15. What further discount would be necessary to make a 10 per cent. discount followed by a 10 per cent. discount equivalent to a 20 per cent. discount?

3. Commission. An **agent** receives a consignment of 1000 bbl. of flour which he is to sell; for making the sale he is to receive 3 per cent. of the **gross proceeds**, *i. e.*, of the sum for which the flour is sold, and to remit to the **consignor** the balance, *i. e.*, the **net proceeds**. Suppose that he sells the flour at \$6·00 a bbl. Then:

The gross proceeds = \$6 × 1000 = \$6000.

The agent's commission = $\frac{3}{100}$ of \$6000 = \$180.

The net proceeds = \$6000 − \$180 = \$5820.

It is plain that the commission is also equal to $\frac{3}{97}$ of the net proceeds.

Next suppose that an agent is instructed by his principal to buy for him 500 bbl. of apples at \$1·25 a bbl. For the purchase \$1·25 × 500 or \$625·00 will be necessary. If the agreement is that the agent is to receive 2 per cent. of the cost of the apples, for making the purchase, his commission will be $\frac{2}{100}$ of \$625·00 or \$12·50. Therefore there should have been sent the agent \$625·00 + \$12·50 or \$637·50. Here the commission is $\frac{2}{100}$ of the sum paid for the apples, or $\frac{2}{102}$ of the sum remitted.

The rate of commission is ordinarily given as a percentage to be charged on the sum for which the agent sells the goods, or on the sum invested by him in goods.

If the agent who sells a consignment, is intrusted with the investment of the proceeds after deducting his commissions, there will occur problems like the following:

An agent whose charge for sales is 3 per cent. and for investments 2 per cent. receives a consignment of flour with instructions to invest the proceeds in lumber, reserving his two commissions. Find to what fraction of the sum received for the flour his commissions will amount.

The first commission $= \frac{3}{100}$ of gross proceeds of sale.

$\therefore$ Net proceeds of sale $= \frac{97}{100}$ " " "

This sum is to be regarded as belonging to the consignor; it is to be employed in purchasing lumber and paying the commission therefor.

$\therefore$ The second commission $= \frac{3}{103}$ of $\frac{97}{100}$ of gross proceeds of sale.

$\therefore$ The two commissions

$= (\frac{3}{100} + \frac{3}{103}$ of $\frac{97}{100})$ of gross proceeds of sale.

$=$ $\frac{6}{103}$ of gross proceeds of sale.

The following solution of the problem is also given:

The first commission $= \frac{3}{100}$ of gross proceeds (of sale).

The second commission $= \frac{3}{100}$ of sum invested (in lumber).

Now the gross proceeds exceed the sum invested by the sum of the two commissions. Therefore, had the second commission been reckoned on the gross proceeds, it would have been greater by $\frac{3}{100}$ of the sum of the two commissions; but in this case the two commissions would have made up $\frac{6}{100}$ of the gross proceeds.

$\therefore$ $\frac{6}{100}$ of gross proceeds $=$ the sum of the two commissions,

$+ \frac{3}{100}$ of the sum of the two commissions,

$= \frac{103}{100}$ of the sum of the two commissions.

We have then

$\frac{103}{100}$ of the sum of two commissions $= \frac{6}{100}$ of gross proceeds.

$\therefore \frac{1}{100}$ " " " $= \frac{1}{103}$ of $\frac{6}{100}$ of gross proceeds.

$\therefore$ sum of two commissions $= \frac{100}{103}$ of $\frac{6}{100}$ " "

$= \frac{6}{103}$ of gross proceeds (of sale).

It is easily seen that the sum of the two commissions $= \frac{6}{97}$ of sum invested.

Exercises

1. A commission merchant sold 450 barrels of flour at $5·85 a barrel. If the rate of commission was 4 per cent., find his commission and the sum remitted to his principal.

2. An agent sells a house for $9000. The rate of commission is 5 per cent.; find his commission and the amount the former owner receives from the sale.

3. An agent sold 3000 tons of hay at $11·40 a ton. The rate of commission being 5 per cent., find his commission and the sum remitted to the consignor.

4. An agent arranges for the purchase of 12000 bushels of wheat at 63 cents a bushel. If his rate of commission is at 2 per cent., what sum must be sent him to complete the purchase and pay the charges?

5. An agent receives $1009·80 to invest in tea at 33 cents a pound. If his commission of 2 per cent. is first to be deducted, find how many pounds it was meant that he should buy.

6. An agent receives $1081·50 to invest in apples at $1·75 a barrel. His commission, which is at 3 per cent., is to be deducted; find how many barrels it was meant that he should buy.

7. An agent sells a house for $8400; the amount received by the former owner from the sale is $7896. Find the rate of commission charged.

8. A commission merchant sells 1200 barrels of flour; his commission, which is at the rate of 2 per cent., amounts to $158·40. For how much a barrel did he sell the flour, and what sum did he send to the consignor?

9. A commission merchant sells a consignment of bacon at 13 cents a pound. His commission at 4 per cent. amounted to $39·00. Find the number of pounds consigned to him, and the sum sent to the consignor.

10. An agent receives for letting a house 5 per cent. of the gross rental, but assumes responsibility for making the collection. He lets the house at $40 a month, and in the course of 3 years loses the rent for one month. What sum did he realize from his commissions in 3 years?

Find also the amount received in rent by the owner of the house, for the time in question.

11. An agent receives a certain percentage of the gross rental for letting a house, and assumes responsibility for collecting the rent. On a house let at $36 a month for three years, he fails to collect one month's rent, and in the three years his net commission is $41·76. Find the percentage allowed him.

12. An agent receives a consignment of 5000 pounds of tea with instructions to sell and to invest the proceeds in flour, having deducted his commissions for the two transactions. The rate of commission for selling is 5 per cent., and for buying $2\frac{1}{2}$ per cent. He sells the tea at 41 cents a pound and buys the flour at $4 a barrel. Find his total commission and the amount of flour bought.

13. An agent sells flour at $4·10 a barrel and buys sugar at $3\frac{1}{2}$ cents a pound, having deducted his charges. The rates of commission are $3\frac{1}{2}$ per cent. for sales and $2\frac{1}{2}$ per cent. for purchases. His total commission is $168. Find the number of barrels of flour sold and the quantity of sugar purchased.

4. **Insurance.** Suppose that the owner of a house worth $6000, wishing to provide against complete loss in case of fire, insures it for $4000. If the Insurance Company charges 2 per cent. of the amount insured for, the insurance being for a term of 3 years, the **premium** paid by the **insured** to the **insurer** (*i.e.*, the Insurance Company) at the beginning of the term is $\frac{2}{100}$ of $4000, or $80·00. In case of complete destruction by fire, within the term, the company will pay the insured $4000. If the destruction is only partial, the company will pay the estimated loss if less than $4000, otherwise $4000.

This example brings out the essential fact in problems in Fire Insurance. The premiums in problems in Life Insurance and in Accident Insurance may be calculated in the same manner when the rate is known.

The rate of insurance is generally given as a percentage on the amount insured, or, which amounts to the same thing, as a sum to be paid for each $100 or $1000 of insurance.

Exercises

1. A building valued at $9000 was insured for a period of three years for $\frac{2}{3}$ of its value, the rate being 2 per cent. Find the premium paid.

2. A merchant's stock was insured for one year for $16,000 at $\frac{3}{4}$ per cent. Find the premium paid.

3. A steamship valued at $200,000 was insured in three companies, in the first for $50,000 at $\frac{3}{4}$ per cent, in the second for $60,000 at $\frac{5}{8}$ per cent., and in the third for $40,000 at [illegible] per cent. Find the total premium.

4. A ship's cargo valued at $48,000 was insured, $\frac{1}{4}$ at the rate of $\frac{3}{8}$ per cent., $\frac{1}{3}$ at the rate of $\frac{3}{4}$ per cent., and $\frac{1}{6}$ at the rate of $\frac{5}{8}$ per cent. Find the premium paid.

5. Find the charge for insuring a house worth $4500 for $\frac{3}{4}$ of its value at $\frac{3}{4}$ per cent. if the agent's fee for issuing the policy is fifty cents.

6. A house was insured for $\frac{2}{3}$ of its value, the rate being $\frac{7}{8}$ per cent. If the premium paid was $42, find the value of the house.

7. A building valued at $12,000 was insured for a period of three years for $\frac{3}{4}$ of its value at the rate of $1\frac{1}{4}$ per cent. Soon after the insurance had been effected, the building was completely destroyed. Find the owner's loss, and the loss to the insurance company.

8. A steamship valued at $150,000 and insured for $\frac{2}{3}$ of its value at $\frac{3}{4}$ per cent. sustained damage to the amount of $24,000. The insurance company's liability being for $\frac{2}{3}$ of the damage, find the company's loss through having carried the risk.

9. A building and its contents valued at $36,000 were insured for $25,000 the rate being $1\frac{1}{4}$ per cent. Soon afterwards the building and contents were completely destroyed. Find the loss to the insurer and to the insured.

10. A company issues a policy of $12,000 on a building the rate being $\frac{7}{8}$ per cent., and reinsures in a second company to the amount of $5000 at 1 per cent. The building is completely destroyed. Find each company's loss.

11. A ship's cargo valued at $60,000 was insured at $\frac{7}{8}$ per cent. so that in case of loss the owner would recover its value and the amount of the premium paid. For what sum was the cargo insured and what was the premium paid?

12. The premium on a policy issued for $4800 was $30. Find the rate of insurance.

13. A house valued at $7500 was insured so that in case of loss there would be recovered $5000 and the amount of the premium. Find the premium paid, the rate being one per cent.

14. A consignment of flour valued at $10,800 was insured so that in case of loss there should be recovered the value of the flour, the cost of insurance, and $400 in addition. Find the premium paid, the rate being $\frac{3}{4}$ per cent.

15. A man insured his house valued at $8000 so that in case of loss he should recover $\frac{4}{5}$ of the value of the house and $\frac{4}{5}$ of the premium paid which was at the rate of $1\frac{1}{4}$ per cent. Find the premium paid.

Find also the loss, in case of complete destruction, to the insurer and to the insured.

16. A shipment of apples was insured at $\frac{3}{4}$ per cent. to cover the value of the apples and the premium paid. The premium paid was $45. Find the value of the apples.

17. A shipment of cattle was insured at 1 per cent. to cover the value, the premium, and $500 additional. The premium was $100; find the value of the cattle.

5. Taxation. The estimated requirements for school purposes for the coming year, in a town in which the rateable property is assessed at $4,500,000, is $18,000. Then on each dollar of such property will be paid

$$\frac{1}{4500000} \text{ of } \$18,000, \text{ or } \$0{\cdot}004.$$

The rate of taxation for school purposes will then be 4 mills on the dollar or $\frac{4}{10}$ of 1 per cent.

The foregoing sufficiently illustrates the way in which the simpler problems of taxation arise.

Exercises

1. Find the tax paid on property assessed for $4000 if the rate is 19¾ mills on the dollar.

2. If the rate of taxation is 19¾ mills on the dollar, and if this rate applies to incomes, find the tax paid by a man whose annual income is $1800 if $700 of this is exempt.

3. The assessed value of the property of a town is $1,800,000 and the rate of taxation is 13 mills on the dollar. If a special collector who receives ½ per cent. is employed, find the town's net receipts from taxation.

4. A village requires $1800 for school purposes for the year. If the assessed value of the rateable property of the village is $450,000, find the rate for school purposes.

5. What rate must be struck on rateable property and income to the amount of $16,000,000 to meet an estimated expenditure of $272,000, for the year?

6. A man whose annual income is $2000 is required to pay taxes on the amount over $700. If his tax bill is for $25·35, find the rate of taxation.

7. A person who pays a rate of 16 mills on the dollar on all his annual income but $700 receives a tax bill for $25·60. Find his income.

8. A man whose income is $2500 finds that his net income after paying the income tax is $2465·15. If $800 was exempt, find the rate of taxation.

9. A man whose income is $3200 finds that his net income after paying the tax of 19½ mills on the dollar is $3151·25. Find how much of his income was exempt.

10. Incomes of not less than $1200 are taxed for all in excess of $500; incomes of less than $1200 are not rated. If the rate of taxation is 18 mills on the dollar, which is the better income, $1200 or $1190?

CHAPTER II

INTEREST

1. If B borrows money from A, actually by a loan, or virtually through not paying a debt when it becomes due, then, in business practice, A will charge B **interest.** The sum charged as interest will depend upon the sum borrowed—the **principal**—, the **time** for which the sum has been borrowed, and the **rate of interest.** The rate of interest is given as the percentage of the principal to be charged as interest when the loan is for one year. For any fractional part of a year, the interest is that fractional part of one year's interest. The following example illustrates how to find the interest when the time is not greater than one year.

Ex. Find the interest on $720·75 for 3 mo. (= $\frac{1}{4}$ yr.) at 6 per cent. per annum.

The interest for 1 yr. = $\frac{6}{100}$ of $720·75.

∴ " " $\frac{1}{4}$ " = $\frac{1}{4}$ of $\frac{6}{100}$ of $720·75

= $10·81 (to the nearest cent).

If we wish to make the computation, we note that, 6 per cent. meaning $\frac{6}{100}$ or ·06, we have only to multiply by 6, move the decimal point two places to the left and then divide by 4.

$$\begin{array}{r} \$720{\cdot}75 \\ 6 \\ \hline 4\overline{)43{\cdot}2450} \\ \hline 10{\cdot}81 \end{array}$$

2. Suppose now that A lends B $750 for 3 years at 5 per cent. per annum. The rule in business is to regard the interest as becoming due at the end of each successive year, dating from the time of making the loan. B's obligation to A may be discharged in two ways:

(1) He may pay A, at the end of the first year, $\frac{5}{100}$ of \$750·00, or \$37·50 interest; at the end of the second year, \$37·50 interest; at the end of the third year, \$37·50 interest, and in addition the principal \$750·00.

(2) He may defer paying the interest until the end of the time. Then, at the end of the first year, the interest having become due, B's indebtedness to A is \$750+\$37·50 or \$787·50. B should then pay interest on this sum for the second year. This interest is $\frac{5}{100}$ of \$787·50 or \$39·38. Therefore, at the end of the second year, B's indebtedness to A is \$787·50+\$39·38 or \$826.88, and this sum should bear interest during the third year. This interest is $\frac{5}{100}$ of \$826·88 or \$41·34, so that, at the end of the 3 years, B should pay A \$826·88+\$41·34, or \$868·22. This last sum is called the **amount**. The total interest earned is \$868·22−\$750·00, or \$118·22.

The two ways are not essentially different, for we can suppose that, in the first case, A may put out at 5 per cent. interest the sums paid as interest at the end of the first and second years; the result to him at the end of the third year would then be the same as in the second case.

By special arrangement, A may agree to receive the interest at the end of the three years as if the interest had not become due at the end of each year, *i. e.*, without charging interest on interest. In that case the money is lent at **simple interest.** If, in the given example, simple interest had been charged, the interest would have been ($\frac{5}{100}$ of \$750)×3, or \$112·50, and the amount \$750·00+\$112·50, or \$862·50.

When the interest becomes due at the end of each year, (or other specified term), and added to the principal, becomes interest-bearing, the loan is said to be at com-

pound interest. This is the normal case and, if nothing to the contrary is said, we must always suppose the interest to be compound, when the time is greater than one year (or other specified term).

3. If we have only to compute the interest, not to explain the process, we may present the work thus :

Ex. Find the compound interest on \$540 for 3 years, at 4 per cent. per annum.

```
$540·00
      4
  21·60
 540
 561·60
      4
  22·464
 561·60
 584·064
      4
  23·36256
 584·064
 607·42656
 540·00
$ 67·43        (Total accrued interest.)
```

It is important to regard an example, as the one just worked, in the following way, the result being essential in later theory.

The int. for 1 yr. $= \frac{4}{100}$ of principal.

∴ The amt. at end of 1 yr. $= \frac{104}{100}$ " "

The int. for 2nd yr. $= \frac{4}{100}$ of this amount.

∴ The amt. at end of 2nd yr. $= \frac{104}{100}$ " "

$= \frac{104}{100}$ of $\frac{104}{100}$ of principal;

$= (\frac{104}{100})^2$ of principal.

Similarly the amt. at end of 3rd yr. $= (\frac{104}{100})^3$ of principal;

$= (\frac{104}{100})^3$ of \$540·00;

$= \$540 \times (1{\cdot}04)^3$.

Hence the total accrued interest $= \$540 \times (1{\cdot}04)^3 - \$540$;

$= \$540\,[(1{\cdot}04)^3 - 1]$.

The factor $(\frac{104}{100})^3$ or $(1{\cdot}04)^3$ appearing in the expression for the amount may be called the **compounding factor** or the **amount factor** for 3 years at 4 per cent. per annum.

4. If money is lent at, say, 6 per cent. per annum, payable half-yearly, it is meant that the rate is 3 per cent. for a half-year and that interest is supposed to become due at the end of each half-year.

In banks and other financial institutions there is little actual computation of interest, tables having been made in order to save time and labor as well as to guard against possible inaccuracy.

Exercises

1. Find the interest on:

(1) \$ 350·00 for 2 months at 4 per cent. per annum;
(2) \$ 943·75 " 3 " 3 " "
(3) \$3725·40 " 4 " 5 " "
(4) \$ 563·84 " 5 " 4½ " "
(5) £135 16s. " 6 " 3½ " "

2. Find the interest on:

(1) \$ 725·00 for 63 days at 4 per cent;
(2) \$ 938·45 " 73 " 5 "
(3) \$ 27·13 " 129 " 4½ "
(4) \$1537·44 " 93 " 6 "
(5) £237 12s. " 146 " 2½ "

3. Find the interest at 3½ per cent. on:

(1) \$ 630·00 from Jan. 7, 1903, to July 13, 1903;
(2) \$7250·00 " Feb. 4, 1902, to March 18, 1902;
(3) \$[illegible]375·64 " July 5, 1902, to Feb. 13, 1903;
(4) \$1720·00 " April 23, 1903, to Sept. 17, 1903;
(5) £340 17s. " Feb. 3, 1896, to May 7, 1896.

4. Find the interest on:

(1) \$1200·00 for 3 years at 5 per cent.;
(2) \$1645·00 " 2 " 4 "
(3) \$1720·15 " 3 " 3 "
(4) \$1[illegible]00·00 " 4 " 3½ "
(5) \$2347·50 " 3 " 2½ "

5. Find the interest on $9600·00 for 3 years at 4 per cent. per annum compounded half-yearly.

6. Find the interest on $1273·50 for 2 years at 5 per cent. compounded half-yearly.

7. Find the interest on $600 from Jan. 17, 1901, to May 21, 1903, at 5 per cent. per annum.

8. Find the interest on $523·50 from Sept. 11, 1901, to July 3, 1903, at 4 per cent. compounded half-yearly.

9. Find the interest, supposed simple, on:

(1)	$1250·00	for	3 yr. 4 mo.	at 5	per cent.;
(2)	$ 763·50	"	2 yr. 93 days	at 4	"
(3)	$ 35·50	"	6 yr. 146 days	at 5	"
(4)	$2371·40	"	1 yr. 95 days	at 4½	"
(5)	$ 395·19	"	2 yr. 70 days	at 3½	"

10. What principal will yield $5·34 interest in 43 days at 5 per cent.?

11. What principal will in 2½ years yield $64·50 interest at 4 per cent., simple interest?

12. At 3½ per cent. simple interest what principal will amount to $[illegible]·30 in 2 yr., 3 mo.?

13. In 49 days the interest on $375·00 was $1·89; find the rate.

14. In 89 days the interest on $480·00 was $5·27; find the rate.

15. At what rate will $1240·00 amount to $1245·35 in 45 days?

16. In what time will $720 at 4 per cent. yield $2·60 interest?

17. In what time at 5 per cent. simple interest will $503 amount to $558·47?

18. Write down the amount factor for:

(1) 3 years at $4\frac{1}{2}$ per cent.;
(2) 2 years at 4 per cent. payable half-yearly;
(3) 2 years, 3 months at 5 per cent.;
(4) 2 years, 146 days at $3\frac{1}{2}$ per cent.;
(5) 2 years, 10 months at 5 per cent. payable half-yearly.

19. In 3 years \$600 amounted to \$694·575; find the rate.

20. In what time will \$4000 amount to \$4564·66 at $4\frac{1}{2}$ per cent.?

21. Find the difference between the simple and the compound interest on \$1640·00 for 2 years at 4 per cent.

22. By what fraction of the principal does the compound interest exceed the simple interest for 3 years at 5 per cent.?

23. The difference between the compound and the simple interest on a sum of money for 2 years at 5 per cent. is \$12; find the sum.

24. A man with \$7000 to invest has a choice of two investments, each for 3 years, one yielding 5 per cent. simple interest, the other yielding $4\frac{1}{2}$ per cent. compound interest. What will be the advantage, at the end of the time, in choosing the better investment?

25. A deposits \$500 in a savings bank at the beginning of each year; if the interest allowed is 4 per cent. compounded half-yearly, what sum is standing to A's credit at the end of three years?

26. A's money exceeds B's by \$800, A's money is invested at 4 per cent. per annum and B's at 5 per cent. per annum; if B's annual income exceeds A's by \$20, find the investments of each.

27. If money is lent at 6 per cent. per annum compounded half-yearly, find the effective rate per annum.

28. What rate compounded half-yearly is effectively 6 per cent. per annum?

CHAPTER III

DISCOUNT

1. On the 19th of January, 1903, as the result of a business transaction, John Gray gave to James White the **promissory note** here copied :

$400 TORONTO, JANUARY 19, 1903.

Three months after date, I promise to pay James White, or order, the sum of Four Hundred Dollars.

(*Signed*) JOHN GRAY.

On the 13th of February, James White, wishing to obtain money at once by means of this note, presents it at a bank to be **discounted**. For the banker, supposed satisfied as to the genuineness of the note and John Gray's ability to meet it, the essential questions are:

(1) *In how many days will the note become* **legally** *due?*

(2) *What amount will John Gray pay when the note becomes due?*

The note declares itself to be **nominally** due April 19; as three **days of grace** are always to be allowed, the note will be **legally** due April 22, the **date of maturity**. Hence the note becomes due in (15+31+22), or 68, days after being presented at the bank. Also, as the sum mentioned on the face of the note, $400—the **face value** of the note—does not bear interest, this is the amount that will be paid at the date of maturity. Thus, if in exchange for a certain sum of money the note becomes the property of the bank, at the end of 68 days the bank will receive from John Gray $400. For the accommodation and for the use of the money advanced, the bank charges a **rate of discount**, say 6 per cent. per annum.

$$\text{The discount} = \tfrac{68}{365} \text{ of } \tfrac{6}{100} \text{ of } \$400;$$
$$= \$4{\cdot}47 \text{ (to the nearest cent).}$$

The banker gives James White $400·00—$4·47 or $395·53, the **proceeds**. James White, the **payee** or person to whom or to whose order the note was to be paid, **indorses** the note by writing his name across the back, and it becomes the property of the bank. At the end of 68 days the bank collects $400 from John Gray, the **maker** of the note.

2. Next suppose that, on the 22nd of January, 1903, A wishes to borrow for three months a certain sum from the bank. The security being satisfactory, he gives the bank a promissory note at three months and the bank **discounts** the note. Suppose the note is for $500 and that the rate of discount is 5 per cent. The note matures on April 25, *i. e.*, at the end of 9+28+31+25 or 93 days.

$$\text{The discount} = \tfrac{93}{365} \text{ of } \tfrac{5}{100} \text{ of } \$500$$
$$= \$6{\cdot}37 \text{ (to the nearest cent).}$$
$$\text{The proceeds} = \$500{\cdot}00 - \$6{\cdot}37 = \$493{\cdot}63.$$

A will receive from the bank $493·63, and, on April 25, will be called upon to pay the bank $500·00. In such a case it is frequently said that A borrows $500·00 from the bank, paying the interest in advance at the rate of 5 per cent. per annum.

3. The preceding examples of discounting are typical. Normally, the time for which a note is discounted does not exceed 3 or 4 months, and, in Canada, days of grace are

always to be taken into account. When it is a question of discounting for a longer period, the *discount or sum to be deducted from the amount of the note*, is as a rule determined from the *rate of interest*; in this case there is no *rate of discount*. Questions involving the rate of interest will be treated in the next chapter.

Exercises

1. Find the discount and the cash proceeds in the case of:

(1) A note drawn Feb. 3, 1903, at 90 days for $540 and discounted March 19, 1903, at the rate of 6 per cent.

(2) A note made Dec. 13, 1902, at 3 months for $624 and discounted Jan. 5, 1903, at the rate of 5 per cent.

(3) A note made Jan. 15, 1903, at 60 days for $412·50 and discounted Jan. 19, 1903, at 6 per cent.

2. The note here copied:

$750·00. KINGSTON, JANUARY 3, 1903.

Three months after date, I promise to pay James Baird, or order, the sum of Seven Hundred and Fifty $^{00}/$ Dollars, with interest at 5 per cent. per annum, value received.

(*Signed*) JOHN KEARNS.

was discounted Feb. 1, 1903, at the rate of 6 per cent. per annum. Find the proceeds.

3. Find the discount and the cash proceeds in the case of:

(1) A note drawn March 15, 1903, at 90 days for $1020 with interest at 4 per cent. and discounted April 12, 1903, at 5 per cent.

(2) A note drawn May 1, 1903, at 3 months for $1760·50 with interest at $4\frac{1}{2}$ per cent. and discounted May 1, at 5 per cent.

(3) A note made March 1, 1903, at 90 days for $960·00 with interest at 6 per cent. and discounted March 19, at 6 per cent.

4. A note drawn June 17, 1902, at 60 days for $640 with interest at $5\frac{1}{2}$ per cent. was discounted July 3, 1902, at 6 per cent. What rate of interest did the banker make on the money advanced?

5. On April 15, 1903, A, by giving his note at 3 months, borrows from the bank $500 cash. For what sum was the note made if the bank's rate of discount is 6 per cent.?

6. Find the face value of a note made Aug. 13, 1901, at 60 days, and discounted the same day at 5 per cent., which had for proceeds $234·60.

7. A note for $600 was discounted 43 days before maturity and the proceeds were $596·11. Find the rate of discount.

8. A note made July 13, 1902, at 60 days for $540 with interest at 5 per cent., was discounted Aug. 22, 1902, and the proceeds were $542·60. Find the rate of discount.

9. A note made December 13, 1899, at 90 days for $480 with interest at 5 per cent. was discounted January 4, 1900, at 6 per cent. Find the rate of interest made by the bank on the money advanced.

10. The proceeds of a 60 day note for $720 discounted May 5, 1903, at 6 per cent. were $717·04. Find when the note became nominally due.

11. A note is drawn June 11th at 90 days with interest at 6 per cent.; it is discounted July 4 at 5 per cent. Find

(1) What fraction the interest is of the face value;

(2) What fraction the discount is of the amount of the note;

(3) What fraction the discount is of the proceeds.

12. The discount on a note, made February 27, 1903, at 3 months for $1080 with interest at 5 per cent. and discounted March 18, was $13·12. Find the rate of discount.

CHAPTER IV

PRESENT WORTH

1. If \$100 is put out at interest for 1 year at 5 per cent., at the end of the year it will come back as \$100+\$5 or \$105. On this account, \$100 *now* is said to be the *equivalent of \$105 to be paid at the end of one year*. While one man wishing to have his money invested would prefer the latter, and another man requiring cash would prefer the former, the equivalence, as a matter of finance, is not affected by these preferences.

Ex. A holds against B a claim of \$600 to be paid at the end of one year. B wishes to discharge the claim now; if money is worth 4 per cent. per annum, what sum should B pay A ?

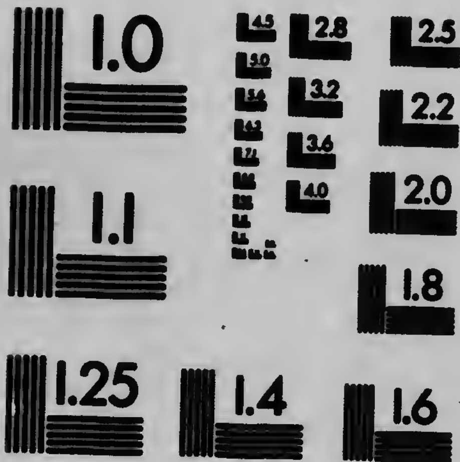
\$104 due in 1 yr. has for equivalent now \$100.
∴ \$600 due in 1 yr. has for equivalent now $\frac{600}{104}$ of \$100 or \$576·92.
Hence B should pay A \$576·92.

In the preceding example, \$576·92 is called the **present worth** of \$600 due at the end of 1 year, the rate of interest being 4 per cent. It is plain that \$576·92 put out for 1 year at this rate would amount to \$600. The difference \$600—\$576·92, or \$23·08, being an *allowance off* \$600 for immediate payment, may be called a *discount*. It is to be remembered that it is determined by the *rate of interest*, not by a *rate of discount*; it is called the **true discount**. When this somewhat misleading term is employed it is to be understood that we have to do with the rate of interest.

2. Next suppose that a sum of money is put out at interest for 3 years at 5 per cent. The amount at the end of the time is $(\frac{105}{100})^3$ of the sum. Therefore, a sum now has for equivalent, at the end of 3 years, $(\frac{105}{100})^3$ of that sum; and a sum due at the end of three years has for equivalent now $(\frac{100}{105})^3$ of that sum. The factor $(\frac{100}{105})^3$

MICROCOPY RESOLUTION TEST CHART

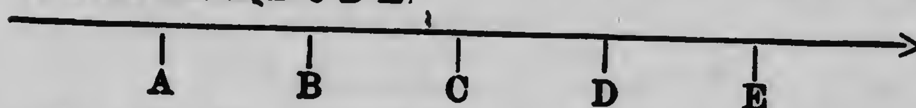
(ANSI and ISO TEST CHART No. 2)



may be called the **present worth factor** for 3 years, interest at 5 per cent.

If by special agreement simple interest is to be employed, the present worth factor for three years, interest at 5 per cent., is $\frac{100}{115}$ or $\frac{1}{1.15}$.

3. It is now plain that the **value** of a sum of money depends upon the time when it became or will become due, and upon the rate of interest; and that when these facts are given, the equivalent of this sum at any later or earlier time may be found. This dependence is rendered somewhat striking by representing measured time on a straight line, equal intervals on the line denoting equal intervals of time. Thus the line A B C D E.



where AB=BC=CD=DE may represent 4 years, each interval denoting 1 year. Suppose that the rate of interest is 5 per cent., and that the sum of money becomes due at the time indicated by the point D; then at the time indicated by the point B—*i. e.*, two years earlier—the **value** of this sum is $(\frac{100}{105})^2$ of the sum. In like manner the value at time E, of a sum which became due at time B, is $(1\cdot05)^3$ of that sum, the rate of interest being 5 per cent.

4. In chapter III, it was pointed out that, in the actual discounting of notes, the rate of discount is supposed given, and that, in normal business, the time does not exceed 3 or 4 months. We can now find what the discount and the proceeds would be if the rate of interest were given, this rate being supposed the same as the rate of discount in the actual case. It is readily seen that the discount in the supposed case is less, and therefore the proceeds greater, than in the actual case. The difference, however, is not great, and the computation in the actual case is much more

easily made. An expression for the difference may be found. For,

The (*bank*) *discount* = the given percentage, for the given time, of *the sum* ;

Also,

The *true discount* = the given percentage, for the given time, of the *present value*.

But, the *sum* = the *present value* + the *true discount*.

Therefore,

The (*bank*) *discount exceeds the true discount by the given percentage, for the given time, of the true discount.*

This is usually stated thus : *The difference between the true and the bank discount is equal to the interest on the true discount.*

Exercises

1. Find the present worth of :
   (1) \$540, due 7 months hence, the rate of interest being 5 per cent.;
   (2) \$129·50, due 1 year hence, the rate of interest being $4\frac{1}{2}$ per cent.;
   (3) \$1000, due 5 years hence, the rate of interest being 4 per cent.;
   (4) \$1750, due $3\frac{1}{2}$ years hence, the rate of interest being $4\frac{1}{2}$ per cent.

2. Find the present worth factor for :
   (1) 2 years, the rate being 5 per cent.;
   (2) 10 months, the rate being $4\frac{1}{2}$ per cent.;
   (3) $2\frac{1}{2}$ years, the rate being 4 per cent.;
   (4) 300 days, the rate being 6 per cent.

3. In example 2, to what fraction of the sum is the true discount equal in each case ?

4. A owes B \$500 to be paid at the end of 8 months, \$600 to be paid at the end of 10 months, and \$900 to be paid at the end of 12 months. If the rate of interest is $4\frac{1}{2}$ per cent., find what sum paid now would discharge these obligations.

5. A is under obligation to pay B \$400 at the end of each year for the next four years. If the rate of interest is 5 per cent., find what sum paid now would be an equivalent.

6. A man wishes his son to receive $600 at the end of each year for the next three years; if the rate of interest charged by bankers is 5 per cent., find for what sum paid now a banker would undertake the payments.

7. The present worth of $151·20 due a certain number of days hence, the rate of interest being 5 per cent., is $150·00. Find the number of days.

8. The present worth of $138·15 due a certain number of months hence is $135·00. If the rate of interest is 4 per cent., find the number of months.

9. A owes B $600 to be paid at the end of 90 days, and $600 to be paid at the end of 30 days. Find when A might equitably discharge his indebtedness to B by paying $1200 (*i. e.*, the amount of the two debts), supposing the *rate of discount* to be 5 per cent.

The result is called the *equated time of payment.*

Shew that the result is independent of the rate of discount.

If the *rate of interest* had been given, say, 5 per cent., would the result have been the same?

10. Find the equated time of payment of $600 due 90 days hence and $1200 due 45 days hence.

Find also when the debts might be discharged by a payment of $1800, on the supposition that money is worth 5 per cent.

11. A holds against B a note for $250 which matures in 75 days; find the difference between the proceeds from discounting at 5 per cent., and the present worth if the rate of interest is 5 per cent.

12. The true discount on a sum of money due at the end of one year is $5·00; the interest for the same time on an equal sum at the same rate of interest would be $5·20. Find the sum and the rate.

13. The true discount on a sum of money due at the end of two years is $164·00; the interest on an equal sum for the same time and at the same rate of interest is $180·81. Find the sum and the rate.

14. How large must be the amount of a note, which matures in 93 days, for the difference in the proceeds from discounting at 5 per cent. discount and at 5 per cent. interest to be as much as 1 cent?

CHAPTER V

PARTIAL PAYMENTS

On July 15, 1902, A gave his note, payable on demand, for $750 with interest at 6 per cent. per annum, to B. When, on March 20, 1903, B calls for settlement, the following payments are found indorsed on it:

July 30, 1902, $ 50;
Dec. 17, 1902, $ 10;
Jan. 12, 1903, $200;
Feb. 20, 1903, $120.

It is required to find the amount that A should pay B when the settlement is called.

The rule followed in such a case is to devote the payment to the discharge of the interest due when the payment is made if it is sufficient to meet this interest, the balance, if any being employed to reduce the principal. If the payment is not sufficient to meet the interest, it is simply added to the next payment, or to the next two payments, &c. until the total of payments is sufficient to meet the interest due at the time the last payment considered is made.

Here the interest on $750 from July 15 to July 30, the time of the first payment, is found to be $1·85. The payment $50, meets the interest and reduces the principal to $701·85.

The interest on $701·85 from July 30 to Dec. 17, is found to be in excess of $10, the payment made on Dec. 17.

The interest on $701·85 from July 30 to Jan. 12 is $19·15, and the payments made in this time are $10 and $200. Therefore, the interest is paid and the principal reduced to

$$(\$701{\cdot}85 + \$19{\cdot}15) - (\$10 + \$200) \text{ or } \$511{\cdot}00.$$

The interest on $511·00 from Jan. 12 to Feb. 20 is found to be $3·28. The payment of $120 meets this and reduces the principal to $394·27.

The interest on $394·28 from Feb. 20 to March 20 is found to be $1·81.

Hence on March 20, A should pay B $394·28 + $1·81 or $396·09.

The work may be presented thus:

Principal July 15....	$750·00
Interest to July 30....	1·85
Due July 30....	751·85
Paid July 30....	50·00
Reduced principal July 30....	701·85
Interest to Dec. 17....	16·15
Interest Dec. 17 to Jan. 12....	3·00
Due Jan. 12....	721·00
Paid Dec. 17 and Jan. 12, $200+$10	210·00
Reduced principal Jan. 12....	511·00
Interest Jan. 12 to Feb. 20....	3·28
Due Feb. 20....	514·28
Paid Feb. 20....	120·00
Reduced principal Feb. 20....	394·28
Interest Feb. 20 to Mar. 20....	1·81
Due Mar. 20....	$396·09.

Exercises

1. A note, drawn Aug. 13, 1902, for $1000 on demand, with interest at 5 per cent., has indorsed on it the following payments:

Sept. 20, 1902, $ 75·60;
Nov. 17, 1902, $ 90·00;
March 20, 1903, $ 10·00;
April 15, 1903, $150·00.

What sum on May 29, 1903, will meet the note?

2. A demand note, made Jan. 17, 1903, for $800 with interest at 4½ per cent., has indorsed on it the following payments:

Jan. 31, 1903, $100;
Feb. 28, 1903, $200;
Mar. 31, 1903, $300.

What sum was due on the note on April 30, 1903?

3. A mortgage for $3000, dated March 15, 1899, and bearing interest at 5 per cent., has indorsed on it the following payments:

March 15, 1900, $500;
March 15, 1901, $500;
March 15, 1902, $500;
March 15, 1903, $500.

What sum would discharge the mortgage on June 15, 1903?

4. A mortgage for $4000, dated June 13, 1900, and bearing interest at 5½ per cent., has indorsed on it the following payments:

Dec. 13, 1900, $ 600;
June 13, 1901, $ 700;
Dec. 13, 1901, $ 800;
June 13, 1902, $ 900;
Dec. 13, 1902, $1000.

What sum would discharge the mortgage on June 13, 1903?

CHAPTER VI

STOCKS

1. A company is formed to construct and control a street railway. To enter upon the undertaking it is found to be desirable to have in hand a sum of $3,000,000, and capitalists are invited to furnish the money on the understanding that the profits from the management of the road are to be distributed among those who supply the money, and in proportion to the sums supplied. The amount $3,000,000 is divided into 30,000 **shares** of $100. A person who puts $5000 into the enterprise receives a **certificate** to the effect that he is the holder of fifty shares of one hundred dollars each, in the company's **stock**. He is said to have subscribed to 50 shares, and becomes a **shareholder**. His money, with that of the other shareholders, is employed in constructing and equipping the railway.

Suppose now that the road has been completed and that, as a result of a year's management, the company finds that the profits are such as to allow $6 to be given the shareholders on every $100 stock held. A **dividend** of 6 per cent. is declared and paid.

If now money is worth only 4 per cent., a man with money to invest sees that $150 at 4 per cent. will bring him in each year $6, *i. e.*, only as much as $100 stock has produced for the shareholders in the railway. Thus, to find investment for his money, he might be willing to pay $150 for a share of $100 railway stock. Other considerations, as a belief in the increasing prosperity of the company and therefore in the prospect of a higher dividend, may lead him to offer even more, say $160, for $100 stock. A holder of the stock may at the same time think it well to sell his stock and, with it, his claims to dividends. Thus it comes about that a share of stock of nominal value $100 is bought and sold at varying prices.

2. Next suppose that the Government of a country wishes to borrow somewhat more than $20,000,000 for a term of years, say 25. If money is worth 4 per cent. per annum, the Government may announce its willingness to pay 5 per cent. per annum on 200,000 shares of $100. A capitalist, or a company, whose money can find investment at only 4 per cent., so that $125 yields each year $5, may regard the Government Loan as a safe and desirable investment, and may offer $125 for each share of $100. If this offer is the best, the Government accepts it. The Government will receive from the capitalist or company $25,000,000 and each year it will pay on the stock $1,000,000.

If in the course of time the rate of interest paid on money should decline, the $100 share, continuing to claim each year $5, would command in the stock markets a price higher than $125. Should the rate of interest advance the price would decline.

3. The preceding are typical cases and the general features of stocks may be seen in them.

Stocks are *handled*, *i.e.*, bought and sold, on the stock exchange by brokers who charge the persons buying or selling stocks, a certain percentage on the nominal value of the stock, not on the sum for which it is bought or sold.

