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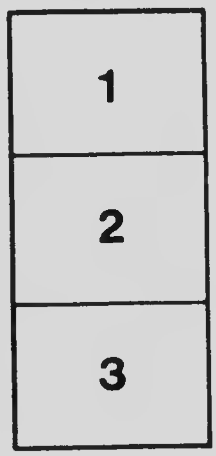
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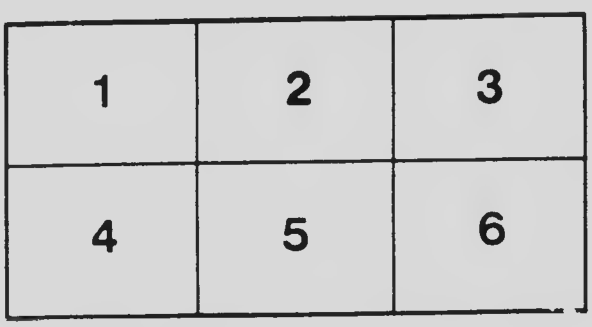
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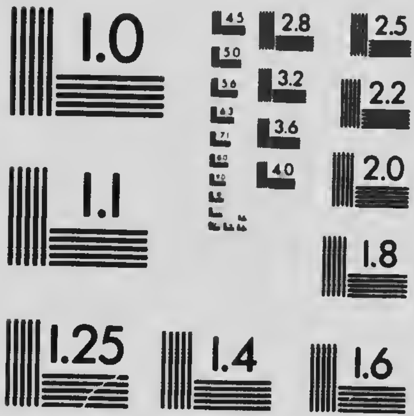

W. J. Gage & Co.'s Mathematical Series

ELEMENTS OF ALGEBRA

BY

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TORONTO:
W. J. GAGE & CO., LIMITED
1908

PREFACE

The justification for putting before the teachers of the country a new Elementary Algebra is the belief that, in spite of the multiplicity of books, there is still room for improvement in the method of treating the subject.

The aim of the present book is to give an elementary presentation of the fundamental principles and operations of Algebra in such a way as to relate each step to the previous knowledge of the student. Starting with the statement that algebra is a kind of advanced arithmetic, we have, especially in the early chapters, endeavoured to keep the student in constant touch with familiar facts.

In the treatment of the various topics we have kept the class-room constantly in mind, and have tried, both in language and illustration, to present the facts as a good teacher would do were he before a class.

While as far as possible we have used the inductive method, leading the student by illustration to see the steps by which new ideas and processes become possible, we have endeavoured to make it sound in theory by taking the student out of the concrete example to a realization of the processes in relation to number alone. Further, to this end, we have not hesitated to introduce, where necessary, a new idea by means of a definition. In the treatment of negative numbers, for example, we deemed it wiser to begin with a definition, and from considerations based on this definition, to derive our definition of Concrete Quantity, Art. 30. This, we believe, will be found to be both simple and accurate.

In the treatment of Multiplication (Arts. 63 and 64), we have preferred to build upon the arithmetical definition as the most logical way and to show that multiplication by a fraction is really consistent with it (Art. 182). This is not only sound in theory but easily understood, multiplication by a fraction involving both multiplication and division.

One other feature in the general treatment of the subject will, we think, be specially helpful to the student, viz., the constantly placing in juxtaposition opposite processes. The best illustration is seen in the treatment of factoring. The various types of factoring will be found carefully classified and placed side by side with corresponding types in multiplication.

Following factoring will be found a chapter on solving equations of the second degree. This not only affords additional practice in factoring, but makes the approach to quadratic equations simple and direct. The treatment of quadratics will be found in harmony with this chapter.

In the treatment of H. C. F. and L. C. M., exercises worked by factoring are separated from those worked by division. Practically, the latter method is of little consequence as compared with the former. We commend to teachers the treatment of the general problem in Arts. 154–156.

Two short chapters on Graphs have been introduced, one at the end of simple equations and the other at the end of quadratics. Graphs in elementary algebra can be justified because of the assistance they give in understanding algebraical processes, and their practical application; the limited space given to them in this book can be justified on the same ground. This book is an Algebra, not an Analytic Geometry.

The examples have been carefully graded throughout. An effort has been made to have the examples at the beginning of each exercise illustrate individual points and to combine these in the later examples. We have by careful arrangement tried to avoid an excess of examples, the tendency of which is to create dislike for the subject. At the same time we have not lost sight of the fact that "practice makes perfect." We hope we have struck a happy mean. In general we would say that two-thirds of the examples in each exercise will be found sufficient. The more difficult ones at the end may be left for second reading.

With regard to the topics treated a word may be said. We have tried to draw a reasonable line between an elementary and an advanced course in the subject. The latter must be based upon the former. Topics like Inequalities, Imaginaries (except as required to interpret quadratic equations), Complex Numbers, Special Problems in the Theory of Quadratics, Indeterminate Forms, have been omitted. These, we believe, belong to an advanced course, as generalizations of some complexity are involved. These, with other topics, we propose to treat in a separate volume, which will also be bound up with this book for those who so desire it. We have put in a section on Cube Root, not because of its value, but because it is asked for in courses of study. We recommend its omission.

The chapters on Indices and Surds are put before Quadratics, but they are self-contained chapters and can be read later if desired. Our reason for so placing them is that they assist in understanding some problems in quadratics.

Not least amongst the features which we believe will commend this book, is the form in which it is published. For this we wish to thank the publishers.

We also desire to express our thanks to Prof. Cochrane, of the University of Manitoba, for many helpful suggestions and criticisms.

THE AUTHORS.

May, 1908.

CONTENTS

ELEMENTS OF ALGEBRA

CHAPTER I

DEFINITIONS AND NOTATION

1. Algebra. Algebra is a kind of advanced Arithmetic in which letters are employed as well as figures to represent numbers.

2. Numbers. Numbers as used in Algebra may be either **particular** or **general**; the former are represented by figures, the latter by letters.

Figures and letters used to represent numbers are called **numerical symbols.**

3. Particular Numbers. A particular number is any one definite number which may be specified. Such numbers are represented as in arithmetic by figures, since a figure, 3 for example, has always the same meaning or value.

4. General Numbers. A general number is one to which no definite value has been assigned. It is frequently some one number, at present unknown, whose precise value we wish to find. Such a number is represented by a letter, a, b, c, etc., to which we may assign any value whatever; but each letter is assumed to represent the same value throughout any one example or problem.

5. Signs. The signs $+$, $-$, $\times$, $\div$, $=$, used in Arithmetic to denote the operations of addition, subtraction, multiplication and division and the sign of equality, are used without change of meaning in Algebra.

These signs, excepting $=$, together with others to be explained, are called **symbols of operation.**

6. The sign of multiplication is usually omitted between two letters or between a figure and a letter, and the two symbols are placed side by side.

Thus ab means $a \times b$; $5abc$ means $5 \times a \times b \times c$, etc.

7. The sign for division, $\div$, is not very frequently used in algebra; the dividend is usually written over the divisor in the form of a fraction instead.

Thus $a \div b$ is usually written $\frac{a}{b}$, each expression indicating that the number denoted by a is to be divided by the number denoted by b.

8. Addends. Numbers represented by letters or by figures and connected by the sign $+$, denoting addition, are called **addends,** and the result of the addition is called the **sum.** Thus $a + b$ denotes that the numbers represented by a and b are to be added; a and b are consequently called addends; similarly x and 5 in the expression $x + 5$, are addends.

9. Factors. Two or more numbers, represented by letters or figures which are to be multiplied, are called **factors,** and the result of the multiplication is called the **product.**

Thus ab consists of two factors; $5abc$ consists of four factors, etc.

10. Addends repeated. When a number represented by a letter is to be taken two or more times as an addend, we write the letter once with a figure before it to show how many times it is to be so taken.

Thus $a+a=2a$, $a+a+a=3a$, and so on to any extent. If $a=5$, then $2a=10$, $3a=15$, etc.

11. Coefficient. A figure used to show how many times another number is to be taken as an addend, is called a **coefficient** of that number.

Thus the 2 and the 3 in the preceding Art. are coefficients of the number a.

A coefficient may also be regarded as a multiplier, and if so, it and the following number are then called factors.

When a letter, considered as an addend, has no coefficient written, 1 is always to be understood.

Where a letter is used to represent a multiplier it is called a **literal coefficient.**

12. Factors repeated. When a number represented by a letter or a figure is to be taken two or more times as a factor, we write the number once with a small figure above and to the right of it, to show how many times it is to be so taken.

Thus $a\times a=a^2$, $a\times a\times a=a^3$, and so on to any extent. If $a=5$, then $a^2=25$, $a^3=125$, etc.

13. Exponent. A number used to show how many times another is to be taken as a factor, is called an **exponent** or **index.**

When a letter, considered as a factor, has no exponent written, 1 is always to be understood.

Thus the 2 and the 3 in the preceding Art. are exponents of the number a.

14. Square and Cube. The product obtained by two equal factors is called a **square** because the area of a square is the product of the two equal factors representing two adjacent sides.

The product of three equal factors is called a **cube** because the volume of a cube is the product of the three equal factors representing three adjacent edges.

The expressions a^2 and a^3 are read "a squared" and "a cubed."

15. Expression. A collection of algebraic symbols representing numbers is called an **expression.**

Thus $3a^2$, $5ab$, $2x - 3y$, etc., are algebraic expressions.

16. Terms and Signs. The parts of an algebraic expression connected by the signs + and - are called **terms.** Each term has a **sign** and is usually composed of factors. Where no sign is written before the first term of an expression, the sign + is understood.

Thus $4x^2 - 5xy + 6y^2$ is an algebraic expression consisting of three terms; each term consists of a coefficient or numerical factor and two literal factors.

The first and third terms have the sign +, and the second has the sign - ; if the order of the terms be changed, each term must be preceded by its own sign. The preceding expression might have been written $6y^2 + 4x^2 - 5xy$ without change of meaning.

17. Like Terms. Like terms are those which differ only in their numeral coefficients.

Thus $4ax$ and $7ax$, $3b^2y$ and $5b^2y$ are pairs of like terms, but $3ax$ and $3ay$, $5a^2b$ and $7ab^2$ are pairs of unlike terms.

18. Names. An expression consisting of but one term is called a **monomial.** Expressions consisting of two or more terms are usually called **multinomials** or **polynomials.** Sometimes, however, the words **binomial** and

trinomial are used to specify expressions of two and three terms respectively.

19. Brackets. Brackets are pairs of symbols used to combine two or more separate terms into a single term, or a single factor of a term.

Thus $a + (b - c)$ is an expression consisting of two terms of which the first term is a, and the second is $(b - c)$; $x^2 - (a + b)x + ab$ is an expression of three terms each of which has two factors, $(a + b)$ being a single factor of the second term.

A second pair of brackets may be used to enclose terms, one or more of which is enclosed by a first pair, and so on to any extent. The two parts composing a pair are of the same shape. The forms ( ), } { and [ ], are those in general use.

A line, called a Vinculum, drawn over a number of terms, serves the purpose of a pair of brackets.

Thus $a - \overline{b - c}$ means the same as $a - (b - c)$.

20. Sign with brackets. A letter or a figure written beside a bracket, or two pairs of brackets written with no sign between them, indicates multiplication. Thus $3(a + b)$ indicates that the sum of a and b is to be multiplied by 3; $(a + b)(x + y)$ means that the sum of x and y is to be multiplied by the sum of a and b. Each of the preceding expressions is a monomial consisting of two factors.

21. Order of performing operations. When several operations are to be performed, it is necessary to observe the proper order in performing them.

The multiplication of the factors comprising the several terms must precede the addition or subtraction indicated by the signs between the terms.

Thus in the expression $a+bc$ two operations are ind
cated, one of addition, indicated by the sign + betw
a and b, and one of multiplication indicated by writing th
letters b and c side by side, but the multiplication mus
precede the addition.

Thus if $a=2,\ b=3,\ c=5,$ then $a+bc=2+15=17.$

If we desired to have the addition performed before th
multiplication it should be written thus $(a+b)c$, and th
result would then be $(2+3)5=25$.

Again, $3a^2$ denotes two operations, squaring the a an
multiplying the result by 3. These operations reverse
should be written $(3a)^2$.

Thus if $a=5$, then $a^2=25$ and $3a^2=75$, whilst $(3a$
$=(15)^2=225$.

22. Examples on symbols 1 and 0. Two symbols of
number, 1 and 0, deserve careful attention. As a factor,
produces no effect and may consequently be omitted, bu
as an addend it must be counted. As an addend,
produces no effect and may be omitted, but as a factor
produces 0 as product.

Ex. 1. If $a=1$, then $5a=5$, $3ab=3b$, but $a+5=6$
$a^2=1\times 1=1$, etc.

Ex. 2. If $m=0$, then $5m=0$, $mab=0$, but $a+m=a$

Ex. 3. If $a=1,\ b=2,\ c=3,\ m=0,$

then
$$\begin{aligned}a^3+a^2(b+c)+5(b^2-ac)-mb^2c^2 &= 1+1(5)+5(4-3)-0(4)(9)\\ &= 1+5+5-0\\ &= 11.\end{aligned}$$

Ex. 4.
$$\begin{aligned}b^2c^2\left(\frac{1}{a}+\frac{1}{b}-\frac{2}{c}\right) &= (4)(9)\left(\frac{1}{1}+\frac{1}{2}-\frac{2}{3}\right)\\ &= 36\left(\frac{5}{6}\right)=30.\end{aligned}$$

EXERCISE I

1. If $a = 4$, write down the values of

$$2a,\ a^2,\ 3a,\ a^3,\ 3a^2,\ (3a)^2.$$

2. If $a = 3$, $b = 5$, find the value of

$$ab,\ a^2b,\ ab^2,\ a^2b^2,\ a^2 + b^2,\ (a + b)^2.$$

3. If $a = 1$, $b = 5$, find the value of

$$a + b,\ ab,\ 2a + b,\ 2(a + b),\ 3a^2b,\ 5(b - 3a)^2.$$

4. If $a = 1$, $b = 2$, $c = 3$, find the values of

$$a + bc,\ ab + c,\ (a + b)c,\ a(b + c),\ abc.$$

If $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$, $m = 0$, find the value of

5. $3a + b - c + d$.
6. $2b + c - d + 5$.
7. $ab + bc + ca$.
8. $2bc - cd + de$.
9. $a^2 + b^2 + c^2 - ae$.
10. $abc + bcd + cde$.
11. $3b^2 + 2e^2 - b^2e^2$.
12. $b^2(c - a) + b(c - a)^2$.
13. $2(b + c)^2 - (m + e)^2$.
14. $4e^2 - \{(e - c)^2 + bd\}$.
15. $(3b - e)(bc - a^2 + m)$.
16. $(c^2 - bcd)(c^2 + d^2 - e^2)$.
17. $\left(\frac{1}{a} - \frac{1}{c}\right)\left(\frac{1}{b} - \frac{1}{d}\right)$.
18. $bcd\left(\frac{1}{b} + \frac{2}{c} - \frac{3}{d}\right)$.
19. $\frac{1}{a}(b + c) + \frac{1}{b}(c + a) + \frac{1}{c}(a + b)$.
20. $\frac{a + b}{c} + \frac{b + c}{d} - \left(\frac{a}{b} + \frac{b}{a}\right)\left(\frac{1}{a} - \frac{1}{b} - \frac{1}{a + b}\right)$.
21. $\frac{8a^2 + 3b^2}{a^2b^2} + \frac{4c^2 + 6b^2}{c^2 - b^2} - \frac{c^2 + d^2}{c^2}$.
22. $\frac{(a + b)(c + d)}{ab + cd} + \frac{(e - a)(c - b)}{b(e - c)} - \frac{(a + b + c)d}{c + d + e} + \frac{bcde}{abcd}$.

POWERS AND ROOTS

23. Powers. A **power** of a number is the product obtained by taking the given number two or more times as a factor.

Thus $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, etc., are the successive powers of 2, and are named the "square," "cube," "fourth power," "fifth power," etc.

24. Roots. A **root** of a number is one of two or more equal factors whose product is the given number.

25. Square Root. The **square root** of a number is one of two equal factors whose product is the given number and is indicated by the sign $\sqrt{\ }$ placed over the number.

Thus since $3^2 = 9$, $\sqrt{9} = 3$; $5^2 = 25$, $\sqrt{25} = 5$, etc.

26. Cube Root. The **cube root** of a number is one of three equal factors whose product is the given number and is indicated by the sign $\sqrt[3]{\ }$ placed over the number.

Thus since $2^3 = 8$, $\sqrt[3]{8} = 2$; $5^3 = 125$, $\sqrt[3]{125} = 5$, etc.

The fourth and higher roots are defined and indicated in a manner similar to the preceding, but they are not of frequent occurrence. The sign $\sqrt{\ }$ is a corruption of *r* in "radix" and is called the **Radical Sign.**

27. It will be observed that whilst the square, cube, or any power of a given number may be found by successive multiplication, but few numbers have exact roots. The various powers of the smaller numbers as indicated in the following exercise, should be written out and learned, and then the corresponding roots will be known at sight.

EXERCISE II

1. Write the squares and the cubes of all the whole numbers from 1 to 12 inclusive.

2. Of what numbers are 27, 125, 343, and 512 the cubes?

3. Write down the values of the following:

$$\sqrt{49},\ \sqrt{121},\ \sqrt[3]{216},\ \sqrt[3]{729},\ \sqrt[3]{1728}.$$

4. If $a=3$, $b=4$, $c=5$, find the values of

$$\sqrt{a^2+b^2},\ \sqrt{c^2-a^2},\ \sqrt[3]{a^3+b^3+c^3},\ \sqrt{(a+b)^2-c^2+1}.$$

5. Write out and learn the fourth powers of the whole numbers from 1 to 5; also the fifth and sixth powers of 1, 2 and 3.

6. Express 64 as a power of 4 and as a power of 2.

7. Express 81 and 729 as powers of 3, and of 9.

8. Express 625 and 256 as fourth powers.

9. If $x=2$, $y=3$, find the values of

$$x^5,\ 5^x,\ x^{y+1},\ (x+1)^y,\ (y-x)^x,\ (x+y)^{xy}.$$

10. Divide 16 into two equal addends, and into two equal factors.

11. What values of x will make

$$2x=64,\ x^2=64,\ 3x=27,\ x^3=27?$$

12. How many twos make 8 if the twos are addends? if they are factors?

13. What values of x will make

$$2x=8,\ 2^x=8,\ 3x=81,\ 3^x=81?$$

14. If $3x=27$, find the value of $4x$.

15. If $3^x=27$, find the value of 4^x.

16. If $x = 10$, $y = 12$, $a = 5$, $b = 1$, find the values of

$$(x + a)(\sqrt{3y + b}) + (y - b)(\sqrt[3]{a^2 + 2b + x})$$

and $$\sqrt{xy + b^2} - \sqrt[4]{ax + by} + \tfrac{1}{2}(a - b).$$

17. If $2s = a + b + c$, find the value of

$\sqrt{s(s - a)(s - b)(s - c)}$, when $a = 3$, $b = 4$, $c = 5$, and when $a = 5$, $b = 12$, $c = 13$.

18. If $2s = a + b + c + d$, find the value of

$\sqrt{(s - a)(s - b)(s - c)(s - d)}$, when $a = 1\frac{1}{2}$, $b = 3\frac{1}{2}$, $c = 4\frac{1}{2}$, $d = 5\frac{1}{2}$.

GENERAL NUMBERS

28. The object of the following exercise is to familiarize the learner with the representation of numbers by letters. The brevity and simplicity of this new mode of expressing numerical relations, and its power to assist in the solution of problems, will soon be evident.

Such expressions as "the sum of any two numbers," "the product of any two numbers," etc., may be very briefly expressed by algebraic symbols, using a and b to denote numbers, as shown by the following examples:

Ex. 1. The sum of any two numbers is $a + b$.

Ex. 2. The product of any two numbers is ab.

Ex. 3. The sum of the squares of any two numbers = $a^2 + b^2$.

Ex. 4. The square of the sum of any two numbers is $(a + b)^2$.

Ex. 5. The square of the difference of any two numbers is $(a - b)^2$.

The student should verify the truth of the preceding statement by substituting for a and b any numerical values, giving the larger value to a, to make the subtraction possible.

EXERCISE III

1. Write the sum and the product of x and 5.

2. How much greater is 10 than 7? 10 than x?

3. A boy is n years old: how old will he be in 2 years? in x years?

4. A father is n years older than his son; how old is he when his son is 5 years old? when his son is y years old?

5. Tom has x marbles. Dick has as many and two more. How many have both together?

6. A rectangle is a inches long and b inches wide. How many inches around it? How much greater is its length than its width?

7. A person having \$$m$ in cash buys two articles worth \$$p$ and \$$q$ respectively. Write down the number of dollars he has left if he pays for them in succession. If he pays for them together.

8. How much will n books cost at \$3 each? at \$$x$ each?

9. A man works n days at \$2 per day and p days at \$3 per day: how many dollars has he in all?

10. How many cents in \$$x$? How many dollars in x cents?

11. A man having \$$b$ in cash buys n articles worth x cents each. How many cents has he left? How many dollars?

12. How many inches in x feet and y inches? In x yards and y feet?

13. A train runs m miles per hour. How many mil will it run in 5 hours? In x hours? In p minutes?

14. How long would it take to walk m miles at 4 mi per hour? at x miles per hour?

15. A man works q hours a day for n days and hours a day for m days. How many hours does he wo and how many dollars will he receive for it at x cents pe hour?

16. A rectangle is x inches long and y inches wid How many feet in its perimeter? How many squa inches in its area and how many square feet?

17. A square is x inches on a side, and a rectangl 3 inches longer and 2 inches narrower than the squar Find the perimeter of the rectangle in inches and feet.

18. A boy is x years old and his brother is y yea old; find the sum and the difference of their ages aft 5 years, the former being the elder.

19. Write a number consisting of 7 tens and 5 unit Write one containing x tens and y units.

20. What value of x will make

$$2x + 3 = 11, \quad 5x - 2 = 53, \quad 3x^2 + 1 = 76?$$

21. A boy has x ten-cent pieces, as many quarters an 2 more; how many cents has he in all?

22. In the preceding example, if the coins mention are together worth \$4, what number does x represent?

23. The edge of a cube is x inches; find the sum of th areas of all its faces. If the sum of the areas of the fac is 54 square inches, find the value of x.

24. A block is x feet long, y feet wide and z feet thic How many cubic feet in it? How many square feet in a its faces? How many feet in the sum of the lengths of all its edges?

25. If n stands for any whole number, then $2n$ represents an even number. Why? Write two expressions containing n, each of which will be an odd number.

26. The divisor is x, the quotient is y, and the remainder is r; what is the dividend?

Express in algebraic symbols the following statements:

27. The square of the sum of any two numbers is equal to the sum of their squares together with twice their product.

28. The square of the difference of any two numbers is equal to the sum of their squares less twice their product.

29. The difference of the cubes of any two numbers, divided by the difference of the numbers, is equal to the sum of the squares of the same number together with their product.

30. The product of the sum and the difference of any two numbers is equal to the difference of their squares.

31. Verify the truth of each of the four preceding statements by substituting numerical values for each of the two letters in each.

POSITIVE AND NEGATIVE NUMBERS

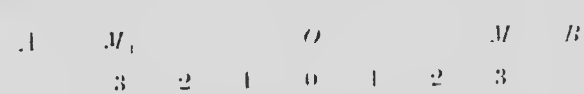
29. So far, the numbers with which we have been concerned are the ordinary numbers used in Arithmetic, or letters used to represent them. These numbers are 1, 2, 3, 4, 5, etc., counting from zero upward. These are called positive numbers and may be written with the positive sign attached to them. In order to perform the operations of algebra, however, we make use of another set of numbers, found by counting from zero in the **opposite direction.** These numbers are called negative numbers and have the sign, −, prefixed to them. The two sets of numbers when arranged in a series each differing from the adjacent ones by unity appear as follow:

$$\ldots -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3,\ +4,\ +5 \ldots$$

30. While these negative numbers have no significance in ordinary arithmetic, a few concrete examples will show how they may be interpreted in algebra.

Ex. 1. If *A* possesses $10 and by investing it makes a gain of $5, thus increasing his $10 to $15, the $5 gain may be considered a positive quantity, as it **increases** the amount possessed by *A*. If, on the other hand, he loses $5 by the investment, thus diminishing his $10 to $5, the $5 loss may be considered a negative quantity, as it **decreases** the amount possessed by *A*. The $5 loss acts upon the original $10 in the **opposite direction** to the $5 gain. If, therefore, gain be considered positive, loss may be considered negative.

Ex. 2. If we desire to measure a distance of 3 inches along the line *AB*, from a fixed point *O*, the zero point for measurement,



we may measure in the direction *OB* or the **opposite direction** *OA*. To distinguish between the two, we may consider the direction *OB* positive, and the direction *OA* negative. *OM* is thus equal to +3 inches and OM_1 to −3 inches.

31. From the examples of the preceding Art. it will be observed that in the complete representation of concrete quantities by numbers, three elements must be clearly specified: the unit, the number of units, and the mode or direction of measurement.

Thus 5 dollars gain, 3 inches to the right, are examples of concrete quantities accurately specified. The units are "dollars" and inches; the numbers of units are 3 and 5; and the directions of measurement are indicated by the phrases, "gain" and "to the right."

Conversely if a dollar gain be the unit, $+5$ means 5 dollars gain, -3 means 3 dollars loss; if a mile to the north is the unit, $+7$ means 7 miles north, -4 means 4 miles south. Thus concrete quantities may always be represented by numbers, and numbers (taken with a unit of measurement) represent concrete quantities.

32. When performing the operations of algebra we may think of the numbers only, without reference to the concrete units, dollar, inch, and we have the series of numbers given above, viz.:

$$\ldots\ 5,\ -4,\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3,\ +4,\ +5\ldots$$

In this series observe

1. We can start at any point and count without limit in either direction, whilst with purely arithmetical numbers we must stop at 0.

2. The direction in which the positive numbers increase is the positive direction, and the opposite is the negative direction. From -5 to -4, -3, etc., the direction is positive; similarly from $+5$ to $+4$, $+3$, etc., the direction is negative.

3. When magnitude alone is considered, a negative

number and the quantity it represents are exactly equal to the corresponding positive number and quantity it represents: -3 is the same number of units as $+3$, and three dollars loss is the same amount of money as 3 dollars gain.

4. When direction alone is considered, it is customary to say that the numbers increase when proceeding in the positive direction from whatever point in the series we begin. This is equivalent to assuming that algebraically -5 is less than -4 and that any negative number whatsoever is less than zero. "Less than nothing" with reference to magnitude is absurd; with reference to position in the series of numbers the meaning is clear.

Ex. If the line *A*, one inch long, drawn to the right, be the unit, then the line *B*, 2 inches long, drawn in the same direction, is denoted by $+2$, and the line *C*, 3 inches long, drawn in the *opposite* direction, is denoted by -3. Had the unit been one inch, drawn to the left, then the lines *B* and *C* would have been represented by -2 and $+3$, respectively.

EXERCISE IV

1. If a line two inches long, drawn from left to right, be the unit, what numbers will represent 3 ft. to the right? 5 yd. to the left?

2. If the unit be 3 ft. to the north, what will be represented by $+10$, -5, $-2\frac{1}{2}$, $+\frac{1}{3}$?

3. What is the unit of measurement when a tree 50 ft. high is represented by $+10$? by -5?

4. If a dollar gain be the unit, what will be represented by $+2\frac{1}{2}$, $-3\frac{1}{4}$?

5. In the preceding example what will represent a loss of \$2.75? a gain of \$3.40? \$4.12½ cash in hand? a debt of \$2.50?

6. If one day forward be the unit of time, what number would refer to *yesterday?* The day after to-morrow?

7. If the letter *a* represents a line of any given length drawn to the right, what would represent a line twice as long drawn to the left? three times as long drawn to the right?

8. When a dollar gain is the unit, what will represent the sum of \$3 debt and \$7 debt? \$5 cash and \$2 debt? \$7 loss and \$3 gain?

9. What must be added to \$5 debt to produce \$7 cash? to \$3 cash to produce \$2 debt?

Represent each of these sums of money by the appropriate number, a dollar cash being the unit.

10. Berlin is 13½ degrees east longitude and St. Petersburg 30½ degrees east. What will represent in degrees a journey from Berlin to St. Petersburg? St. Petersburg to London?

11. A toy balloon can just sustain a weight of six ounces. If attached to a paper basket weighing 4 oz., what would the two combined *weigh?* What weight must be attached to the balloon to make the two weigh 8 oz.? 4 oz.?

12. What change takes place in the number representing a concrete quantity when the unit is doubled? halved? direction changed?

CHAPTER II

ADDITION AND SUBTRACTION

ADDITION

33. Meaning of addition. The operations of addition and subtraction are employed in a more extended sense in Algebra than in Arithmetic, inasmuch as algebra employs two sets of numbers, the positive and the negative, whilst arithmetic employs but one. The fundamental ideas, however, in the two cases are the same, that of addition being the operation of collecting into one number that which formerly existed as two or more separate numbers, whilst subtraction is simply the process of addition reversed. This will be evident from a few simple concrete examples.

Ex. 1. A gain of \$5 in one business transaction, followed by a loss of \$2 in a second transaction, gives a net result of \$3 gain on the whole. The process of combining these two separate items into one is addition, and may be expressed thus:

$$\$5 \text{ gain} + \$2 \text{ loss} = \$3 \text{ gain}.$$

Ex. 2. A journey of five miles north, followed by a journey of 7 miles south, leaves the traveller 2 miles south from the starting point. The addition of the two journeys may be thus expressed:

$$5 \text{ mi. N.} + 7 \text{ mi. S.} = 2 \text{ mi. S.},$$

i.e., we have found a single journey equivalent in result to two specified journeys.

34. Representation of addition. If now we replace the words "gain" and "loss," "miles north" and "miles south" by the signs + and −, these examples of addition may be conveniently expressed thus:

$$(+5)+(-2) = +3$$

$$(+5)+(-7) = -2.$$

The signs + and − when used as above to denote positive and negative quantities are enclosed in brackets, with the numbers to which they refer, to distinguish them from their use to denote the operations of addition and subtraction.

When the quantities to be added are both positive or both negative, the process is the same as in arithmetic, thus:

$5 gain + $3 gain = $8 gain.

$5 loss + $3 loss = $8 loss,

which, expressed in algebraic symbols, becomes

$$(+5)+(+3) = +8$$

$$(-5)+(-3) = -8.$$

35. Mode of addition. The truth of the preceding additions is readily perceived when the concrete units "gain," "loss," etc., are expressed. When only the signs + and − are given we proceed thus:

To add +5 and −2, we find +5 on the scale of numbers, Art. 32, and from it count 2 units in the *negative* direction, as indicated by −2; the result is +3.

To add +5 and −7, begin with +5 and count 7 units in the negative direction, the end of the counting being −2, which is the sum required.

Similarly any algebraic numbers may be added.

From these special examples the definition of the following Art. will be readily understood.

36. Definition. The **sum** of two or more numbers is the number obtained by counting in succession the number of units in the several addends, each in the direction indicated by its sign, beginning the count from zero. The process of finding the sum is **addition**.

37. Addends may be taken in any order. If in the examples of Art. 34 we reverse the order of the addends, we get

$$(-2)+(+5)=+3$$
$$(-7)+(+5)=-2$$

as before. For beginning with -2, and counting 5 units in the positive direction, the end of the count is $+3$; similarly the same result is obtained with any pair of numbers with whichever one we begin. If, then, we have several addends, it will be easiest to combine all the positive numbers and all the negative numbers separately, which is merely arithmetical addition, and then combine the two results.

38. Rule for addition. *To add two or more numbers having* **like signs,** *add the numbers as in arithmetic, and prefix the common sign.*

To add two numbers having **unlike signs,** *take their difference as in arithmetic, and prefix the sign of the greater.*

39. Positive and negative numbers may be used as coefficients of literal expressions, with meanings derived from the definition of Art. 29.

Thus if a denotes any number, $2a$ means twice as large a number, counted in the same direction, on the scale of numbers, as a; but $-3a$ means three times as large a number, counted in the opposite direction.

The sum of any positive number and the same number taken negatively, is evidently zero.

40. If in the examples of Art. 33 we represent a "dollar gain" or "a mile to the north" by the letter a, we get by the reasoning previously given

$$(+5a)+(-2a)=+3a,\quad (+5a)+(-7a)=-2a.$$
$$(+5a)+(+3a)=+8a,\quad (-5a)+(-3a)=-8a.$$

from which we derive the rule for adding like terms.

41. Addition of terms. To add **like terms** we take the algebraical sum of the coefficients and affix the common literal factors. These literal factors may be single letters, as above, or expressions in brackets.

Thus $\qquad 5(a+b)+3(a+b)=8(a+b)$, etc.

Unlike terms can be added only by writing them with the sign of addition between them.

Thus the sum of $2a$ and $3b$ is $2a+3b$. The sum of $2x$ and $-3y$ is $2x+(-3y)=2x-3y$. These sums cannot be expressed as single terms.

42. Omission of brackets. In practice, the brackets used to distinguish the use of + and − as signs of positive and negative number, from their use as signs of addition and subtraction, are usually omitted, and the four examples of algebraical addition in Art. 40 become

1. $5-2=3$. $\qquad$ **2.** $5-7=-2$.

3. $5+3=8$. $\qquad$ **4.** $-5-3=-8$.

When written thus the first example becomes the same as an example in subtraction, which shows that to subtract 2 from 5 is the same as to add −2. The second example, though written like an example in subtraction, cannot be so considered; for 7 is greater than 5 and cannot be taken from it. This shows that the addition of a negative number, according to the definition given, is possible even when the corresponding arithmetical subtraction is impossible.

43. The addition of Algebra appears in some cases to contradict well-known facts of Arithmetic, but it is only in appearance. In Ex. 2, Art. 33, a journey of 5 miles followed by a journey of 7 miles is said to be equivalent to a single journey of 2 miles. In Arithmetic we should say the two journeys are together equivalent to one of 12 miles. Both statements are equally true. The *algebraical addition gives the position of a traveller at the end of his journey; the arithmetical addition gives the distance he has travelled.*

44. The learner should become familiar with addends written in either of the following forms:

Ex. 1. Simplify $(-3a) + (+2a) + (-5a) + (4a)$.

Combine the positive and negative coefficients separately and then combine the results.

Thus $2 + 4 = 6$, $-3 - 5 = -8$, then $6 - 8 = -2$.

The given expression thus becomes $-2a$.

Ex. 2. $3a + 5b - 2b + 4a - 7b - a$

$$= (3 + 4 - 1)a + (5 - 2 - 7)b = 6a - 4b.$$

EXERCISE V

Add the following:

1.	$+3$	-3	$+3$	-3	$+7$	-7
	$+5$	-5	-5	$+5$	-10	$+10$
2.	$+14$	-14	$+14$	-14	$+14$	-14
	$+16$	-16	-16	$+16$	$+16$	-16
	$+20$	-20	$+20$	-20	-20	$+20$
3.	$3a$	$5b$	$7a^2$	$12ab$	$11xy$	$3m$
	$5a$	$6b$	$3a^2$	$4ab$	$2xy$	$-8m$
	$-7a$	$10b$	$8a^2$	$2ab$	$15xy$	$12m$

4. $4x^2 + (-3x^2) + (-7x^2) + (+11x^2) + (-x^2)$.

5. $(-3x^2) + (-5x^2) + (-3y^2) + (+7y^2) + (+x^2) + (-y^2)$.

6. $3m^2 - 5n^2 - 6n^2 - 8m^2 + n^2 + m^2 - 2m^2$.

7. $-2ab - 5ab + 3ab + 7ab + ab - 6ab - ab$.

8. $5a^2b + 3ab^2 - 6ab^2 - 7a^2b + a^2b - ab^2$.

9. $3(a+b) + 5(a+b) - 7(a+b) - 6(a+b) - (a+b)$.

10. If $a = 2$, $b = -3$, $c = -7$, find the value of

$$a+b+c, \quad a+b-c, \quad b-c-a.$$

11. A traveller takes three successive journeys, 10 miles east, 25 miles west and 5 miles east. Add his journeys both algebraically and arithmetically and give the meaning of the result in each case.

ADDITION OF POLYNOMIALS

45. When two or more polynomials are to be added, it is convenient to arrange the terms in columns, so that like terms shall stand in the same column. When a term is moved to a different position its sign must be taken with it. The sign of a first term may be omitted when positive, but if another term be placed before it, the sign must be restored. The columns should be added in succession, beginning at the left.

Ex. 1.

$$\begin{array}{r} 2a + 3b - 4c \\ 4a - 5b + 2c \\ -3a + b - c \\ 5a + 2b + 3c \\ \hline 8a + b \end{array}$$

Ex. 2.

$$\begin{array}{rrrr} 4x^2 - xy + 2y^2 & & \\ 3x^2 + 5xy - 7y^2 & & \\ -x^2 - xy + 3y^2 & & \\ -6x^2 \qquad + y^2 & - z^2 \\ \hline 3xy - y^2 & - z^2 \end{array}$$

Ex. 3. Find the sum of $3(a+2b-c)$ and $4(2a-b+3c)$.

From Art. 11. $3(a+2b-c) = 3a + 6b - 3c$

$4(2a-b+3c) = 8a - 4b + 12c$

Their sum $= 11a + 2b + 9c$.

EXERCISE VI

Add:

1. $3a + b - c$, $4a - 2b + 3c$, $-a + 5b - 6c$.
2. $5a - b + 2c$, $3b + 4c - 2a$, $5c - a + 2b$.
3. $6a + 2b - 5c$, $4a + 5c - 3b$, $2c - a - 4b$.
4. $7a - 4b - 3c$, $a + b + x$, $b - c - 5x$.
5. $4ab - ac + bc$, $bc - 4ab - ac$, $4ac + 5ab$.
6. $x^2 - ax - 2bx$, $2cx - 2x^2 + ax - 2bx$, $bx - cx - ax + x$
7. $a + b - 2c$, $b + c - 2d$, $c + d - 2a$, $d + a - 2b$.
8. $3(a + b)$, $5(a + b)$, $2(a + b)$, $a + b$.
9. $a + b - c$, $2(a + b - c)$, $-(a + b - c)$, $-(a + b - c)$.
10. $5(a^2 + b) + 2c$, $3(a^2 + b) - 7c + d$, $2c - 4d$, $3d - 6(a^2 + b)$, $3c + 7(a^2 + b)$.
11. $4a(b + c) - 5d$, $a(b + c) + 7d$, $-3a(b + c)$, $3d$, $2d - 5e$, $6e - 2a(b + c)$, $6a(b + c) - 7d + e$.
12. $7a - 3b + 5c - 10d$, $2b - 3c + d - 4e$, $5c - 6a - 4e + 2d$, $-3b - 8c + 7a - e$, $21e - 16c + a - 5d$.
13. $3(a^2 + b^2) + 2ab$, $a^2 - 5ab + b^2$, $10ab - 5(a^2 + b^2)$, $3a^2 + 6ab + 3b^2$, $a^2 + b^2$.
14. $a^3 - 3a^2b + 2ab^2$, $b^3 - 3ab^2 + 5a^2b$, $2a^2b + 5ab^2$, $a^3 + b^3 + 2a^2b - 5ab^2$, $7a^2b - a^3 - 2b^3 - 5a$, $3a^2b - 2a^3 - ab^2 + a^2b$.
15. $a^4 + a^3b - 2a^2b^2$, $3a^3b - 5a^2b^2 - 6b^4$, $ab^3 - 3a^2b^2$, $2a^2b^2 - 5a^3b + a^4 + b^4$, $4ab^3 - 2b^4 + 3a^2b^2$.
16. $3(x + y + z)$, $4(x - y + z)$, $5(x - y - z)$, $3x - y + z$.
17. $4(2x - y + 3z)$, $5(y - 2z - x)$, $7(3z - x - 2y)$.
18. If $x = a + 2b + 3c$, $y = b + 2c - 3a$, $z = c - 2a + 3b$, find the value of $x + y + z$.
19. In the preceding example, find the value of $x + 2y + 3z$.

SUBTRACTION

46. The meaning of Subtraction in Algebra follows directly from the meaning assigned to addition. In addition two addends are given and their sum is to be found. In subtraction the sum and one addend are given and the remaining addend is to be found. Thus to subtract 7 from 10 means that we are to find the number 3, which must be added to 7 to make 10. In arithmetic we cannot subtract 10 from 7, because there is no arithmetical number which being added to 10 will make 7. The negative number −3 when added to 10 makes 7; we therefore say that 10 from 7 leaves −3.

47. Definitions. When the sum of two numbers and one of them are given, subtraction is the process by which the other is found. The sum of the two numbers is called the **Minuend,** the given number is called the **Subtrahend** and the number to be found is called the **Difference.**

The difference is, therefore, the number which must be added to the subtrahend to make it equal to the minuend.

48. Let it be required to perform the following subtractions:

From	7	−3	−1	2	0
Take	−2	−5	4	5	−5
Result	9	2	−5	−3	5

To find what must be added to the subtrahend to make the minuend in each case we reason as follows:

7 is the sum of two addends, one of which, −2, is to be removed; −2 will be cancelled, or removed, by adding ... making 9, the other addend required. The sum of −2 and 9 is 7, which proves the work correct.

Similarly −5 is one of two addends which together make −3; to −3 add +5, thus cancelling the given addend, and we obtain +2, the remaining addend or difference required.

49. Rule for subtraction. ***To subtract one number from another, change the sign of the subtrahend and add it to the minuend.***

50. The truth of the several subtractions of Art. 48 will be evident to the eye by reference to the scale of positive and negative numbers, Art. 32.

Thus from

−2 to	+7 is	9 units in the	positive	direction	= 9
−5 "	−3 "	2 "	"	"	= 2
4 "	−1 "	5 "	negative	"	= −5
5 "	2 "	3 "	"	"	= −3
−5 "	0 "	5 "	positive	"	= 5

which are the results formerly obtained.

51. The truth of the rule given for subtraction is also readily perceived by observing that the distance between any two numbers on the scale is not changed by adding the same to each of the given numbers. If, then, we add to each the subtrahend with its sign changed, the new subtrahend is 0, and the new minuend is therefore the result required. The operation here described is precisely the rule given.

52. The subtraction of like terms follows immediately from the subtraction of positive and negative numbers.

Thus

From	$5x$	$3a^2b$	$-2(a+b)$	$3a$
Take	$2x$	$-2a^2b$	$-4(a+b)$	$-5b$
Result	$3x$	$5a^2b$	$2(a+b)$	$3a+5b$

53. Subtraction of terms. Like terms *are subtracted by the algebraical subtraction of their coefficients and annexing their common literal factors.*

Unlike terms *can be subtracted only by connecting the terms by the proper signs.*

54. The double use of the sign −, to denote both a negative number and the operation of subtraction, is somewhat confusing to a beginner. This is especially the case when a change is made from one meaning to the other in the same example. The two meanings, however, lead always to the same result, and are in fact only two ways of thinking and speaking of the same facts. In this connection it is important to observe the truth of the three following statements:

1. The subtraction of a number of positive units is equivalent to the addition of the same number of negative units.

2. The subtraction of a number of negative units is equivalent to the addition of the same number of positive units.

3. The negative of a negative is positive. Expressed in symbols these statements become:

1. $a-(+b)=a+(-b)=a-b.$

2. $a-(-b)=a+(+b)=a+b.$

3. $-(-b)=+b.$

55. Illustrations of the preceding, from actual experience, are numerous:

1. A decrease in a man's income produces the same effect as an equivalent increase in his expenses.

2. A decrease in expenses is equivalent to a corresponding increase in income.

3. The negative of income is expense, the negative of expense is income, *i.e.*, the negative of a negative is positive.

The student should carefully study the theory of algebraic numbers and the illustrations as here given, but when working examples *he should think of nothing but the Rule.*

EXERCISE VII

Subtract

1.	5	-3	-5	-3	10
	-3	5	3	-5	7
2.	-8	12	0	-7	-11
	1	2	3	0	-7
3.	$4x$	0	$14a^2$	$-2ab$	$3a$
	$-6x$	y	$5a^2$	$7ab$	$5b$

4. $7ax-(+3ax)$, $2by-(-5by)$, $-a-(-5a)$.

5. $5x^2-7x^2$, $-2y^2-8y^2$, $-3xy+5xy$.

6. $3a^2-11a^2+7a^2-9a^2$, $5x^3+(-2x^3)-(+4x^3)-(-x^3)$.

7. $4a-(-2b)+8b-(+7a)-6a-5b$.

8. If $a=3$, $b=5$, $c=7$, find the value of

$$a-b+c,\quad a-(b+c),\quad (a+b+c).$$

9. If $a=2$, $b=4$, $c=-7$, find the value of

$$a-b,\quad b-c,\quad c-a,\quad a-(b-c).$$

10. Simplify

$$4(x-y)^2-5(x-y)^2+3(x-y)^2-11(x-y)^2-(x-y)^2.$$

11. Toronto is 44 degrees north latitude and Rio Janeiro is 23 degrees south latitude. Find by algebraical subtraction the number of degrees Toronto is north of Rio Janeiro.