If $100 stock sells for $105 it is said to be at a **premium** of 5, if for $100 it is said to be at **par**, and if for $93 it is said to be at a **discount** of 7. Whatever be the nominal value of a share of stock, the prices quoted refer to $100 stock.

The following examples illustrate the way in which problems in stocks are treated.

Ex. 1. A man having $10,000 to invest instructs his broker to buy Bank of Commerce stock at market prices. The broker bought at 162½ and his charge was ⅛ per cent. Find the quantity of stock bought, the shares being $100.

Cost of 1 share to investor=$162½+$⅛ or $162⅝

The number of shares bought, shares not being broken

$$=\text{integral part of } \frac{\$10,000}{\$162\frac{5}{8}}$$
$$=61.$$

∴ Quantity of stock bought=61 shares or $6100 stock.

The original $10,000 is now represented by 61 shares of stock and $10,000−$162⅝×61 or $79·87 cash.

Ex. 2. If in *Ex. 1* at the end of three months from date of purchase the half-yearly dividend of 3 per cent. is paid, find the rate of interest the investor makes on the money invested.

The dividend received=$3×61=$183.

The sum invested=$162⅝×61=$9920·13.

The time in question=3 mos.=¼ yr.

∴ rate per cent. per annum

$$=\frac{183}{9920\cdot13}\times100\times4=7\cdot4 \text{ to the nearest } \tfrac{1}{10} \text{ of 1 per cent.}$$

Ex. 3. A speculator bought 500 shares, $100 each, of Canadian Pacific Railway stock at 132½ and sold it the next day at 131¼. Brokerage in each case being ⅛ per cent., find his loss.

Cost of 1 share to speculator=$132½+$⅛=$132⅝.

Receipts from sale of 1 share=$131¼−$⅛=$131⅛.

∴ on 1 share his loss=$132⅝−$131⅛=$1½.

∴ on 500 shares his loss=$1½×500=$750.

Exercises

1. Find the cost of:

(1) 120 shares, $10 each, of Canada Permanent stock at 123¼, brokerage ⅛ per cent.

(2) 75 shares, $200 each, of Bank of Montreal stock at 252¼, brokerage ¼ per cent.

(3) 25 shares, $100 each, Twin City Street Railway stock at 119$\frac{3}{8}$, brokerage $\frac{1}{4}$ per cent.

(4) 60 shares, $50 each, Bank of Commerce stock at 161$\frac{1}{4}$, brokerage $\frac{1}{8}$ per cent.

(5) 75 shares, $100 each, Huron and Erie Loan Company stock at 181, brokerage $\frac{1}{8}$ per cent.

2. Find the proceeds of the sale of:

(1) 70 shares, $100 each, of the Crow's Nest Coal Company's stock at 290, brokerage $\frac{1}{8}$ per cent.

(2) $25,000 stock of the Bank of Ottawa at 219, brokerage $\frac{1}{8}$ per cent.

(3) 80 shares, $100 each, of the Canadian Salt Company's stock at 121, brokerage $\frac{1}{8}$ per cent.

(4) 45 shares, $100 each, of the London Street Railway stock at 105$\frac{1}{8}$, brokerage $\frac{1}{8}$ per cent.

(5) 90 shares, $100 each, Commercial Cable stock at 161$\frac{3}{8}$, brokerage $\frac{1}{4}$ per cent.

3. Find the rate of interest received on investments made in the following stocks:

(1) Imperial Life Assurance Co. stock (6 per cent.), bought at 149, brokerage $\frac{1}{4}$ per cent.

(2) Traders' Bank stock, (6 per cent.), bought at 140, brokerage $\frac{1}{8}$ per cent.

(3) Bank of Hamilton stock, (10 per cent.), bought at [illegible], brokerage $\frac{1}{4}$ per cent.

(4) Sao Paulo Tram., (5 per cent.), bought at 94$\frac{5}{8}$, brokerage $\frac{1}{4}$ per cent.

(5) National Trust Co. stock, (6 per cent.), bought at [illegible], brokerage $\frac{1}{8}$ per cent.

4. Find the actual price of a 5 per cent. stock, which yields 4 per cent. on an investment.

If the brokerage is $\frac{1}{8}$ per cent., find the market quotations of this stock.

5. Which is the better investment, a six per cent. stock, at 123$\frac{1}{8}$, or a five per cent. stock at 104$\frac{1}{8}$, brokerage in either case being $\frac{1}{8}$ per cent.?

6. A man instructs his broker to purchase for him 50 shares of Bank of Commerce stock at market prices. The broker buys at 161⅛; if the broker's charges are ⅛ per cent., find the amount of the bill sent to the investor.

7. A man leaves with his broker an indorsed bank cheque for $4000 with instructions to buy as near to this amount as possible, Traders' Bank stock shares at market prices. The broker purchases at 141⅜ and charges ⅛ per cent. brokerage. Find the amount of stock purchased, and the amount of the cheque sent to the investor to complete the transaction.

8. A person sells out 3 per cent. consols at 94⅛, and invests the proceeds in bank stock which sells at 225, and pays yearly dividends of 8½ per cent. If his income is changed to the extent of $57, how much money had he invested?

9. If $11,250 of 3 per cent. stock be sold at 84, and the proceeds invested in 6 per cent. stock at 126, find the change in annual income.

10. A man buys 60 shares of C. P. R. stock at 129⅜ and sells at once at 131. If brokerage in each case is ¼ per cent., find his gain.

11. A man buys 40 shares of Dominion Coal Co. stock at 107, and sells at once at 105. If the brokerage in each case is ¼ per cent., find his loss.

12. A broker buys for himself 100 shares of Bank of Ottawa stock at 212, and sells at once at 212⅛; find his gain.

13. What is the price of stock when $6000 stock can be purchased for $7500?

14. A person transferred $7000 of 5½ per cent. stock at market price 112¾, to a 6 per cent. stock at 122¾; if the brokerage in each case is ¼ per cent., find the change in annual income.

15. A person sells a certain amount of 5 per cent. stock at 115⅞, and invests in a 6 per cent. stock at 137⅞; if brokerage in each case is ⅛ per cent., and if his annual income is increased $4, find the amount of stock held in each case, and the broker's charges.

16. A man sells $15,000 of 5 per cent. stock at 111⅜, and invests in a 6 per cent. stock. If brokerage in each case is ⅛ per cent. and if his income is unchanged, find the quotation for the latter stock.

CHAPTER VII

EXCHANGE

1. A merchant in Toronto purchases goods to the value of £2000 from a merchant in London, England. To make the payment the purchaser might obtain and remit Bank of England notes, or gold coinage, to this amount. This method would be inconvenient, expensive and unsafe. To simplify such payments, bankers or brokers, with houses or agencies in different countries, issue **bills of exchange.** In the case cited, the Toronto merchant would apply at a bank for such a bill. Suppose that the pound sterling is quoted at \$4·84½ when he applies. To meet this amount he would require \$4·84½ × 2000 or \$9690·00. If the commission is ⅛ per cent., the banker's charges are $\frac{1}{800}$ of \$9690·00, or \$12·11, so that the cost of the bill of exchange is \$9690·00 + \$12·11, or \$9702·11. Two, or it may be three, bills are made out, and the *first* (bill) *of exchange* is sent to the London merchant; in case the first is lost the *second of exchange* is sent. On receipt of the bill of exchange, the London merchant presents the bill at the London office or agency of the Toronto bank and receives £2000.

2. Jnst as the London agency is employed to pay Canadian accounts in England, the Canadian agency is employed to pay English acconnts in Canada, and the sending of actual money forward and back is in great measure obviated. If Canadian accounts to be paid in England are in excess of English accounts to be paid in Canada, *i.e.*, if the **balance of trade** is against Canada, the value of the ponnd in Canada will be correspondingly high. Thus the **rate** or **course** of foreign exchange, *i.e.*, the value of the unit of money in one country in terms of the money of another will differ from time to time.

3. For exchange between Canada and Britain the statutory or par value of the pound is $\$4{\cdot}86\frac{2}{3}$. The **old par** of exchange was given by the equation £9 = \$40 or £1 = $\$4{\cdot}44\frac{4}{9}$. The **new par** being $\$4{\cdot}86\frac{2}{3}$ is at a premium of $9\frac{1}{2}$ per cent. on the old par. Quotations are still made on the old par. Thus the quotations, March 18, 1903, as given in the financial columns of the daily papers, are:

BETWEEN BANKS

Sterling	Buyers	Sellers
60 days' sight	$8\frac{7}{8}$	$8\frac{15}{16}$
Demand	$9\frac{5}{8}$	$9\frac{11}{16}$

For exchange between the United States and Britain the quotations in New York give the exchange value of the pound sterling in dollars. For March 18, 1903, the quotations are:

New York	Posted	Actual
Stg. 60 days' sight	$4{\cdot}84\frac{1}{2}$	$4{\cdot}83\frac{5}{8}$
do Demand	4·88	$4{\cdot}83\frac{5}{8}$ to $4{\cdot}87\frac{1}{8}$

Between two cities within a country there may be a like system of meeting accounts by means of bills of exchange. This is **domestic exchange** as contrasted with **foreign exchange.**

The quotations are to be understood as including the commission or brokerage, if nothing to the contrary is stated.

EXERCISES

1. Find the cost of a bill of exchange on London for £241.7*s*., when sterling exchange is quoted at $8\frac{3}{4}$.

2. A Montreal merchant buys goods in England to the amount of £3250. If exchange is at $9\frac{3}{4}$ find what he must pay for a draft (bill of exchange) for this amount.

3. Find the cost in New York of a draft for £1250, exchange being quoted at 4·84¼.

4. A Canadian merchant pays $3924 for a draft on London for £810. What is the quotation for sterling exchange?

5. Find the cost, in Toronto, of a draft for $1750 on Vancouver at ⅛ % premium.

6. A Canadian merchant buys a draft to pay an account of 1600 francs in Paris. If exchange is quoted at 5·19 (*i.e.*, 1 dollar = 5·19 francs) find the cost of the draft.

7. A Canadian merchant buys a draft to pay an account of 6400 marks in Berlin. If exchange is quoted at 96 (*i.e.*, 4 marks = 96 cents), find the cost of the draft.

8. If in New York demand-bills are sold at 4·87½ and bills at 60 days' sight at 4·83¾, what is the rate of discount?

9. If sterling exchange is at 9¾, and exchange between London and Paris is 25·28½ francs, what should a Toronto merchant pay for a bill on London to pay a debt in Paris of 6000 francs?

10. If sterling exchange is at 8⅞, and exchange between London and Berlin is 20·27 marks on the pound, what should a Montreal merchant pay for a bill on London to pay a debt of 3500 marks in Berlin?

11. Explain the different ordinary ways in which a remittance may be made:

(1) From one town in Canada to another in Canada;
(2) From a town in Canada to a town in the United States;
(3) From a town in Canada to a town in England.

SECTION II

MENSURATION

Mensuration is concerned with the measnrement of lengths, areas, and volumes. If the problem is one of the measurement of length, a linear unit, as 1 foot, 1 metre, is supposed given; as in the case of all measurements, the length is known when the number of units contained in it has been found. The area of a surface is in general found indirectly, this measnrement being obtained through linear measurements; the nnit adopted is the area of the square whose side is the linear unit, and the area of a surface is known when the nnmber of snch units contained in it, or equivalent to it, has been fonnd. So also in the case of the determination of a volume the measnre is in general fonnd through linear measurements, and the nnit is the volume of the cnbe whose edge is the linear unit.

CHAPTER I

PLANE RECTILINEAL FIGURES

1. **The Rectangle.** If two sides of a rectangle are 3 inches and 4 inches in length, and if at intervals of 1 inch on two adjacent sides straight lines are drawn at right angles to these sides, the rectangle is divided into squares of sides 1 inch in length. Along the side 3 inches, there is a set of three such squares; the other side being 4 inches

long, there are 4 such sets, and therefore in all 3×4 or 12 square inches.

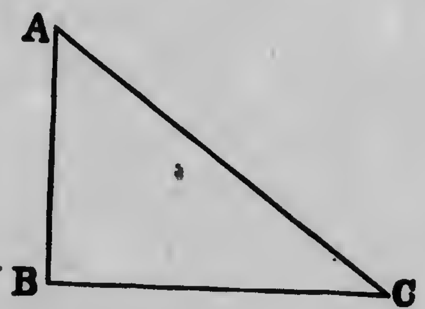
In like manner if the sides of a rectangle are $2\frac{1}{2}$ inches and $3\frac{1}{2}$ inches, we see by reference to a figure that the area is $(2\frac{1}{2} \times 3\frac{1}{2})$ or $8\frac{3}{4}$ square inches.

It is thus evident that, for all rectangles whose sides are measured by integers or fractions, there holds the following rule:

The area of a rectangle is measured by the product of the measures of two adjacent sides.

This rule will be assumed to hold in the case of rectangles one or both of whose adjacent sides have irrational measures.

2. The Right-Angled Triangle. Let ABC be a right-angled triangle, with the angle B a right angle. Then if the rectangle with AB and BC as adjacent sides be constructed, it is evident that the area of the given triangle is one-half that of the rectangle.



Hence, *the area of a right-angled triangle is measured by one-half the product of the measures of the sides containing the right angle.*

Geometry teaches that:

In any right-angled triangle the square on the side opposite the right angle (the hypotenuse) is equal to the sum of the squares on the sides.

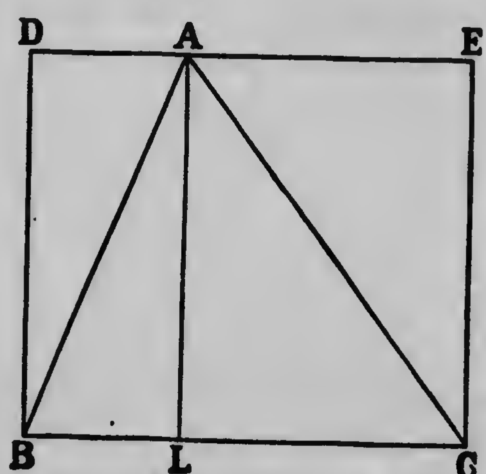
Therefore, if the measures of AB, BC are 3, 4, the square on AC measures $3^2 + 4^2$ which is 25. Therefore AC measures 5.

Similar reasoning shews that if the side of a square measures 1, the measure of its diagonal is $\sqrt{2}$, and we have a concrete representation of this irrational number.

3. The General Triangle. Let ABC be any triangle. Describe the rectangle DBCE on the *base* BC, with the side DE passing through A. Draw AL perpendicular to BC. Then manifestly the area of the triangle ABC is one-half that of the rectangle DBCE.

Since AL—called the *altitude* of the triangle—is equal to DB or EC, it follows then that:

The area of a triangle is measured by one-half the product of the measures of the altitude and the base.



Suppose the three sides given and let the measures of AB, BC, CA be 13, 14, 15. Let AL and BL measure h and k; then LC measures $14 - k$. Consequently since ALB and ALC are right-angled tri[illegible]les we have

$$h^2 + k^2 = 13^2;$$
$$h^2 + (14 - k)^2 = 15^2.$$
$$\therefore\ (14 - k)^2 - k^2 = 15^2 - 13^2;$$
$$\therefore\ 196 - 28k = 56;$$
$$\therefore\ 28k = 140, \text{ or } k = 5.$$
$$\therefore\ h^2 + 5^2 = 13^2;$$
$$\therefore\ h^2 = 144, \text{ or } h = 12.$$

Therefore, the measures of the altitude and the base being known, it follows that:

$$\text{the measure of the area} = \frac{12 \times 14}{2} = 84.$$

If the measures of the sides of a triangle are a, b, c, a process of reasoning, similar to that followed in the example, leads to the result:
The *measure of the area* of a triangle $= \sqrt{s(s-a)(s-b)(s-c)}$ *where* $2s = a+b+c$, *so that* s *is one-half the sum of the measures of the sides.*

4. The Parallelogram. A reference to the figure shews that the parallelogram is equal in area to the rectangle

on the same base and between the same parallels. If then we call the side of the rectangle the altitude of the rectangle, or of the parallelogram, we have the result:

The measure of the area of a parallelogram is equal to the product of the measures of the base and the altitude.

5. The Trapezoid. It is seen from the figure that the area of the trapezoid is equal to the areas of the two triangles

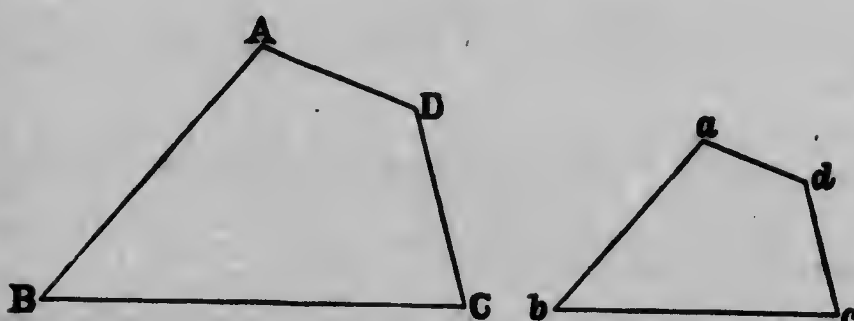
of the same altitude as the trapezoid and with its parallel sides as bases.

It follows then that:

The measure of the area of a trapezoid is equal to the product of the measure of the altitude and the half-sum of the measures of the parallel sides.

6. Similar Figures. Let ABCD, *abcd*, be two quadrilaterals such that the angles A, B, C, D, are equal to the angles *a*, *b*, *c*, *d*, and that

$$\frac{AB}{ab} = \frac{BC}{bc} = \frac{CD}{cd} = \frac{DA}{da}$$



Then it is seen that *abcd* is merely a reduced copy of ABCD.

By measurement AB, BC, CD, DA are found to be 33·5 mm., 42 mm., 20·5 mm., 16·75 mm. in length; also *bc* is found to be 28 mm. long. Hence, *without measurement*, we know that the lengths of *ab*, *cd*, *da* are

$\frac{28}{42}$ of 33·5 mm., $\frac{28}{42}$ of 20·5 mm., $\frac{28}{42}$ of 16·75 mm.;

or, correctly to hundredths,

22·33 mm., 13·67 mm., 11·17 mm.

Exercises

1. Find the areas of the rectangles the lengths of whose adjacent sides are:

(1) 7 in. and 5 in.; (2) 15 dm. and 13 dm.; (3) 19 m. and 17 m., giving in each case the complete explanation.

2. The area of a rectangle is 13 ares; one side is 25 metres long. Find the length of an adjacent side and the length of a diagonal.

3. The area of a rectangular field is 3 acres. If one side is 165 yards long, find the length of an adjacent side and the length of a diagonal.

4. Shew that the area of the floor of a room 5·6 m. long and 4·8 m. wide is equal to that of a strip of carpet 42 m. long and 0·64 m. wide.

5. Shew that the area of the floor of a room 18 ft. by 15 ft. is equal to that of a strip of carpet 30 in. wide and 36 yd. long.

6. Find the cost of carpeting a room 16 ft. by 12 ft. with carpet 27 in. wide at $1·20 a yd.

7. A courtyard 72 yd. by 60 yd. has a gravel walk 2 yd. wide around it; find the area of the walk.

8. One rectangle has its sides 15·93 m. and 13·37 m., and a second rectangle has its sides twice as long as those of the first. Compare the areas of the rectangles.

9. A square and a rectangle have the same perimeter; the sides of the rectangle are 48 in. and 36 in. in length. Compare the areas of the two figures.

10. The adjacent sides of a rectangle are 9 m. and 16 m. long. Find the length of a diagonal and the side and diagonal of the square equal in area to the rectangle.

11. Find the lengths of the hypotenuses of the right-angled triangles whose sides are:

(1) 5 in. and 12 in.; (2) 3·9 m. and 5·2 m.; (3) 37 dm. and 53 dm.

12. If the hypotenuse of a right-angled triangle is 91 yd. long, and if one side is 35 yd. long, find the length of the remaining side.

13. In a right-angled triangle the hypotenuse and one side measure 81 m. and 57 m.; find the length of the remaining side.

14. The sides of a right-angled triangle are 20 ft. and 21 ft. long; find the length of the straight line joining the right angle to the middle point of the opposite side.

15. Taking 1 inch as the unit construct lines whose lengths are measured by:

$$(1)\ \sqrt{2};\ (2)\ \sqrt{3};\ (3)\ \sqrt{5}.$$

16. The sides of a right-angled triangle measure 44 m. and 117 m.; find the length of the perpendicular from the right angle to the opposite side.

17. One side of a right-angled triangle measures 28 chains, and the distance from the right angle to the middle point of the opposite side is 26·5 chains. Find the area of the triangle.

18. If ABC is a triangle with the angle B a right angle, and if BM is the perpendicular from B to AC, then ABC, BMC, AMB are similar triangles.

Hence if the length of AB and BM are 40 m. and 24 m., find the remaining side and the hypotenuse of the triangle.

19. Find the area of the equilateral triangle the length of whose side is 20 yd.

20. The sides of a right-angled triangle are 10 dm. and 24 dm. in length; find the areas of the equilateral triangles described on the sides and the hypotenuse of the triangle, pointing out any relations among these areas.

21. Find the areas of the triangles the lengths of whose sides are:

(1) 50 in., 58 in., 72 in.; (2) 15 cm., 37 cm., 44 cm.; first determining the length of the perpendicular to the longest side from the opposite angle. Apply also the general formula.

22. The sides of a triangle are 15 ft., 20 ft. and 25 ft. Their middle points are joined. Find the area of the triangle thus formed.

23. The side of a rhombus is 13 inches long and one diagonal is 24 inches long; find the length of the other diagonal and the area of the rhombus.

24. Find the area of a trapezoid whose parallel sides are 57 in. and 33 in. long and whose altitude is 24 in.

25. Find the area of the trapezoid whose parallel sides are 84 ft. and 177 ft. in length, and whose non-parallel sides are 34 ft. and 65 ft. in length.

26. Find the length of a side of an equilateral triangle whose area is equal to that of a square whose side is 17 inches long.

27. A ladder 45·5 ft. long with its foot on the street reaches to a height of 44·1 ft. on the wall of a house on one side, and when turned reaches to a height of 36·4 ft. on the wall of a house on the other side. Find the width of the street.

28. On a map a square of which the area is 4·5 sq. in. represents 4050 sq. mi.; find the length of a river represented by a line 13·6 in. long.

29. The length of one diagonal of a rhombus is double that of the other: the area is 16 square inches; find the length of each side.

30. Find the area of a parallelogram whose adjacent sides are 45 m. and 36 m. long, an angle between two adjacent sides being equal to an angle of an equilateral triangle (60°).

31. There is a foot-path along two sides of a square ten-acre field. How much would a man gain by crossing the field from one corner to a point in the path 16½ feet from the opposite corner, instead of following the foot-path?

32. The length of a field is to its width as 3:2. Its area is 15 acres. Find the length of the field.

33. A farm of 70 acres is $1\frac{3}{4}$ as long as it is wide. Find the cost of fencing it with wire fencing at 15 cents a yard; the posts are placed 2 rods apart and cost 40 cents each.

34. The length of a rectangular field is to its breadth as 6:5; one-sixth of the area is wooded, and the remainder, 625 square rods, is under cultivation. What are the dimensions of the field?

35. A rectangular court 90 yd. by 80 yd. has paths 8 ft. wide joining the middle points of the opposite sides and also a path the same width running around it. The remainder is sodded at a cost of 65 ct. a sq. yd. The laying of the paths costs $1·25 a sq. yd. Find the total expense of laying out the court.

36. The sides AB, BC, CA of a triangle are 25 m., 29 m., 36 m. in length; find its area.

Shew that the angle C is acute.

If BM is the perpendicular from B to CA, find the length of CM, and shew that the square on AB (a side opposite an acute angle) is less than the sum of the squares on BC and CA (the sides containing the acute angle) by twice the rectangle contained by AC and CM.

This property is general.

37. The sides AB, BC, CA of a triangle are 11 dm., 13 dm., 20 dm. in length; shew that the angle B is obtuse.

If CM is drawn perpendicular to AB produced, find BM, and shew that the square on CA (the side opposite the obtuse angle) exceeds the sum of the squares on AB and BC (the sides containing the obtuse angle) by twice the rectangle contained by AB and BM.

This property is general.

CHAPTER II

THE CIRCLE

1. **Ratio of Circumference to Diameter.** When the **radius** of a circle is known the magnitude of the circle is determined. The **diameter** is twice the radius. If a square is circumscribed to the circle, each side of the square is equal to a diameter, and, since the perimeter of the square is greater than that of the circle, it is seen that the **circumference** of the circle is less than four diameters. In like manner, if a regular hexagon is inscribed to the circle, it is seen that each side is equal to a radius and that the circumference is greater than three diameters. The exact ratio of the circumference to the diameter cannot be expressed by a finite nnmber of figures, though, by means of formulæ that cannot here be derived, it can be computed to any degree of accuracy. In practice we employ an approximate value $3\frac{1}{7}$, or 3·14159, or 3·1416. In the statement of theorems and results the exact value is denoted by π. Hence, if r, d, c measure the radius, the diameter, and the circumference of a circle, we have the relations:

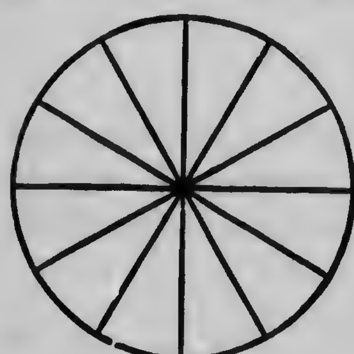
$$\begin{aligned} d &= 2r, \\ c &= \pi d = 2\pi r, \\ &= 3\tfrac{1}{7}d = 6\tfrac{2}{7}r, \text{ (approximately).} \end{aligned}$$

2. **Area.** It is shewn in more advanced works on metrical geometry that, if a denotes the measure of the area of a circle, the area is given by the formula,

$$a = \pi r^2.$$

The argument is too refined to be given complete statement here. However, it is well to look into the problem and see how its solntion is approached.

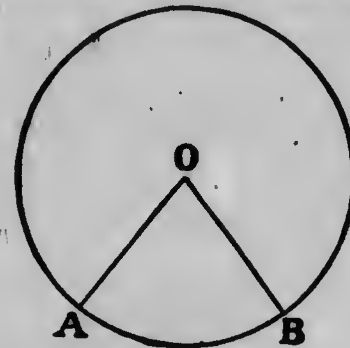
Suppose the circle divided into a large number of equal sectors. Let the number be even and suppose the equal sectors, making up a semi-circle, cut out and arranged as in the figure. Now let the number of sectors be made greater and greater. Then each sector becomes more and more nearly an isosceles triangle of altitude equal to the radius; the bases of the sectors as arranged in the figure become more and more nearly a straight line of length



equal to the semi-circumference; and the aggregate of sectors, tends to an arrangement which presents them as making up one-half of a rectangle whose base is the semi-circumference and altitude the radius. The area, *a* of the whole circle, being twice that of the semi-circle, would thus appear to be given by the formula:

$$a = \tfrac{1}{2} rc = \tfrac{1}{2} r . 2\pi r = \pi r^2.$$

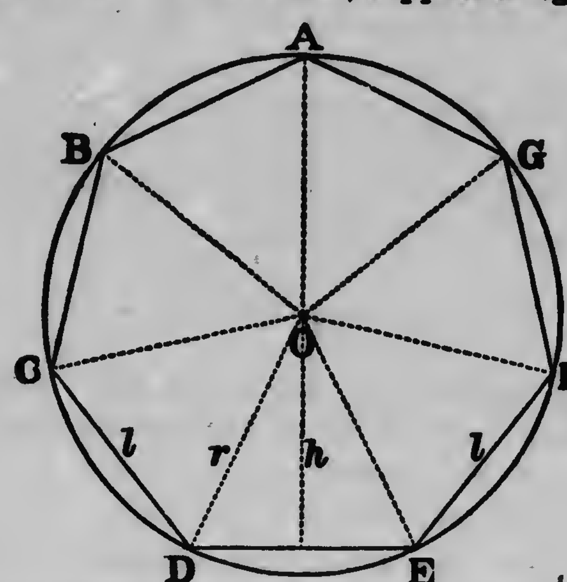
Or we may regard the problem in the following way. The area of a *triangle* is given by one-half the product of the measures of the altitude and the base, and from the vertex to the base only one line, the perpendicular, can be drawn which can be called the altitude. Now if we consider the *sector* OAB as a *sort of triangle*, it has the property that every straight line drawn from the vertex to the base is perpendicular to the base, or that every such line is its altitude. We are thus made to think that the area of the sector is given by one-half the product of



the measures of its base and its altitude, *i. e.*, of its arc and the radius. Then letting the sector enlarge itself so as to make up the whole circle we are directed again to the result:

$$a = \frac{1}{2}rc = \pi r^2.$$

A third method of approaching the question may be found instructive. In the circle, suppose a regular polygon



of n sides to be inscribed. Join the centre to the angular points, dividing the polygon into n triangles. Let l measure the length of a side of the polygon, p the perimeter of the polygon, $\bar{a}$ its area, and h the altitude of each triangle.

The measure of the area of one of these triangles $= \frac{hl}{2}$.

$\therefore$ The measure of the area of the polygon $= \frac{nhl}{2}$.

$$\therefore \bar{a} = \frac{ph}{2}, \text{ (since } p = nl).$$

Now let the number of sides of the polygon be made greater and greater. Then the area of the polygon becomes more and more nearly equal to that of the circle; p its perimeter becomes more and more nearly equal to the circumference, and h becomes more and more nearly equal to the radius. Thus the relation $\bar{a} = \frac{ph}{2}$ seems to shade into the relation:

$$a = \frac{cr}{2} = \pi r^2.$$

Exercises

NOTE.—The approximation 3·1416 is to be employed for the the ratio $c:d$, if nothing is said to the contrary.

1. Find the circumferences of the circles whose radii measure:

8 in., 3·5 m., 14 yd., 9 dm.;

(1) taking the ratio $c:d$ as $3\frac{1}{7}$ and giving fractional results as fractions, and as decimals correct to hundredths;
(2) taking this ratio as 3·1416 and giving results correct to hundredths.

2. Find the radii of the circles whose circumferences measure:

22 m., 7·7 yd., 64·3 dm., 75·1 in.;

(1) taking the ratio $c:d$ as $3\frac{1}{7}$ and giving fractional results as fractions, and as decimals correct to hundredths;

(2) taking this ratio as 3·1416 and giving results correct to hundredths;

(3) taking 0·3183 as an approximation to the ratio $d:c$, and giving results correct to hundredths.

3. One circle has a radius twice as great as another; compare their circumferences. (Has the result of the comparison been already assumed ?)

4. Find the side of a square whose perimeter measures the same as the circumference of a circle whose radius is 1 metre.

5. Taking the ratio $c:d$ as $\frac{22}{7}$, find the number of revolutions made by a carriage wheel, whose radius measures 2 ft., in going 1 mile.

6. Taking the ratio $c:d$ as $\frac{22}{7}$, find the number of revolutions made by a carriage wheel whose radius measures 0·5 m., in going 1 kilometre.

7. A wheel 3 feet in diameter made 8439 revolutions in a journey from one town to another. Find the distance between the towns.

8. The radius of a circle is 40 inches; find the length of an arc which subtends an angle of 40 degrees at the centre.

9. A degree of longitude in Toronto measures 264613·3 feet. Find in miles the length of the parallel of latitude passing through Toronto.

10. Two circles, each of radius 1 metre long, pass each through the centre of the other; find the perimeter of the area common to the two circles.

11. A wheel of radius 2·5 feet rotates on a fixed axle, making 40 revolutions in a minute; through what distance does a point on the rim of the wheel pass in 1 hour ?

12. The perimeter of a semi-circle is 48 in.; taking the ratio $c:d$ as $\frac{22}{7}$, find the radius of the semi-circle.

13. A circle is described about an equilateral triangle of side 0·32 m. long; taking the ratio $c:d$ as 3·14159, find to two places of decimals the length of the circumference of the circle.

14. Find the areas of the circles whose radii measure:

7 m., 3·5 in., 6 yd.

(1) taking the ratio $c:d$ as $\frac{22}{7}$ and giving results as fractions, and as decimals correct to tenths;
(2) taking the approximation 3·1416 and working to hundredths.

15. Find the area of a circle whose circumference measures 45 inches.

16. If the area of a circle measures 2·37 ares, find the length of its radius and of its circumference.

17. Employing the formula for the area of a circle, shew that the areas of two circles are to one another as the squares of (the measures of) their radii.

18. The cost of fencing a circular plot of ground at $1·50 a yard was $1650. Find the area of the plot.

19. Two equal circles of radius 21 dm. pass each through the centre of the other. Find the area common to the two circles taking $\frac{22}{7}$ as the ratio $c:d$.

20. The common chord of two equal circles of radius 15·4 cm. is equal to the radius. Find the area common to the two circles taking $\frac{22}{7}$ as the ratio $c:d$.

21. Shew that the semi-circle described on the hypotenuse of a right-angled triangle is equal to the sum of the semi-circles described on the two sides of the triangle.

22. A road runs around a circular pond; the outer circumference is 440 yd., and the width of the road is 20 yd. Find the area of the pond.

23. A circular court of 100 ft. diameter is to have a walk 10 ft. wide around it on the inside. The remainder is to be sodded. Find the total cost, if the pavement costs $1·30 a sq. yd. and the sodding 25 ct. a sq. yd.

24. The area of a circle whose circumference is 27 inches is divided into two equal parts by a circle described about the same centre. Find correctly to three decimal places the circumference of the latter circle.

25. Find the circumference of a circle whose area is equal to that of a square, the diagonal of which is 35 feet.

26. A rectangle, a semi-circle and an isosceles triangle have equal bases and equal altitudes; shew that their areas are as $4 : \pi : 2$.

27. Three circles each 3 feet in diameter touch each other. Find the area of the enclosed figure.

28. A square is inscribed in a circle of radius 11·2 in. Taking the ratio $c:d$ as $\frac{22}{7}$, find the area of the part of the circle without the square.

29. If in example 28 the radius measures r, find in terms of π and r, the measure of the area in question.

30. A square is inscribed in a circle whose circumference is $13\frac{3}{4}$ yards; find its area.

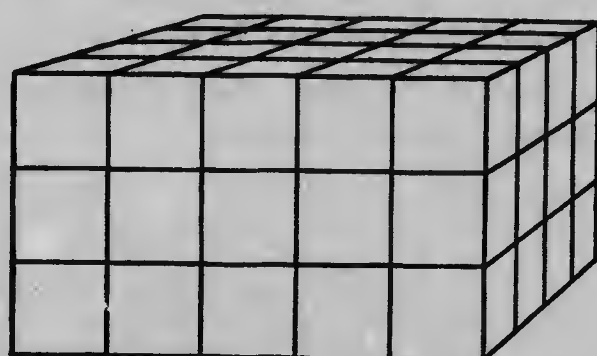
31. A regular hexagon is inscribed in a circle of radius 12·6 cm. Find the area of the part of the circle without the hexagon.

32. If in example 31 the radius measures r, find in terms of π and r the measure of the area in question.

CHAPTER III

THE SIMPLER SOLIDS

1. **The Cuboid.** If a parallelepiped, *i. e.*, a solid figure contained by six parallelograms of which every opposite two are parallel, has all its faces rectangles, it will be called a **cuboid.** The cuboid is then a rectangular parallelepiped.



If the dimensions of a cuboid are 3 in., 4 in., 5 in., then, referring to the figure, we see that it can be divided into $(3\times4\times5)$ c. in. The cases, in which one or more of these dimensions involve fractional or irrational numbers, are to be regarded as in the analogous case of the rectangle, and we have the following rules:

The volume of a cuboid is measured by the product of the measures of its length, breadth and height; or, by the product of the measures of its height and the area of its base.

A diagonal of the cuboid may easily be found as it is seen to be the hypotenuse of a right-angled triangle whose sides are the altitude of the cuboid and a diagonal of its base.

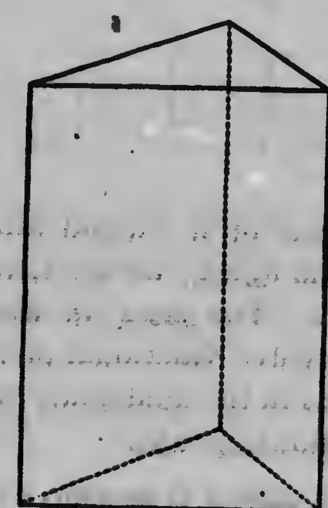
2. **The Rectangular Prism.** On the base of a cuboid suppose a polygon to be traced; then that part of the

cuboid which stands vertically above the polygon is called a **rectangular**, or **right, prism.** The base and the upper surface are parallel and equal; the vertical faces are rectangles.

It is readily seen that the volumes of the cuboid and the right prism are to each other as their bases. Hence we have:

The volume of a right prism is measured by the product of the measures of its height and the area of its base.

In what follows, the word prism, will mean right prism, if nothing to the contrary is stated.



The figure shown is that of a **triangular prism,** *i. e.*, a prism standing on a triangular base.

3. **The Cylinder.** If on the base of a cuboid a circle be traced, then that part of the cuboid standing vertically above the circle is called a **right circular cylinder.** As in the case of the prism, it is seen that the volumes of the

cylinder and the cuboid are to each other as their bases. Therefore we have:

The volume of a cylinder is measured by the product of the measures of its height and the area of its base.

In what follows, the word cylinder, will mean right circular cylinder, if nothing to the contrary is stated.

In the figure, OD is the radius of the base and OP the height or altitude. The measures of these lines are denoted by r and h. Therefore if v measures the volume of the cylinder and a the area of its base, we have:

$$v = ah$$
$$= \pi r^2 h.$$

The cylinder may be regarded as having been generated by the rotation of the rectangle ODBP about the side OP.

Suppose the curved surface of the cylinder to be a sheet; then if this sheet is slit along DB, it may be opened up and laid on a plane, where it is seen as a rectangle with DB as one side and the circumference of the base as an adjacent side. Therefore if c measures the circumference of the base, the measure of the area of the curved surface is ch which is equal to $2\pi rh$.

Exercises

1. Find the volumes, and the areas of the surfaces, of the cuboids whose dimensions are:

(1) 7 ft., 11 ft., 13 ft.
(2) 9 in., 23 in., 25 in.
(3) 2·3 m., 5·7 m., 6·1 m.
(4) 17·1 cm., 18·3 cm., 21·5 cm.
(5) 13·2 mm., 15·1 mm., 18·7 mm.

2. The volume of a cuboid measures 506 cubic feet, and its height is 8 feet; find the area of its base.

3. The volume of a cuboid on a square base is 507 c. ft., and its height is 3 ft.; find the length of the side of the square base, the area of the surface of the cuboid, and the total length of its edges.

4. The volume of a cuboid whose breadth is twice, and its length three times, its height, is 2058 cubic feet; find the dimensions of the cuboid.

5. The dimensions of a room are 16 ft., 20 ft., 25 ft.; find the length of the diagonal of the room.

6. Construct a line the measure of whose length is $\sqrt{3}$.

7. Cubes of metal the length of whose edges are 3 cm., 4 cm., 5 cm., are melted and cast into a single cube. Find the length of its edge.

8. The external dimensions of a rectangular covered box, made of 1 inch stuff, are 7 ft., 8 ft., 9 ft.; find the capacity of the box and the quantity of lumber in it.

9. If a cubic foot of water weighs 1000 ounces and ice expands 10 per cent. in freezing, what volume of ice will weigh 28,800 pounds?

10. If each dimension of a cubical block of iron is increased $\frac{1}{10}$ by heating, what is the percentage of increase in volume?

11. Find the length of a pipe which will hold 180 gallons of water, if the cross-section is a rectangle 6 inches by 3 inches, given that the volume of 1 gallon is 277·463 cubic inches.

12. Find the volume of a prism whose base is an equilateral triangle the length of whose side is 30 in., and which is 20 in. high.

13. Find the volume of a prism standing on a triangular base the lengths of whose sides are 21 cm., 20 cm., 29 cm., and which is 34 cm. high.

14. Find the volume of a prism whose base is a regular hexagon the length of whose side is 20 in., and which is 30 in. high.

15. The volume of a prism whose base is an equilateral triangle is 640 c. in.; if its height is 25 in., find the length of a side of the base.

16. Find the cost of digging a ditch $1\frac{1}{4}$ mi. long, 4 ft. deep, $3\frac{1}{2}$ ft. wide at the bottom, and $5\frac{1}{4}$ ft. wide at the top, at 15 ct. a c. yd.

17. Find the surface and volume of a cylinder 60 feet long, the radius of the base being 8 feet.

18. Find the surface and volume of a cylinder of height 23 cm., the radius of the base being 5 cm.

19. The volume of a cylinder is 160 cubic feet and its height 5 feet; find the radius of its base.

20. Find the area of the surface of a cylinder of height 15 inches, the radius of the base being 3 inches.

21. A cylindrical tank 5 feet in diameter and 8 feet deep is filled with water; find the number of gallons of water, given that a gallon contains 277·5 cubic inches.

22. The height of a cylinder is to the diameter of its base as 3:2; if its volume is 320 cubic inches find its height.

23. The circumference of a circular plate of metal is 22 in. and the thickness is 2 in. Find its thickness after it has been extended by hammering until its area is 308 sq. in.

24. The area of the curved surface of a cylinder is 135 sq. cm. and its height is 4·5 cm.; find its volume.

25. The area of the curved surface of a cylinder is to the total area of its surface as 3:4; if the area of its base is 35 sq. in., find its volume.

26. From a bar of wood whose cross section is a hexagon of side 1 dm. and whose length is 1 metre, the largest possible cylindrical bar is turned. What volume is converted to shavings?

27. A lead tube 3 feet long and of internal diameter 2 inches is melted and cast into a cube. What is the edge of the cube if the lead is $\frac{1}{4}$ of an inch thick?

28. How many washers $\frac{1}{4}$ of an inch thick, of 1 inch internal diameter and $1\frac{1}{4}$ inch external diameter, can be cast from a rod of iron 6 feet long and 1 inch in radius, allowing 6 per cent. for waste?

29. What is the weight of a closed iron cylinder filled with mercury, if the height is 25 cm. and the external diameter 10 cm. supposing the iron to be 1 cm. thick. 1 c. cm. of iron weighs 7·8 g. and 1 c.cm. of mercury weighs 13·6 g.

SECTION III

MISCELLANEOUS APPLICATIONS

1. **Averages.** A merchant's sales for a week are given by the table:

Monday....	\$312·50
Tuesday ...	\$290·23
Wednesday	\$330·19
Thursday..	\$304·10
Friday......	\$298·18
Saturday...	\$480·26

The total sales for the week are seen to be \$2015·46. This total is the same as if on each day he had sold one-sixth of this amount, or \$335·91. Then \$335·91 is called the **average** of the daily sales for the week.

The following examples will serve to develop the idea of average.

Exercises

1. A traveller's expenses for a week were as follows: Monday, \$8·70; Tuesday, \$7·50; Wednesday, \$7·40; Thursday, \$9·60; Friday, \$13·25; Saturday \$12·10. Find his average daily expenses.

2. The net profits of an enterprise for three consecutive years were \$3500, \$4320, \$2120; find the average yearly profit.

3. The circulation of a newspaper for a certain week was declared to be: Monday, 45,384; Tuesday, 46,329; Wednesday, 46,482; Thursday, 44,297; Friday, 44,693; Saturday, 52,370. Find the average daily circulation for the week.

4. Of a force of 100 policemen 7 are of height 6 ft. 2 in.; 15 of height 6 ft. 1 in.; 32 of height 6 ft.; 29 of height 5 ft. 11 in.; and the remainder of height 5 ft. 10 in. Find the average height. How many fall below the average ?

5. A speculator bought 5 sections of land, and sold 2 sections at an advance of 25 per cent., 2 sections at an advance of 40 per cent., and the remaining section at a gain of 10 per cent. Find the average gain.

6. Find the average of the numbers:

5, 8, 11, 14, 17, 20, 23, 26, 29, 32.

7. On 5 examination papers of 100 marks each, a candidate obtained an average of 77 marks. On the first two papers he obtained 54 and 82 marks. What was the average on the remaining papers?

8. Find the average of the first twenty integers.

9. A speculator's gains and losses for a week are as follows: Monday, gain $26·00; Tuesday, loss $4·60; Wednesday, gain $277·25; Thursday, gain $72·15; Friday, loss $84·80; Saturday, gain $192·40. What is his average daily gain?

10. Find the average area of 3 circles of radii 35 in., 42 in., and 49 in. What is the area of the circle of the average radius? Take $\frac{22}{7}$ as the approximation to the ratio *c*:*d*.

11. If a train travels $\frac{1}{3}$ of a certain distance at the rate of 32 mi. per hour, $\frac{1}{4}$ of the remainder at 40 mi. per hour, and the remaining distance at the rate of 36 mi. per hour, find the average rate in mi. per hour.

12. A grain dealer buys 2000 bu. of wheat at 63 ct. a bu. He sells 200 bu. at 65 ct. a bu.; 1200 bu. at 64 ct. a bu.; 360 bu. at 63$\frac{1}{2}$ ct. a bu.; and the remainder at 62$\frac{1}{2}$ ct. a bu.; find the average gain per bu.

13. The distance of the three vertices of a triangle from a straight line in its plane are 7 in., 10 in., and 15 in. Find the average distance from this line.

14. A grocer bought 1000 bbl. of flour at $4·90 a bbl. He sells 200 bbl. at $5·40 a bbl.; 40 per cent. of the remainder at $5·30 a bbl.; 230 bbl. at $5·05 a bbl.; 20 per cent. of the remainder at $4·86 a bbl.; and the remainder at a loss of 10 per cent. Find the average gain a bbl.

15. A owes B $60 at the end of 3 months and 60 dollars at the end of 9 months. Find the average term of credit.

16. A owes B $60 at the end of 30 days; $25 at the end of 40 days; $15 at the end of 45 days; find the average term of credit.

17. The average of 7 results is 13; the average of the first 3 is 10, and that of the last 3 is 15. Find the fourth number.

2. Proportionate Distribution. The following examples will illustrate the method of solving problems which call for a division of a quantity into parts in accordance with certain stated conditions.

Ex. 1. Divide $360 among A, B and C in the proportion of 3:4:5.

The meaning is that, out of every $12, A is to receive $3, B $4, and C $5. Therefore they will receive $\frac{3}{12}$, $\frac{4}{12}$, and $\frac{5}{12}$ respectively, of the sum to be divided. Their shares are then $90, $120, and $150.

Ex. 2. Divide $545 among A, B and C, so that A may have $35 less than twice as much as B, and B $40 more than C.

In such problems it is well, as a rule, to choose the share of one of the persons as a unit, and to express the shares of the others in terms of this.

Here choose C's share as this unit.

Then B's share = C's share + $40,

and A's share = 2 B's share − $35

= 2 C's share + $80 − $35

= 2 C's share + $45.

∴ The three shares make up 4 C's share + $85 which must equal $545.

∴ 4 times C's share must equal $460,

∴ C's share = ¼ of $460 = $115;

∴ B's share = $115 + $40 = $155;

∴ A's share = $155 × 2 − $35 = $275.

Exercises

1. Divide \$631·20 among 6 men, 12 boys and 18 girls, so that each man may receive twice as much as each girl, and each boy 10 cents more than each girl.

2. Divide 220 into three parts in the proportion of 2: 7: 13.

3. Divide 527 acres among A, B, C and D so that A's share is to B's share as 2:3; B's share is to C's share as 2:3; and C's share is to D's share as 3:4.

4. Divide \$100 among A, B and C so that A's share is to B's share as 2:1, and $16\frac{2}{3}$ per cent. of A's share is equal to C's share.

5. A and B form a partnership. A contributes \$600 at the beginning of the year, \$200 at the end of 3 months and \$400 at the end of 8 months; B contributes \$300 at the beginning of the year, \$400 at the end of 6 months and \$400 at the end of 8 months. How should a profit of \$900 be divided at the end of the year?

6. A father divides \$12,000 between his two sons, aged 18 and 20 years, so that if their shares be invested at 5 per cent. per annum they will have equal shares when they become of age. What sum does he give to each?

7. Divide \$720 between A and B so that $\frac{1}{2}$ of A's share exceeds $\frac{1}{3}$ of B's by \$10.

8. Divide \$108 among four boys in such a way that the second may have one-half as much as the first, the third one-half as much as the first two, and the fourth one-half as much as the first three.

9. Divide \$180 among three men so that the second may have \$44 more than one-third as much as the third, and the first \$52 more than one-third as much as the second.

10. Divide \$11·60 among 13 men and 17 boys in such a way that 3 men may receive as much as 5 boys.

11. Two men invest capital in the ratio of $4\frac{2}{3}$ to 5. At the end of 3 months the first withdraws $\frac{1}{3}$ of his capital and the second withdraws $\frac{2}{5}$ of his capital. The year's gains are \$6500. How should this be divided?

12. Divide \$2160 among 12 men, 18 women, 27 boys and 30 girls in such a way that each man receives $\frac{2}{3}$ as much as each woman, each woman $\frac{2}{3}$ as much as each boy, and each boy \$2 more than each girl.

3. Mixtures. The following examples are typical illustrations of problems that may be included under this heading.

Ex. 1. In what proportion should tea worth 40 ct. a lb. and tea worth 60 ct. a lb., be mixed to make a mixture worth 45 ct. a lb.?

When we say that the mixture is worth 45 ct. a lb., we are in a way saying that part of the tea is worth 5 ct. more than its value, and that part is worth 15 ct. less than its value, the proportions however being such as to balance the excess and the defect.

Now if $\frac{1}{5}$ of a lb. of the tea worth 40 ct. be taken, the excess is 1 ct., while if $\frac{1}{15}$ of a lb. of tea worth 60 ct. be taken the defect is 1 ct. Therefore if the two teas are mixed in the proportion

$$\tfrac{1}{5} : \tfrac{1}{15}, \textit{ i.e.}, 3:1$$

the balance of excess and defect is secured.

A rule for such problems has been given under the name of *alligation*: Write down the values of each kind and of the mixture as below:

$$\begin{array}{ccc} 40 & & 15 \\ & 45 & \\ 60 & & 5 \end{array}$$

Subtract 40 from 45 and place the difference in line with 60 as shewn; subtract 45 from 60 and place the difference in line with 40 as shewn. The proportion is 15:5.

Ex. 2. One gal. of spirits containing 15 per cent. of water is mixed with 2 gal. of spirits containing 20 per cent. of water. Find the strength of the mixture.

The quantity of water in the first $= \frac{15}{100}$ of 1 gal.

The quantity of water in the second $= \frac{20}{100}$ of 2 gal.

$= \frac{40}{100}$ of 1 gal.

$\therefore$ Total quantity of water in mixture $= (\frac{15}{100} + \frac{40}{100})$ gal.

$= \frac{55}{100}$ gal.

$= \dfrac{18\frac{1}{3}}{100}$ of 3 gal.

$\therefore$ Since in the mixture there are 3 gal., $18\frac{1}{3}$ per cent. of the mixture is water, or, the mixture is $81\frac{2}{3}$ per cent. strong.

Exercises

1. An alloy contains 40 per cent. of lead. How much lead must be added to 1 gramme of the alloy to make it contain 50 per cent. of lead?

2. Two tea-chests contain black and green tea in the proportion of 2:5, and 3:7; if the contents of both chests are mixed, find the proportion of black and green tea in the mixture, if the first chest is three times as large as the second.

3. Coffee worth 40 ct. a lb. was mixed with chicory worth 15 ct. a lb. in such a proportion that when the mixture was sold at 24 ct. a lb. there was realized a gain of 10 per cent. Find the proportion.

4. Mercury and zinc are mixed in the proportion of 9:1 to form an amalgam. How much mercury must be added to 1 gramme of the mixture to form an amalgam one-tenth as strong?

5. Wine and water are mixed in the proportion of 7:1. How much water must be added to 50 gal. of the mixture in order that there may be a gain of 30 per cent. in selling at the cost price of the wine?

6. Find the quantities of tea at 25 ct. and 37 ct. a lb. respectively, which must be mixed with 25 lb. at 32 ct. a lb. to make up a total of 47 lb. which may be sold at 45 ct. a lb. at a gain of 50 per cent.

7. A milk-can contains 50 quarts of milk; after each quart is dipped out a quart of water is added. What per cent. of milk does the fifth customer receive?

8. Three gal. of spirits 80 per cent. strong, 8 gal. 75 per cent. strong and 12 gal. 70 per cent. strong are mixed with 2 gal. of water. What is the percentage strength of the mixture?

9. Equal volumes of lead, tin and zinc are melted together. How many g. of tin and zinc must be added to 100 g. of the alloy to make equal proportions by weight of the three metals, given that, 1 c. cm. of lead weighs 11·4 g., 1 c. cm. of tin weighs 7·3 g., and 1 c. cm. of zinc weighs 7·1 g.?

10. Four gal. of spirits 90 per cent. strong, are poured into a ten-gal. keg, containing 3 gal. of spirits 88 per cent. strong; if the keg is then filled with water, what is the strength of the mixture?

11. A grocer mixed together two kinds of tea and sold the mixture, 144 lb., at an advance of 20 per cent. on cost, receiving for it \$62·10. Had he sold each kind of tea at the same price per lb. as he sold the mixture he would have gained 15 per cent. on the one and 25 per cent. on the other. How many lb. of each kind were there in the mixture, and what was the cost of each per lb.?

12. Two kinds of sulphur, 95 per cent. and 90 per cent. pure are mixed, and the resulting mixture is $91\frac{2}{3}$ per cent. pure. In what proportions were they mixed?

13. A grocer mixed three kinds of sugar worth 3 ct., $3\frac{1}{2}$ ct. and 4 ct. a lb. in the proportion of 5:3:2; if he sold the mixture at 5 ct. a lb., what was his gain per cent.?

14. A tobacconist mixed 4 kinds of tobacco worth 20 ct., 30 ct., 40 ct. and 50 ct. a lb. in the proportion of 4:3:2:1; if he sold the mixture at 40 ct. a lb., what was his gain per cent.?

15. A grocer mixed three kinds of tea worth 25 ct., 30 ct. and 35 ct. a lb., in the proportion of 4:3:2, and sold the mixture at 40 ct. a lb., thereby gaining \$5·00. What quantity of each kind of tea was sold?

4. **Work.** In problems involving the combining or separating of the work done by different agents, the student should bear in mind that it is the *work* done that is combined or separated.

Ex. A can do a piece of work in 24 days, and B in 18 days; in what time could they together do the piece of work?

In 1 day A can do $\frac{1}{24}$ of the work.

In 1 day B can do $\frac{1}{18}$ of the work.

∴ In 1 day A and B can do $\frac{1}{24}+\frac{1}{18}$ of the work.

Now $\frac{1}{18}+\frac{1}{24}=\frac{7}{72}$.

A and B together do $\frac{7}{72}$ of the work in 1 day.

∴ A and B together do $\frac{72}{72}$ of the work in $\frac{72}{7}$ or $10\frac{2}{7}$ days.

Exercises

1. One man can do a piece of work in 6 days, another can do it in 8 days and a third can do it in 9 days. How long will it take the three working together to do the work?

2. A man can mow 1 acre of hay in 1 day, and a boy can mow $\frac{3}{4}$ acre in 1 day. How long will it take 3 men and 4 boys, working together, to mow 24 acres of hay?

3. If a man can do $\frac{3}{2}$ as much work as a boy, how long will it take 6 men and 9 boys, who are joined after 4 days, by 2 men and 3 boys, to do a piece of work that would take 10 men 25 days to finish?

4. Eight men promised to do a piece of work in 4 days; but at the end of each day one man left off work; how long did it take to complete the work?

5. A man can do a certain piece of work in 56 days; a boy can do the same piece of work in 64 days; how long will it take 6 men and 10 boys to do the work?

6. One man can do a piece of work in 12 days; another can do it in 16 days; after working together for three days, the second man left off work; how long did it take the first man to finish the work?

7. A and B could plough a field in 6 days, B and C could plough it in 7 days, and C and A could plough it in 8 days, how long would it take the three, working together, to plough the field?

8. A builder undertook to build a house in 42 days, and engaged 30 men to do the work. But after 20 days he found it necessary to engage 20 men more, and then he accomplished the work 2 days too soon. How many days behind-hand would he have been if he had not engaged the 20 additional men?

9. A and B can do a piece of work in $2\frac{8}{11}$ days, B and C can do it in $3\frac{3}{13}$ days, and A, B and C can do it in $1\frac{103}{107}$ days. How long will it take B to do it?

10. Twenty men can perform a piece of work in 12 days; find how many men will perform a piece of work half as large again, in a fifth part of the time, supposing that they work the same nnmber of hours in the day, but that two of the second set can do as much work as three of the first set.

11. A can do 10 per cent. of a piece of work in 6 days of 10 hours each; B can do 15 per cent. of it in 10 days of 8 hours each. If both men work together and the whole work is worth $102, what does each man get?

12. Five men mow $\frac{1}{3}$ of a field of hay in 4 days; then 6 boys work with them for 1 day. If it takes the men $2\frac{1}{3}$ days to finish the work, how long would it have taken 12 boys to mow the field?

13. Ten men or 16 boys can do a piece of work in 80 days; if they all work together for 20 days, and if then 2 men and 6 boys leave off work, how long will it take the rest to finish the work?

14. If 2 men and 4 boys can do a piece of work in $9\frac{3}{8}$ days, and 3 men and 3 boys can do an equal amount of work in $9\frac{1}{11}$ days, how long will it take 5 men and 6 boys to do twice the amount of work?

15. If 10 men, or 14 boys, or 18 girls can do a piece of work in 30 days, in what time will the work be completed if they all work together for 8 days, and then all of the girls, 7 boys and 2 men leave off working?

16. Four men or 5 boys could do a piece of work in 24 days. If 6 men and 8 boys were employed at the piece of work, and 3 days before it was completed 2 men and 3 boys left off working, how long did it take to do the work?

5. Velocities. The following examples will serve to bring out what is meant by *relative velocity*.

Ex. 1. Two trains of lengths 100 yd. and 120 yd. are moving on parallel tracks in opposite directions at the rates of 30 mi. and 40 mi. an hr.; in what time will they pass each other?

After the engines meet they *separate* at the rate of 30 mi.+40 mi., or 70 mi., an hr., *i.e.*, their *relative velocity* is 70 mi. an hr. The trains will have passed each other when the engines will have become separated 100 yd.+120 yd., or 220 yd.

$$\therefore \text{ the time required} = \left(\frac{220\times60\times60}{70\times1760}\right)\text{ sec.}$$

$$= 6\tfrac{3}{7}\text{ sec.}$$

Ex. 2. At what time after 2 o'clock is the minute-hand first opposite the hour-hand?

The minute-hand moves through 60 minute spaces (on the dial) in 60 minutes (time).

The hour-hand moves through 5 minute spaces in 60 minutes.
∴ The minute-hand *gains*, on the hour-hand, 55 minute spaces in 60 minutes.

The minute-hand is first opposite the hour-hand when it has gained 10+30, or 40 minute spaces.

$$\therefore \text{ the time required} = \tfrac{40}{55} \text{ of } 60 \text{ minutes},$$
$$= 43\tfrac{7}{11} \text{ minutes}.$$

Exercises

1. A boy starts out from home walking at the rate of 3 mi. per hr., and $1\frac{3}{4}$ hr. later his father starts out in the same direction, walking at the rate of 4 mi. per hr. In what time will the father overtake his son and how far from the starting point will they be?

2. A man can row 1 mi. up stream in 20 min. and 1 mi. down stream in 15 min. At what rate is the stream flowing?

3. Two men start together from the same point to run around a circular track 1 mile in length. If one runs 18 rods while the other runs 13 rods, at what point in the track will they first be together again, and how far will each have gone?

4. When first after 3 o'clock are the hour-hand and the minute-hand of a watch together?

5. Two trains, 180 feet and 220 feet long respectively, pass each other in $2\frac{1}{4}$ sec. in going in opposite directions. In going the same direction one passes the other in 12 sec. Find the rates of the trains (in miles per hour).

6. When first after 6 o'clock are the hour-hand and the minute-hand of a clock at right angles?

7. A leaves home at 1 p. m. walking at the rate of 3 miles per hour. After walking 2 hours he rests half an hour. At 3.15 p. m. B starts out after him walking at the rate of 4 miles per hour. At what time will B overtake A?

8. When first after 4 o'clock are the hour-hand and the minute-hand of a clock opposite ?

9. In a mile race between a bicycle and a tricycle their rates were as 5 to 4; the latter had half a minute's start and was beaten by 173 yards. Find the actual rate of each.

10. A train 110 yards long overtakes A who is going at the rate of 4 miles an hour, and passes him in 9 seconds. Ten minutes after leaving A the train meets B and passes him in 7½ seconds. In what time after meeting the train will B meet A ?

11. A and B, two bicyclists, start abreast on a circular race-track 35 yards in width. If A rides on the inside of the track at the rate of 6 miles an hour and B rides on the outside of the track at the rate of 10 miles an hour, in how many hours will they be abreast again if the internal diameter of the race-track is 350 yards, taking $\frac{22}{7}$ as the approximation to the ratio $c:d$?

12. Four men start together from the same point, and run in the same direction around a ring at different uniform speeds. The first runs at the rate of 10 miles, the second at the rate of 10¾ miles, the third at the rate of 11¼ miles, and the fourth at the rate of 12½ miles, each per hour. At what part of the ring will they be first all together again after starting ?

13. If A can beat B by 4 yards in 100 yards, and B runs 100 yards while C runs 104 yards, and C runs 100 yards while D runs 95 yards, how many yards start must A give D in a 100 yards race that they may come in abreast ?

14. A watch which is set correctly at noon indicates 10 minutes after 9 the same evening when the true time is 10. What is the true time when the watch indicates 11 on the same evening?

15. A ship 78 miles from shore springs a leak which admits 2¼ tons of water in 6¼ minutes. If the pumps throw out 15 tons an hour, and 68 tons are sufficient to sink her, find the average rate of sailing that she may just reach the shore as she begins to sink.

SECTION IV

MISCELLANEOUS EXAMPLES

I

1. Find the value of:

$$\frac{2\frac{1}{4}}{5\frac{1}{4}} + \frac{\frac{4}{5}\text{ of }\frac{5}{6} - \frac{2}{3}\text{ of }\frac{1}{12}}{(\frac{3}{5}+\frac{2}{3}) \div (\frac{2}{3}+\frac{4}{5})} - \frac{1}{13}\text{ of }\frac{6}{7} + \frac{1}{11},$$

expressing the result as a decimal correct to thousandths.

2. Shew that when any integer is divided by 3 the remainder is the same as when the sum of its digits is divided by 3.

3. Divide \$300 among A, B, C, D, so that A's share will be twice B's, B's share twice C's, and C's share twice D's.

4. Find the interest on \$590·75 from Dec. 13, 1902, to May 7, 1903, at $3\frac{1}{2}$ per cent. per annum.

5. In a triangle whose sides measure 26 ft., 40 ft., 42 ft., find the length of the perpendicular drawn from each angle to the opposite side.

II

1. Simplify:

$$\frac{3}{5}(3\frac{1}{3} + 1\frac{1}{4})£ + \frac{1\frac{1}{3} - \frac{1}{3}\text{ of }1\frac{4}{5}}{\frac{1}{16}\text{ of }3\frac{1}{3} + \frac{4}{15}} \times 0·95\text{ of }5\,s. + \frac{8·4}{0·012}\,d.$$

expressing the result as the decimal of £1.

2. Shew that 11 is a factor of:

(*a*) Every integer expressed by an even number of digits 9; *e.g.*, 999999.

(*b*) Every integer expressed by the digit 1 in the first and the last place, with an even number of intervening zeros.

3. Two boys are to receive the same sum; but, if one were to receive \$15 more and the other \$9 less, the one would receive three times as much as the other. What sum are they to receive?

4. If simple interest is allowed in what time will \$360 amount to \$382·50 at $2\frac{1}{2}$ per cent per annum?

5. A landlord after paying 6 pence in the pound on his rental, for property tax, and 4 per cent. on a mortgage of £13,250, had left £2044. What was his rental?

III

1. Find the value of:

$$\left\{ \frac{2\frac{1}{2}}{16} + \frac{3\frac{1}{2}}{12} \text{ of } 3\frac{1}{4} - \left(\tfrac{7}{18} \text{ of } 1\tfrac{5}{7} - \tfrac{1}{2}\right) \right\} \div \left\{ \frac{1\frac{1}{4}}{14} - \tfrac{3}{4} \text{ of } \tfrac{1}{3} \right\}$$

2. Shew that a number is divisible by 11 if the difference between the sum of the digits in the odd places and the sum of the digits in the even places is divisible by 11.

Shew that the converse also is true.

3. Divide \$360 among 4 men, 5 women and 6 children so that each man will have three times as much as a woman, and each woman twice as much as a child.

4. A note drawn Sept. 7, at 90 days, for \$1050 is discounted Sept. 20 at 6 per cent; find the proceeds.

5. A rectangle whose sides are in the ratio of 5:7 has an area of 61740 square metres. Find its sides.

IV

1. Simplify:

$$\left\{\frac{\frac{3}{4}+\frac{5}{6}+\frac{7}{8}+\frac{11}{12}}{\frac{2}{3}-\frac{1}{2}}\times\frac{1}{34\frac{1}{2}}\right\}+\left\{\frac{7\frac{1}{2}}{6\frac{1}{4}}+\frac{11\frac{1}{3}-2\frac{2}{3}}{11\frac{1}{3}+2\frac{2}{3}}\times10\frac{2}{13}-7\frac{1}{2}\right\}$$

2. The sum of the digits of each of two given numbers is divisible by 9. The numbers are added; shew that the sum of the digits in the result is divisible by 9.

3. A person by selling goods for \$291 loses 3 per cent. What should they have been sold for to gain 4 per cent.?

4. Find the proceeds of a note made Jnne 17 at 3 months for \$1200·00 with interest at $4\frac{1}{2}$ per cent. and discounted June 29 at 6 per cent.

5. The sum of \$600 is borrowed at the beginning of the year at a certain rate, and at the end of 8 months an additional \$300 is borrowed at double the previous rate. At the end of the year the interest on both loans is \$24. Find the rate on the earlier loan.

V

1. What number divided by $(\frac{2}{15}+\frac{1}{15})\div(3-\frac{1}{3})\times(\frac{1}{3}+\frac{1}{2})$ will give $\frac{3}{14}$ of $\dfrac{4\frac{4}{5}}{6\frac{1}{2}}$ of $\dfrac{6\frac{2}{11}}{11\frac{1}{4}}$ of 247?

2. The difference between the sums of the digits in the odd places and those in the even places of each of two

given numbers is divisible by 11. The numbers are added; shew that the sum also possesses this property.

3. A merchant sells cloth at 90 cents a yard, thereby gaining 20 per cent. By what per cent. must he increase his marked price in order to gain 40 per cent.?

4. Find the amount of $420 for 2 years 3 months at 5 per cent.

5. An equilateral triangle and a square have the same perimeter; find the ratio of their areas.

VI

1. Find the value of:

$$\frac{(0{\cdot}025)^3 - (0{\cdot}0125)^3}{(0{\cdot}025)^2 - (0{\cdot}0125)^2}$$

2. The L. C. M. of 63, 84, 99, 156 and another number prime to each of them is 684684. Find the other number.

3. What must be the marked price of goods which cost $12·00 in order that the merchant may allow a discount of 20 per cent. and still make a profit of 25 per cent.?

4. If simple interest is allowed, find at what rate $840 will produce $132·30 in 4½ years.

5. A boy is engaged for 20 days at $1·20 a day on the understanding that for every day he is idle, instead of receiving his wage, he is to pay 50 cents for his board. At the end of the time he received $17·20; how many days did he work?

VII

1. Find the value, correct to four places of decimals, of 29·37956 × 3·73564.

2. Find the least number which divided by 19 leaves the remainder 13, and divided by 31 leaves the remainder 25.

3. A boy is one-third as old as his father and he has a brother whose age is one-sixth of his own; the ages of all three amount to 50 years. Find the age of each.

4. For what sum must a ninety-day note be drawn to realize $150 when discounted at 6 per cent.?

5. A square and a regular hexagon have the same perimeter; find the ratio of their areas.

VIII

1. Find to four decimal places the value of:

$$1{\cdot}3276 \times 13{\cdot}39 \times 7{\cdot}25643.$$

2. The sum of two numbers is 4225, and their G. C. M. is 845. Shew that there are two pairs of numbers satisfying these conditions and find them.

3. A sum of money is to be divided among 17 boys and 9 men, and 8 boys are to receive as much as 5 men. After 4 boys and 5 men have received their shares, find what percentage of the sum remains.

4. The proceeds of a note for $1460, discounted 53 days before maturity are $1447·28. Find the rate of discount.

5. Black tea costing 51 cents a pound is mixed with green tea costing 34 cents a pound in the proportion of 10 to 7. Find the gain per cent. made in selling the mixture at 55 cents a pound.

IX

1. Find correct to four places of decimals the value of

$$(3·14159)^2 \times (2·78183)^2.$$

2. The G. C. M. of two integers is 17 and their L. C. M. is 4199. Shew that there are two pairs of such numbers and find them.

3. A farmer is to receive $168 from A, B and C for pasturage. A has had in pasture 15 cattle for $2\frac{1}{2}$ months, B, 8 cattle for 3 months, and C, 10 cattle for $2\frac{1}{2}$ months. Find what each should pay.

4. A man saves $400 a year and at the end of the year invests it at 5 per cent. per annnm. To what sum will his savings amount at the end of 4 years?

5. A circle and a regular hexagon have the same perimeter; find the ratio of their areas.

X

1. Find the prodnct of 5·3827 and 4·5938. Also by the contracted method find the prodnct correct to four places of decimals, and indicate the unnecessary work in the earlier mnltiplication if a result to four decimal places is sufficient.

2. The G. C. M. of two integers is 357 and their L.C.M. is 12852. Find the integers if they are each of four digits.

3. Divide $12·95 among A, B and C, giving A $1·50 less than three times as much as B, and B 80 cents more than twice C's share.

4. A note made Jan. 13, 1903, at 90 days for $1500 is discounted Feb. 20, at 6 per cent. Find to the nearest thousandth of 1 per cent. the rate per cent. made by the banker on his money.

Shew that the sum $1500 in no way affects the result.

5. How many pounds of coffee at 24 cents a pound must be mixed with 6 pounds at 36 cents a pound that a gain of 33⅓ per cent. may be realized by selling the mixture at 40 cents a pound?

XI

1. Divide 5·398727 by 7·329548 to five places of decimals without employing the contracted method, and then indicate the unnecessary work.

2. In England computers have frequently to express a sum given in pounds, shillings and pence, as pounds and a a decimal of a pound. Much use is made of the rule: For each florin (2 *s.*) write £·1; for each shilling write £·05; for each sixpence write £·025 and for each remaining farthing write £·001. If, however, the number of remaining farthings is 12, or is greater than 12 add an additional ·001. The result will be correct to three places of decimals.

Illustration: £17. 13 *s.* 10¾ *d.*

£17·	£17·65
·6	·025
·05	·020
·025	
·020	
£17·695	£17·695

Establish this rule.

3. A merchant sold 500 yards of cloth for $690, part of it at $1·50 a yard and the remainder at $1·30 a yard. How many yards were sold at each rate?

4. A holds against B two notes, one, a ninety-day note for \$600, and the other, a sixty-day note for \$900. The notes are exchanged for a single note for \$1500; for how many days must it be drawn if the proceeds from discounting it at 6 per cent. would equal the proceeds of the two original notes discounted at the same rate?

5. A well 7 ft. in diameter and 28 ft. deep is to have a lining of special bricks, fitting close together without mortar, 7 inches thick; find in tons the weight of the bricks, supposing one cubic inch of brick to weigh $\frac{3}{4}$ of an ounce., and 1 cwt.=112 lbs. (Take the approximation $\frac{22}{7}$ as the ratio *c:d.*)

XII

1. Find the value, correct to four places of decimals, of:

$$3·14159 \div 2·7182818.$$

2. (*a*) Find the G. C. M. of:

$$1\frac{8}{45},\ 6\frac{4}{11},\ 7\frac{3}{5},\ 9\frac{1}{15}.$$

(*b*) Find the L. C. M. of:

$$6\frac{1}{4},\ 7\frac{1}{2},\ 9\frac{1}{11},\ 12\frac{1}{2}.$$

3. How much water must be mixed with 75 gallons of alcohol worth \$1·00 a gallon, to make a mixture worth 75 cents a gallon?

4. The difference between the simple and the compound interest on a certain sum of money for three years at 3 per cent. is $\$5·45\frac{2}{5}$; find the sum.

What would the result have been if the difference had been given to the nearest cent., namely, \$5·45?

5. A merchant uses a yard-measure which is $\frac{1}{4}$ in. too short. How much does he cheat a customer who buys to the amount of \$72?

XIII

1. Find, correct to four places of decimals, the value of:

[illegible].

2. The sum of two numbers is 3773, and their difference is 861. Find the numbers.

3. How many pounds of chicory worth 8 cents a pound must be mixed with 15 pounds of coffee worth 40 cents a pound, so that by selling the mixture for $7·90 there will be made a gain of 25 per cent.?

4. On June 17 a merchant sells flour the cash value of which is $1500; to oblige his customer he accepts in payment his note for three months for that sum with interest at 6 per cent. He immediately has the note discounted at 6 per cent; find his loss through his customer's not having paid cash.

5. Three circles, each of radius 12 yards, touch one another; find the area and perimeter of the figure bounded by the arcs between the points of contact.

XIV

1. Find the value, correct to four places of decimals, of:

$$3 \cdot 2\dot{5}\dot{7} \div 9 \cdot \dot{3}\dot{9}.$$

2. The sum of the second and the third of three numbers is 114; of the third and the first is 98; and of the first and the second is 90. Find the numbers.

3. A farmer has cows worth $60 each, and sheep worth $4·50 each; the number of cows and sheep being 28 and their value $570, find the number of each.

4. An annual deposit of $250 is made with a loan company which pays 4 per cent. per annum on deposits, compounded half-yearly. Find the amount of all these deposits when the fourth has been made.

5. Two qualities of tea are mixed to the amount of 184 pounds, in the proportion of 11 to 12. Their prices are as 9 to 10 and the value of the mixture is $87·60. Find the number of pounds of each kind and the price of each kind.

XV

1. Find, correct to seven places of decimals, the value of the reciprocal of 2·30258509.

2. The product of the second and the third of three numbers if 4189; of the third and the first is 2911; and of the first and the second is 2419. Find the numbers.

3. Two-thirds of a number increased by two-ninths of the number and this increased by 80 will give one-third more than the number. Find the number.

4. A note for $230, drawn on Jan. 2, 1896, at 3 months, and bearing interest at 8 per cent. per annum, is discounted on Feb. 1, at 7 per cent. Find the proceeds.

5. ABCD is a quadrilateral whose sides AB, BC are each 40 rods in length. The angle B is 120°; the side AD is double the diagonal AC, and the side CD is three times the side BC. Find the area of the quadrilateral and the length of the diagonal BD.

XVI

1. Find, correct to four places of decimals, the value of:

$$(0{\cdot}8218127)^2+(0{\cdot}5697577)^2.$$

2. Shew that the square of an integer whose last (the units) digit is 5 may be found by writing down 25 preceded by the product of the total number of tens and a number one greater.

Ex. 1. $65^2=4225$, noting that $42=6\times7$.

Ex. 2. $115^2=13{,}225$, noting that $132=11\times12$.

3. A horse dealer sold two horses at $150 each, on the one gaining 25 per cent. and on the other losing 25 per cent. Find the gain or loss on the two sales.

4. On Jan. 1, 1890, a person borrowed $2417·50 at $6\frac{1}{4}$ per cent. simple interest, promising to return it as soon as it amounted to $2582·50. On what day did the loan expire?

5. A merchant bought 23 yards of silk and 15 yards of satin for $41. The satin cost 20 cents a yard more than the silk. What was the price per yard of each?

XVII

1. Find the value, correct to five places of decimals, of:

$$3{\cdot}1415926\times57{\cdot}2957795.$$

2. Shew that the sum of a number expressed by two digits and the number formed by writing the digits in reverse order is divisible by 11.

3. A man bought 138 acres of land at a certain price an acre. He at once sold 75 acres at a gain of 10 per cent. and the remainder at a gain of 5 per cent. The gain

realized was \$591 less than if he had sold all at a gain of 12 per cent. Find the price paid for the land.

4. A holds against B a note for \$730, drawn Feb. 14, 1897, at three months, bearing interest at 5 per cent. per annum. On Mar. 17, this note is discounted at 6 per cent. per annum. Find

(*a*) the proceeds;

(*b*) the rate of interest made by the bank on the money advanced.

5. One circle has its centre on the circumference of another circle and cuts it at the extremities of a diameter; the radius of the smaller circle being 20 inches, find the area common to the two circles.

XVIII

1. Find the value, correct to four places of decimals, of:

$$\frac{7\cdot3549\times3\cdot2756}{11\cdot2985}.$$

2. Find how many numbers there are, less than a thousand, which contain both 7 and 11 exactly.

3. In 8 days A can do as much work as B can do in $8\frac{3}{4}$ days, or and C can do in $8\frac{1}{3}$ days. If \$354·85 is paid for a piece of work done by them working together, how should this sum be divided?

4. A man puts \$350 in a Savings' Bank each year, making his first deposit Dec. 31, 1890. How much will there be to his credit Jan. 1, 1895, the Bank adding 4 per cent. per annum?

5. Vinegar and water are mixed in the proportion of 5 to 1. How much water must be added to 6 gallons of the mixture in order that the proportion may be 4·5 to 1?

XIX

1. Find, correct to three places of decimals, the value of:

$$\frac{10^6}{453{\cdot}59 \times 30{\cdot}48^3}.$$

2. Shew that the difference between two numbers, which are prime to each other, is prime to each of the numbers.

3. A merchant bought a certain number of barrels of flour for $4400. He reserved 40 barrels for his own use and sold ⅘ of the remainder for $3952, which was $608 more than cost. Find the number of barrels bought.

4. Find to the nearest cent the proceeds of a note for $637·50, drawn on May 13, at 60 days, and discounted on June 5, at 8 per cent.

5. ABCD is a quadrilateral with the angles A and C each right angles; AB, BC, CD are 25 rods, 39 rods and 52 rods in length respectively. Find the length of DA, the area of the quadrilateral and the lengths of the diagonals.

XX

1. Find the value, correct to five places of decimals, of:

$$2{\cdot}3759 \times 3{\cdot}3895 \times 7{\cdot}3594.$$

2. How many integers are there which require three figures for their expression, and of these how many have zero as their last figure?

3. In what proportions must coffee which cost 40 cents a pound and chicory which cost 20 cents a pound be mixed, in order that when the mixture is sold at 35 cents a pound

there is realized a gain of 20 per cent. on the chicory and 25 per cent. on the coffee?

Find also the gain per cent. realized on the whole outlay.

4. A teacher's salary of $1000 is paid in four quarterly payments at the end of each quarter. What sum at the beginning of the year is equivalent to these payments, reckoning compound interest at 2 per cent. per quarter?

5. A vessel contains a mixture of 12 gallons of brandy and 8 gallons of water. How many gallons of the mixture must be drawn off so that when the quantity withdrawn has been replaced by water the mixture may consist of equal parts of brandy and water?

XXI

1. Find the square root of:

(*a*) 12581209;

(*b*) 0·00290521.

2. Of the following fractions indicate those that will yield circulating decimals:

[illegible], [illegible], [illegible], [illegible], [illegible].

In each case give the reason, and state the limit to the number of figures in the period.

3. A man sold two houses for equal sums, on one gaining $12\frac{1}{2}$ per cent. and on the other losing $12\frac{1}{2}$ per cent. If on the two sales the total loss was $200, find the cost of each house and the percentage of loss on the two sales.

4. Divide \$916·00 among A, B and C, so that 5 per cent. of A's share may equal 7½ per cent. of B's, and 12½ per cent. of B's may equal 20 per cent. of C's.

5. The fore and hind wheels of a carriage are 10 feet and 12 feet respectively in circumference. How far will the carriage have gone when the fore wheel has made 80 revolutions more than the hind wheel ?

XXII

1. Find the cube root of:

(*a*) 45499293;

(*b*) 12·812904.

2. For each of the following recurring decimals give the complete work of reduction to an equivalent vulgar fraction, verifying the rule to this end:

$$0{\cdot}5\dot{3}\dot{7},\ 0{\cdot}23\dot{5}7\dot{9},\ 7{\cdot}2\dot{5}38\dot{7}.$$

3. A's money is three times as much as B's. They each receive \$500 and then three times A's money is equal to seven times B's. How much money has each now ?