56. Subtraction of polynomials. When the subtrahend contains two or more terms, the subtraction is performed by subtracting each term in succession, i.e. by changing the sign of each term of the subtrahend and adding it to the minuend. Like terms should be placed under each other as in addition.

Ex. 1.

From	$4a^3 - 3a^2b + 7ab^2 - b^3$
Take	$a^3 - 5a^2b + 8ab^2 + 2b^3$
Result	$3a^3 + 2a^2b - ab^2 - 3b^3$

The signs of the terms in the lower line should be changed in thought only and added to those above. Thus we say -1 and 4 make 3; $+5$ and -3 make 2; -8 and 7 make -1; -2 and -1 make -3, thus giving the required coefficients to which the literal factors are to be affixed.

Ex. 2. What must be added to $a^2 - b^2 - c^2 + 2ab - ac$, to make $a^2 + ab + bc + ac$?

The first expression is evidently the subtrahend; write it below the other, like terms under each other so far as possible.

From	$a^2 + ab + bc + ca$
Take	$a^2 - b^2 - c^2 + 2ab - ac$
Result	$b^2 + c^2 - ab + bc + 2ac$

EXERCISE VIII

1. From $4a - 3b + c$ take $2a + b - 3c$.
2. From $3a + 2b - 5c - 6d$ take $4a - 2b - 5c - 7d$.
3. From $-a - b + 2c$ take $b - c - a + x$.
4. From $3x^3 - 5x^2y + y^3$ take $2x^2y - 3xy^2 - x^3 + y^3$.
5. From $1 - 2x + 3x^2 - 5x^3$ take $x^3 + 3x^2 - 5x - 1$.
6. From $a^2 - b^2 - c^2 + 2bc$ take $b^2 - c^2 - a^2 + 2ac$.

7. From $4x^2 - 3xy + 7y^2 - 5xz + 6yz - z^2$
take $x^2 - y^2 - z^2 + xy - 7xz + 7yz$.

8. From $x^2 + y^2 - z^2 + 2xy - 3xz + yz$
take the sum of $yz - x^2$, $xy - z^2$, and $y^2 - 3xz$.

9. From the sum of $2x^3 - 3x^2y + y^3$ and $2xy^2 - x^3 - 4y$
take the sum of $x^2y + 2xy^2$ and $x^3 + y^3 - 3x^2y$.

10. What must be added to $a^2 + b^2 - c^2 + 2ab - ac - bc$
to make $ab + bc + ca$?

11. What must be subtracted from $1 - a + b + a^2b + ab$
to leave $a^2b - b + c + 1$?

12. What must be subtracted from the sum of
$4x^3 + 3x^2y - y^3$, $4x^2y - 3x^3$ and $7x^2y + 9y^3 - 2xy$
to leave $2x^3 - 3x^2y + y^3$?

13. From $5(a - b) + 6(x - y)$ take $2(a - b) - 7(x - y)$.

14. From $3(a + b) - 4(c + d) - 5(x - y) + p$
take $a + b - 5(c + d) - 7(x - y) - q$.

15. From $3(a + b - c) + 5(a - b + c) + 3(a - b - c)$
take $2a - 3b + 2(b + c - a)$.

16. From $a - b + 2c - 3d$ take the sum of

$2a + 3b - c + 4e$ $\quad$ $b + 3c - 4d + 5e$

$2c + d - e - 5a$ $\quad$ $d - 2e + 3a - b$.

Verify the result by subtracting in succession each expression separately.

17. From $9a - 4b - 17c - 12d + 12e$ take the sum of

$7a - 3b + 5c - 10d$ $\quad$ $2b - 3c + d - 4e$

$5c - 6a - 4e + 2d$ $\quad$ $-3b - 8c + 7a - e$

and verify result as in preceding example.

18. If $x = 2a + b - c$, $y = 3b + c - a$, $z = c + a - b$,
find the value of $2x - 3y - 4z$.

BRACKETS

57. Removal of brackets. The sign + preceding a bracket, indicates that the terms contained are to be added to what precedes. Now terms are added by connecting them in succession, each preceded by its own sign. Therefore

A bracket preceded by the sign + may be removed, each term retaining its sign unchanged.

58. The sign − preceding a bracket, indicates that the terms contained are to be subtracted from what precedes. Now subtraction requires that the sign of each term subtracted be changed and the result added. Therefore

A bracket preceded by the sign − may be removed, providing the sign of every term within be changed.

59. The truth of each of the following equalities should be carefully considered and verified by assuming a suitable value for each letter.

1. $a+(b+c)=a+b+c.$
2. $a+(b-c)=a+b-c.$
3. $a-(b+c)=a-b-c.$
4. $a-(b-c)=a-b+c.$
5. $a+(-b+c)=a-b+c.$
6. $a-(-b+c)=a+b-c.$

Observe that each term within a bracket has its own sign, but the term + is omitted before the first term inside a bracket. The sign preceding a bracket belongs to the expression as a whole, and is no longer needed when the bracket is removed.

60. Two or more pairs of brackets are frequently used in the same expression, one pair enclosing a portion of the terms enclosed by another pair. In such cases the two parts forming one pair must be carefully observed. It is the simplest to remove them one pair at a time, taking always the innermost. A little experience, however, will enable the student to take them in any order, and to remove several pairs at one operation. At each step like terms should be combined to save labor in writing.

Ex. 1. $a - \{b - (c - a) + (b - c) - a\}$

$$= a - \{b - c + a + b - c - a\}$$
$$= a - 2b + 2c.$$

Ex. 2. $x - [x - a - (2a - 2x) + \{a - (a - x)\}]$

$$= x - x + a + (2a - 2x) - \{a - (a - x)\}$$
$$= a + 2a - 2x - a + (a - x)$$
$$= 2a - 2x + a - x$$
$$= 3a - 3x.$$

In Ex. 2 the outer brackets were removed each time, and in both examples like terms were combined after the removal of each pair.

61. The rules for the insertion of brackets follow immediately from the rules for their removal.

Thus $a - b + c - d - e + f = (a - b) + (c - d) - (e - f)$

$$= a - (b - c) - (d + e) + f$$
$$= a + (-b + c) + (-d - e + f)$$
$$= \{a - (b - c) - (d + e) + f\}, \text{ etc.}$$

It will be observed that a term placed in a bracket preceded by the sign + retains its own sign, but when the sign − precedes the bracket, the sign of each term is changed. The identity of the above expressions should be verified by removing the brackets from each of them.

EXERCISE IX

Remove the brackets from the following expressions and combine like terms.

1. $(a + b) + (a - b)$.
2. $(a + b) - (a - b)$.
3. $(a + b - c) - (b + c - a)$.
4. $a - (b - c) + (c - a)$.

5. $2a - c - (b + c - a) + (-c + 2b + 3a)$.

6. $3a - (b - 2c) - \{c - (a - b) + (a - c)\} - (2b - 2c)$.

7. $(2a - 3b) - (-b + c) + \{a - (b + c - a) + (c - 2a) - b\}$.

8. $x - \{a - 1 - (x - a + 1) + 2 - (a + 3)\} - (-3a + x)$.

9. $2x + [-x + a - (2a + 2x) + \{a - (a - x) + x - 2a\} - a]$.

10. $-\{(x - 2y) - (x + 3)\} - \{2 - (x - 3y) + 2x\}$
$- \{3 - (x - y)\}$.

11. $a - [2b + \{a - 2b - (a - b + c) + b\} - a - (b - c)]$.

12. $x - [y - \{z - (x - y) + z\} - (x - y + z)]$.

13. $\{(3a - 2b) + (2c - a)\} - \{a - (b - 2a) - c\}$
$+ \{a - (b + c)\}$.

14. $a - [b - (a - b) - \{a - (b - a) - b\}$
$- \{a - (b - a - b - a)\}]$.

15. $a - [b - \{a - (b - a - b - a) - b\} - a]$.

16. Arrange the terms of $a - b + c + d - e - f$ in alphabetical order in brackets; two terms in each pair of brackets; three terms in each pair.

17. In the same expression, place b, c, d in one pair and e, f in another pair, with the sign $-$ in front of each pair; the sign $+$ in front of each pair.

18. In the same expression enclose b and c, d and e, in small brackets, and then enclose these groups with f in an outer pair.

19. Verify the work of the preceding example by first inserting the outer pair of brackets and then inserting the two inner pairs.

20. Add $a - \{b - (c + d) + e\}$, $a - \{b + (c - d) - e\}$,
$a - [b - \{c - (d - e)\}]$ and $-\{(a - b) - (c - d)\} - e$,
and from their sum take $a - \{b - (c + a) + b\} + c$.

CHAPTER III

MULTIPLICATION AND DIVISION

MULTIPLICATION

62. In Arithmetic, when one number is multiplied by another, the former is called the **Multiplicand** and the latter the **Multiplier.** The result of the multiplication is called the **Product.**

The same terms are used in Algebra.

63. In Aritimetic the **process** of multiplication is defined as follows:

One number is multiplied by another when the former is used as an addend as many times as the number indicated by the latter.

For example, $5 \times 3 = 5 + 5 + 5 = 15.$

We have simply to extend this definition to include negative numbers, to define the process of multiplication in Algebra.

64. The use of negative numbers gives rise to three new cases of multiplication, each of which must be clearly understood:

I. A **negative** multiplicand with a **positive** multiplier.

II. A **positive** multiplicand with a **negative** multiplier.

III. A **negative** multiplicand with a **negative** multiplier

I. The first of these is easily understood.

For example, $(-5) \times 3 = (-5) + (-5) + (-5) = -15.$

Here the sign, −, presents no difficulty: it merely shows the direction in which the 5 is counted. Counting from zero to −5 on the scale of numbers and repeating the counting in the same direction three times brings you to −15.

To take a concrete example: If a man travels 5 miles in the direction selected as negative, then continues 5 miles further, and again continues 5 miles further, he will finally be 15 miles in a negative direction from his starting point.

The distinction, therefore, between 5×3 and $(-5) \times 3$ is simply that the product of 5 and 3 in the one case is counted in the positive direction, and in the other case in the negative direction.

Hence $$(-5) \times 3 = -(5 \times 3) = -15. \qquad (2)$$

11. In performing a multiplication by a negative multiplier we have only to keep in mind the fact stated in the definition of a negative number, namely, that the presence of the minus sign changes the direction of counting, and the meaning of the process is quite clear.

For example, $5 \times (-3)$ means that 5 is to be multiplied by 3 and the sign of the product changed, that is, the product, 15, is to be counted in the direction opposite to that in which the multiplicand, 5, is counted.

Hence $$5 \times (-3) = -(5 \times 3) = -15. \qquad (3)$$

The same result is at once apparent if we assume the arithmetical law, that the multiplier and multiplicand can be interchanged without changing the product.

Thus $$5 \times (-3) = (-3) \times 5,$$
then from (2) $$= -(3 \times 5)$$
$$= -15.$$

Hence the rule: *To multiply a positive number by a negative number perform the ordinary arithmetical multiplication without regard to sign and give the negative sign to the product.*

III. The same principle applies when a negative multiplicand is to be multiplied by a negative multiplier.

For example, $(-5)\times(-3)$ means that the multiplicand (-5) is to be multiplied by 3, giving -15, see (2), and then the direction of counting is to be changed, making the product 15.

This may be stated thus. $(-5)\times(-3) = -(-5)\times 3$

$$= -(-15)$$

$$= +15. \qquad (4)$$

65. The signs of the products in the four examples of the preceding Art. do not in any way depend upon the numerical value of the particular multiplicands or multipliers used, but upon their signs alone. We have, therefore, for any numbers whatsoever,

1. $(+a)\times(+b) = +ab$. **2.** $(-a)\times(+b) = -ab$.

3. $(+a)\times(-b) = -ab$. **4.** $(-a)\times(-b) = +ab$.

That is, the sign of the product of two numbers is + when both numbers have the same sign, and − when they have different signs. More briefly expressed, this becomes what is known as the

Rule of Signs. *Like signs give* +, *unlike signs give* −.

The sign of the product of three or more numbers may be obtained by a repeated use of this rule. Thus:

1. The product of any number of **positive** factors is **positive.**

2. The product of any *even* number of **negative** factors is **positive.**

3. The product of any *odd* number of **negative** factors is **negative.**

4. If the sign of one factor be changed the sign of the product is changed.

66. The student should observe that while the multiplicand may be either a concrete quantity or a simple number, *the multiplier must always be abstract.* It is simply a number used to count addends in one of two directions of measurement.

67. Factors may be taken in any order without change of product.

Thus $2 \times 3 \times 5 = 2 \times 5 \times 3 = 5 \times 3 \times 2 =$, etc., $= 30$

Similarly $abc = bca = cab =$, etc.

This principle enables us to combine the numerical factors from two different expressions whose product is to be found.

Thus $3a \times 4b = 3 \times a \times 4 \times b = 3 \times 4 \times a \times b = 12ab.$

When a figure and one or more letters are factors it is customary to place the figure first, and the letters in the order of the alphabet, as in the preceding example.

68. The product of two powers of the same number is formed as follows:

Since $a^2 = aa$ and $a^3 = aaa$,

we have $a^2 \times a^3 = aa \times aaa = aaaaa = a^5.$

Similarly $a^m \times a^n = a^{m+n}$, $a^m \times a^n \times a^p = a^{m+n+p}$, etc., where m, n, p are any positive whole numbers.

The exponent of a letter in a product is equal to the sum of the exponents of that letter in the factors multiplied.

69. The preceding rules enable us to immediately write the product of any number of monomials.

Ex. 1. $-3ab \times 5bc = -15ab^2c.$

Ex. 2. $4a^2b \times -3bc \times -5ac = 60a^3b^2c^2.$

Ex. 3. $-3x^2 \times 2xy \times -4ax \times -5by = -120abx^4y^2.$

In forming such products, three things require attention:

1. **The Sign.** This is + for positive factors and for an even number of negative factors; − for an odd number of negative factors.

2. **The Coefficient.** This is the product of the numerical factors formed as in arithmetic without regard to signs.

3. **The Literal Factors.** These consist of all the letters which occur, each letter having for exponent the sum of its exponents in the several expressions to be multiplied.

EXERCISE X

Multiply

1.	3	5	-3	-11	-8
	-5	-3	-5	7	5
2.	$3x$	$-2ab$	$4xy$	$-3a^2b$	$5a$ [illegible]
	$-5x$	$3ac$	$-5x$	$-2ab^2$	a^2b [illegible]
3.	x^3y	$13x^2yz$	$-x^2y^3$	$8m^2n$	$7a\ b$ [illegible]
	$-y$	$7y^2z$	$-xy$	$-3nx$	$-5abc$ [illegible]

4. $2a^2bc \times \ 3ab^2c \times 5abc^2.$

5. $5ab \times -3bx \times 2ax \times -4abx.$

6. Find the values of $(-5)^2$, $(-1)^2 \times 8$, $(-3)^3 \times -2$.

7. Find the values of $2^3 + (-2)^3$, $2^4 + (-2)^4$.

8. Simplify $(3-4)(-5+2) - 2^2$, $(-1)^3(5-7) - 3(-1$ [illegible]

9. Simplify $(3a^3)^2$, $(2a^2)^3$, $(-2ab^2)^4$, $(-3a^2b^3)$[illegible].

10. If $a = 2$, $b = -5$, find the values of

$$a + b,\ a - b,\ ab,\ ab(a - b),\ a^2 + b^2.$$

11. If $a = 2$, $b = -3$, $c = 5$, find the value of

$$ab(a - b) + bc(b - c) + ca(c - a).$$

MULTIPLICATION OF A POLYNOMIAL BY A MONOMIAL

70. If either of two factors be multiplied by any number their product will be multiplied by the same number.

Thus $3 \times 5 = 15$; $6 \times 5 = 30$ and $3 \times 10 = 30$;

that is, when either the 3 or the 5 is doubled their product 15 is also doubled.

71. If both of two addends be multiplied by any number their sum will also be multiplied by the same number.

Thus $3 + 5 = 8$; $6 + 10 = 16$;

that is, when both the 3 and the 5 are doubled their sum, 8, is also doubled.

72. The very important principles of Arts. 70, 71 are made evident to the eye by the following diagrams:

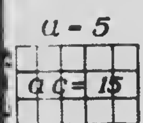


Fig. 1.

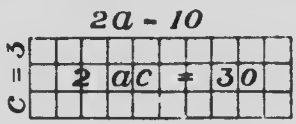


Fig. 2.

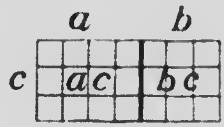


Fig. 3.

73. The area of a rectangle is the product of two factors, the length and the breadth. In Fig. 2 the length is double the length in Fig. 1, and its area is evidently also double, the width being the same in each. This illustrates the truth expressed in Art. 70.

In Fig. 3 the length is $a+b$, the width is c, and consequently the area is $c(a+b)$. If we now divide the rectangle into two rectangles whose lengths are a and b, their areas are ac and bc respectively.

Therefore $$c(a+b) = ac + bc,$$

which illustrates the truth of Art. 71.

74. The principle of Art. 71 may be extended to any number of addends, each of which may be positive or negative and consist of any number of factors. The multiplier may also contain two or more factors and be either positive or negative, *i.e.*, the multiplicand may be any **polynomial** and the multiplier any **monomial.** This gives us the following

Rule: *To multiply a polynomial by a monomial we multiply each term in succession and connect the partial products by the proper signs.*

Ex. 1. Multiply $3a^2 - 4ab + 5b^2$ by $2a$.

Arrange thus
$$\begin{array}{l} 3a^2 - 4ab + 5b^2 \\ 2a \\ \hline 6a^3 - 8a^2b + 10ab^2 \end{array}$$

Begin at the left and work towards the right.

Ex. 2. Simplify $3a(2a^2 + ab - 2b^2) - 5b(a^2 - ab + 3b^2)$

This example consists of two multiplications similar to the preceding, followed by the addition of like terms of the two products.

Now $$3a(2a^2 + ab - 2b^2) = 6a^3 + 3a^2b - 6ab^2$$
and $$-5b\ (a^2 - ab + 3b^2) = -5a^2b + 5ab^2 - 15b^3$$
Adding, we get $$6a^3 - 2a^2b - ab^2 - 15b^3$$
the required result in its simplest form.

It will be observed that the negative sign connecting the two expressions was taken with the $5b$ as part of the multiplier, and the two products were then *added.* We might have taken $+5b$ as the multiplier and then the second product would have been *subtracted* from the first. The method given is usually the better one to follow.

EXERCISE XI

Multiply

1. $x^2 - 2x + 3$ by $2x$.
2. $3x^2 + 4x - 2$ by $-3x$.
3. $x^2 - 2xy + y^2$ by $2y$.
4. $2a^2 - 3ab + b^2$ by $-4ab$.
5. $1 - 2x + 3x^2$ by $-x$.
6. $xy + yz - xz$ by xyz.
7. $7a^2x - 2aby - 3xy^2 + 4b^2y$ by $-3abx$.
8. $2a^2 - 3b^2 - c^2 - 2ac - 4bc + 2ab$ by $-3abc$.
9. $1 - a + b - ac + bc - abc$ by a^2b.

Simplify

10. $3x(2x^2 - 5x + 6) + 2x(x^2 + 2x - 3)$.
11. $2x(x^2 - 2x + 3) - 5x(x^2 + 3x - 1) + 3x^2 - 4$.
12. $2a(a - b) + 3b(2a - 3b) - 2(a^2 - ab + 2b^2)$.
13. $a(2a^2 - 3ab - b^2) - 2b(a^2 - ab + 2b^2) - 3ab(2a - 3b)$.
14. $3(a - b + 2c) - 2(2a + 3b + 5c) + 5(b - 2c + 3a)$.
15. $a(a + b - c) - b(b + c - a) + c(c + a - b) - (c^2 + 2ab - b^2)$.
16. $2x\{3x - 2(x - 2y)\} - 3y\{2(x + 2y) - x\} + 5xy$.
17. $(px + qy) + (x + y) + (p - 1)x - (q + 1)y$.
18. $(a + b)x + (b + c)y - \{(a - b)x - (b - c)y\}$.
19. $(m + n)x + (m - n)y - m(x + y) - n(x - y)$.
20. $(a - b)x + (b - c)y + (c - a)z - a(x - y) - b(y - z)$
$c(z - x) + bx + cy + az$.

MULTIPLICATION OF POLYNOMIALS BY POLYNOMIALS

75. In Art. 73 we have shown that $c(a+b)=ac+cb$, in which $a+b$ is multiplied by c. If now we take $a+b$ for the multiplier, the product in either case being the area of the rectangle, must remain unchanged.

That is $$(a+b)c=ac+bc.$$

Similarly $$(a-b)c=ac-bc,$$

as may easily be shown by a similar diagram.

If now we replace c by an expression of two addends $c+d$ or $c-d$, we get the following:

$$(a+b)(c+d)=a(c+d)+b(c+d)$$
$$=ac+ad+bc+bd$$

and

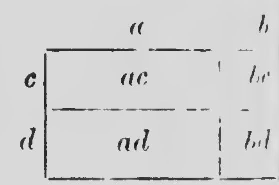
$$(a-b)(c-d)=a(c-d)-b(c-d)$$
$$=ac-ad-bc+bd,$$

which gives the rule for the multiplication of polynomials.

To multiply a polynomial by a polynomial.

Multiply each term of the multiplicand by each term of the multiplier and connect the partial products by the proper signs.

76. The process of the preceding Art. may be made evident to the eye by drawing a rectangle whose length is $a+b$ and width $c+d$ and dividing into four smaller rectangles as in the figure.



The area of the rectangle taken as a whole is

$$(a+b)(c+d).$$

The sum of the areas of the several parts is

$$ac+ad+bc+bd.$$

The two expressions must therefore be equal.

77. Two different classes of polynomials are of frequent occurrence, those whose terms consist of successive powers of one letter, and those in which two letters occur, the sum of whose exponents in each term is constant. Before multiplying such expressions they should be arranged so that the exponents of one of the letters in the successive terms will be either in descending or in ascending order of magnitude, as in the following examples.

Ex. 1. Multiply $2x^3 + x^2 - 3x - 4$ by $3x^2 - 2x + 1$.

$$\begin{array}{lrrrrrr}
\text{Arrange thus} & & & 2x^3 & +x^2 & -3x & -4 \\
 & & & & 3x^2 & -2x & +1 \\
 & 6x^5 & +3x^4 & -9x^3 & -12x^2 & & \\
 & & -4x^4 & -2x^3 & +6x^2 & +8x & \\
 & & & 2x^3 & +x^2 & -3x & -4 \\
\text{Product} & 6x^5 & -x^4 & -9x^3 & -5x^2 & +5x & -4
\end{array}$$

Having arranged the terms with their exponents in order of magnitude, we begin at the left, multiplying each term of the multiplicand by $3x^2$; then by $-2x$ placing each term of the product in the second line under the like term in the first line; similarly with the third line; finally the sum of the three lines is the product required.

Ex. 2. Multiply $a^2 + ab + b^2$ by $a^2 - ab + b^2$.

$$\begin{array}{lrrrrr}
\text{Arrange as before} & & & a^2 & +ab & +b^2 \\
 & & & a^2 & -ab & +b^2 \\
 & a^4 & +a^3b & +a^2b^2 & & \\
 & & -a^3b & -a^2b^2 & -ab^3 & \\
 & & & a^2b^2 & +ab^3 & +b^4 \\
\text{Product} & a^4 & & +a^2b^2 & & +b^4
\end{array}$$

The two examples here given belong to the classes specified at the beginning of this Art. Such examples may always be worked by the brief method shown in the next example.

Ex. 3. Multiply $x^4 - 2x^3 + 3x^2 + 5$ by $x^3 + 2x^2 - 3$.

Arrange the coefficients of multiplicand and multiplier in the usual order for multiplication, but note that in each expression there is one term wanting, and place a cypher in the vacant place. The work of multiplication may then be arranged as follows :

$$\begin{array}{rrrrrrrr}
1 - 2 & + 3 & + 0 & + 5 & & & & \\
1 + 2 & + 0 & - 3 & & & & & \\
\hline
1 - 2 & + 3 & + 0 & + 5 & & & & \\
& 2 & - 4 & + 6 & + 0 & + 10 & & \\
& & - 3 & + 6 & - 9 & - 0 & - 15 & \\
\hline
1 + 0 & - 1 & + 3 & + 11 & + 1 & - 0 & - 15 &
\end{array}$$

Result $\qquad x^7 - x^5 + 3x^4 + 11x^3 + x^2 - 15.$

The student will observe that the purpose of the cypher is to keep the other coefficients in their proper columns. The highest exponent, 7, is obtained by taking the sum of the highest exponents in multiplicand and multiplier. Polynomials of either class described in this Art. can be multiplied by this method, which is called "multiplying by detached coefficients."

EXERCISE XII

Multiply

1. $2x^2 - x + 3$ by $3x - 2$.
2. $x^2 + 2x - 3$ by $2x - 1$.
3. $4x^2 - 5x - 2$ by $5x + 3$.
4. $3x^2 + x - 5$ by $-x + 2$.
5. $x^2 + 2x + 4$ by $x - 2$.
6. $a^2 - a + 1$ by $a + 1$.
7. $a^2 + ab + b^2$ by $a - b$.
8. $a^2 - ab + b^2$ by $a + b$.
9. $a^2 - 2a + 3$ by $a^2 + 2a - 3$.
10. $2a^2 - 5ab + 3b^2$ by $2a^2 + 5ab + 3b^2$.
11. $2x^3 - x^2 + 3x - 1$ by $3x^2 + x - 2$.

12. $3x^3 - 7x^2 - 4x + 5$ by $2x^2 + 3x - 1$.

13. $1 - 2x + 3x^2 + 4x^4$ by $1 + 2x - 3x^2$.

14. $2 - x^3 + 3x^2 - 4x + x^4$ by $1 - 2x + x^2$.

15. $4x - 2x^2 + 3x^3 - 1$ by $2x - x^2 + 3$.

16. $x^5 - x^4 + x^3 - x^2 + x - 1$ by $1 + x$.

17. $a^4 + 2a^3 + 3a^2 + 2a + 1$ by $a^4 - 2a^3 + 3a^2 - 2a + 1$.

18. $x^5 - 3x^4 + 2x^2 - 5x + 7$ by $x^3 - 2x + 3$.

19. $x^2 + xy + y^2 - x + y + 1$ by $x - y + 1$.

20. $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.

21. $a^2 + 2ab + b^2 - c^2$ by $c^2 - a^2 + 2ab - b^2$.

22. $x^2 + 4y^2 + z^2 + 2xy + 2yz - xz$ by $x - 2y + z$.

23. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ by
$a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.

24. $x^8 + x^7y - x^5y^3 - x^4y^4 - x^3y^5 + xy^7 + y^8$ by
$x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

Simplify

25. $(x - 1)(2x + 3) + (2x - 1)(x + 3) - (3x + 2)(x - 5)$.

26. $2(1 - x)(1 + x + x^2) + 3(1 + x)(1 - x + x^2)$
$- x(x + 1)(x - 1)$.

27. $(x + 1)(x + 2)(x - 3) - (x - 1)(x - 2)(x + 3)$.

28. $(a + b)(a - b) - (b - c)(b + c) - (c - a)(c + a)$.

29. $(a - b)(a + b - c) + (b - c)(b + c - a) + (c - a)(c + a - b)$.

30. $a(a + b)^2 - b(a - b)^2 - ab(a + 3b)$.

31. $(a^2 + ab + b^2)(a^2 - ab + b^2) - a^2(a + b)(a - b)$.

32. $(a - 2)(a + 2)(a^2 + 4) + (b - 2)(b + 2)(b^2 + 4)$
$- (a^2 + b^2)(a^2 - b^2)$.

DIVISION

78. The meaning of Division of numbers and the rules for performing it are derived directly from multiplication, as the following simple examples will show:

$$\frac{20}{4}=5, \qquad \frac{-20}{4}=-5, \qquad \frac{20}{-4}=-5, \qquad \frac{-20}{-4}=5.$$

The first example is that of arithmetic, in which we seek a number which multiplied by 4 will make 20; the multiplication table gives the required number 5. In the next example we seek a multiplier of 4 which gives -20; as before, the absolute number is 5, but the rule of signs in multiplication requires it to be negative, viz., -5. Similarly for the other two examples, observe that the number to be divided is the product of two factors, one of which is the number by which we divide, and the remaining factor is the quotient.

79. Definition. When the product of two factors and one of them are given, the process of finding the remaining factor is called **Division.** The given product is called the **Dividend,** the given factor is called the **Divisor** and the factor to be found is called the **Quotient.**

Since $(+a)\times(+b)=+ab \quad \therefore \quad \frac{+ab}{+b}=+a$ Art. 65

$$(-a)\times(+b)=-ab \quad \therefore \quad \frac{-ab}{+b}=-a$$

$$(+a)\times(-b)=-ab \quad \therefore \quad \frac{-ab}{-b}=+a$$

$$(-a)\times(-b)=+ab \quad \therefore \quad \frac{+ab}{-b}=-a$$

That is, the sign of the quotient of two numbers is + when both numbers have the same sign, and − when they have different signs. From this we have, as in multiplication, the

Rule of Signs. *Like signs give* $+$, *unlike signs give* $-$.

80. In the previous Art. the dividend is represented by just two factors and the divisor by one, but both dividend and divisor may have any number of factors. In such cases each of the factors of the divisor must be removed from the dividend; the remaining factors constitute the quotient.

Thus $$\frac{abcd}{ab} = cd, \qquad \frac{10xy(x-y)}{5(x-y)} = 2xy, \text{ etc.}$$

81. The quotient of two powers of the same factor is formed thus:

Since $a^5 = aaaaa$, and $a^2 = aa$,

Therefore $$\frac{a^5}{a^2} = \frac{aaaaa}{aa} = aaa = a^3 = a^{5-2}.$$

Similarly $\dfrac{a^m}{a^n} = a^{m-n}$, in which m and n may be any positive whole numbers.

82. Index law in division. *The exponent of a letter in a quotient is obtained by subtracting the exponent of the divisor from the exponent of the dividend.*

83. The close analogy between division and subtraction should be carefully observed.

To remove an addend from an algebraic expression is subtraction; to remove a factor is division. Thus removing a from $a+b$ is subtraction; removing a from ab is division.

Again, to change $5a$ to $3a$ is to subtract $2a$; to change a^5 to a^3 is to divide by a^2.

The difference of two expressions remains unchanged if a new addend be introduced or removed from each; the quotient remains unchanged if a new factor be introduced or removed from each.

Thus $(a+b)-a=b$ and $(a+b+x)-(a+x)=b$

$$\frac{ab}{a}=b \text{ and } \frac{abx}{ax}=b, \text{ etc.}$$

84. Division of polynomials by a monomial. From Art. 74, it is evident that a polynomial is divided by a monomial by dividing each of its terms in succession and connecting the partial quotients by the proper signs.

Ex. 1. $\dfrac{10abc}{-5a}=-bc,\quad \dfrac{-24a^3xy^2}{-3ay^2}=8a^2x,\quad \dfrac{(a+b)^5}{(a+b)^2}=(a+b)^3.$

Ex. 2. $\dfrac{4a^3b-6a^2b^2-10ab^3}{2ab}=2a^2-3ab-5b^2.$

EXERCISE XIII

1. $\dfrac{20}{-5},\quad \dfrac{-20}{5},\quad \dfrac{-20}{-5},\quad \dfrac{-64}{-16}.$
2. $\dfrac{(-6)^3}{(-2)^2},\quad \dfrac{-750}{(-5)^2},\quad \dfrac{75}{(-3)(-5)},\quad \dfrac{84}{(-1)^{34}}.$
3. $\dfrac{8a^2b^3}{2ab},\quad \dfrac{12a^3b^2c}{-3abc},\quad \dfrac{-14x^2y}{-2xy},\quad \dfrac{-18m^2n^5}{3m^2n}.$
4. $\dfrac{84x^5y^{10}}{-4xy^5},\quad \dfrac{-40a^2x^3y}{10axy},\quad \dfrac{75b^4c^5}{-15bc^5},\quad \dfrac{-72a^5b^2c^3}{12a^2c}.$
5. $6a^2b\times 5ab^2$ by $10a^2b^2$; $2ax\times -3by$ by $-6xy$.
6. $3xy\times -5yz\times -6xz$ by $2x^2\times -3y$.
7. $4x^3-6x^2+8x$ by $2x$.
8. $9y^3+12y^2-6y$ by $-3y$.
9. $8a^4-16a^3b+24a^2b^2$ by $8a^2$.

10. $25a^3b^2 - 50a^2b^3 - 100ab^2c$ by $-25ab^2$.

11. $-3x^4y + 5x^3y^2 - 6x^2y^3 - xy^4$ by $-xy$.

12. $-49x^5yz^3 + 63x^4y^2z - 56x^3y^3z^2$ by $-7x^3yz$.

13. $6(a+b)^2 - 8(a+b)^3 + 10(a+b)^4$ by $2(a+b)^2$.

14. $x^2y(x-y) - yz(x-y)^2 + y^3(x-y)$ by $y(x-y)$.

15. $(3ab^2 - 3a^2b + 6ab^2)(2a^2b + 2ab^2)$ by $6a^2b^2$.

DIVISION OF POLYNOMIALS BY POLYNOMIALS

85. The rule for the division of one polynomial by another is obtained by closely observing the mode of multiplying one polynomial by another and then reversing the process.

$$
\begin{array}{ll}
\text{Suppose the divisor to be} & 3x^2 - 5x + 7 \\
\text{and the quotient} & \underline{2x - 4} \\
 & 6x^3 - 10x^2 + 14x \\
 & \underline{\quad\; - 12x^2 + 20x - 28} \\
\text{Then the dividend is} & 6x^3 - 22x^2 + 34x - 28
\end{array}
$$

Now observe:

1. The first term of the dividend, $6x^3$, is the product of the first term of the divisor, $3x^2$, and the first term of the quotient.

2. Therefore, the first term of the quotient, $2x$, is to be obtained by dividing the first term of the divisor, $3x^2$, into the first term of the dividend, $6x^3$.

86. The dividend is the sum of the products of the divisor by the several terms of the quotient. Therefore, if from the dividend we subtract the product of the divisor by $2x$, the first term of the quotient, the remainder must be the product of the divisor and the remaining term of the quotient (or the sum of such products when more than one term of the quotient remains to be found).

The work may therefore be arranged as follows:

$$\begin{array}{r|l} 3x^2 - 5x + 7 & 6x^3 - 22x^2 + 34x - 28 \;|\; 2x - 4 \\ & \underline{6x^3 - 10x^2 + 14x} \\ & \quad - 12x^2 + 20x - 28 \\ & \quad \underline{- 12x^2 + 20x - 28} \end{array}$$

87. To divide a polynomial by a polynomial.

Arrange the terms of divisor and dividend both in descending or both in ascending powers of a common letter.

Divide the first term of the dividend by the first term of the divisor; the result will be the first term of the quotient.

Multiply each term of the divisor by the first term of the quotient and subtract the product from the dividend.

If there be a remainder, consider it a new dividend and proceed as before.

88. It is essential that the terms in the several remainders be kept in the same order with regard to the exponents of the letter of reference. If a remainder occurs in which the highest exponent of the letter of reference is lower than the highest exponent of that letter in the divisor, the division cannot be exactly performed. Such examples will be further considered in the chapter on Fractions.

The following are additional examples:

Ex. 1. Divide $x^3 - 9x^2 + 23x - 30$ by $x - 6$.

$$\begin{array}{r|l} x - 6 & x^3 - 9x^2 + 23x - 30 \;|\; x^2 - 3x + 5 \\ & \underline{x^3 - 6x^2} \\ & \quad - 3x^2 + 23x \\ & \quad \underline{- 3x^2 + 18x} \\ & \qquad\qquad 5x - 30 \\ & \qquad\qquad \underline{5x - 30} \end{array}$$

Ex. 2. Divide $a^3 + b^3$ by $a + b$.

$$
\begin{array}{l|l|l}
a+b & a^3 + b^3 & a^2 - ab + b^2 \\
 & a^3 + a^2b & \\
 & \quad - a^2b + b^3 & \\
 & \quad - a^2b - ab^2 & \\
 & \qquad ab^2 + b^3 & \\
 & \qquad ab^2 + b^3 &
\end{array}
$$

Ex. 3. Divide $a^3 - b^3$ by $a^2 + ab + b^2$.

$$
\begin{array}{l|l|l}
a^2 + ab + b^2 & a^3 \qquad\qquad - b^3 & a - b \\
 & a^3 + a^2b + ab^2 & \\
 & \quad - a^2b - ab^2 - b^3 & \\
 & \quad - a^2b - ab^2 - b^3 &
\end{array}
$$

Ex. 4. Divide $1 - 4x^2 + 16x^3 - x^4 - 12x^5$ by $1 + 2x - 3x^2$.

$$
\begin{array}{l|l|l}
1 + 2x - 3x^2 & 1 \qquad - 4x^2 + 16x^3 - x^4 - 12x^5 & 1 - 2x + 3x^2 + 4x^3 \\
 & 1 + 2x - 3x^2 & \\
 & \quad - 2x - x^2 + 16x^3 & \\
 & \quad - 2x - 4x^2 + 6x^3 & \\
 & \qquad 3x^2 + 10x^3 - x^4 & \\
 & \qquad 3x^2 + 6x^3 - 9x^4 & \\
 & \qquad\quad 4x^3 + 8x^4 - 12x^5 & \\
 & \qquad\quad 4x^3 + 8x^4 - 12x^5 &
\end{array}
$$

When a term of the regular series in the dividend is wanting, as in Exs. 3 and 4, it is convenient to leave a vacant space in order to permit like terms to be placed in the same column.

If both divisor and dividend are not already in their simplest forms, as in the preceding examples, they must be simplified by performing any indicated multiplications and collecting like terms before attempting to perform the division.

EXERCISE XIV

Divide

1. $x^2 + 10x + 21$ by $x + 3$.
2. $x^2 - 11x + 24$ by $x - 8$.
3. $x^2 - x - 56$ by $x + 7$.
4. $x^2 + x - 90$ by $x + 10$.
5. $4x^2 - 9$ by $2x - 3$.
6. $x^3 - 7x + 6$ by $x - 2$.
7. $a^2 - b^2$ by $a - b$.
8. $a^3 - b^3$ by $a - b$.
9. $a^3 + b^3$ by $a + b$.
10. $a^3 - b^3$ by $a^2 + ab + b^2$.
11. $x^3 - 7x - 6$ by $x - 3$.
12. $4x^3 + 5x + 21$ by $2x + 3$
13. $2x^3 + 7x^2 + 5x + 100$ by $2x^2 - 3x + 20$.
14. $a^5 - 5a^3 + 7a^2 + 6a + 1$ by $a^2 + 3a + 1$.
15. $3a^4 - 5a^3b + a^2b^2 + 13ab^3 + 4b^4$ by $a^2 - 3ab + 4b^2$.
16. $4x^5 + 7x^3 - 6x - 12x^4 + 5x^2 + 3$ by $2x^2 + 3 - x$.
17. $19x^4 - x^2 + 10 + 3x^6 - 11x^5 - 13x^3$ by $3 + x^2 - 2x$.
18. $a^4 + a^2b^2 + b^4$ by $a^2 + ab + b^2$.
19. $a^8 + a^4b^4 + b^8$ by $a^4 - a^2b^2 + b^4$.
20. $a^{15} + b^{15}$ by $a^5 + b^5$ and by $a^3 + b^3$.
21. $a^{15} - b^{15}$ by $a^{10} + a^5b^5 + b^{10}$ and by $a^{12} + a^9b^3 + a^6b^6 + a^3b^9 + b^{12}$
22. $x^6 - 2x^3 + 1$ by $x^2 - 2x + 1$ and by $x^4 - x^3 - x + 1$.
23. $x^7 - x$ by $x^3 - x$ and by $x^4 - 2x^3 + 2x^2 - x$.
24. $a^5 - b^5 + a^2b^3 - a^3b^2$ by $a^3 - b^3 - 2a^2b + 2ab^2$.
25. $x^5 - y^5 - x^3y^2 + x^2y^3$ by $x^2 + y^2 + 2xy$.
26. $8a^6 - b^6 + 21a^3b^3 - 24a^5b$ by $3ab - a^2 - b^2$.
27. $x^4 - px^3 + px^2 - p^2x$ by $x - p$.
28. $x^4 + 2mx^3 + m^2x^2 - n^2$ by $x^2 + mx + n$.
29. $a^3x^3 + a^2bx^2y - ab^2xy^2 - b^3y^3$ by $ax - by$.
30. $6a^4x - 17a^3x^2 + 44a^2x^3 - 48ax^4$ by $2a^2 - 3ax$.
31. Divide the product of $x^2 + x - 2$ and $2x^2 - 7x + 6$ by $2x^2 + x - 6$.

32. Divide $x^7 - 64x$ by $x(x+2)+4$.

33. Divide $x^2(x^4-5)+5x-1$ by $x(x-2)+1$.

34. Divide $a(a+b)^2 - b(a-b)^2 + 2b(a^2+b^2)$ by $a+b$.

35. Divide $(a+b+1)(a+b-2)-10$ by $a+b-4$.

36. Divide $(a^2+ab+b^2)(a^2-ab+b^2)+a^2b^2$
by $(a+b)^2-2ab$.

37. Divide $(a+b)(a^2-ab+b^2)+3ab-1$ by $a+b-1$.

38. Divide a^6-6a+5 by $(a^2+a+1)^2+2(a+2)$.

39. Divide $(x+1)^2(x^3-x+5)+(2x-1)(x^2-2x+3)$
by $(x-1)^2+1$.

40. Divide $(x-1)^2(x^3-x^2+1)-(3x-2)(x^2+x-1)$
$+18x+1$ by $(x+1)^2+2$.

LITERAL COEFFICIENTS

89. **Meaning of Coefficients extended.** The definition f a coefficient in Art. 11 tacitly implies that it is a whole umber and expressed in figures, and coefficients of this ind are the only ones thus far brought into use. The efinition is there given in simple form for beginners and xpresses only a part of the truth. A coefficient is a ctor or multiplier, and as such may be any algebraical xpression, integral or fractional, and expressed by letters r figures.

Thus in ax, by, $(a+b)x$, $(a-b)y$, the literal factors a, b, $+b$, $a-b$ may be considered literal coefficients of the ctors which follow them.

It is equally true that x and y might be considered coficients of the factors which precede them.

90. **Use of Brackets.** By the aid of the preceding Art. e sum of two or more unlike terms, having at least one teral factor in common, may be expressed as a single

term by considering the unlike factors as literal coefficients and enclosing their sum in a bracket.

For just as $2x + 3x = (2+3)x = 5x$,

so $ax + bx = (a+b)x$,

the only distinction being that in the first example we have a single symbol, 5, to take the place of $2+3$, but we have no single symbol to stand in place of $a+b$, and consequently the expression $(a+b)x$ cannot be further simplified.

The following are additional examples:

Ex. 1.

To	ax	$(a-b)y$	ay	$(a+b-c$
add	$-bx$	$2by$	y	$(b-2c$
Sum	$(a-b)x$	$(a+b)y$	$(a+1)y$	$(a+2b-c$

Ex. 2. From $(a+b)m-(a+b)n+(a-b)p-(a-b)q$
take $(a-b)m-(a-c)n-(b-c)p+(b+c)q$
Result $2bm-(b+c)n+(a-c)p-(a+c)q$

91. In subtracting terms with literal coefficients, we follow the ordinary rule, but we have a choice of two ways of applying it. When the signs preceding the brackets are alike, as in the case of the coefficients of m and n, we change the sign of each term *within* the bracket and add, placing the common sign before the result. The coefficients of p and q have unlike signs preceding the bracket, and in such cases it is best to change the sign which precedes the bracket of the lower terms and add the terms within the brackets as they stand.

92. The following are important examples in multiplication and division, in which the terms containing the same power of x are combined:

Ex. 1. Find the continued product of

$$x+a,\ x+b \text{ and } x+c.$$

$$\begin{array}{l}
x + a \\
x + b \\
\hline
x^2 + ax \\
\qquad bx + ab \\
\hline
x^2 + (a+b)x + ab \\
x + c \\
\hline
x^3 + (a+b)x^2 + abx \\
\qquad\qquad cx^2 + (ac+bc)x + abc \\
\hline
x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc
\end{array}$$

Ex. 2. Divide $x^3 - (a+b+c)x^2 + (ab+bc+ac)x - abc$ by $x - a$.

$$\begin{array}{l}
x - a)\ x^3 - (a+b+c)x^2 + (ab+bc+ac)x - abc\ (\ x^2 - (b+c)x + bc \\
\quad x^3 - ax^2 \\
\hline
\quad -(b+c)x^2 + (ab+bc+ac)x \\
\quad -(b+c)x^2 + (ab+ac)x \\
\hline
\qquad\qquad bcx - abc \\
\qquad\qquad bcx - abc
\end{array}$$

EXERCISE XV

Collect the coefficients of x and y in Exs. 1-8.