4. A merchant imported goods paying a duty of 10 per cent. Circumstances force him to sell them at a loss of 4½ per cent.; had he sold them a month earlier he would have made \$500 more than he did, and have cleared 2½ per cent. on the whole transaction; what was the cost of the goods?

5. A mixture contains 4 parts spirits and 3 parts water. A certain part of the mixture is drawn off and replaced by water. If the mixture now contains 3 parts spirits and 4 parts water, what fraction of the original mixture was withdrawn ?

XXIII

1. Extract the square root of:

(*a*) 63409369;
(*b*) 24·950025.

2. Express the fractions:

$$\frac{4}{7},\ \frac{2}{11},\ \frac{8}{13}$$

as equivalent fractions with denominators in which appear only the figure 9.

3. The discount on a note for $3650, which matured on Aug. 21, and was discounted on June 24, was $40·60. Find the rate of discount.

4. In what proportion must two kinds of tea, which cost 50 ct. and 65 ct. a lb. respectively, be mixed, so that when sold at 60 ct. a lb. there may be a gain of $11\frac{1}{9}$ per cent.?

5. Find the number of acres in a field in the form of a trapezoid, whose parallel sides are 20 rods and 34 rods, and whose slant sides are 13 rods and 15 rods.

XXIV

1. Extract the cube root of:

(*a*) 182284263;
(*b*) 66430·125.

2. Express the fractions:

$$\frac{8}{15},\ \frac{11}{12},\ \frac{13}{18}$$

as equivalent fractions with denominators in which appear some number of figures 9 followed by some number of figures 0.

3. A, who owes B $1000 to be paid at the end of four years, wishes to pay the debt in four equal annual instalments. The rate of interest being 5 per cent. per annum, find the amount of the annual payment.

4. A man has 4 hours at his disposal; how far may he drive out into the country at a rate of 6 miles an hour, so that, walking back at the rate of 3 miles an hour, he may have 15 minutes free before the expiration of the 4 hours?

5. Of a mixture of wine and water, $\frac{3}{4}$ is wine. When 6 gallons of water are added the wine is $\frac{7}{10}$ of the mixture. Find the number of gallons of wine and of water in the original mixture.

XXV

1. Find, correct to three places of decimals, the square root of:

(a) 2;
(b) $\frac{4}{5}$.

2. One-half the product of a certain two consecutive integers is a square; shew that one of the integers is a square and that one-half the other is a square.

Illustrations: 1, 2; 8, 9; 49, 50.

3. A man invests $10,800 in 3 per cent. stock at 75; he sells out at 80 and invests $\frac{1}{2}$ of the proceeds in $3\frac{1}{2}$ per cent. stock at 96 and the remainder at 5 per cent. par. Find the change in his income.

4. A starts to walk from P to Q at the rate of three miles an hour, and one hour afterwards B starts from P and overtakes A in four hours. Walking on, B arrives at Q two hours before A. Find the distance from P to Q.

5. A person wishes to determine the height of a flag-pole. A lamp-post 9 feet high stands at a distance of 400 feet from the foot of the flag-pole. When the observer, the height of whose eye from the ground is $5\frac{1}{2}$ feet, stands at a point 14 feet from the lamp-post, his eye is in line with the top of the flag-pole and the top of the lamp-post. Find the height of the flag-pole.

XXVI

1. Find, correct to two places of decimals, the cube root of:

(*a*) 2;

(*b*) $\frac{1}{2}$.

2. A bookseller deducts 10 per cent. from the marked price of his books, and after this has a gain of 25 per cent. He sells a book for \$7·20. Find the cost price of the book, and what per cent. the marked price is in advance of the cost price.

3. What sum of money deposited in a bank at the end of each year for the next three years, will amount to the same sum as \$5000 deposited now, banks paying 4 per cent. per annum, interest added yearly?

4. Three pounds of tea and four pounds of coffee cost \$4. If tea were to rise 20 per cent. in cost and coffee to decrease 20 per cent. the same quantities would cost \$4·16. Find the price per pound of each.

5. The gross rental of an estate is £13,500 and deductions arising from rates and taxes at 3 *s*. in the pound, and interest on a mortgage of £3000, amount to 16 per cent. of this rental. What is the rate of interest paid on the mortgage?

XXVII

1. Find, correct to four places of decimals, the square root of:

(*a*) 18;
(*b*) [illegible].

2. From the list price of a line of goods a purchaser is allowed a trade discount of 20 per cent.; a further discount of 10 per cent. off the trade price for taking a quantity, and a still further discount of 5 per cent. off his bill for cash. Find his gain per cent. by selling at 10 per cent. less than the list price.

3. A and B are two railway companies that pay respectively 4½ per cent. and 1½ per cent. per annum on their $100 shares. When the price of a share in A is 101¼ and and in B is 32¼, what is the amount of money which, when invested in one rather than in the other, would give rise to a difference of income of $31·50?

4. Two wheels of a carriage are 3 ft. 9 in. and 4 ft. 8 in. respectively in diameter. How far will the carriage have gone when one wheel has gained 12 revolutions on the other?

5. The two sides of a right-angled triangle measure 5 in. and 12 in. On the sides and hypothenuse are described semi-circles towards the outside; find the area of the figure bounded by the three semi-circles.

XXVIII

1. Find the cube root, correct to two places of decimals, of:

(*a*) 8;
(*b*) [illegible].

2. A merchant makes a purchase of cloth, marks it at an advance of 20 per cent. on cost, and, after selling one-half of it, finds that one third of the remainder is so damaged as to sell for only one-half of its cost. What advance must be made on the marked price of what now remains so that on the whole there may be a gain of 15 per cent.?

3. A man rents a farm for 2 years at $441·00 per annum, the rent for any year being supposed to be paid at the end of that year. Money being worth 5 per cent. per annum, compound interest, find what sum would now pay the 2 years' rent.

4. A and B invest capital in the proportion of 7 to 8. After 5 months, A takes away one-half of his capital, and B two-thirds of his capital. At the end of the year they have gained $3545; what is the share of each?

5. A merchant marks his goods at 40 per cent. in advance of cost, and in selling uses a lb. weight ½ oz. too light. If he throws off 10 per cent. of his marked price, find his gain per cent.

XXIX

1. Find the square root, correct to four places of decimals, of:

(*a*) 1·7;

(*b*) $0\cdot 2\dot{1}\dot{3}$.

2. A buys 600 yd. of silk at 95 ct. a yd., and sells it at once receiving in payment a ninety-day note for $700·00, which he at once discounts at a bank at 6 per cent. per annum. Find the gain.

3. What must be the market value of 6 per cent. stock, so that after paying an income tax of 16 mills on the dollar, it may yield 5 per cent. on the investment?

4. If $65 is needed to pay for seed and the labour of 20 men who plant a field of 180 yards square in 2 days, how much would it cost and how long would it take to plant a field of 40,000 square yards, employing 50 men?

5. The sides AB, BC, CA, of a triangle are 40 ft., 45 ft., 50 ft, in length. From a point M in AB, 25 ft. from the vertex A, a line MN is drawn parallel to the base. Find the areas of the two similar triangles, and, taking the difference of the areas, verify the rule for finding the area of a trapezoid.

XXX

1. Find the cube root, correct to three places of decimals, of:

(*a*) 0·573;

(*b*) 0·0159.

2. Which is more profitable; to buy wheat at 85 ct. a bushel at 8 months (money being worth 4½ per cent. per annum), or for 82 ct. a bushel cash?

3. An agent received a consignment of wheat, which he sold at 80 cents a bushel, charging a commission of 2 per cent. With the net proceeds, after reserving a commission of 2 per cent. for investing, he purchased lumber at $23.00 a thousand. If the former commission exceeded the latter by $9·20, find the quantities of wheat sold and lumber bought.

4. A 27-gallon keg is half full of wine $\frac{4}{5}$ pure; 10 per cent. is drawn out and the keg filled with water. What is the percentage of its purity now?

5. A merchant sold 2400 lb. of sugar at 5 ct. a lb., gaining 25 per cent.; 1600 lb. at 6 ct. a lb., gaining 20 per cent.; and 1600 lb. at $4\frac{1}{2}$ ct. a lb. If on the whole he gained 20 per cent., find at what advance on cost the last sale was made.

XXXI

1. Express the gramme as a decimal of the ounce Troy.

2. A bought a drug store and a grocery store for $7000, and received $550 rent per annum for the two. He made 7 per cent. on the cost of the drug store and 9 per cent. on the cost of the grocery store. Find the cost of each.

3. A owes B $400 due in 1 year, $300 due in 2 years, and $200 due in 3 years. What sum paid now would cancel the debts, money being worth 5 per cent. per annum compound interest?

4. Five long-distance riders go round a circular track 8, 9, 10, 11 and 12 times respectively in one hour. If all start together when will all be together again? How many seconds start should the fastest man give each of the others that all may finish together in a race of 20 times around the track?

5. Find the cost of fencing on both sides of a race-track 7 yards wide, around a circular piece of land, the radius of which is 154 yards, at $12\frac{1}{2}$ cents a yard. (Take $\frac{22}{7}$ as the approximation to the ratio $c:d$.)

XXXII

1. The eagle weighs 258 grains, nine-tenths pure gold; 1869 sovereigns weigh 480 ounces Troy, eleven-twelfths pure gold. Find the value of one sovereign in the terms of the dollar.

2. A man has the choice of lending his money at 7½ per cent. compound interest, or at 8 per cent. simple interest, money and interest to be paid at the end of 3 years. Show which is the better investment.

3. A dealer shipped 200 barrels of apples to Liverpool; the average cost of the apples was $3·75 a barrel; for what sum must he have the apples insured at ¼ per cent. premium to guard against all loss, in case of shipwreck, his other expenses being $75?

4. If a cask contains 4 parts vinegar and 1 part wine, how much of the mixture must be drawn off and water substituted to make the cask contain vinegar in proportion to water as 3:2?

5. The length of a room is 1¼ times its breadth and the breadth is 1⅓ times its height. The room contains 1620 cubic feet, find its dimensions.

XXXIII

1. The velocity of light is 186,337 miles a second; find the velocity in kilometres a second.

2. A man after paying an income-tax of 9 *d.* in the pouud has £4812. 10 *s.* left. Find his income.

3. A man invests $6000 in 5 per cent. stock at 120; at the end of one year, having just received the yearly

dividend, he sells at 121⅛. How much better off is he than if he had loaned his money at 5 per cent. per annum ?

4. A clock loses 3 seconds in every hour. At 4 p.m. on Monday it is 10 minutes fast; find when it will indicate the right time.

5. Two concentric circles have radii of lengths 11 feet and 9 feet; find the radius of the concentric circle whose circumference bisects the area between their circumferences.

XXXIV

1. Find the average length of the calendar year.

2. A person has a note for $100, payable in 2 years, and one for $50, payable in 3 years; he takes $135 for them; when should the money be paid to him so as to allow 6 per cent. compound interest for the money and what is the present value of the notes ?

3. An agent, A, insures a cargo for $80,000 at ¼ per cent. B takes ½ of A's risk at ⅜ per cent. and C takes ⅓ of the remainder at ⅔ per cent., while D takes ⅓ of B's risk at ⅓ per cent. In case the ship is safe find the profit or loss of each agent.

4. How much gold 90 per cent. pure must be mixed with 24 ounces 65 per cent. pure, so that the alloy may be 80 per cent. pure ?

5. Find the cost of fencing a plot of ground which is in the form of a rectangle with a semi-circle at each end, if the length of a side of the rectangle is twice the length of an end and the area of the plot is 4456·64 square rods, the cost of fencing being $1·00 a rod.

XXXV

1. Given that a cubic inch of distilled water weighs 252·286 grains, find the weight of a cubic centimetre of distilled water in grammes.

2. A man after paying an income tax of 19 mills on the dollar on that part of an income in excess of $700 has left $2367·70. Find his income.

3. A person bought stock at 95⅛, and after receiving a half-yearly dividend of 7 per cent. per annum sold out at 92⅛, brokerage each way being ⅛ per cent. If his net gain was $25, how much stock did he buy?

4. If ⅔ of A's money equals ⅘ of B's, and ⅘ of B's equals ½ of C's, and the interest on all their money for 3 years 8 months at 6 per cent. is $5225; how much money has each?

5. The cross section of a water pipe is a regular hexagon whose side is 1 decimetre. At what rate must water flow through the pipe in order to fill in 15 hours a cylindrical reservoir the radius of whose base is 100 metres, and whose depth is 3 metres?

XXXVI

1. The pressure of the atmosphere is 14·7 pounds on the square inch; find the pressure in grammes on the square centimetre.

If 1 cubic centimetre of mercury weighs 13·59 grammes find the height in millimetres of the column of mercury that this pressure will support.

2. A certain percentage of $2700 together with the percentage of $3500 at a rate 1 per cent. higher amounts to $345. Determine the percentage in each case.

3. A building worth $6000 is insured so that in case of fire there may be recovered $\frac{3}{4}$ of the value of the house and $\frac{3}{4}$ of the premium paid. Find the premium, the rate being 3 per cent.

4. A woman buys 3 cwt. of a mixture of Manitoba flour and Ontario flour, for making bread, paying therefor $8·60. She buys the same amount of flour with the proportions interchanged, for making pastry, and the cost is $7·90. If the Manitoba flour is worth $3 a cwt., find the price of Ontario flour.

5. What length of copper wire 1 mm. in diameter will weigh 1 Kg., if 1 c. cm. of copper weighs 8·85 g.?

XXXVII

1. A cubic foot of liquid silver weighs 593 pounds; find the weight of a cubic centimetre of liquid silver in grammes.

2. A man whose income is $2800 pays an income tax of 19½ mills on the dollar on that part of his income which is not exempt from taxation. If his net income is $2759·05 find how much of his income was exempt.

3. A man invested a certain sum in 3 per cent. stock at 75 and another sum greater by $3000 in 5 per cent. stock at 120. If the income from the latter exceeds that from the former by $134, find the sums invested.

4. Two trains start from the same place, one at 1 p. m. and the other at 2.30 p. m. The latter overtakes the former at 7.30 p. m. If the former had been 8 miles

further than it was when the latter started, it would not have been overtaken till 9 p. m. Find the rates of the trains.

5. The diameter of a circular plate of lead is 13 inches. from this is cut out a circular plate of radius 6 inches, and the remainder of the lead is moulded into the form of a circular plate one-fourth as thick as the former. Find the diameter of this plate.

XXXVIII

1. A cubic foot of wrought iron will weigh from 485 pounds to 493 pounds. Find the limits, in grammes, of the weight of a cubic centimetre of wrought iron.

2. A borrows from B a sum of money and agrees to pay him by three annual payments of $200 each. If money is worth 5 per cent. per annum compound interest, find the sum borrowed.

3. A building is insured for $400 more than ⅚ of its cost at 4 per cent. If destroyed the loss will be $216. Find the cost of the building.

4. A speculator bought two houses, the first costing ⅔ as much as the second. In selling he gained 20 per cent. on the first and lost 5 per cent. on the second. His net gain was $160. Find his net gain per cent.

5. Two wheels fixed on parallel shafts 12 feet apart revolve in the same plane. If the radii of the wheels are 2 feet and 4 feet in length, find the length of belting required to pass around the wheels, supposing the belting to cross itself between the wheels.

XXXIX

1. Given that 1 gallon of distilled water weighs 10 pounds, and that 1 cubic foot of distilled water weighs 62·2786 pounds, find the number of cubic inches in 1 gallon.

2. From 1870 to 1880 the population of a town increased 30 per cent.; from 1880 to 1890 it decreased 30 per cent. The population in 1870 exceeded that in 1890 by 2781. Find the population in 1880.

3. A man holds 15,600 stock worth 60, and if he transfers it to 4 per cent. stock at 78 he can increase his annual income $12; before he could effect the transfer each stock increased 2 in price; find how his income is now altered.

4. A man loaned $800, part of it at 5 per cent. and the remainder at 7 per cent. If his annual income from both investments amounts to $49·40, find the sums lent at the different rates.

5. Two circles, the radius of each of which is 24 inches long, touch each other. A common tangent is drawn which with the two circles encloses a sort of triangular figure. Find the areas of the parts into which this figure is divided by a circle whose centre is in the common tangent and which touches the two circles.

XL

1. A standard metre, *i.e.*, a rod of length 39·37079 inches at 32° F., is made of brass. If a brass rod expands 10·5 millionths of its length at 32° F. for each rise in temperature of 1° F., shew that at 62° F. the length of this standard metre is somewhat greater than 39·382 inches.

NOTE.—In Canada the legal equivalent of the metre is very nearly 39·382 inches.

2. A man borrows \$12,000 for a year at 8 per cent. and loans it at 2 per cent. per quarter-year, compounding interest at the end of each quarter. How much money will he have made at the end of the year?

3. An agent receives a consignment of flour and is instructed to invest the proceeds in lumber having reserved his two commissions. The two commissions amount to \$250, the former exceeding the latter by \$7·50; find the rate of commission supposed the same in each case.

4. A increases his capital 50 per cent. yearly less \$200. At the end of 3 years his capital is \$2425; what was it originally?

5. The sides of a right-angled triangle are 6 feet, 8 feet and 10 feet in length. On the hypotenuse is described a triangle whose other sides are 17 feet and 21 feet in length; triangles similar to this are described on the other two sides of the given triangle. Find the areas of the triangles thus formed and shew that the sum of the areas of the two latter is equal to that of the former.

XLI

1. A wine merchant mixes three qualities of wine in the proportion of $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{4}$ with 10·7 litres of brandy. If the brandy formed $\frac{1}{13}$ of the mixture find the number of litres of each wine.

2. A holds against B a ninety-day note for \$540; B offers A immediate payment. Find what sum he should pay supposing,

(a) the rate of *discount* to be 6 per cent. per annum;
(b) the rate of *interest* to be 6 per cent. per annum.

3. A contractor in building a house paid 2½ times as much for material as for labor; had the latter cost 8 per cent. more, and the former 10 per cent. more, the whole cost would have been $5745. Find the actual cost.

4. A man invested $5000 in 3 per cent. stock at 75, and $6000 in another stock at 90. If his income from the latter exceeded that from the former by $100, find what rate was paid by the latter stock.

5. The base of a cistern is 8 feet by 9½ feet and the cistern contains 3 feet of water. If 180 gallons of water are added what depth of water will there then be? A gallon of water weighs 10 pounds and a cubic foot of water weighs 1000 ounces.

XLII

1. A man bought 100 yd. of one kind of cloth and 150 yd. of another, the total cost being $475·00. The former was sold at an advance of 20 per cent. and the latter at an advance of 25 per cent., and the sum received for both was $585·00. Find the cost price a yd. of each kind of cloth.

2. A wine merchant mixed three kinds of wine worth 50 ct., 60 ct. and 70 ct. a qt. in the proportion of 3:2:1, and to the mixture added 2 gal. of water; if he sold the mixture at 70 ct. a qt. and gained 30 per cent., how much of each kind of wine did he sell?

3. At what time after 8 o'clock are the minute-hand and the hour-hand of a watch first at right angles?

4. A man secures a net income of $2312·20 from a fixed salary, and the rent of a house. On the house, which rents for $50 a month, there is a mortgage of $2000 at 6 per cent.

per annum, $4000 insurance at 1¼ per cent. premium, taxes at the rate of 19 mills on the dollar on an assessment of $5000, and on his salary a tax of 19 mills with $800 exempt. What is his salary?

5. The external diameter of a hollow steel shaft is 20 inches and the internal diameter 12 inches. Find the weight of a piece of this shafting 30 feet long, being given that a cubic foot of steel weighs 490 pounds.

XLIII

1. A train 160 yd. long moving at the rate of 24 mi. an hr. overtakes a train 148 yd. long on a parallel track and passes it in 1¾ min. How long would it take the trains to pass each other going in opposite directions?

2. A merchant bought 200 yards of cloth at $1·50 a yard, payable in three months, and sold it one month after at $1·75 a yard, payable in four months. To pay the purchase money he borrowed for the necessary time at the rate of 6 per cent. per annum. Find his gain on the transaction.

3. If 10 men, or 14 boys, or 18 girls can do a piece of work in 30 days, in what time will the work be completed if they all work together until 2 days before it is finished, when all the girls, 7 boys and 2 men leave off working?

4. A teacher's salary increases ⅕ every year; each year his expenses are ¾ of his salary; at the end of each year he puts the balance in a bank which pays 4 per cent. At the end of the third year he has $376·96; what was his initial salary?

5. A rectangular solid is hammered until its length is increased 10 per cent., and its width 15 per cent.; by what per cent. has its thickness been diminished?

XLIV

1. Two trains 150 yd. and 180 yd. long respectively, pass each other moving in the same direction, in $2\frac{1}{4}$ min., and moving in opposite directions, in $\frac{1}{4}$ min. Find the rates of the trains.

2. A man having $2496 in cash invested it in stock at 78 which paid $3\frac{1}{2}$ per cent.; afterwards when it had risen to $102\frac{3}{4}$ he sold ont and invested his money in a mortgage at 4 per cent. If he paid an income tax of 16 mills on the dollar what was the change in his income?

3. A person annually increases his capital 20 per cent. less a yearly expenditure of $500. At the end of 4 years his capital amounts to $18,052; find his original capital.

4. A merchant marks his goods at an advance of 25 per cent. on cost. After selling $\frac{1}{2}$ of the goods he finds that some of the goods in hand are damaged so as to be worthless; he marks the saleable goods at an advance of 10 per cent. on the marked price and finds in the end that he has made 20 per cent. on cost. What part of the goods was damaged?

5. A bar of lead whose cross-section is a regular hexagon of side 3 cm. and whose length is 20 cm. is melted down with a rod of zinc whose diameter is 1 cm. and whose length is 15 cm. What is the edge of a cube of zinc which when melted down with this alloy gives an alloy consisting of equal parts of lead and zinc, if 1 c.cm. of lead weighs 11·3 g. and 1 c.cm. of zinc weighs 7·1 g.?

XLV

1. A could do a piece of work in 36 days, B in 24 days and C in 16 days. They worked together, but A left off work 2½ days and B 1½ days before it was completed. Find the time occupied; if $72 was given for the work what should each man get?

2. A mortgage for $1800 dated April 1, 1889, and bearing interest at 6 per cent., has endorsed upon it the following payments:

Oct. 12, 1889, $300;
Sept. 15, 1890, $450;
Nov. 1, 1891, $250.

How much would pay off the mortgage on Nov. 1, 1892, each payment to cover the interest to date?

3. The pressure of compressed air varies inversely as its volume. If the pressure on the inner surface of a cylinder fitted with a piston be 20 pounds on the square inch, and when the piston is forced in 2 inches the pressure becomes 30 pounds on the square inch, what is the length of the cylinder?

4. A merchant bought 3885 yd. of cloth and marked it at an advance of 33⅓ per cent. on cost; in selling the first half of it he gave only 35 in. for a yd., but in selling the remainder he gave 37 in. for a yd. He gained on the whole transaction $3897. What did the cloth cost him a yd.?

5. A, B and C, whose rates of walking are 3½, 4 and 5 miles an hour respectively, walk on circular tracks whose circumferences are 8, 10 and 15 miles respectively, and

whose centres are in the same straight line. At the same instant they start from points on this line, and on the same side of the centres. Find (1) when first they will be all on this line at the same time; (2) all at the same time at the points from which they started; (3) whether they will ever be all at the same instant on the straight line at the points on opposite sides of the circles to the starting points.

XLVI

1. A coal merchant bought 1600 tons of coal at $4·95 a ton. He sells it at an advance of 20 per cent. and in selling uses false scales. If he gains $1650 find the weight of his ton.

2. A merchant buys a quantity of goods and sells $\frac{1}{3}$ of it at an advance of $33\frac{1}{3}$ per cent. and $\frac{1}{4}$ of it at an advance of 25 per cent. He then finds that $\frac{2}{5}$ of the quantity on hand being damaged will sell at only $\frac{1}{2}$ of cost; at what advance on cost must he sell the remainder so that on the whole he may gain $12\frac{1}{2}$ per cent.?

3. A person invests money

(*a*) In bank stock at 128 paying half-yearly dividends of 4 per cent., subject to an income tax of 18 mills on the dollar; and

(*b*) In city property yielding a rental of 10 per cent., costing him one-fifth of the rent for insurance and repairs, and $18\frac{1}{2}$ mills on the assessed value (90 per cent. of the cost) for taxes.

If the whole amount invested is $4989 how shall he divide it so that the net income from the two investments may be the same?

4. A gallon contains 277·274 cubic inches; a cubic foot of water weighs 62·5 pounds. If mercury weighs 13·5 times as much as water, how many gallons of mercury will weigh a ton?

5. Water is flowing at the rate of 10 miles an hour through a pipe 14 in. in diameter into a rectangular reservoir 187 yards by 96 yards. In what time will the surface be raised 1 inch, taking $\frac{22}{7}$ as the approximation to the ratio $c:d$?

XLVII

1. A person's coal-bill for the year is $100·80. If coal had cost him 10 per cent. less he would have been able, with the same sum, to purchase 2 tons more than he did. Find the price of coal a ton.

2. A merchant sells tea at an advance of 25 per cent. on cost and employing faulty scales sells $15\frac{3}{4}$ oz. as a lb. Find his gain on the sale of 2000 lb. of tea purchased at 25 ct. a lb. Find also what his gain would have been had the scales been accurate.

3. On Dec. 3, a man bought 30 shares of a certain 5 per cent. stock at $118\frac{3}{8}$, brokerage $\frac{1}{4}$ per cent., paying therefor a cheque. On Dec. 15, he received the quarterly dividend, which he deposited in the bank; on Dec. 28, he sold out at $119\frac{1}{8}$, brokerage $\frac{1}{4}$ per cent., depositing the proceeds in the bank. If the bank allows 3 per cent. per annum, the interest being paid on the minimum monthly balance, find by what amount his bank balance on Dec. 30, has been increased through the stock transaction.

4. The money deposited in a savings' bank during the year 1885 was 5 per cent. greater than that deposited in 1884.

In 1886 the deposits were $33\frac{1}{3}$ per cent. greater than in 1885, while the amount deposited in 1887 exceeded the average of the three previous years by 20 per cent. The aggregate for the four years was \$150,937·50. Find the amount deposited in each year.

5. A circular plate of lead 14 cm. in radius and 3 cm. thick is divided into three equal parts and these parts are cast into the forms of a cube, a cylinder of radius 1 cm., and a prism whose base is an equilateral triangle of side equal to the height of the prism. Find the side of the cube, the height of the cylinder and the height of the prism, taking $\frac{22}{7}$ as the approximation to the ratio $c:d$.

XLVIII

1. A can do a piece of work in 18 hr., B in 15 hr. and C in 12 hr. They work at it in succession each 3 hr. in the order A, B, C, A. . . At the end of what time will the work be finished and by whom ?

2. A note bearing interest at 6 per cent. per annum, and having two years to run is offered for sale. What per cent. advance on its face value will a purchaser offer if he wishes to make 5 per cent. on his money ?

3. The profits of a loan company for a year were sufficient to enable the directors to add \$20,000 to a reserve fund, to pay \$5965 for cost of management, to pay two half-yearly dividends of $3\frac{1}{2}$ per cent. on a paid-up capital stock of \$309,056, and to have still on hand \$4236. Find the profits for the year.

4. A man with an income of \$800 a year spends one-tenth of it upon goods imported under an average duty of

30 per cent., and sold to him at a profit of 25 per cent. What fraction of his income is taken by this indirect taxation, and at what per cent. above actual cost is he paying for the imported goods?

5. A, B and C run around a circular track whose internal diameter is 840 yards. Their rates of running are 10, 9¾ and 9½ miles an hour. If C runs on the inside of the track, B two yards and A four yards from the inside, what start should A, running from the starting point, give B and C that they may all in one round come in abreast at the starting point. (Take $\frac{22}{7}$ as the approximation to the ratio *c*:*d*.)

XLIX

1. The hour, minute and second hands of a watch are on concentric axes. When first, after 12 o'clock will the direction of the second hand produced backwards bisect the angle between the hour and the minute hands?

2. The capital of a railway company is $20,000,000 and in addition it has borrowed $12,000,000 at 4 per cent. per annum. Its gross receipts for the year are $3,000,000 and the working expenses 40 per cent. of the gross receipts. What dividend to the nearest half of 1 per cent. can be declared, if at least $50,000 has to be placed aside for the reserve fund?

3. A invested in 7 per cent. stock at 78⅛, and having received a half-year's dividend sold out at 79⅜, paying ⅛ per cent. brokerage on each transaction, and increased his capital altogether by $292·50. How much did he invest?

4. A merchant buys a quantity of goods and sells ¾ of it at an advance of 15 per cent., and ¼ of it at an advance

of 20 per cent. He now discovers that 10 per cent. of his goods are quite unsaleable. What per cent. profit must he obtain on the remainder that he may gain 15 per cent. on the whole transaction?

5. If a lead pipe 75 metres long weighs 340 Kg., and if its internal diameter is 3 cm., calculate the thickness, being given that 1 c.cm. of lead weighs 11·35 g.

L

1. Find in centimetres the edge of a cubic block of lead which weighs a ton, given that a cubic centimetre of lead weighs 11·35 grammes.

2. A person invested equal sums in 4 per cent. stock at $115\frac{3}{8}$ and $3\frac{1}{2}$ per cent. stock at $98\frac{7}{8}$, brokerage $\frac{1}{8}$ per cent. After paying an income tax of 16 mills on the dollar, his net income was $477·24. He then sold these stocks at the same quotations and invested, to the nearest share, the proceeds and his net income in 4 per cent. stock at $109\frac{7}{8}$, brokerage in each case being $\frac{1}{8}$ per cent. If the rate on incomes has, in the meantime, been changed to 18 mills on the dollar find his net income from the new stock.

3. A bath can be filled by the cold-water pipe in 9 min., and by the hot-water pipe in $11\frac{1}{4}$ min. A person leaves the bath-room after turning on both taps simultaneously, and returns at the moment when the bath should be full. Finding, however, that the waste-pipe has been open, he now closes it. In $3\frac{3}{4}$ min. more the bath is full. In what time would the waste-pipe empty it?

4. A dealer has 1000 hats for sale; at first he sells so as to gain 50 per cent. on the cost price, but after a time he

lets the remainder go for what he can get and finds he loses on these latter sales 10 per cent. If his total gain be 29 per cent., how many hats did he sell at a gain of 50 per cent.?

5. The weight of 500 feet of round copper wire is $6\frac{1}{4}$ lb.; find its diameter if a cubic foot of copper weighs 555 lb.

The same wire is cast into the form of a hollow cylinder of $\frac{3}{4}$ in. internal diameter and 8 in. long; find its external diameter if 1 cubic foot of this cast copper weighs 554 lb.

PART III

CHAPTER I

NOTES ON THE SIMPLER THEORY

1. **Multiplication of Fractions.** The introduction of fractional numbers into arithmetic makes it desirable to extend to those numbers the operations admitted when integers only were considered. The meanings attached to addition and subtraction made it at once possible to carry those operations over into the field of fractional numbers. A meaning might have been given to the multiplication of integers which would have allowed that operation to be similarly extended; for if we say that 4×3 means three of the fours, we may say that $\frac{2}{3}\times\frac{4}{5}$ means four-fifths of the two-thirds, and the multiplication of fractions becomes a possibility; this is probably the most direct way of introducing the subject. However the meaning usually assigned to the multiplication of 4×3 is four taken three times and as already stated (see p. 22) when the operation $\frac{2}{3}\times\frac{4}{5}$ is proposed a difficulty arises.

Two ways of overcoming this difficulty offer themselves. We may either devise a definition for the multiplication of integers which shall suggest a meaning for the multiplication of fractions, say for $\frac{2}{3}\times\frac{4}{5}$; or we may seek independently of the multiplication of integers for a rule of multiplication of fractions which shall give the results of the rule of multiplication of integers when the fractions are equal to integers, through having their numerators divisible by their denominators, and which shall make the multiplication of fractions subject to the same laws as the multiplication of integers. Thus we want a definition of multiplication of fractions which when applied to $\frac{6}{1}\times\frac{14}{2}$ shall give 42 as result, and which moreover shall be subject to the three laws of multiplication stated on p. 5.

Of the two ways the first is to be rejected because we cannot frame a definition which shall be adapted to all possible extensions of a notion such as multiplication, for we do not know beforehand what they may be. The second way is the one always adopted in mathematics.

It is to be noticed that the definition of multiplication frequently given: Multiplication is that process by which having given two numbers we form a third number composed of either of the given numbers in the same manner as the other given number is composed of unity, is one which admits of application to the multiplication of fractions but which would lead to incorrect results if applied to the multiplication of irrational numbers.

2. Approximations. (*a*) When for the number

$$3{\cdot}14159$$

we take the approximations 3·1415 or 3·1416 we can say that the **error** is less than 0·0001, *i. e.*, that the approximation differs from the exact value by less than 0·0001. So in actual measurements, where it is impossible to obtain exact results, a limit to the amount of error is generally known, and when the limit to the error in each measurement is known it is possible to find a limit to the error in any result obtained from them.

Although an error of 1 in measuring 100, is the same *absolutely* as an error of 1 in measuring 1000, yet it is very *different* from the point of view of exact measurement. We are thus led to the idea of the **relative** error, *i. e.*, *the* *ratio of the error to the quantity measured.* Thus in taking 3·1416 as an approximation to 3·14159 ... the relative error is $\frac{0\cdot00001}{3\cdot14159\ldots}$, or is less than $\frac{1}{3\times10^5}$.

(*b*) In the treatment of contracted multiplication and division, the approximation was always expressed in terms of the decimal point. Since in measurements the amount of error is given by stating the number of significant figures known to be correct, it is more and more the practice to call for multiplications and divisions correct to a stated number of significant figures.

The following examples will make clear what is meant.

Ex. 1. Find the product 17·2915×0·3729 correct to four significant figures.

```
17·2915
  9·273
-------
 5·1874
 1·2104
    346
    155
-------
  6·448
```

We begin the multiplication with 3, the first significant digit of the multiplier and place it under 1 where multiplication by 3 should begin if we are to have five figures in the result; for the same reason 7 is placed under 9, etc. We work to five significant figures to be sure of four.

Ex. 2. Find the quotient 29·372345÷2538·2719 correct to five significant figures.

```
 ×× ××
2538·2719)29·372345(0·011572
          25·3827
          -------
           3·9896
           2·5383
          -------
           1·4513
           1·2691
          -------
             1822
             1777
             ----
               45
```

If the result is to have five significant figures, we start the division with six figures in the divisor for reasons stated in the treatment of the contracted method of division in Chapter IV of Part I. When the five figures have been found the decimal point is to be introduced.

3. Square Root. It is shewn in works on algebra that when $n+1$ figures of a square root have been found, n more may be found by mere division.

Ex. Find the square root of 191·7936 to seven significant figures.

```
                  1'91·79'36(13·84896
                  1
                  ─────
   23              91
                   69
                  ─────
   26·8            22·79
                   21·44
                  ──────
   27·64            1·3536
                    1·1056
                  ────────
             27·68)2480
                   2214
                   ────
                    266
                    248
                    ───
                     18
```

Here four figures are found in the ordinary way. Then when the part of the root already found is doubled, instead of taking down two zeros, finding the next figure and adding it to the figures of the diviser, we mark out one figure of the divisor and divide. The division is by the contracted process.

4. Cube Root. In the process of extracting the cube root as given in Chapter V of Part I the chief difficulty, after the process is understood, consists in forming the number which is to be subtracted as each new figure is formed. By the method illustrated below, this difficulty is overcome.

Ex. Find the cube root of 91733851.

4		91'733'851(451
4	16	64
8	32	27733
4	4800	27125
120	625	608851
5	5425	608851
125	650	
5	607500	
130	1351	
5	608851	
1350		
1		
1351		

The student is referred to p. 50 for the extraction of the root of this number in the ordinary way, and should compare the ways in which the successive subtrahends are formed.

Here 4 is seen to be the first figure of the root; 4 is written down, then its square 16 and then its cube 64, which is subtracted from 91. Then 4 is written below the 4 already written and added to it; the sum is multiplied by 4 and placed under the 16 and added to it with result 48; 4 is next written below 8 and added to it. We have thus the numbers 12 ($=4\times3$), 48 ($=4^2\times3$). To 12 we add one zero, to 48 two zeros and take down with the remainder 27 the period 733. Then 4800 is the trial divisor and the next figure of the root is seen to be 5. We then place 5 below 120 and add; the sum is multiplied by 5 and the result is added to 4800 and this sum, in turn, is multiplied by 5 and taken from 27733. Next 5 is placed below 125 and added to it; the sum is multiplied by 5 and the result is added to 5425; 5 is then placed below 130 and added to it. We then add one zero to 135, two zeros to 6075, and the next period is taken down. The trial divisor is now 607500 and the next figure of the root is 1; the amount to be subtracted being the same as 608851, the operation terminates.

The operation may now be shortened for at each step the addition (or subtraction) may be made without writing down the number to be added (or subtracted).

		91'783'851(451
4	16	64
8	4800	27783
125	5425	27125
1[illegible]	607500	608851
[illegible]51	608851	608851

It is shewn in works on algebra that if $n+2$ figures of a cube root have been found $n-1$ or n more figures may be found by mere division.

Ex. Find the cube root of 2937·569387 to six significant figures.

		2'937·569'387(14·3217
1	1	1
2	300	1 937
34	436	1 744
38	58800	193·569
423	60069	180·196
426	6134700	13·371387
4292	6143284	12·286568
	×× ×	
4294	6151872	1·084 819
		615
		470
		431

Here four figures are found in the way above described. Then instead of adding two zeros to the trial divisor 6151872 and taking down three zeros as the next period on the number, we mark out the last figure of the trial divisor. Then since by division we can obtain one or at most two figures, we divide as if 615 were the divisor and 1084 the dividend. The last figure, 7, is correct.

Exercises

1. Find, to seven places of decimals, the square roots of the following numbers:

$$2,\ 3{\cdot}1415926536,\ 7,\ 13{\cdot}215,\ 3,\ 2{\cdot}7182818285.$$

2. Find, to six places of decimals, the cube roots of the following numbers:

$$2,\ 9{\cdot}372956384,\ 7,\ 11,\ 13,\ 24.$$

3. Shew that, if it is assumed that, for all numbers, a series of multiplications and divisions may be taken in any order, then

$$\tfrac{3}{5}\times\tfrac{4}{7}=\tfrac{3\times4}{5\times7}=\tfrac{12}{35}.$$

4. Find the following products by the contracted method, in each case correct to four places of decimals:

$$42{\cdot}7724\times6{\cdot}71237,\ 0{\cdot}216236\times14{\cdot}764359,$$

$$0{\cdot}243\times5{\cdot}\dot{7}\dot{2},\ 0{\cdot}43\dot{6}\times0{\cdot}\dot{2}4\dot{7},\ 3{\cdot}14159\times\sqrt{2}.$$

5. Make the following divisions by the contracted method correct to 5 places of decimals:

$$0{\cdot}707106\div3{\cdot}1415926;\ 27{\cdot}634625\div0{\cdot}02379;\ \sqrt{5}\div\sqrt{3};$$

$$1\div3{\cdot}14159265;\ 278{\cdot}\dot{4}\dot{2}\div21{\cdot}\dot{6}0\dot{4};\ 0{\cdot}\dot{2}1\dot{6}\div4{\cdot}22\dot{1}\dot{7}.$$

6. In the ordinary way it is found that

$$\tfrac{1}{7}=0{\cdot}\dot{1}4285\dot{7}.$$

(*a*) Shew that before the division was made it was known that the result would be a pure recurring decimal with at most six figures in the period.

(*b*) Shew that when three figures were found the remaining three figures might have been obtained by multiplication.

(*c*) Write down the recurring decimals equivalent to

$$\tfrac{2}{7},\quad \tfrac{3}{7},\quad \tfrac{4}{7},\quad \tfrac{5}{7},\quad \tfrac{6}{7}.$$

making only one partial division in each case.

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)

1.0 2.8 2.5 3.2 2.2 3.6 2.0 4.0 1.1 1.8 1.25 1.4 1.6

4.5 5.0 5.6 6.3 7.1

7. Express $\frac{1}{17}$ as a recurring decimal and construct and answer questions similar to those of example 6.

8. Determine the least number by which 1696 must be multiplied to be exactly divisible by 1113.

9. Two numbers are expressed by the same digits in different orders; shew that their difference is divisible by 9.

10. Two fractions in their lowest terms have different denominators; shew that their sum cannot be an integer.

11. If a is a number prime to 30, then a number can be found which is a multiple of a and in the writing of which the figure 1 is the only one employed.

12. By means of the relation

$$\frac{a}{b} = \frac{1}{q+1} + \frac{a-r}{b(q+1)}$$

where q and r are the quotient and the remainder when b is divided by a, shew that any proper fraction may be expressed as the sum of fractions with numerators 1.

Illustration:

$$\begin{aligned} \tfrac{5}{13} &= \tfrac{1}{3} + \tfrac{2}{39}, \\ &= \tfrac{1}{3} + \tfrac{1}{20} + \tfrac{1}{780}. \end{aligned}$$

13. Shew that no integer which is not prime to 10 can have as a multiple a number in the writing of which the digit 9 is the only one employed; and that every integer prime to 10 has among its multiples numbers requiring for their expression, only the digit 9.

14. If b is a prime number and if the fraction $\frac{a}{b}$ (in its lowest terms) generates a recurring decimal with $2n$ figures in its period, shew that the sum of the numbers formed by the first n and by the last n figures of the period will be a number expressed by n figures all nines.

15. Shew that we may obtain an approximation to a number which differs from it by not more than $1 \div 10^n$ by stopping at the nth decimal place.

16. The only integers that can be added to the two terms of a fraction in its lowest terms without changing its value are equimultiples of those terms.

17. If we arrange in order of magnitude the fractions in their lowest terms which are less than unity, whose denominators are less than a given number, the fractions at equal distances from the extreme ones have the same denominator and their sum is equal to unity.

18. Any fraction less than unity can be represented, to as close a degree of approximation as we choose, by a sum of fractions whose numerators are all equal to unity and whose denominators are equal to different powers of 2.

19. Find, correct to eight places of decimals, the product,

$$57{\cdot}295779513 \times 3{\cdot}1415926536.$$

20. Find, to three significant figures, the error in taking $\frac{4}{3} \times \frac{22}{7} \times (6{\cdot}223)^3$ instead of $\frac{4}{3} \times 3{\cdot}1416 \times (6{\cdot}2227)^3$.

21. If we write down the series of numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\ldots$$

such that each is equal to the sum of the two preceding, shew that the sum of any number of these terms beginning at the first is one less than the second number after the last one summed.

Illustration:

$$0+1+1+2+3+5+8=21-1.$$

22. The following rule is sometimes given to divide by 3·14159: Multiply by 7, divide by 11, then by 2, and add 4 ten-thousandths of the result. Find, to three significant figures, the error committed in obtaining 15 ÷ 3·14159 by this rule.

23. Find, correct to six places of decimals, the value of:

$$\frac{1}{1\times3}+\frac{1}{2\times3\times3}+\frac{1}{3\times3\times3\times3}+\frac{1}{4\times3\times3\times3\times3}+\frac{1}{5\times3\times3\times3\times3\times3}$$

24. If it is uncertain whether the third decimal place in 11·23 ··· is 4 or 5, find the limits of the error committed by using 11·234 to compute $\frac{1}{11{\cdot}23\ldots}$.

25. Can the square of a number ending in 5 end in 125?

CHAPTER II

SERIES

1. **Series.** A sequence of numbers, as for example,

1, 5, 9, 13, 17, ...

or, 1, 3, 9, 27, 81, ...

in which the successive numbers are formed according to some **law**, is called a **series**. When the law of a series is known, the series may be continued as far as we please.

As a rule the numbers of a series will be connected by the sign + (or, it may be −); the individual numbers will be called **terms** of the series.

2. **Arithmetical Series.** Consider the series:

$$1+4+7+10+13+16+19+22+\ldots\ldots$$

Here each term is made from the term immediately preceding it, by the addition of 3, *i. e.*, consecutive terms differ by the same number, or, in other words, the difference between consecutive terms is **constant**. Thus the law of the series is known. Such a series is called an **arithmetical series**, or an **arithmetical progression**. The number added to any term to make the next term, is called the **common difference**.

Any term of a given arithmetical progression may be found without finding all the terms preceding it. Thus in the series considered, it is readily seen that the *seventeenth* term, being *sixteen* terms in advance of the first term, is $1+3\times16$, or 49.

The sum of any number of terms of an arithmetical series may be found without actual addition, and indeed without writing down all the terms.

Ex. 1. Find the sum of eleven terms of the arithmetical series:

$$1 + 4 + 7 + 10 + \cdots\cdots$$

The common difference is here 3, and we have then to find the sum:

$$1 + 4 + 7 + 10 + 13 + 16 + 19 + 22 + 25 + 28 + 31.$$

The sum of the 1st term and the 11th (the last) term, is $1 + 31$ or 32, and the average is 16.

The sum of the 2nd term and the 10th (the next to the last) term, is also 32, since $(1 + 3) + (31-3) = 1 + 31$, and the average of these terms is 16.

In like manner the average of the 3rd and 9th terms, of the 4th and 8th terms, of the 5th and 7th terms, is seen to be 16, while the 6th, (the middle) term is the average of the 5th and 7th terms.

Thus the sum is the sum of 11 terms of average 16 and is equal to 16×11 or 176.

Since the average was found by adding the first and last terms, we see that the sum is equal to the product of the number of terms by one-half the sum of the first and last terms.

Ex. 2. Find the sum of 8 terms of the arithmetical series:

$$9 + 17 + 25 + 33 + \cdots\cdots$$

Here the common difference is 8, and we have to find the sum:

$$9 + 17 + 25 + 33 + 41 + 49 + 57 + 65.$$

As in *Ex. 1*, the average of 9 and 65, is the same as that of 17 and 57, of 25 and 49, of 33 and 41. Here then also there is an average value of the terms, though there is no one middle term equal to the average.

As in *Ex. 1*, the sum is seen to be

$$\frac{9+65}{2} \times 8 = 296,$$

or to be equal to the product of the number of terms by one-half the sum of the first and last terms.

The rule brought out in the preceding examples for finding the sum is easily seen to be general, and there is no further need of writing down all the terms whose sum is sought. Thus take the example:

Ex. 3. Find the sum of 63 terms of the series:

$$3 + 10 + 17 + \cdots\cdots$$

Here the common difference is 7, and the 63rd term (the last of those considered) is $3 + 7 \times 62$ or 437.

$$\therefore \text{ Average of terms} = \tfrac{437+3}{2} = 220.$$

$$\therefore \text{ Sum required} = 220 \times 63 = 13{,}860.$$

The reasoning above followed, is general, and may be given in general form by employing algebraic symbols. Thus:

Ex. 4. Sum to n terms the series:

$$a + (a + d) + (a + 2d) + \cdots\cdots$$

Here the common difference is d. Let l denote the nth (the last) term. Then having in mind all the terms, though they are not actually written, let

$$s = a + (a + d) + \cdots\cdots\cdots + (l - d) + l;$$
$$\therefore s = l + (l - d) + \cdots\cdots\cdots \quad (a + d) + a;$$

where in the latter equation the terms are written in reverse order.

Then by addition

$2s = (a + l) + (a + l) + \cdots \quad \cdot + (a + l) + (a + l) = n(a + l)$, since there is one term $(a + l)$ for each term of the series

$$\therefore s = \frac{n(a + l)}{2} = n\left(\frac{a + l}{2}\right).$$

3. Geometrical Series. Consider the series:

$$1 + 2 + 4 + 8 + 16 + \cdots$$

Here each term is made from the term immediately preceding it by multiplying by 2, *i. e.*, each term bears to the term just preceding it the same ratio, or the ratio of consecutive terms is constant. Thus the law of the series is known. Such a series is called a **geometrical series** or a **geometrical progression.** The factor which multiplied into any term gives the next term is called the **common ratio.**

Any term of a given geometrical progression may be found without finding all the terms preceding it. Thus, in the series considered, it is readily seen that the *seventeenth* term, the *sixteenth* in advance of the first term, is 1×2^{16} or 2^{16}.

The sum of any number of terms of a geometrical series may be found without actual addition.

Ex. 1. Find the sum of 9 terms of the series:

$$3+6+12+24+\cdots\cdots$$

Here the common ratio is 2, and we have to find the sum of

$$3+6+12+24+48+96+192+384+768.$$

Denote the sum sought by s.

$$\therefore \ s = 3+6+12+24+48+96+192+384+768.$$

$$\therefore 2s = \quad 6+12+24+48+96+192+384+768+1536.$$

The second equation is formed from the first by multiplying each term by 2, and setting it one place to the right. Then by subtracting the numbers in the first equation from those in the second equation we have

$$s = 1536 - 3 = 1533.$$

Ex. 2. Find the sum of 35 terms of the series:

$$2+\tfrac{2}{3}+\tfrac{2}{9}+\tfrac{2}{27}+\cdots\cdots$$

Here the common ratio is $\frac{1}{3}$, and the 35th term is $2\times(\frac{1}{3})^{34}$ or $\frac{2}{3^{34}}$.

Then having in mind the terms not written, let

$$s=2+\tfrac{2}{3}+\tfrac{2}{9}+\cdots\cdots+\frac{2}{3^{33}}+\frac{2}{3^{34}}.$$

$$\therefore \tfrac{1}{3}s= \quad \tfrac{2}{3}+\tfrac{2}{9}+\cdots\cdots+\frac{2}{3^{33}}+\frac{2}{3^{34}}+\frac{2}{3^{35}}.$$

Subtracting the numbers in the second from those in the first equation, we have

$$\tfrac{2}{3}s=2-\frac{2}{3^{35}}.$$

$$\therefore s=\tfrac{3}{2}\left(2-\frac{2}{3^{35}}\right)=3\left(1-\frac{1}{3^{35}}\right).$$

Ex. 3. Find the sum of n terms of the series:

$$a+ar+ar^2+\cdots\cdots$$

Here the common ratio is r, and the nth term is ar^{n-1}. Then let

$$s=a+ar+ar^2+\cdots\cdots+ar^{n-2}+ar^{n-1}.$$

$$\therefore rs= \quad ar+ar^2+\cdots\cdots+ar^{n-2}+ar^{n-1}+ar^n.$$

Then (1), if r is greater than 1, we have by subtraction,

$$rs-s=ar^n-a;$$

$$\therefore s(r-1)=a(r^n-1);$$

$$\therefore s=a\frac{r^n-1}{r-1}.$$

and (2), if r is less than 1,

$$s-rs=a-ar^n;$$

$$\therefore s(1-r)=a(1-r^n);$$

$$\therefore s=a\frac{1-r^n}{1-r}.$$

In the results of *Ex. 3* are to be found the rules for writing down the sum of any number of terms in a geometrical progression.

Consider now the series:

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots$$

Quoting the result of *Ex. 3* we see that

$$\text{The sum of 20 terms} = 1\cdot\frac{1-(\frac{1}{2})^{20}}{1-\frac{1}{2}} = 2-\frac{1}{2^{19}}.$$

$$\text{The sum of 60 terms} = 2-\frac{1}{2^{59}}.$$

$$\text{The sum of } n \text{ terms} = 2-\frac{1}{2^{n-1}}.$$

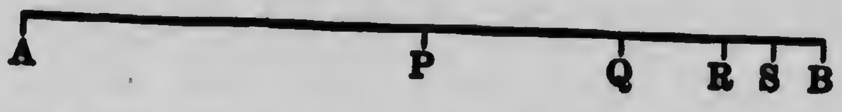
It is then seen that however many terms be taken the sum is less than 2, but that, as more and more terms are taken, the sum becomes nearer in value to 2. For the sum of five terms is less than 2 by the number $\frac{1}{2^4}$; the sum of six terms is less than 2 by the number $\frac{1}{2^5}$, that is differs from 2 by only one-half as much as the sum of five terms differs from 2, and so with the addition of each term we diminish the difference between the sum and 2 by one-half of itself and therefore we can, by taking a sufficiently large number of terms, make the sum to differ from 2 by a number which is less than any given number, however small. Otherwise stated, there is no number less than 2 which the sum cannot be made to exceed. It follows that the non-terminating or **infinite** series,

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots$$

has 2 as the **limit** of the sum of n of its terms as n is indefinitely increased, and it is said to have 2 as its sum.

The series may be looked upon as a perfectly definite though not simple way of giving the number 2.

That the sum of n terms may, by increasing n, be made to differ from 2 by as little as we please is rendered very striking if we represent the terms taken by lengths measured on a straight line.



AB measures 2; AP, 1; PQ, $\frac{1}{2}$; QR, $\frac{1}{4}$; RS, $\frac{1}{8}$; etc. Then AQ measures the sum of 2 terms; AR of 3 terms; AS of 4 terms; etc. Not many terms need be taken to make the sum practically 2, though the sum is never actually 2.

In like manner if we had considered the series,

$$a + ar + ar^2 + \cdots$$

where r is less than 1, we should have had the sum of n terms

$$= a\frac{1-r^n}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Now $\frac{ar^n}{1-r}$ is the product of $\frac{a}{1-r}$ by r^n, and r is less than 1. But we know that successive powers of a number less than 1 are each smaller than the preceding ones and that by increasing the exponent sufficiently we can make the power as small as we choose. For example, consider the powers of $\frac{2}{3}$: $\frac{2}{3}$, $\frac{4}{9}$, $\frac{8}{27}$ $\cdots$ $(\frac{2}{3})^{11}$ (= $\frac{2048}{177147}$ which is less than $\frac{1}{86}$); they diminish steadily and it is evident that if we take the exponent large enough we can make the power as small as we like. Hence by increasing n we can diminish $\frac{ar^n}{1-r}$ as much as we like, that is we can make the sum of n terms of the series differ from $\frac{a}{1-r}$ by as little as we like; hence the limit of the sum of n terms as n is increased indefinitely is $\frac{a}{1-r}$, and the sum to infinity is $\frac{a}{1-r}$.

The recurring decimal is an interesting example of an infinite series.

Ex. It has been seen that,

$$\tfrac{1}{3} = 0{\cdot}3\tfrac{1}{3} = 0{\cdot}333\tfrac{1}{3} = 0{\cdot}33333\cdots \text{ (non-terminating)}.$$

Now, starting with the so-called decimal,

$$0{\cdot}33333\cdots \text{ (non-terminating)},$$

we see that that it is nothing other than the infinite series:

$$\tfrac{3}{10} + \tfrac{3}{100} + \tfrac{3}{1000} + \tfrac{3}{10000} + \cdots$$

This is a geometrical progression with the common ratio $\frac{1}{10}$. The sum is therefore

$$\frac{\frac{3}{10}}{1-\frac{1}{10}} = \tfrac{3}{9},\ i.\ e.,\ \tfrac{1}{3}.$$

Exercises

1. Find the sum of (1) 20 terms, (2) n terms, of each of the following series:

(1) $1 + 2 + 3 + 4 + 5 + \cdots$

(2) $1 + 3 + 5 + 7 + 9 + \cdots$

(3) $30 + 29 + 28 + 27 + 25 + \cdots$

(4) $8 + 17 + 26 + 35 + \cdots$

(5) $11 + 25 + 39 + 53 + \cdots$

(6) $99 + 94 + 89 + 84 + \cdots$

2. Find the sum of (1) 30 terms, (2) n terms, of each of the following series:

(1) $1 + 3 + 9 + 27 + \cdots$

(2) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

(3) $7 + 14 + 28 + 56 + \cdots$

(4) $5 + 15 + 45 + 135 + \cdots$

(5) $2 + \frac{6}{5} + \frac{18}{25} + \frac{54}{125} + \cdots$

(6) $1 + \frac{1}{(1 \cdot 04)} + \frac{1}{(1 \cdot 04)^2} + \frac{1}{(1 \cdot 04)^3} + \cdots$

3. Find the arithmetical mean of 4 and 16; of 7 and 13; of 5 and 18; of -5 and -2; of 8 and -8; of 9 and -7.

4. Find the geometrical mean of 4 and 9; of 6 and 24; of 5 and 18.

5. Find the sum of each of the following infinite series:

(1) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

(2) $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots$

(3) $1 + \frac{1}{1 \cdot 05} + \frac{1}{(1 \cdot 05)^2} + \frac{1}{(1 \cdot 05)^3} + \cdots$

(4) $1 + \frac{1}{1 \cdot 045} + \frac{1}{(1 \cdot 045)^2} + \frac{1}{(1 \cdot 045)^3} + \cdots$

(5) $\frac{3}{2} - \frac{6}{5} + \frac{24}{25} - \frac{96}{125} + \cdots$

6. In the case of each series given in example 5, find the sum of the infinite series of terms following the tenth term.

7. The middle points of the sides of a square are joined forming a second square; the middle points of the sides of this square are joined to form a third square, etc. Shew that the sum of the areas of the squares that can be thus formed can never exceed twice the area of the original square.

8. The first term of an arithmetical progression is 6, the difference is 13; find the sum of 21 terms.

9. The first term of an arithmetical progression is $2\frac{1}{2}$, the difference is $\frac{1}{2}$; find the sum of 50 terms.

10. Insert (1) seven arithmetic means (2) three geometric means between 1 and 49.

11. Find the sum of 120 terms of the arithmetical progression

$$-5+4+\cdots$$

12. The sum of 20 terms of an arithmetical progression whose difference is 2, is equal to 420. Find the first term.

13. The sum of 50 terms of an arithmetical progression whose difference is 5, is 6275. Find the first term.

14. The sum of 12 terms of an arithmetical progression whose first term is 4, is 180. Find the difference.

15. The sum of 160 terms of an arithmetical progression whose first term is -159 is equal to zero. Find the difference.

16. The sum to infinity of a geometrical progression whose first term is 11, is 22. Find the ratio.

17. Sum the arithmetical progression whose first term is 15 and whose last term is 160, the number of terms being 25.

18. Find the sum of 60 terms of the arithmetical progression whose twelfth term is 29 and whose twenty-third term is 51.

19. The fourteenth, twentieth and last terms of an arithmetical progression are 251, 413 and 548; find the first term, the number of terms and the sum.

20. Find an arithmetical progression such that the sum of the first five terms is one-fourth the sum of the following five terms, the first term being unity.

21. Shew that the sum of $2n+1$ consecutive integers is divisible by $2n+1$.

22. In a geometrical progression shew that the product of any two terms equidistant from a given term is the same.

23. Find the values of the following recurring decimals:

$$0\cdot\dot{4},\ 0\cdot\dot{9},\ 0\cdot\dot{1}5\dot{0},\ 0\cdot1\dot{2}6\dot{4},\ 0\cdot2\dot{1}3\dot{3},\ 0\cdot36\dot{4}2\dot{8}.$$

24. Shew that

$$\left(\tfrac{4}{10}+\tfrac{4}{100}+\cdots\right)^{\frac{1}{2}}=\tfrac{2}{3}.$$

25. Sum to infinity the series,

$$\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{8}-\tfrac{1}{18}+\tfrac{1}{32}-\tfrac{1}{108}+\cdots$$

26. Prove that

$$\left(\tfrac{4}{10}+\tfrac{4}{100}+\cdots\right)\left(\tfrac{6}{10}+\tfrac{6}{100}+\cdots\right)=\tfrac{8}{27}.$$

27. Find the sum of 50 terms of the series,

$$5+55+555+\cdots.$$

28. Find a geometrical progression continued to infinity such that each term is ten times the sum of all which follow it.

29. Find the sum of 20 terms of the series,

$$0\cdot\dot{3}+3\cdot\dot{3}+33\cdot\dot{3}+333\cdot\dot{3}+\cdots.$$

30. In a circle a square is inscribed; in this square a circle is inscribed; in this latter circle a square is inscribed, etc. Shew that the limit to the sum of the areas of the circles is twice the area of the original circle, and the limit to the sum of the areas of the squares is twice the area of the first square.

31. In a circle a regular hexagon is inscribed; in this hexagon a circle is inscribed; in this latter circle a regular hexagon is inscribed, etc. Shew that the limit to the sum of the areas of the circles is four times the area of the original circle, and the limit to the sum of the areas of the regular hexagons is four times the area of the first hexagon.

32. Employ the identity, $(x+1)^3 - x^3 = 3x^2 + 3x + 1$, to find the sum,

$$1^2+2^2+3^2+4^2+\cdots+(n-1)^2+n^2.$$

CHAPTER III

LOGARITHMS

1. Indices. The laws of indices are illustrated by the equations:

$$10^m \times 10^n = 10^{m+n} \qquad \text{I}$$

$$\left.\begin{aligned} \frac{10^m}{10^n} &= 10^{m-n} \quad (m > n) \\ \frac{10^m}{10^n} &= \frac{1}{10^{n-m}} \quad (n > m) \end{aligned}\right\} \qquad \text{II}$$

$$(10^m)^r = 10^{mr} \qquad \text{III}$$

Here m, n, r are integers. There is a real gain in introducing exponents that are not positive integers, namely, **fractional**, **negative** and **zero** exponents. To give such exponents a meaning we *agree* that they are to obey the laws of indices established for positive integral exponents. Then:

(1) By Law III, $(10^{\frac{1}{3}})^3 = 10^{\frac{1}{3}\times 3} = 10^1 = 10$,

or $10^{\frac{1}{3}} = \sqrt[3]{10}$;

(2) By Law I, $10^5 \times 10^0 = 10^{5+0} = 10^5 = 10^5 \times 1$,

or $10^0 = 1$;

(3) By Laws I and II, $10^6 \times 10^{-5} = 10^{6-5} = 10^1 = 10^6 \times \frac{1}{10^5}$

or $10^{-5} = \frac{1}{10^5}$.

2. Logarithms. In the relation $10^4 = 10{,}000$ the index 4 is called the **logarithm** of 10,000 to the **base** 10. The relation is written briefly in such a way as to give prominence to the number 4 thus:

$$4 = \log_{10} 10{,}000.$$

In like manner, the relation $3^4 = 81$ leads to the statement, $\log_3 81 = 4$. Hence:

The logarithm of a number is the index of the power to which a given number called the base must be raised to yield that number.

In what follows the base will be supposed to be 10, in the absence of any statement to the contrary. Hence log 10,000 will mean $\log_{10} 10{,}000$.

From the table:

$$10^0 = 1$$

$10^1 = 10$	$10^{-1} = 0{\cdot}1$
$10^2 = 100$	$10^{-2} = 0{\cdot}01$
$10^3 = 1000$	$10^{-3} = 0{\cdot}001$
$10^4 = 10{,}000$	$10^{-4} = 0{\cdot}0001$
etc.	etc.

we derive at once the table:

$$\log 1 = 0$$

$\log 10 = 1$	$\log 0{\cdot}1 = -1$
$\log 100 = 2$	$\log 0{\cdot}01 = -2$
$\log 1000 = 3$	$\log 0{\cdot}001 = -3$
$\log 10{,}000 = 4$	$\log 0{\cdot}0001 = -4$
etc.	etc.

The logarithms of rational numbers other than those indicated in the table just given, are not rational, *i. e.*, cannot be expressed by a finite number of figures. We have then to be satisfied with approximations. Thus the logarithm of 72 correct to seven places of decimals is found to be 1·8573325.

Into the method of computing the approximate logarithms we shall not here enter, but shall suppose such logarithms as will be employed, to be given.

The integral part of the logarithm of a number is called the **characteristic**, and the decimal part the **mantissa.**

3. Important Facts. An examination of the table in art. 2 leads to the following conclusions:

(1) *The growth of the logarithm is not proportionate to the growth of the number.*

Thus as the number grows from 1 to 10 the logarithm grows from 0 to 1, while a growth from 10 to 100 of the number—a range ten times as great—gives only an equal growth of the logarithm, namely from 1 to 2.

(2) *The characteristic of the logarithm of a number greater than unity may be determined by inspection.*

For example take the number 3729·3. This number lies between 10^3 and 10^4; its logarithm therefore lies between 3 and 4, so that the characteristic is 3.

It is thus seen that, *the characteristic of the logarithm of a number greater than unity is one less than the number of figures before the decimal point.*

(3) *The characteristic of the logarithm of a number less than unity may be determined by inspection.*

For example, take the number 0·00375. This number lies between 10^{-3} and 10^{-2}. Its logarithm therefore lies between -3 and -2, *i.e.*, the logarithm is -3 plus a decimal, *or* -2 minus a decimal. We shall *agree to keep the mantissa positive* and hence say that the logarithm is -3 plus a decimal. The characteristic here then is -3.

It is now easily seen that, *the characteristic of the logarithm of a number less than unity is the number of the place of the first significant figure in that number expressed as a decimal.*

4. Interpolation. From the tables given at the end of the book, we have (after supplying the characteristics) to six places of decimals:

$$\log 7325 = 3{\cdot}864808$$
$$\log 7326 = 3{\cdot}864867$$
$$\log 7327 = 3{\cdot}864926$$
$$\log 7328 = 3{\cdot}864985$$

For each advance of 1 in the number there is an advance of 59 (in the sixth place) in the logarithm. This seems to be at variance with one of the conclusions of the preceding article, but it is to be noted that here the growth 1, 2 or 3, relatively to the number 7325, is small. If we examine the tables further, we find that what we observe here is general, and we can say that the growth in the number over a relatively small range is always attended with a proportionate growth in the logarithm. This fact enables us to solve problems like the following:

Ex. 1. Given log 7325 = 3·864808, and log 7326 = 3·864867, find log 7325·64.

A growth of 1 in 7325 gives a growth of 59 (in the sixth place) in the logarithm.

∴ a growth of 0·64 in 7325 gives a growth of $\frac{64}{100}$ of 59 (in the sixth place) in the logarithm.

Now $\frac{64}{100}$ of 59 = 38, to the nearest integer.

∴ log 7325·64 = 3·864808 + 38 (in the sixth place),
= 3·864846.

Ex. 2. Referring to the logarithms given above, find what number has 3·864890 as its logarithm.

$$\log 7326 = 3{\cdot}864867$$
$$\log 7327 = 3{\cdot}864926$$

∴ the number belonging to 3·864890 lies between 7326 and 7327.

The given logarithm exceeds the logarithm of 7326 by 23 in the sixth place.

Now a growth of 59 in the logarithm means a growth of 1 in the number.

Therefore a growth of 23 in the logarithm means a growth of $\frac{23}{59}$ in the number.

Also $\frac{23}{59} = 0{\cdot}39$ to the nearest hundredth.

$\therefore$ the given logarithm belongs to 7326·39.

5. **Theorems.** We shall now prove the following theorems:

(1) *The logarithm of the product mn of any two numbers m and n is equal to the sum of the logarithms of m and n.*

For let $x = \log m$, and $y = \log n$.

$\therefore m = 10^x$, and $n = 10^y$.

$\therefore mn = 10^x \times 10^y = 10^{x+y}$.

$\therefore \log (mn) = x + y = \log m + \log n$.

It is now easily shewn that the logarithm of the product of any number of factors is the sum of the logarithms of the factors.

(2) *The logarithm of the quotient $\frac{m}{n}$ is equal to the difference between the logarithms of the dividend and the divisor.*

As before put $x = \log m$, and $y = \log n$.

Then $m = 10^x$, and $n = 10^y$.

$$\therefore \frac{m}{n} = \frac{10^x}{10^y} = 10^{x-y}.$$

$$\therefore \log \left(\frac{m}{n}\right) = x - y = \log m - \log n.$$

(3) *The logarithm of the power m^p is p times the logarithm of m.*

Let $x = \log m$, so that $m = 10^x$.

$\therefore m^p = (10^x)^p = 10^{px}$.

$\therefore \log (m^p) = px = p \log m$.

(4) *The logarithm of $\sqrt[r]{m}$ is $\frac{1}{r}$ of the logarithm of m.*

Let $x = \log m$, so that $m = 10^x$.

$\therefore \sqrt[r]{m}$, which equals $(m)^{\frac{1}{r}}$, $= (10^x)^{\frac{1}{r}} = 10^{\frac{x}{r}}$.

$\therefore \log \sqrt[r]{m} = \frac{x}{r} = \frac{1}{r} \log m$.

6. **Important Fact.** From the tables we find,

$$\log 7325 = 3{\cdot}864808.$$

$$\begin{aligned} \text{Now } \log 73{\cdot}25 &= \log \tfrac{7325}{100} = \log 7325 - \log 100 \\ &= 3{\cdot}864808 - 2, \text{ since } 2 = \log 100, \\ &= 1{\cdot}864808. \end{aligned}$$

$$\begin{aligned} \text{Again, } \log 732500 &= \log (7325 \times 100) = \log 7325 + \log 100 \\ &= 3{\cdot}864808 + 2 \\ &= 5{\cdot}864808. \end{aligned}$$

$$\begin{aligned} \text{Also, } \log 0{\cdot}0007325 &= \log \tfrac{7325}{10000000} = \log 7325 - \log 10000000 \\ &= 3{\cdot}864808 - 7 \\ &= -4 + 0{\cdot}864808, \text{ (the mantissa being kept positive).} \end{aligned}$$

This last result is written $\bar{4}{\cdot}864808$, the notation indicating that the minus sign effects only the 4.

Hence with the convention that the mantissæ of the logarithms of numbers less than unity are to be positive, the logarithms of the numbers 7325, 73·25, 732500, 0·0007325 have the same mantissa.

It is evident then that:

Numbers which differ only in the position of the decimal point have logarithms with the same mantissa.

Now the characteristics of the logarithms are known by inspection of the numbers. Therefore in a table of logarithms it is sufficient to give the mantissæ.

7. Applications. We are now in a position to make use of logarithms to simplify and shorten the work of computation.

Ex. 1. Find the product, $3{\cdot}217 \times 0{\cdot}1389 \times 7{\cdot}513$.

The logarithm of the product

$= \log 3{\cdot}217 + \log 0{\cdot}1389 + \log 7{\cdot}513$

$= 0{\cdot}507451 + \bar{1}{\cdot}142702 + 0{\cdot}875813$, (as given in the tables)

$= 0{\cdot}525966$.

Now from the tables $\log 3{\cdot}357 = 0{\cdot}525951$, which is 15 (in the sixth place) less than the logarithm found. Also 129 is the difference in the logarithm for a growth of 1 (in the fourth significant place) in the number. Therefore the logarithm found, namely 0·525966, belongs to the number $3{\cdot}357 + \frac{15}{129}$ of 0·001, *i. e.*, to 3·3571.

∴ the product required is 3·3571.

This does not mean that the product is exactly 3·3571, but that this number is the product, correct to five significant figures. If the logarithm had been given to a higher degree of accuracy the result would have been nearer to the exact result, or would have been exact.

If we had only to make the computation the necessary work would have been as follows:

$$\begin{array}{ll} & 0{\cdot}507451 \\ & \bar{1}{\cdot}142702 \\ & \underline{0{\cdot}875813} \\ 3{\cdot}357 & 0{\cdot}525966 \\ \underline{\quad\quad 1} & \quad\quad \underline{51} \\ & 129)\overline{150}(1 \end{array}$$

Ex. 2. Find the quotient, $3{\cdot}279 \div 2{\cdot}594$.

$$\begin{array}{rr} (\log 3{\cdot}279 =) & 0{\cdot}515741 \\ (\log 2{\cdot}594 =) & \underline{0{\cdot}413970} \\ (\log (\text{quotient}) =) & 0{\cdot}101771 \\ (\log 1{\cdot}264 =) & \underline{47} \\ & 343)\overline{240}(1 \end{array}$$

∴ quotient = 1·2641 (approximately).

Ex. 3. Find the value of $(1{\cdot}237)^5$.

$$
\begin{aligned}
\log 1{\cdot}237 &= 0{\cdot}092370 \\
\log (1{\cdot}237)^5 &= 0{\cdot}461850 \\
\log 2{\cdot}896 &= \phantom{0{\cdot}46}\underline{1799}
\end{aligned}
$$

$$150)\overline{510}(3$$

$\therefore\ (1{\cdot}237)^5 = 2{\cdot}8963$, (approximately).

Ex. 4. Find the fifth root of 0·9734.

$$\log 0{\cdot}9734 = \bar{1}{\cdot}988291.$$

$$
\begin{aligned}
\therefore\ \log (0{\cdot}9734)^{\frac{1}{5}} &= \tfrac{1}{5} \text{ of } \bar{1}{\cdot}988291 \\
&= -(\tfrac{1}{5} \text{ of } 5) + \tfrac{1}{5} \text{ of } 4{\cdot}988291 \\
&= \bar{1} + 0{\cdot}997658 \\
&= \bar{1}{\cdot}997658, \text{ when the mantissa is positive.}
\end{aligned}
$$

Now $\log 0{\cdot}9946 = \bar{1}{\cdot}997648$, and as in the earlier examples we find that $\bar{1}{\cdot}997658$ belongs to the number 0·99462 so that this latter is the root (approximate) sought.

Exercises

1. Find the numbers whose logarithms are:

1·093772; 3·701010; 2·713265; $\bar{2}{\cdot}113113$; $\bar{4}{\cdot}172658$; $\bar{7}{\cdot}423268$.

2. Find the following products:

263×721; 2·317×7·231; 3·014×1·032×0·3789;
7·389×15·27×0·3718; 1·795×1·237×0·3694;
2·397×7·213×0·793×0·594.

3. Find the value of:

$$\frac{31{\cdot}37\times1{\cdot}359\times2{\cdot}374}{1{\cdot}379\times5{\cdot}293}\ ;\quad \frac{0{\cdot}3794\times0{\cdot}5938\times0{\cdot}7925}{0{\cdot}1378\times0{\cdot}0739}$$

4. Find to four significant figures the square roots of:

1·37, 4·39, 2, 7, 13, 0·17.

5. Find to four significant figures the cube roots of:

1·29, 73·5, 2, 3, 5, 0·13.

6. Find to four significant figures the fifth roots of:

$$3{\cdot}29,\ 7{\cdot}5,\ 3,\ 5,\ 7,\ 0{\cdot}379.$$

7. Find to four significant figures the values of the following:

$$\left\{27{\cdot}01\ (12{\cdot}93)^{\frac{1}{3}}\right\}^{\frac{1}{4}};\quad \left\{\frac{(15{\cdot}7)^{\frac{1}{3}}(14{\cdot}23)^{\frac{1}{4}}}{(10{\cdot}25)^{\frac{1}{5}}}\right\}^{\frac{1}{3}}.$$

8. Given log 2, shew how to find log 0·05 and log $\sqrt{1{\cdot}25}$.

9. How many digits are there in

$$2^{32},\ 7^{7},\ 2{\cdot}5^{24}.$$

10. Find the place of the first significant figure in $(0{\cdot}\dot{3})^{6\frac{1}{2}}$, $(\frac{1}{3})^{21}$, $(\frac{2}{7})^{18}$.

11. Find the values of:

$$\log_3 81,\ \log_2 1024,\ \log_7 343.$$

12. Find the least power to which $\frac{1}{3}$ must be raised to give a result less than $\frac{1}{84500}$.

13. Find the values of x which satisfy the following equations:

$$2^x = 32;\ 5^x = \tfrac{1}{5};\ 10^x = 1;$$
$$10^x = 23;\ 13^x = 117;\ 3^{2x+1} = 5^{3x-1}.$$

14. Employing the formula for the measure of the area of a triangle in terms of its sides, find the areas of the triangles the lengths of whose sides are:

(1) 15 yd., 17 yd., 23 yd.;
(2) 17·35 m., 29·47 m., 37·38 m.;
(3) 119·3 dm., 275·9 dm., 354·6 dm.;
(4) 79·5 in., 99·7 in,, 113·8 in.

15. Find the values of:

$$1000\ (\tfrac{1000}{1045})^{9},\ 380{\cdot}50\ (\tfrac{100}{104})^{2}\ (\tfrac{1000}{1045}).$$

16. Find the present worth of:

(1) $900 due 7 years hence, the rate of interest being 5 per cent.

(2) $1200 due 11 years hence, the rate of interest being 5 per cent., compounded half-yearly.

(3) $721·50 due in 19 years, the rate of interest being 3½ per cent., compounded quarterly.

17. Find the amount of $600 in 7 years, 3 months, at 4½ per cent., compounded quarterly.

18. If in 7 years, 6 months, $700 amounts to $974·03, find the rate per cent. per annum.

19. At a certain rate compounded quarterly $1200 amounts in 10 years to $1972·23. Find the rate.

20. Find the volume of a cylinder whose height is 32·62 cm. and the radius of whose base is 4·23 cm.

21. A man borrows $15,000 at 4½ per cent. per annum, compound interest, for 10 years, and lends it at 4½ per cent., compounded quarterly. Find his gain.

22. The sides of a triangle are 12·63 cm., 13·74 cm. and 17·82 cm. Find the lengths of the perpendicular lines drawn from each angle to the opposite side.

23. A borrows $10,000 from B at 3½ per cent. per annum compounded yearly for 7 years. He loans ½ of it at 3½ per cent., compounded half-yearly, ⅓ of it at 3 per cent., compounded yearly, and the remainder at 4 per cent., compounded quarterly. Find his gain or loss.

24. A man deposits $100 in a savings' bank. At the end of each year for 3 years he deposits $25 more than the preceding deposit. If interest is at 5 per cent., compounded half-yearly, find what sum he has in the bank at the end of the third year.

25. A man deposits $500 in a savings' bank. At the end of each year for 6 years he deposits nine-tenths of the preceding deposit. If interest is at 5 per cent., compounded quarterly, find what sum he has in the bank at the end of the sixth year.

CHAPTER IV

ANNUITIES

1. If A is under obligation to pay $500 to B at the end of each year for the next three years, we have seen that it is possible to find the present value of these payments, given the rate of *interest*. If this rate is 5 per cent., then the present value of these three payments is equal to

$$\left(\frac{1}{1\cdot05}\right) \text{ of } \$500 + \left(\frac{1}{1\cdot05}\right)^2 \text{ of } \$500 + \left(\frac{1}{1\cdot05}\right)^3 \text{ of } \$500.$$

By making use of the Interest Tables at the end of the book we find that the present value is

$$\$476\cdot19 + \$453\cdot51 + \$431\cdot06 = \$1360\cdot76.$$

A payment, like the above, recurring at stated intervals, is called an **annuity**; the annuity in question is said to **begin** now and to **run** for three years.

An annuity is said to be **deferred** for a number of years when it begins at the end of that number of years, *i.e.*, when the first payment is made at the end of one more than that number of years.

The interval between successive payments is generally one year, but it may be any period.

2. The addition of the present value of the different payments may, when they are numerous, be a very tedious operation. We shall find a formula that will allow the value to be more directly computed.

Suppose that A is to pay B an annuity of $400 beginning now and running for 20 years, the rate of interest being 5 per cent. Then the present value of all the payments is

$$\frac{100}{105} \text{ of } \$400 + \left(\frac{100}{105}\right)^2 \text{ of } \$400 + \cdots + \left(\frac{100}{105}\right)^{20} \text{ of } \$400.$$

This is a geometrical progression of 20 terms, with common ratio $\frac{100}{105}$ or $\frac{1}{1\cdot05}$. Its sum is, therefore,

$$\frac{100}{105} \text{ of } \$400 \times \frac{1-(\frac{100}{105})^{20}}{1-\frac{100}{105}};$$

$$\text{or } \$400 \times \frac{100}{5} \times \{1-(\tfrac{100}{105})^{20}\};$$

$$\text{or } \$400 \times \frac{1}{\cdot05} \times \{1-(\tfrac{1}{1\cdot05})^{20}\}.$$

3. For the general case in which the annuity is $\$A$ and the rate r on the unit, the annuity beginning now and running for n years, the present value is easily seen to be

$$\frac{\$A}{1+r} + \frac{\$A}{(1+r)^2} + \cdots + \frac{\$A}{(1+r)^n}.$$

This is a geometrical progression whose first term is $\frac{\$A}{1+r}$ and common ratio, $\frac{1}{1+r}$, the number of terms being n. Its sum is therefore,

$$\frac{\$A}{1+r} \cdot \frac{1-\dfrac{1}{(1+r)^n}}{1-\dfrac{1}{1+r}}$$

which reduces to

$$\frac{\$A}{r} \cdot \left\{ 1-\frac{1}{(1+r)^n} \right\}.$$

4. If in art. 3 we suppose the number of payments to be increased indefinitely the series giving the present value is an infinite series and the sum is

$$\frac{\$A}{r}$$

That this is the present value is readily seen otherwise for $\frac{\$A}{r}$ put out at interest, the rate being r on the unit, would continue to bring in $\frac{\$A}{r} \times r$, or $\$A$, every year.