1. $ax + by + mx + ny + px + qy.$
2. $3ax - 2by + ax + cx - by + 7y.$
3. $mx + nx - my - ny - x + y + my - nx.$
4. $a^2x + a^2y + abx - 2aby + b^2x - b^2y.$
5. $ax + by + bx + ay - (a+1)x - (b-1)y + 2x + 3y.$
6. $(a - 2b)x + (b - 3c)y + 2bx - by + (m - a)x - (m - 3c)y.$
7. $mx + ny + (m - n)x + (m - n)y - 2mx - my.$
8. $(a - b)x + (b - c)y + (b - c)x + (c - a)y + (a - c)x - (a - b)y.$

9. From $ax^2 + bxy - cy^2$ take $px^2 - qxy + ry^2$.

10. From $(a - b)x^2 + (b + c)xy - (c - a)y^2$
take $(a - c)x^2 - (a - c)xy + (a - b)y^2$.

11. From the sum of
$$(a - b)x + (b - c)y + (c - a)z \text{ and } bx - cy + z$$
take the sum of $(b - c)x + y + z$ and $(c + 1)x - 2cy + (c - 1)z$.

Arrange the products according to the powers of x in Exs. 12–20.

12. $(x + a)(x + b)$.

13. $(x - a)(x - b)$.

14. $(x + a)(x - b)$.

15. $(x - a)(x + b)$.

16. $(x + a)(x + b)(x + c)$.

17. $(x - a)(x - b)(x - c)$.

18. $(x + a)(x - b)(x + c)$.

19. $(x - a)(x + b)(x - c)$.

20. $(x - a)(x + b) + (x - b)(x + c) + (x - c)(x + a)$.

21. Divide $a^2 - b^2 - c^2 + 2bc$ by $a - b + c$.

22. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

23. Divide $8a^3 - b^3 + c^3 + 6abc$ by $2a - b + c$.

24. Divide $x^2 + xy + 2xz - 2y^2 + 7yz - 3z^2$ by $x - y + 3z$.

25. Divide $x^2 - (a + b)x + ab$ by $x - a$.

26. Divide $x^2 - (a + b + c)x + a(b + c)$ by $x - (b + c)$.

27. Divide $x^3 - (a + m)x^2 + (am + mn)x - amn$ by $x - a$.

28. Divide $x^3 + (a + b + c)x^2 + (ab + bc + ac)x + abc$
by $x + a$ and by $x + b$.

29. Divide $x^3 - (a + b + c)x^2 + (ab + bc + ac)x - abc$
by $x - b$ and by $x - c$.

30. Divide $x^3 + (a + b - c)x^2 + (ab - bc - ac)x - abc$
by $x^2 + (a - c)x - ac$.

31 Divide $a^2(b + c) + a(b^2 + c^2) - 2bc(b + c)$ by $a + b + c$.

32. Divide $2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4$
by $2ab - a^2 - b^2 + c^2$ and by $2bc - b^2 - c^2 +$

CHAPTER IV

SIMPLE EQUATIONS: ONE UNKNOWN QUANTITY

(ELEMENTARY)

93. Definition. An Equation is a statement that two lgebraical expressions are equal, *i.e.*, they represent the me number.

Thus $2x+3x=5x$, $2(x-5)=2x-10$, $2x-7=x+3$ are quations.

94. Sides of an equation. The expressions on opposite les of the sign of equality, =, are called the **sides,** or **embers,** of the equation.

95. Two expressions may represent the same number different ways.

1. They may represent the same number for all values the letters which they contain.

Thus $2x+3x=5x$, and $2(x-5)=2x-10$, whatever value ay be assigned to x, *i.e.*, x represents a general number.

2. They may represent the same number for only one more particular values of the letter.

Thus $2x-7=x+3$ only when $x=10$, and then each side the equation becomes 13. For all other values of x e two expressions are unequal, *i.e.*, x represents a parular number.

96. Identities. Two expressions which are equal for values of the letters contained form an **identity** or, as is sometimes called, an **identical equation.**

An identity always consists of two expressions, of which the one may be derived from the other by the ordinary rules of algebra.

97. Conditional equations. Two expressions which are equal for one or more particular values form a **conditional equation.** The word "conditional," however, is usually omitted and the name equation alone used. It is with this class of equation that elementary algebra is chiefly concerned.

98. Unknown quantity. The letter to which a particular value must be given to make the sides equal, is called the **unknown quantity.**

99. Solving an equation. To solve an equation is to find the value of the unknown quantity which makes the two sides equal. This value is called a **Root** of the equation and is said to satisfy the equation, *i.e.*, to make the two sides equal.

100. Simple equation. A simple equation is one which does not contain the square or any higher power of the unknown quantity. Such equations have but one root.

Thus $3x - 1 = 2(x+3)$ is a simple equation and is satisfied by but one value of x, viz., $x = 7$.

But $x^2 - x = 6$ is not a simple equation. It is satisfied when $x = 3$ or -2, but for no other value.

101. Axioms used in solving equations.

1. If equals be added to equals the sums are equal.

2. If equals be taken from equals the remainders are equal.

3. If equals be multiplied by equals the products are equal.

4. If equals be divided by equals (not zero) the uotients are equal.

102. Illustrations. If two rods have the same length ey will still be equal if an inch be added to or subacted from each; also if each be multiplied or divided the same number. A pair of scales, having equal ms, balances when the weights in the two scale-pans are ual. If the same additional weight be placed in, or moved from each pan, the balance will not be destroyed.

103. *Ex.* Solve the equation $5x-7=3x+11$ and rify the result.

SOLUTION

The given equation is	$5x-7=3x+11$	
Add 7 to each side	$5x=3x+11+7$	(Ax. 1.)
Subtract $3x$ from each	$5x-3x=11+7$	(Ax. 2.)
Combine like terms	$2x=18$	
Divide by coefficient of x	$x=9$	(Ax. 4.)

VERIFICATION

When $x=9$, $5x-7=5(9)-7=45-7=38$
d $3x+11=3(9)+11=27+11=38$,
ich proves the solution correct.

104. In the preceding solution observe:

1. Adding 7 to each side caused the -7 to disappear m the first side of the equation, and $+7$ to appear stead upon the opposite side.

2. Subtracting $3x$ from each side caused $+3x$ to disear from the second side and to reappear as $-3x$ on e first side.

3. The object in making these changes was to colle[ct] all the terms containing x on one side and all other term[s] on the other side of the equation.

These observations furnish the mode of solution of [a] simple equation and the reasoning upon which it [is] founded.

105. Transposing terms. *A term may be transpo[sed] from one side of an equation to the other, without destroying t[he] equality, providing the sign of the term be changed.*

With this statement compare the first two statements [of] Art. 104.

106. Rule for solving. *Remove brackets, if any occ[ur]; transpose all terms containing the unknown quantity to t[he] first side, and all remaining terms to the second side of [the] equation; combine like terms, and divide both sides by [the] coefficient of the unknown quantity.*

Ex. 1. Solve the equation $2x - 3(x - 2) = 4 + 2(x - 5$

Remove brackets $\quad 2x - 3x + 6 = 4 + 2x - 10$

Transpose terms $\quad 2x - 3x - 2x = 4 - 10 - 6$

Combine terms $\quad -3x = -12$

Divide by -3 $\quad x = 4$ (Ax.

Verification $\quad 2x - 3(x - 2) = 2(4) - 3(2) = 8 - 6$

$\quad 4 + 2(x - 5) = 4 + 2(-1) = 4 - 2$

Ex. 2. Solve equation $2(x + 3)^2 - 13 = (x - 1)^2 + (x - 2)^2$

Simplify each side $\quad 2x^2 + 12x + 5 = 2x^2 - 6x + 5$

Transpose $\quad 2x^2 - 2x^2 + 12x + 6x = 5 - 5$

Combine $\quad 18x = 0$

Divide by 18 $\quad x = 0$

Verification $\quad 2(x + 3)^2 - 13 = 2(3)^2 - 13 \quad 5$

$\quad (x - 1)^2 + (x - 2)^2 = (-1)^2 + (-2)^2 \quad 5$

Ex. 3. Solve equation $a(x-a)+b(x-b)=2ab$.

Remove brackets $ax-a^2+bx-b^2=2ab$

Transpose $ax+bx=a^2+2ab+b^2$

Combine terms in x $(a+b)x=a^2+2ab+b^2$

Divide by $a+b$ $x=a+b$

Verification $a(a+b-a)+b(a+b-b)=ab+ab$

$$=2ab$$

EXERCISE XVI

107. Solve the following equations and verify each result.

1. $2x-5=x+7$.
2. $3x+4=x+22$.
3. $3(x-5)=2(x+1)+10$.
4. $2(x+7)=3(x-11)$.
5. $x-(5-2x)=4(1-x)+5$.
6. $2(x+2)+3(x-5)+1=0$.
7. $x-(4x-5)+3(x-7)=x$.
8. $3x+7-(5x-2)-11=0$.
9. $2(x-2)+3(x-3)-4(x-4)=0$.
10. $5x-2(7-3x)+4(2x-5)-(x+2)=0$.
11. $8-2(3x+5)-2(2-5x)=8x-(6-11x)$.
12. $x-7(4x-11)=14(x-5)-19(8-x)-61$.
13. $5x+6(x+1)-7(x+2)-8(x+3)-2(x+8)=0$.
14. $(x+3)(x+7)=(x+2)(x-12)$.
15. $(x-1)(x-9)=(x+2)(x-11)$.
16. $(x+8)(x-11)-2(x+3)(x-7)=x(4-x)+52$.
17. $(x-2)(7-x)+(x-5)(x+3)-2(x-1)+12=0$.
18. $(2x-3)(x+7)-(x-5)(2x+3)=x(x-2)-x^2$.
19. $(2x-7)(x+5)=(9-2x)(4-x)+229$.
20. $(x-1)(x-3)(x-5)=(x-2)(x-3)(x-4)$.

21. $(x-1)^2+(x-3)^2=2(x-5)^2-12.$

22. $(x-1)^3+(x-2)^3+(x-3)^3=3(x-1)(x-2)(x-3).$

23. $5(a+x)+3(a-x)=3x.$

24. $a(x-a)+b(x-b)=a(x+b)-a^2.$

25. $(x+a)(x-b)-(x-a)(x+b)=(a-b)x.$

26. $a(x-a)=b(x-b).$ 27. $(a+b)x+(a-b)x=a^2.$

28. $(a+b)x-(a-b)x=b^2.$ 29. $ax-bx=a+bx-2b.$

30. $a(x+a)+b(b-x)=2ab.$ 31. $a(x+a)=b(x-b)+2a$

32. $(x-a)^2-(x-a)(x-b)+(x-b)^2=x(x-a)+a^2.$

33. $x(a+b)+(a+b)^2=c(c-x).$

34. $x(x-a)+b(x-b)=(x-a)^2+2ab.$

35. $(a+b)(x-c)+(b+c)(x-a)=(c-a)(x+b).$

36. $a(x-a)+b(x-b)=c(x-c)=2ab.$

PROBLEMS

108. Algebra is extensively used in the solution of problems in which one or more numbers are to be found from their connection with other numbers already known. The mode of proceeding can best be learned from a few simple examples.

Ex. 1. A purse contains dimes and quarter-dollars, 15 more dimes than quarters, and 51 coins in all. Find the number of each.

Let	x	= number of quarter-dollars,
then	$x+15$	= " dimes.
Adding	$x+(x+15)$	= total number of coins
		= 51, the number given.
Therefore	$2x+15$	= 51,
from which	x	= 18, the number of quarters,
and	$x+15$	= 33, " " dimes.

Ex. 2. In the preceding example, if the total value of he coins had been given, \$7.80, instead of their number, he solution would have been as follows:

Let x = number of quarter-dollars,
hen $x + 15$ = " dimes,
nd $25x$ = " cents in 25 quarters,
" $10(x+15)$ = " " $x + 15$ dimes.
dding $25x + 10(x+15)$ = total number of cents
= 780 cents, the number given.
Therefore $35x + 150 = 780$,
rom which $x = 18$, the number of quarters,
nd $x + 15 = 33$, " " dimes.

Ex. 3. A father, 40 years of age, has 3 children of 10, and 6 years of age. In how many years will the sum of he ages of his children be double his own age?

Let x = the number of years required,
hen $40 + x$ = father's age after x years
nd $10 + x$ = eldest child's age
" $8 + x$ = second "
" $6 + x$ = youngest "
" $24 + 3x$ = sum of children's ages,
" $2(40 + x)$ = twice father's age.
Therefore $24 + 3x = 2(40 + x)$
olving $x = 36$, the number of years required.

Ex. 4. A yard of velvet is worth 50 cents more than yard of silk; 10 yards of silk and 12 yards of velvet e together worth \$61. Find the value of a yard of ach.

Let	$x =$	number of cents for a yard of silk.
then	$x+50 =$	" " " velvet
"	$10x =$	" " 10 yards of silk;
"	$12(x+50) =$	" " 12 " velvet.
Adding	$10x+12(x+50)=6100.$	
Solving equation	$x=250.$	
and	$x+50=300.$	

The values are, therefore, \$2.50 and \$3 respectively.

109. No rules can be given for the solution of problems, but the following observations may be of some assistance as a general guide:

1. Let x stand for the number from which the other numbers connected with the problem can be most easily found.

2. Find from the problem two different expressions, each of which represents the same number. These will form the two sides of the equation.

3. When concrete quantities occur, they must all be expressed in units of the same denomination.

4. Be careful to specify clearly the units which x is used to count; x must stand for an *abstract* number.

EXERCISE XVII

1. Find the number which when multiplied by 3 and with 11 added to the product makes 47.

2. The sum of two numbers is 75 and their difference 17. Find the numbers.

3. The double of a certain number is greater by 3 than the number itself with 7 added. Find the number.

4. Two boys together have \$1.25 and one of them has 17 cents more than the other. How much has each?

5. Two boys had each the same number of cents. One of them lost 25 cents and the other earned 7 cents, and then the latter had twice as many as the former. How many had each at first?

6. The length of a room is 3 times its width. If it were 10 ft. shorter and 12 ft. wider it would be square. Find its width.

7. The length of a room is 8 ft. more than its width and its perimeter is 60 ft. Find its length.

8. *A* and *B* have each the same amount of money; *A* gains $17 and *B* loses $3 and now *A* has 3 times as much as *B*. How much had each at first?

9. *A* is three times as old as *B*, but in 10 years he will be only twice as old. How old is *A* at present?

10. Two boys have each the same number of marbles; one of them wins 24 from the other and now the former has 4 times as many as the latter. How many had each at first?

11. Divide 56 into three parts such that the first part may exceed the second part by 8 and the third part by 11.

12. Divide 75 into two parts such that 12 times one part may equal 13 times the other part.

13. At an election 875 votes were cast and the successful candidate had 29 majority. How many votes did each receive?

14. Divide 89 into three parts such that the first part may be less than the second by 5 and less than the third by 12.

15. Divide $7.64 between *A*, *B* and *C*, giving *A* 5 cents less than *B*, and *C* 2 cents more than *A* and *B* together.

16. Two men have each the same amount of money. One of them gives the other \$50 and now the latter has 3 times as much as the former. How much had each at first?

17. The ages of two men differ by 20 years, and 15 years ago the elder was twice as old as the younger. Find the age of each at present.

18. A father 40 years of age has 3 sons, the sum of whose ages is 20 years. In how many years will the sum of the sons' ages equal their father's age?

19. The age of a father is 3 years more than 5 times the age of his son, and the sum of their ages is 33 years. Find their ages.

20. There are four more girls than boys in a certain class, and three times the number of boys is greater by 9 than twice the number of girls. How many girls are there?

21. A parent divides \$2500 between 2 sons and 3 daughters, giving each son \$100 more than each daughter. Find the share of each.

22. A franc is worth 5 cents less than a mark; 5 francs and 7 marks are together worth \$2.63. Find the value of each coin in cents.

23. A rouble is worth 14 cents more than a guilder; 10 roubles and 5 guilders are together worth \$7.40. Find the value of each coin in cents.

24. A pound of tea is worth 5 cents more than 2 pounds of coffee; 4 lbs. of tea are worth 9 lbs. of coffee. Find the value of a pound of coffee.

25. A bushel of barley is worth 2 bushels of oats, and a bushel of wheat is worth 20 cents more than a bushel of barley; 2 bushels of wheat, 5 of barley and 10 of oats are worth \$10. Find the value of a bushel of oats.

26. A pound of tea is worth 6 cents more than 2 pounds of coffee; 4 lbs. of tea and 5 lbs. of coffee are together worth \$2.45. Find the value of a pound of tea.

27. Divide \$1.69 between *A*, *B* and *C*, giving *B* 15 cents less than twice as much as *C*, and *A* 1 cent more than *B* and *C* together.

28. *A*, *B*, *C* and *D* together have \$7.70. *A* and *B* have together \$4.70; *A* and *C*, \$5.45; *A* and *D*, \$3.95. How much has *A*?

29. A purse contains quarter-dollars and 10-cent pieces, 42 coins in all; the total value is \$6.45. Find the number of quarters.

30. A purse contains a number of 10-cent pieces, as many francs and six more, each worth 19 cents. The francs are together worth \$2.40 more than all the 10-cent pieces. How much are all the francs together worth?

31. A merchant bought 100 yards of cloth at \$2.50 per yard. He sold part of it at \$2.75 per yard and the remainder at \$3 per yard, gaining on the whole \$40. How many yards did he sell at \$3 per yard?

32. A workman worked 40 days, part of the time at \$1.60 per day and the remainder of the time at \$1.80 per day. For the former period he received \$13 more than for the latter period. How much did he receive in all?

33. A workman was engaged for 60 days on condition that he should receive \$1.50 for each day he worked but should forfeit 75 cents for each day he was idle. At the end of the period he received \$38.25. How many days was he idle?

34. A gentleman gave a number of children 10 cents each and had a dollar left. To have given them 15 cents

each he would have required a dollar more than he possessed. How much money had he?

35. A wine merchant has two kinds of wine, one worth 60 cents and the other 75 cents a quart. From these he wishes to make a mixture of 100 gallons worth $2.75 a gallon. How many gallons of the former kind must he take?

36. Two casks contain equal quantities of water. From the first cask 40 quarts are drawn and from the second 35 gallons. One cask now contains twice as much as the other. How much did each cask at first contain?

37. A rectangle is 1 foot longer and 6 inches narrower than the side of a square. The area of the square is 48 sq. in. less than the area of the rectangle. Find the area of the square.

38. The sides of two squares differ by 3 inches and their areas differ by 117 square inches. Find the area of the smaller square.

39. The length of a field is twice its width; if 10 yards were added to its width and the same amount subtracted from its length, the area would be increased by 700 square yards. Find its original area.

EQUATIONS WITH FRACTIONS

110. Equations frequently occur with fractional coefficients. They may always be solved by the methods of the following examples.

Ex. 1. Solve the equation $\frac{x}{5} - 5 = 3\frac{1}{3} - \frac{x-1}{5}$.

Multiply each side by 15; this will not destroy the equality and will cause the fractions to disappear.

Therefore $\quad 5x = 75 + 50 - 3(x - 1)$

$\quad\quad = 125 - 3x + 3.$

Then $\quad 8x = 128,$

and $\quad x = 16.$

Since letters in algebra stand for numbers, the multiplication of fractions containing letters follows the ordinary rules of arithmetic.

Thus $\frac{x}{3} \times 15 = 5x$; $\frac{x - 1}{5} \times 15 = 3(x - 1)$, etc.

Observe carefully the negative sign before the fraction containing two terms in the numerator. When the denominator has been removed by multiplication the negative sign causes the sign of each term in the numerator to be changed as in the example given.

The multiplier which will cause all the fractions to disappear is evidently the least common multiple of the denominators. This process is known as "clearing an equation of fractions."

Ex. 2. Solve equation $\frac{1}{2}(x - 3) - \frac{1}{3}(x - 5) = 1 - \frac{1}{12}(x - 8).$

Multiply by 12, $\quad 6(x - 3) - 4(x - 5) = 12 - (x - 8).$

Remove brackets, $\quad 6x - 18 - 4x + 20 = 12 - x + 8,$

then $\quad 3x = 18,$

and $\quad x = 6.$

Observe the two different forms in which fractions containing literal expressions may be written, with the fractional part as a coefficient as in this example, or with the denominator written beneath the numerator as in Ex 1. The meaning is the same in both cases. Such terms are multiplied by multiplying the fractional coefficient only. Art. 70.

The student should verify each solution.

Thus when $x = 6$,

$$\frac{1}{2}(x-3) - \frac{1}{3}(x-5) = \frac{1}{2}(6-3) - \frac{1}{3}(6-5) = \frac{3}{2} - \frac{1}{3} = 1\frac{1}{6}$$

$$1 - \frac{1}{12}(x-8) = 1 - \frac{1}{12}(6-8) = 1 - \left(-\frac{1}{6}\right) = 1\frac{1}{6},$$

which proves the solution correct.

EXERCISE XVIII

Solve and verify each result:

1. $2x + \frac{x-5}{3} = \frac{37}{3}$.

2. $\frac{x-1}{2} + \frac{x-2}{3} = 5$.

3. $\frac{x-4}{4} = 6 - \frac{x}{3}$.

4. $\frac{x-1}{2} - \frac{x-2}{3} = \frac{2}{3} - \frac{x-3}{4}$.

5. $\frac{x+1}{2} + \frac{x+2}{3} + \frac{x-9}{4} = 0$.

6. $\frac{x-5}{2} - \frac{x-4}{3} = \frac{6-2x}{3}$.

7. $\frac{5x+3}{3} - \frac{3x-7}{2} = 10(5x-1) - \frac{19}{9}$.

8. $\frac{2x-1}{3} - \frac{5-4x}{4} = x + \frac{5}{6}$.

9. $\frac{2x-11}{7} + \frac{5x-3}{2} = x + \frac{1}{2}$.

10. $\frac{8x-15}{3} - \frac{11x-1}{7} = \frac{7x+2}{13}$.

11. $\frac{3x+5}{8} - \frac{21+x}{2} = 5(x-3)$.

12. $\frac{x}{4} - \frac{x-1}{3} = \frac{x-5}{12} + \frac{1}{2}$.

13. $\frac{2x-3}{12} + \frac{2-3x}{5} + \frac{3-4x}{8} = 0$.

14. $\frac{2x+7}{3} + \frac{3-x}{8} = \frac{2(x+5)}{11}$.

15. $\frac{3x-5}{2}+\frac{4x-11}{3}-\frac{5x-37}{6}=0.$

16. $\frac{x}{4}+\frac{5}{3}(11-x)=\frac{1}{12}(34-11x).$

17. $\frac{3}{5}+\frac{2}{x}+\frac{7}{3x}-\frac{1}{5x}=\frac{8}{3}.$

18. $\frac{x+1}{2}-\frac{3}{x}=\frac{x}{3}-\frac{5-x}{6}.$

19. $\frac{3-x}{4}-\frac{2}{x}+\frac{x+6}{3}=\frac{x}{12}.$

20. $\frac{x}{2}+\frac{2x-5}{5}=\frac{9x^2-40}{10x}.$

21. $\frac{1}{3}(1-2x)-\frac{1}{4}(7-2x)+\frac{1}{6}(11-2x)+\frac{7}{12}=0.$

PROBLEMS

111. The solution of problems resulting in fractional equations is of the same general nature as the solutions already given.

Ex. 1. Bought a number of apples at 2 for a cent and twice as many more, lacking 5, at 5 for a cent. Sold them all at 5 for 2 cents, gaining 17 cents in all. How many did I buy?

Let x = number at 2 for a cent,

then $2x-5$ = " 5 "

and $\frac{x}{2}$ = number of cents for first lot,

$\frac{2x-5}{5}$ = " " second lot.

Also $3x-5$ = total number bought,

and $\frac{2}{5}(3x-5)$ = number of cents received.

Then $\frac{x}{2}+\frac{2x-5}{5}=\frac{2}{5}(3x-5)-17,$

solving $x=60.$

Then $3x-5=175$, the total number bought.

Ex. 2. A saves 20% of his income; B has $1\frac{1}{2}$ times as large an income and saves 25% of it. A spends \$20 more in 6 mos. than B saves in a year. Find their incomes.

Let $x = A$'s income in dollars,

then $\frac{3x}{2} = B$'s " "

and $\frac{4x}{5}$ A's expenses " per annum.

" $\frac{3x}{8} = B$'s savings per annum.

Therefore $\frac{2x}{5} - 20 = \frac{3x}{8}$,

from which $x = 800$, and $\frac{3x}{2} = 1200$.

Their incomes are, therefore, \$800 and \$1200 respectively.

Ex. 3. A rectangle is $\frac{2}{3}$ as wide as it is long, and its area is 25 sq. ft. less than the area of a square of equal perimeter. Find the area of the square.

Let $2x =$ the width of the rectangle in feet,

then $3x =$ " length " " "

and $10x =$ " perimeter of rectangle,

" $\frac{5x}{2} =$ side of square of equal perimeter.

" $6x^2 =$ area of rectangle in square feet.

" $\left(\frac{5x}{2}\right)^2 =$ " square "

Then $6x^2 + 25 = \left(\frac{5x}{2}\right)^2$,

$\therefore$ $24x^2 + 100 = 25x^2$.

$\therefore$ $x^2 = 100$, and $x = 10$.

Then area of square $= \left(\frac{5x}{2}\right)^2 = (25)^2 = 625$ sq. ft.

EXERCISE XIX

1. Find a number whose half exceeds its third by $2\frac{5}{6}$.

2. Two numbers differ by a unit, and if the larger be divided by 3 and the smaller by 5 the sum of the quotients will be 11. Find the larger number.

3. Two numbers differ by 9 and one of them is $\frac{3}{4}$ of the other. Find the larger number.

4. Divide 17 into two parts, such that a third of one part and a quarter of the other may together equal 5.

5. Find the number whose third part is as much less than 48 as its double is greater than 29.

6. It requires $6\frac{1}{4}$ minutes longer for a boy to walk to school at 3 miles per hour than at 4 miles per hour. How far is it to school?

7. Find a number such that when diminished by 7, one-third the remainder is greater by 3 than one-fifth of the original number.

8. A boy bought a number of marbles at 3 for a cent, and having lost 5 he sold the remainder at a half cent each, gaining on the whole 4 cents. How many did he buy?

9. A post stands with one-third of its length in the earth, one-half in the water and 6 feet above the surface of the water. Find its length.

10. A boy spent one-fifth of his money for candy, one-half the remainder for oranges and had 5 cents more than one-third of his original sum left. What had he at first?

11. The number of boys in a certain class is one more than [illegible] half the whole, and the number of girls is 6 less than [illegible] How many in the class?

12. Bought a number of apples at 2 for a cent and twice as many lacking one at 3 for a cent; the cost of the second lot was 8 cents more than the cost of the first lot. How many in all did I buy?

13. A journey of $26\frac{1}{2}$ miles was performed in $5\frac{1}{2}$ hours, part of it at 4 miles per hour and the remainder at 7 miles per hour. How many miles at each rate?

14. How far can a boy ride his wheel at 12 miles per hour and return on foot at 4 miles per hour to a point [illegible] miles from where he started, being $5\frac{1}{2}$ hours in all on the journey?

15. Divide $1000 into two parts, such that the interest on part of it at 5% and on the remainder at 6% may be $1.50 more than the interest on the whole at $5\frac{1}{2}\%$.

16. Divide $780 into two parts, such that the simple interest on the first part for 3 months at 5% may be $1.50 less than the simple interest on the remainder for 5 months at 6%.

17. A purse contains dimes and quarters, 35 coins in all. The total value of all the quarters is $1\frac{2}{3}$ times the total value of the dimes. How many dimes in the purse?

18. One-third of a rouble is worth 1 cent less than a franc, and 4 francs are together worth 5 cents less than $1\frac{1}{2}$ roubles. Find the value of a rouble in cents.

19. A franc is worth 3 cents more than $\frac{2}{3}$ of a mark; a mark and a franc are together worth 43 cents. Find the value of each coin in cents.

20. Five years ago Ann was half as old as Mary; at present Mary is 3 times as old as she was when Ann was born. How old is Ann?

21. A rectangle is $\frac{3}{4}$ as wide as it is long and its area is $2\frac{1}{4}$ sq. ft. less than that of a square of equal perimeter. Find the area of the rectangle.

22. A rectangle is $\frac{3}{5}$ as wide as it is long, and if an inch is added to both length and width its area is increased by 81 square inches. Find its perimeter in feet.

23. Two workmen, A and B, have each the same income of which A saves 5% and B 10% annually. B spends $15 more per month than A saves in a year. Find their income.

24. A teacher was able to save 15% of his salary, but his rent having been raised $5 per month, his whole annual expenses are now 6 times what he formerly saved. Find his income.

25. A saves 20% of his income; B has $1\frac{1}{2}$ times as large an income and saves 25% of it. A spends $20 more in 5 months than B spends in 4 months. How much does A save in a year?

26. A workman worked 64 days for $148, part of the time receiving $2 per day and for the remainder $2.50 per day. How much money did he earn at $2 per day?

27. A workman saves 10% of his income. His pay is increased 10% and his expenses rise $12\frac{1}{2}$%; his weekly savings are now 15 cents less than before. Find his original weekly wage.

28. A teacher gets an increase of $50 per annum for two years. The first year he saves $\frac{1}{4}$ of his salary, the second year he saves a third, and the last year he saves one half. His whole saving is equal to his salary for a year and two months at the original rate. Find his first year's salary.

CHAPTER V

SIMPLE EQUATIONS: TWO UNKNOWN QUANTITIES

(ELEMENTARY)

112. In the solution of problems in which two unknown numbers are to be found, we may use two letters, one to represent each number. We must then obtain from the problem two distinct statements, each of which will furnish an equation. From these equations the unknown numbers are to be found.

Ex. Find the two numbers whose sum is 10 and difference 2.

Let $x =$ the larger number.

and $y =$ " smaller "

Then $x + y = 10$ (1)

and $x - y = 2.$ (2)

Adding, $2x = 12,$ (Ax. 1)

or $x = 6.$

In (1) for x write 6, $6 + y = 10,$

or $y = 4.$

Then 6 and 4 are the numbers required.

Observe carefully the two facts given in this problem, (1) the sum is 10, (2) the difference is 2, and note how each equation expresses one fact in algebraical symbols.

113. In the preceding example, if but one statement had been made regarding the two numbers no definite result could have been obtained.

For if $\quad x + y = 10$ alone be given,

then $\quad x = 7,\ x = 6,\ x = 12,$ etc.,

$\qquad y = 3,\ y = 4,\ y = -2,$

all satisfy the required condition, since the sum of each pair is 10.

Similarly if $\quad x - y = 2$ alone is given,

we have $\quad x = 8,\ x = 6,\ x = 1,$ etc.,

$\qquad y = 6,\ y = 4,\ y = -1,$

each pair of which satisfy the given equation. But when both equations are to be satisfied *at the same time, i.e.*, by the same pair of numbers, there is but one pair, 6 and 4, which can be chosen.

114. Independent equations. Two equations which express different facts, *i.e.*, two facts, one of which cannot be inferred from the other, are said to be **independent.** The two equations of Art. 110 are independent, since from the fact that the sum of two numbers is 10 we cannot infer that their difference is 2. But the two equations

$$x + y = 10$$
$$2x + 2y = 20$$

are not independent: the second equation is a mere repetition of the first.

115. Simultaneous equations. Two independent equations which are to be satisfied by the same values of two unknown quantities are called **simultaneous equations.** The two equations of Art. 112 are simultaneous equations.

116. Elimination. From two simultaneous equations containing two unknown quantities it is generally possible to obtain a new equation containing but one unknown quantity. The quantity which does not appear in the new equation is said to be **eliminated,** and the process by which the new equation is obtained is called **elimination.**

By adding the two equations of Art. 112 the y was eliminated and the new equation contained x alone. Its value was then easily found.

117. Solution of simultaneous equations. The solution of simultaneous equations is effected by eliminating one of the unknown quantities and solving the resulting equation by the methods already given.

Ex. 1. Solve the equation $2x + 3y = 21,$ (1)

$5x + 2y = 25.$ (2)

Multiplying the first equation by 2 and the second by 3 we get

$4x + 6y = 42.$ (3)

$15x + 6y = 75.$ (4)

Subtracting (3) from (4), $11x = 33.$

or $x = 3.$

Similarly multiplying (1) by 5 and (2) by 2

we get $10x + 15y = 105.$ (5)

$10x + 4y = 50.$ (6)

Subtracting (6) from (5), $11y = 55.$

or $y = 5.$

Having found the value of x, we might have substituted its value, 3, in either of the given equations, and then would have been easily found. Thus for x write 3 in equation (1).

we have $6 + 3y = 21$,

from which $y = 5$ as before.

This process, known as **substitution**, is usually the most satisfactory method.

Ex. 2. Solve $\dfrac{x+y}{7} - \dfrac{2y-x}{3} = 3$, (1)

$$\frac{3y+2x}{4} + \frac{9(x-1)}{8} = \frac{x}{2}. \qquad (2)$$

Multiply (1) by 21, $3x + 3y - 14y + 7x = 63$,

collecting terms, $10x - 11y = 63$. (3)

Multiply (2) by 8, $6y + 4x + 9x - 9 = 4x$,

collecting terms, $9x + 6y = 9$,

or $3x + 2y = 3$. (4)

Multiplying (3) by (2) and (4) by 11,

$20x - 22y = 126$, (5)

$33x + 22y = 33$. (6)

Adding (5) and (6), $53x = 159$,

or $x = 3$.

Substituting 3 for x in (4), $9 + 2y = 3$,

or $y = -3$.

VERIFICATION

$$\frac{x+y}{7} - \frac{2y-x}{3} = \frac{3-3}{7} - \frac{-6-3}{3} = 0 - (-3) = 3,$$

and $\dfrac{3y+2x}{4} + \dfrac{9(x-1)}{8} = \dfrac{-9+6}{4} + \dfrac{18}{8} = -\dfrac{3}{4} + \dfrac{9}{4} = \dfrac{3}{2} = \dfrac{x}{2}$,

which proves the values found for x and y to be correct.

118. The ability to solve simultaneous equations quickly and correctly is obtained only by observation and experience, but the following general directions may be of service to the learner:

1. Clear each equation of fractions, remove brackets, collect the terms and strike out any factor which may be common to all the terms in either equation.

2. To eliminate a letter, find the L. C. M. of its coefficients in the two equations and multiply each equation by the quotient obtained by dividing the L. C. M. by the coefficient of that letter.

3. Subtract or add the resulting equations according as the signs of the coefficients of the letter to be eliminated are alike or different.

4. Substitute the value of the letter thus found for that letter in one of the preceding equations, choosing the one in the simplest form and with the smallest coefficients, and thus find the value of the remaining letter.

EXERCISE XX

Solve the equations and verify the results obtained.

1. $x+y=20$, $x-y=4$.

2. $x+y=25$, $x+2y=28$.

3. $2x+y=35$, $x+2y=37$.

4. $3x-y=16$, $2x+5y=141$.

5. $5x-2y=25$, $2x+3y=-9$.

6. $8x+3y=37$, $12x+5y=59$.

7. $2(x-1)+y=12$, $4x+3(y-1)=29$.

8. $3(x-5)+6=5y$, $4(y+1)+8=3x$.

9. $2(x+3)+7=y+23,$
$3(x-y)-5=4x-3.$

10. $x-y=7(x+y),$
$5x+7y=1.$

11. $\frac{x}{2}+\frac{y-1}{3}=6,$
$\frac{x-y}{4}+\frac{x}{3}=1.$

12. $\frac{x-y}{3}+1=\frac{x+y}{5}-\frac{7}{3},$
$\frac{x}{3}-\frac{y}{2}=0.$

13. $\frac{2x}{3}+\frac{y-1}{2}=\frac{x+y}{2},$
$\frac{x-1}{2}-\frac{y}{5}=\frac{y-x}{2}-1.$

14. $2(x-1)+\frac{2y}{5}=21,$
$3(y+1)-\frac{x}{11}=x-\frac{3}{2}.$

15. $\frac{x-1}{3}+\frac{y+1}{5}=\frac{x+y}{3},$
$x+5y=0.$

16. $\frac{y-3}{7}+\frac{2y-3}{5}=\frac{y}{2},$
$2(x-y+5)=3(y-x).$

17. $\frac{x+2y}{5}-\frac{y}{10}=\frac{2x+3y}{15},$
$\frac{x+5}{2}-1=0.$

18. $12(x-y)=\frac{3x+4y}{2},$
$\frac{x+1}{2}=4y-x.$

19. $\frac{1}{3}(x+7)+\frac{1}{5}(y+2)=3\frac{1}{2},$
$\frac{1}{5}(x+y)-\frac{1}{3}(x-y)=0.$

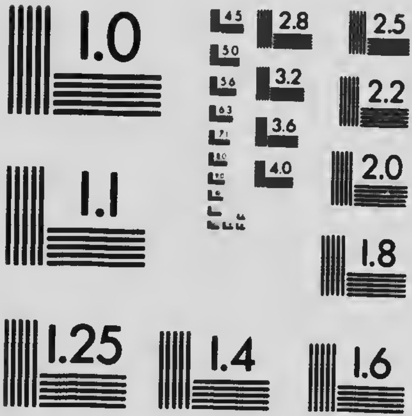
20. $\frac{1}{7}(3x+2y)=x-10,$
$\frac{1}{9}(3x-2y)=y+\frac{9x}{16}.$

21. $\frac{1}{3}(5x+7y+2)-\frac{1}{4}(3x+4y+7)=x,$
$\frac{1}{4}(7x+3y+4)-\frac{1}{5}(6x+5y+7)=y.$

22. $\frac{1}{3}(2x-3y-1)+\frac{1}{5}(x+y-2)=2x-\frac{y}{5},$
$\frac{1}{2}(5x+2y+1)-\frac{1}{3}(3-x-y)=\frac{1}{6}(y-2x).$

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



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PROBLEMS

119. The following are additional examples of the use of two letters in the solution of problems. In many cases the solution may also be effected by the skilful use of a single letter. The learner is recommended to occasionally solve the same problem by each method and to compare the various steps in the two solutions.

Ex. 1. A bill of \$5.55 was paid in quarter-dollars and 10-cent pieces, 27 coins in all. How many of each were used?

Let x = the number of quarter-dollars,
and y = " 10-cent pieces.
Then $25x$ = the value of the quarters in cents,
and $10y$ = " 10-cent pieces.
Then $x + y = 27$, the number of coins,
and $25x + 10y = 555$, their value in cents.

Solving these equations in the usual way, $x = 19$, and $y = 8$, the numbers required.

Ex. 2. A number consisting of two digits is equal to 5 times the sum of its digits, and if 9 be added, the resulting number consists of the same digits interchanged. Find the numbers.

Let x = the tens' digit and y = the units' digit.
Then $x + y$ = the sum of the digits,
$10x + y$ = the original number,
and $10y + x$ = the second "
The equations are $10x + y = 5(x + y)$,
and $10x + y + 9 = 10y + x$.
Solving, we get $x = 4$ and $y = 5$.
The required number is, therefore, 45.

Ex. 3. *A* saves $\frac{1}{5}$ of his daily pay and *B* saves $\frac{1}{4}$ of his; together they save \$1.25 per day. *A* receives an increase of 10% and *B* an increase of $12\frac{1}{2}$%, and now *A* saves 4 cents per day more than *B*. Find the daily wage of each.

Let x = number of cents received per day by *A*,
and y = " " " " " " " *B*.

Then $$\frac{x}{5}+\frac{y}{4}=125. \qquad (1)$$

and $$\frac{1}{5}\left(\frac{11x}{10}\right)-\frac{1}{4}\left(\frac{9y}{8}\right)=4. \qquad (2)$$

Simplifying $$4x+5y=2500 \qquad (3)$$

$$176x-225y=3200. \qquad (4)$$

Multiplying (3) by 45, $180x+225y=112500$.

Adding (3) and (4), $356x=115700$,

or $x=325$,

then $y=240$.

Their incomes are, therefore, \$3.25 and \$2.40 respectively.

EXERCISE XXI

1. The sum of two numbers is 47, and 3 times the smaller number is greater by a unit than twice the larger. Find the numbers.

2. The sum of two numbers is 5 times their difference, and the double of the smaller is greater than the larger by 4. Find the numbers.

3. Two pounds of tea and 3 pounds of coffee are together worth \$1.43, while 3 pounds of tea and 2 pounds of coffee are worth \$1.62. Find the value of a pound of each.

4. Two men together earn 90 cents more per day than 3 boys, and 4 men together earn 60 cents more per day than 7 boys. Find the daily wage of a man and a boy.

5. Two apples cost one-half as much as 3 oranges; a dozen apples and 21 oranges together cost a dollar. Find the price per dozen of apples and oranges.

6. Two bushels of oats weigh 8 lbs. more than one bushel of wheat; 2 bushels of wheat weigh 18 lbs. more than 3 bushels of oats. By how much does the weight of 3 bushels of wheat exceed the weight of 5 bushels of oats?

7. A and B play for a stake of $10, to be furnished by the loser. If A wins he will then have twice as much as B, but if B wins he will have 3 times as much as A. How much money had each at first?

8. Tom saves 25% of his week's wages and Dick saves 20% of his; together they save $6 per week. Tom's expenses are $\frac{3}{4}$ of Dick's expenses. Find their weekly wage.

9. A saves half his income, B saves one-third of his income; together they save $1.80 per day. If their incomes were interchanged and each saved the same fraction of his income as before, A would save 10 cents a day more than B. How much does each save per day?

10. Paid a dollar for some apples at 3 cents each and some oranges at 5 cents each. Sold two-fifths of the apples and one-fourth of the oranges at cost for 34 cents. How many of each did I buy?

11. A and B together earn $5.75 per day. If A's wages were reduced 20% and B's raised 20%, A would still have 10 cents a day more than B. How much does each earn per day?

12. The sum of the ages of A and B is $\frac{2}{3}$ of the sum of the ages of C and D. Two years ago the sum of the ages of A and B was one-half the sum of the ages of C and D. Find the sum of the ages of all four at present.

13. A farmer sold wheat at a dollar a bushel and barley at 80 cents, the average price for the whole being 88 cents. The total value of the barley was $10 more than the total value of the wheat. How many bushels did he sell in all?

14. A purse contains quarters and half-dollars, $13.75 in all. The total value of the half-dollars is greater by $5.25 than the total value of the quarters. How many coins are in the purse?

15. A number consisting of two digits is 4 times the sum of its digits. If 27 be added the resulting number will consist of the original digits interchanged. Find the number.

16. A number consisting of two digits is greater by 2 than 3 times the sum of its digits. The sum of its digits is double their difference. Find the number.

17. Prove that the sum of any two numbers consisting of the same pair of digits interchanged is divisible by 11, and the difference of the numbers is divisible by 9. State in words the quotient in each case.

18. Show that a number consisting of two digits, whose sum is three times their difference, is equal to either 4 times the sum of its digits or to 7 times their sum. Distinguish between the two cases.

19. A certain fraction becomes equal to $\frac{1}{2}$ when a unit is added to the numerator, but equal to $\frac{1}{3}$ if a unit be added to the denominator. Find the fraction.

20. The value of a certain fraction becomes $\frac{1}{4}$ when a unit is added to its numerator and to $\frac{1}{8}$ if 8 be added to its denominator. Find the fraction.

21. A bill is exactly paid by 8 marks and 12 guilders or by 13 marks and 9 guilders. How many marks would pay the bill?

22. A freight car carries 13 bales of cotton and 33 casks of wine as a full load. When 9 casks and 5 bales have been removed the car is still two-thirds filled. How many bales would fill the car?

23. The area of a rectangle will be unchanged if its width be increased 3 ft. and its length diminished 4 ft.; but if its width be diminished by 3 ft. and its length increased 5 ft., the area will be reduced by 15 sq. ft. Find its original length and breadth.

24. A boy can ride a bicycle $1\frac{1}{2}$ miles further in three hours than he can walk in 7 hours. He can ride from home to school in 15 minutes and return on foot in 36 minutes. How far is it to school?

25. A journey was performed in $4\frac{1}{2}$ hours, a part of it at 4 miles per hour and the remainder at 10 miles per hour. If the distances travelled at the two rates were interchanged, the time required would be 27 minutes greater. Find the whole distance travelled.

EXAMINATION PAPERS

I

1. Find the value of $(x-a)^2+(x-b)^2-2(x-a)(x-b)$ when $x=5$, $a=4$, $b=0$.

2. Draw a rectangle whose length is x feet and width y feet. Write in three different ways its perimeter in feet. Write its area in square feet and in square inches.

3. From the sum of $2a-3b+c$, $2(b-c)-a$ and $3b-2\left(c+\frac{1}{2}a\right)$ take the sum of $2a-3(b-c)$ and $c+2(a-b)$.