Such an annuity is called a **perpetuity.**

5. If an annuity of $750 is deferred 10 years and is to run 15 years, the rate of interest being 4 per cent., its present value is

$$\frac{\$750}{(1{\cdot}04)^{11}} + \frac{\$750}{(1{\cdot}04)^{12}} + \cdots + \frac{\$750}{(1{\cdot}04)^{25}}.$$

There are here 15 terms in geometrical progression and the sum is

$$\frac{\$750}{(1{\cdot}04)^{11}} \cdot \frac{1-\left(\frac{1}{1\cdot04}\right)^{15}}{1-\frac{1}{1\cdot04}},$$

which reduces to

$$\frac{\$750}{0{\cdot}04} \times \frac{1}{(1{\cdot}04)^{10}} \cdot \left\{ 1 - \frac{1}{(1{\cdot}04)^{15}} \right\}$$

For the general case in which the annuity is $\$A$, deferred m years and running n years, the rate being r on the unit, the result is

$$\frac{\$A}{r} \cdot \frac{1}{(1+r)^m} \cdot \left\{ 1 - \frac{1}{(1+r)^n} \right\}.$$

Exercises

Note.—In the following examples the student is recommended to work out all results, employing the Interest Tables and the Tables of Logarithms at the end of the book.

1. Find the present value of an annuity of $120 to be paid at the end of each year for 10 years, money being worth 4½ per cent. per annum.

2. An annuity of $600 is to run for 4 years. If the rate of interest is 5 per cent.,

(1) Find the present value of the annuity.

(2) Find the value of each payment at the end of the 4 years, sum these values, and find the present value of the sum.

3. Treat as in example 2 an annuity of $500 to run for 3 years, the rate of interest being 4½ per cent.

4. Write down the expressions for, and then compute, the present values of,

(1) An annuity of $200 running 5 years, interest at 4 per cent.

(2) An annuity of $725 running 6 years, interest at 5 per cent.

(3) An annuity of $75 running 4 years, interest at 4½ per cent.

5. A payment of $84 is to be made every half-year for the next three years. If interest is at 5 per cent., payable half-yearly, find the present value of all the payments.

6. Find the present value of an annuity of $150 to be paid at the end of each half-year for the next 7 years, interest being at the rate of 6 per cent. per annum.

7. Find the present value of an annuity of $900 for the coming 11 years, interest being at the rate of 4 per cent., payable half-yearly.

8. Find the present value of an annuity of $120 deferred 2 years and running 3 years, the rate of interest being 5 per cent.

9. An annuity of $200 deferred 2 years is to run 3 years. If the rate of interest is 5 per cent.,

(1) Find the present value of the annuity.

(2) Find the value of each payment at the time of the last payment, sum their values and find the present value of the sum.

10. Treat as in example 9 an annuity of $830 deferred 1 year and running 4 years, the rate of interest being 3½ per cent.

11. Find the present value of an annuity of $600, deferred 3 years and running for 11 years, interest at 5½ per cent. per annum.

12. Write down the expressions for, and then compute, the present values of:

(1) An annuity of \$72 deferred 3 years, running 4 years, interest at $3\frac{1}{2}$ per cent.

(2) An annuity of \$225 deferred 2 years, running 3 years, interest at $4\frac{1}{2}$ per cent.

(3) An annuity of \$96 deferred 4 years, running 5 years, interest at 6 per cent.

13. A man wishes to purchase an annuity of \$300 for 10 years. If money is worth 6 per cent. per annum, what sum will be required for the purchase?

14. Find what annuity for the next 12 years can be bought for \$10,500, the rate of interest being 4 per cent.

15. If the rate of interest is 4 per cent. find the present value of a perpetuity of \$90.

16. If the rate of interest is $4\frac{1}{2}$ per cent. find the present value of a perpetuity of \$210.

17. Find the present value of a perpetuity of \$125 deferred 5 years, the rate of interest being 3 per cent.

18. A perpetuity of \$250 is sold for \$6250. Find at what rate the interest is calculated.

19. A school section borrows \$4500 to build a school house; this sum is to be repaid in ten equal annual instalments. If money is worth 6 per cent. per annum, find the amount of the instalment.

20. A loan of \$5000 is to be repaid in 7 years in equal half-yearly instalments, interest at the nominal rate of 4 per cent. per annum. Find the amount of each instalment.

21. A man borrows \$8000 from a loan company agreeing to pay principal and interest in eight equal annual instalments. If money is worth 5 per cent., find the annual payment.

22. Twenty years ago a man insured his life for \$10,000 paying an annual premium of 2 per cent. During the first ten years money could have been invested at 6 per cent., and at 4 per cent. during the next ten years. If he should die now which would have been the more profitable investment for his family?

23. A man deposits $600 at the beginning of each year for 15 years in a savings' bank which allows 3 per cent. per annum compounded half-yearly, on deposits. What sum will be standing to his credit at the end of the fifteenth year?

24. A mortgage on a farm for $4000 with interest at 6 per cent. has 4 years to run. It is offered for sale; what sum should a man seeking investment for his money at 5 per cent. offer for it?

25. A town is under obligation to pay at the end of each year for 4 years the interest on $10,000 at 6 per cent. and at the end of the 4 years to pay this sum. What tax for this purpose must be collected each year in order that the interest may be paid each year, and that each year a deposit which will be sufficient to meet the payment of $10,000 may be made in a bank which allows 4 per cent. per annum interest.

26. A father buys for his son on his twelfth birthday an annuity of $520 to be paid on his fifteenth birthday and each successive birthday to the twenty-first inclusive. If money is worth $3\frac{1}{2}$ per cent. payable half-yearly, find the sum the father pays for the annuity.

27. A mortgage of $8000, dated Jan. 1, 1903, payable in four annual instalments of $2000 each, interest reckoned at 6 per cent. payable half-yearly, is sold on July 1, 1903. What sum must the purchaser pay so that the investment may be worth 8 per cent.?

28. A town issues 20 year debentures for $20,000 bearing interest at 5 per cent. (*i.e.*, declares its readiness to pay 5 per cent. of $20,000 each year, and $20,000 at the end of 20 years). What should an investment company wishing to have its money invested at $4\frac{1}{2}$ per cent. offer in cash for the debentures?

29. A man deposits in a bank $100, and at the end of each year deposits 10 per cent. more than the previous year. If interest is paid at the rate of 4 per cent. compounded half-yearly, find what sum he has in the bank when he has made his tenth deposit.

30. A sum of $2000 is lent to be repaid with interest at 4 per cent. beginning with $80 at the end of the first year, and increasing 50 per cent. each year on the last preceding payment. Find when the debt will be paid off.

31. A person who has a capital of $20,000 for which he receives interest at 5 per cent., spends every year $2500. Find in how many years he will have all of his money spent.

CHAPTER V

MENSURATION

1. **The Pyramid.** A **pyramid** is a solid bounded by plane faces, all but one of which meet in a point, called the **vertex** of the pyramid, the face opposite the vertex being called its **base**.

The length of the perpendicular from the vertex to the plane of the base is called the **altitude** of the pyramid.

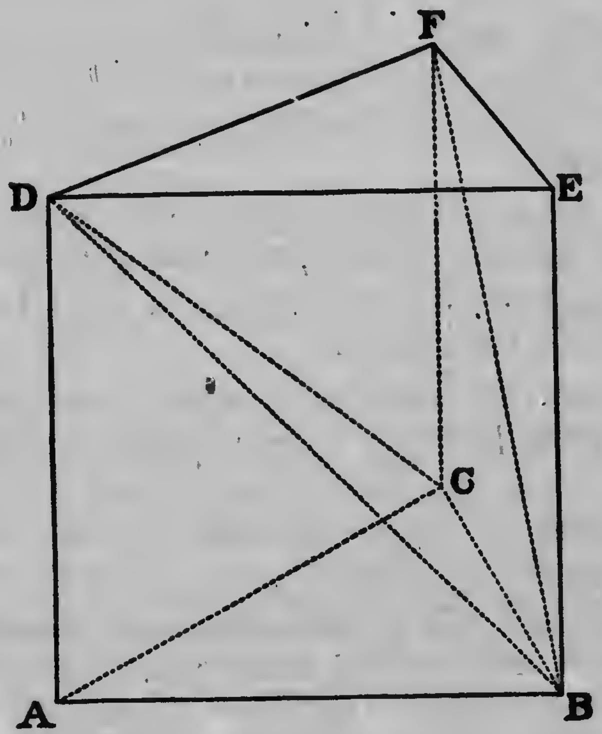
A pyramid with a triangular base is called a **triangular pyramid**, or a **tetrahedron.**

It is shewn in works on geometry that pyramids, of equal altitudes, standing on equal bases are of the same volume.

An expression for the measure of the volume of a pyramid may now be found.

Let A B C D E F be a right prism on a triangular base. If now the prism be divided by the plane D B C, one part, D A B C, is a pyramid; if the other part be divided by the plane D F B, the two parts D F B C and D B E F are pyramids. Thus the prism has been divided into three pyramids. Now the pyramids D A B C, B E F D (*i.e.* D B E F) are equal, having equal bases and equal altitudes; for the same reason the pyramids D F B C and D B E F are equal. Therefore the three pyramids into which the prism has been divided are equal. The volume of the pyramid D A B C is then one-third that of the prism, or is measured by one-third of the product of the measures of the area of the base A B C and the altitude.

The pyramid D A B C is special in that the base is triangular, and the edge D A is the altitude. But in the case of any other pyramid, if a perpendicular be drawn from the vertex to the base, and the foot of the perpendicular joined to the angular points of the base, the pyramid is seen to



be the sum (possibly, too, a difference) of a number of such pyramids. Hence we have the following theorem :

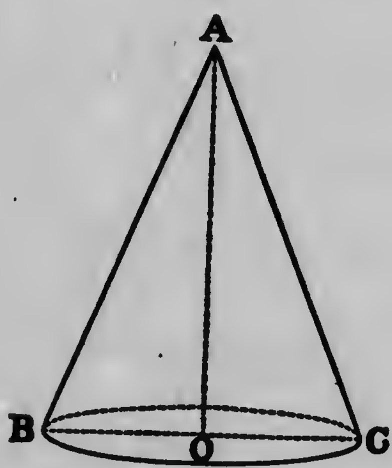
The volume of a pyramid is measured by one-third of the product of the measures of the area of its base and its altitude.

2. **The Cone.** Let the right-angled triangle A O C make a complete revolution about the side A O as axis; the solid thus generated is a **right circular cone.**

In what follows the word cone will be taken to mean right circular cone.

The point A is its **vertex**, and the length of A O its **altitude.** The side A C of the triangle has generated the **curved surface** or **mantel** of the cone, and the side O C of the triangle, the circular base of the cone.

Since the vertex A is at the same distance from every point in the circumference of the base, it is plain that if



the mantel, regarded as a sheet, be cut along the line A C, and placed on a plane, it will assume the form of a sector of a circle; the radius of this sector is equal to the **slant** height of the cone. Hence if r, h and l measure the radius of the base of the cone, its altitude and its slant height, we have the relation

$$l=\surd(r^2+h^2),$$

and the area of the curved surface is measured by

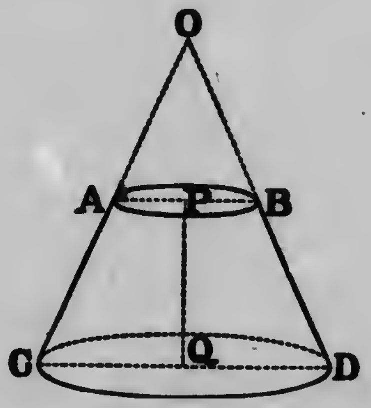
$$\pi\, rl \text{ or } \pi\, r\surd(r^2+h^2).$$

To find the measure or the volume of a cone.

Suppose a regular polygon inscribed in its circular base, and consider the pyramid with the same vertex as the cone, and with this polygon as base.

The volume of the pyramid is measured by one-third of the product of the measures of its altitude and the area of its base.

If now, as in the case of the circle, we suppose the number of sides of the polygon increased, the area of the



base of the pyramid becomes more and more nearly equal to the area of the base of the cone, and the volume of the pyramid becomes more and more nearly equal to the volume of the cone; we are thus led to suppose that the volume of a cone is measured by one-third of the product of the measures of its altitude and the area of its base. In more advanced works this relation is shewn to be exact, and we have the formula:

$$v = \frac{1}{3}\pi r^2 h.$$

where v denotes the measure of the volume.

A solid such as APBDQC—the part of the cone OCQD remaining after a conical part O A P B has been cut off—is called the **frustum** of a cone.

Let a, b be the measures of the lengths of QD, PB—the radii of the base and the top of the frustum; let h and k be the measures of the lengths PQ, BD—the *altitude* and the *slant height* of the frustum.

To find the area of the curved surface of the frustum, we have:

The measure of the area of the curved surface of the cone O C Q D equals $\pi\, a.\text{OD}$, and the measure of the area of the curved surface of the cone O A P B equals $b\,\pi\,.\text{OB}$, where OD, OB denote the measures of the lengths of the lines OD, OB.

$\therefore$ The measure of the area of the curved surface of the frustum$=\pi\,(a.\text{OD}-b.\text{OB})$.

Now, from the similar triangles OPB, OQD,

$$\frac{\text{OB}}{\text{OD}}=\frac{b}{a}$$

$$\therefore\ \frac{\text{OB}}{\text{BD}}=\frac{b}{a-b}\,;$$

$$\text{whence OB}=\frac{kb}{a-b}\,.$$

$$\text{Similarly OD}=\frac{ka}{a-b}\,.$$

Therefore, the measure of the area of the curved surface of the frustum

$$=\pi\left(\frac{ka^2}{a-b}-\frac{kb^2}{a-b}\right),$$

$$=\pi\ k\ (a+b)=2\,\pi\,k\cdot\frac{a+b}{2}\,.$$

Thus it is seen that:

The measure of the area of the curved surface of the frustum of a cone is equal to that of a cylinder whose altitude is equal to the slant height of the frustum, and the radius of whose base is equal to one-half the sum of the radii of the ends of the frustum.

To find the measure of the volume of the frustum we have.

The measure of the volume of the cone O C Q D

$$= \tfrac{1}{3} \pi a^2 \cdot OQ.$$

The measure of the volume of the cone O A P B

$$= \tfrac{1}{3} \pi b^2 \cdot OP.$$

∴ The measure of the volume of the frustum

$$= \tfrac{1}{3} \pi (a^2 \cdot OQ - b^2 \cdot OP).$$

Now, as before, we have

$$\frac{OP}{OQ} = \frac{b}{a};$$

$$\therefore \frac{OP}{PQ} = \frac{b}{a-b};$$

whence $OP = \dfrac{hb}{a-b}$.

Similarly $OQ = \dfrac{ha}{a-b}$.

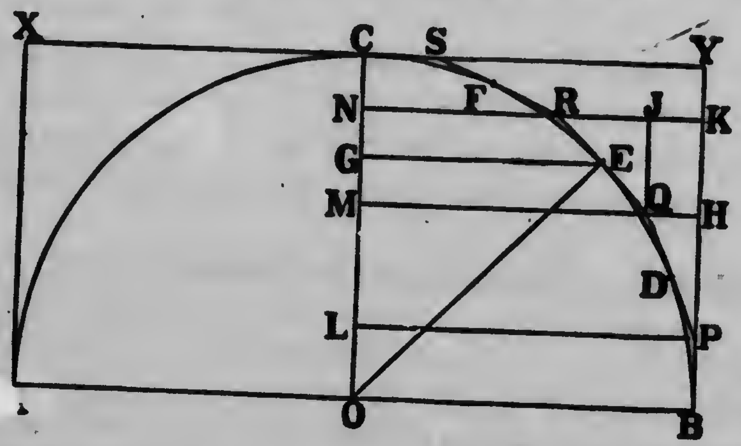
Therefore, the measure of the volume of the frustum

$$= \tfrac{1}{3} \pi \left(\frac{ha^3}{a-b} - \frac{hb^3}{a-b}\right).$$

$$= \tfrac{1}{3} \pi h (a^2 + ab + b^2).$$

3. **The Sphere.** If a semi-circle makes a complete revolution about its base, the solid generated is a **sphere.** The circumference of the semi-circle generates the surface of the sphere.

Let CB be a quadrant of a circle, and let it make a complete revolution about the radius OC. The figure generated is a hemi-sphere whose curved surface is generated by the quadrant. Let BPQRSC be part of a regular polygon supposed circumscribed to the circle of which CB is a quadrant, BP and CS being half-sides.



Now consider the frustum whose curved surface is generated by RQ in the revolution. The area of its mantel is measured by

$$2\pi \cdot \tfrac{1}{2}(NR+MQ)\,RQ,$$

denoting by NR, MQ, RQ the measures of the lengths of these lines. Now $(NR+MQ)=2\,EG$. Therefore the area of the mantel in question is measured by

$$2\pi\,EG.\,RQ.$$

But from the similar triangles EGO, QJR, where EG and QJ are drawn perpendicular to OC and RK, it is seen that $EG.\,RQ = EO.\,QJ = EO.\,HK = MH.\,HK.$

Therefore the area of the mantel generated by R Q is measured by

$$2\pi\, MH.\, HK,$$

and is therefore equal to the area of the cylindrical surface generated by H K. Similar results hold for the mantels generated by P Q and R S. Therefore we have the result:

The area generated by the partial polygon BPQRSC in a revolution about the axis O C, is equal to the cylindrical surface generated by the rotation of B Y about the same axis.

If now the number of sides of the polygon be increased, the surface generated becomes more and more nearly equal to that generated by the revolution of the quadrant, *i. e.*, to the area of the hemisphere and we are led to suppose that,

The area of the hemisphere generated by BC in a revolution, is equal to that of the cylindrical surface generated by B Y.

The relation is shewn to be exact in more advanced treatises on mensuration.

Hence if s denotes the measure of the surface of a sphere, and r that of its radius, we have the formula

$$s=4\pi r^2.$$

From the investigation we are led to suppose that the area of the zone generated by an arc as D E is equal to that of the corresponding part of the cylindrical surface. This relation also is exact so that the area of a zone is measured by the product of $2\pi r$ and the height of the zone.

To find the volume of a sphere, we may suppose the surface divided into a large number of parts, and each part to be taken as the base of a sort of conical or pyramidal solid with its vertex at the centre. If the number of parts be increased, the bases are more and more nearly plane, and the altitude of each of these pyramidal solids more and more nearly equal to the radius of the sphere. Hence we are led to suppose that the volume is measured by one-third of the product of the radius, and the totality of the bases (*i.e.*, the area of the surface of the sphere) or, that, as a formula,

$$v = \tfrac{1}{3} r \times 4\pi r^2 = \tfrac{4}{3}\pi r^3.$$

This formula is exact.

EXERCISES

1. Find the volume of the pyramid whose height is 9 feet and whose base is a triangle of sides 6 feet, 8 feet and 10 feet.

2. Find the volume of the tetrahedron each of whose edges is one inch in length.

3. By means of a drawing shew that a cube may be divided into three pyramids with square bases and equal in all respects.

4. Shew that a cube may be divided into six equal pyramids with vertices at its centre.

Employ this fact to verify the formula for the volume of a pyramid.

5. Find the volume of the tetrahedron each of whose faces is an equilateral triangle of area one square centimetre.

6. If a plane, parallel to the base of a pyramid, cut the pyramid, the part between the plane and the base is called the *frustum of a pyramid.* Shew that the area of the sides of the frustum of a pyramid is the sum of the areas of a number of trapezoids.

7. A regular pyramid whose base is a square of side 1 inch and whose height is 6 inches is cut by a plane which bisects its height. Find the area of the sides and the volume of the frustum so formed.

8. Find the volume and the area of a cone whose height is 12 centimetres and the radius of whose base is 2 centimetres.

9. Find the volume and the area of a cone whose slant height is 10 inches and the diameter of whose base is 6 inches.

10. A military tent is 9 feet high; its shape is that of a cone standing on a cylinder whose diameter is 12 feet and height 3 feet. Find the cost of the canvas, 27 inches wide at 15 cents a yard, allowing 3 yards for seams and waste.

11. What is the volume generated by the revolution about its hypotenuse, of a right-angled triangle whose sides are 21 feet and 28 feet in length.

12. The lengths of the sides of a triangle are as 3:4:5. Show that the volumes of the solids generated by the revolutions of the triangle about the sides are as 4:3: $\frac{12}{5}$.

13. The radius of the base of a cone is 5 inches, its height is 12 inches; find the volume and the area of the frustum cut off [illegible] of the height from the base.

14. The height of a cone is 7 inches and the area of its base is 3 square inches. Find the volume and the area of the frustum contained between the base and the plane which is parallel to the base and which bisects the height of the cone.

15. A right circular cone was measured. The method of measurement was such that it was known only that the diameter of the base is not less than 6·22 m. nor more than 6·24 m., and the slant side is not less than 9·42 m. nor more than 9·44 m. Find the slant area of the cone, taking (1) the lesser dimensions, (2) the greater dimensions.

Express half the difference of the two answers as a percentage of the mean of the two.

If, in calculating the area, the computer gives 10 significant figures in his answer, how many of these are unnecessary?

16. Two buckets, one cylindrical of 7 inches diameter, the other a frustum of a cone with the diameters of its ends 6 inches and 8 inches, are of the same depth, 9 inches. Compare their volumes.

17. Find the volume and the area of a sphere whose radius is 16·25 metres.

18. A spherical shell, internal diameter 14 inches, is filled with water. Its contents are poured into a cylindrical vessel whose internal radius is 14 inches; find the depth of the water in the cylinder.

19. The volume of a sphere is found by multiplying the cube of the radius by 4·1888; and the area of the circle, by multiplying the square of the radius by 3·1416. Find the area of a circle which by rotating about a diameter will describe a sphere whose volume is a cubic foot.

20. A sphere of radius 7 inches is equal in volume to a cone of height 14 inches. Find the radius of the base of the cone, taking $\frac{22}{7}$ as the approximation to the ratio, $c:d$.

21 The surface of a sphere is equal to one-half of that of a right circluar cone; the radius of the base of the cone is 1 foot and its height is $\sqrt{3}$ feet. Find the volume of the sphere.

22. How many cubic inches of wood are there in a hollow wooden ball 10 inches in diameter, the wood being $2\frac{1}{2}$ inches thick?

23. If the diameter of a 9-pound shot is 3 inches find the diameter of an 18-pound shot.

24. The base of a pyramid is a regular hexagon of which each side is 36 feet; find the height of the pyramid if its volume is equal to that of a sphere of which the radius is 16·25 feet.

25. The surface of a sphere is 1386 sq. in. and that of a cube is 1536 sq. in. Find which has the greater volume.

26. A sphere and a cube have the same surface area. Compare the diameter of the sphere with an edge of the cube, working correctly to three decimal places.

27. A rectangle, a semi-circle, and an isoceles triangle have equal bases and equal altitudes; shew that the volumes generated by revolving them about their bases are as 3:2:1.

28. Find the radius of that sphere the number of square centimetres of whose surface equals the number of cubic inches of its volume.

29. From a sphere of radius 1 inch the largest possible cube is cut. What fraction of the volume of the sphere is cut away?

30. A hemisphere, a cylinder and a cone stand on the same base. If their heights are the same compare their volumes and their areas.

TABLES

OF

WEIGHTS, MEASURES AND VALUES

1. CANADIAN WEIGHTS AND MEASURES

a. Linear Measures

The unit of linear measure is the yard. In the Dominion Weights and Measures Act of 1879 it is defined as follows:

"The straight line or distance between the centres of the two gold plugs or pins . . . . in the bronze bar by this Act declared to be the Dominion standard for determining the Dominion standard yard, measured when the bar is at a temperature of sixty-one degrees and ninety-one hundredths of Fahrenheit's thermometer, ... shall be the legal standard measure of length, and shall be called the Dominion standard yard and shall be the only unit or standard measure of extension from which all other measures of extension, whether linear, superficial or solid, shall be ascertained."

The bronze bar in question is deposited in the Department of the Interior at Ottawa.

The multiples and sub-multiples of the yard in actual use or to which reference is made from time to time are the inch; the foot; the yard; the rod, pole, or perch; the furlong; and the mile. The relations among these measures are shewn in the following table:

Inches (in.)	Feet (ft.)	Yards (yd.)	Rods (rd.)	Furlongs (fur.)	Miles (mi.)
12	1				
36	3	1			
198	16.5	5.5	1		
7920	660.	220.	40	1	
63360	5280.	1760.	320	8	1

Other measures are:

The chain = 100 links } = 22 yards.
The fathom = 6 feet.
The hand = 4 inches.
The nautical mile = 6077 feet.
The league = 3 miles.

There may also be employed, in those parts of Quebec originally held by seigniorial tenure, the following:

The French or Paris foot = 12.79 inches.
The arpent (acre) = 180 French feet.
The perch = 18 French feet.

b. Surface or Square Measures

The unit of square measure is the square yard. Its relations to the principal derived measures are given in the table:

Square Inches (sq. in.)	Square feet (sq. ft.)	Square yards (sq. yd.)	Square rods (sq. rd.)	Roods (r.)	Acres (A.)
144	1				
1296	9	1			
	272.25	30.25	1		
	10890.	1210.	40	1	
	43560.	4840.	160	4	1

Note.—1 square mile = 640 acres.

There may also be employed, in those parts of Quebec originally held by seigniorial tenure, the following:

The arpent = 32400 square French feet.
The perch = 324 square French feet.

c. Solid or Cubic Measures

The unit of solid measure is the cubic yard. Its relations to the principal derived units are shewn in the table:

Cubic inches (c.in.)	Cubic feet. (c.ft.)	Cubic yards. (c.yd.)
1728	1	
46656	27	1

Other measures of volume are:

The board-measure foot=1 square foot × 1 inch thickness,
=144 cubic inches.

The cord (of wood or stone)=8 feet length × 4 feet breadth × 4 feet height,
=128 cubic feet.

d. Measures of Weight

The unit of weight or mass is the pound avoirdupois. By the Dominion Weights and Measures Act it is defined to be the Imperial pound, which, by the Weights and Measures Act of 1878 of the Parliament of Great Britain and Ireland, is declared to be the weight or mass of a certain lump of platinum deposited in the Standards Department of the Board of Trade at Westminster.

The following table gives the relations of the pound to the derived units:

Drams (dr.)	Ounces (oz.)	Pounds (lb.)	Hundred-weight (cwt.)	Tons (T.)
16	1			
256	16	1		
	1600	100	1	
		2000	20	1

The grain is defined by the relation,

7000 grains=1 pound;

and the ounce Troy by the relation,

480 grains=1 ounce Troy.

The following measures are given though some of them are now seldom if ever used:

The long ton	=2240 pounds.
The quarter	=25 pounds.
The stone	=14 pounds.
The pennyweight	=24 grains.
The ounce Troy	=20 pennyweights.
The pound Troy	=12 ounces (Troy).

By the Dominion Act already cited,

"All articles sold by weight shall be sold by avoirdupois weight, except that gold, silver, platinum, and precious stones, and articles made thereof, may be sold by the ounce Troy or by any decimal part thereof."

e. Measures of Capacity

The unit of measure of capacity is the gallon. It is thus defined in the Act of 1879:

"The unit or standard measure of capacity from which all other measures of capacity, as well for liquids as for dry goods, shall be derived, shall be the gallon containing ten imperial standard pounds weight of distilled water weighed in air against brass weights, with the water and the air at the temperature of sixty-two degrees of Fahrenheit's thermometer, and with the barometer at thirty inches."

The relations of the gallon to the derived measures are shewn in the table:

Pints (pt.)	Quarts (qt.)	Gallons (gal.)	Pecks (pk.)	Bushels (bu.)	Barrels (bbl.)
2	1				
8	4	1			
16	8	2	1		
64	32	8	4	1	
200	100	25	....	....	1

NOTE.—A cubic foot of distilled water at 62° F., the barometer standing at 30 inches, weighs 62.2786 pounds.

Certain substances sold nominally by the bushel are sold actually by weight. In such cases the Act has declared what weight shall be regarded as a bushel. The table is a follows:

Blue grass seed........14 lb.
Oats..................34 lb.
Malt..................36 lb.
Castor beans..........40 lb.
Hemp seed............44 lb.
Barley, buckwheat and
timothy seed........48 lb.

Onions............ ...50 lb.
Indian corn, rye and
flax seed ....56 lb.
Wheat, peas, beans and
clover seed..........60 lb.
Potatoes, turnips, carrots, parsnips, beets..60 lb.

Other measures are:

The quarter......................... = 8 bushels.
The barrel of flour..................... = 196 pounds.
The barrel of pork or of beef........... = 200 pounds.
The pint.......................... = 20 fluid ounces.
" .... = 160 fluid drachms.
" .... = 9600 fluid minims.
The gill............................ = 0.25 pints.

2. THE METRIC SYSTEM OF MEASURES AND WEIGHTS

The fundamental units in the Metric System of measures and weights are the **metre** and the **kilogramme.**

The metre, originally meant to be the ten-millionth part of the distance from the equator to the pole of the earth, is now actually defined by the standard metre, the length of a platinum bar preserved in the national archives of France. In countries where the metric system has been adopted, the virtual material standards are copies of the standard metre. In a country, as Canada, where the metric system is permissive, the effective definition of the metre is the legal equivalent of the metre in terms of the unit in general use.

The kilogramme is the mass or weight of a piece of platinum deposited in the archives of France, copies of which have been made and distributed among the governments which have legalized the metric system. The standard of weight was connected with the standard of length by being made as nearly as possible of the same weight as that of the amount of distilled water at the temperature of 4° C., contained in a cube each edge of which is one-tenth of a metre in length. In the terminology of the metric system the gramme, which is the thousandth part of the kilogramme, is treated as the unit of weight.

The metric system is a decimal system, the derived units being formed by division or multiplication by powers of ten. The multiple units are named by employing the Greek numeral-prefixes, *deka-*, *hecto-*, *kilo-*, *myria-*, to denote *ten*, *one-hundred*, *one-thousand*, *ten-thousand times* (the associated unit). The sub-multiple units are named by employing the Latin numeral-prefixes, *deci-*, *centi-*, *milli-*, to denote *one-tenth*, *one-hundredth*, *one-thousandth of* (the associated unit). Sometimes also the prefixes, *mega-*, *micro-*, are employed to denote the millionth multiple and the millionth part.

Manifestly then, if a measure is written in terms of one unit, it is possible to pass to any derived unit simply by moving the decimal point; thus 537.29 metres = 53.729 dekametres = 53729 centimetres.

The following tables give the equivalents of the metric measures and weights in terms of the measures and weights in general use in Canada:

a. MEASURES OF LENGTH

Metric Denominations	Values in Metres	Equivalents in Denominations in Use
Myriametre (Mm.)	10,000	6.21382 miles
Kilometre (Km.)	1000	0.62138 miles
Hectometre (Hm.)	100	109.36331 yards
Dekametre (Dm.)	10	10.93633 yards
Metre (m.)	1	1.09363306 yds. 3.28089917 feet 39.37079 inches
Decimetre (dm.)	0.1	3.93708 inches
Centimetre (cm.)	0.01	0.39371 inches
Millimetre (mm.)	0.001	0.03937 inches

b. MEASURES OF SURFACE

Metric Denominations	Values in Square Metres	Equivalents in Denominations in Use
Hectare (Ha.)	10,000	2.47115 acres
Are (a.)	100	119.60333 square yards
Centiare (ca.)	1	1.19603 square yards 10.76430 square feet
Square Decimetre (sq. dm.)	0.01	15.50059 square inches
Square Centimetre (sq.cm.)	0.0001	0.15501 square inches
Square Millimetre (sq.mm.)	0.000001	0.00155 square inches

c. MEASURES OF CAPACITY OR VOLUME

Metric Denominations and Values			Equivalents in Denominations in Use
Names	No. of Litres	Cubic Measure	
Kilolitre (Kl.) or stere (st.)...	1000	1 cubic metre.	3.43901 quarters 35.31658 cubic ft.
Hectolitre (Hl.)...	100	0.1 cubic metre.....	2.75121 bushels 3.53166 cubic ft.
Dekalitre (Dl.) ...	10	10 cubic decimetres.	2.20097 gallons 0.35317 cubic ft.
Litre (l.).........	1	1 cubic decimetre .	1.76077 pints 61.02705 cubic in.
Decilitre (dl.)	0.1	0.1 cubic decimetres.	0.17608 pints 6.10270 cubic in.
Centilitre (cl.)....	0.01	10 cubic centimetres	0.61027 cubic in.
Millilitre (ml.)....	0.001	1 cubic centimetre.	0.06103 cubic in.

d. MEASURES OF WEIGHT

Metric Denominations	Values in Grammes	Equivalents in Denominations in Use
Millier or tonneau (Megagramme)	1,000,000	
Quintal........................	100,000	
Myriagramme (Mg.)..	10,000	22.04621 lb.
Kilogramme or Kilog (Kg.)......	1000	2.20462125 lb. 15432.34874 grains
Hectogramme (Hg.)............	100	3.52739 oz.
Dekagramme (Dg.)	10	5.64383 drams
Gramme (g.)	1	15.43235 grains
Decigramme (dg.)...............	0.1	1.54324 grains
Centigramme (cg.)..............	0.01	0.15432 grains
Milligramme (mg.)	0.001	0.01543 grains

3. VALUES

a. Canadian Money

The unit of Canadian money is the dollar. The relations of the dollar to its derived units are shewn in the table:

1000 mills = 100 cents (ct.) = 1 dollar ($).

The term mill is rarely employed for any purpose other than the quotation of the rate of taxation.

The dollar is defined in terms of the pound, the unit of British money. The relation between these units is:

1 pound = $4{\cdot}86\frac{2}{3}$ dollars.

b. United States Money

The unit of United States money is the dollar. The derived units are as those of the Canadian unit, except that in the United States there are, in addition, the dime which equals ten cents, and the eagle which equals ten dollars.

The dollar as a gold coin is required to be made of gold of nine-tenths fineness, and to weigh 25·8 grains. This condition secures that the United States dollar and the Canadian dollar are of practically the same value.

c. British Money

The unit of British money is the pound sterling. Its relation to the derived units are given in the table:

960 farthings = 240 pence (d.) = 20 shillings (s.) or (/-)
= 1 pound (£).

The sovereign, as a gold coin, is required to be of gold of $\frac{11}{12}$ fineness and to be such that 1869 sovereigns weigh 480 ounces Troy.

4. MEASURES OF TIME

The unit of time is the mean solar day, which is defined approximately by the relation:

365·242216 mean solar days=1 year.

The relations of the (mean solar) day to its derived units are shewn in the table:

Seconds (sec.)	Minutes (min.)	Hours (hr.)	Days (da.)	Weeks (wk.)
60	1			
3600	60	1		
86,400	1440	24	1	
	10,080	168	7	1

In order to avoid fractional parts of a day in the statement for a year, *calendar* years of two kinds have been adopted, the common year consisting of 365 days, and the leap year of 366 days. Every year whose date-number is a multiple of 4 is a leap year, except those whose date-numbers are multiples of 400.

The calendar year is divided into 12 months of unequal length. The months, January, March, May, July, August, October, December, consist each of 31 days; the months, April, June, September, November, each of 30 days; February, of 28 days, in the common year, and of 29 days, in the leap year.