4. Multiply $(a+b)^2$ by $(a-b)^2$, add $a^2b^2-b^4$ to the product and divide the final result by $a-b$.

5 Divide $a^2(a-b) - b^2(a-b) + ab(a+b)$ by $a+b$.

6 Solve the equation

$$5x - [5 - \{2x + 3(1-x) + 2(x-3)\}] = 10.$$

7. A fish was caught whose tail weighed 5 lbs.: its head weighed as much as its tail and $\frac{1}{3}$ of its body, and its body weighed as much as its head and tail. Find weight of the fish.

II

1. Find the value of $\frac{a^2+bc^2}{ax-b-c^2}$ when $a=3$, $b=5$, $c=2$, $x=1$.

2. If $2^x = 16$, find the values of $3x$, x^3 and 3^x.

3. Add $a^2 - 2ab - \frac{10b^2}{15}$, $2b^2 - 3a\left(a+\frac{1}{3}b\right)$, $\frac{2b}{4}(a-2b) + 2a^2$.

4. Simplify

$$(2x+5)(x-3) + (1-2x)(3x+1) - 3(2x-1)(2-x).$$

5. Divide

$$(a-2b)(a-3b) + 2b(a-38b) \text{ by } 2(a+3b) - (a-b).$$

6. Solve the equation $\frac{1}{3}\left(\frac{3x}{4}+6\right) - \frac{4\frac{1}{2}-x}{3} = \frac{x}{2}\left(\frac{4}{x}+\frac{1}{6}\right)$.

7. Tom saves \$10 per week and Dick saves \$12 per week; Harry saves \$15 per week but starts two weeks after the other two. When will Harry's savings be as much greater than Tom's as they are less than Dick's savings, and how much money will each then have?

III

1. Find the value of $\sqrt{1+\frac{x-1}{1-y}} - \sqrt{\frac{1}{6}\left(\frac{1}{1-x}+\frac{1}{y-1}\right)}$ when $x=\frac{1}{3}$, $y=\frac{1}{4}$.

2. A rectangle is 6 feet longer than it is wide, and a square has the same perimeter as the rectangle. How much greater is the area of the square than that of the rectangle?

3. Express in words the result when the difference of any two numbers is (1) added to their sum, (2) subtracted from their sum.

4. Simplify $\frac{1}{2}\left(\frac{2x}{3}+4\right)-\frac{1}{3}\left(7\frac{1}{2}-x\right)-\frac{x}{2}\left(\frac{6}{x}-1\right)$ and find the value of the result when $x=3$, and when $x=5$.

5. Divide

$$(a-x)(a-2x)+(x-a)(x-2a)-2(a-x)^2 \text{ by } x-a.$$

6. Solve equation $\frac{x}{2}(2-x)-\frac{x}{4}(3-2x)=\frac{x+10}{6}$ and verify the result.

7. A and B play marbles. At the end of the first game A had twice as many as he had at first. At the end of the second game B had twice as many as he had at the close of the first game, and then each had the same number. If they together had 80, how many had each at first?

IV

1. Find the value of $a^2b-\frac{a}{b^2}-\left(\frac{a}{b}\right)\left\{\frac{1}{a^2}-\frac{1}{ab}+\frac{1}{b^2}\right\}$ when $a=-1$, $b=2$.

2. A rectangle is $2x+3$ feet long and width 8 feet less, find its perimeter, its area and the area of a square of equal perimeter.

3. Simplify $a(a+b-1)-b(a-b+2)-(a-b)(a+b-3\ldots$

4. Multiply $(a-b)^2+(b-c)^2+(c-a)^2$ by $a+b+c$.

5. Divide $(a^2+b^2)^2-a^2b^2$ by $(a+b)^2-ab$.

6. Solve equation $\frac{1}{2}\left\{\frac{1}{2}\left(\frac{x}{2}-\frac{1}{2}\right)-\frac{1}{2}\right\}-\frac{1}{2}=1.$

7. At a baseball game Tom made 5 more runs than he was years old; Dick, who was two years older, made twice as many runs as he was years old, beating Tom by 9 runs. How old was Tom?

V

1. Find the value of $ab(a-b)+bc(b-c)+ca(c-a)$ when $a=2$, $b=-3$, $c=5$.

2. If $x=a+b$, $y=a-b$, find the value of x^2-xy+y^2.

3. Divide

$$(x-1)^3+(x-2)^3+(x-3)^3-3(x-1)(x-2)(x-3) \text{ by } 9(2-x).$$

4. Simplify

$$(a-b)(a-b-x)(a+2b-2x)+b(b-x)(3a-2b-2x)$$

when for x we write a.

5. If $\frac{1}{2}(x-1)+\frac{1}{3}(x-2)=\frac{1}{4}(x-3)$, find the value of

$$\frac{1}{2}(x+1)+\frac{1}{3}(x-2).$$

6. Solve equations $x-\frac{1}{4}(y-2)=5$, $4y-\frac{1}{3}(x+10)=3$.

7. A boy spent 50 cents in buying apples at the rate of 3 for 5 cents and oranges at 40 cents a dozen. Had the numbers of apples and oranges been interchanged the cost would have been 20 cents more. How many of each did he buy?

VI

1. If $\frac{2}{3}(6x-5)=\frac{x}{6}-\frac{1}{2}(2-3x)$, find the value of

$$\frac{1}{3}(1-2x)+\frac{1}{4}(x-1)^2.$$

2. Find the algebraical expression which when divided

by $a^2 - ab + b^2$ gives $a^2 + ab + b^2$ for quotient with b ... remainder.

3. Simplify

$$(1-a)(1-b)(1-c)+a(1-b)(1-c)+b(1-c)+c.$$

4. Divide $(a+b)(b+c)(c+a)+abc$ by $a+b+c$.

5. Solve equation $\frac{1}{5}(2x+1)^2-\frac{1}{20}(4x-1)^2=\frac{15}{8}+\frac{3(x-1)}{10}$.

6. Solve equations $\frac{6x-2y}{5}-\frac{y+x}{10}=\frac{2x-1}{10}$,

$$y-\frac{1}{3}+\frac{x}{4}=10-\frac{y-2x}{3}.$$

7. Five francs are together worth one cent more than 2 florins, and if the value of a franc were decreased one cent, 3 francs would be worth 7 cents more than a florin. Find the value of each coin in cents.

VII

1. If $x=\frac{1}{3}$, $y=-\frac{1}{2}$ and $\frac{2}{3x}-\frac{3}{2y}+\frac{5}{z}=6$, find the value of z.

2. Simplify $(x+1)\{x(x-1)+1\}-(x-1)\{1+x(1+x)\}$.

3. Multiply $(a+b)^2+(b+c)^2-(a-c)^2$ by $a-b+c$.

4. Divide $(1+b)(1-b)+a+ab(a+b)$ by $1+a+b$.

5. Solve equation $a(x-a^2)+b(x-b^2)=0$ and verify the result.

6. The perimeter of a room is 44 feet. If it were 2 feet longer and 1 foot wider the area of the floor would be 34 sq. ft. greater. Find its length and width.

7. Two-thirds of a shilling is worth 3 cents less than a franc, and a half a franc is worth half a cent more than ⅓ of a shilling. Find the value of each coin in cents.

VIII

1. Simplify

$(a+b)\{2a-3(a-2b)-b\}-(a-b)\{2(b-3a)+2a-3b\}$.

2. Multiply $x^2-(a+b)x+ab$ by $x^2+(a-b)x-ab$ and divide the result by $(x-b)^2$.

3. Divide

$(x-a)^2+(y-b)^2-(ay-bx)^2+(a^2+b^2-1)(x^2+y^2-1)$
by $ax+by-1$.

4. If $x=a+2b-3c$, $y=a-b+3c$, find the value of $x^2+xy-2y^2$ in terms of a, b and c.

5. Solve the equation $\frac{2x}{3}\left(1-\frac{5}{x}\right)+\frac{3x}{4}\left(1-\frac{1}{x}\right)=\frac{5}{4}(x-1)$, and verify the result.

6. A number consisting of two digits is greater by 2 than 5 times the units' digit. If the digits be reversed the resulting number will be greater by 3 than 7 times the sum of its digits. Find the number.

7. A loses $\frac{1}{4}$ of his money and B gains an amount equal to $\frac{1}{5}$ of what A had at first, and now they have equal sums. If A should now give B \$50 he would have only one-half as much left as B would then have. How much had each at first?

CHAPTER VI

FACTORING

120. In multiplication two factors, the multiplicand and the multiplier, are given and from them their product is to be found. Division partially reverses this process, by giving the product and one factor, and from these the remaining factor is to be found. Factoring is the complete converse of multiplication, since from the product alone we seek to find both the factors from which it has been formed.

121. Certain algebraical expressions occur so frequently that it is desirable to be able to write their product or quotient without performing all the work of the ordinary operations. For this purpose it is necessary to learn what may be called the algebraical multiplication table, *i.e.*, the product of certain simple factors, and from them to obtain the product or quotient of more complicated expressions. This process reversed enables us to proceed from a given product to the factors which produce it. These two processes, placed side by side, form the subject matter of this chapter.

122. Monomial factors. The factors of a monomial are evident upon inspection.

Ex. 1. The factors of $3a^2b$ are 3, a, a and b.

Ex. 2. The factors of $15ab(a+b)$ are 3, 5, a, b and $a+b$.

123. Polynomials with monomial factors.

By multiplication we have $m(a+b) = ma + mb$.

Reversing the process, we have $ma + mb = m(a+b)$.

When each term of a polynomial contains the same factor the whole expression may be divided by this factor; the quotient will be the other factor.

Ex. 1. $2a + 2b + 2c = 2(a + b + c)$.

Ex. 2. $3a^2 + 3ab + 3ac = 3a(a + b + c)$.

Ex. 3. $5ax^2y + 5bxy^2 + 5cxyz = 5xy(ax + by + cz)$.

124. Groups of terms having a common factor.

By multiplication, $(a+b)(x+y) = a(x+y) + b(x+y)$
$= ax + ay + bx + by$.

Reversing the process, $ax + ay + bx + by = a(x+y) + b(x+y)$
$= (a+b)(x+y)$.

Similarly many expressions may be factored by separating them into groups containing a common monomial factor.

Ex. 1. $ac - bc - ad + bd = c(a-b) - d(a-b) = (c-d)(a-b)$.

Ex. 2. $2x^2 + ax - 6bx - 3ab = x(2x+a) - 3b(2x+a)$
$= (x-3b)(2x+a)$.

Ex. 3. $x^2 - ax + bx - cx + ac - bc = x(x-a+b) - c(x-a+b)$
$= (x-c)(x-a+b)$.

The terms must be grouped in such a way that when the monomial factors have been removed the resulting expressions within the brackets may be exactly alike. If the expression contains only *one* power of a particular letter, the terms containing that letter should be collected into a single group.

EXERCISE XXII

Resolve into factors:

1. $3a - 3b + 3c$.
2. $ab + b^2 - bc$.
3. $5ac - 10bc - 15c^2$.
4. $6x^3 - 9x^2 + 15x$.
5. $7p^3 - 35p^2 + 63p$.
6. $22m^2 - 33mn + 77n^2$.
7. $5a^2b - 15ab^2$.
8. $6a^2bx - 8ab^2y + 6abxy$
9. $12a^4b - 24a^3b^2 + 36a^2b^3$.
10. $26ax^2y^2 - 39bxy^3$.
11. $(a+b)x + (a+b)y$.
12. $a(x-y) - b(x-y)$.
13. $ac - ad + bc - bd$.
14. $ab - ac - bd + cd$.
15. $am + m + an + n$.
16. $ma - mb - a + b$.
17. $a^2 + ab + ac + bc$.
18. $ab + b^2 - ac - bc$.
19. $ax - bx + ab - x^2$.
20. $bc - a^2 + ab - ac$.
21. $x^3 - 3x^2 + 2x - 6$.
22. $a^3 - ab - 2a^2 + 2b$.
23. $2a^3 - a^2 + 2a - 1$.
24. $1 + a + b + ab$.
25. $1 - a - b + ab$.
26. $a^3 + a^2 + a + 1$.
27. $a^5 + a^4 - a^3 - a^2 + a + 1$.
28. $abx + aby - cdx - cdy$
29. $2ax^2 - 3bxy - 2axy + 3by^2$.
30. $a^2x + b^2y - ab(x+y)$.
31. $2ax + 3ay - 2bx - 3by - 5a + 5b$.
32. $a^2cx - abxy + abcx - b^2xy$.
33. $a^2b^2 - 3abc^2 - 3a^3c + b^3c$.
34. $x^2 + ax - bx - cx - ac + bc$.

125. Square of a binomial: Product of sum and difference. The following are three important examples in multiplication:

1.
$$\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \quad\quad \\ ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

2.
$$\begin{array}{r} a - b \\ a - b \\ \hline a^2 - ab \quad\quad \\ - ab + b^2 \\ \hline a^2 - 2ab + b^2 \end{array}$$

3.
$$\begin{array}{r} a + b \\ a - b \\ \hline a^2 + ab \quad\quad \\ - ab - b^2 \\ \hline a^2 \quad\quad - b^2 \end{array}$$

These results should be memorized in symbols in the following forms:

1. $(a+b)^2 = a^2 + 2ab + b^2.$

2. $(a-b)^2 = a^2 - 2ab + b^2.$

3. $(a-b)(a+b = a^2 - b^2.$

Their exact meaning should also be memorized in words, remembering that a and b stand for any numbers or algebraical expressions whatever.

1. The square of the sum of any two numbers is equal to the sum of their squares together with twice their product.

2. The square of the difference of any two numbers is equal to the sum of their squares less twice their product.

3. The product of the sum and difference of any two numbers is equal to the difference of their squares.

126. Definition. A general truth expressed in algebraic symbols is called a **formula.** The three equalities of Art. 125 are formulæ. They may also be called identities or identical equations. Art. 96.

127. From the formulæ of Art. 125 many results in multiplication may be written without performing the ordinary operation.

Ex. 1. From $(a+b)^2 = a^2 + 2ab + b^2$

we get $(2x+3y)^2 = 4x^2 + 12xy + 9y^2$

by writing $2x$ for a and $3y$ for b.

Ex. 2. From $(a-b)^2 = a^2 - 2ab + b^2$

we get $(3x-5)^2 = 9x^2 - 30x + 25$

by writing $3x$ for a and 5 for b.

Ex. 3. From $(a-b)(a+b)=a^2-b^2$

we get $(2x-3y)(2x+3y)=4x^2-9y^2$ as before.

Results in the multiplication of arithmetical factors may also be sometimes conveniently obtained.

Ex. 1. $65\times 65=(60+5)^2=(60)^2+2(5)(60)+5^2$

$=3600+600+25$

$=4225.$

Ex. 2. $49\times 49=(50-1)^2=(50)^2-2(50)+1$

$=2500-100+1$

$=2401.$

Ex. 3. $69\times 71=(70-1)(70+1)=4900-1$

$=4899.$

EXERCISE XXIII

Perform the operations indicated.

1. $(x+y)^2$.
2. $(x-y)^2$.
3. $(x+y)(x-y)$.
4. $(a+3b)^2$.
5. $(a-3b)^2$.
6. $(a-3b)(a+3b)$.
7. $(2a+3b)^2$.
8. $(2a-3b)^2$.
9. $(2a-3b)(2a+3b)$.
10. $(3x-5)^2$.
11. $(2x+5)^2$.
12. $(5x-3)(5x+3)$.
13. $(1+3a)^2$.
14. $(5b-1)^2$.
15. $(4a+1)(1-4a)$.
16. $(a^2+b^2)^2$.
17. $(b^2-a^2)^2$.
18. $(a^2-b^2)(b^2+a^2)$.
19. $(ax-by)^2$.
20. $(2a^2-3ab)^2$.
21. $(a^2+ab)(ab+a^2)$.
22. $(-1+2x)^2$.
23. $(-1-2x)^2$.
24. $(-1+2x)(-1-2x)$.
25. $(m^2n-mn^2)^2$.
26. $(2p^2-5pq^2)^2$.
27. $(2x-3y)(-2x-3y)$.
28. 81×81.
29. 87×93.
30. 78×82.
31. $(a+b)^2+(a-b)^2$.
32. $(a+b)^2-(a-b)^2$

33. $(a-2b)(a+2b)(a^2+4b^2)$.

34. $(2x-3y)(2x+3y)(4x^2+9y^2)$.

35. $(a-b)(a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8)$.

128. Trinomials which are exact squares.

Since $$a^2+2ab+b^2=(a+b)^2,$$
and $$a^2-2ab+b^2=(a-b)^2,$$
we observe that

1. A trinomial is the square of a binomial when it consists of two positive square terms plus or minus twice the product of the square roots of those terms.

2. The square roots of the square terms give the terms of the binomial, and the sign of the remaining term gives the connecting sign of the binomial.

Ex. 1. $$4x^2+12xy+9y^2=(2x)^2+2(2x)(3y)+(3y)^2$$
$$=(2x+3y)^2.$$

Ex. 2. $$x^4-6x^2+9=(x^2-3)^2.$$

Ex. 3. $$12x^2-60xy+75y^2=3(4x^2-20xy+25y^2)$$
$$=3(2x-5y)^2.$$

129. Difference of squares. Since $a^2-b^2=(a+b)(a-b)$, the difference between any two square terms is the product of the sum and the differences of the square roots of those terms.

Ex. 1. $$4a^2-9b^2=(2a+3b)(2a-3b).$$

Ex. 2. $$1-25x^2=(1+5x)(1-5x).$$

Ex. 3. $$50x^2-98y^4=2(25x^2-49y^4)$$
$$=2(5x+7y^2)(5x-7y^2).$$

Each result should be tested by multiplying together the factors obtained.

EXERCISE XXIV

Express in factors:

1. $x^2 - y^2$.
2. $x^2 - 2xy + y^2$.
3. $x^2 + 2x + 1$.
4. $4x^2 - 9y^2$.
5. $4a^2 + 12x + 9$.
6. $1 + 6a + 9a^2$.
7. $16x^2 - 1$.
8. $4a^2 - 4a + 1$.
9. $4a^2 + 20ab + 25b^2$.
10. $16x^4 - 81y^4$.
11. $x^6 - 10x^3 + 81$.
12. $9x^2 + 25y^2 - 30xy$.
13. $5a^4 - 20b^4$.
14. $3x^4 - 12x^2 + 12$.
15. $24x^2 + 54y^2 + 72xy$.
16. $a^2b^4 - 100c^6$.
17. $a^2b^2 + 25 - 10ab$.
18. $(a - b)^2 + 4ab$.
19. $162x^4 - 512y^4$.
20. $25a^4x^2 - 30a^2b^2xy + 9b^4y^2$.
21. $9a^2x^2 + 49b^2x^2 - 42abx^2$.
22. $3x^2y^2 - 18axy^3 + 27a^2y^4$.
23. $81a^2x^4 - a^2$.
24. $48x^5y^2 - 3xy^2$.
25. $(2a + 3b)^2 - 24ab$.
26. $(a + 2b)^2 - 4b(a + 2b)$.
27. $a^2c^2 - a^2d^2 - b^2c^2 + b^2d^2$.
28. $1 - a^2 - b^2 + a^2b^2$.
29. $a^2b^2 - b^2 - a^2c^2 + c^2$.
30. $(a + 2b)^2 - 2b(a + 2b) - b^2$.

130. Binomials having first terms alike. To form the product of two binomials which differ only in their second terms, the coefficients of their first terms being unity and positive.

Ex. 1.	*Ex. 2.*	*Ex. 3.*	*Ex. 4.*
$x + 2$	$x - 2$	$x + 2$	$x - 2$
$x + 3$	$x - 3$	$x - 3$	$x + 3$
$x^2 + 2x$	$x^2 - 2x$	$x^2 + 2x$	$x^2 - 2x$
$3x + 6$	$-3x + 6$	$-3x - 6$	$+3x - 6$
$x^2 + 5x + 6$	$x^2 - 5x + 6$	$x^2 - x - 6$	$x^2 + x - 6$

In these examples observe:

1. The first term of the product is the square of the first term of the two factors.

2. The coefficient of the second term is the algebraic sum of the second terms of the factors.

3. The last term of the product is the product of the second terms in the two factors.

131. The preceding results are all contained in the single formula

$$(x+a)(x+b) = x^2 + (a+b)x + ab;$$

but it is instructive to observe the results from actual multiplication with various numbers.

Ex. 1. For a and b write $2y$ and $-5y$ respectively.

Then $(x+2y)(x-5y) = x^2 + (2y-5y)x - 10y^2$

$= x^2 - 3xy - 10y^2.$

Ex. 2. Let $x = m+n,\ a = 3,\ b = -5.$

Then $(m+n+3)(m+n-5) = (m+n)^2 + (3-5)(m+n) - 15$

$= m^2 + 2mn + n^2 - 2m - 2n - 15.$

Ex. 3. $(3a-5)(3a+7) = (3a)^2 + 2(3a) - 35$

$= 9a^2 + 6a - 35.$

EXERCISE XXV

Perform the operations indicated.

1. $(x+3)(x+5)$. **2.** $(x-3)(x-5)$. **3.** $(x+3)(x-5)$.

4. $(y+5)(y-6)$. **5.** $(y-6)(y-1)$. **6.** $(y+11)(y-9)$.

7. $(a-3)(a-8)$. **8.** $(b+15)(b-5)$. **9.** $(c-10)(c+20)$.

10. $(x+2y)(x-3y)$. **11.** $(x-20y)(x+y)$.

12. $(x+13y)(x-3y)$. **13.** $(x^2-7)(x^2-2)$.

14. $(a^3-3)(a^3-5)$. **15.** $(b^4+25)(b^4-5)$.

16. $(x+ab)(x-3ab)$. **17.** $(1-3x)(1+10x)$.

18. $(a-7b^2)(a-b^2)$. **19.** $(x^2-2yz)(x^2+12yz)$.

20. $(a^2-5ab)(a^2+20ab)$. **21.** $(2x-5y)(2x-10y)$.

22. $(3a - 4b)(3a + 11b)$. **23.** $(x + a)(x + b)$.

24. $(3x + a)(3x - b)$. **25.** $(a + b + 5)(a + b - 3)$.

26. $(x^2 + x - 6)(x^2 + x + 4)$. **27.** $(x - 1)(x - 2)(x - 3)(x - 4)$.

28. $(x + 1)(x - 2)(x - 3)(x - 6)$.

29. $(x - 3)(x + 3)(x - 4)(x + 4)$.

30. $(a - b)(a - 5b)(a + b)(a + 5b)$.

132. Trinomials; coefficient of first term, unity and positive.

The formula $x^2 + (a + b)x + ab = (x + a)(x + b)$,

which is that of Art. 131 reversed, enables us to resolve a trinomial into factors providing it has factors of the form there given. This will be the case:

1. If the exponent of the leading letter in the first term be double that in the second, and

2. If two numbers can be found whose algebraic sum is the coefficient of the second term and whose product is the third term.

In other words, to factor a trinomial of this class we seek expressions which, when written in place of x, a and b in this formula, make the part on the left agree with the trinomial to be factored.

Resolve the following examples into binomial factors:

Ex. 1. Factor $x^2 - 8x + 15$. The first term of each factor is evidently x. The product of the second terms of the two factors is $+15$, therefore these terms have the same sign; and since the sign of the middle term is negative, both connecting signs of the factors are negatives; -3 and -5 are evidently the numbers.

Therefore $x^2 - 8x + 15 = (x - 3(x - 5)$.

Ex. 2. Factor $x^2 - 2x - 35$. The product of the second terms is -35, therefore they have unlike signs; the sum of these terms is -2, therefore the negative factor is the greater; 5 and -7 are evidently the numbers.

Therefore $$x^2 - 2x - 35 = (x + 5)(x - 7).$$

Ex. 3. Factor $a^4 + 2a^2b^2 - 24b^4$. In this example write x in place of a^2.

Then
$$\begin{aligned} a^4 + 2a^2b^2 - 24b^4 &= x^2 + 2b^2x - 24b^4 \\ &= (x + 6b^2)(x - 4b^2) \\ &= (a^2 + 6b^2)(a^2 - 4b^2) \\ &= (a^2 + 6b^2)(a + 2b)(a - 2b). \end{aligned}$$

Ex. 4. Factor $5x^2 - 5x - 30$. Each term evidently contains the factor 5. Removing it, we have
$$\begin{aligned} 5x^2 - 5x - 30 &= 5(x^2 - x - 30) \\ &= 5(x - 6)(x + 5). \end{aligned}$$

Ex. 5. Write y in place of $x^2 + x$.

Then $$y^2 - 8y + 12 = (y - 6)(y - 2).$$

Therefore
$$\begin{aligned} (x^2 + x)^2 - 8(x^2 + x) + 12 &= (x^2 + x - 6)(x^2 + x - 2) \\ &= (x + 3)(x - 2)(x + 2)(x - 1). \end{aligned}$$

EXERCISE XXVI

Resolve into factors:

1. $x^2 + 8x + 15$.
2. $x^2 + 8x + 12$.
3. $x^2 + 8x + 7$.
4. $y^2 - 7y + 12$.
5. $y^2 - 8y + 12$.
6. $y^2 - 13y + 12$.
7. $a^2 + 2a - 15$.
8. $a^2 - 2a - 15$.
9. $a^2 - 2a - 35$.
10. $b^2 + 2b - 35$.
11. $b^2 - 11b - 12$.
12. $b^2 + 11b - 12$.
13. $m^2 - 20m + 51$.
14. $m^2 - 20m - 300$.
15. $m^2 + 19m - 20$.

16. $x^4 - 4x^2 - 21$. 17. $x^4 - 5x^2 + 4$. 18. $x^4 - 10x^2 + 9$.

19. $x^6 + x^3 - 12$. 20. $x^8 - 14x^4 - 32$. 21. $x^4 - 41x^2 + 400$.

22. $x^2y^2 - 14xy - 32$. 23. $x^4y^2 + 15x^2y - 100$.

24. $x^2 + 20xy + 75y^2$. 25. $m^2 + 5mnp - 84n^2p^2$.

26. $a^2b^2 - 18abc + 56c^2$. 27. $1 + 5x^2 - 6x^4$.

28. $4x^2 + 4x - 24$. 29. $3a^2 - 3a - 216$.

30. $5x^4 + 30x^2 - 200$. 31. $2ax^6 - 10a^2x^3 - 28a^3$.

32. $a^2b + 18a^2bx - 19a^2bx^2$. 33. $x - 5x^2 - 6x^3$.

34. $x^2 + (2a + b)x + 2ab$. 35. $x^2 - (2a + 3b)x + 6ab$.

36. $x^4 + (a^2 - b^2)x^2 - a^2b^2$. 37. $x^4 - (4a^2 + 9b^2)x^2 + 36a^2b^2$.

38. $(3a)^2 - 2(3a) - 15$. 39. $(4a^2)^2 - 17(4a^2) + 16$.

40. $(x^2 - 2x)^2 - 2(x^2 - 2x) - 3$. 41. $(x^2 + 4x)^2 - 2(x^2 + 4x) - 15$.

133. Trinomials; first coefficient, not unity. Trinomials which do not come under either of the classes already considered, may sometimes be resolved into two binomial factors.

Consider the following examples in multiplication.

$3x - 5$	$4x - 3y$
$2x - 3$	$x + 6y$
$6x^2 - 10x$	$4x^2 - 3xy$
$\quad - 9x + 15$	$\quad 24xy - 18y^2$
$6x^2 - 19x + 15$	$4x^2 + 21xy - 18y^2$

Such products may easily be written from inspection.

1. The first term of the product in each case is the product of the first terms of the factors, and the last term is the product of the last terms of the factors.

2. The middle term of the product in each case is the algebraic sum of the product of each first term in one factor, and the last term in the other factor.

Thus $2(-5) + 3(-3) = -19$ and $1(-3) + 6(4) = 21$.

When the numbers are not too large these processes can be performed mentally and the result obtained without written work.

The preceding process reversed will generally enable us to obtain the factors of a trinomial of this class, providing it has binomial factors.

Ex. Resolve $4x^2 + 5xy - 6y^2$ into factors.

We observe:

1. The coefficients of the first terms of the two factors must be either 2, 2, or 4, 1; those of the last terms must be 2, 3, or 6, 1, one coefficient being negative in the latter case.

2. From these pairs of numbers coefficients must be selected by trial, which by cross-multiplication, as in Art. 133, will give the coefficient, 5, of the middle term.

3. It is useless to try such a combination as $2x - 4y$, since that would give 2 as a factor of the whole expression, which is not the case.

By trial we find $4x^2 + 5xy + 6y^2 = (4x - 3y)(x + 2y)$.

Similarly $\quad 4x^2 - 23xy - 6y^2 = (4x + y)(x - 6y)$.

The more difficult cases in the factoring of trinomials will be considered at a subsequent stage of the work.

EXERCISE XXVII

Write from inspection the product of the factors

1. $(2x + y)(x + 2y)$.
2. $(2x - y)(x - 2y)$.
3. $(3x + 2y)(x + y)$.
4. $(3x - 2y)(x + y)$.
5. $(4x - 3y)(x - 2y)$.
6. $(3x - 5y)(2x + 3y)$
7. $(4x - y)(x + 6y)$
8. $(6x^2 - y^2)(x^2 - 2y^2)$

9. $(9x + 8y)(8x - y)$. **10.** $(5x^2 - y)(x^2 - 5y)$.

11. $2(3x - 4y)(2x + y)$. **12.** $a(6x + y)(x - 6y)$.

Resolve into factors:

13. $2x^2 - 5x + 2$. **14.** $2x^2 + 5xy + 3y^2$.

15. $2x^2 - 7xy + 3y^2$. **16.** $2x^2 - 11x + 12$.

17. $2x^2 + 5x - 12$. **18.** $12x^2 - x - 20$.

19. $12x^2 - 25xy + 12y^2$. **20.** $6ax^2 - 37axy + 6ay^2$.

21. $10a^2x^2 - 7a^2xy - 33a^2y^2$. **22.** $ax^2 + (a^2 + 1)xy + ay^2$.

23. $4x^4 - 17x^2y^2 + 4y^4$. **24.** $9x^4 - 82x^2y^2 + 9y^4$.

134. Cube of a binomial. From Art. 123 we have

$$(a + b)^2 = a^2 + 2ab + b^2$$

Multiplying by $a + b$

$$\begin{array}{rl} & a^2 + 2ab + b^2 \\ & a + b \\ \hline & a^3 + 2a^2b + ab^2 \\ & \quad\quad a^2b + 2ab^2 + b^3 \\ \hline (a + b)^3 = & a^3 + 3a^2b + 3ab^2 + b^3 \end{array} \quad (1)$$

Similarly
$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 \quad (2)$$

It is apparent at once that (2) can be derived from (1) by substituting $-b$ for $+b$ throughout.

The student should commit to memory results (1) and (2).

135. The results (1) and (2) may also be written thus

$$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$$
$$(a - b)^3 = a^3 - b^3 - 3ab(a - b)$$

These forms are sometimes more convenient than (1) and (2).

136. Homogeneous expressions. When each term of an expression has the same number of literal factors it is said to be **homogeneous**. The number of literal factors in each term is called its **degree** or **dimensions.** If the expression be not homogeneous it is said to have the same degree as its highest term.

Thus $a^2 + 2ab + b^2$ is a homogeneous expression of the second degree; $a^3 + 3a^2b + 3ab^2 + b^3$ is a homogeneous expression of the third degree; $a^2x^2 + by$ is an expression of the fourth degree but not homogeneous.

EXERCISE XXVIII

Perform the operations indicated:

1. $(x+y)^3$. 2. $(x-y)^3$. 3. $(x+1)^3$. 4. $(x-1)^3$.

5. $(a+2b)^3$. 6. $(a-2)^3$. 7. $(2a+1)^3$. 8. $(2a-b)^3$.

9. $(a+3b)^3$. 10. $(3a-b)^3$. 11. $(2a+3b)^3$. 12. $(5a-2b)^3$.

13. $(a+b)^3+(a-b)^3$. 14. $(a+b)^3-(a-b)^3$.

15. Divide $a(a+2b)^2 - b(2a+b)^2$ by $a-b$.

16. Divide $a(a-2b)^2 + b(2a-b)^2$ by $a+b$.

17. Divide $(a+b)^3 + c^3$ by $a+b+c$.

18. Divide $a^3 - (b-c)^3$ by $a-b+c$.

19. Divide $(a+2b)^3 - b^3$ by $a+b$.

20. Divide $(a+b)^3 - 8b^3$ by $a-b$.

21. Divide $a(a+2b)^3 - b(2a+b)^3$ by $(a-b)^3$.

22. Divide $a(a-2b)^3 + b(2a-b)$ by $(a+b)^3$.

23. Find the value of $x^3+y^3+z^3-3xyz$ when

$$x = a-b,\quad y = b-c,\quad z = c-a.$$

137. Sum and difference of cubes. The rules for factoring have thus far been derived from the process of multiplication. The factoring of binomials consisting of the sum or the difference of two cubes is more readily obtained from division.

By actual division we easily obtain

$$\frac{a^3+b^3}{a+b} = a^2 - ab + b^2, \qquad \frac{a^3-b^3}{a-b} = a^2 + ab + b^2$$

And since the dividend is always equal to the produ[ct of] the divisor and quotient, we have

$a^3+b^3 = (a+b)(a^2-ab+b^2)$, $\quad a^3-b^3 = (a-b)(a^2+ab+$ ...

from which the factors of the sum or the difference of two cubes may be readily obtained.

Ex. 1. $8x^3+27y^3 = (2x)^3+(3y)^3$

$= (2x+3y)(4x^2-6xy+9y^2)$.

To obtain this result we observe:

1. $8x^3$ is the cube of $2x$ or $(2x)^3$; $27y^3$ is the cub[e of] $3y$ or $(3y)^3$.

2. Let $a = 2x$, $\quad b = 3y$,

then $a^2 = 4x^2$, $\quad b^2 = 9y^2$ and $ab = 6xy$.

Therefore $a+b = 2x+3y$ and $a^2-ab+b^2 = 4x^2-6xy+$ ...

Making these changes for a and b in the factors of ... we obtain the required factors for the given expression.

Ex. 2. $x^6+y^6 = (x^2)^3+(y^2)^3$

$= (x^2+y^2)(x^4-x^2y^2+y^4)$.

Note that while $2^3 = 8$, $(x^2)^3 = x^6$, therefore $(2x^2)^3 = 8x^6$; *i.e.*, to cube a monomial we cube the coefficient and multiply the exponent by 3.

Ex. 3. $x^6-y^6 = (x^3+y^3)(x^3-y^3)$. Art. 1..

$= (x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$

Ex. 4. $(a-b)^3+c^3 = \{(a-b)+c\}\{(a-b)^2-(a-b)c+$ ...

$= (a-b+c)(a^2+b^2+c^2-2ab-$ ...

EXERCISE XXIX

Resolve into factors:

1. $x^3 + y^3$.
2. $x^3 + 8$.
3. $x^3 + 1$.
4. $x^3 + 64$.
5. $x^3 - y^3$.
6. $1 - y^3$.
7. $1 - 8y^3$.
8. $8x^3 + 125y^3$.
9. $64x^3 - 1000y^3$.
10. $x^6 + y^6$.
11. $x^6 + 1$.
12. $27 + y^6$.
13. $x^6 - 64$.
14. $8a^3 + 27b^3c^3$.
15. $125x^3y^3 - 343z^3$.
16. $27a^4 - 64a$.
17. $250a - 16a^4$.
18. $x^6 + 8y^3$.
19. $a^9 + b^9$.
20. $8a^6 + 27b^6$.
21. $64a^6b^6 - 1$.
22. $a^{12} + b^{12}$.
23. $a^{12} - b^{12}$.
24. $(a - b)^3 - 1$.
25. $(a - b)^3 + 8b^3$.
26. $(a + b)^3 - 8b^3$.
27. $(a + 2b)^3 - 27b^3$.
28. $(x^2 - x)^3 - 8$.
29. $(x^2 + x)^3 - 8$.
30. $1 + a^3 + b^3 + a^3b^3$.
31. $1 - a^3 - b^3 + a^3b^3$.

138. Square of Polynomials. Consider the following examples in multiplication:

$$\begin{array}{l} a + b + c \\ a + b + c \\ \hline a^2 + ab + ac \\ \quad\;\; ab + b^2 + bc \\ \quad\;\; ac + bc + c^2 \\ \hline a^2 + 2ab + 2ac + b^2 + 2bc + c^2 \\ = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc \end{array} \qquad \begin{array}{l} a + b - c \\ a + b - c \\ \hline a^2 + ab - ac \\ \quad\;\; ab + b^2 - bc \\ \quad\; - ac - bc + c^2 \\ \hline a^2 + 2ab - 2ac + b^2 - 2bc + c^2 \\ = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc \end{array}$$

These results consist in each case of two sets of terms:

1. The squares of the several terms, each being positive.

2. Twice the product of each pair of terms, each product being positive or negative according as the terms which produced it had like or unlike signs.

The same laws of formation would evidently hold good in forming the square of any polynomial.

139. To square a polynomial. *To the sum of the squares of each term add twice the product of each term into each of the terms which follow it, the sign of each product being obtained from the signs of the terms which produced it.*

These results may also be obtained from the squares of a binomial (Art. 125) by enclosing two of the terms in a bracket.

Ex. 1. $(a+b+c)^2 = \{a+(b+c)\}^2 = a^2+2a(b+c)+(b+c)^2$
$= a^2+2ab+2ac+b^2+2bc+c^2$
$= a^2+b^2+c^2+2ab+2ac+2bc$ as before.

Ex. 2. $(a-b-c)^2 = \{a-(b+c)\}^2 = a^2-2a(b+c)+(b+c)^2$
$= a^2-2ab-2ac+b^2+2bc+c^2$
$= a^2+b^2+c^2-2ab-2ac+2bc.$

The learner should practise both methods and be careful of a negative sign preceding a bracket.

EXERCISE XXX

Write from inspection :

1. $(x+y+z)^2$. **2.** $(x+y-z)^2$. **3.** $(x-y+z)^2$.

4. $(x-y-z)^2$. **5.** $(-x+y-z)^2$. **6.** $(a-2b+c)^2$.

7. $(2a-b-3c)^2$. **8.** $(a-2b+5c)^2$. **9.** $(a-b+c-d)^2$.

10. $(a^2+ab+b^2)^2$. **11.** $(a^2-ab+b^2)^2$.

12. $(1-x+x^2)^2$. **13.** $(x^2+x-1)^2$.

14. $(x^3-x^2+x-1)^2$. **15.** $(ab+ac+bc)^2$.

Express the following as squares :

16. $(a+b)^2+2a(b+c)+c^2$. **17.** $a^2-2a(b-c)+c^2$.

18. $(a+b)^2-4b(a+b)+4b^2$. **19.** $(a-b)^2+4b(a-b)+4b^2$.

20. $a^2-2a(a+b)+(a+b)^2$. **21** $a^2-2a(a-b)+(a-b)^2$.

22. $1 + 4a^2 + b^2 + 4a - 2b - 4ab$.

23. $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$.

24. $a^2 + 4b^2 + 9c^2 - 4ab - 6ac + 12bc$.

140. Extension of the sum and difference formula in multiplication. The product of two expressions which are alike, excepting the sign of one or more terms, may always be obtained from the sum and difference formula, as shown in the following examples:

Ex. 1. Find the product of $x + y + z$ and $x + y - z$.

In the formula $\qquad (a + b)(a - b) = a^2 - b^2$,

writing $x + y$ in place of a, and b in place of z,

we have $\qquad \{(x + y) + z\}\{(x + y) - z\} = (x + y)^2 - z^2$;

that is $\qquad (x + y + z)(x + y - z) = x^2 + 2xy + y^2 - z^2$.

Ex. 2.

$$\begin{aligned}(a + b - c)(a - b + c) &= \{a + (b - c)\}\{a - (b - c)\}\\ &= a^2 - (b - c)^2\\ &= a^2 - b^2 + 2bc - c^2.\end{aligned}$$

In Ex. 1 the terms $x + y$ have the same signs in the two expressions; they are, therefore, placed in a bracket and take the place of a single letter.

In Ex. 2 the only term having the same sign in the two expressions is a, which, therefore, stands alone. The b and c having *opposite* signs in the two expressions, have the *same* sign after being placed in brackets with a negative sign preceding the bracket in the second expression.

Ex. 3.

$$\begin{aligned}(a^2 + ab + b^2)(a^2 - ab + b^2) &= \{(a^2 + b^2) + ab\}\{(a^2 + b^2) - ab\}\\ &= (a^2 + b^2)^2 - a^2b^2\\ &= a^4 + 2a^2b^2 + b^4 - a^2b^2\\ &= a^4 + a^2b^2 + b^4.\end{aligned}$$

Note that in this example the product of two trinomials is a trinomial, and observe many similar examples in the following exercise.

EXERCISE XXXI

Simplify the following by the sum and difference method:

1. $(a+b+c)(a+b-c)$.
2. $(a+b-c)(a-b-c)$.
3. $(2a-b+c)(2a-b-c)$.
4. $(a+2b-c)(a-2b+c)$.
5. $(x-2y+3z)(2y+3z-x)$.
6. $(x^2+x+1)(x^2-x+1)$.
7. $(x^2+xy+y^2)(x^2-xy+y^2)$.
8. $(x^2+3x+9)(x^2-3x+9)$.
9. $(2x^2+3x+3)(2x^2-3x+3)$.
10. $(x^2+2ax+2a^2)(x^2-2ax+2a^2)$.
11. $(2a^2+2a+1)(2a^2-2a+1)$.
12. $(a^4+a^2+1)(a^4-a^2+1)$.
13. $(a+b+c+d)(a+b-c-d)$.
14. $(a-b+c-d)(a+b-c+d)$.
15. $(a^2+2ab+b^2)(a^2-2ab+b^2)$.
16. $(a^2+b^2-c^2+2ab)(c^2-a^2-b^2+2ab)$.
17. $(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$.

141. Extension of the sum and difference formula in factoring.

Ex. 1. Factor $(a-2b)^2-9c^2$.

This expression is the difference of two squares.
The first term is the square of $a-2b$.
The second term is the square of $3c$.
The sum of these terms is $a-2b+3c$.
The difference of these terms is $a-2b-3c$.
Therefore $(a-2b)^2-9c^2=(a-2b+3c)(a-2b-3c)$.

Ex. 2.

$$(5x-2)^2-(x-4)^2=\{(5x-2)+(x-4)\}\{(5x-2)-(x-4)\}$$
$$=(5x-2+x-4)(5x-2-x+4)$$
$$=(6x-6)(4x+2)$$
$$=6(x-1)2(2x+1)$$
$$=12(x-1)(2x+1).$$

Ex. 3. $a^2-4b^2-9c^2+12bc=a^2-(4b^2-12bc+9c^2)$
$$=a^2-(2b-3c)^2$$
$$=(a+2b-3c)(a-2b+3c).$$

The term $12bc$ must evidently be taken with $4b^2$ and $9c^2$ to make a square, but since these latter terms are negative, we place the whole three in a bracket preceded by the negative sign, thus forming an exact square.

EXERCISE XXXII

Resolve into factors:

1. $(a+b)^2-c^2$.
2. $a^2-(b-c)^2$.
3. $a^2-(b+c)^2$.
4. $(a-2b)^2-c^2$.
5. $4a^2-(a+b)^2$.
6. $4a^2-(2a-b)^2$.
7. $(x^2-5)^2-16$.
8. $(x^2-10)^2-36$.
9. $(x^2-x)^2-36$.
10. $(2a-3b)^2-(a-2b)^2$.
11. $(3a-5b)^2-(a+b)^2$.
12. $a^2+2ab+b^2-9c^2$.
13. $a^2-b^2-4c^2+4bc$.
14. $b^2+c^2-a^2-2bc$.
15. $2bc-b^2-c^2+a^2$.
16. $a^2-2ab+b^2-4$.
17. $a^2x^2+b^2y^2+2abxy-1$.
18. $1-9a^2-25b^2+30ab$.
19. $4ab-4a^2-b^2+25c^2$.
20. $a^2+2ab+b^2-c^2-2cd-d^2$.
21. $a^2-b^2+c^2-d^2+2ac+2bd$.
22. $a^2+b^2-c^2-d^2-2ab-2cd$.
23. $a^2-b^2+c^2-d^2-2ac-2bd$.

24. $(x^2 - 2x - 9)^2 - 36.$

25. $(x^2 - 5x + 5)^2 - 1.$

26. $(x^2 - 25)^2 - 4(x + 5)^2.$

27. $(2x^2 + 3x - 5)^2 - (x^2 - 9x - 40)^2.$

142. Trinomials which are the difference of squares. Trinomials having two terms which are exact squares may sometimes be expressed as the difference of squares and thus be resolved into factors.

Ex. 1.
$$\begin{aligned} a^4 + a^2b^2 + b^4 &= a^4 + 2a^2b^2 + b^4 - a^2b^2 \\ &= (a^2 + b^2)^2 - a^2b^2 \\ &= (a^2 + b^2 + ab)(a^2 + b^2 - ab) \\ &= (a^2 + ab + b^2)(a^2 - ab + b^2). \end{aligned}$$

The student will observe that the above is simply Ex. 3, Art. 140, reversed.