5. ANGULAR MEASURE

The unit of angular measure is the complete revolution or, as it is sometimes called, the *circle*. The relations of this unit to the derived units are shown in the table:

Seconds (″)	Minutes (′)	Degrees (°)	Right Angles	Circles
60	1			
3600	60	1		
	5400	90	1	
		360	4	1

INTEREST TABLES

½ Per Cent.			¾ Per Cent.			
Years.	Amount of 1 dollar at the end of a certain number of years.	Present value of 1 dollar payable at the end of a certain number of years.	Years.	Amount of 1 dollar at the end of a certain number of years.	Present value of 1 dollar payable at the end of a certain number of years.	Years.
1	1.005000	0.995025	1	1.007500	0.992556	1
2	1.010025	0.990074	2	1.015056	0.985167	2
3	1.015075	0.985149	3	1.022669	0.977833	3
4	1.020150	0.980248	4	1.030339	0.970554	4
5	1.025251	0.975371	5	1.038067	0.963330	5
6	1.030378	0.970518	6	1.045852	0.956158	6
7	1.035529	0.965690	7	1.053696	0.949040	7
8	1.040707	0.960885	8	1.061599	0.941975	8
9	1.045910	0.956105	9	1.069561	0.934963	9
10	1.051140	0.951348	10	1.077583	0.928003	10
11	1.056396	9.946615	11	1.085664	0.921095	11
12	1.061678	0.941905	12	1.093807	0.914238	12
13	1.066986	0.937219	13	1.102010	0.907432	13
14	1.072321	0.932567	14	1.110276	0.900677	14
15	1.077683	0.927917	15	1.118603	0.893972	15
16	1.083072	0.923301	16	1.126992	0.887318	16
17	1.088486	0.918707	17	1.135445	0.880712	17
18	1.093929	0.914136	18	1.143960	0.874156	18
19	1.099398	0.909588	19	1.152540	0.867649	19
20	1.104895	0.905063	20	1.161184	0.861190	20
21	1.110420	0.900560	21	1.169893	0.854779	21
22	1.115972	0.896080	22	1.178667	0.848416	22
23	1.121552	0.891622	23	1.187507	0.842100	23
24	1.127160	0.887186	24	1.196414	0.835831	24
25	1.132795	0.882772	25	1.205387	0.829609	25
26	1.138459	0.878380	26	1.214427	0.823434	26
27	1.144152	0.874010	27	1.223535	0.817304	27
28	1.149872	0.869662	28	1.232712	0.811220	28
29	1.155622	0.865335	29	1.241957	0.805181	29
30	1.161400	0.861030	30	1.251272	0.799187	30
31	1.167207	0.856746	31	1.260656	0.793238	31
32	1.173043	0.852483	32	1.270111	0.787333	32
33	1.178908	0.848243	33	1.279637	0.781472	33
34	1.184803	0.844023	34	1.289234	0.775654	34
35	1.190726	0.839823	35	1.298904	0.769880	35
36	1.196680	0.835645	36	1.308645	0.764149	36
37	1.202664	0.831488	37	1.318460	0.758460	37
38	1.208677	0.827351	38	1.328349	0.752814	38
39	1.214720	0.823235	39	1.338311	0.747210	39
40	1.220794	0.819139	40	1.348349	0.741648	40

	1 Per Cent.			1½ Per Cent.		
Years.	Amount of 1 dollar at the end of a certain number of years.	Present value of 1 dollar payable at the end of a certain number of years.	Years.	Amount of 1 dollar at the end of a certain number of years.	Present value of 1 dollar payable at the end of a certain number of years.	Years.
1	1.010000	0.990099	1	1.012500	0.987654	1
2	1.020100	0.980296	2	1.025156	0.975461	2
3	1.030301	0.970590	3	1.037971	0.963418	3
4	1.040604	0.960980	4	1.050945	0.951524	4
5	1.051010	0.951466	5	1.064082	0.939777	5
6	1.061520	0.942045	6	1.077383	0.928175	6
7	1.072135	0.932718	7	1.090850	0.916716	7
8	1.082857	0.923483	8	1.104486	0.905398	8
9	1.093685	0.914340	9	1.118292	0.894221	9
10	1.104622	0.905287	10	1.132271	0.883181	10
11	1.115668	0.896324	11	1.146424	0.872277	11
12	1.126825	0.887449	12	1.160755	0.861509	12
13	1.138093	0.878663	13	1.175264	0.850873	13
14	1.149474	0.869963	14	1.189955	0.840368	14
15	1.160969	0.861349	15	1.204829	0.829993	15
16	1.172579	0.852821	16	1.219890	0.819746	16
17	1.184304	0.844377	17	1.235138	0.809626	17
18	1.196147	0.836017	18	1.250577	0.799631	18
19	1.208109	0.827740	19	1.266210	0.789759	19
20	1.220190	0.819544	20	1.282037	0.780008	20
21	1.232392	0.811430	21	1.298063	0.770379	21
22	1.244716	0.803396	22	1.314288	0.760868	22
23	1.257163	0.795442	23	1.330717	0.751474	23
24	1.269735	0.787566	24	1.347351	0.742197	24
25	1.282432	0.779768	25	1.364193	0.733034	25
26	1.295256	0.772048	26	1.381245	0.723984	26
27	1.308209	0.764404	27	1.398511	0.715046	27
28	1.321291	0.756836	28	1.415992	0.706218	28
29	1.334504	0.749342	29	1.433692	0.697500	29
30	1.347849	0.741923	30	1.451613	0.688889	30
31	1.361327	0.734577	31	1.469759	0.680038	31
32	1.374941	0.727304	32	1.488131	0.671984	32
33	1.388690	0.720103	33	1.506732	0.663688	33
34	1.402577	0.712973	34	1.525566	0.655494	34
35	1.416603	0.705914	35	1.544636	0.647402	35
36	1.430769	0.698925	36	1.563944	0.639409	36
37	1.445076	0.692005	37	1.583493	0.631515	37
38	1.459527	0.685153	38	1.603287	0.623719	38
39	1.474123	0.678370	39	1.623328	0.616018	39
40	1.488864	0.671653	40	1.643619	0.608413	40

1½ Per Cent.			1¾ Per Cent.			
Years.	Amount of 1 dollar at the end of a certain number of years.	Present value of 1 dollar payable at the end of a certain number of years.	Years.	Amount of 1 dollar at the end of a certain number of years.	Present value of 1 dollar payable at the end of a certain number of years.	Years.
1	1.015000	0.985222	1	1.017500	0.982801	1
2	1.030225	0.970662	2	1.035306	0.965898	2
3	1.045678	0.956317	3	1.053424	0.949285	3
4	1.061364	0.942184	4	1.071850	0.932958	4
5	1.077284	0.928260	5	1.090617	0.916913	5
6	1.093443	0.914542	6	1.109702	0.901142	6
7	1.109845	0.901027	7	1.129122	0.885646	7
8	1.126493	0.887711	8	1.148882	0.870413	8
9	1.143390	0.874592	9	1.168987	0.855441	9
10	1.160541	0.861667	10	1.189444	0.840728	10
11	1.177949	0.848933	11	1.210260	0.826269	11
12	1.195618	0.836387	12	1.231439	0.812058	12
13	1.213552	0.824027	13	1.252990	0.798091	13
14	1.231756	0.811849	14	1.274917	0.784365	14
15	1.250232	0.799851	15	1.297228	0.770875	15
16	1.268986	0.788031	16	1.319929	0.757616	16
17	1.288020	0.776385	17	1.343028	0.744586	17
18	1.307341	0.764912	18	1.366531	0.731780	18
19	1.326951	0.753607	19	1.390445	0.719194	19
20	1.346855	0.742470	20	1.414778	0.706825	20
21	1.367058	0.731498	21	1.439537	0.694668	21
22	1.387564	0.720688	22	1.464729	0.682720	22
23	1.408377	0.710037	23	1.490361	0.670978	23
24	1.429503	0.699544	24	1.516443	0.659438	24
25	1.450945	0.689206	25	1.542981	0.648096	25
26	1.472710	0.679020	26	1.569983	0.636950	26
27	1.494800	0.668986	27	1.597457	0.625995	27
28	1.517222	0.659099	28	1.625413	0.615228	28
29	1.539981	0.649359	29	1.653858	0.604647	29
30	1.563080	0.639762	30	1.682800	0.594248	30
31	1.586526	0.630308	31	1.712249	0.584027	31
32	1.610324	0.620993	32	1.742213	0.573982	32
33	1.634479	0.611816	33	1.772702	0.564110	33
34	1.658996	0.602774	34	1.803725	0.554408	34
35	1.683881	0.593866	35	1.835290	0.544873	35
36	1.709140	0.585090	36	1.867407	0.535502	36
37	1.734777	0.576443	37	1.900087	0.526292	37
38	1.760798	0.567924	38	1.933338	0.517240	38
39	1.787210	0.559531	39	1.967172	0.508344	39
40	1.814018	0.551262	40	2.001597	0.499601	40

Years.	2 Per Cent. Amount of 1 dollar at the end of a certain number of years.	2 Per Cent. Present value of 1 dollar payable at the end of a certain number of years.	Years.	2½ Per Cent. Amount of 1 dollar at the end of a certain number of years.	2½ Per Cent. Present value of 1 dollar payable at the end of a certain number of years.	Years.
1	1.020000	0.980392	1	1.025000	0.975610	1
2	1.040400	0.961169	2	1.050625	0.951814	2
3	1.061208	0.942322	3	1.076891	0.928599	3
4	1.082432	0.923845	4	1.103813	0.905951	4
5	1.104081	0.905731	5	1.131408	0.883854	5
6	1.126162	0.887971	6	1.159693	0.862297	6
7	1.148686	0.870560	7	1.188686	0.841265	7
8	1.171659	0.853490	8	1.218403	0.820747	8
9	1.195093	0.836755	9	1.248863	0.800728	9
10	1.218994	0.820348	10	1.280085	0.781198	10
11	1.243374	0.804263	11	1.312087	0.762145	11
12	1.268242	0.788493	12	1.344889	0.743556	12
13	1.293607	0.773032	13	1.378511	0.725420	13
14	1.319479	0.757875	14	1.412974	0.70[illegible]727	14
15	1.345868	0.743015	15	1.448298	0.690466	15
16	1.372786	0.728446	16	1.484506	0.673625	16
17	1.400241	0.714163	17	1.521618	0.657195	17
18	1.428246	0.700159	18	1.559659	0.641166	18
19	1.456811	0.686431	19	1.598650	0.625528	19
20	1.485947	0.672971	20	1.638616	0.610271	20
21	1.515666	0.659776	21	1.679582	0.595386	21
22	1.545980	0.646839	22	1.721571	0.580865	22
23	1.576899	0.634156	23	1.764611	0.566697	23
24	1.608437	0.621721	24	1.808726	0.552875	24
25	1.640606	0.609531	25	1.853944	0.539391	25
26	1.673418	0.597579	26	1.900293	0.526235	26
27	1.706886	0.585862	27	1.947800	0.513400	27
28	1.741024	0.574375	28	1.996495	0.500878	28
29	1.775845	0.563112	29	2.046407	0.488661	29
30	1.811362	0.552071	30	2.097568	0.476743	30
31	1.847589	0.541246	31	2.150007	0.465115	31
32	1.884541	0.530633	32	2.203757	0.453771	32
33	1.922231	0.520229	33	2.258851	0.442703	33
34	1.960676	0.510028	34	2.315322	0.431905	34
35	1.999890	0.500028	35	2.373205	0.421371	35
36	2.039887	0.490223	36	2.432535	0.411094	36
37	2.080685	0.480611	37	2.493349	0.401067	37
38	2.122299	0.471187	38	2.555682	0.391285	38
39	2.164745	0.461948	39	2.619574	0.381741	39
40	2.208040	0.452890	40	2.685064	0.372431	40

3 Per Cent.			3½ Per Cent.			
Years.	Amount of 1 dollar at the end of a certain number of years.	Present value of 1 dollar payable at the end of a certain number of years.	Years.	Amount of 1 dollar at the end of a certain number of years.	Present value of 1 dollar payable at the end of a certain number of years.	Years.
1	1.030000	0.970874	1	1.035000	0.966184	1
2	1.060900	0.942596	2	1.071225	0.933511	2
3	1.092727	0.915142	3	1.108718	0.901943	3
4	1.125509	0.888487	4	1.147523	0.871442	4
5	1.159274	0.862609	5	1.187686	0.841973	5
6	1.194052	0.837484	6	1.229255	0.813501	6
7	1.229874	0.813091	7	1.272279	0.785991	7
8	1.266770	0.789409	8	1.316809	0.759412	8
9	1.304773	0.766417	9	1.362897	0.733731	9
10	1.343916	0.744094	10	1.410599	0.708919	10
11	1.384234	0.722421	11	1.459970	0.684946	11
12	1.425761	0.701380	12	1.511069	0.661783	12
13	1.468534	0.680951	13	1.563956	0.639404	13
14	1.512590	0.661118	14	1.618695	0.617782	14
15	1.557967	0.641862	15	1.675349	0.596891	15
16	1.604706	0.623167	16	1.733986	0.576706	16
17	1.652848	0.605016	17	1.794676	0.557204	17
18	1.702433	0.587395	18	1.857489	0.538361	18
19	1.753506	0.570286	19	1.922501	0.520156	19
20	1.806111	0.553676	20	1.989789	0.502566	20
21	1.860295	0.537549	21	2.059431	0.485571	21
22	1.916103	0.521892	22	2.131512	0.469151	22
23	1.973589	0.506692	23	2.206114	0.453286	23
24	2.032794	0.491934	24	2.283328	0.437957	24
25	2.093778	0.477606	25	2.363245	0.423147	25
26	2.156591	0.463695	26	2.445959	0.408838	26
27	2.221289	0.450189	27	2.531567	0.395012	27
28	2.287928	0.437077	28	2.620172	0.381654	28
29	2.356566	0.424346	29	2.711878	0.368748	29
30	2.427262	0.411987	30	2.806794	0.356278	30
31	2.500080	0.399987	31	2.905031	0.344230	31
32	2.575083	0.388337	32	3.006708	0.332590	32
33	2.652335	0.377026	33	3.111942	0.321343	33
34	2.731905	0.366045	34	3.220860	0.310476	34
35	2.813862	0.355383	35	3.333590	0.299977	35
36	2.898278	0.345032	36	3.450266	0.289833	36
37	2.985227	0.334983	37	3.571025	0.280032	37
38	3.074783	0.325226	38	3.696011	0.270562	38
39	3.167027	0.315753	39	3.825372	0.261412	39
40	3.262038	0.306557	40	3.959260	0.252572	40

4 Per Cent.			4½ Per Cent.			
Years.	Amount of 1 dollar at the end of a certain number of years.	Present value of 1 dollar payable at the end of a certain number of years.	Years.	Amount of 1 dollar at the end of a certain number of years.	Present value of 1 dollar payable at the end of a certain number of years.	Years.
1	1.040000	0.961538	1	1.045000	0.956938	1
2	1.081600	0.924556	2	1.092025	0.915730	2
3	1.124864	0.888996	3	1.141166	0.876297	3
4	1.169859	0.854804	4	1.192519	0.838561	4
5	1.216653	0.821927	5	1.246182	0.802451	5
6	1.265319	0.790314	6	1.302260	0.767896	6
7	1.315932	0.759918	7	1.360862	0.734828	7
8	1.368569	0.730690	8	1.422101	0.703185	8
9	1.423312	0.702587	9	1.486095	0.672904	9
10	1.480244	0.675564	10	1.552969	0.643928	10
11	1.539454	0.649581	11	1.622853	0.616199	11
12	1.601032	0.624597	12	1.695881	0.589664	12
13	1.665074	0.600574	13	1.772196	0.564272	13
14	1.731676	0.577475	14	1.851945	0.539973	14
15	1.800944	0.555264	15	1.935282	0.516720	15
16	1.872981	0.533908	16	2.022370	0.494469	16
17	1.947900	0.513373	17	2.113377	0.473176	17
18	2.025817	0.493628	18	2.208479	0.452800	18
19	2.106849	0.474642	19	2.307860	0.433302	19
20	2.191123	0.456387	20	2.411714	0.414643	20
21	2.278768	0.438834	21	2.520241	0.396787	21
22	2.369919	0.421955	22	2.633652	0.379701	22
23	2.464716	0.405726	23	2.752166	0.363350	23
24	2.563304	0.390121	24	2.876014	0.347703	24
25	2.665836	0.375117	25	3.005434	0.332731	25
26	2.772470	0.360689	26	3.140679	0.318402	26
27	2.883369	0.346817	27	3.282010	0.3[illegible]01	27
28	2.998703	0.333477	28	3.429700	0.291[illegible]1	28
29	3.118651	0.320651	29	3.584036	0.279015	29
30	3.243398	0.308319	30	3.745318	0.267000	30
31	3.373133	0.296460	31	3.913857	0.255502	31
32	3.508059	0.285058	32	4.089981	0.244500	32
33	3.648381	0.274094	33	4.274030	0.233971	33
34	3.794316	0.263552	34	4.466362	0.223896	34
35	3.946089	0.253415	35	4.667348	0.214254	35
36	4.103933	0.243669	36	4.877378	0.205028	36
37	4.268090	0.234297	37	5.096860	0.196199	37
38	4.438813	0.225285	38	5.326219	0.187750	38
39	4.616366	0.216621	39	5.565899	0.179665	39
40	4.801021	0.208289	40	5.816365	0.171929	40

5 Per Cent.			6 Per Cent.			
Years.	Amount of 1 dollar at the end of a certain number of years.	Present value of 1 dollar payable at the end of a certain number of years.	Years.	Amount of 1 dollar at the end of a certain number of years.	Present value of 1 dollar payable at the end of a certain number of years.	Years.
1	1.050000	0.952381	1	1.060000	0.943396	1
2	1.102500	0.907029	2	1.123600	0.889996	2
3	1.157625	0.863838	3	1.191016	0.839619	3
4	1.215506	0.822702	4	1.262477	0.792094	4
5	1.276282	0.783526	5	1.338226	0.747258	5
6	1.340096	0.746215	6	1.418519	0.704960	6
7	1.407100	0.710681	7	1.503630	0.665057	7
8	1.477455	0.676839	8	1.593848	0.627412	8
9	1.551328	0.644609	9	1.689479	0.591898	9
10	1.628895	0.613913	10	1.790848	0.558395	10
11	1.710339	0.584679	11	1.898299	0.526787	11
12	1.795856	0.556837	12	2.012196	0.496969	12
13	1.885649	0.530321	13	2.132928	0.468839	13
14	1.979932	0.505068	14	2.260904	0.442301	14
15	2.078928	0.481017	15	2.396558	0.417265	15
16	2.182875	0.458111	16	2.540352	0.393646	16
17	2.292018	0.436297	17	2.692773	0.371364	17
18	2.406619	0.415521	18	2.854339	0.350344	18
19	2.526950	0.395734	19	3.025599	0.330513	19
20	2.653298	0.376889	20	3.207135	0.311805	20
21	2.785963	0.358942	21	3.399564	0.294155	21
22	2.925261	0.341849	22	3.603537	0.277505	22
23	3.071524	0.325571	23	3.819750	0.261797	23
24	3.225100	0.310068	24	4.048935	0.246978	24
25	3.386355	0.295303	25	4.291871	0.232999	25
26	3.555673	0.281241	26	4.549383	0.219810	26
27	3.733456	0.267848	27	4.822346	0.207368	27
28	3.920129	0.255094	28	5.111687	0.195630	28
29	4.116136	0.242946	29	5.418388	0.184557	29
30	4.321942	0.231377	30	5.743491	0.174110	30
31	4.538039	0.220359	31	6.088101	0.164255	31
32	4.764941	0.209866	32	6.453387	0.154957	32
33	5.003189	0.199872	33	6.840590	0.146186	33
34	5.253348	0.190355	34	7.251025	0.137911	34
35	5.516015	0.181290	35	7.686087	0.130105	35
36	5.791816	0.172657	36	8.147252	0.122741	36
37	6.081407	0.164436	37	8.636087	0.115793	37
38	6.385477	0.156605	38	9.154252	0.109239	38
39	6.704751	0.149148	39	9.703507	0.103055	39
40	7.039989	0.142046	40	10.285718	0.097222	40

	7 Per Cent.			8 Per Cent.		
Years.	Amount of 1 dollar at the end of a certain number of years.	Present value of 1 dollar payable at the end of a certain number of years.	Years.	Amount of 1 dollar at the end of a certain number of years.	Present value of 1 dollar payable at the end of a certain number of years.	Years.
1	1.070000	0.934579	1	1.080000	0.925926	1
2	1.144900	0.873439	2	1.166400	0.857339	2
3	1.225043	0.816298	3	1.259712	0.793832	3
4	1.310796	0.762895	4	1.360489	0.735030	4
5	1.402552	0.712986	5	1.469328	0.680583	5
6	1.500730	0.666342	6	1.586874	0.630170	6
7	1.605781	0.622750	7	1.713824	0.583490	7
8	1.718186	0.582009	8	1.850930	0.540269	8
9	1.838459	0.543934	9	1.999005	0.500249	9
10	1.967151	0.508349	10	2.158925	0.463193	10
11	2.104852	0.475093	11	2.331639	0.428883	11
12	2.252192	0.444012	12	2.518170	0.397114	12
13	2.409845	0.414964	13	2.719624	0.367698	13
14	2.578534	0.387817	14	2.937194	0.340461	14
15	2.759032	0.362446	15	3.172169	0.315242	15
16	2.952164	0.338735	16	3.425943	0.291890	16
17	3.158815	0.316574	17	3.700018	0.270269	17
18	3.379932	0.295864	18	3.996019	0.250249	18
19	3.616528	0.276508	19	4.315701	0.231712	19
20	3.869684	0.258419	20	4.660957	0.214548	20
21	4.140562	0.241513	21	5.033834	0.198656	21
22	4.430402	0.225713	22	5.436540	0.183940	22
23	4.740530	0.210947	23	5.871464	0.170315	23
24	5.072367	0.197147	24	6.341181	0.157699	24
25	5.427433	0.184249	25	6.848475	0.146018	25
26	5.807353	0.172195	26	7.396353	0.135202	26
27	6.213868	0.160930	27	7.988061	0.125187	27
28	6.648838	0.150402	28	8.627106	0.115914	28
29	7.114257	0.140563	29	9.317275	0.107327	29
30	7.612255	0.131367	30	10.062657	0.099377	30
31	8.145113	0.122773	31	10.867669	0.092016	31
32	8.715271	0.114741	32	11.737083	0.085200	32
33	9.325340	0.107235	33	12.676050	0.078889	33
34	9.978114	0.100219	34	13.690134	0.073045	34
35	10.676581	0.093663	35	14.785344	0.067634	35
36	11.423942	0.087535	36	15.968172	0.062625	36
37	12.223618	0.081809	37	17.245626	0.057986	37
38	13.079271	0.076457	38	18.625276	0.053690	38
39	13.994820	0.071455	39	20.115298	0.049713	39
40	14.974458	0.066780	40	21.724521	0.046031	40

9 Per Cent.			10 Per Cent.			
Years.	Amount of 1 dollar at the end of a certain number of years.	Present value of 1 dollar payable at the end of a certain number of years.	Years.	Amount of 1 dollar at the end of a certain number of years.	Present value of 1 dollar payable at the end of a certain number of years.	Years.
1	1.090000	0.917431	1	1.100000	0.909091	1
2	1.188100	0.841680	2	1.210000	0.826446	2
3	1.295029	0.772183	3	1.331000	0.751315	3
4	1.411582	0.708425	4	1.464100	0.683013	4
5	1.538624	0.649931	5	1.610510	0.620921	5
6	1.677100	0.596267	6	1.771561	0.564474	6
7	1.828039	0.547034	7	1.948717	0.513158	7
8	1.992563	0.501866	8	2.143589	0.466507	8
9	2.171893	0.460428	9	2.357948	0.424098	9
10	2.367364	0.422411	10	2.593742	0.385543	10
11	2.580426	0.387533	11	2.853117	0.350494	11
12	2.812665	0.355535	12	3.138428	0.318631	12
13	3.065805	0.326179	13	3.452271	0.289664	13
14	3.341727	0.299246	14	3.797498	0.263331	14
15	3.642482	0.274538	15	4.177248	0.239392	15
16	3.970306	0.251870	16	4.594973	0.217629	16
17	4.327633	0.231073	17	5.054470	0.197845	17
18	4.717120	0.211994	18	5.559917	0.179859	18
19	5.141661	0.194490	19	6.115909	0.163508	19
20	5.604411	0.178431	20	6.727500	0.148644	20
21	6.108808	0.163698	21	7.400250	0.135131	21
22	6.658600	0.150182	22	8.140275	0.122846	22
23	7.257874	0.13[illegible]1	23	8.954302	0.111678	23
24	7.911083	0.126405	24	9.849733	0.101526	24
25	8.623081	0.115968	25	10.834706	0.092296	25
26	9.399158	0.106393	26	11.918177	0.083905	26
27	10.245082	0.097608	27	13.109994	0.076278	27
28	11.167140	0.089548	28	14.420994	0.069343	28
29	12.172182	0.082155	29	15.863093	0.063039	29
30	13.267678	0.075371	30	17.449402	0.057309	30
31	14.461770	0.069148	31	19.194342	0.052099	31
32	15.763329	0.063438	32	21.113777	0.047362	32
33	17.182028	0.058200	33	23.225154	0 043057	33
34	18.728411	0.053395	34	25.547670	0.039142	34
35	20.413968	0.048986	35	28.102437	0.035584	35
36	22.251225	0.044941	36	30.912681	0.032349	36
37	24.253835	0.041231	37	34.003949	0.029408	37
38	26.436680	0.037826	38	37.404343	0.026735	38
39	28.815982	0.034703	39	41.144778	0.024304	39
40	31.409420	0.031838	40	45.259256	0.022095	40

TABLES

OF

LOGARITHMS OF NUMBERS

FROM 1 TO 10,000

TABLES

OF

LOGARITHMS OF NUMBERS

FROM 1 TO 10,000

LOGARITHMS 1—100

No.	Log	No.	Log	No.	Log	No.	Log
1	0.000000	26	1.414973	51	1.707570	76	1.880814
2	0.301030	27	1.431364	52	1.716003	77	1.886491
3	0.477121	28	1.447158	53	1.724276	78	1.892095
4	0.602060	29	1.462398	54	1.732394	79	1.897627
5	0.698970	30	1.477121	55	1.740363	80	1.903090
6	0.778151	31	1.491362	56	1.748188	81	1.908485
7	0.845098	32	1.505150	57	1.755875	82	1.913814
8	0.903090	33	1.518514	58	1.763428	83	1.919078
9	0.954243	34	1.531479	59	1.770852	84	1.924279
10	1.000000	35	1.544068	60	1.778151	85	1.929419
11	1.041393	36	1.556303	61	1.785330	86	1.934498
12	1.079181	37	1.568202	62	1.792392	87	1.939519
13	1.113943	38	1.579784	63	1.799341	88	1.944483
14	1.146128	39	1.591065	64	1.806180	89	1.949390
15	1.176091	40	1.602060	65	1.812913	90	1.954243
16	1.204120	41	1.612784	66	1.819544	91	1.959041
17	1.230449	42	1.623249	67	1.826075	92	1.963788
18	1.255273	43	1.633468	68	1.832509	93	1.968483
19	1.278754	44	1.643453	69	1.838849	94	1.973128
20	1.301030	45	1.653213	70	1.845098	95	1.977724
21	1.322219	46	1.662758	71	1.851258	96	1.982271
22	1.342423	47	1.672098	72	1.857332	97	1.986772
23	1.361728	48	1.681241	73	1.863323	98	1.991226
24	1.380211	49	1.690196	74	1.869232	99	1.995635
25	1.397940	50	1.698970	75	1.875061	100	2.000000

No	0	1	2	3	4	5	6	7	8	9	D
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	*0300	*0724	*1147	*1570	*1993	*2415	424
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616	419
104	7033	7451	7868	8284	8700	9116	9532	9947	*0361	*0775	416
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	*0195	*0600	*1004	*1408	*1812	*2216	*2619	*3021	404
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	*0207	*0602	*0998	396
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	389
112	9218	9606	9993	*0380	*0766	*1153	*1538	*1924	*2309	*2694	386
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524	382
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	*0320	379
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	372
117	8186	8557	8928	9298	9668	*0038	*0407	*0776	*1145	*1514	369
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	9181	9543	9904	*0266	*0626	*0987	*1347	*1707	*2067	*2426	360
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	9905	*0258	*0611	*0963	*1315	*1667	*2018	*2370	*2721	*3071	352
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	*0026	346
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	341
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	*0253	338
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	3943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	*0245	330
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	*0012	323
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	315
138	9879	*0194	*0508	*0822	*1136	*1450	*1763	*2076	*2389	*2702	314
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
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141	9219	9527	9835	*0142	*0449	*0756	*1063	*1370	*1676	*1982	307
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
144	8362	8664	8965	9266	9567	9868	*0168	*0469	*0769	*1068	301
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055	299
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895	293
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689	289
151	8977	9264	9552	9839	*0126	*0413	*0699	*0985	*1272	*1558	287
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	*0051	281
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
157	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
158	8657	8932	9206	9481	9755	*0029	*0303	*0577	*0850	*1124	274
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
160	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556	271
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
162	9515	9783	*0051	*0319	*0586	*0853	*1121	*1388	*1654	*1921	267
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	*0193	256
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742	254
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
173	8046	8297	8548	8799	9049	9299	9550	9800	*0050	*0300	250
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790	249
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	*0176	245
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
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181	7679	7918	8158	8398	8637	8877	9116	9355	[illegible]	[illegible]	239
182	260071	0310	0548	0787	1025	1263	1501	1739	[illegible]	[illegible]	238
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
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185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
186	9513	9746	9980	*0213	*0446	*0679	*0912	*1144	*1377	*1609	233
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	8754	8982	9211	9439	9667	9895	*0123	*0351	*0578	*0806	228
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034	222
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
199	8853	9071	9289	9507	9725	9943	*0161	*0378	*0595	*0813	218
200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980	217
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
204	9630	9843	*0056	*0268	*0481	*0693	*0906	*1118	*1330	*1542	212
205	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656	211
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012	207
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
213	8380	8583	8787	8991	9194	9398	9601	9805	*0008	*0211	203
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236	202
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
218	8456	8656	8855	9054	9253	9451	9650	9849	*0047	*0246	199
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222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	*0054	194
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989	193
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
229	9835	*0025	*0215	*0404	*0593	*0783	*0972	*1161	*1350	*1539	189
230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424	188
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
234	9216	9401	9587	9772	9958	*0143	*0328	*0513	*0698	*0883	185
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728	184
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
239	3398	8580	8761	8943	9124	9306	9487	9668	9849	*0030	181
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837	181
241	2017	2197	2377	2557	2737	2917	[illegible]	3277	3456	3636	180
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	178
245	9166	9343	9520	9698	9875	*0051	*0228	*0405	*0582	*0759	177
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521	176
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
250	7940	8114	8287	8461	8634	8808	8981	9154	9328	9501	173
251	9674	9847	*0020	*0192	*0365	*0538	*0711	*0883	*1056	*1228	173
252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949	172
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
257	9933	*0102	*0271	*0440	*0609	*0777	*0946	*1114	*1283	*1451	169
258	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
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262	8301	8467	8633	8798	8964	6129	9295	9460	9625	9791	165
263	9956	*0121	*0286	*0451	*0616	*0781	*0945	*1110	*1275	*1439	165
264	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
269	9752	9914	*0075	*0236	*0398	*0559	*0720	*0881	*1042	1203	161
270	431364	1525	1685	1846	2007	2167	2328	2488	2649	2809	161
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
273	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	159
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
275	9333	9491	9648	9806	9964	*0122	*0279	*0437	*0594	*0752	158
276	440909	1066	1224	1381	1538	1695	1852	2009	2166	2323	157
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
280	7158	7313	7468	76[illegible]	7778	7933	8088	8242	8397	8552	155
281	8706	8861	9015	917[illegible]	9324	9478	9633	9787	9941	*0095	154
282	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633	154
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
288	9392	9543	9694	9845	9995	*0146	*0296	*0447	*0597	*0748	151
289	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248	150
290	2398	2548	2697	2847	2997	3146	3296	3445	3594	3744	150
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
295	9822	9969	*0116	*0263	*0410	*0557	*0704	*0851	*0998	*1145	147
296	471292	1438	1585	1732	1878	2025	2171	2318	2464	2610	146
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145
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301	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144
302	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299	144
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
308	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
309	9958	*0099	*0239	*0380	*0520	*0661	*0801	*0941	*1081	*1222	140
310	491362	1502	1642	1782	1922	2062	2201	2341	2481	2621	140
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	139
312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	138
316	9687	9824	9962	*0099	*0236	*0374	*0511	*0648	*0785	*0922	137
317	501059	1196	1333	1470	1607	1744	1880	2017	2154	2291	137
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
320	5150	5286	5421	5557	5693	5828	5964	6099	6234	6370	136
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
322	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	135
323	9203	9337	9471	9606	9740	9874	*0009	*0143	*0277	*0411	134
324	510545	0679	0813	0947	1081	1215	1349	1482	1616	1750	134
325	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	133
326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4415	133
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	132
330	8514	8646	8777	8909	9040	9171	9303	9434	9566	9697	131
331	9828	9959	*0090	*0221	*0353	*0484	*0615	*0745	*0876	*1007	131
332	521138	1269	1400	1530	1661	1792	1922	2053	2183	2314	131
333	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	130
334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	130
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210	129
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	129
337	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788	129
338	8917	9045	9174	9302	9430	9559	9687	9815	9943	*0072	128
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342	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167	127
343	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432	126
344	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693	126
345	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951	126
346	9076	9202	9327	9452	9578	9703	9829	9954	*0079	*0204	125
347	540329	0455	0580	0705	0830	0955	1080	1205	1330	1454	125
348	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701	125
349	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944	124
350	4068	4192	4316	4440	4564	4688	4812	4936	5060	5183	124
351	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419	124
352	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	123
353	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	123
354	9003	9126	9249	9371	9494	9616	9739	9861	9984	*0106	123
355	550228	0351	0473	0595	0717	0840	0962	1084	1206	1328	122
356	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
357	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
358	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	121
359	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182	121
360	6303	6423	6544	6664	6785	6905	7026	7146	7267	7387	120
361	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589	120
362	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787	120
363	9907	*0026	*0146	*0265	*0385	*0504	*0624	*0743	*0863	*0982	119
364	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
365	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362	119
366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
368	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
369	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
370	8202	8319	8436	8554	8671	8788	8905	9023	9140	9257	117
371	9374	9491	9608	9725	9842	9959	*0076	*0193	*0309	*0426	117
372	570543	0660	0776	0893	1010	1126	1243	1359	1476	1592	117
373	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
374	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	116
375	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	116
376	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226	115
377	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	115
378	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	115
379	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669	114
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381	580925	1039	1153	1267	1381	1495	1608	1722	1836	1950	114
382	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085	114
383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218	113
384	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348	113
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	113
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	112
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	112
388	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838	112
389	9950	*0061	*0173	*0284	*0396	*0507	*0619	*0730	*0842	*0953	112
390	591065	1176	1287	1399	1510	1621	1732	1843	1955	2066	111
391	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175	111
392	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282	111
393	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386	110
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	110
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586	110
396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	110
397	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	109
398	9883	9992	*0101	*0210	*0319	*0428	*0537	*0646	*0755	*0864	109
399	600973	1082	1191	1299	1408	1517	1625	1734	1843	1951	109
400	2060	2169	2277	2386	2494	2603	2711	2819	2928	3036	108
401	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118	108
402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	108
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274	108
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348	107
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	107
406	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	107
407	9594	9701	9808	9914	*0021	*0128	*0234	*0341	*0447	*0554	107
408	610660	0767	0873	0979	1086	1192	1298	1405	1511	1617	106
409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	106
410	2784	2890	2996	3102	3207	3313	3419	3525	3630	3736	106
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792	106
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	105
413	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895	105
414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	105
415	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	105
416	9093	9198	9302	9406	9511	9615	9719	9824	9928	*0032	104
417	620136	0240	0344	0448	0552	0656	0760	0864	0968	1072	104
418	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	104
419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	104
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421	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210	103
422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	103
423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	103
424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	102
425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	102
426	9410	9512	9613	9715	9817	9919	*0021	*0123	*0224	*0326	102
427	630428	0530	0631	0733	0835	0936	1038	1139	1241	1342	102
428	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	101
429	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	101
430	3468	3569	3670	3771	3872	3973	4074	4175	4276	4376	101
431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383	100
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388	100
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390	100
434	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389	100
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436	9486	9586	9686	9785	9885	9984	*0084	*0183	*0283	*0382	99
437	640481	0581	0680	0779	0879	0978	1077	1177	1276	1375	99
438	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366	99
439	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354	99
440	3453	3551	3650	3749	3847	3946	4044	4143	4242	4340	98
441	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324	98
442	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306	98
443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	98
445	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	97
446	9335	9432	9530	9627	9724	9821	9919	*0016	*0113	*0210	97
447	650308	0405	0502	0599	0696	0793	0890	0987	1084	1181	97
448	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	97
449	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	97
450	3213	3309	3405	3502	3598	3695	3791	3888	3984	4080	96
451	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042	96
452	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002	96
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960	96
454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	96
455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	95
456	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	95
457	9916	*0011	*0106	*0201	*0296	*0391	*0486	*0581	*0676	*0771	95
458	660865	0960	1055	1150	1245	1339	1434	1529	1623	1718	95
459	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663	95
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461	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548	94
462	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	94
463	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	94
464	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	94
465	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293	93
466	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	93
467	9317	9410	9503	9596	9689	9782	9875	9967	*0060	*0153	93
468	670246	0339	0431	0524	0617	0710	0802	0895	0988	1080	93
469	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	93
470	2098	2190	2283	2375	2467	2560	2652	2744	2836	2929	92
471	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	92
472	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	92
473	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	92
474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	92
475	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
476	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	91
477	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	91
478	9428	9519	9610	9700	9791	9882	9973	*0063	*0154	*0245	91
479	680336	0426	0517	0607	0698	0789	0879	0970	1060	1151	91
480	1241	1332	1422	1513	1603	1693	1784	1874	1964	2055	90
481	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	90
482	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857	90
483	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756	90
484	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	90
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	89
486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	89
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
488	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	89
489	9309	9398	9486	9575	9664	9753	9841	9930	*0019	*0107	89
490	690196	0285	0373	0462	0550	0639	0728	0816	0905	0993	89
491	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	88
492	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
493	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	88
494	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	88
495	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	88
496	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	87
497	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	87
498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	87
499	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	87
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501	9838	9924	*0011	*0098	*0184	*0271	*0358	*0444	*0531	*0617	87
502	700704	0790	0877	0963	1050	1136	1222	1309	1395	1482	86
503	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	86
504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	86
505	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065	86
506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	86
507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	86
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
509	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	85
510	7570	7655	7740	7826	7911	7996	8081	8166	8251	8336	85
511	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	85
512	9270	9355	9440	9524	9609	9694	9779	9863	9948	*0033	85
513	710117	0202	0287	0371	0456	0540	0625	0710	0794	0879	85
514	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	84
515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566	84
516	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	84
517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	84
518	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	84
519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	84
520	6003	6087	6170	6254	6337	6421	6504	6588	6671	6754	83
521	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587	83
522	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	83
523	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
524	9331	9414	9497	9580	9663	9745	9828	9911	9994	*0077	83
525	720159	0242	0325	0407	0490	0573	0655	0738	0821	0903	83
526	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	82
527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
528	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
529	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
530	4276	4358	4440	4522	4604	4685	4767	4849	4931	5013	82
531	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
532	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	82
533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
534	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	81
536	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
537	9974	*0055	*0136	*0217	*0298	*0378	*0459	*0540	*0621	*0702	81
538	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
539	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	81
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542	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720	80
543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519	80
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	80
545	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	80
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
549	9572	9651	9731	9810	9889	9968	*0047	*0126	*0205	*0284	79
550	740363	0442	0521	0600	0678	0757	0836	0915	0994	1073	79
551	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
560	8188	8266	8343	8421	8498	8576	8653	8731	8808	8885	77
561	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	77
562	9736	9814	9891	9968	*0045	*0123	*0200	*0277	*0354	*0431	77
563	750508	0586	0663	0740	0817	0894	0971	1048	1125	1202	77
564	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
566	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
567	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
570	5875	5951	6027	6103	6180	6256	6332	6408	6484	6560	76
571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	76
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578	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
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583	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
584	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
586	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
587	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
588	9377	9451	9525	9599	9673	9746	9820	9894	9968	*0042	74
589	770115	0189	0263	0336	0410	0484	0557	0631	0705	0778	74
590	0852	0926	0999	1073	1146	1220	1293	1367	1440	1514	74
591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
592	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
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598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
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611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
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625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
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632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	69
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
639	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	68
640	6180	6248	6316	6384	6451	6519	6587	6655	6723	6790	68
641	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	68
642	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
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645	9560	9627	9694	9762	9829	9896	9964	*0031	*0098	*0165	67
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647	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
649	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
650	2913	2980	3047	3114	3181	3247	3314	3381	3448	3514	67
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653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
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655	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
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666	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
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671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	65
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675	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	64
676	9947	*0011	*0075	*0139	*0204	*0268	*0332	*0396	*0460	*0525	64
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678	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
679	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
680	2509	2573	2637	2700	2764	2828	2892	2956	3020	3083	64
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684	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	63
685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
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687	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	63
688	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	63
689	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
690	8849	8912	8975	9038	9101	9164	9227	9289	9352	9415	63
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693	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	63
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695	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
696	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
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704	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	62
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
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714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	61
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	60
720	7332	7393	7453	7513	7574	7634	7694	7755	7815	7875	60
721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	60
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746	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
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749	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
750	5061	5119	5177	5235	5293	5351	5409	5466	5524	5582	58
751	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	58
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765	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	57
766	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	57
767	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
769	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	57
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771	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	56
772	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	56
773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	56
774	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
775	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
776	9862	9918	9974	*0030	*0086	*0141	*0197	*0253	*0309	*0365	56
777	890421	0477	0533	0589	0645	0700	0756	0812	0868	0924	56
778	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	56
779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
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782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	56
783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	55
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
790	7627	7682	7737	7792	7847	7902	7957	8012	8067	8122	55
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
793	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	55
794	9821	9875	9930	9985	*0039	*0094	*0149	*0203	*0258	*0312	55
795	900367	0422	0476	0531	0586	0640	0695	0749	0804	0859	55
796	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	55
797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
799	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
800	3090	3144	3199	3253	3307	3361	3416	3470	3524	3578	54
801	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	54
802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
810	8485	8539	8592	8646	8699	8753	8807	8860	8914	8967	54
811	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	54
812	9556	9610	9663	9716	9770	9823	9877	9930	9984	*0037	53
813	910091	0144	0197	0251	0304	0358	0411	0464	0518	0571	53
814	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
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822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
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826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	9078	9130	9183	9235	9287	9340	9392	9444	9496	9549	52
831	9601	9653	9706	9758	9810	9862	9914	9967	*0019	*0071	52
832	920123	0176	0228	0280	0332	0384	0436	0489	0541	0593	52
833	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
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835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
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840	4279	4331	4383	4434	4486	4538	4589	4641	4693	4744	52
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	52
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
847	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
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849	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
850	9419	9470	9521	9572	9623	9674	9725	9776	9827	9879	51
851	9930	9981	*0032	*0083	*0134	*0185	*0236	*0287	*0338	*0389	51
852	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898	51
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
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862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	50
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866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
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871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467	50
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	50
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874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	50
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
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879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	49
880	4483	4532	4581	4631	4680	4729	4779	4828	4877	4927	49
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	49
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	49
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
890	9390	9439	9488	9536	9585	9634	9683	9731	9780	9829	49
891	9878	9926	9975	*0024	*0073	*0121	*0170	*0219	*0267	*0316	49
892	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803	49
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
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903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	48
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
910	9041	9089	9137	9185	9232	9280	9328	9375	9423	9471	48
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914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	47
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
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920	3788	3835	3882	3929	3977	4024	4071	4118	4165	4212	47
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
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933	9882	9928	9975	*0021	*0068	*0114	*0161	*0207	*0254	*0300	47
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935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
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942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
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944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	7724	7769	7815	7861	7906	7952	7998	8043	8089	8135	46
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412	45
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
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958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
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961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
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965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
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968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
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971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
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973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
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975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
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981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
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998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
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1001	0434	0477	0521	0564	0608	0651	0694	0738	0781	0824	43
1002	0868	0911	0954	0998	1041	1084	1128	1171	1214	1258	43
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1004	1734	1777	1820	1863	1907	1950	1993	2036	2080	2123	43
1005	2166	2209	2252	2296	2339	2382	2425	2468	2512	2555	43
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1007	3029	3073	3116	3159	3202	3245	3288	3331	3374	3417	43
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1010	4321	4364	4407	4450	4493	4536	4579	4622	4665	4708	43
1011	4751	4794	4837	4880	4923	4966	5009	5052	5095	5138	43
1012	5181	5223	5266	5309	5352	5395	5438	5481	5524	5567	43
1013	5609	5652	5695	5738	5781	5824	5867	5909	5952	5995	43
1014	6038	6081	6124	6166	6209	6252	6295	6338	6380	6423	43
1015	6466	6509	6552	6594	6637	6680	6723	6765	6808	6851	43
1016	6894	6936	6979	7022	7065	7107	7150	7193	7236	7278	43
1017	7321	7364	7406	7449	7492	7534	7577	7620	7662	7705	43
1018	7748	7790	7833	7876	7918	7961	8004	8046	8089	8132	43
1019	8174	8217	8259	8302	8345	8387	8430	8472	8515	8557	43
No	0	1	2	3	4	5	6	7	8	9	D