Ex. 2.
$$\begin{aligned} a^4 + 2a^2b^2 + 9b^4 &= (a^2 + 3b^2)^2 - 4a^2b^2 \\ &= (a^2 + 3b^2 + 2ab)(a^2 + 3b^2 - 2ab) \\ &= (a^2 + 2ab + 3b^2)(a^2 - 2ab + 3b^2). \end{aligned}$$

The square terms a^4 and $9b^4$ suggest $(a^2 + 3b^2)^2$. This includes $6a^2b^2$ as a middle term; the given expression has only $2a^2b^2$, therefore $4a^2b^2$ must be subtracted.

EXERCISE XXXIII

Factor:

1. $x^4 + x^2y^2 + y^4.$
2. $x^4 + 2x^2 + 9.$
3. $9a^4 + 2a^2b^2 + b^4.$
4. $a^4 + 5a^2b^2 + 9b^4.$
5. $4a^4 + 11a^2 + 25.$
6. $a^4 - 3a^2b^2 + b^4.$
7. $a^4 + 4b^4.$
8. $4a^4 + 1.$
9. $4x^4 + 3x^2y^2 + 9y^4.$
10. $a^4 + 64b^4.$
11. $4x^4 + 4x^2 + 25.$
12. $a^4 - 10a^2b^2 + 9b^4.$

13. $49x^4 - 44x^2y^2 + 4y^4$.

14. $25a^4 + 71a^2x^2 + 64x^4$.

15. $4x^4 - 37x^2y^2 + 9y^4$.

16. $(a+b)^4 + 4(a-b)^4$.

17. $(a+b)^4 + (a^2-b^2)^2 + (a-b)^4$.

18. $(a-1)^4 + 4$.

19. $a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2a^2c^2$.

MISCELLANEOUS EXAMPLES

143. The principles already expressed may be combined in various ways and extended to complicated expressions. The grouping method of Art. 124 may be extended to groups of terms containing a common binomial or trinomial factor.

The student should, however, reserve the greater part of the following exercise for a second reading of the subject.

Ex. 1. Factor $x^2 - y^2 + ax - ay$.

The terms may be arranged in two groups, each containing the factor $x - y$.

Thus $$x^2 - y^2 + ax - ay = (x^2 - y^2) + a(x - y)$$
$$= (x - y)(x + y + a).$$

The principle of Art. 122 may be extended to the product of trinomial factors.

Ex. 2. Factor $9x^4 + 15x^3 - 5x - 1$.

We have $$9x^4 + 15x^3 - 5x - 1 = (9x^4 - 1) + 5x(3x^2 - 1)$$
$$= (3x^2 - 1)(3x^2 + 1) + 5x(3x^2 - 1)$$
$$= (3x^2 - 1)(3x^2 + 1 + 5x)$$
$$= (3x^2 - 1)(3x^2 + 5x + 1).$$

Ex. 3. Factor $2x^2 - 6y^2 + 8z^2 - xy - 8xz + 2yz$.

First factor the trinomial containing x and y.

Thus $$2x^2 - xy - 6y^2 = (2x + 3y)(x - 2y).$$

To each of these factors attach a term containing z. The product of the two terms added must be $8z^2$, and from the term $-8z^2$ we see that they must both be negative. By trial we find that $-4z$ added to the first factor and $-2z$ to the second give the required product.

Thus $(2x+3y-4z)(x-2y-2z)$ are the factors required.

EXERCISE XXXIV

Factor:

1. $a^2 - b^2 - ac + bc$.
2. $a^2 - a - b^2 + b$.
3. $x^2 + ax + a - 1$.
4. $a^2 + 3a - 4b^2 + 6b$.
5. $4a^4 + 6a^3 - 3a - 1$.
6. $a^4 - b^4 - 2ab(a^2 - b^2)$.
7. $a^3 - 1 + a(a - 1)$.
8. $a^3 + 1 - a(a + 1)$.
9. $(y + 1)^3 - 3y(y + 1)$.
10. $y^3 + 1 + a(y^2 - y + 1)$.
11. $(ab + 1)^2 - (a + b)^2$.
12. $(ab - 1)^2 - a^2 - b^2 + 2ab$.
13. $x^4 - 3x^2 - 4$.
14. $x^4 - 3x^2 + 1$.
15. $a^4 - 7a^2b^2 + b^4$.
16. $a^4 - 11a^2b^2 + b^4$.
17. $x^4 + 10x^2y^2 + 9y^4$.
18. $x^4 + 10x^2y^2 + 49y^4$.
19. $8a^3 - 2a^2 - 6a$.
20. $12x + 3x^2 - 9x^3$.
21. $a^4 + a^2 + \frac{1}{4}$.
22. $a^4 + \frac{1}{4}$.
23. $(a - 1)^4 + 4$.
24. $(a + 1)^4 - 16$.
25. $a^7b - ab^7$.
26. $3a^7 + 3ab^6$.
27. $m^4 + n^4 - mn(m^2 + n^2)$.
28. $m^4 + n^4 + mn(m^2 + n^2)$.
29. $a^4 - 4b^4 + ab(2a^2 + ab)$.
30. $x^4 - y^4 + xy(2x^2 + xy)$.
31. $a(1 - c^2) - c(1 - a^2)$.
32. $(1 + a)^2(1 + c^2) - (1 + c)^2(1 + a^2)$.
33. $x^4 + (2b^2 - a^2)x^2 + b^4$.
34. $x^4 + 2ax^3 + a^2x^2 - b^4$.
35. $x^2y^3 - x^2 - y^3 + 1$.
36. $a^3b^2 - 4a^3 - b^2 + 4$.
37. $1 + y - x^2(1 - y) - 2xy$.
38. $1 + y - x^2(1 - y) + 2x$
39. $(x^2 - 2x - 25)^2 - 100$.
40. $x^6 + 6abx^4 + 8a^2b^2x^2$.

41. $(a-b)^6+6ab(a-b)^4+8a^2b^2(a-b)^2$.

42. $(a+b)^6-6ab(a+b)^4+8a^2b^2(a+b)^2$.

43. $x(x+2y)^2-y(2x+y)^2$.

44. $x(x+2y)^3-y(2x+y)^3$

45. $x(x-2y)^3+y(2x-y)^3$.

46. $4a^2b^2-(a^2+b^2-c^2)^2$.

47. $4b^2c^2-(a^2-b^2-c^2)^2$.

48. $(a^2-b^2)(x^2-y^2)+4abxy$.

49. $(a^2-b^2)(x^2-y^2)-4abxy$.

50. $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$

51. $(a^2-b^2-c^2+d^2)^2-4(ad-bc)^2$.

52. $a^2(a^2-1)-b^2(b^2-1)$.

53. $ab(x^2-y^2)+xy(a^2-b^2)$.

54. $a^3+b^3-c^3+3ab^2+3a^2b$.

55. $a^3-b^3+c^3-3a^2b+3ab^2$.

56. $(a+b)^2-3(a+b)c-4c^2$.

57. $a^2+b^2-12c^2-2ab-ac+bc$.

58. $6x^2+3y^2-8z^2+11xy+8xz-2yz$.

59. $(a-b)(b^2-c^2)-(a^2-b^2)(b-c)$.

60. $(a-b)(b^3-c^3)-(a^3-b^3)(b-c)$.

61. $(1+y)^2-2x^2(1-y^2)+x^4(1-y)^2$

62. $(1+y)^2-2x^2(1+y^2)+x^4(1-y)^2$

63. $x^2(x^2-a^2)-y^2(y^2-a^2)+2xy(x^2-y^2)$.

64. $\{(a-b)(a-c)-2a(b+c)\}^2-(a+b)^2(a+c)^2$.

65. $\{(a-b)(c-d)+2(ab+cd)\}^2-(a+b)^2(c+d)^2$.

CHAPTER VII

EQUATIONS SOLVED BY FACTORING

144. Equations containing the square or higher powers of the unknown quantity occur in many departments of mathematical work. Their solution in simple cases is conveniently effected by factoring. The following axioms are employed:

√ **1.** Zero multiplied by any number whatsoever gives zero as product.

2. If the product of any number of factors is zero, one (at least) of the factors is zero.

Expressed in symbols these axioms become:

(1) If x denote any number, then $x \times 0 = 0$;

(2) If $xy = 0$, then either $x = 0$, or $y = 0$, or if $(x-2)(x-3) = 0$ then either $x - 2 = 0$ or $x - 3 = 0$.

Ex. 1. Solve the equation $x^2 - 2x = 15$.

SOLUTION

Bring all the terms to the first side of the equation.

Then $\qquad x^2 - 2x - 15 = 0.$

Factoring, $\qquad (x-5)(x+3) = 0.$

Therefore either $\qquad x - 5 = 0$ or $x + 3 = 0$;

that is $\qquad x = 5$ or $x = -3$.

The meaning of the double result is that two different values of x satisfies the equation. In other words the equation has two roots.

VERIFICATION

When $x = 5$, $x^2 - 2x = 5^2 - 2(5) = 25 - 10 = 15$.
" $x = -3$, $x^2 - 2x = (-3)^2 - 2(-3) = 9 + 6 = 15$,

which proves that each value found is a root of the given equation.

Ex. 2. Solve $x(x-1) + 3(x-2) = (2x-3)^2$.

Removing brackets, $x^2 - x + 3x - 6 = 4x^2 - 12x + 9$

Collecting terms, $-3x^2 + 14x - 15 = 0$.

Dividing by -1, $3x^2 - 14x + 15 = 0$.

Factoring, $(x-3)(3x-5) = 0$.

From which $x = 3$ or $\frac{5}{3}$

EXERCISE XXXV

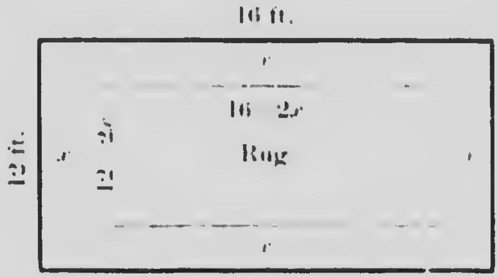
Solve equations and verify each root.

1. $x^2 + x = 6$.
2. $x^2 - x = 12$.
3. $x^2 - 6x = 40$.
4. $x^2 + 2x = 35$.
5. $x^2 + 4x = 60$.
6. $x^2 - 32 = 4x$.
7. $2x^2 - 4x - 6 = 0$.
8. $(x-3)^2 = x - 1$.
9. $(x-2)^2 = 3(x-2)$.
10. $(x-4)^2 = 5(x-4)$.
11. $3(x^2 - 10) = x^2 + 2$.
12. $(x-3)^2 + (x-4)^2 = (x-2)^2$.
13. $9x^2 - 25 = 0$.
14. $3x^3 - 27x = 0$.
15. $(2x-3)^2 = 3x^2 + (x-4)^2$.
16. $(x+3)^2 + (x+4)^2 = (x+5)^2$.
17. $x^4 + 9 = 10x^2$.
18. $(x^2 - 10)^2 = 9x^2$.
19. [illegible] $x = 0$.
20. $x^3 + x^2 = x + 1$.
21. [illegible] $2x^2 = x - 2$.
22. $x^2 - ax - 2a^2 = 0$.
23. [illegible] $ax + bx = ab$.
24. $ax - a^2 + ab = bx$.
25. [illegible] $- 2ax + a^2 - b^2 = 0$.
26. $(a+b)x^2 = (a-b)(a^2 - b^2)$.

PROBLEMS

145. Equations of the second degree are of frequent occurrence, especially in the solution of geometrical problems.

Ex. 1. A floor 16 feet long and 12 feet wide is covered by a rug, excepting a uniform border, whose area is 52 square feet. Find the width of the border.



Let x = the width of the border in feet,

then $12 - 2x$ = " " " rug "

and $16 - 2x$ = " length " " "

Now area of rug + area of border = area of floor,

that is $(12 - 2x)(16 - 2x) + 52 = 192$

Simplifying, $4x^2 - 56x + 52 = 0,$

or $x^2 - 14x + 13 = 0.$

Factoring, $(x - 1)(x - 13) = 0,$

from which $x = 1$ or $13.$

The value 13 must evidently be rejected as inapplicable to the problem. The width of the border is, therefore, 1 foot.

Ex. 2. If a square and a rectangle have the same perimeter, show that the square has the greater area

Let x = a side of the square,

and $x + y$ = the length of the rectangle;

then, $x - y$ = " width " "

Now the area of the square = x^2,

and the area of the rectangle = $(x + y)(x - y)$

$= x^2 - y^2$;

therefore the area of the square is the greater by y^2.

No unit of measurement is mentioned in the solution but it is tacitly assumed that all measurements are based on the same unit.

EXERCISE XXXVI

1 Find two numbers whose product is 150, one of the numbers being two-thirds of the other.

2 The sum of two numbers is 15 and the sum of their squares is 117. Find the numbers.

3. The difference of two numbers in 2 and the sum of their squares is greater than their product by 124. Find the numbers.

4 The difference of two numbers is 2 and the square of their sum is greater than the sum of their squares by 70. Find the numbers.

5 The difference of two numbers is 6, and if 47 be added to twice the square of the less, the sum will equal the square of the greater.

6 The area of a square will be doubled by adding 6 feet to the length of one side and 4 feet to the other. Find its present area.

7 The number of square feet in the area of a square is greater than the number of linear feet in its perimeter by 96. Find the side of the square.

8. A rectangle is 2 feet longer than wide, and if 4 feet be added to both length and width, its area will be doubled lacking 48 square feet. Find its length.

9. The perimeter of a rectangle is 70 feet and its area is 300 square feet. Find its length.

10. A picture, including its frame, is 20 inches long and 16 inches wide; the area of the frame is 68 square inches. Find the width of the frame.

11. The perimeter of one square exceeds that of another by 36 feet, and the area of the greater is 1 square yard more than 3 times the area of the smaller. Find their sides in feet.

12. The diagonal of a rectangle is 1 foot longer than one of the sides and 8 feet longer than the other. Find the area.

13. The diagonal of a rectangle is 25 feet and the sum of the other two sides is 31 feet. Find the sides.

14. What must be taken from both length and width of a field 40 rods long and 30 rods wide to leave just half the original area?

15. The difference in the cubes of two consecutive integers is 331. Find the numbers.

16. The difference in the edges of two cubes is 1 foot and the number of cubic feet in the difference of their volumes is greater by 25 than the number of square feet in the difference of their surfaces. Find the edge of the smaller cube.

17. The sum of the squares of two numbers is equal to twice their product. Show that the two numbers must be equal.

18 A square and a rectangle have the same perimeter, and the length of the rectangle is 5 feet greater than a side of the square. By how much does the area of the square exceed the area of the rectangle?

19 A rectangle is 8 inches longer than wide and its area is greater by 2 square inches than half the area of the square whose perimeter equals the perimeter of the rectangle. Find the width of the rectangle.

20 A picture is 20 inches long and 12 inches wide. What width of frame must be put upon it so that the area of the frame may be $\frac{3}{5}$ of the area of the picture?

21 If the difference of two numbers is n, show that the difference of their squares is n times their sum.

22 If the difference of two numbers is a unit, show that the difference of their cubes is equal to the sum of their squares together with their product

CHAPTER VIII

COMMON FACTORS AND MULTIPLES

HIGHEST COMMON FACTOR

146. Common Factor. An algebraical expression which is a factor of each of two or more expressions, is called a **common factor.**

Thus $3ab$ is a common factor of $3a^2b^3$ and $6a^2bc$.

147. Prime expressions. An expression which has no factors but itself and 1, is a **prime expression,** and two expressions which have no common factor are said to be **prime** to each other.

Thus $a+b$, a^2+b^2, are prime expressions, and ab and cd, $x+y$ and $x-y$ are pairs of expressions prime to each other.

148. Highest Common Factor. The highest common factor of two or more algebraical expressions is the expression of the highest degree and greatest numerical coefficients which will exactly divide each of them. It is evidently the product of all their common prime factors.

The letters H. C. F. are used as an abbreviation for highest common factor.

Ex. 1. Find the H. C. F. of $30a^2b^3c$, $45ab^2c$ and $60b^2c^2$.

$$30a^2b^3c = 5 \times 2 \times 3 \times a^2b^3c$$

$$45ab^2c = 5 \times 3 \times 3 \times ab^2c$$

$$60b^2c^2 = 5 \times 2 \times 2 \times 3 \times b^2c^2$$

Now the factors 5 and 3 are common to all the numerical coefficients; b^2 is the highest power of b, and c the highest power of c common to all the expressions. The H. C. F. is therefore $15b^2c$.

Ex. 2. Find the H. C. F. of a^2+ab, $ab+b^2$ and a^2-b^2.

Factoring, we have
$$a^2+ab=a(a+b)$$
$$ab+b^2=b(a+b)$$
$$a^2-b^2=(a+b)(a-b).$$

The only factor common to all three expressions is $a+b$, which is, therefore, the H. C. F. required.

149. To find the H. C. F. of two or more expressions.

Resolve each expression into its prime factors; and the product of all the common factors, each taken with the lowest exponent which occurs in any of the expressions, is the H. C. F. required.

150. Highest Common Factor in Algebra corresponds to Greatest Common Measure in Arithmetic. A different name is used, because the word "greatest" is not appropriate in referring to literal expressions. For instance a^2 is of higher degree than a but not necessarily greater. If $a=\frac{1}{2}$, $a^2=\frac{1}{4}$, and consequently a^2 is less than a. Similar remarks apply to Least Common Multiple.

EXERCISE XXXVII

Find the H. C. F. of

1. a^2b and b^2c^2d.
2. $6a^2b^2c$ and $9abcd$.
3. $5a^3b^3$, $10a^2bc^2$ and $20a^2b^2$.
4. $70a^2b^3$, $28a^4b^4$ and $42a^2b^5$.
5. $36x^2y^2$, $60x^3y^4$ and $72x^2y^5$.
6. $5x^2y^2$ and $30xy(a+b)$.
7. $10(a-b)$ and $5ab(a-b)$.

8. $12a^2b^2c$ and $4a^3bx + 8ab^2y$.

9. $14m^2nx$ and $21m^3p - 7mx$.

10. $3m^2x + 3mny$ and $mxy + ny^2$.

11. $5a^4 + 5a^2b^2$ and $7a^2b^2 + 7b^4$.

12. $a^3b^2 - a^2b^3$ and $abd(ac - bc)$.

13. $a^2 + ab - 2b^2$ and $a^2 + 4ab - 5b^2$.

14. $a^2 - 5a - 14$ and $a^2 + a - 56$.

15. $a^2 - 4$ and $a^2 + 5a - 14$.

16. $x^3 - 1$ and $3x^2 + 3x + 3$.

17. $x^4 - y^4$ and $x^6 - y^6$.

18. $x^4 + x^2y^2 + y^4$ and $x^3 + y^3$.

19. $x^6 + y^6$ and $x^8 + x^4y^4 + y^8$.

20. $x^3 + 8$ and $x^4 + 4x^2 + 16$.

21. $8x^3 + 1$ and $4x^2 + 4x + 1$.

22. $x^5 - x$ and $x^3 + x$.

23. $x^3 + 1$ and $x^3 + 2x(x + 1) + 1$.

24. $a^3 - b^3$ and $(a + b)^2 - ab$.

25. $125a^3 + 64b^3$ and $50a^2 - 40ab + 32b^2$.

26. $a(a + 1)^2$, $a^2(a^2 - 1)$ and $2a(a^2 - a - 2)$.

27. $p(p^2 - q^2)$, $q(p - q)^2$ and $pq(p^2 + pq - 2q^2)$.

28. $6(a - b)(a + b)^2$, $8(a^2 - b^2)^2$, $10(a^4 - b^4)$.

29. $(a + b)^2 - c^2$, $(b + c)^2 - a^2$, and $b^2 - (a + c)^2$.

30. $ab(a - c)(b - c)$ and $bc(b - a)(c - a)$.

31. $2a^3 - 2a^2b + ab^2 - b^3$ and $6a^4 + 5a^2b^2 + b^4$.

32. $3x^3 - 3x^2y + xy^2 - y^3$ and $x^3 - x^2y + xy^2 - y^3$.

33. $x^4 - x^2 - 20$ and $3x^4 + 3x^3 - 15x - 75$.

LOWEST COMMON MULTIPLE

151. Common Multiple. A **common multiple** of two or more algebraical expressions is any expression of which each of the given expressions is a factor.

Thus $12a^2bc$ is a common multiple of $2a$, $3bc$, $6abc$, ...

152. Lowest Common Multiple. The **lowest common multiple** of two or more algebraical expressions is the expression of lowest degree and smallest numerical coefficients of which each of the given expressions is a factor. It evidently contains all the prime factors which any one of the given expressions contains, but no others.

The letters L. C. M. are used as an abbreviation for lowest common multiple.

Ex. 1. Find the L. C. M. of $2ab$, $3bc$ and $6ac$.

The expression $6abc$ contains all the factors found in any one of the given expressions, but no other factor. It is, therefore, the L. C. M. required.

Ex. 2. Find the L. C. M. of $8a^2b$, $12ab^3$, $20b^2c^4$.

The literal factors are evident by inspection; the highest powers of each which occur in any one term are a^2, b^3, c^4. The numerical coefficients expressed as prime factors are 2^3, $2^2 \times 3$ and $2^2 \times 5$, the highest powers of each being 2^3, 3, 5. The product of all the factors selected, $120a^2b^3c^4$, is the L. C. M. required.

Ex. 3. Find the L. C. M. of $a^2b - ab^2$, $a^3 - a^2b$ and $a^2b^2 - b^4$.

Factoring, we have
$$a^2b - ab^2 = ab(a - b)$$
$$a^3 - a^2b = a^2(a - b)$$
$$a^2b^2 - b^4 = b^2(a + b)(a - b).$$

The different prime factors, each taken to the highest power occurring in any one expression, are a^2, b^2, $a + b$, and $a - b$. Their product $a^2b^2(a^2 - b^2)$ is the L. C. M. required.

153. To find the L. C. M. of two or more expressions. *Resolve the expressions into their prime factors; and the product of all the prime factors, each taken to the highest power occurring in any one expression, will be the L. C. M. required.*

EXERCISE XXXVIII

Find the L. C. M. of

1. ab, bc, ac.
2. $2a^2bc$, $3ab^2c^2$, $6abc$…
3. $6m^3n^2$, $15n^3p$, $20m^2p^3$.
4. $21l^2m^2$, $28m^3n^4$, $56n$…
5. a^2+ab, $ab-b^2$, a^2-b^2.
6. $ax-x^2$, a^2-ax, a^2x-ax^2.
7. $x-1$, x^2-1, x^2-x.
8. $2x-2$, $3x+3$, $4x-4$.
9. $a+2b$, $a-2b$, a^2-4b^2.
10. $(a+1)^2$, a^2+a, b…
11. $1-x$, $1-x^2$, $(1-x)^2$.
12. $(x-y)^2$, $x^2+xy-2y$…
13. $2x$, $5x^2+10x$, $3x^2+6x$.
14. $1-2x$, $1+2x$, 1…
15. a^2-ab, b^2-ab, $(a-b)^2$.
16. a^3-b^3, a^2+ab+b^2, $a-b$.
17. $4(x^2-y^2)$, $6(x-y)^2$, $10(x+y)^2$.
18. $3(x-x^2)$, $4(x^2-x^3)$, $6x^2(x-1)^2$.
19. x^3-xy^2, xy^2+y^3, x^3-x^2y, xy^2-x^2y.
20. x^2-3x+2, x^2-5x+6, x^2-4x+3.
21. $a(x-1)^2$, $ab^2(x^2-1)$, $3b(x^2+x-2)$, $6ab(x^2+3x+2)$.
22. $x^2-(a+b)x+ab$, $x^2-(b+c)x+bc$, $x^2-(a+c)x+ac$.
23. $a+b$, $a-b$, a^2-b^2, a^3-b^3, $a^4+a^2b^2+b^4$, a^6-b^6.
24. x^2-5x+6, x^2-4, x^2-9, x^2-x-6, x^2+x-6.
25. $(a^2+ab)^2$, $(ab-b^2)^2$, $a^2b^2(a^2-b^2)^2$.
26. $12x^2-x-20$, $12x^2-25x+12$, $16x^2+8x-15$.
27. $a^2-b^2+c^2+2ac$, $a^2+b^2-c^2+2ab$, $b^2+c^2-a^2+2bc$.
28. $2(x^3-1)-7x(x-1)$, $3x(x+1)-2(x^3+1)$, x^2…

HIGHEST COMMON FACTOR

(CONTINUED)

154. When the factors of the given expressions cannot readily be found in the ordinary way, their common factors, if any, may be found by the aid of the following principles:

1. If one expression is a factor of another, it will also be a factor of any multiple of the other.

Thus x is a factor of $3x$; it is also a factor of $15x$, $3ax$, etc.

2. If one expression is a factor of two others, it will also be a factor of the sum or the difference of any multiples of the others.

Thus x is a factor of ax and bx, take any multiples of these, as max and nbx.

Then $max + nbx = x(ma + nb)$ and $max - nbx = x(ma - nb)$.

3. If one expression be divided by another, the remainder, if any, will be the difference between the dividend and a multiple of the divisor, and will therefore contain all the factors of the original expressions.

Thus if $17x$ be divided by $3x$ the remainder is $2x$, which contains the common factor x.

4. The common factors of two expressions will not be changed by multiplying or dividing either of them by a factor not contained in the other.

Thus abc, bcd have the common factors bc; remove d, which is not a factor of abc, or multiply abc by x; in each case the same common factors, bc, remain.

155. The general mode of applying the preceding principles is to obtain from the given expressions other expressions of a lower degree, which still retain all the common factors, as shown at length in the following examples.

Ex. 1. Find the H. C. F. of $12x^2 - x - 20$ and $12x^2 - 25x + 12$.

By division, $12x^2 - 25x + 12)\,12x^2 - x - 20\,(1$

$$\underline{12x^2 - 25x + 12}$$

Remainder, $24x - 32 = 8(3x - 4)$

The remainder contains all common factors; it also contains the factor 8, which is evidently *not* a common factor and may therefore be rejected. The only remaining factor $3x - 4$, must be the common factor if there is one. By actual division we find it divides $12x^2 - 25x + 12$; it also divides $24x - 32$, therefore it divides their sum, which is the other given expression and is the H. C. F. required.

Ex. 2. Find the H. C. F. of $4a^2 + a - 5$ and $4a^3 - 2a^2 - 3a + 1$.

$$4a^2 + a - 5)\,4a^3 - 2a^2 - 3a + 1\,(a$$
$$\underline{4a^3 + a^2 - 5a}$$
$$-3a^2 + 2a + 1$$

Multiply by 4. $\quad 4 \qquad$ Art. 154, 4

$$-12a^2 + 8a + 4\,(-3$$
$$\underline{-12a^2 - 3a + 15}$$

Reject the factor 11. $\quad 11a - 11 = 11(a - 1)$. Art. 154, 3

$$a - 1)\,4a^2 + a - 5\,(4a + 5$$
$$\underline{4a^2 - 4a}$$
$$5a - 5$$
$$\underline{5a - 5}$$

The H. C. F. is, therefore, $a - 1$.

The first remainder is multiplied by 4 to render its first term exactly divisible by the first term of the divisor and thus avoid all fractional coefficients. This does not introduce a new common factor, since 4 is not a factor of the divisor. Again, 11 is rejected from the second remainder; this does not remove any common factor, since 11 is not a factor of either of the original expressions.

Ex. 3. Find the H. C. F. of $3x^3 - 13x^2 + 23x - 21$ and $6x^3 + x^2 - 44x + 21$.

The following is a convenient arrangement of the work:

$$\begin{array}{r|rl|r}
x & 3x^3 - 13x^2 + 23x - 21 & 6x^3 + x^2 - 44x + 21 & 2 \\
 & 3x^3 - 10x^2 + 7x & 6x^3 - 26x^2 + 46x - 42 & \\
1 & -3x^2 + 16x - 21 & 9\,)\,27x^2 - 90x + 63 & \\
 & -3x^2 + 10x - 7 & 3x^2 - 10x + 7 & x - 1 \\
-2\,) & 6x - 14 & 3x^2 - 7x & \\
 & 3x - 7 & -3x + 7 & \\
 & & -3x + 7 &
\end{array}$$

From the first remainder the factor 9 is rejected; the result is used as divisor, the original divisor becoming dividend and the two terms of the resulting quotient, x and -1, being placed on the left. The process is thus continued from left to right and from right to left alternately until a zero remainder occurs, when the last divisor $3x - 7$, is the H. C. F. required.

Ex. 4. Find the H. C. F. of $4ax^3 - 10ax^2 + 6ax$ and $3ax^2y + 18axy - 21ay$.

Each expression contains a monomial factor.

Thus $$4ax^3 - 10ax^2 + 6ax = 2ax(2x^2 - 5x + 3)$$
$$3ax^2y + 18axy - 21ay = 3ay(x^2 + 6x - 7).$$

Evidently a is a common factor, but $2x$ and $3y$ are not common factors and may be rejected. The H. C. F. of the resulting expressions may be found in the usual way to be $x - 1$. Therefore $a(x - 1)$ is the H. C. F. required.

156. It has been shown (Art. 150) that each remainder contains all the common factors. We shall now show that the last divisor contains nothing but common factors and is therefore the H. C. F.

Let A and B be any two algebraical expressions; divide A by B, let p be the quotient and R the remainder, as indicated in the margin.

$$B)\,A\,(p$$
$$\underline{pB}$$
$$R = A - pB$$

Then $$R = A - pB. \quad (1)$$

and $$A = R + pB. \quad (2)$$

Now from (1) every factor of A and B is a factor of R and B,
and " (2) " " " R " B " " A " B.

Therefore the common factors of A and B are the same as those of R and B.

In the next step of the process R becomes divisor and B dividend; it therefore follows that the common factors of each divisor and its dividend are the same.

When a zero remainder occurs, the divisor is a factor of the dividend and consequently the H. C. F. of itself and the dividend. And since the H. C. F. of each divisor and its own dividend is the same, this last divisor is the H. C. F. of the original expressions.

157. The H. C. F. of two algebraical expressions does not correspond to the G. C. M. of the numerical values of the expressions for all values of the letters involved.

The H. C. F. of $x^2 + 8x + 7$ and $x^2 + 6x + 5$ is $x + 1$.

Now the numerical value of $x + 1$ will always be a factor of the numerical values of the given expressions for all values of x, but it may not be their greatest common measure. If x be 2, the expressions become 27 and 21, and $x + 1 = 3$, and this is their G. C. M. But if $x = 3$, we get 40 and 32 and $x + 1 = 4$, which is a common factor but not their G. C. M. The explanation lies in the fact that the remaining factors of the two expressions $x + 7$ and $x + 5$ are prime to each other algebraically but not arithmetically. In other words, they have no common factor for all values of x, but have a common factor for particular values, *i.e.*, when x is an odd number.

Again, expressions which have factors algebraically, may become prime numerically.

Thus $x^2 - 1 = (x + 1)(x - 1)$ for all values of x, but if $x = 2$, $x^2 - 1 = 3$, a prime number. In this case the factors become 3 and 1 respectively, forming the same prime number.

158. The process for finding the H. C. F. is frequently tedious. Every artifice possible should be employed to prevent the introduction of large numerical multipliers, and to reject from each remainder any factor which is evidently not a part of the H. C. F. Two points deserve careful attention:

1. A multiplication may sometimes be avoided by eliminating the lowest power of the leading letter, instead of the highest. If this be done, a literal factor may always be struck out of the remainder. See examples 5, 9, 12 in the following exercise.

2. If a remainder occurs which can be factored, the factors may easily be tested in succession. Examples 13 and 15 are especially adapted to this mode of solution.

159. The H. C. F. of three expressions. The H. C. F. of three expressions may be found by finding the H. C. F. of two of them; then the H. C. F. of the result and the third will be the H. C. F. of all.

Let A, B, C be the three expressions, D the H. C. F. of A and B; then D is the H. C. F. of A, B, and C. For every factor of D and C is a factor of A, B, and C, and every factor of A, B, and C is a factor of D and C. Therefore D is the H. C. F. of A, B, and C. The process may evidently be extended to any number of expressions.

EXERCISE XXXIX

Find the H. C. F. of

1. $4x^2+12x+9$ and $2x^2-5x-12$.
2. $x^3-4x^2+9x-10$ and $x^3+2x^2-3x+20$.
3. $x^3+3x^2+5x+15$ and x^3+4x^2+5x+6.
4. x^3+2x^2+x-4 and x^3+x^2-2x-8.
5. x^3+3x^2+5x+3 and $2x^3+3x^2+4x-3$.
6. $x^3+2x^2-10x-8$ and $2x^3+5x^2-17x-20$.
7. $3x^3-17x^2-2x+8$ and $3x^3-20x^2+21x-6$.
8. $2x^3+5x^2-22x+15$ and $5x^3+18x^2-33x+10$.
9. $6x^3-19x^2+11x+6$ and $10x^3-19x^2+2x+6$.
10. $2x^3-15x^2+10x-21$ and $3x^3-22x^2+5x+14$.
11. x^4+5x^3-x-5 and $x^5+3x^4+x^3-x^2-3x-1$.
12. $4x^3-18x^2+20x-6$ and $2x^4-12x^3+19x^2-5x+6$.
13. $2x^5-11x^2-9$ and $4x^5+11x^4+81$.
14. $x^4-6x^3+13x^2-12x+4$ and $x^5-4x+8x^3-16x\ \ldots\ 16$
15. $9x^5+11x^3-2$ and $81x^5+11x+4$.
16. $x^5-55x+21$ and $21x^5-55x^4+1$.
17. $2x^4+9x^3+14x+3$ and $3x^4+14x^3+9x+2$.
18. $2ax^2-6axy+4ay^2$ and $3ax^2y+3axy^2-6ay^3$.
19. $6x^3+3x^2y-18xy^2$ and $12ax^2-12axy-9ay^2$.
20. $a^3x^2+3a^2bx+2ab^2$ and $2a^2bx^2-ab^2x-3b^3$.

21. x^2+2x-3, x^3+3x^2-x-3 and x^3+4x^2+x-6.

22. $x^3-6x^2+11x-6$, $x^3-9x^2+26x-24$ and $x^3-8x^2+19x-12$.

23. x^4-10x^2+9, $x^4+10x^3+20x^2-10x-21$ and $x^4+4x^3-22x^2-4x+21$.

24. x^3-a^3 and $x^3-ax^2+ax-a^2$.

LEAST COMMON MULTIPLE

(CONTINUED)

160. The L. C. M. of two expressions which separately cannot readily be resolved into factors may be obtained by first finding their H. C. F.

Ex. Find the L. C. M. of x^3-x+6 and $2x^3-x^2+9$.

The H. C. F. found by the ordinary method is x^2-2x+3.

Then by division $\quad x^3-x+6=(x^2-2x+3)(x+2)$,

and $\quad 2x^3-x^2+9=(x^2-2x+3)(2x+3)$.

The L. C. M. is evidently $\quad (x^2-2x+3)(x+2)(2x+3)$.

161. *The L. C. M. of two algebraical expressions is their product divided by their H. C. F.*

Let A and B denote the two expressions, D their H. C. F.

Let a and b denote the remaining factors of A and B.

Then $\quad A=aD$, and $B=bD$,

and therefore, abD is their L. C. M.

Then $$\frac{AB}{D}=\frac{aD\times bD}{D}=abD$$

which, expressed in words, is the proposition.

In practice it is most convenient to obtain and express the result in factors, as indicated in the example of the preceding Art.

Another mode of expressing the same proposition is the following:

The product of the H. C. F. and the L. C. M. of any two expressions is equal to the product of the given expressions.

162. The L. C. M. of three or more polynomials may be obtained by a repeated application of the principle already exemplified. Each pair of the several expressions must be tested for common factors. Where common factors are found, the other factors may be found by division. If two expressions have no common factor, their product must be taken as factors in the L. C. M.

EXERCISE XL

Find the L. C. M. of

1. x^3+2x^2+2x+1 and x^3-x^2-x-2.
2. $2x^3+3x^2-8x+3$ and x^3+4x^2+x-6.
3. $2x^3+9x^2+7x-3$ and $3x^3+5x^2-15x+4$.
4. $9x^3-x-2$ and $3x^3-10x^2-7x-4$.
5. $6x^3-x^2y-11xy^2+6y^3$ and $6x^3-11x^2y-xy^2+6y^3$.
6. $a^3+2a^2b-ab^2-2b^3$ and $a^3-2a^2b-ab^2+2b^3$.
7. $a^3-7ab^2-6b^3$ and $a^3-4a^2b+4ab^2-3b^3$.
8. x^3-7x-6, x^3-7x+6 and $x^3-6x^2+11x-6$.
9. $2a^2+ab-6b^2$, $3a^2+8ab+4b^2$ and $6a^2-5ab-6b^2$.
10. $a^4+2a^2b^2+9b^4$, $a^3-ab^2-6b^3$ and $a^3-ab^2+6b^3$.
11. $a^2x+ax^2-6x^3$, $a^3-a^2x-2ax^2$ and $a^3+2a^2x-5ax^2-6x^3$.
12. x^4-3x^3+x-3, $2x^4-5x^3-5x^2+5x+3$ and $2x^4-3x^3+2x^2-1$.
13. $2x^3+3x^2-3x-2$, $2x^3-3x^2-3x+2$ and x^3+2x^2-5x-6.

CHAPTER IX

FRACTIONS

163. How fractions arise. The division of one number by another cannot always be performed. If it is required to divide 17 by 3 we say that the quotient is $5\frac{2}{3}$, the real meaning, however, of this operation is that 17 has been separated into two addends, 15 and 2, of which the former has been divided, giving 5 as quotient, while the remaining 2 is written with the divisor beneath it to show that division is impossible.

164. Quantity always divisible. The distinction with reference to division between a concrete quantity and the number which represents it should be carefully noted. A rod 3 feet long may be divided into two equal parts, but three dots cannot be divided into two equal groups for the reason that the number 3 cannot be divided by 2.

165. Definition. A fraction is the indicated quotient of two numbers when the division cannot really be performed.

Thus $\frac{2}{3}$, $\frac{3}{5}$, $\frac{a}{b}$, $\frac{a^2+b^2}{a+b}$ are fractions; but $\frac{10}{5}$, [illegible] are fractions in form but not in reality.

The division in an algebraic fraction may become possible for special values of the letters involved.

Thus if $a = 6$, $b = 3$, the fraction $\frac{a^2+b^2}{a+b} = \frac{36+9}{6+3} = 5$.

166. Terms. The number to be divided is called the **Numerator,** the divisor is called the **Denominator**; together they are called the **terms** of the fraction.

167. Meaning of a fraction. The division indicated by a fraction becomes possible when the fraction is used as a coefficient of a number which is a multiple of the denominator or of a concrete quantity capable of division. The numerator then becomes a multiplier of the quantity concerned, and the denominator a divisor of the product.

Thus $\frac{2}{3}$ of 6 means that 6 is to multiplied by 2 and the result divided by 3.

Similarly $\frac{2}{3}$ of a given line means that a line twice as long is to be divided into three equal parts. The division by the denominator may precede the multiplication by the numerator when convenient, the result being the same in either case.

168. Negative numerators and denominators. When either or both the terms of a fraction are negative, the multiplication and division, as defined in the previous Art., may be readily performed. For the meaning of multiplication by a negative multiplier, given in Art. 64, II, is applicable to both numerical and concrete quantities. The meaning of division by a negative divisor readily follows.

Thus a line 6 inches long, *drawn to the right*, when divided by -3 gives a line 2 inches long, *drawn to the left*: since the latter when multiplied by -3 gives the original line.

Hence $\frac{1}{-3}$ of a line 6 inches long, drawn to the right, is a line 2 inches long, drawn to the left.

169. Fundamental property of a fraction. Since the effect of any number used as a multiplier is cancelled by using the same number as a divisor of the result, a factor common to both terms of a fraction may be introduced or rejected.

Thus $\frac{2}{3}$ and $\frac{2x}{3x}$ are equal, whatever be the value of x.

For the effect of the factor, x, used as a multiplier in the numerator is cancelled by the factor, x, used as a divisor in the denominator.

170. Sign of a fraction. A sign, + or −, written before a fraction opposite the dividing line, is the sign of the fraction as a whole, and must not be confused with the sign of numerator or denominator, taken separately.

From the rule of signs for division we have

$$\frac{a}{b}=\frac{-a}{-b}=-\frac{-a}{b}=-\frac{a}{-b},$$

which shows that any *two* of the three signs may be changed without affecting the fraction as a whole, but if any one be changed the sign of the whole is changed.

Similarly $\frac{a}{b-c}=\frac{-a}{c-b}=-\frac{a}{c-b}$; $\frac{x-y}{a-b}=\frac{y-x}{b-a}$, etc.

Changing the signs of both numerator and denominator is equivalent to multiplying each by −1.

REDUCING FRACTIONS TO LOWEST TERMS

171. Definition. When numerator and denominator contain no common factor, a fraction is said to be in its **lowest terms.**

172. To reduce a fraction to its lowest terms. *Resolve numerator and denominator into their prime factors, reject all factors common to each, and the result will be the fraction in its lowest terms.*

Ex. 1. $\frac{a^3b}{ab^3} = \frac{a^2}{b^2}$, by striking out the common factors ab.

Ex. 2. $\frac{a^2 - ab}{ab - b^2} = \frac{a(a-b)}{b(a-b)} = \frac{a}{b}$, by removing the common factor $a - b$.

Ex. 3. $\frac{bcx - bc}{bc - bcx^2} = \frac{bc(x-1)}{bc(1-x^2)} = \frac{x-1}{(1+x)(1-x)} = \frac{-1}{1+x}$.

EXERCISE XLI

Reduce to their lowest terms:

1. $\frac{5a^2b}{10ab^2}$.
2. $\frac{6a^2bc}{10ab^2d}$.
3. $\frac{15a^2xy^2}{25axy}$.
4. $\frac{36b^2c^3x^5}{48ab^2cx^3}$.
5. $\frac{240p^3q^4r^6}{360p^5q^4r^7}$.
6. $\frac{52l^6m^4n^2}{91l^3m^2n}$.
7. $\frac{a^2}{a^2 - ab}$.
8. $\frac{8a^2}{16ab - 24ac}$.
9. $\frac{6ab}{3a^2b - 6ab^2}$.
10. $\frac{a^2 + ab}{am + mb}$.
11. $\frac{bl + bm}{acl + acm}$.
12. $\frac{abx - aby}{x^2y - xy^2}$.
13. $\frac{l^2 - lm}{m^2n - lmn}$.
14. $\frac{b^2cx - b^2c}{b^2c - b^2cx^2}$.
15. $\frac{a^2 + ab}{(a+b)^2}$.
16. $\frac{a^2 - ab}{a^2c - b^2c}$.
17. $\frac{6x^2y - 6xy^2}{6x^3y - 9xy^3}$.
18. $\frac{3a^4x}{5a^4y} \quad \frac{3ab^2x}{5ab^2y}$.
19. $\frac{6a^2 + 9ab}{8a^2b - 18b^3}$.
20. $\frac{a^4 - a^2b^2c^2}{b^2c^4 - a^2c^2}$.
21. $\frac{a^3 + b^3}{a^3 + a^2b}$.

22. $\dfrac{a^4 - b^4}{a^6 - b^6}$.
23. $\dfrac{a^4 - a^2b^2 + b^4}{a^6 + b^6}$.
24. $\dfrac{a^4 + 4}{(a+1)^2 + 1}$.

25. $\dfrac{a^2 - ab - 2b^2}{a^2 + ab - 6b^2}$.
26. $\dfrac{a^2 - (b - c)^2}{(a + c)^2 - b^2}$.
27. $\dfrac{a^2 + ab - ac}{(a - c)^2 - b^2}$.

28. $\dfrac{c^2 - a^2 - b^2 + 2ab}{b^2 - c^2 - a^2 + 2ac}$.
29. $\dfrac{x^3 + 1}{x^3 - 2x^2 + 2x - 1}$.

30. $\dfrac{1 - a^2}{(1 + ax)^2 - (a + x)^2}$.
31. $\dfrac{x^3 - ax^2 + b^2x - ab^2}{x^3 - ax^2 - b^2x + ab^2}$.

32. $\dfrac{(a+b)^2 + (a-b)^2}{a^3 - a^2b + ab^2 - b^3}$.
33. $\dfrac{(a+b)^3 + (a-b)^3}{a^5 + a^3b^2 - 6ab^4}$.

34. $\dfrac{(1 - 3a)(1 - 2a) - a(5a - 7)}{(1 - 3a)^2 + a(a - 3)^2}$.

35. $\dfrac{(a^2 - b)^2 - (b^2 - a)(1 - ab)}{(b^2 - a)^2 - (1 - ab)(a^2 - b)}$.

36. $\dfrac{x^3 - 5x^2 + 10x - 8}{2x^3 - 9x^2 + 17x - 12}$.
37. $\dfrac{x^3 - 3x^2 + 3x - 2}{x^3 - x^2 - x - 2}$.

38. $\dfrac{x^3 - x^2 + 2}{x^3 - 3x^2 + 4x - 2}$.
39. $\dfrac{3x^3 - 16x^2 + 25x - 12}{3x^3 - 8x^2 - 7x + 12}$.

40. $\dfrac{(a - b + c)\{(a + b)^2 - c^2\}}{4a^2b^2 - (a^2 + b^2 - c^2)^2}$.

41. $\dfrac{a^2 - b^2 - c^2 + 2bc + a + b - c}{c^2 - a^2 - b^2 + 2ab + b + c - a}$.

CHANGES IN THE FORM OF FRACTIONS

173. **Changing a fraction to a mixed expression.** The division indicated by a fraction may frequently be partially performed, giving rise to a **mixed expression**, the quotient being considered integral and the remainder forming the numerator of a fraction with the divisor as denominator.

Ex. 1. $\dfrac{a^4+b^4}{a+b} = a^3 - a^2b + ab^2 - b^3 + \dfrac{2b^4}{a+b}.$

Ex. 2. $\dfrac{x^3-2x+2}{x^2+x+1} = x - 1 + \dfrac{-2x+3}{x^2+x+1} = x - 1 - \dfrac{2x-3}{x^2+x+1}.$

In Ex. 1 the remainder, $2b^4$, being positive, becomes the numerator, and the fraction is joined to the quotient by the sign $+$.

In Ex. 2 the remainder consists of two terms, of which the first is negative. In this case the fraction may be preceded by either the positive or the negative sign, as written above, but when the negative sign is used the sign of each term in the numerator must be changed. In Arithmetic the fraction is placed after the quotient with no sign between, but in Algebra this is not admissible, since when no sign is written multiplication is understood.

174. Changing a mixed expression to a complete fraction A mixed expression may be reduced to a complete fraction by reversing the process of the preceding Art., that is:

Multiply the integral expression by the denominator, to the product annex the numerator, and under the result write the denominator.

Ex. $x - 1 - \dfrac{2x-3}{x^2+x+1} = \dfrac{(x-1)(x^2+x+1) - (2x-3)}{x^2+x+1}$

$$= \frac{x^3-2x+2}{x^2+x+1}.$$

175. A fraction whose numerator consists of several terms may be separated into an equivalent number of separate fractions connected by the signs which connect the terms.

Ex.
$$\frac{8a^2-12ab+20b^2}{24ab}=\frac{8a^2}{24ab}-\frac{12ab}{24ab}+\frac{20b^2}{24ab}$$
$$=\frac{a}{3b}-\frac{1}{2}+\frac{5b}{6a}.$$

That is, to divide a polynomial by a monomial we divide each term in succession, Art. 84

EXERCISE XLII

Reduce the following fractions to mixed expressions:

1. $\dfrac{2x^2-5x+3}{x-2}$.
2. $\dfrac{a^2+b^2}{a+b}$.
3. $\dfrac{x^3}{x+y}$.
4. $\dfrac{a^2+ab+b^2}{a-b}$.
5. $\dfrac{2x^3-5x^2+6x-3}{x^2-x+1}$.
6. $\dfrac{x^3-2x-1}{x^2+1}$.
7. $\dfrac{x^3-x+6}{x^2+x+1}$.
8. $\dfrac{5x^3}{5x^2-5x+3}$.
9. $\dfrac{x^3-a^2x}{x^2-ax-b}$.

Reduce to complete fractions:

10. $1-\dfrac{x}{1+x}$.
11. $1-a+\dfrac{a^2}{1+a}$.
12. $x+y+\dfrac{y^2}{x-y}$.
13. $1+\dfrac{a-b}{a+b}$.
14. $1-\dfrac{a-b}{a+b}$.
15. $2x-y-\dfrac{2xy}{x+y}$.
16. $x+y+\dfrac{(x-y)^2}{x+y}$.
17. $a+7-\dfrac{2a-3}{a-5}$.
18. $\dfrac{a-5}{a+3}-a+2$.
19. $ab+1+\dfrac{a^2b^2}{1-ab}$.
20. $a^2+ab+b^2+\dfrac{2b^2}{a-b}$.
21. $1-x+x^2-\dfrac{x^4}{1+x+x^2}$.
22. $\dfrac{y(x-a)}{x+a}-x+a-y$.
23. $x-1-\dfrac{x^3+x+2}{(x+1)^2}$.
24. $\dfrac{x^3-7x+6}{x^2-x-2}+x-1$.

25. $1 + x + x^2\left(1 + x + \dfrac{x^2}{1 - x}\right)$.

26. $a - ab + b^2\left(a - ab + \dfrac{ab^2}{1 + b}\right)$.

Separate into fractions with a single term in the numerator:

27. $\dfrac{ab + bc + ca}{abc}$.

28. $\dfrac{x^4 - 4x^2y^2 + 8y^4}{6x^2y^2}$.

29. $\dfrac{25x^3 - 40x^2 - 75x + 100}{200x^2}$.

30. $\dfrac{(a^2 + ab + b^2)(a^2 - ab + b^2)}{a^4b^4}$.

31. $\dfrac{x(a + b) - y(x + y)}{ax + ay + bx + by}$.

32. $\dfrac{a(x + y) - b(x - 2y)}{a^2 + ab - 2b^2}$.

ADDITION AND SUBTRACTION OF FRACTIONS

176. Two or more fractions having different denominators may be replaced by equivalent fractions having the same denominator.

For, let $\dfrac{a}{b}$ and $\dfrac{c}{d}$ be any two fractions.

Then $\dfrac{a}{b} = \dfrac{ad}{bd}$, and $\dfrac{c}{d} = \dfrac{bc}{bd}$. Art. 169.

This is the converse of the process of reducing a fraction to its lowest terms.

177. Two or more fractions having the same denominator may be combined into a single fraction having that denominator.

Thus $\dfrac{a}{n} + \dfrac{b}{n} - \dfrac{c}{n} = \dfrac{a + b - c}{n}$,

which is the converse of the rule for the division of polynomials. Art. 84.

178. The two preceding principles enable us to combine into a single fraction two or more fractions connected by the signs of addition or subtraction.

Ex. 1.
$$\frac{a}{3b} - \frac{1}{2} + \frac{5b}{6a} = \frac{8a^2}{24ab} - \frac{12ab}{24ab} + \frac{20b^2}{24ab}$$
$$= \frac{8a^2 - 12ab + 20b^2}{24ab}.$$

The numerator and denominator of the first fraction were multiplied by $8a$, those of the second by $12ab$, and the last by $4b$, to change all into equivalent fractions with the common denominator $24ab$. These multipliers were obtained by dividing the L. C. M. of the denominators of the given fractions by each denominator in succession. See example in Art. 175.

Ex. 2. Simplify $\dfrac{a}{ab + b^2} - \dfrac{b}{a^2 + ab}$.

Factoring the denominators and taking the L. C. M. for the common denominator, we have

$$\frac{a}{b(a+b)} - \frac{b}{a(a+b)} = \frac{a^2 - b^2}{ab(a+b)} = \frac{a-b}{ab}.$$

Ex. 3. $\dfrac{a+b}{ab - b^2} + \dfrac{a-b}{a^2 + ab} - \dfrac{3a - b}{a^2 - b^2}$.

The L. C. M. of the denominators is $ab(a^2 - b^2)$.

Dividing the L. C. M. by the denominators in succession, we get $a(a+b)$, $b(a-b)$, ab.

Then
$$\frac{a+b}{b(a-b)} + \frac{a-b}{a(a+b)} - \frac{3a-b}{a^2-b^2}$$
$$= \frac{a(a+b)^2 + b(a-b)^2 - ab(3a-b)}{ab(a^2-b^2)}$$
$$= \frac{a^3 + b^3}{ab(a^2 - b^2)} = \frac{a^2 - ab + b^2}{ab(a-b)}.$$

Observe the effect of a negative sign before a fraction having two or more terms in the numerator.

Fractions not in their lowest terms should be reduced before being added or subtracted.

179. To add fractions. *Find the L. C. M. of the denominators. Divide the L. C. M. by the denominator of each fraction and multiply its numerator by the quotient. Connect the several products by the signs connecting the fractions, and under the result write the L. C. M. Reduce the resulting fraction to its simplest form.*

EXERCISE XLIII

Simplify

1. $\frac{a}{2}+\frac{2a}{3}-\frac{3a}{4}$.

2. $\frac{a+b}{3}+\frac{a-2b}{4}+\frac{b}{6}$ [illegible].

3. $\frac{x-1}{5}+\frac{x-3}{10}+\frac{3x-5}{15}$.

4. $\frac{6a-b}{8}-\frac{3a-2b}{4}+\frac{b}{8}$.

5. $\frac{4a-5b}{3}-\frac{2a-b}{6}$.

6. $\frac{2a-1}{2}+\frac{3a-1}{3}+\frac{5}{6}$.

7. $\frac{1}{a}+\frac{1}{2a}+\frac{1}{3a}+\frac{1}{6a}$.

8. $\frac{2}{5x}+\frac{1}{2x}+\frac{1}{10x}+$ [illegible].

9. $\frac{x-1}{2}-\frac{3}{x}-\frac{x}{3}+\frac{5-x}{6}$.

10. $\frac{x-1}{3x}-\frac{x-5}{5x}-\frac{2}{15}$.

11. $\frac{a-b}{ab}+\frac{b-c}{bc}+\frac{c-a}{ca}$.

12. $\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}$.

13. $\frac{a-x}{x}+\frac{a+x}{a}-\frac{a^2-x^2}{ax}$.

14. $\frac{2}{xy}-\frac{3y^2-x^2}{xy^3}+$ [illegible].

15. $\frac{a}{a+b}+\frac{b}{a-b}$.

16. $\frac{a+y}{a(a-y)}-\frac{ay+}{a^2(a-}$ [illegible].

17. $\frac{2a}{a^2-b^2}-\frac{1}{a-b}$.

18. $\frac{a^2+b^2}{a^2-b^2}+\frac{b}{a+b}$.

19. $\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}$.

20. $\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{y}{x-y}$.

21. $\dfrac{a}{b(a-b)}-\dfrac{b}{a(a-b)}$.

22. $\dfrac{a+b}{2(a-b)}-\dfrac{a^2+b^2}{2(a^2-b^2)}$.

23. $\dfrac{m+n}{mn-n^2}-\dfrac{m+n}{m^2-mn}$.

24. $\dfrac{m+n}{m(m-n)}-\dfrac{mn+n^2}{m^2(m-n)}$.

25. $\dfrac{1}{a+b}+\dfrac{3ab}{a^3+b^3}$.

26. $\dfrac{a}{a+b}+\dfrac{b}{a-b}-\dfrac{2ab}{a^2-b^2}$.

27. $\dfrac{b}{a^2+ab}+\dfrac{a}{ab+b^2}+\dfrac{2}{a+b}$.

28. $\dfrac{a^2+ax}{ax+x^2}+\dfrac{ax+x^2}{a^2-x^2}-\dfrac{a^2}{a^2-ax}$.

29. $\dfrac{a^2}{a^2-ab}-\dfrac{b^2}{ab-b^2}+\dfrac{a^2-b^2}{(a-b)^2}$.

30. $\dfrac{1}{x+5}+\dfrac{3}{x-5}-\dfrac{2x}{x^2-25}$.

31. $\dfrac{x+1}{2x-1}-\dfrac{x-1}{2x+1}-\dfrac{3}{4x^2-1}$.

32. $\dfrac{a^3-ab^2}{a+b}+2ab+\dfrac{a^2b-b^3}{a-b}$.

33. $\dfrac{a^2+ab+b^2}{a^3-b^3}-\dfrac{a^2-ab+b^2}{a^3+b^3}$.

34. $\dfrac{1+x}{1+x+x^2}-\dfrac{1-x}{1-x+x^2}$.

35. $\dfrac{x^2+xy+y^2}{x+y}-\dfrac{x^2-xy+y^2}{x-y}$.

36. $\dfrac{x^3+1}{x-1}-\dfrac{x^3-1}{x+1}-\dfrac{4x}{x^2-1}$.

37. $\dfrac{x^2+x+1}{x^2-x+1}-\dfrac{x^2-x+1}{x^2+x+1}$.

38. $\dfrac{x-1}{x-2}+\dfrac{x-2}{x-3}-\dfrac{2x-5}{x^2-5x+6}$.

39. $\dfrac{x}{x^2-5x}+\dfrac{2(x-1)}{x^2-8x+7}+\dfrac{2}{x^2-12x+35}$.

40. $\dfrac{1}{x^2-9x+20}+\dfrac{1}{x^2-11x+30}$.

41. $\dfrac{1}{x^2-7x+12}-\dfrac{1}{x^2-5x+6}$.

42. $\dfrac{1}{2x^2-x-1}-\dfrac{1}{2x^2+x-3}$.

43. $\dfrac{1}{2x^2-5x+2}-\dfrac{2}{4x^2-1}$.

44. $\dfrac{1}{x^2+3x+2}+\dfrac{2x}{x^2+4x+3}+\dfrac{1}{x^2+5x+6}$.

45. $\dfrac{1}{x+y} + \dfrac{x+y}{x^2 - xy + y^2} - \dfrac{xy}{x^3 + y^3}$.

46. $\dfrac{x+y}{x-y} + \dfrac{x^2 - y^2}{x^2 + xy + y^2} - \dfrac{xy(x+y)}{x^3 - y^3}$.

47. $\dfrac{x+y}{x^2 + xy + y^2} + \dfrac{x-y}{x^2 - xy + y^2} + \dfrac{2y^3}{x^4 + x^2y^2 + y^4}$.

48. $\dfrac{1-2x}{3(x^2 - x + 1} + \dfrac{1}{6(x+1)} + \dfrac{1+x}{2(x^2+1)}$.

49. $\dfrac{1+2x}{3(x^2 + x + 1)} - \dfrac{1}{6(x-1)} + \dfrac{1-x}{2(x^2+1)}$.

50. $\dfrac{x-1}{x+1} + \dfrac{x+1}{x-1} + \dfrac{3x-2}{x+3} + \dfrac{3x-4}{x-2}$.

51. $\dfrac{x+1}{x-1} + \dfrac{x-1}{x+1} + \dfrac{3x+2}{x-3} + \dfrac{3x+4}{x+2}$.

180. A proper grouping of the fractions to be combined frequently facilitates the work.

Ex. 1. Simplify $\dfrac{1}{x-1} + \dfrac{1}{x+1} + \dfrac{2x}{x^2+1} + \dfrac{4x^3}{x^4+1}$.

We have $$\frac{1}{x-1} + \frac{1}{x+1} = \frac{2x}{x^2-1};$$

then $$\frac{2x}{x^2-1} + \frac{2x}{x^2+1} = 2x\left(\frac{1}{x^2-1} + \frac{1}{x^2+1}\right) = \frac{4x^3}{x^4-1},$$

and $$\frac{4x^3}{x^4-1} + \frac{4x^3}{x^4+1} = 4x^3\left(\frac{1}{x^4-1} + \frac{1}{x^4+1}\right) = \frac{8x^7}{x^8-1}.$$

181. A rearrangement of the binomial factors containing a negative sign facilitates the simplification of certain expressions.

Ex. 1. $$\frac{x}{a-x} + \frac{a}{x-a} = \frac{x}{a-x} - \frac{a}{a-x} = -\frac{x-a}{a-x} = 1.$$

Ex. 2. $$\frac{1}{1-x}+\frac{x}{x^2-1}=\frac{1+x}{1-x^2}-\frac{x}{1-x^2}=\frac{1}{1-x^2}.$$

Ex. 3. $$\frac{1}{(x-1)(x-2)}+\frac{3}{(2-x)(x-3)}+\frac{2}{(1-x)(3-x)}$$
$$=\frac{(x-3)-3(x-1)+2(x-2)}{(x-1)(x-2)(x-3)}=\frac{-4}{(x-1)(x-2)(x-3)}.$$

Observe that

$$2-x=-(x-2) \text{ and } (1-x)(3-x)=(x-1)(x-3).$$

Therefore the L. C. M., when divided by the second denominator, gives $-(x-1)$ as quotient, but there is no change of sign in connection with the last division.

The final result may be written in either of the forms

$$\frac{4}{(x-1)(x-2)(3-x)} \text{ or } \frac{4}{(1-x)(2-x)(3-x)}.$$ See Art. 170.

EXERCISE XLIV

1. $\frac{1}{a-1}+\frac{1}{a+1}+\frac{2a^2}{1-a^2}.$

2. $\frac{1}{1-a}-\frac{1}{1+a}+\frac{2a}{1+a^2}.$

3. $\frac{3}{x-1}-\frac{x+1}{x-1}-\frac{x^2}{1-x^2}.$

4. $\frac{x}{1-x}-\frac{x^2}{1-x^2}+\frac{x}{1+x^2}.$

5. $\frac{1}{x-2}+\frac{1}{x+2}+\frac{4}{4-x^2}.$

6. $\frac{3}{1-2x}-\frac{7}{1+2x}-\frac{4-20x}{4x^2-1}.$

7. $\frac{1}{x-y}-\frac{1}{x+y}-\frac{2y}{x^2+y^2}.$

8. $\frac{x}{x-y}-\frac{y}{x+y}-\frac{x^2-y^2}{x^2+y^2}.$

9. $\frac{a}{a-x}+\frac{3a}{a+x}-\frac{2ax}{a^2-x^2}.$

10. $\frac{a}{x+a}+\frac{2x}{x-a}+\frac{a(3x-a)}{a^2-x^2}.$

11. $\frac{1}{x+2}+\frac{2x+3}{x-2}+\frac{x(16-x)}{4-x^2}.$

12. $\frac{x-a}{x-b}+\frac{x-b}{x-a}+\frac{(a-b)^2}{(a-x)(x-b)}.$

13. $\dfrac{x+y}{2y} - \dfrac{x}{x+y} + \dfrac{x^2y - x^3}{2y(x^2-y^2)}$.

14. $\dfrac{x-y}{y} + \dfrac{2x}{x-y} - \dfrac{x^3+xy^2}{y(x^2-y^2)}$.

15. $\dfrac{1}{3(x-1)} + \dfrac{2x+1}{3(x^2+x+1)} - \dfrac{1}{x(x^3-1)}$.

16. $\dfrac{1}{3(x+1)} + \dfrac{2x-1}{3(x^2-x+1)} + \dfrac{1}{x(x^3+1)}$.

17. $\dfrac{1}{4(x-1)} - \dfrac{3}{4(x+1)} - \dfrac{1}{x^2+1} + \dfrac{x^2}{x^4-1}$.

18. $\dfrac{a}{(a-b)(a-c)} + \dfrac{b}{(b-c)(b-a)} + \dfrac{c}{(c-a)(c-b)}$.

19. $\dfrac{1}{(x-2)(x-3)} + \dfrac{2}{(x-3)(1-x)} + \dfrac{1}{(1-x)(2-x)}$.

20. $\dfrac{2}{(x-2)(x-3)} + \dfrac{3}{(3-x)(x-5)} + \dfrac{1}{(5-x)(2-x)}$.

21. $\dfrac{c(a+b)}{ab(a-c)(b-c)} + \dfrac{a(b+c)}{bc(b-a)(c-a)}$.

22. $\dfrac{a-b}{a^2-b^2-ac-bc} + \dfrac{a+b}{a^2-b^2+ac+bc}$.

23. $\dfrac{3a+b}{a^2+3ab+2b^2} - \dfrac{a+7b}{a^2+5ab+6b^2} + \dfrac{2b}{a^2+4ab+3b^2}$.

24. $\dfrac{x-1}{x^2-5x+6} - \dfrac{2(x-2)}{x^2-4x+3} + \dfrac{x-3}{x^2-3x+2}$.

25. $\dfrac{a^2-b^2}{a^2-b^2+ac-bc} + \dfrac{b^2-c^2}{b^2-c^2+ab-ac} + \dfrac{c^2-a^2}{c^2-a^2+bc-ab}$.

26. $\dfrac{x^2-(y-z)^2}{(z+x)^2-y^2} + \dfrac{y^2-(z-x)^2}{(x+y)^2-z^2} + \dfrac{z^2-(x-y)^2}{(y+z)^2-x^2}$.

27. $\dfrac{1}{1-x} - \dfrac{1}{1+x} - \dfrac{2x}{1+x^2} - \dfrac{4x^3}{1+x^4} - \dfrac{8x^7}{1+x^8}$.

28. $\dfrac{2x}{6x} - \dfrac{3}{4} + \dfrac{3x-1}{6x+4} - \dfrac{x^2-2x+1}{9x^2-12x+4} + \dfrac{2x^2-5}{36x^2-16}$.

MULTIPLICATION OF FRACTIONS

182. Meaning of Multiplication. Multiplication as defined in Art. 63 requires the multiplier to be a whole number. Art. 167 shows that the numerator of a fraction is to be considered a multiplier, and the denominator a divisor of any number or concrete quantity, to which the fraction may be applied. With this meaning multiplication by a fractional multiplier becomes readily intelligible, since it consists of the successive performance of two operations, each of which has already been defined.

183. Product of two fractions. If two fractions, $\frac{4}{5}$ and $\frac{2}{3}$ for example, be used in succession as multipliers of any quantity capable of being multiplied and divided, four successive operations are to be performed: (1) multiply by 4, (2) divide by 5, (3) multiply by 2, (4) divide by 3. Now since factors, and therefore divisors, may be taken in any order, we may replace the two multipliers, 2 and 4, by the single multiplier 8, and the two divisors, 3 and 5, by the single divisor 15. That is, to multiply the quantity by $\frac{4}{5}$ and the result by $\frac{2}{3}$, is equivalent to multiplying by $\frac{8}{15}$.

Hence $$\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}.$$

Similarly $$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

If negative numbers occurred in numerators or denominators, the preceding reasoning would be equally applicable.

184. Definition. Art. 183 may be summed up in the following definition: The product of two fractions is the single fraction which, when used as a multiplier on any multiplicand, produces the same effect as the original fractions when used in succession as multipliers on the same multiplicand.

185. Rule: *To find the product of two or more fractions, multiply all the numerators for the numerator of the product, and all the denominators for the denominator of the product.*

186. The proof of the preceding rule is based on the definition of multiplication and the assumption that the order of multiplication and division may be changed. Expressed in symbols the reasoning may be given thus:

Let $\frac{a}{b}$ and $\frac{c}{d}$ be two fractions whose product is required.

Since b and d are divisors, they will be cancelled by the same numbers used as multipliers.

Therefore, $\frac{a}{b} \times b = a$ and $\frac{c}{d} \times d = c$.

Multiplying these equals, $\frac{a}{b} \times b \times \frac{c}{d} \times d = ac$.

Changing the order, $\frac{a}{b} \times \frac{c}{d} \times bd = ac$. Art. 67.

Dividing by bd, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

The same reasoning is evidently applicable to any number of fractions.

187. Factors common to a numerator and a denominator should be removed before multiplication, thus shortening the process and giving the product in its lowest terms.

Ex. 1. $\frac{2a^2}{3bc} \times \frac{6b^2}{5ca} \times \frac{5c^2}{16ab} = \frac{2 \times 6 \times 5 \times a^2b^2c^2}{3 \times 5 \times 16 \times a^2b^2c^2} = \frac{1}{4}$.

Ex. 2. $\dfrac{a^2-ab}{ab+b^2}\times\dfrac{a+b}{a^2+ab-2b^2}=\dfrac{a(a-b)}{b(a+b)}\times\dfrac{a+b}{(a-b)(a+2b)}$

$$=\frac{a}{b(a+2b)}=\frac{a}{ab+2b^2}.$$

Ex. 3. $\dfrac{x^2-5x+6}{x^2-16}\times\dfrac{x^2-2x-8}{3x-x^2}\times\dfrac{x^2+4x}{x^2+3x+2}.$

$$\frac{(x-2)(x-3)}{(x+4)(x-4)}\times\frac{(x-4)(x+2)}{x(3-x)}\times\frac{x(x+4)}{(x+2)(x+1)}=\frac{2-x}{x+1}.$$

Observe that the quotient of $x-3$ by $3-x$ is -1, which, multiplied by $x-2$, gives $2-x$.

DIVISION OF FRACTIONS

188. Reciprocal. If the product of two numbers is unity, or 1, each number is said to be the **reciprocal** of the other.

Thus 2 and $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{2}$, are pairs of reciprocal numbers, the product of each pair being 1.

Similarly $\dfrac{a}{b}$ and $\dfrac{b}{a}$ are reciprocals for all values of a and b.

189. To divide $\dfrac{a}{b}$ by $\dfrac{c}{d}$.

Denote the required quotient by x. From the definition of division we have

$$\text{Dividend} = \text{Divisor} \times \text{Quotient},$$

that is
$$\frac{a}{b}=\frac{c}{d}\times x.$$

Multiply by $\dfrac{d}{c}$,
$$\frac{a}{b}\times\frac{d}{c}=\frac{d}{c}\times\frac{c}{d}\times x$$

that is
$$\frac{ad}{bc}=x,\ \text{since}\ \frac{d}{c}\times\frac{c}{d}=1,$$

which, expressed in words, gives the rule for division.

190. Rule: *To divide any expression by a fraction, multiply the dividend by the reciprocal of the divisor.*

Ex. 1. $\dfrac{15a^2}{16b^2} \div \dfrac{5a}{8b} = \dfrac{15a^2}{16b^2} \times \dfrac{8b}{5a} = \dfrac{3a}{2b}.$

Ex. 2. $\dfrac{a^2-4b^2}{a^2+4ab} \div \dfrac{a^2-2ab}{ab+4b^2} = \dfrac{a^2-4b^2}{a^2+4ab} \times \dfrac{ab+4b^2}{a^2-2ab}.$

$$= \frac{(a+2b)(a-2b)}{a(a+4b)} \times \frac{b(a+4b)}{a(a-2b)} = \frac{b(a+2b)}{a}.$$

Ex. 3. $\left(\dfrac{a}{a-b} - \dfrac{b}{a+b}\right) \div \left(\dfrac{a+b}{a-b} + \dfrac{a-b}{a+b}\right)$

$$= \frac{a^2+b^2}{a^2-b^2} \div \frac{2(a^2+b^2)}{a^2-b^2} = \frac{a^2+b^2}{a^2-b^2} \times \frac{a^2-b^2}{2(a^2+b^2)} = \frac{1}{2}.$$

EXERCISE XLV

Simplify

1. $\dfrac{3ax}{5by} \times \dfrac{10b^2}{21ax^2}.$

2. $\dfrac{2a^2}{3bc} \times \dfrac{6b^2}{5ac} \times \dfrac{15c^2}{12ab}.$

3. $\dfrac{3a^2}{7b^2y} \times \dfrac{14cd^2}{15ab} \times 4by.$

4. $\dfrac{5a^2b^2c}{11xy^2z} \times \dfrac{33xyz^2}{25abc^2}.$

5. $\dfrac{12ab}{35x^2} \div \dfrac{18a^2}{45xy}.$

6. $\dfrac{7p^2q^2}{11r^2y} \div 14pqy.$

7. $\dfrac{8a}{8(b-c)} \div \dfrac{4a}{b^2-c^2}.$

8. $11a^2bc \div \dfrac{22abc^2}{5x}.$

9. $\dfrac{3a^2-6ab}{10xy} \times \dfrac{15xy}{9a^2-18ab}.$

10. $\dfrac{a^2-4}{a^2+5a} \times \dfrac{a^2-25}{a^2+2a}.$

11. $\dfrac{3a^2+3ab}{5ab-5b^2} \div \dfrac{6ab+6b^2}{10a^2-10ab}.$

12. $\dfrac{4ab-6b^2}{a(a+b)} \div \dfrac{2b^2}{a^2+ab}.$

13. $\dfrac{(a^2-b^2)^2}{2x} \times \dfrac{x^2}{(a+b)(a-b)^2}.$

14. $\dfrac{(a^2-ab)^2}{a^2b+ab^2} \div \dfrac{a^2(a-b)}{2ab+2b^2}.$

15. $$\frac{x^2-3x+2}{x^2-5x+6} \times \frac{x^2-7x+12}{x^2-9x+20} \times \frac{30-11x+x^2}{42-x-x^2}.$$

16. $$\frac{a^2-2ab+b^2}{a^2-ab-2b^2} \times \frac{a^2+2ab+b^2}{a^2+ab-2b^2} \times \frac{2a^2+5ab+2b^2}{2a^2-ab-b^2} \div \frac{a+b}{a-b}.$$

17. $$\frac{a^6-b^6}{a^2-b(2a-b^2)} \times \frac{a-b}{(a+b)^2-ab} \times \frac{a}{(a-b)^2+ab} \div \frac{a}{(a-b)^2+4ab}.$$

18. $$\frac{a^2-x^2}{a^2-3ax+2x^2} \times \frac{ax-2x^2}{a^2+ax} \times \frac{a^2-ax}{(a-x)^2} \div \frac{2(a+x)^2+ax}{(a+x)^2-x(a+3x)}.$$

19. $$\frac{x^3+216}{x^2-x-42} \times \frac{x^3-3x^2}{x^4-12x^3+36x^2} \div \frac{x^2+2x-15}{2x^2-98} \times \frac{x^2+x-42}{(x-6)^2+6x}.$$

20. $$\frac{(x-y)^2-z^2}{(x+y)^2-z^2} \times \frac{z^2-(x+y)^2}{(z-y)^2-x^2} \times \frac{x^2-(y-z)^2}{x^2-(y+z)^2} \times \frac{x+y+z}{x-y+z}.$$

21. $$\frac{(x+m)^2-(y+n)^2}{(x+y)^2-(m+n)} \div \frac{(x-y)^2-(n-m)^2}{(x-m)^2-(n-y)^2}.$$

22. $$\left(\frac{1}{a}-\frac{1}{ab}\right)\left(\frac{1}{b}+\frac{1}{a-b}\right)+\frac{1}{4ab}\left(\frac{a-b}{a+b}-\frac{a+b}{a-b}\right).$$

23. $$\left(\frac{a^2-ab}{a^3-b^3}\right)\left(\frac{a^2+ab+b^2}{a+b}\right)\left(\frac{a}{a-b}+\frac{b}{b-a}\right).$$

24. $$\frac{1}{2b}\left(\frac{1}{a-b}-\frac{1}{a+b}\right) \div \frac{1}{2a}\left(\frac{1}{a-b}+\frac{1}{a+b}\right).$$

25. $$\frac{1}{y}\left(\frac{x+y}{x}-\frac{x}{x+y}\right)+\frac{1}{x}\left(\frac{x-y}{y}+\frac{y}{x+y}\right).$$

26. $$\frac{x}{2a}\left(\frac{2}{x}-\frac{1}{a+x}+\frac{1}{a-x}\right)+\frac{a}{2x}\left(\frac{2}{a}-\frac{1}{a+x}+\frac{1}{x-a}\right).$$

27. $$\frac{1-a^2}{(1+ax)^2-(a+x)^2} \div \frac{1-x^2}{(1-ax)^2-(a-x)^2}.$$

28. $$\frac{2(a^2-ab+b^2)}{2a+3b} \div \left(\frac{2a-3b}{a^2-b^2} \div \frac{4a^2-9b^2}{a^3+b^3}\right).$$

29. $\dfrac{a(1-b^2)+b(1-a^2)}{(1-a^2)(1-b^2)-4ab} \times \left(\dfrac{1-ab}{a+b} - \dfrac{a+b}{1-ab}\right).$

30. $\dfrac{1}{x}\left(\dfrac{1}{a}+\dfrac{1}{y}\right)\left(\dfrac{1}{a-x}-\dfrac{1}{a-y}\right) \div \dfrac{1}{y}\left(\dfrac{1}{a}+\dfrac{1}{x}\right)\left(\dfrac{1}{a-y}-\dfrac{1}{a-x}\right).$

191. Special methods in multiplication and division. Expressions consisting of two or more fractional terms may sometimes be multiplied or divided like ordinary polynomials. The various artifices used in Chapter VI to abbreviate the work of multiplication, and the various formulæ used in factoring, may also be occasionally employed with advantage in connection with fractions.

Ex. 1. $\left(\dfrac{x^2}{2}-\dfrac{x}{3}+\dfrac{1}{4}\right)\left(\dfrac{x}{3}+\dfrac{1}{4}\right) = \dfrac{x^3}{6}+\dfrac{x^2}{8}-\dfrac{x^2}{9}-\dfrac{x}{12}+\dfrac{x}{12}+\dfrac{1}{16}$

$$= \frac{x^3}{6}+\frac{x^2}{72}+\frac{1}{16}.$$

Ex. 2. $\left(\dfrac{a}{b}+\dfrac{b}{a}\right)^2 = \left(\dfrac{a}{b}\right)^2 + 2\left(\dfrac{a}{b}\right)\left(\dfrac{b}{a}\right) + \left(\dfrac{b}{a}\right)^2 = \dfrac{a^2}{b^2}+2+\dfrac{b^2}{a^2}.$

Ex. 3. $\left(\dfrac{a^3}{b^3}+\dfrac{b^3}{a^3}\right) \div \left(\dfrac{a}{b}+\dfrac{b}{a}\right) = \left(\dfrac{a}{b}\right)^2 - \left(\dfrac{a}{b}\right)\left(\dfrac{b}{a}\right) + \left(\dfrac{b}{a}\right)^2$

$$= \frac{a^2}{b^2} - 1 + \frac{b^2}{a^2}.$$

The result in Ex. 2 is derived from $(x+y)^2 = x^2 + 2xy + y^2$ by writing $\dfrac{a}{b}$ and $\dfrac{b}{a}$ in place of x and y.

Similarly the result in Ex. 3 is derived from

$$\frac{x^3+y^3}{x+y} = x^2 - xy + y^2.$$

EXERCISE XLVI

Simplify

1. $\left(x+\frac{1}{x}\right)^2$.
2. $\left(x-\frac{1}{x}\right)^2$.
3. $\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)$.
4. $\left(\frac{a}{b}+\frac{x}{y}\right)\left(\frac{a}{b}-\frac{x}{y}\right)$.
5. $\left(\frac{1}{x}-1\right)\left(\frac{1}{x^2}+\frac{1}{x}+1\right)$.
6. $\left(\frac{1}{x}+x\right)\left(\frac{1}{x^2}-1+x^2\right)$.
7. $\left(\frac{a}{b}+1\right)\left(\frac{a^2}{b^2}-\frac{a}{b}+1\right)$.
8. $\left(x^2-\frac{1}{x^2}\right)\div\left(x-\frac{1}{x}\right)$.
9. $\left(1-\frac{1}{x^3}\right)\div\left(1-\frac{1}{x}\right)$.
10. $\left(\frac{a^3}{b^3}-\frac{b^3}{a^3}\right)\div\left(\frac{a}{b}-\frac{b}{a}\right)$.
11. $\left(x^3-\frac{1}{x^3}\right)\div\left(x^2+1+\frac{1}{x^2}\right)$.
12. $\left(1-\frac{1}{x}\right)\left(\frac{1}{x^2-1}\right)$.
13. $\left(x-\frac{1}{x}\right)\left(1+\frac{1}{x-1}\right)$.
14. $ab\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{a}-\frac{1}{b}\right)$.
15. $a\left(\frac{1}{a}-\frac{1}{ab}\right)\left(\frac{b}{1-b^2}\right)$.
16. $\left(\frac{a}{x^2}-\frac{x}{a^2}\right)\div\left(\frac{x}{a}+1+\frac{a}{x}\right)$.
17. $\left(\frac{a^2}{x^2}+\frac{x}{a}\right)\div\left(\frac{1}{x}-\frac{1}{a}+\frac{x}{a^2}\right)$.
18. $\left(1-\frac{x}{1+x}\right)\left(1+\frac{x}{1-x}\right)$.
19. $\frac{1}{x^3-1}\div\left(\frac{x}{1-x}+1\right)$.
20. $\left(\frac{2x}{3}+1\right)\left(x^2-\frac{x}{2}+\frac{1}{3}\right)$.
21. $\left(\frac{x^3}{6}+\frac{x^2}{72}+\frac{1}{16}\right)\div\left(\frac{x}{3}+\frac{1}{4}\right)$.
22. $\left(y-x+\frac{x^2}{y}\right)\left(\frac{1}{x}+\frac{1}{y}\right)$.
23. $\left(1-\frac{2y^3}{x^3+y^3}\right)\div\left(1-\frac{2y}{x+y}\right)$.
24. $\left(x^2+1+\frac{1}{x^2}\right)\left(x^2-1+\frac{1}{x^2}\right)$.

25. $\left(\frac{x^4}{y^4}+1+\frac{y^4}{x^4}\right)\div\left(\frac{x^2}{y^2}+1+\frac{y^2}{x^2}\right).$

26. $\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}+2\right)\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}-2\right).$

27. $\left(\frac{a}{b}+\frac{b}{c}-\frac{c}{a}\right)\left(\frac{a}{b}-\frac{b}{c}+\frac{c}{a}\right).$

28. $\left(\frac{a^2}{b^3}+\frac{1}{a}\right)\div\left(\frac{a}{b^2}-\frac{1}{b}+\frac{1}{a}\right).$

29. $\left(a^4+\frac{1}{a^2}\right)\div\left(1+\frac{1}{a^4-a^2}\right).$

30. $\left(a^2+2+\frac{2}{a^2}\right)\left(a^2-2+\frac{2}{a^2}\right).$

31. $\left(\frac{a^4}{2}+\frac{2}{a^4}\right)\div\left(a^2+2+\frac{2}{a^2}\right).$

32. $\left(x-2+\frac{1}{x}\right)\left(x+1+\frac{1}{x}\right)^2.$

33. $\left(\frac{x}{2}+2+\frac{2}{x}\right)\left(\frac{x}{2}-1+\frac{2}{x}\right).$

34. $\left(x-\frac{a^2-xy}{x-y}\right)\left(y-\frac{a^2-xy}{y-x}\right)+\left(\frac{a^2-xy}{y-x}\right)^2.$

35. $\left(\frac{x^5}{6}-\frac{x^4}{8}-\frac{7x^3}{48}+\frac{41x^2}{12}-\frac{27x}{4}+\frac{5}{3}\right)\div\left(\frac{x^2}{2}+\frac{3x}{4}-\frac{5}{2}\right).$

COMPLEX FRACTIONS

192. The division of one fractional expression by another is frequently indicated by writing the former as numerator and the latter as denominator in fractional form. Such an expression is called a **complex fraction** and is simplified by performing the division indicated.

Ex. $$\frac{a-\frac{ab}{a+b}}{a+\frac{ab}{a-b}}=\frac{\frac{a^2}{a+b}}{\frac{a^2}{a-b}}=\frac{a^2}{a+b}\times\frac{a-b}{a^2}=\frac{a-b}{a+b}.$$

Many complex fractions are most easily simplified by multiplying both numerator and denominator by a factor which will cause the small fractions in numerator and denominator to disappear.

Ex. 1. $$\frac{2\frac{1}{4} - \frac{1}{2}(x-3)}{\frac{2x-1}{3} - \frac{x-3}{4}} = \frac{27 - 6(x-3)}{4(2x-1) - 3(x-3)} = \frac{45-6x}{5x+5}.$$

Ex. 2. $$\frac{\frac{a+2b}{a+b} + \frac{a}{b}}{\frac{a+2b}{b} - \frac{a}{a+b}} = \frac{b(a+2b) + a(a+b)}{(a+2b)(a+b) - ab}$$

$$= \frac{a^2 + 2ab + 2b^2}{a^2 + 2ab + 2b^2} = 1.$$

In Ex. 1 the multiplier used was 12; in Ex. 2 it was $b(a+b)$. The L. C. M. of the denominators of the small fractions is evidently the multiplier required.

EXERCISE XLVII

Simplify

1. $\dfrac{\frac{x}{2} - 3}{2\frac{1}{3}}$.

2. $\dfrac{\frac{1}{2}(x-1)+2}{\frac{2}{3}\left(x - \frac{1}{2}\right) - 1}$.

3. $\dfrac{3\frac{1}{2}x - \frac{2}{3}}{\frac{5}{6}(4-21x)}$.

4. $\dfrac{3x - \frac{2}{3}\left(x + \frac{1}{2}\right)}{1\frac{2}{3} - \frac{1}{2}(2x-3)}$.

5. $\dfrac{\frac{x}{2}(x-2)}{\frac{x}{3}(3x-1) - 3\frac{1}{3}}$.

6. $\dfrac{\left(1 + \frac{2}{x}\right)^2 - 4}{\left(1 - \frac{1}{x}\right)^2 + \frac{1}{x^2}}$.

7. $\dfrac{1 + \frac{1}{a-1}}{1 - \frac{1}{a+1}}$.

8. $\dfrac{\frac{a+b}{b} - \frac{a+b}{a}}{\frac{1}{b} - \frac{1}{a}}$.

9. $\dfrac{\dfrac{1}{a}-\dfrac{1}{b+c}}{\dfrac{b+c}{a}-\dfrac{a}{b+c}}.$

10. $\dfrac{1+\dfrac{(a-b)^2}{4ab}}{\left(\dfrac{1}{a}+\dfrac{1}{b}\right)^2}.$

11. $\dfrac{a-\dfrac{a-b}{1+ab}}{1+\dfrac{a(a-b)}{1+ab}}.$

12. $\dfrac{\dfrac{1+x}{1-x}-\dfrac{1-x}{1+x}}{\dfrac{1+x}{1-x}+\dfrac{1-x}{1+x}}.$

13. $\dfrac{\dfrac{a+b}{2a}-\dfrac{2b}{a+b}}{\dfrac{a+b}{4b}-\dfrac{a}{a+b}}.$

14. $\dfrac{\dfrac{a^2+1}{2a+1}-\dfrac{a}{2}}{2+\dfrac{5a}{1-2a}}.$

15. $\dfrac{\dfrac{1}{bc}+\dfrac{1}{ca}+\dfrac{1}{ab}}{\dfrac{b}{c+a}-\dfrac{c+a}{b}}.$

16. $\dfrac{1}{1-\dfrac{x}{x-1}}-\dfrac{1}{\dfrac{x}{x+1}-1}$

17. $\dfrac{1}{(a+b-1)\left(\dfrac{1}{a-1}-\dfrac{1}{b-1}\right)}+\dfrac{1}{\dfrac{a-1}{b}-\dfrac{b-1}{a}}.$

18. $\dfrac{1}{x-1+\dfrac{1}{1+\dfrac{x}{4-x}}}.$

19. $\dfrac{3}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}.$

20. $\dfrac{1}{1-\dfrac{1}{1-x-\dfrac{1}{1+x}}}.$

21. $\dfrac{\dfrac{(x-2y)^2}{y(x+y)}+\dfrac{(2x-y)}{x(x+y)}}{\dfrac{(x+2y)^2}{y(x-y)}-\dfrac{(2x+y)}{x(x-y)}}$

22. $\dfrac{\dfrac{1+x}{1-x}-\dfrac{1-x}{1+x}+\dfrac{4x}{1+x^2}+\dfrac{8x}{1+x^4}}{\dfrac{1+x^2}{1-x^2}-\dfrac{1+x^2}{1+x^2}+\dfrac{4x^2}{1+x^4}}.$

CHAPTER X

SIMPLE EQUATIONS

(ADVANCED)

ONE UNKNOWN QUANTITY

193. The following chapter is a continuation of Chapter IV, and contains examples of a greater degree of difficulty.

Ex. 1. Solve the equation $\frac{2x+8}{5}-\frac{5x-2}{2x+7}=\frac{6x-7}{15}$.

Rearranging, $$\frac{2x+8}{5}-\frac{6x-7}{15}=\frac{5x-2}{2x+7}.$$

Combining, $$\frac{31}{15}=\frac{5x-2}{2x+7}.$$

Clearing of fraction, $$62x+217=75x-30,$$

from which $$13x=247,$$

and $$x=19.$$

When both simple and compound expressions occur in the denominators of the fractions, the solution will be more easily obtained by combining the fractions having simple denominators before clearing the equation of fractions.

Ex. 2. Solve the equation $\frac{x-6}{2x+3}+\frac{x-3}{4(x-9)}=\frac{3}{4}$.

The L. C. M. of the denominators is $4(2x+3)(x-9)$.