No	0	1	2	3	4	5	6	7	8	9	D
1020	008600	8643	8685	8728	8770	8813	8856	8898	8941	8983	42
1021	9026	9068	9111	9153	9196	9238	9281	9323	9366	9408	42
1022	9451	9493	9536	9578	9621	9663	9706	9748	9791	9833	42
1023	9876	9918	9961	*0003	*0045	*0088	*0130	*0173	*0215	*0258	42
1024	010300	0342	0385	0427	0470	0512	0554	0597	0639	0681	42
1025	0724	0766	0809	0851	0893	0936	0978	1020	1063	1105	42
1026	1147	1190	1232	1274	1317	1359	1401	1444	1486	1528	42
1027	1570	1613	1655	1697	1740	1782	1824	1866	1909	1951	42
1028	1993	2035	2078	2120	2162	2204	2247	2289	2331	2373	42
1029	2415	2458	2500	2542	2584	2626	2669	2711	2753	2795	42
1030	2837	2879	2922	2964	3006	3048	3090	3132	3174	3217	42
1031	3259	3301	3343	3385	3427	3469	3511	3553	3596	3638	42
1032	3680	3722	3764	3806	3848	3890	3932	3974	4016	4058	42
1033	4100	4142	4184	4226	4268	4310	4353	4395	4437	4479	42
1034	4521	4563	4605	4647	4689	4730	4772	4814	4856	4898	42
1035	4940	4982	5024	5066	5108	5150	5192	5234	5276	5318	42
1036	5360	5402	5444	5485	5527	5569	5611	5653	5695	5737	42
1037	5779	5821	5863	5904	5946	5988	6030	6072	6114	6156	42
1038	6197	6239	6281	6323	6365	6407	6448	6490	6532	6574	42
1039	6616	6657	6699	6741	6783	6824	6866	6908	6950	6992	42
1040	7033	7075	7117	7159	7200	7242	7284	7326	7367	7409	42
1041	7451	7492	7534	7576	7618	7659	7701	7743	7784	7826	42
1042	7868	7909	7951	7993	8034	8076	8118	8159	8201	8243	42
1043	8284	8326	8368	8409	8451	8492	8534	8576	8617	8659	42
1044	8700	8742	8784	8825	8867	8908	8950	8991	9033	9075	42
1045	9116	9158	9199	9241	9282	9324	9366	9407	9449	9490	42
1046	9532	9573	9615	9656	9698	9739	9781	9822	9864	9905	42
1047	9947	9988	*0030	*0071	*0113	*0154	*0195	*0237	*0278	*0320	41
1048	020361	0403	0444	0486	0527	0568	0610	0651	0693	0734	41
1049	0775	0817	0858	0900	0941	0982	1024	1065	1107	1148	41
1050	1189	1231	1272	1313	1355	1396	1437	1479	1520	1561	41
1051	1603	1644	1685	1727	1768	1809	1851	1892	1933	1974	41
1052	2016	2057	2098	2140	2181	2222	2263	2305	2346	2387	41
1053	2428	2470	2511	2552	2593	2635	2676	2717	2758	2799	41
1054	2841	2882	2923	2964	3005	3047	3088	3129	3170	3211	41
1055	3252	3294	3335	3376	3417	3458	3499	3541	3582	3623	41
1056	3664	3705	3746	3787	3828	3870	3911	3952	3993	4034	41
1057	4075	4116	4157	4198	4239	4280	4321	4363	4404	4445	41
1058	4486	4527	4568	4609	4650	4691	4732	4773	4814	4855	41
1059	4896	4937	4978	5019	5060	5101	5142	5183	5224	5265	41
No	0	1	2	3	4	5	6	7	8	9	D

No	0	1	2	3	4	5	6	7	8	9	D
1060	025306	5347	5388	5429	5470	5511	5552	5593	5634	5674	41
1061	5715	5756	5797	5838	5879	5920	5961	6002	6043	6084	41
1062	6125	6165	6206	6247	6288	6329	6370	6411	6452	6492	41
1063	6533	6574	6615	6656	6697	6737	6778	6819	6860	6901	41
1064	6942	6982	7023	7064	7105	7146	7186	7227	7268	7309	41
1065	7350	7390	7431	7472	7512	7553	7594	7635	7676	7716	41
1066	7757	7798	7839	7879	7920	7961	8002	8042	8083	8124	41
1067	8164	8205	8246	8287	8327	8368	8409	8449	8490	8531	41
1068	8571	8612	8653	8693	8734	8775	8815	8856	8896	8937	41
1069	8978	9018	9059	9100	9140	9181	9221	9262	9303	9343	41
1070	9384	9424	9465	9506	9546	9587	9627	9668	9708	9749	41
1071	9789	9830	9871	9911	9952	9992	*0032	*0073	*0114	*0154	41
1072	030195	0235	0276	0316	0357	0397	0438	0478	0519	0559	40
1073	0600	0640	0681	0721	0762	0802	0843	0883	0923	0964	40
1074	1004	1045	1085	1126	1166	1206	1247	1287	1328	1368	40
1075	1408	1449	1489	1530	1570	1610	1651	1691	1732	1772	40
1076	1812	1853	1893	1933	1974	2014	2054	2095	2135	2175	40
1077	2216	2256	2296	2337	2377	2417	2458	2498	2538	2578	40
1078	2619	2659	2699	2740	2780	2820	2860	2901	2941	2981	40
1079	3021	3062	3102	3142	3182	3223	3263	3303	3343	3384	40
1080	3424	3464	3504	3544	3585	3625	3665	3705	3745	3786	40
1081	3826	3866	3906	3946	3986	4027	4067	4107	4147	4187	40
1082	4227	4267	4308	4348	4388	4428	4468	4508	4548	4588	40
1083	4628	4669	4709	4749	4789	4829	4869	4909	4949	4989	40
1084	5029	5069	5109	5149	5190	5230	5270	5310	5350	5390	40
1085	5430	5470	5510	5550	5590	5630	5670	5710	5750	5790	40
1086	5830	5870	5910	5950	5990	6030	6070	6110	6150	6190	40
1087	6230	6269	6309	6349	6389	6429	6469	6509	6549	6589	40
1088	6629	6669	6709	6749	6789	6828	6868	6908	6948	6988	40
1089	7028	7068	7108	7148	7187	7227	7267	7307	7347	7387	40
1090	7426	7466	7506	7546	7586	7626	7665	7705	7745	7785	40
1091	7825	7865	7904	7944	7984	8024	8064	8103	8143	8183	40
1092	8223	8262	8302	8342	8382	8421	8461	8501	8541	8580	40
1093	8620	8660	8700	8739	8779	8819	8859	8898	8938	8978	40
1094	9017	9057	9097	9136	9176	9216	9255	9295	9335	9374	40
1095	9414	9454	9493	9533	9573	9612	9652	9692	9731	9771	40
1096	9811	9850	9890	9929	9969	*0009	*0048	*0088	*0127	*0167	40
1097	040207	0246	0286	0325	0365	0405	0444	0484	0523	0563	40
1098	0602	0642	0681	0721	0761	0800	0840	0879	0919	0958	40
1099	0998	1037	1077	1116	1156	1195	1235	1274	1314	1353	39
No	0	1	2	3	4	5	6	7	8	9	D

ANSWERS

EXERCISES. Pages 10-12.

5. 3, 4, 0, 5, 7, 6. **9.** 1499, 4301, 188, 1479. **10.** 29685, 989675, 59245875. **12.** 23. **13.** 7. **14.** 29, 8. **16.** 48438.

EXERCISES. Pages 17-18.

1. 193. **5.** 360+13, 720+13, 1080+13, etc. **6.** 5. **7.** 66. **8.** 45. **9.** 100. **10.** 15, 16, 17, 18.

EXERCISES. Pages 36-37.

1. [illegible]. **2.** [illegible]. **3.** 3[illegible]. **9.** 3[illegible]. **10.** 7. **11.** 9.

EXERCISES. Pages 41-42.

1. (1) 1726·8021; (2) 44·951; (3) 0·05191; (4) 19·802; (5) 10·3515.
2. (1) 3·2875; (2) 4·2054; (3) 42·500; (4) 7·0935; (5) 0·27248.
3. (1) 0·875; (2) 0·44; (3) 0·10625; (4) 0·234375; (5) $0\cdot\dot{4}3\dot{2}$; (6) $0\cdot\dot{6}$; (7) $0\cdot\dot{1}4285\dot{7}$; (8) $0\cdot\dot{9}2307\dot{6}$; (9) $0\cdot\dot{5}9\dot{2}$; (10) $0\cdot\dot{6}4\dot{8}$; (11) $0\cdot8\dot{3}$; (12) $0\cdot58\dot{3}$; (13) $0.\dot{3}\dot{2}\dot{7}$; (14) $0\cdot44\dot{7}\dot{2}$; (15) $0\cdot39236\dot{1}$; (16) $0\cdot86\dot{1}$. **4.** $1\cdot\dot{0}4262137925\dot{5}$. **5.** $0\cdot60\dot{7}83333896482\dot{7}0$.
6. (1) [illegible]; (2) [illegible]; (3) [illegible]; (4) 13[illegible].
7. $\frac{1}{14000}$ ($=0\cdot0000\dot{7}1428\dot{5}$), *i.e.*, between 0·00007 and 0·00008.
8. 2·8835. **9.** (1) 0·2027326; (2) 0·0911608; **10.** 1·7893.
11. 0·329.

EXERCISES. Pages 52-54.

1. 13, 17, 19, 21, 32, 54, 78, 99. 2. 11, 12, 17, 21, 25, 29, 47, 74. 3. 123, 234, 304, 609, 758, 895. 4. 111, 234, 475, 738. 5. 2·646, 3·606, 5·385, 8·544, 11·269. 6. 2·22, 2·84, 4·63, 5·35, 8·01. 7. 2·9, 5·3, 1·34, 3·27, 0·67. 8. 1·3, 2·3, 1·32, 0·86. 9. 0·837, 0·265, 0·177, 0·056. 10. 0·67, 0·37, 0·61, 0·28, 0·13. 11. 63, 78, 165, 225; 21, 35, 72, 88. 12. 1·6, 1·5. 13. 0·83, 0·76, 0·79, 0·72. 14. 0·94, 0·82, 0·80, 0·85.

EXERCISES. Pages 56-59.

1. [illegible] 2. 25°/., 37½°/. or 37·5°/., 31¼°/. or 31·25°/., 40°/., 58⅓°/. or 58·33°/., 17$\frac{23}{32}$°/. or 17·71°/., 54$\frac{1}{10}$°/. or 54·10°/., 33$\frac{3}{152}$°/. or 33·02°/., 3°/., 37·5°/., 3·5%, 315%. 3. 45·36%. 4. 91·44%. 5. 76°/., 62%, 65%, 68%. 6. 27·03%. 7. $1·98. 8. s. 35%, c. 34·5%, i. 30.5%. 9. $84,000. 10. $2·00, 11. $48.00. 12. $102·00. 13. 300. 14. 20%. 15. 55·2%. 16. $3937·50. 17. 47½%. 18. 28⅘%. 19. 5%. 20. 1$\frac{117}{153}$% (=1·767%). 21. $15. 22. 14$\frac{2}{7}$%. 23. 23⅓%; cost price $95·02, marked price $123·53. 24. $56·00. 25. 38⅛%, $\frac{5}{16}$, c.p. $12·42, s.p. $15·52. 26. 1·008 (The merchant who would wish this result would mark goods at $1·01, sell at 91 ct., and gain 8⅓%). 27. 25%. 28. $5400. 29. 80ct., 10ct. 30. $360. 31. 4, 6. 32. $1000. 33. $1·25 a yd.

EXERCISES. Pages 60-61.

1. $134·75. 2. $28·13. 3. $317·00. 4. $513·24, $524·04. 5. $518·40. 6. $36. 7. 28%. 9. $37·76. 10. 21·83%. 11. 27·1%. 12. 12½%. 13. 33⅓%. 14. 10%. 15. 1·2346%.

EXERCISES. Pages 64-65.

1. $105·30, $2527·20. 2. $450, $8550. 3. $1710, $32,490. 4. $7711·20. 5. 3000 lb. 6. 600 bbl. 7. 6%. 8. $6·60. 9. 7500 lb., $936. 10. $32, $1368. 11. 6%. 12. $150, 475 bbl. 13. 700 bbl., 77,200 lb.

EXERCISES. Pages 66-67.

1. $120. 2. $120. 3. $1275. 4. $360. 5. $11·75. 6. $7200. 7. $3135, $6865. 8. $15,250. 9. $24,687·50, $11,312·50. 10. $6045, $4950. 11. $70,617·91, $617·91. 12. $\frac{1}{4}$%. 13. $50·51. 14. $70·44. 15. $90·91, $5969·70, $2030·30. 16. $5055. 17. $9400.

EXERCISES. Page 68.

1. $78. 2. $21.45. 3. $23,283. 4. 4 mills on the dollar. 5. 17 mills. 6. 19$\frac{1}{2}$ mills. 7. $2300. 8. 20$\frac{1}{2}$ mills. 9. $700. 10. The latter by $2.60.

EXERCISES. Pages 72-74.

1. (1) $2·33; (2) $7·08; (3) $62·09; (4) $10·57; (5) £2.7s.6$\frac{1}{4}$d. 2. (1) $5·01; (2) $9·38; (3) $0·43; (4) $23·50; (5) £23.15s.2$\frac{1}{4}$d. 3. (1) $11·30; (2) $29·20; (3) $179·10; (4) $24·24; (5) £3. 1s. 5$\frac{1}{4}$d. 4. (1) $389·15; (2) $134·23; (3) $159·50; (4) $265·54; (5) $180·50. 5. $1211·16. 6. $132·21. 7. $72·74. 8. $38·98. 9. (1) $208·33; (2) $68·86; (3) $11·36; (4) $134·48; (5) $30·32. 10. $906·56. 11. $645. 12. $593·56. 13. 3$\frac{1}{4}$%. 14. 4$\frac{1}{2}$%. 15. 3$\frac{1}{2}$%. 16. 33 da. 17. 2 yr. 75 da. 18. (1) $(1·045)^3$; (2) $(1·02)^4$; (3) $(1·06)^2 \times (1·0125)$; (4) $(1·035)^2$ $(1·014)$; (5) $(1·025)^5$ $(1·016)$. 19. 5%. 20. 3 yr. 21. $2·62. 22. $\frac{61}{8000}$(=0·007625). 23. $4800. 24. The former by $61·84. 25. $1624·50. 26. $6000, $5200. 27. 6·09%. 28. 2·9563%.

EXERCISES. Pages 77-78.

1. (1) $4·35, $535·65; (2) $5·68, $518·02; (3) $4·00, $408·50. 2. $751·56. 3. (1) $9·17, $1021·23; (2) $23·18, $1757·94; (3) $12·02, $962·66. 4. 6·047%. 5. $609·42. 6. $236·64. 7. 5$\frac{1}{2}$%. 8. 6%. 9. 6.07%. 10. May 27. 11. (1) $\frac{111}{36500}$; (2) $\frac{1}{730}$; (3) $\frac{1}{732}$. 12. 6%.

EXERCISES. Pages 81-82.

1. (1) $524·70; (2) $123·92; (3) $621·93; (4) $1400·77. 2. (1) $1 + (1·05)^6$; (2) [illegible]; (3) $1 + [(1·04)^2(1·02)]$; (4) [illegible]. 3. (1) [illegible]; (2) [illegible]; (3) [illegible]; (4) [illegible]. 4. $1924·90. 5. $1451·96. 6. $1633·95. 7. 59 da. 8. 7 mo. 9. 61 da. 10. 60 da., 61 da., 11. Set. 12. $130, 4%. 13. $1704, 5%. 14. $62·40.

EXERCISES. Pages 84-85.

1. $708·53. 2. $206·49. 3. $1510·10 (or $1510·26 if the last int. is taken for 92 da., not for 3 mo.). 4. $404·72.

EXERCISES. Pages 88-90.

1. (1) $1483·50; (2) $37,856·25; (3) $2996·88; (4) $4848·75; (5) $13,584·38. 2. (1) $20,291·25; (2) $54,718·75; (3) $9670·00; (4) $4741·88; (5) $14,501·25. 3. (1) 4·002%; (2) 4·282%; (3) 4·236%; (4) 5·284%; (5) 4·267%. 4. 125, 124¼. 5. The former. 6. $4040·63. 7. $2800 stock, $31. 8. $9450. 9. $112·50. 10. $60. 11. $60. 12. $50. 13. $125. 14. $1. 15. $10,000, $8400, $46. 16. 133⅓.

EXERCISES. Pages 96-98.

1. $1166·53. 2. $15,852·78. 3. $6056·25. 4. 9. 5. $1754·38. 6. $308·29. 7. $1536. 8. 4·835% (=4[illegible]%). 9. $1157·47. 10. $835·53.

EXERCISES. Pages 100-106.

1. (1) 35 sq. in.; (2) 195 sq. dm.; (3) 323 sq. m. 2. 52m., $\sqrt{3329}$=57·697m. 3. 88 yd., 187 yd. 6. $34·13. 7. 512 sq. yd. 8. 1:4. 9. 49:48. 10. 18·3576m., 12m., 16·9706m. 11. (1) 13 in.; (2) 6·5m.; (3) 64·637 dm. 12. 84 yd. 13. $\sqrt{3312}$ (=57·55m.). 14. 14·5 ft. 16. 41·184m. 17. 630 sq. chains. 18. BC=30m., AC=50m. 19. 173·205 sq. yd. 20. $25\sqrt{3}$, $144\sqrt{3}$, $169\sqrt{3}$ sq. dm. *or* 43·301, 249·415, 292·717 sq. dm.; sum of first and second is equal to third.

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)

1.0 1.1 1.25 1.4 1.6 1.8 2.0 2.2 2.5 2.8 3.2 3.6 4.0 4.5 5.0 5.6 6.3 7.1

21. (1) 1440 sq. in., p = 40 in.; (2) 264 sq. cm., p = 12 cm. **22.** 37·5 sq. ft. **23.** 120 sq. in. **24.** 1080 sq. in. **25.** 2088 sq. ft. **26.** 12·917 in. **27.** 38·5 ft. **28.** 408 mi. **29.** 4·472 in. **30.** 140·296 sq. in. **31.** 381·713 ft. **32.** 60 rd. **33.** $451. **34.** 30 rd. by 25 rd. **35.** $5457·60. **36.** 360 sq. m., CM = 21m.

EXERCISES. Pages 107-110.

1. (1) $50\frac{2}{7}$ in. or 50·29 in., 22m., 88 yd., $56\frac{4}{7}$ dm. or 56·57 dm.; (2) 50·27 in., 21·99m., 87·96 yd., 56·55 dm. **2.** (1) $3\frac{1}{2}$ m. or 3·5m., $1\frac{9}{40}$ yd. or 1·23 yd., $10\frac{101}{440}$ dm. or 10·23 dm., $11\frac{417}{440}$ in. or 11·95 in.; (2) 3·50m., 1·23 yd., 10·23 dm., 11.95 in.; (3) 3·50m., 1·23 yd., 10·23 dm., 11·95 in. **3.** Twice as great; implied in formula $c = 2\pi r$. **4.** 1·5708m. **5.** 420. **6.** $318\frac{2}{11}$. **7.** 5 mi. 112 yd. **8.** 27·925 in. **9.** 18,041·8 mi. **10.** 4·1888m. **11.** 7 mi. 739·2 ft. **12.** $9\frac{1}{3}$ in. **13.** 1·16m. **14.** (1) 154 sq. m., $38\frac{1}{2}$ sq. in. or 38·5 sq. in., $113\frac{1}{7}$ sq. yd. or 113·1 sq. yd.; (2) 153·94 sq. m., 38·48 sq. in., 113·20 sq. yd. **15.** 161·14 sq. in. **16.** 8·68m., 54·60m. **18.** 96250 sq. yd. or 96288·5 sq. yd. as $\frac{22}{7}$ or 3·1416 is taken for π. **19.** 542·08 sq. dm. **20.** 43·07 sq. cm. **22.** 7857·1 or 7862·8 as $\frac{22}{7}$ or 3·1416 is taken for π. **23.** $706·86. **24.** 19·092 in. **25.** 87·7 ft. **27.** 0·36 sq. ft. **28.** 143·36 sq. in. **29.** $r^2(\pi - 2)$. **30.** 81·06. **31.** 86·03. **32.** $r^2(\pi - 1{\cdot}5\sqrt{3})$.

EXERCISES. Pages 114-116.

1. (1) 1001 c.ft.; (2) 5175 c.in.; (3) 79·971 c.m.; (4) 6727·995 c.cm.; (5) 3727·284 c.cm. **2.** 63·25 sq. ft. **3.** 13 ft., 494 sq. ft., 64 ft. **4.** 7 ft. by 14 ft. by 21 ft. **5.** 35·79 ft. **6.** The diagonal of a cube whose side measures 1. **7.** 6 cm. **8.** 472·83 c. ft., 31·17 c. ft. **9.** 506·88 c. ft. **10.** 19·94%. **11.** 231·22 ft. **12.** 7794·2 c. in. **13.** 7140 c. cm. **14.** 31,176·9 c. in. **15.** 7·69 in. **16.** $641·67. **17.** 3418·06 sq. ft., 12,063·74 c. ft. **18.** 879·648 sq. cm., 1806·42 c. cm. **19.** 3·19 ft. **20.** 339·29 sq. in. **21.** 978·14 gal. **22.** 6·476 in., 9·714 in. **23.** $\frac{1}{4}$ in., taking $\frac{22}{7}$ for π. **24.** 322·3 c. cm. **25.** 350·5 c ft. **26.** 2·418 c. dm. **27.** 3·992 in., say 4 in. **28.** 1732. **29.** 22020·7 g.

EXERCISES. Pages 117-119.

1. \$9·76. **2.** \$3313·33. **3.** 46,592·5. **4.** 5 ft. 11·66 in. **5.** 28%. **6.** 18·5. **7.** 83. **8.** 10·5. **9.** \$79·23. **10.** 5646·66 sq. in., 5544 sq. in. **11.** 35·52 mi. an hour. **12.** 0·8ct. a bu. **13.** $10\frac{3}{8}$ in. **14.** $16\frac{1}{4}$ct. **15.** 6 mo. **16.** 36 da. **17.** 16.

EXERCISES. Page 120.

1. Man \$30, boy \$15·10, girl \$15. **2.** 20, 70, 130. **3.** 68a., 102a., 153a., 204a. **4.** \$60, \$30, \$10. **5.** \$524·16, \$375·84. **6.** \$5707·49, \$6292·51. **7.** \$300, \$420. **8.** \$32, \$16, \$24, \$36. **9.** \$48, \$60, \$72. **10.** Man 50ct., boy 30ct. **11.** \$3278·76, \$3221·24. **12.** Girl \$18, boy \$20, woman \$30, man \$45.

EXERCISES. Pages 122-123.

1. 0.5 g. **2.** 81:199. **3.** 3:8. **4.** 9 g. **5.** $6\frac{7}{8}$ gal. **6.** 17 lb. and 5 lb.; many answers. **7.** 92·237%. **8.** 67·2%. **9.** 15·89 g., 16·66 g. **10.** 62·4%. **11.** 25 lb. at $34\frac{1}{2}$ ct.; 23 lb. at $37\frac{1}{2}$ ct. **12.** 3:5. **13.** 49·25°/.. **14.** $33\frac{1}{3}$°/.. **15.** 20 lb., 15 lb., 10lb.

EXERCISES. Pages 124-125.

1. $2\frac{14}{23}$ da. **2.** 4 da. **3.** $16\frac{5}{8}$ da. **4.** $5\frac{2}{3}$ da. **5.** $3\frac{47}{59}$ da. **6.** $6\frac{3}{4}$ da. **7.** $4\frac{4}{7}$ da. **8.** $11\frac{1}{3}$ da. **9.** 6 da. **10.** 100 men. **11.** \$54, \$48. **12.** 5 da. **13.** $28\frac{4}{57}$ da. **14.** 10 da. **15.** $24\frac{12}{13}$ da. **16.** $8\frac{35}{51}$ da.

EXERCISES. Pages 126-127.

1. $5\frac{1}{4}$ hr., 21 mi. **2.** $\frac{1}{2}$ mi. an hr. **3.** 192 rods from starting point; 3 mi. 192 rd., 2 mi. 192 rd. **4.** 3 h. $16\frac{4}{11}$ min. **5.** $65\frac{10}{11}$ and $34\frac{1}{11}$ mi. an hr. **6.** 6 h. $16\frac{4}{11}$ min. **7.** 9.30 p.m. **8.** 4 h. $54\frac{6}{11}$ min. **9.** 15 and 12 mi. an hr. **10.** 40 min. **11.** $\frac{5}{12}$ hr. **12.** At a point $\frac{2}{3}$ of way round from starting point, after the faster has made $32\frac{2}{3}$ rounds. **13.** 5·152 yd. **14.** 12 midnight. **15.** $7\frac{1}{2}$ mi. an hr.

EXERCISES. Miscellaneous Pages 156-166.

I. **1.** $\frac{217}{1150}$, 0·100. **3.** $160, $80, $40, $20. **4.** $8·21. **5.** 38·77 (=38$\frac{7}{9}$) ft., 25·2 ft., 24 ft.

II. **1.** £6·821. **3.** $21. **4.** 2¼ yr. **5.** £2640.

III. **1.** 14. **3.** $54, $18, $9. **4.** $1036·19. **5.** 210 m., 294 m.

IV. **1.** 0·742. **3.** $312. **4.** $1197·46. **5.** 3 °/ₒ.

V. **1.** $\frac{2}{11}$. **3.** 16⅔ °/ₒ. **4.** $468·84. **5.** 1:1·299.

VI. **1.** 0·0291$\dot{6}$. **2.** 19. **3.** $18·75. **4.** 3½ °/ₒ. **5.** 16 da.

VII. **1.** 109·7514. **2.** 583. **3.** 36 yr., 12 yr., 2 yr. **4.** $152·33. **5.** 1:1·1547.

VIII. **1.** 128·9944. **2.** 845, 3380; 1690, 2535. **3.** ($\frac{97}{157}$=) 61·78 °/ₒ. **4.** 6 °/ₒ. **5.** 25 °/ₒ.

IX. **1.** 76·3766. **2.** 17, 4199; 221, 323. **3.** $70, $48, $50. **4.** $1724·05. **5.** 1·10:1.

X. **1.** 24·72704726, 24·7270. **2.** 1428, 3213. **3.** $8·40, $3·30, $1·25. **4.** 6·055 °/ₒ. **5.** 6 lb.

XI. **1.** 0·73657. **3.** 200 yd. at $1·50, 300 yd. at $1·30. **4.** 72 da. **5.** 10·5875 tons.

XII. **1.** 1·15573. **2.** (a) $\frac{7}{440}$; (b) 9900. **3.** 25 gal. **4.** $2000·00, $1998·53. **5.** $1·50.

XIII. **1.** 0·2531. **2** 2317, 1456. **3.** 4 lb. **4.** $1499·63. **5.** 23·221 sq. yd., 37·699 yd.

XIV. **1.** 0·3468. **2.** 37, 53, 61. **3.** 8 cows, 20 sheep. **4.** $1062·13. **5.** 88 lb. and 96 lb.; 45 ct. and 50 ct.

XV. **1.** 0·4342945. **2.** 41, 59, 71. **3.** 180. **4.** $231·86. **5.** 4849·74 sq. rods, 144·22 rods.

XVI. **1.** 1·0000. **3.** $20. **4.** Jan. 6, 1891. **5.** $1·00, $1·20.

XVII. 1. 180·0000. 3. $100. 4. $731·79, 6·059°/₀. 5. 856·636 sq. in.

XVIII. 1. 2·1323. 2. 12. 3. $126·03, $115·22, $113·60. 4. $1895·71. 5. $\frac{1}{8}$ gal.

XIX. 1. 2·373. 3. 800 bbl. 4. $632·33. 5. 60 rd., 1764 sq. rd., 65 rd., 56 rd. (Note:—Rectangle contained by the diagonals of a quadrilateral in a circle is equal to the sum of the rectangles contained by the opposite sides).

XX. 1. 59·26608. 2. 900, 90. 3. 11:15; 22·973°/₀. 4. $951·93. 5. $3\frac{1}{3}$ gal.

XXI. 1. (a) 3547; (b) 0·0539. 2. $\frac{13}{18}$, $\frac{23}{28}$, $\frac{11}{12}$; 8, 28, 12. 3. $5600, $7200; 1·5625°/₀. 4. $439·68, $293·12, $183·20. 5. 1600 yd.

XXII. 1. (a) 357; (b) 2·34. 3. $3500, $1500. 4. $6198·35. 5. $\frac{1}{4}$.

XXIII. 1. (a) 7963; (b) 4·995. 2. $\frac{714285}{999999}$, $\frac{27}{99}$, $\frac{692307}{999999}$. 3. 7°/₀. 4. 11:4. 5. 2·025A.

XXIV. 1. (a) 567; (b) 40·5. 2. $\frac{48}{90}$, [illegible], [illegible]. 3. $232·01. 4. $7\frac{1}{2}$ mi. 5. $33\frac{3}{4}$ gal. wine, $20\frac{1}{4}$ gal. water.

XXV. 1. (a) 1·414; (b) 0·845. 3. $92 increase. 4. 45 mi. 5. 109 ft.

XXVI. 1. (a) 1·26; (b) 0·89. 2. $5·76, $38\frac{2}{3}$%. 3. $1801·74. 4. 80ct., 40ct. 5. $4\frac{1}{2}$%.

XXVII. 1. (a) 3·6056; (b) 0·9199. 2. $31\frac{11}{16}$%. 3. $174·15. 4. 720 ft. (Take $\frac{22}{7}$ for π). 5. 295·4652 sq. in.

XXVIII. 1. (a) 1·44; (b) 0·95. 2. $16\frac{2}{3}$%. 3. $820. 4. $1785, $1760. 5. 27% nearly.

XXIX. 1. (a) 4·1231; (b) 0·4617. 2. $119·30. 3. $118·08. 4. $80·25, $\frac{22}{21}$ da. 5. 854·93 sq. ft., 534·33 sq. ft., 320·60 sq. ft.

XXX. **1.** (a) 0·831; (b) 0·251. **2.** The former. **3.** 14,662$\frac{1}{3}$ bu., 490M. **4.** 60%. **5.** 12$\frac{1}{2}$%.

XXXI. **1.** 0·0325. **2.** \$4000, \$3000. **3.** \$825·83. **4.** In 1 ar.; 3000 sec., 2000 sec., 1200 sec., 545$\frac{5}{11}$ sec. **5.** \$247·50.

XXXII. **1.** \$4·8666. **2.** The former. **3.** \$831·23. **4.** $\frac{1}{4}$. **5.** 15 ft. by 12 ft. by 9 ft.

XXXIII. **1.** 299877 Km. **2.** £5000. **3.** \$25. **4.** Midnight on Wed. of the next week. **5.** $\sqrt{101}$ (=10·05) ft.

XXXIV. **1.** 365·2475 da. **2.** In 187 da., \$130·98. **3.** A loses \$75, B gains \$50, C gains \$125, D gains \$100. **4.** 36 ounces. **5.** \$285·66.

XXXV. **1.** 0·99766 g. **2.** \$[illegible]400. **3.** \$4000. **4.** A, \$9204·15; B, \$8628·89; C, \$5916·96. **5.** 241·84 Km.

XXXVI. **1.** 1033·5 g., 760·5 mm. **2.** 5% and 6%. **3.** \$146·94. **4.** \$2·50. **5.** 16·3869 Km.

XXXVII. **1.** 9·500g. **2.** \$700. **3.** \$5400, \$8400. **4.** 17$\frac{1}{7}$ and 23$\frac{1}{3}$ mi. an hr. **5.** 10 in.

XXXVIII. **1.** 7·769 g. to 7·898 g. **2.** \$544·65. **3.** \$3000. **4.** 5$\frac{4}{7}$%. **5.** 45·917 ft.

XXXIX. **1.** 277·463 c. in. **2.** 30,900. **3.** \$15·60 increase. **4.** \$330, \$470. **5.** 155·26 sq. in., 22·99 sq. in., 22·99 sq. in., 44·98 sq. in.

XL. **2.** \$29·18. **3.** 3%. **4.** \$1000. **5.** 84 sq. ft., 30 24 sq. ft., 53·76 sq. ft.

XLI. **1.** 58·8 l., 49·0 l., 42·0 l. **2.** \$531·74, \$531·87. **3.** \$5250. **4.** 4$\frac{1}{2}$%. **5.** 4·547 in.

XLII. **1.** \$1·75, \$2·00. **2.** 19$\frac{1}{11}$ gal., 12$\frac{8}{11}$ gal., 6$\frac{4}{11}$ gal. **3.** 27$\frac{3}{11}$ min. **4.** \$2000. **5.** 20525·12 lb.

XLIII. **1.** $\frac{1}{25}$ min. **2.** \$47. **3.** $11\frac{3}{8}$ da. **4.** \$400. **5.** 20·95%.

XLIV. **1.** 25 mi. an hr., 20 mi. an hr. **2.** \$17.10. **3.** \$10,000. **4.** $\frac{16}{125}$. **5.** 9·014 c. cm.

XLV. **1.** $6\frac{3}{11}$ da.; A, \$12·16; B, \$21·24; C, \$38·60. **2.** \$980·62. **3.** 6 in, **4.** \$3. **5.** 120 hr.; 240 hr.; they will not.

XLVI. **1.** 1986·2 lb. **2.** $13\frac{5}{9}$%. **3.** In (a) \$2455, in (b) \$2534. **4.** 14·7724 gal. **5.** 14·3065 sec.

XLVII. **1.** \$5·60. **2.** \$145·16. \$125·00. **3.** \$36·09. **4.** \$31,250·00, \$32,812·50, \$43,750·00, \$43,125·00. **5.** 8·51 cm., 11·25 cm., 196 cm.

XLVIII. **1.** 15 hr. 12 min.; by C. **2.** 1·914%. **3.** \$51,834·92. **4.** 2·308%, $62\frac{1}{2}$%. **5.** $27\frac{1}{15}$ yd., $54\frac{2}{15}$ yd.

XLIX. **1.** 30·5516 sec. **2.** 6%. **3.** \$5086·25. **4.** 56% (or $23\frac{17}{21}$% if the 10% refers to what remained). **5.** 0·377 cm.

L. **1.** 43·08 cm. **2.** \$510·64. **3.** $6\frac{2}{3}$ min. **4.** 650. **5.** 0·0655 in.; 2·84 in.

EXERCISES. Pages 173-175.

1. 1·4142136, 1·7724539, 2·6457513, 3·6352441, 1·7320508, 1·6487213. **2.** 1·259921, 2·108428, 1·912931, 2·223980, 2·351335, 2·884499. **4.** 287·1042, 3·1926, 1·3917, 0·1080, 4·4429. **5.** 0·22508, 1161·60677, 1·29099, 0·31831, 12·88726, 0·05122. **7.** 0·0588235294117647. **8.** 21. **19.** 180·00000000. **20.** 0·404. **22.** 0·000107. **23.** 0·405144. **24.** Less than 0·0000158, *or* less than 2 hundred-thousandths. **25.** It can not.

EXERCISES. Pages 183-186.

1. (1) 210, $n(n+1)\div 2$; (2) 400, n^2; (3) 410, $n(61-n)\div 2$; (4) 1870, $n(9n+7)\div 2$; (5) 2880, $n(7n+4)$; (6) 1030, $n(203-5n)\div 2$. **2.** (1) $(3^{20}-1)\div 2$, $(3^n-1)\div 2$;

(2) [illegible] $-1 \div (2\times 3^{10})$, $3(1-1\div 3^{n})\div 2$: (3) $7\times 2^{20}-7$, $7\times 2^{n}-7$; (4) $5(3^{20}-1)\div 2$, $5(3^{n}-1)\div 2$; (5) $5(1-3^{20}\div 5^{20})$, $5(1-3^{n}\div 5^{n})$; (6) $26(1-1\div 1{\cdot}04^{20})$, $26(1-1\div 1{\cdot}04^{n})$. **3.** 10, 10, 11[illegible], −3[illegible], 0, 1. **4.** 6, 12, $3\sqrt{10}$. **5.** (1) [illegible]; (2) [illegible]; (3) 21; (4) 23[illegible]; (5) [illegible]. **6.** (1) [illegible] $\div 3^{9}$; (2) [illegible] $\div 4^{9}$; (3) $21\div 1{\cdot}05^{10}$; (4) 23[illegible] $\div 1{\cdot}045^{10}$; (5) [illegible] $\times 4^{10}\div 5^{9}$. **8.** 2856. **9.** 575[illegible]. **10.** (1) 7, 13, 19, 25, 31, 37, 42; (2) $\sqrt{7}$, 7, $7\sqrt{7}$. **11.** 63,660. **12.** 2, **13.** 3. **14.** 2. **15.** 2. **16.** [illegible]. **17.** 2187·5. **18.** 3960. **19.** −100, 25, 5600. **20.** 1, −2, −5··· **23.** [illegible], 1, [illegible], [illegible], [illegible], [illegible]. **25.** [illegible]. **27.** $50(10^{20}-1)\div 81-250\div 9$. **28.** Any G. P. with common ratio [illegible]. **29.** $(10^{20}-1)\div 27$.

EXERCISES. Pages 194-196.

1. 12·41, 5023·54, 516·732, 0·012975, 0·00000026501. **2.** 189623, 16·7542, 1·17855, 41·9502, 0·820221, 8·1441. **3.** 13·866, 17·5325. **4.** 1·170, 2·095, 1·414, 2·646, 3·606, 0·4123. **5.** 1·089, 4·189, 1·260, 1·442, 1·710, 0.5066. **6.** 1·269, 1·496, 1·246, 1·380, 1·476 0·8236. **7.** 8·062, 1·448. **9.** 10. 6, 10 (in integral part). **10.** 4th, 7th, 8th. **11.** 4, 10, 3. **12.** 17. **13.** 5, −1, 0, 1·361728, 1·85664, 1·02925. **14.** (1) 127·445 sq. yd.; (2) 249·23 sq. in.; (3) 13877·3 sq. dm.; (4) 3875·73 sq. in. **15.** 672·91, 353·358. **16.** (1) $639·62; (2) $697·03; (3) $372·09. **17.** $829·97. **18.** 4½%. **19.** 5%. **20.** $1833·65. **21.** $172·20 (at end of time). **22.** 13·686, 12·580, 9·7. **23.** $191·18. **24.** $3031·89. **25.** $14,556·53.

EXERCISES. Pages 199-202.

1. $949·53. **2.** $2127·57. **3.** $1374·48. **4.** (1) $890·36; (2) $3679·88; (3) $269·06. **5.** $462·68. **6.** $1699·10, or $1699·47 as different Tables are used. **7.** $7867·45. **8.** $296·41. **9.** $494·01. **10.** $2945·56. **11.** $4134·97 (or $4135·03 by better tables). **12.** (1) $238·53; (2) $566·39; (3) $320·31. **13.** $2208·02. **14.** $1118·80. **15.** $2250. **16.** $4666·67. **17.** $3594·20. **18.** 4°/. **19.** $611·41. **20.** $412·46 (or $412·44). **21.** $1237·77. **22.** Insurance by $3166·44.

23. $11,517·56. **24.** $4141·84. **25.** $2354·90. **26.** $2582·49. **27.** $7731·95. **28.** $21,300·79. **29.** $1858·81 (or $1858·74). **30.** 6 regular payments and a 7th of less amount. **31.** In the 11th year.

EXERCISES. Pages 311-313.

1. 72 c. ft. **2.** 0·11785 c. in. **5.** 0·4136 c. in. **7.** Each slant face 3·0104 sq. in., base 1 sq. in., top 0·25 sq. in., vol. 1·75 c. in. **8.** 50·2656 c. cm.; 89·0048 sq. cm. **9.** 89·907 c. in.; 122·5224 sq. in. **10.** $6·52. **11.** 10344·66 c. ft. **13.** 221·08 c. in.; 226·89 sq. in. **14.** 6¼ c. in.; 19·37 sq. in. **15.** 184·06388 sq. in., 185·05783 sq. in.; 0·0255%; eight. **16.** 147:184. **17.** 3318·31 c. m.; 179741·2 sq. m. **18.** 2¼ in. **19.** 1·209 sq. ft. **20.** 9·899 in. **21.** 5·854 c. ft. **22.** 458·15 c. in. **23.** 4·327 in. **24.** 16·0145 ft. **25.** 4851·98 c. in., 4090 c. in. **26.** 1·382:1 **28.** 61·886 cm. **29.** 63·245%. **30.** 2:3:1.