Then $4(x-6)(x-9)+(x-3)(2x+3)=3(2x+3)(x-9)$,

or $$4x^2-60x+216+2x^2-3x-9=6x^2-45x-81,$$

from which $$-18x=-288,$$

and $$x=16.$$

Ex. 3. Solve the equation $\frac{ax}{x-b} + \frac{bx}{x-a} = a + b.$

Clearing of fraction, $ax(x-a) + bx(x-b)$

$$= (a+b)(x-a)(x-b).$$

Then $ax^2 - a^2x + bx^2 - b^2x = (a+b)\{x^2 - (a+b)x + ab\},$

and $(a+b)x^2 - (a^2+b^2)x = (a+b)x^2 - (a+b)^2x + ab(a+b),$

∴ $\{(a+b)^2 - (a^2+b^2)\}x = ab(a+b),$

∴ $2abx = ab(a+b),$

∴ $x = \frac{a+b}{2}.$

In solving literal equations keep the terms containing the same powers of x bracketed together, as in the above example. The verification of the result forms excellent practice in the simplification of fractional expressions.

$$\text{Thus}\quad \frac{ax}{x-b} + \frac{bx}{x-a} = \frac{\frac{a}{2}(a+b)}{\frac{1}{2}(a+b)-b} + \frac{\frac{b}{2}(a+b)}{\frac{1}{2}(a+b)-a}$$

$$= \frac{a^2+ab}{a-b} + \frac{ab+b^2}{b-a} = \frac{a^2-b^2}{a-b} = a+b.$$

EXERCISE XLVIII

1. $\frac{x+1}{3} + \frac{x-1}{4} = \frac{2x+8}{6}.$

2. $\frac{x-1}{5} + \frac{2x-1}{3}$ [illegible]

3. $\frac{2x+1}{5} - \frac{5x+3}{6} = \frac{1-3x}{10}.$

4. $\frac{4x-5}{11} - \frac{2(x-1)}{5}$ [illegible]

5. $\frac{2}{x} + 5\left(\frac{1}{25} - \frac{1}{x}\right) = \frac{1}{8}.$

6. $\frac{18}{x} + 13\left(\frac{1}{15} - \frac{1}{x}\right)$ [illegible]

7. $\frac{x}{20} + \frac{6}{100}(1000 - x) = 56\frac{1}{2}.$

8. $\frac{x}{80} = \frac{780 - x}{40}$ [illegible]

9. $\dfrac{3x-1}{4}+\dfrac{5-6x}{8}=\dfrac{15}{x-2}$.

10. $\dfrac{3x-5}{3}+\dfrac{4-5x}{5}=\dfrac{x-29}{x-1}$.

11. $\dfrac{3x-1}{5}-\dfrac{6x-7}{10}=\dfrac{3x-5}{2x-3}$.

12. $\dfrac{9}{4}x-\left(1-\dfrac{2-x}{9}\right)=7x$.

13. $\dfrac{9x-1}{4}-\left(x-\dfrac{2x-1}{9}\right)=7$.

14. $\dfrac{2x+1}{5}-\dfrac{7x+4}{4x+3}=\dfrac{2(3x-11)}{15}$.

15. $\dfrac{x+10}{3}-\dfrac{3}{5}(3x-4)+\dfrac{1}{6}(3x-2)(2x-3)=x^2-\dfrac{8}{15}$.

16. $\dfrac{1}{3}(x-1)+\dfrac{1}{20}(16x-3)-\dfrac{1}{8}(7x-6)$

$$=2+\dfrac{1}{2}(x-2)+\dfrac{1}{10}(3x-9).$$

17. $\dfrac{x}{7}-\dfrac{x-5}{11}+5=2x-\left(\dfrac{5x}{77}+1\right)-\dfrac{74}{11}$.

18. $\dfrac{4x+7}{x-1}+\dfrac{2x-3}{2x-2}=\dfrac{31}{2}$.

19. $\dfrac{2x+1}{2x-1}-\dfrac{3x-1}{2(1-2x)}=\dfrac{17}{8}$.

20. $\dfrac{1}{x-3}+\dfrac{1}{x-5}=\dfrac{2}{x}$.

21. $\dfrac{1}{1-3x}+\dfrac{1}{1-5x}=2$.

22. $\dfrac{3}{x+2}+\dfrac{4}{x+3}=\dfrac{24}{x^2+5x+6}$.

23. $\dfrac{3}{2x+4}-\dfrac{5}{3x+6}=\dfrac{1}{6x}$.

24. $\dfrac{x-2}{2x+1}+\dfrac{x-1}{4(x-3)}=\dfrac{3}{4}$.

25. $\dfrac{3x-1}{2x-1}-\dfrac{4x-2}{3x-2}=\dfrac{1}{6}$.

26. $\dfrac{1}{x+1}+\dfrac{1}{x-1}=\dfrac{2}{x-2}$.

27. $\dfrac{2}{x-2}+\dfrac{3}{x-3}=\dfrac{5}{x-5}$.

28. $\dfrac{1}{x-1}+\dfrac{1}{2x-1}=\dfrac{3}{2x-3}$.

29. $\dfrac{2}{4x-3}+\dfrac{1}{x-1}=\dfrac{3}{2x+1}$.

30. $\left(x+\dfrac{5}{2}\right)\left(x-\dfrac{3}{2}\right)=(x+5)(x-3)-\dfrac{3}{4}$.

31. $\left(x - \frac{5}{2}\right)\left(x + \frac{3}{2}\right) = (x - 5)(x + 3) + \frac{93}{4}$.

32. $\frac{3+x}{3-x} + \frac{2+x}{2-x} + \frac{1+x}{x-1} = 1$.

33. $\frac{x+3}{3x+2} - \frac{2x+1}{9x+6} + \frac{x}{3} = \frac{1}{3}$.

34. $\frac{3}{2x-1} - \frac{2x+1}{2x-1} = \frac{4x^2}{1-4x^2}$.

35. $\frac{x-4}{x^2-8x+12} = \frac{2x-13}{x^2-10x+24} +$ [illegible]

36. $\frac{1}{a} + \frac{a}{x+a} = \frac{x+a}{ax}$.

37. $\frac{1}{x}$ [illegible]

38. $\frac{x+a}{x-b} + \frac{x+b}{x-a} = 2$.

39. $\frac{x+a}{x-a}$ [illegible]

40. $\frac{a(a-x)}{b} - \frac{b(b+x)}{a} = x$.

41. $\frac{a(ax-1)}{b}$ [illegible]

42. $\frac{x^2-a^2}{bx} - \frac{a-x}{b} = \frac{2x}{b} - \frac{a}{x}$.

43. $\frac{x-a}{a-b} - \frac{x+a}{a+b} = \frac{2ax}{a^2-b^2}$.

44. $\frac{1}{x} + \frac{b}{x+a} = \frac{1+b}{x+b}$.

45. $\frac{3(x-a)}{2x-b} = \frac{3x+a}{2(x-b)}$.

46. $\frac{1}{x-a} - \frac{1}{x-b} = \frac{a-b}{x^2-ab}$.

47. $\frac{1}{1-ax} - \frac{1}{1-bx} = \frac{(a-b)x}{1-abx}$.

48. $\left(\frac{1}{a} - \frac{1}{b} - x\right)\left(\frac{1}{a+b} - x\right) = \left(\frac{1}{a} - x\right)^2$.

194. Simplification by Division. The solution of fractional equations is sometimes facilitated by the division, so far as is possible, of each numerator by its own denominator. A proper arrangement of the fractions in two groups, each of which is simplified by itself, is also sometimes convenient.

Ex. 1. Solve $\frac{2x^2 - 5x + 9}{x^2 - 3x + 6} = \frac{2x - 7}{x - 4}$.

Dividing each numerator by its own denominator,

we have $$2 + \frac{x - 3}{x^2 - 3x + 6} = 2 + \frac{1}{x - 4},$$

or $$\frac{x - 3}{x^2 - 3x + 6} = \frac{1}{x - 4},$$

or $$(x - 4)(x - 3) = x^2 - 3x + 6,$$

$$x^2 - 7x + 12 = x^2 - 3x + 6,$$

from which $$x = \frac{3}{2}.$$

Ex. 2. Solve $\frac{1}{2x - 5} + \frac{1}{2x - 11} = \frac{1}{2x - 9} + \frac{1}{2x - 7}$.

Rearranging, $$\frac{1}{2x - 5} - \frac{1}{2x - 9} = \frac{1}{2x - 7} - \frac{1}{2x - 11},$$

or $$\frac{-4}{4x^2 - 28x + 45} = \frac{-4}{4x^2 - 36x + 77}.$$

Therefore $$4x^2 - 28x + 45 = 4x^2 - 36x + 77,$$

from which $$8x = 32,$$

or $$x = 4.$$

Ex. 3. Solve $\frac{x}{2} + \frac{\frac{1}{3}(2x - 3) - \frac{1}{4}(3x - 1)}{\frac{1}{2}(x - 1)} = \frac{3x + 4}{6}$.

Multiplying numerator and denominator of the complex fraction by 12 and writing result in simplest form we have

$$\frac{x}{2} - \frac{x + 9}{6x - 6} = \frac{3x + 4}{6},$$

or $$\frac{x + 9}{6x - 6} = \frac{2}{3},$$

from which $$3x + 27 = 12x - 12.$$

Then $$9x = 39, \text{ or } x = 4\tfrac{1}{3}.$$

EXERCISE XLIX

1. $\dfrac{2x-1}{x-1}+\dfrac{x-2}{x-3}=3.$

2. $\dfrac{2x-3}{x-1}+\dfrac{3x-7}{x-2}=5.$

3. $\dfrac{7x-2}{x+1}-\dfrac{2x-5}{x-1}=\dfrac{5x^2+4}{x^2-1}.$

4. $\dfrac{6x+8}{2x+1}-\dfrac{2x+32}{x+10}=1.$

5. $\dfrac{x^2-x+1}{x-1}+\dfrac{x^2+x+1}{x+1}=2x.$

6. $\dfrac{3x^2-5x+1}{3x-5}=\dfrac{2x^2-3x+2}{2x-3}.$

7. $\dfrac{x+4a}{x+a}+\dfrac{4x+2b}{x-b}=5.$

8. $\dfrac{x+2a-b}{x-a+2b}+\dfrac{2x-5a+4b}{2x-2a+b}=2.$

9. $\dfrac{x}{2\frac{1}{2}}+\dfrac{x}{3\frac{1}{3}}=5.$

10. $\dfrac{x}{4\frac{1}{2}}+\dfrac{20-x}{7\frac{1}{3}}=3\frac{1}{2}.$

11. $\dfrac{1}{x-2}-\dfrac{1}{x-6}=\dfrac{1}{x+1}-\dfrac{1}{x-3}.$

12. $\dfrac{x}{x-2}-\dfrac{x+1}{x-1}=\dfrac{x-8}{x-6}-\dfrac{x-9}{x-4}.$

13. $\dfrac{x-3}{x-5}-\dfrac{x-1}{x-3}=\dfrac{x-4}{x-6}-\dfrac{x-2}{x-4}.$

14. $\dfrac{2}{x+2}+\dfrac{4}{x-1}+\dfrac{18}{3x-2}=\dfrac{36}{3x-1}.$

15. $\dfrac{8x-25}{2x-5}+\dfrac{16x-93}{2x-11}=\dfrac{18x-86}{2x-9}+\dfrac{6x-26}{2x-7}.$

16. $7\frac{1}{2}\left(x-\dfrac{2}{3}\right)+\dfrac{3}{8}\left(\dfrac{9x}{4}-\dfrac{1}{2}\right)=11\frac{1}{2}.$

17. $\dfrac{2\frac{1}{2}x-1}{7}+\dfrac{4\frac{1}{2}x-5}{11}=\dfrac{1\frac{1}{2}x-2\frac{1}{3}}{1\frac{2}{3}}.$

18. $\dfrac{x-2}{3} - \dfrac{\frac{x}{3}(x-5)}{x-2\frac{1}{3}} = \dfrac{\frac{1}{2}(x+7)}{3x-7}.$

19. $\dfrac{x}{6\frac{1}{4}} + \dfrac{2\frac{1}{6} + 2\frac{1}{3}x}{9\frac{2}{3}x+10} = x - \dfrac{7x-2}{8\frac{1}{3}}.$

20. $\dfrac{1\frac{1}{2}x - 1}{2\frac{1}{4}x + \frac{1}{2}} - \dfrac{2 - \frac{1}{2}x}{1+x} = \dfrac{4\frac{1}{3}x - 5}{2x+2}.$

21. $\dfrac{5\frac{1}{2}x - 1}{8\frac{3}{4}} - \dfrac{2\frac{1}{3}x - 13\frac{1}{2}}{7\frac{1}{2}} - \dfrac{3\frac{1}{3}x - 2}{10\frac{1}{2}} = \dfrac{x+23}{x-23}.$

22. $\dfrac{4x-17}{9} - \dfrac{3\frac{2}{3} - 22x}{33} = x - \dfrac{6}{x}\left(1 - \dfrac{x^2}{54}\right).$

23. $\dfrac{4x - \frac{2}{3}(1+x)}{4} + \dfrac{\frac{2}{5}(x-1) + 2x}{5\frac{1}{2}} = \dfrac{10 + \frac{x+1}{25}}{4\frac{2}{5}}.$

24. $\dfrac{a+x}{a} - \dfrac{2x}{a+x} + \dfrac{x^2(x-a)}{a(a^2-x^2)} = \dfrac{1}{3}.$

25. $\dfrac{x^2 - ax - bx + ab}{x-a} = \dfrac{x^2 - 2bx + 2b^2}{x-b} - \dfrac{c^2}{x-c}.$

26. $\dfrac{x}{a+b} + abx = a + b + \dfrac{1}{ab}.$

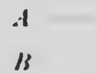
27. $\dfrac{(a+1)x}{b} + \dfrac{(b+1)x}{a} + \dfrac{2ab}{a+b} = a + b + 1.$

28. $\dfrac{x+a}{b} + \dfrac{x+b}{c} + \dfrac{x+c}{a} = \dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a} + 1.$

29. $\dfrac{m(x-a)}{x-b} + \dfrac{n(x-b)}{x-a} = m + n.$

30. $\dfrac{m(x+a)}{x-b} + \dfrac{n(x+b)}{x-a} = \dfrac{(m+n)x}{x-a-b}.$

195. Special Methods. The solution of fractional equations is frequently facilitated by employing a principle, the truth of which will be evident from the following simple example :



Let A and B denote any two concrete quantities, such that when A is divided into three and B into two equal parts, the parts are all equal. The relative magnitudes of A and B may then be expressed in fractional form in either of two ways, thus:

$$\frac{A}{3} = \frac{B}{2} \text{ or } \frac{A}{B} = \frac{3}{2},$$

which are simply two different ways of expressing the same fact.

Again, from the diagram it is evident that

$$\frac{A}{3} = \frac{B}{2} = \frac{A+B}{5} = \frac{A-B}{1}.$$

Also, if the numbers 3 and 2 are replaced by any numbers, m and n, we shall still have

$$\frac{A}{m} = \frac{B}{n} = \frac{A+B}{m+n} = \frac{A-B}{m-n}.$$

A formal proof of the principles here exemplified is given in the following Art.

196. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d} = \frac{a-c}{b-d}$, and $\frac{a}{c} = \frac{b}{d}$.

For, let $\frac{a}{b} = x$, and $\frac{c}{d} = x$.

then $a = bx$, and $c = dx$,

and $a + c = bx + dx$

$= (b+d)x.$

Therefore $$\frac{a+c}{b+d} = x = \frac{a}{b} = \frac{c}{d}.$$

Similarly $$\frac{a-c}{b-d} = x = \frac{a}{b} = \frac{c}{d}.$$

Also $$\frac{a}{c} = \frac{bx}{dx} = \frac{b}{d}.$$

Ex. 1. Solve equation $\left(\frac{x-a}{x-b}\right)^2 = \frac{x-2a}{x-2b}$.

Expand the square of the first fraction, multiply numerator and denominator of the second fraction by x, subtract numerators and denominators as explained above, and we have

$$\frac{x^2-2ax+a^2}{x^2-2bx+b^2} = \frac{x^2-2ax}{x^2-2bx} = \frac{a^2}{b^2};$$

that is $$\frac{x-2a}{x-2b} = \frac{a^2}{b^2}.$$

Clearing of fractions, $b^2x - 2ab^2 = a^2x - 2a^2b$,

from which $$x = \frac{2ab}{a+b}.$$

Ex. 2. Solve $\frac{m}{x-a} + \frac{n}{x-b} = \frac{m+n}{x-a-b}$.

Adding first two fractions and multiplying numerator and denominator of the last fraction by x, we have

$$\frac{(m+n)x - an - bm}{x^2-(a+b)x+ab} = \frac{(m+n)x}{x^2-(a+b)x} = \frac{an+bm}{ab},$$

that is $$\frac{m+n}{x-a-b} = \frac{an+bm}{ab},$$

which can be readily solved in the ordinary way. The result is $$x = \frac{a^2n+b^2m}{an+bm}.$$

If these two examples be solved by the ordinary mode the brevity of the method given will be readily apparent.

197. An equation may sometimes be solved by finding a common factor of the two sides, providing the factor contains the unknown quantity.

Ex. 1. Solve the equation $\frac{x-2}{m} = \frac{x-2}{n}$.

The equation is evidently satisfied when $x - 2 = 0$, or when $x = 2$. Since in that case each side becomes zero for all values of m and n.

Ex. 2. Solve $(x-2a)^3 + (x-2b)^3 = 2(x-a-b)^3$.

The expression $(x-2a)+(x-2b)$ is evidently a factor of the left-hand side of the equation and the half of it, viz., $x-a-b$ is a factor of the other side. If, therefore, $x-a-b=0$, or $x=a+b$, each side becomes zero and the equation is satisfied. If for x we write $a+b$ we have

$$(x-2a)^3 + (x-2b)^3 = (-a+b)^3 + (a-b)^3 = 0,$$

and

$$2(x-a-b)^3 = 2(0)^3 = 0.$$

The factor 2 and the exponent 3 on the right of the equation do not in any way affect the solution.

EXERCISE L

1. $\frac{2x^2-3x+5}{x^2+7x+10} = \frac{2x-3}{x+7}$.

2. $\left(\frac{x+a}{x-b}\right)^2 = \frac{x+2a}{x-2b}$ [illegible]

3. $\frac{1}{x-2} + \frac{2}{x-3} = \frac{3}{x-5}$.

4. $\frac{3}{x+3} - \frac{5}{x-5} + \frac{[illegible]}{x-2} = 0$ [illegible]

5. $\frac{1}{x} - \frac{b}{x+a} = \frac{1-b}{x-b}$.

6. $\frac{a}{x-a} - \frac{b}{x-b} = \frac{a}{x}$ [illegible]

7. $\frac{x}{x-1} + \frac{x+1}{x} = 2$.

8. $\frac{6}{x-2} + \frac{4(x-1)}{x^2-2x} = 3$ [illegible]

9. $\frac{1}{x-a}-\frac{1}{x-b}=\frac{1}{x+b}-\frac{1}{x+a}.$

10. $\frac{a}{b+x}-\frac{b}{a+x}=\frac{a-b}{c+x}.$

11. [illegible] $=\frac{2(a-b)}{x-a-b}.$

12. $\frac{ax}{a+x}+\frac{bx}{b+x}=a+b.$

13. $\frac{a}{x+a}+\frac{c}{x-c}=\frac{a-c}{x+a-c}.$

14. $\frac{a+c}{x+2b}+\frac{b+c}{x+2a}=\frac{a+b+2c}{x+a+b}.$

15. $\frac{ax^2+bx+c}{px^2+qx+r}=\frac{ax+b}{px+q}.$

16. $\frac{x^3-ax^2+bx \text{ [illegible]}}{x^3+ax^2+bx \text{ [illegible]}}=\frac{x^2-ax+b}{x^2+ax+b}.$

17. $\frac{x^2-11x+19}{x^2+x-11}=\frac{x^2-5x+7}{x^2-7x+23}.$

18. $\frac{x-a}{b+c}+\frac{x-b}{c+a}+\frac{x-c}{a+b}=3.$

19. $\frac{x+a}{b+c}+\frac{x+b}{c+a}+\frac{x+c}{a+b}+3=0.$

20. $(x-a)^3+(x+b)^3=(2x-a+b)^3.$

21. $(2x+a-b)^3-(x-a+b)^3=3(x+2a-2b)^3.$

22. $\frac{x-2a}{b+c-a}+\frac{x-2b}{c+a-b}+\frac{x-2c}{a+b-c}=3.$

23. $\frac{x-2a}{b+c-a}+\frac{x-2b}{c+a-b}+\frac{x-2c}{a+b-c}=\frac{3x}{a+b+c}.$

PROBLEMS

198. Problems relating to uniform motion are of frequent occurrence and of great importance in mathematics. Each problem requires the careful consideration of three elements: (1) the space passed over, (2) the time occupied, (3) the velocity or the rate of motion. For example, if a man walks 4 miles per hour for 5 hours, the total space

passed over is 5 times 4 miles, or 20 miles in all. In common language we say

Space = velocity × time.

More accurately this should be expressed thus:

Let s denote the number of units of space passed over,
" t " " " " " time occupied,
and v " " velocity, *i.e.*, the number of units of space passed over in one unit of time.

Then $$s = vt, \quad t = \frac{s}{v}, \quad v = \frac{s}{t},$$

which are but three different ways of expressing the same fact, and show how to find any one of the three elements when the other two are known. Each of these forms is frequently employed in the solution of problems.

Ex. 1. A can run 10 yards per second; B can run 8 yards 2 feet per second and has 20 yards' start. How far must A run before he catches B?

Let P and Q be the positions of A and B at the beginning, R the point where A catches B, *i.e.*, A runs from P to R while B runs from Q to R.

Let x = number of seconds the race lasts.

Then $10x$ = the number of yards in PR,

and $8\frac{2}{3}x$ " " " QR.

Now since $$PR = PQ + QR,$$

we have $$10x = 20 + 8\tfrac{2}{3}x,$$

from which $x = 15$ and $10x = 150$, the number of yards required.

The solution may also be obtained thus:

Let x = the number of yards in PR,

then $x - 20$ = " " " QR.

Therefore $$\frac{x}{10} = \frac{x - 20}{8\frac{2}{3}},$$

from which $x = 150$, the number of yards required.

In the first solution each side of the equation represents the whole *distance* of the race; in the second solution each side represents the *time* occupied.

Ex. 2. A boatman who can row 8 miles per hour in still water, rows for 1 hour and 15 minutes down stream; the return trip requires an hour and 45 minutes. Find the rate of the stream and the distance rowed.

Let x = rate of stream in miles per hour.

Then $8 + x$ = rate of boat down stream in miles per hour,

and $8 - x$ = " " " up " " "

Therefore $1\frac{1}{4}(8 + x) = 1\frac{3}{4}(8 - x)$,

from which $x = 1\frac{1}{3}$, the number of miles per hour,

and $1\frac{1}{4}(8 + x) = \frac{5}{4}(8 + 1\frac{1}{3}) = 11\frac{2}{3}$ miles distance.

Ex. 3. At what time between 3 and 4 o'clock is the long hand twice as far from the figure 7 as the short hand is from the figure 3?

Two solutions are possible, one before the long hand reaches the figure 7 and one after it has passed.

Draw the outline of a clock face with the hands in position at 3 o'clock, and again in the position required by the problem.

Let x = number of minute spaces passed over by the minute hand from 3 o'clock to time required.

Then $\frac{x}{12}$ = number of spaces passed by hour hand.

From the diagram $x = 35 - 2\left(\frac{x}{12}\right)$ for first solution,

and $x = 35 + 2\left(\frac{x}{12}\right)$ " second "

from which $x = 30$ and 42 respectively.

The times are, therefore, 30 minutes and 42 minutes respectively after 3 o'clock.

Ex. 4. A and B can together do a work in 6 days which A alone can do in 10 days. In what time can B alone do it?

Let x = number of days required by B.

Then in 1 day

A does $\frac{1}{10}$ of the work, B does $\frac{1}{x}$, A and B do $\frac{1}{6}$.

Therefore $\frac{1}{10} + \frac{1}{x} = \frac{1}{6}$,

from which $3x + 30 = 5x$,

or $x = 15$, the number of days required.

Ex. 5. A and B can do a work in 36 hours; if A works 2 hours and B 5 hours the work is $\frac{1}{12}$ done. In what time could A alone do it?

Let x = number of days required by A.

Then in 1 day

A and B do $\frac{1}{36}$ of the work, A does $\frac{1}{x}$, B does $\frac{1}{36} - \frac{1}{x}$.

Then $\frac{2}{x} + 5\left(\frac{1}{36} - \frac{1}{x}\right) = \frac{1}{12}$,

from which $x = 54$, the number of days required.

EXERCISE LI

1. Find a fraction whose value is [illegible] and whose numerator is less than its denominator by 9.

2. Find a fraction whose value is [illegible] and whose denominator is less by 8 than twice its numerator.

3. Find the value of $\frac{x}{1+x}$ when $\frac{x}{1-x} = \frac{1}{3}$.

4. If the sum of the fractions $\frac{1}{2x-1}$, and $\frac{x}{x+2}$ be unity, find the value of each fraction separately.

5. A and B begin business with capital in proportion of 3 to 5. A gains \$152 and B loses \$172 and now their capital is proportion 2 to 3. How much had each at first?

6. A person invested a sum of money in 3% stock at 80, and one dollar more than twice as much in 4% stock at 104; his whole income was \$57.25. Find total amount of cash invested.

7. If A can run 8 yards per second while B can run $7\frac{1}{2}$, but has a start of 3 seconds, in what time will A overtake B?

8. In the preceding example, if B had a start of 20 yards, how far must A run to overtake him?

9. A train which travels 32 miles an hour is 35 minutes in advance of another which travels 40 miles per hour. In what time and distance will the latter overtake the former?

10. An express train which travels 45 miles per hour starts 36 minutes after a freight train, which it overtakes in 2 hours and 4 minutes. What is the velocity of the freight train?

11. It requires $19\frac{1}{2}$ minutes longer to walk a certain distance at $3\frac{1}{3}$ miles per hour than to return the same distance at 4 miles per hour. Find the distance.

12. *A* can do a work in 6 days which *B* can do in $7\frac{1}{2}$. In what time can both together do it?

13. *A* and *B* can do a work in 5 days; if *A* works alone 3 days, *B* can finish it in 8 days. In what time could *A* alone do it?

14. *A* can do a work in 40 days; *B* can do [illegible] as much work as *A*; with the help of *C* they can all together do it in $13\frac{1}{3}$ days. In what time could *C* alone do it?

15. A tank could be filled in 15 minutes by two pipes A and B running together. After A has been running for 5 minutes B is also turned on, and in 13 minutes more the tank is filled. In what time could A alone fill it?

16. *A* and *B* can together do a work in 25 days; if *A* works 2 days and *B* 10 days, $\frac{1}{5}$ of the work is done. In what time could *A* alone do it?

17. A boatman who rows 6 miles per hour rows down stream in $3\frac{1}{2}$ hours, a distance which requires 4 hours and 54 minutes to return. Find the distance and the rate of the stream.

18. A steamer descends a river which runs $1\frac{1}{2}$ miles an hour as far in 4 hours as it can return in $4\frac{8}{11}$ hours. Find the rate of the steamer in still water and the distance.

19. A boatman can row a certain distance down stream in 20 minutes which would require half an hour in still water. How long will it take him to return? Compare his rate in still water with the rate of the stream.

20. A boatman rowing with the tide moves 40 miles in 4 hours. Returning he requires 6 hours rowing against tide twice as strong. Find the velocity of the stronger tide.

21 At what times between 6 and 7 o'clock are the hands at right angles to each other?

22 At what time between 3 and 4 o'clock is the long hand (1) as far past the figure 6 as the short hand is past the figure 3, (2) twice as far past the figure 7 as the short hand is past the figure 3?

23 At what times between 5 and 6 o'clock is the long hand twice as far from the figure 7 as the short hand is from the figure 5?

24 Find the times between 7 and 8 o'clock when the hour and minute hands are (1) exactly opposite, (2) at right angles, (3) coincident.

25 At what time after 3 o'clock will the long hand be at a point twice as many minute spaces before the figure 7 as the short hand is past the figure 3.

26 Divide 64 into two parts such that the quotient of one part by the other may be 3, with remainder 5.

27 Find a number such that when divided into parts in the ratio 1 : 2 : 3, the product of the three parts is equal to the original number.

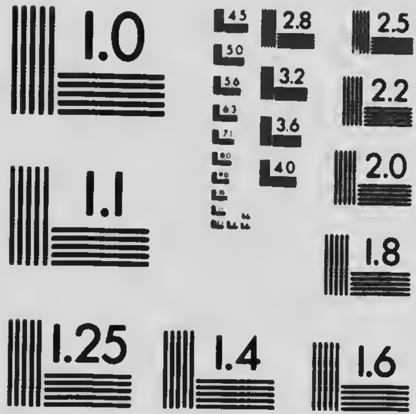
28 Find a number such that when divided into two parts in the ratio 2 : 3, the product of these parts is the same as the product of the three parts when divided in the ratio 1 : 3 : 4.

29 It requires 4 hours to walk a certain distance and 1 hour and 36 minutes to ride the same distance at a rate of 6 miles per hour faster than the former rate. Find the distance.

30 A franc is worth 19 cents; a mark is worth a franc

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



and $\frac{1}{8}$ of a guilder, and a guilder is worth 3 cents less than a franc and a mark. Find the value of each coin in cents.

31. A boy rides his wheel from A to B at $7\frac{1}{2}$ miles per hour and returns on foot by a shorter road at $3\frac{1}{2}$ miles per hour, but taking 20 minutes more time. The whole distance travelled being $8\frac{1}{2}$ miles, find the whole time occupied in the journey.

32. A merchant added yearly to his capital $\frac{1}{3}$ of it, but took from it at the end of each year $2000 for expenses. At the end of the third year, after deducting the last $2000, he had twice his original capital. How much had he at first?

33. A trader maintained himself for 3 years at an expense of $1450 a year, and each year increased the part of his stock which was not so expended by $\frac{1}{4}$ of it. At the end of the third year he had added 50% to his original capital. What had he at first?

34. Tom can run 11 yards while Dick runs 10. How many yards' start must Dick have in a 100-yard race to beat Tom by 2 yards?

35. A man invested $7485 in stock, a part in 3% stock at 90 and the remainder in $3\frac{1}{2}$% stock at 97. The total increase being $250 per annum, find the amount invested in each kind of stock.

36. A and B can do a work in 20 days, A and C in 25 days. When B has worked 8 days and C 10 days the work is one-half done. In what time could A do the work?

37. A boy walks to school at the rate of 11 yards in 8 seconds, and is 40 seconds late. If he had walked 5 yards in 3 seconds he would have been 30 seconds too soon. Find the distance to school and the time in which he should make the journey.

38. An express train leaving Toronto at 9 a.m. arrives in Hamilton at 9.30 a.m., while a freight train leaving Hamilton at 9 a.m. arrives in Toronto at 9.45 a.m. At what hour and at what fraction of the whole distance from Toronto do they meet?

39. *A*, *B* and *C*, starting from the same point, travel in the same direction. *A* starts at 9 a.m. at 10 miles per hour; *B* starts 10 minutes later at 12 miles per hour; *C* starts 20 minutes later still at 15 miles per hour. When will *C* be midway between *A* and *B*?

40. A teacher's savings and expenses are as 9 : 16; he gets an increase of $250 per annum in salary and increases his expenses $100 per annum, and now his savings and expenses are as 2 : 3. Find his original salary.

41. *A*, *B* and *C* can walk $4\frac{1}{2}$ miles, 4 miles and $3\frac{1}{2}$ miles per hour respectively. *A* and *B* start to walk from P to Q at the same time that *C* starts to walk from Q to P. *C* meets *A* 10 minutes before he meets *B*. How far is it from P to Q?

CHAPTER XI

SIMULTANEOUS EQUATIONS

TWO UNKNOWN QUANTITIES

199. The simple cases of Simultaneous Equations with two unknown quantities were considered in Chapter V. We now proceed to the solution of more complicated examples and to explain methods by which the work may frequently be facilitated.

Ex. 1. Solve $17x + 15y = 46$ (1)

$15x + 17y = 50.$ (2)

Adding (1) and (2), $32x + 32y = 96.$

Dividing by 32, $x + y = 3.$ (3)

Subtracting (2) from (1), $2x - 2y = -4,$

or $x - y = -2.$ (4)

From (3) and (4) we easily get $x = \frac{1}{2}$

$y = 2\frac{1}{2}.$

The object of adding and subtracting, as given above, is to obtain equations with smaller coefficients and thus avoid the multiplication of large numbers.

Ex. 2. Solve $\frac{3}{x} + \frac{6}{y} = 8.$ (1)

$\frac{5}{x} + \frac{9}{y} = 13.$ (2)

In this and many similar examples it is best to consider $\frac{1}{x}$ and $\frac{1}{y}$ as the unknown quantities and to eliminate one of them before clearing of fractions. Multiplying (1) by 3 and (2) by 2,

we have $$\frac{9}{x}+\frac{18}{y}=24 \quad (3)$$

$$\frac{10}{x}+\frac{18}{y}=26. \quad (4)$$

Subtracting (3) from (4), $$\frac{1}{x}=2.$$

from which $$2x=1, \text{ or } x=\frac{1}{2}.$$

Then from (1) we easily obtain $$y=3.$$

Ex. 3. Solve $$3x+4y=8xy \quad (1)$$

$$5x-2y=9xy. \quad (2)$$

Dividing each equation by xy

we get $$\frac{3}{y}+\frac{4}{x}=8 \quad (3)$$

$$\frac{5}{y}-\frac{2}{x}=9. \quad (4)$$

Equations (3) and (4) can now be solved by the method of the previous example.

Ex. 4. Solve $$ax+by=c \quad (1)$$

$$lx+my=n. \quad (2)$$

Multiplying (1) by m and (2) by b

we get $$max+mby=mc \quad (3)$$

$$lbx+mby=nb. \quad (4)$$

Subtracting (4) from (3), $$(ma-lb)x=mc-nb,$$

or $$x=\frac{mc-nb}{ma-lb}.$$

Similarly we obtain $$y=\frac{lc-na}{lb-ma}.$$

In this example, the value of x being a complicated expression, it is easier to obtain y by eliminating x from the original equations than to substitute the value of x, as was done in previous examples.

Ex. 5. Solve
$$\frac{x-1}{y+1}=\frac{a}{b} \qquad (1)$$
$$\frac{ax-by}{x+y}=a-b+1. \qquad (2)$$

Clear each equation of fractions, collect the terms in x and y and arrange in usual form.

We have
$$bx-ay=a+b \qquad (3)$$
$$(b-1)x-(a+1)y=0. \qquad (4)$$

Subtract (4) from (3), $\quad x+y=a+b. \qquad (5)$

From (3) and (5) we obtain $\quad x=a+1,$ and $y=b-1.$

Observe the simplification effected by subtracting one equation from the other.

Ex. 6. Solve
$$\frac{28}{x+y}+\frac{30}{x-y}=14 \qquad (1)$$
$$\frac{7}{x+y}+\frac{3}{x-y}=2. \qquad (2)$$

Eliminate $x-y$ by multiplying (2) by 10 and subtracting (1) from it.

Thus $\frac{42}{x+y}=6$, from which $\quad x+y=7.$

Substituting this value in (2) we get $x-y=3.$

The values of x and y may now be easily obtained.

200. Indeterminate and inconsistent equations. Certain peculiarities in the solution of simultaneous equations will be best understood from their application to the solution of a number of simple problems.

Ex. 1. It is required to find a fraction such that if a unit be added to the numerator, or if 4 be subtracted from the denominator, in either case the resulting fraction will equal $\frac{1}{4}$.

Let $\frac{x}{y}$ be the required fraction.

Then $$\frac{x+1}{y}=\frac{1}{4}, \qquad (1)$$

and $$\frac{x}{y-4}=\frac{1}{4}. \qquad (2)$$

Simplifying (1) $$4x+4=y \qquad (3)$$

" (2) $$4x=y-4. \qquad (4)$$

But equation (4) is merely equation (3) with the 4 removed to the other side. It is not an independent equation and consequently gives no additional information regarding the values of x and y. We may give x any value whatever, find a corresponding value of y from either (3) or (4), and thus obtain as many different fractions as we please which satisfy the conditions of the problem.

Thus let $x=1,\ 2,\ 3,\ 7,\ 20$, etc.,

then $y=8,\ 12,\ 16,\ 32,\ 84$, etc.

Each of the fractions $\frac{1}{8}$, $\frac{2}{12}$, etc., possesses the properties stated in the problem.

Equations (1) and (2) are **not independent;** the problem admits of an indefinite number of solutions and is therefore said to be **indeterminate.**

Ex. 2. In the previous problem assume that when 1 is added to the numerator, or 3 subtracted from the denominator, the fraction becomes $\frac{1}{4}$.

Then $\qquad \dfrac{x+1}{y} = \dfrac{1}{4} \qquad$ and $\qquad \dfrac{x}{y-3} = \dfrac{1}{4},$

from which $\qquad y = 4x + 4 \qquad$ and $\qquad y = 4x + 3.$

No values of x and y can satisfy both these equations. The equations are **inconsistent** and the proposed problem is impossible.

EXERCISE LII

Solve

1. $x + y = 37,$
$\dfrac{x}{2} + \dfrac{y}{3} = 15.$

2. $\dfrac{x}{2} + \dfrac{y}{3} = \dfrac{47}{6},$
$\dfrac{x}{3} + \dfrac{y}{2} = \dfrac{43}{6}.$

3. $\dfrac{5x}{2} + \dfrac{10y}{3} = 45,$
$\dfrac{10x}{3} + \dfrac{5y}{2} = \dfrac{85}{2}.$

4. $17x + 19y = 146,$
$16x + 18y = 138.$

5. $11x + 19y = 3,$
$19x + 11y = 27.$

6. $29x + 85y = 31,$
$13x - 43y = 95.$

7. $10x - \dfrac{y}{3} = 47,$
$10y - \dfrac{x}{5} = 89.$

8. $\dfrac{x-1}{5} = \dfrac{y-1}{3} + \dfrac{16}{15},$
$\dfrac{x+y}{9} = \dfrac{x-y}{4}.$

9. $\dfrac{4}{3x-y} = \dfrac{3}{2x-3y},$
$\dfrac{8}{x+y} = \dfrac{11}{x-y+1}.$

10. $\dfrac{x+1}{2} = \dfrac{y+1}{3} = \dfrac{x+y}{4}.$

11. $\dfrac{2x-1}{3} = \dfrac{3x}{5} = \dfrac{4x-y+1}{7}.$

12. $\dfrac{2x-y}{x-2y} = \dfrac{4x+2y+1}{2x-3y} = 5.$

13. $\dfrac{x+2}{2y+1} = \dfrac{4y+3x+36}{2x-y} = 3.$

14. $\dfrac{x-1}{3} - \dfrac{y+1}{4} = \dfrac{2x-3}{5} - \dfrac{13-2y}{7} = 0.$

15. $\dfrac{x-3y}{6} - \dfrac{x-2}{5} + \dfrac{8}{3} = \dfrac{x+y}{11} - \dfrac{y-3}{4} = \dfrac{3}{2}.$

16. $\dfrac{2(15x+13y)}{7} = 7(x-2y+1) = 12\left\{y-3+\dfrac{2}{3}\left(y+7\dfrac{1}{2}\right)\right\}.$

17. $\dfrac{x-1}{x-5} = \dfrac{y+8}{y+4},$
$\dfrac{2x+3y-5}{x-y} = \dfrac{1}{3}.$

18. $\dfrac{3x-2}{5x-1} = \dfrac{3y+7}{5y+7},$
$\dfrac{3x-1}{x+5} = \dfrac{6y-5}{2y-3}.$

19. $x - \dfrac{4y-x}{13-x} = 4 + \dfrac{2x-17}{2},$
$y - \dfrac{2y-1}{x-5} = \dfrac{83}{6} - \dfrac{46-3y}{3}.$

20. $\dfrac{7-6x}{10y-19} = \dfrac{4-3x}{5y-11},$
$\dfrac{6x-10y-17}{3x-5y+2} = \dfrac{4x-14y-5}{2x-7y+12}.$

21. $\dfrac{21}{x} - \dfrac{6}{y} = 19,$
$\dfrac{12}{x} + \dfrac{21}{y} = 19.$

22. $\dfrac{6}{x} - \dfrac{4}{y} = \dfrac{1}{2},$
$\dfrac{14}{x} + \dfrac{8}{y} = \dfrac{10}{3}.$

23. $\dfrac{4}{x} - \dfrac{5}{y} = \dfrac{13}{3},$
$\dfrac{7}{x} + \dfrac{3}{y} = \dfrac{11}{3}.$

24. $\dfrac{11}{x+y} - \dfrac{2}{x-y} = \dfrac{4}{3},$
$\dfrac{3}{x+y} - \dfrac{5}{x-y} = \dfrac{38}{21}.$

25. $\dfrac{2}{3x} + \dfrac{1}{5y} = \dfrac{5}{3},$
$\dfrac{1}{2x} - \dfrac{2}{3y} = -\dfrac{17}{6}.$

26. $\dfrac{x+y}{xy} = \dfrac{1}{y} - \dfrac{1}{x} + \dfrac{4}{7} = \dfrac{3}{7}.$

27. $\dfrac{1}{7x} + \dfrac{y}{18} = 11,$
$\dfrac{1}{9x} + \dfrac{y}{4} = 16.$

28. $7xy = 14x + 12y,$
$11xy = 7x + 10\frac{1}{2}y.$

29. $\dfrac{xy}{x+y} = \dfrac{2xy}{x-y} = 10.$

30. $x + y = 2a,$
$ax - by = a^2 + b^2.$

31. $ax + y = b,$
$x + by = a.$

32. $\dfrac{x}{a} + \dfrac{y}{b} = \dfrac{x}{b} - \dfrac{y}{a} = 1.$

33. $\dfrac{x}{a+b} - \dfrac{y}{a-b} = \dfrac{1}{a+b}$,

$\dfrac{x}{a+b} + \dfrac{y}{a-b} = \dfrac{1}{a-b}$.

34. $\dfrac{a+b}{x} + \dfrac{a-b}{y} = a^2 + b^2$,

$\dfrac{a-b}{x} + \dfrac{a+b}{y} = a^2 - b^2$.

35. $b(x+y) + a(x-y) = 4$,

$\dfrac{x+y}{a} - \dfrac{x-y}{b} = 0$.

36. $x - y = a - b$,

$\dfrac{x-a}{b} + \dfrac{y-b}{a} = a + b$.

37. $\dfrac{x}{a} + \dfrac{y}{b} = b - a$,

$\dfrac{x}{b-c} + \dfrac{y}{c-a} = a + b$.

38. $\dfrac{x}{a} + \dfrac{y}{b} = c + d$,

$\dfrac{x}{c} + \dfrac{y}{d} = a + b$.

39. $a(x+y) + b(x-y) = 2a$,

$\dfrac{x+y}{2(a^2+b^2)} = \dfrac{1}{a^2-b^2}$.

40. $\dfrac{x}{a} + \dfrac{y}{b} = \dfrac{x}{b} - \dfrac{y}{a} = \dfrac{1}{a} + \dfrac{1}{b}$.

41. $\dfrac{a}{bx} + \dfrac{b}{ay} = a + b$,

$\dfrac{b}{x} + \dfrac{a}{y} = a^2 + b^2$.

42. $\dfrac{x}{a+b} + \dfrac{y}{a-b} = \dfrac{2}{ab}$,

$\dfrac{x}{a-b} - \dfrac{y}{a+b} = \dfrac{1}{a^2-b^2}$.

43. $\dfrac{x+y-1}{x-y+1} = a$,

$\dfrac{y-x+1}{x-y+1} = b$.

44. $\dfrac{x}{a+b} + \dfrac{y}{a-b} = 2$,

$(x-a)^2 + (y-b)^2$
$= (x-b)^2 + (y-a)^2$.

THREE UNKNOWN QUANTITIES

201. When three unknown quantities have to be found, it is usually most convenient to represent each of them by a separate letter and then three independent equations containing these letters must be given, from which their values may be found by methods illustrated by the following examples :

Ex. 1. Solve the equations

$$2x + y + 3z = 22 \qquad (1)$$
$$x + 2y - z = 3 \qquad (2)$$
$$3x + 5y + 4z = 41. \qquad (3)$$

Multiplying (2) by 3 and adding (1)

we get $$5x + 7y = 31. \qquad (4)$$

Multiplying (2) by 4 and adding (3)

we get $$7x + 13y = 53. \qquad (5)$$

Multiplying (4) by 7, (5) by 5 and subtracting

we get $$16y = 48, \text{ or } y = 3.$$

In (1), for y substitute its value 3.

Then $7x + 39 = 53$, from which $x = 2$.

In (1), for x and y substitute their values, 2 and 3.

Then $4 + 3 + 3z = 22$, from which $z = 5$.

In this solution note the following facts:

1. From three equations three unknown numbers, each represented by a letter, were to be found.

2. From each of two pairs of these equations the same unknown number, z, was eliminated, giving two independent equations, (4) and (5), containing two unknown numbers, x and y.

3. From these two equations, one unknown number, x, was eliminated, and from the resulting equation the remaining number, y, was found.

4. The value of the letter, y, was substituted in an equation containing two unknowns, and thus the value of a second letter, x, was found.

5. The values of the two letters already found were substituted in one of the original equations, and thus the remaining unknown number was found.

6. The z was eliminated first, but the letters might have been eliminated in any order. In practice the letter having the smallest coefficients should usually be taken first.

Ex. 2. Solve

$$2x - 3y = 4 \quad (1)$$
$$5y - 4z = 6 \quad (2)$$
$$4z - 2x = -6. \quad (3)$$

Equation (1) does not contain z; if then we eliminate z from (2) and (3) we shall get an additional equation containing only x and y.

We have then $\quad 2x - 3y = 4 \quad (4)$

and $\quad -2x + 5y = 0. \quad (5)$

from which $\quad 2y = 4$, or $y = 2$ and $x = 5$,

then from either (2) or (3) we get $z = 1$.

Ex. 3. Solve equations $\quad 3x + 5y - 4z = 3 \quad (1)$

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{5} \quad (2)$$

From (2) we have $\quad x = \frac{2z}{5}, \quad y = \frac{3z}{5}.$

Substituting these values in (1)

we get $\quad \frac{6z}{5} + 3z - 4z = 3,$

from which $\quad z = 15$, then from (2) $x = 6$, and $y = 9$.

When the values of two of the letters can be conveniently expressed in terms of the third, the solution can be most conveniently effected by substitution, as in this example.

202. Indeterminate and inconsistent equations. Various peculiarities may arise in the connection with equations containing three unknown quantities. The following examples illustrate two of the most important:

Ex. 1. A quantity of grain consisting of wheat, barley and oats, 45 bushels in all, weighs 2000 lbs. and ... worth

\$31.35. The weights of a bushel of each are 60 lbs., 48 lbs. and 34 lbs.; their values are 93 cents, 75 cents and 54 cents respectively. Find the number of bushels of each.

Let x, y, z represent the number of bushels.

Then

$$x + y + z = 45 \qquad (1)$$

$$60x + 48y + 34z = 2000 \qquad (2)$$

$$93x + 75y + 54z = 3135. \qquad (3)$$

From (1) and (2) $\quad 26x + 14y = 470 \qquad (4)$

" (1) " (3) $\quad 39x + 21y = 705. \qquad (5)$

If now we divide (4) by (2) and (5) by 3, in each case we get

$$13x + 7y = 235, \qquad (6)$$

which shows that (4) and (5) are not independent, and consequently an indefinite number of values of x and y will satisfy the equations; and then for each such pair of values we can, from (1), find a corresponding value for z.

Thus

$$x = 3, \ 10, \ 15, \ 17,$$

$$y = 28, \ 15, \ 12\tfrac{1}{7}, \ 2,$$

$$z = 14, \ 20, \ 17\tfrac{6}{7}, \ 26, \text{ etc.},$$

will be found to satisfy all three equations. The problem is therefore **indeterminate.**

The cause of this peculiarity arises from the fact that the three original equations are not all independent. Equation (5), obtained from (1) and (3), is the same as (4) obtained from (1) and (2), and consequently (3) gives no information not contained in (1) and (2).

In fact equation (3) may be directly derived from equations (1) and (2) by multiplying the former by 3 and the latter by $\frac{3}{2}$ and adding the results. The given problem, therefore, contains three unknown quantities but gives only two statements concerning them, and these are not sufficient for the purpose.

Ex. 2. In Ex. 1 assume the total value to be \$32, the other quantities remaining the same as before.

Proceeding as before,

Equation (4) remains $26x + 14y = 470,$ (7)

" (5) becomes $39x + 21y = 770.$ (8)

Dividing (7) by 2 $13x + 7y = 235,$ (9)

" (8) by 3 $13x + 7y = 256\frac{2}{3}.$ (10)

Now (9) and (10) are clearly **inconsistent** and consequently the problem as now stated is impossible.

EXERCISE LIII

Solve

1. $x + y + z = 11,$
$2x - y + 2z = 16,$
$3x + 2y + 3z = 31.$

2. $x - y + 2z = 1.$
$3x + 2y - z = 2.$
$2x + 3y + 4z = 22.$

3. $3x - 2y + z = 10,$
$4x + 3y + 2z = 28,$
$5x + 2y - 3z = 4.$

4. $5x - 4y - 3z = 2.$
$3x + 7y - 5z = 52.$
$2x - 5y + 4z = 16.$

5. $x - 2y + 3z = 2,$
$2x - 4y + z = -1,$
$3x - 2y + 2z = 7.$

6. $4x + 3y + 5z = 20.$
$2x - 9y + 3z = 25.$
$6x + 12y - 7z = 55.$

7. $5x = 3y + 10z - 4,$
$6y = 8z - 2x - 28,$
$4z = 3x + 5y + 10.$

8. $2(x - y) + 3(y$ [illegible]$) = -9.$
$4(x - z) - 2(y -$ [illegible]$) = 2.$
$5(x + y) - 3(x +$ [illegible]$) = -17.$

9. $x + y = 12,$
$y + z = 14.$
$z + x = 20.$

10. $2x - 3y = 5,$
$3y - 4z = 7.$
$4z + 3x = 8.$

11. $x + y + z = 8,$
$2x + 3y = 8,$
$2y - 3z = -13.$

12. $3x + y = 1 - 2z,$
$2y + 3z = x - 10,$
$z - 3x = 4(y - 1).$

13. $2(x + y) = x - z,$
$x + 1 = 3(y + z),$
$z - 2y = 5x.$

14. $3(x - 1) = 2(y - 2),$
$2(y - 3) = 4(x - 2),$
$y = 2(x + z).$

15. $\frac{x}{5} + \frac{y-1}{3} = \frac{x+y+z}{3},$
$\frac{x-1}{4} - \frac{y-2}{5} = \frac{z+3}{10},$
$\frac{2y-5}{3} + \frac{1-z}{2} = \frac{y-z}{2}.$

16. $\frac{x-y}{2} + \frac{y-z}{3} = \frac{x}{5},$
$\frac{x+y}{2} + \frac{x+z}{3} = 2(x - y + z)$
$\frac{y+z-x}{3} = \frac{z-1}{4}.$

17. $2x - 3y = 3y - 5z = x + y + z = 31.$

18. $\frac{3x+4y}{5} = \frac{4y+5z}{10} = \frac{5z+3x}{15} = \frac{x+y+z+3}{4}.$

19. $\frac{2}{x} + \frac{4}{y} = 5,$
$\frac{6}{y} - \frac{8}{z} = -6,$
$\frac{6}{z} - \frac{8}{x} = 5.$

20. $\frac{2}{x} + \frac{3}{y} = -6,$
$\frac{3}{x} + \frac{1}{z} = \frac{17}{2},$
$\frac{1}{y} + \frac{2}{z} = 5.$

21. $\frac{9}{2x} - \frac{1}{y} = 4,$
$\frac{5}{z} - \frac{3}{x} = 8,$
$\frac{7}{2y} + \frac{15}{4z} = 4.$

22. $\frac{3}{x} - \frac{2}{3y} + \frac{1}{z} = 8,$
$\frac{1}{2x} + \frac{3}{2y} + \frac{2}{z} = \frac{27}{2},$
$\frac{3}{4x} + \frac{1}{y} + \frac{3}{z} = \frac{33}{2}.$

23. $\frac{1}{x}\left(\frac{5}{y} + \frac{7}{z}\right) = \frac{2}{3},$
$\frac{1}{y}\left(\frac{7}{z} + \frac{3}{x}\right) = \frac{2}{5},$
$\frac{1}{z}\left(\frac{3}{x} + \frac{5}{y}\right) = \frac{2}{7}.$

24. $y+z=2a,$
$z+x=2b,$
$x+y=2c.$

25. $bz+cy=a,$
$az+cx=b,$
$ay+bx=c.$

26. $x+y+z=0,$
$ax+by+cz=0,$
$c(x+c)-b(y+a)=0.$

27. $x+y+z=0,$
$ax+by+cz=0,$
$a^2x+b^2y+c^2z=1.$

28. $x-ay+a^2z=a^3,$
$x-by+b^2z=b^3,$
$x-cy+c^2z=c^3.$

29. $\frac{x-1}{3}=\frac{y-2}{5}=\frac{z-3}{7},$
$x+y+z=21.$

30. $\frac{x+m}{n+p}=\frac{y+n}{p+m}=\frac{z+p}{m+n},$
$x+y+z=m+n+p.$

31. $x+y+z=a+b+c,$
$bx+cy+az=cx+ay+bz,$
$=a^2+b^2+c^2.$

32. $x+y+z=a+b+c,$
$bx+cy+az=cx+ay+bz,$
$=ab+bc+ca.$

EXERCISE LIV

PROBLEMS

1. The sum of two numbers is 3 times their difference, but if 5 be added to each their sum will be 4 times their difference. Find the numbers.

2. One-half the sum of two numbers and one-third of their difference together make 13. If 3 be added to each, one result is 3 times the other. Find the numbers.

3. If a unit be added to numerator and denominator of a fraction it becomes $\frac{5}{7}$; if 14 be subtracted from each it becomes $\frac{1}{2}$. What must be subtracted from each to make it $\frac{2}{3}$?

4. A number consisting of two digits, of which the units' digit is the greater, when divided by the sum of its digits gives 3 as quotient with 7 remainder. When divided by the difference of its digits the quotient is 9 and remainder 1. Find the number.

5. The double of a number consisting of two digits is greater by 4 than the number with its digits reversed. If the tens' digit be halved and the resulting number doubled the result will be greater than the original number by 9. Find the first number.

6. A man has two watches and a chain worth \$22. The first watch and chain are together worth $1\frac{1}{3}$ times as much as the second watch, while the second watch and the chain are together worth $1\frac{2}{3}$ times as much as the first. Find the value of each watch.

7. *A* saves $\frac{1}{5}$ of his daily pay and *B* saves $\frac{1}{3}$ of his; together they save \$1.44. *A* receives an increase of $12\frac{1}{2}\%$ and *B* an increase of 10%, and now *B* saves 16 cents a day more than A. Find the daily wage of each.

8. A bushel of wheat and 2 bushels of barley are together worth \$2.18. If the barley were to rise in value $12\frac{1}{2}\%$ and the wheat were to fall 10%, 8 bushels of wheat would be worth 9 bushels of barley. Find the value of a bushel of each.

9. The value of $3\frac{1}{2}$ yards of velvet and $12\frac{3}{4}$ yards of silk is the same as that of $4\frac{1}{2}$ yards of velvet and 5 yards of silk, the total value in each case being \$63.80. Find the value of a yard of each.

10. Three boys have together 120 marbles. The first gives $\frac{1}{6}$ of his marbles and the second gives $\frac{1}{9}$ of his to the third, and now all three have the same number. How many had each at first?

11. If *A* should receive $\frac{1}{3}$ of *B*'s money he would have $\frac{1}{2}$ as much again as before. If *A* should now return \$6 to *B*, the former would still have twice as much as the latter. What had each at first?

12. *A* and *B* can together do a work in $13\frac{1}{3}$ days. If *A* works 8 days and *B* 10 days, $\frac{2}{3}$ of the work will be done. In what time could each separately perform it?

13. *A* and *B* reap a field of wheat in 10 days, *B* and *C* in 12 days, and *A* can do twice as much as *C* in the same time. In what time would *C* alone do it, and in what time could all three do it?

14. A cistern has three pipes A, B, and C. By A and B it can be filled in 36 minutes; if C be now opened and B closed it will be emptied in $1\frac{1}{2}$ hours. If B be again opened it will be filled again in 3 hours. In what time could A and B separately fill it and C alone empty it?

15. A man can row 20 miles down stream and back again in $7\frac{1}{2}$ hours. He can row 2 miles down in the same time as 1 mile up stream. Find his rate of rowing in still water and the rate of the stream.

16. A boatman can row 13 miles down stream and back again in 4 hours $13\frac{1}{2}$ minutes; in still water he could row the same distance in 4 hours. How much longer did it take to row down than to row back?

17. In a 480-yard race *A* gives *B* a start of 15 yards and beats him by 2 seconds. In a second trial *A* gives *B* a start of 15 yards in a 360-yard race and beats him by 1 second. How many yards does each run per second?

18. A river flows 3 miles per hour and a steamer in going down stream requires 18 seconds to pass a given point, whilst in returning it requires 30 seconds. Find the length of the boat and its rate in still water.

19. The front wheel and the hind wheel of a carriage together make 19 revolutions in 88 feet. If the circumference of the front wheel were doubled it would then

make 15 revolutions less than the hind wheel in 176 yards. Find the circumference of each wheel.

20. A man and his two sons could dig a ditch in 6 days; if the man and either son work 7 days, the other son can complete the work in 2 days. In what time can each alone do the work?

21. Two men took turns in rowing a boat; the one could cover the whole distance in 10 hours, the other in 14 hours; the journey was completed in 12 hours. How many hours did each one row?

22. A quantity of water sufficient to fill 3 jars of different sizes will fill the smallest jar 4 times, or the largest one twice with 4 gals. to spare, or the second jar 3 times with 2 gals. to spare. What is the capacity of each jar?

23. *A* gave to *B* and to *C* as much money as each already had; *B* then gave to *A* and *C* as much as each then had; *C* then gave to *A* and *B* as much as each then had, after which each had $8. How much had each at first?

24. Three boys had each a bag of nuts. Each boy gave each other boy $\frac{1}{5}$ of what he had originally in his own bag and then they severally had 740, 580, 380. How many had each at first?

25. *A* saves $\frac{1}{5}$ of his income and *B* saves $\frac{2}{3}$ of his; *B* spends $50 less per annum than twice what *A* saves, and *A* spends $150 more than *B* saves. Find their incomes per annum.

26 *A* and *B* can do a work in 12 days, *B* and *C* in 15 days; *A* does as much work in 1 day as *C* does in 2. In what time could each one alone and also all together perform it?

27. A grocer has 25 lbs. of tea worth 40 cents a pound; he wishes to mix it with other teas at 50 and 60 cents a pound to make 100 pounds in all worth 54 cents a pound. How many pounds of each kind must he take?

28. If a traveller were to make $1\frac{1}{2}$ miles more per hour he would require 40 minutes less time. If he were to make $1\frac{1}{2}$ miles per hour less he would require one hour more. Find the whole distance.

29. The incomes of two men are as $3:2$ and their expenditures are as $5:3$. Each saves \$500 per annum. Find their incomes and expenditures.

30. The sum of the three digits of which a number is composed is 15, and the tens' digit is $\frac{2}{3}$ of the sum of the other two. If the hundreds' digit be removed the resulting number is greater by 15 than twice the number formed by removing the units' digit. Find the number.

31. If the length of a rectangle were diminished by $2\frac{1}{2}$ feet and the width increased 4 feet the area would be increased by 100 square feet. Its perimeter is 107 feet; find its area.

32. If a unit be added to both numerator and denominator of a fraction its value is increased by $\frac{1}{22}$; if a unit be subtracted from each its value will be diminished by $\frac{3}{55}$. Find the fraction.

33. In a quarter of a mile race A can beat B by 11 seconds or by 121 feet. Find their rates in feet per second.

34. In a mile race A gives B a start of 44 yards and is beaten by 1 second. In a second trial A gives B a start of 6 seconds and beats him by $9\frac{5}{9}$ yards. Find the number of yards each runs per second.

35. One cask contains wine and water in the ratio $3:5$.

another in ratio 8 : 7. How many gallons must be taken from each cask to make 41 gallons of wine and 43 gallons of water?

36. The area of a certain triangle would remain unchanged if its altitude were increased by 1 foot and its base reduced 4 inches, also if its altitude were decreased $1\frac{2}{3}$ feet and its base increased by 8 inches. Find its base and altitude.

37. A teacher after paying his taxes has \$2058.40 remaining. The tax rate is 18 mills on the dollar for the value of his house and for the excess of his salary above \$1000. There is also a frontage tax on his grounds, which is $\frac{1}{6}$ of the tax on his house and \$9.60 less than the tax on his salary. Find his salary and the value of his house.

EXAMINATION PAPERS

I

1. Find the value of $\frac{x+a}{x-b}+\frac{x+b}{x-a}$ when $x=\frac{a+b}{2}$.

2. Multiply $a^2-ab+b^2-\frac{b}{a}+\frac{1}{a^2}-1$ by $a+b+\frac{1}{a}$.

3. Divide $\frac{x^3}{2}+\frac{1}{3}ax^2-\frac{2}{3}a^2x+\frac{1}{6}a^3$ by $\frac{x}{2}-\frac{a}{6}$.

4. Find the H. C. F. and L. C. M. of

$$2x^2-7x-4 \text{ and } 2x^3-7x^2+2.$$

5. Solve equations $\frac{x-2y}{5}+\frac{5y-2x}{10}=\frac{1}{10}$,

$$\frac{1}{x-y}+\frac{2}{x-2y}=\frac{3}{x-3y}.$$

6. How many children in a family in which each boy has an equal number of brothers and sisters, but each girl has twice as many brothers as she has sisters?

7. If the price per yard of a piece of cloth had been one-third less, 25 yards more could have been bought for the same money. How many extra yards might have been bought if the price had been three-fourths of what was really paid?

II

1. Reduce to lowest terms

$$\frac{x^3+5x^2-28}{x^4+x^2-20} \text{ and } \frac{(x^2+2)^2+(x^2-2)^2}{2(x+1)^2+2}.$$

2. Simplify $\left(1-\frac{4y^2}{x^2}\right)\left(1+\frac{4y}{x-2y}\right)\div\left(\frac{(x+2y)^2}{x^3-2x^2y}\right).$

3. Solve $\frac{c}{x-c}+\frac{a-c}{x+a-c}=\frac{a}{x+a}$, and find the value of each side of the equation in terms of a and c.

4. If $x=\frac{c-a}{2}$, find the value of $\frac{a+x}{c-x}+\frac{a(a+2x)}{c(c-2x)}.$

5. The H. C. F. of two expressions is $x-1$, and their L. C. M. is $x^3-6x^2+11x-6$. Find the expressions.

6. A sum of money was equally divided among a number of persons by giving the first $10 and one-fourth of the remainder, the second $20 and one-fourth of what then remained, and so on. Find the sum divided and the number of persons.

7. The side of a square is 16 inches less than the length of a rectangle, and its perimeter is 4 inches less than the perimeter of the rectangle. The area of the square is 16 square inches greater than the area of the rectangle. Find the side of the square.

III

1. Divide

$$c(x+a)(x+a-c)+(a-c)(x+a)(x-c)-a(x-c)(x+a-c)$$
by $2x+a-c$.

2. If $x=a^2-bc,\ y=b^2-ca,\ z=c^2-ab$, show that

$$\frac{x^2-yz}{a}=\frac{y^2-zx}{b}=\frac{z^2-xy}{c}=ax+by+cz.$$

3. Simplify

$$\left(\frac{a^2+ab}{a^3+b^3}\right)\left(\frac{a^2-ab+b^2}{a-b}\right)\div\left(1-\frac{2a^3}{a^3-b^3}\right)\left(1+\frac{2ab}{a^2-ab+b^2}\right)$$

4. Solve equations $\dfrac{xy}{x+y}=a-b,\ \dfrac{xy}{x-y}=a+b.$

5. Find the L. C. M. of

$$5(a-b)^2,\ 10(a^2-b^2)^2,\ 15(a^2+ab)^2,\ 20(ab-b^2)^2.$$

6. *A* can do a piece of work in 7 days less than *B*, and both together can do it in 1 day more than half the time required by *A*. In what time can both together do it?

7. A pedestrian's rate of walking up hill, on the level and down hill are $3\frac{1}{2}$, 4 and $4\frac{1}{2}$ miles per hour respectively. He walks to a certain point and returns by the same road in $5\frac{4}{7}$ hours. On a level road he would walk the same number of miles in $5\frac{1}{2}$ hours. How many miles of each kind of road in the journey?

IV

1. Find the value of $\dfrac{ar(r^n-1)}{(r-1)^2}-\dfrac{an}{r-1}$ when $n=2$ and when $n=3$.

2. The sum of two algebraic expressions is

$$2x^3 + 2x^2 - 8x + 12,$$

and their difference is $2x^2 + 8x + 6$. Find their H.C.F.

3. Simplify

$$\left(1 + \frac{x-1}{x^3+1}\right)\left(\frac{x^6-1}{(x^3-x)^2}\right)\left(\frac{(x^2+x)^2}{(x^2+1)^2-x^2}\right)\left(\frac{1-3x+3x^2-x^3}{1-x+x^2-x^3}\right).$$

4. Find the value of $\frac{1}{a-x} + \frac{1}{x-c} + \frac{1}{x-a}$ when $x = \frac{2ac}{a+c}$.

5. Solve equations
$$x - \tfrac{1}{2}(y-z) = 1$$
$$y - \tfrac{1}{3}(z-x) = 2$$
$$z - \tfrac{1}{4}(x-y) = 3.$$

6. The sum of the ages of a family of children is one-fourth the sum of the ages of their parents. At the end of 4 years the sum of the ages of the children will be doubled, and in $5\frac{1}{3}$ years the sum of the ages of the parents will be double the sum of the ages of their children. How many children in the family?

7. Three travellers, A, B, C, set out from the same point in the same direction. A and B start together at 10 and 12 miles per hour respectively. C starts 2 hours later at 15 miles per hour. In what time after starting will C be midway between A and B?

V

1. Divide $\frac{a^3+1}{b^3} - \frac{3a}{b^2} + 1$ by $\frac{a^2-a+1}{b^2} - \frac{a+1}{b} + 1$.

2. Simplify $\left(\frac{a^2}{b^2} - 1\right)\left(1 - \frac{a}{a-b}\right) \div \left(\frac{a^3}{b^3} + 1\right)\left(\frac{1}{\frac{a^2}{b^2} - \frac{a}{b} + 1}\right)$

3. If $\frac{x}{2x+1}+\frac{1}{x-2}=\frac{1}{2}$, find value of $\frac{1}{x-1}+\frac{2}{x+2}$.

4. If $a-b=1$, show that $(a^2-b^2)^2=a^3-b^3+ab$, and state the whole problem in words alone.

5. Solve equations $\frac{x+1}{y+1}=\frac{a}{b}$, $\frac{ax-by}{x+y}=a-b-1$.

6. A and B together do a work in a certain time. If each did one-half the work separately, A would work 2 days less and B 4 days more than before. In what time could each separately do the work?

7. Three thalers are worth a halfpenny more than 11 francs; 5 francs are worth a halfpenny more than 2 florins; 1 thaler is worth twopence more than a franc and a florin together. Find the value of each coin in pence.

VI

1. Find the value of $\frac{a(a^n-1)-n(a-1)}{a^n(a-1)^2}$ when $n=2$ and when $a=3$.

2. Show that $(x+a)^2+2(x+a)(y+b)+(y+b)^2$
$$=(x+b)^2+2(x+b)(y+a)+(y+a)^2,$$
and write a third expression similar in form and equal to each of them.

3. The H. C. F. of two expressions is x^2-1 and their L. C. M. is x^4-10x^2+9. Find the expressions.

4. Simplify $\left(a-\frac{1-ab}{a-b}\right)\left(b+\frac{ab-1}{b-a}\right)\div\frac{(1-ab)^2}{(a-b)(b-a)}$.

5. Solve the equation $\frac{p}{q}\{m-(x-n)\}=\frac{q}{p}\{n-(x-m)\}$. Do the values of p and q affect the value of x? Examine the effect of making $p=q$ before solving the equation.

6. A boatman rowing with the tide makes m miles in t hours. Returning he makes $\frac{3}{5}$ of the distance in the same time against a tide three-fourths as strong. Find the velocity of the stronger tide.

7. Gold loses $\frac{1}{19}$ of its weight and silver $\frac{1}{10}$ of its weight when weighed in water. Find the amount of each metal in a mass of gold and silver which weighs 98 lbs. in air and 90 lbs. in water.

VII

1. Find the values of

$$2^{n-1}(2^n - 1),\ 2^n(2^n - 1) \text{ and } 2^{n-1}(2^{n+1} - 2),$$

when $n = 2$, when $n = 3$, and when $n = 4$. Why are the values of the second and third expressions in each case just double the value of the first?

2. Find the value of

$$\frac{x^3 + y^3 + z^3 - 3xyz}{a^3 + b^3 + c^3 - 3abc} \text{ when } x = b + c,\ y = c + a,\ z = a + b.$$

3. Find the H. C. F. and L. C. M. of

$$x^3 - (a^2 - b)x + ab, \text{ and } x^3 - 2ax^2 + (a^2 + b)x - ab.$$

4. Simplify $\quad \dfrac{1}{x - 3} - \dfrac{3}{x - 1} + \dfrac{3}{x + 1} - \dfrac{1}{x + 3}.$

5. Solve $\quad \dfrac{a}{x - a} + \dfrac{b}{y + b} = \dfrac{b}{x - a} + \dfrac{a}{y + b} = 1,$

and verify the values obtained.

6. Find the price of oranges per dozen when 4 less for a dollar raises the price 20%.

7. A crew can row 11 miles up stream and 1[illegible] miles down in 6 hours; they can also row 21 miles up and $19\frac{1}{2}$ down in 11 hours. Find their rate of rowing and the rate of the stream.

CHAPTER XII

ELEMENTARY GRAPHS

203. Representation of a point on a line. Let AB be a line divided into any number of equal parts. Fig. 1.

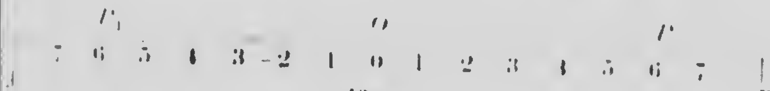


Fig. 1

Taking O as the point from which we measure distances, we may regard, as shown in Art. 29, the distances measured to the right as positive and those measured to the left as negative.

Now if a point be represented by the number which represents its distance and direction from O, then to every real number, positive or negative, there corresponds a point on the line, and to every point on the line there corresponds a real number.

Denoting the distances measured from O by x, we have for $x = 6$ the distance OP, and for $x = -6$ the distance OP_1, and for the distance between the two points P and P_1, 12 $= 6 - (-6) = 12$.

204. Representation of a point in a plane. Now let two straight lines X_1OX and Y_1OY (Fig. 2) be at right angles to each other and let each be divided into equal parts. Let the distances along X_1OX measured to the right be considered positive, and the distances measured to the left negative.

Similarly consider the distances measured upward from O along OY as positive, and the distances measured downward from O as negative.

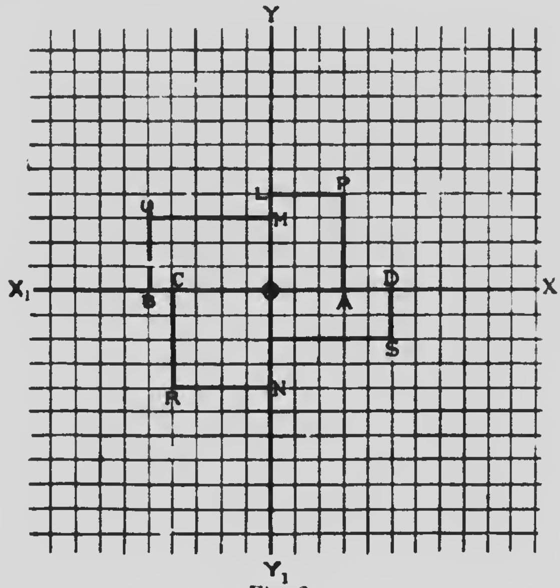


Fig. 2.

Denote the distances measured along $X_1\,OX$ by x, and the distances along $Y_1\,OY$ by y. Call the line $X_1\,OX$ the axis of x, and $Y_1\,OY$ the axis of y.

At the point where $x = 3$ draw a line parallel to $Y_1\,OY$, and at the point where $y = 4$ draw a line parallel to $X_1\,OX$. Let these two lines meet in P. The position of the point P in the plane is determined by stating the distances of these two lines from the axes, viz., OA, OL, or what is the same thing, OA and AP.

The distances OA and AP are called the co-ordinates of the point. AP is called the ordinate of the point, and OA the abscissa.

The point for which $x = 3$ and $y = 4$ is usually denoted by (3, 4); the point for which $x = a$ and $y = b$ is similarly denoted by (a, b).

The co-ordinates of Q in Fig. 2 are $x = -5$, $y = 3$, the point being denoted by $(-5, 3)$. Similarly the point R is $(-4, -4)$ and S is $(5, -2)$.

If now the whole plane be divided by drawing lines parallel to XOX_1 and YOY_1, using the same linear unit, it is at once seen that the position of any point on the plane can be represented by means of a pair of co-ordinates.

EXAMPLES

1. Plot the points (2, 3), (−2, 3), (2, −3), (−2, −3).
2. Draw the line joining the points (2, 3), (−2, −3), and find the ordinate corresponding to the abscissa 4.
3. Draw the lines joining (5, 1), (3, 3) and (3, 1), (2, −4). At what point do these lines intersect?

205. Representation of an algebraical expression. Take the expression $2x - 6$. The value of the expression will depend upon the value assigned to x.

Assume that $\qquad 2x - 6 = y.$

Then when $\qquad x = 1, \; y = -4;$

" $\qquad x = 2, \; y = -2;$

" $\qquad x = 4, \; y = 2;$

" $\qquad x = -2, \; y = -10.$

These results may be tabulated thus:

$x = 1$	2	4	−2
$y = -4$	−2	2	−10

If now the points (1, −4), (2, −2), (4, 2), (−2, −10) be plotted, we get the points P_1, P_2, P_3, P in Fig. 3.

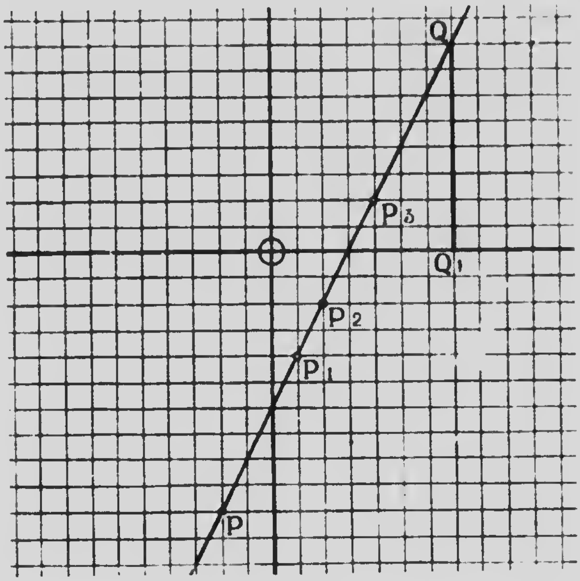


Fig. 3.

These points when joined all lie in a straight line.

The value of $2x - 6$, corresponding to any particular value for x, can now be determined from the diagram.

For example:

When $x = 7$ or OQ_1, $2x - 6 = 8$, or QQ_1.

This can at once be verified by calculation.

All algebraical expressions of the first degree, when plotted as above, are straight lines, and hence it is only necessary to determine two corresponding values for x and y to determine all points on the line.

The line shows that when $x = 3$, $y = 0$, *i.e.*, 3 is the root of the equation $2x - 6 = 0$. This is at once verified by solving the equation.

206. Representation of a linear equation involving two unknown quantities. Take the expressions

$$3x + 4y - 5 = 0.$$

Solving for y in terms of x, we get

$$y = \frac{5 - 3x}{4}.$$

Now when $x = 3$, $y = -1$,

" $x = -5$, $y = 5$.

Since the expression is of the first degree, these two points are sufficient to determine the line, every point on which satisfies the given equation.

For example, from Fig. 4, when $x = 7$, $y = -4$, or the point P, $(7, -4)$, is on the line.

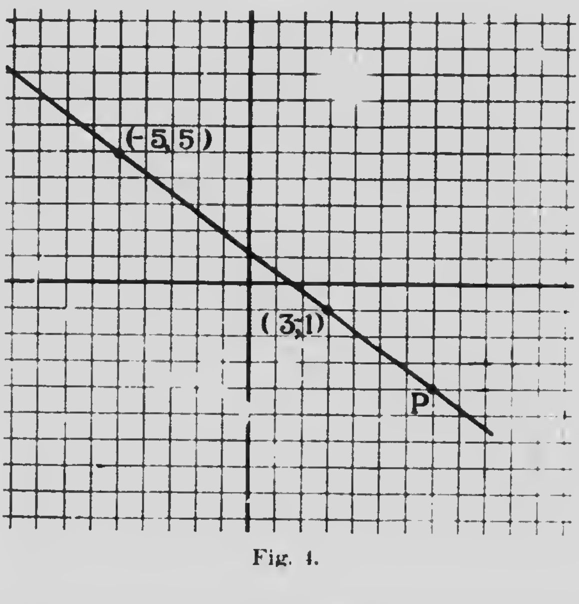


Fig. 4.

This is at once verified by substituting in the equation the values 7 and -4 for x and y respectively.

207. The solution of simple simultaneous equations of two unknown quantities by graphs. Take the equations

$$x - y = -5, \qquad (1)$$

$$x + 3y = 3. \qquad (2)$$

In equation (1) $y = 5 + x$.

When $x = 1$, $y = 6$; and $x = -3$, $y = 2$.

The line through the points (1, 6), (−3, 2) is a line all points of which satisfies equation (1), the line AB, Fig. 5

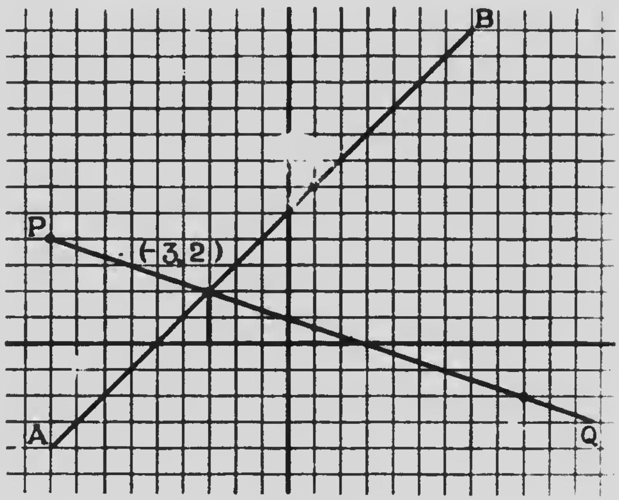


Fig. 5.

In equation (2) $y = \frac{3 - x}{3}$.

When $x = 9$, $y = -2$; and $x = -9$, $y = 4$.

The line through the points (9, −2), (−9, 4) is a line all points of which satisfies equation (2), the line PQ. Fig. 5.

The diagram at once shows that the two lines have one point in common, viz., (−3, 2), that is, that $x = -3$, $y = 2$ are the values of x and y, which satisfies the two given equations.

The student can at once verify the result by solving in the ordinary way.

EXERCISE LIV

1. Plot the following points:

(1) $(1, 4)$, (5) $(4, 0)$,
(2) $(-3, -4)$, (6) $(0, 4)$,
(3) $(3, -7)$, (7) $(0, -3)$,
(4) $(-7, 3)$, (8) $(-3, 0)$.

2. Find the intersection of the lines joining the following pairs of points.

(1) $(5, 2)$, $(-2, -5)$ and $(-5, 0)$, $(0, -5)$,
(2) $(-5, -1)$, $(7, 5)$ and $(-9, 2)$, $(6, 3)$,
(3) $(-3, 7)$, $(7, -3)$ and $(-5, -3)$, $(1, 3)$.

3. Draw the graphs of the following expressions.

(1) $2x - 3$, (3) $\frac{4x + 5}{3}$, (5) $\frac{1}{2}x$,
(2) $3x - 2$, (4) $\frac{3}{5}x + 2$, (6) $2x$.

4. Draw the graphs of the following linear equations.

(1) $3x + 2y = 6$, (3) $x = 5$, (5) $3x - 2y = -5$,
(2) $8x + 15y = -6$, (4) $y = 4$, (6) $3x - 4y = 22$.

5. Solve graphically the following equations.

(1) $3x - 2y = 7$, $4x - y = 16$.
(2) $2x + 3y = 2$, $3x + y = 7$.
(3) $6x + 7y = 16$, $5x + 3y = 2$.
(4) $2x - y = 11$, $x - 2y = 10$.
(5) $\frac{1}{2}x + \frac{1}{3}y = 6$, $\frac{1}{3}x + \frac{1}{2}y = -6\frac{1}{2}$.
(6) $\frac{x}{3} + \frac{y}{4} = 2$, $3x + 4y = 25$.

CHAPTER XIII

SQUARE AND CUBE ROOT

SQUARE ROOT

208. Definition. The Square Root of a given algebraical expression is an expression which, when multiplied by itself, produces the given expression.

Thus the square root of $9a^2$ is $+3a$, or $-3a$.

For $(+3a)^2 = 9a^2$, and $(-3a)^2 = 9a^2$.

Similarly the square root of $a^2 + 2ab + b^2$ is either $a+b$, or $-a-b$.

For $(a+b)^2 = (-a-b)^2 = a^2 + 2ab + b^2$.

From the foregoing it is seen at once that every algebraical expression has **two** square roots, the sign of the one being opposite to that of the other. We shall only consider the one square root, viz., that beginning with a positive sign.

As stated in Art. 25, the **Radical Sign**, $\sqrt{\quad}$, when written over an expression, indicates the square root of the expression.

209. The square root of a monomial. In the first example given above (Art. 208), $\sqrt{9a^2} = 3a$, 3 being the square root of 9, and a being the square root of a^2.

Similarly $\sqrt{9a^2b^2} = 3ab$, 3, a, and b being, respectively, the square root of 9, a^2 and b^2.

To find the square root of a monomial, therefore, take the square root of each factor; multiply these square roots together, and the result is the square root of the expression.

Ex. $\sqrt{100a^4b^6c^2} = 10a^2b^3c.$

210. The square root of a trinomial expression. The square root of a trinomial expression may be found by expressing it in simple factors. This is usually done by observation.

Thus: (1) $\sqrt{a^2 - 2ab + b^2} = \sqrt{(a-b)(a-b)} = a - b.$

(2) $\sqrt{a^2 + 8a + 16} = \sqrt{(a+4)(a+4)} = a + 4.$

(3) $\sqrt{36x^2 - 84xy + 49y^2} = \sqrt{(6x-7y)(6x-7y)}$
$= (6x - 7y).$

211. The process of finding the square root of a trinomial expression can be exhibited as follows :

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$= a^2 + (2a + b)b.$$

Therefore $\sqrt{a^2 + (2a+b)b} = a + b.$ (1)

Consider the expression under the radical.

$$a^2 + (2a+b)b.$$

1. The square root of the first term is a, the first term on the right-hand side of (1).

2. If the square of this term, a^2, be subtracted from the whole expression, the remainder is

$$(2a+b)b. \quad (2)$$

3. If now $2a$, twice the term of the root already obtained, be divided into the first term of the remainder (2), the quotient is b, the second term on the right-hand side of (1).

4. If b be added to $2a$, and the whole, $2a+b$, be multiplied by b and subtracted from (2), there is no remainder.

Tabulated, the process appears thus:

$$\begin{array}{lll} & \underline{a^2 + 2ab + b^2} & a + b \text{ Ans.} \\ & a^2 & \\ \underline{2a + b} & \overline{2ab + b^2} & \text{remainder (2).} \\ & 2ab + b^2 & \end{array}$$

No remainder.

The above may be directly stated thus:

Take the square root of a^2, a; square this term and subtract from the whole expression. The remainder is $2ab + b^2$.

To get the new divisor, multiply a, the part of the root already found, by 2, making $2a$, the first term of the divisor. Divide $2a$ into the first term of the remainder $2ab + b^2$, obtaining b. To complete the divisor add b and make it the second term of your answer.

Now multiply the divisor by b and subtract. There is no remainder.

The answer is $a + b$.

Take example (3) Art. 210. Find the square root of $36x^2 - 84xy + 49y^2$.

The process is as follows:

$$\begin{array}{lll} & 36x^2 - 84xy + 49y^2 & 6x - 7y \text{ Ans.} \\ & 36x^2 & \\ & \overline{-84xy + 49y^2} & \\ 12x - 7y & -84xy + 49y^2 & \end{array}$$

No remainder.

212. The square root of a multinomial. The method is an extension of Art. 211.

We have seen (Art. 139) that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$
$$= a^2 + (2a + b)b + (2a + 2b + c)c.$$

Therefore $\sqrt{a^2 + (2a + b)b + (2a + 2b + c)c} = a + b + c$. (1)

Consider the expression under the radical.

$$a^2 + (2a + b)b + (2a + 2b + c)c.$$

1. The square root of the first term is a, the first term on the right-hand side of (1).

2. If the square of this term, a^2, be subtracted from the whole expression, the remainder is

$$(2a + b)b + (2a + 2b + c)c. \qquad (2)$$

3. If now $2a$, twice the term of the root already found, be divided into the first term of the remainder (2), when expanded, the result is b, the second term of the root shown in (1).

4. If b be added to $2a$, and the whole, $2a + b$, be multiplied by b, and the product be subtracted from (2), the remainder is

$$(2a + b + c)c. \qquad (3)$$

5. If now the portion of the root already found, viz., $a + b$, be doubled and the first term, $2a$, be divided into the first term of the remainder (3), we obtain c, the last term of the root shown in (1).

6. If to $2a + 2b$ we add c, and the whole, $2a + 2b + c$, be multiplied by c and the product subtracted from (3), there is no remainder.

The whole process may be exhibited as follows:

$$\begin{array}{lll}
 & a^2 + (2a + b)b + (2a + 2b + c)c & \underline{a + b + c} \quad \textit{Ans.} \\
 & a^2 & \\
 & (2a + b)b + (2a + 2b + c)c & \text{1st remainder.} \\
2a + b & (2a + b)b & \\
 & (2a + 2b + c)c & \text{2nd remainder.} \\
2a + 2b + c & \underline{(2a + 2b + c)c} & \\
 & \text{No remainder.} &
\end{array}$$

The above may be directly stated thus: Take the square root of a^2, a; square this and subtract from the whole expression.

The remainder is $(2a+b)b+(2a+2b+c)c$.

To get the new divisor, multiply a, the part of the root already found, by 2, making $2a$ the first term of the divisor. Divide $2a$ into $2ab$, the first term of the remainder, obtaining b. To complete the divisor, add b and make it also the second term of your answer.

Now multiply the divisor by b and subtract again.

The remainder is $(2a+2b+c)c$.

To again get a new divisor, multiply the part of the root already obtained, $a+b$, by 2. Divide $2a$ into the first term of the remainder, obtaining c. Add c to $2a+2b$ to complete the divisor. Multiply the whole by c, the third term of your answer.

Subtract the product and there is no remainder.

The answer is $a+b+c$.

It will be noticed that the terms in the original expression are arranged in descending powers of a, and in the order a, b, c.

Ex. Find the square root of $4x^4-3x^2+2x-4x^3+1$.

Rearranging in descending powers of x, we obtain

$$\begin{array}{lrl}
 & 4x^4-4x^3-3x^2+2x+1 & \underline{2x^2-x-1} \quad Ans. \\
 & 4x^4 & \\
 & -4x^3-3x^2+2x+1 & \text{1st remainder.} \\
\underline{4x^2-x} & -4x^3+x^2 & \\
 & -4x^2+2x+1 & \text{2nd remainder.} \\
\underline{4x^2-2x-1} & -4x^2+2x+1 &
\end{array}$$

The student will have no difficulty in applying the method to more general expressions.

EXERCISE LV

Extract the square root of the following:

1. $4a^2b^2c^2$.
2. $25a^4b^6c^8$.
3. $64a^{16}b^{14}$.
4. $\frac{49a^4b^2}{81c^4}$.
5. $\frac{100c^8b^2a^{10}}{9c^4d^2}$.
6. $\frac{36a^2b^2}{16c^2d^2}$.
7. $x^2+12x+36$.
8. $x^2-10x+25$.
9. $x^2-8x+16$.
10. $x^2-36x+324$.
11. $81a^2-18ab+b^2$.
12. $25x^2-70xy+49y^2$.
13. $a^4-10a^3x+33a^2x^2-40ax^3+16x^4$.
14. $1+4x+10x^2+12x^3+9x^4$.
15. $289-374x+121x^2+34x^3-22x^4+x^6$.
16. $x^2+4xy+4y^2+9z^2+6xz+12yz$.
17. $25x^4y^2-30x^3y^3+29x^2y^4-12xy^5+4y^6$.
18. $9x^4+6x^3y-47x^2y^2-16xy^3+64y^4$.
19. $\frac{x^4}{16}+\frac{x^3}{4y}+\frac{3x^2}{20y^2}-\frac{x}{5y^3}+\frac{1}{25y^4}$.
20. $a^6-12a^3+38-\frac{12}{a^3}+\frac{1}{a^6}$.
21. $(x^2+xy+y^2)^2-4xy(x^2+y^2)$.
22. $(ab+bc+ca)^2-4abc(a+c)$.
23. $a^4+b^4+c^4+d^4-2a^2(b^2+d^2)-2b^2(c^2-d^2)+2c^2(a^2-d^2)$.

CUBE ROOT

213. Definition. The Cube Root of an algebraical expression is one of the three equal factors which, multiplied together, produce the expression.

Thus: (1) $8=2\times 2\times 2$,

therefore $\sqrt[3]{8}=2$.

(2) $-8 = (-2)\times(-2)\times(-2)$,

therefore $\sqrt[3]{-8} = -2$.

(3) $27a^3 = (3a)(3a)(3a)$,

therefore $\sqrt[3]{27a^3} = 3a$.

214. It follows from the above that the cube root of a monomial is found by taking the cube root of the different factors which compose it, and multiplying them together.

It will be noticed also that the cube root of a positive monomial is positive, while the cube root of a negative monomial is negative.

215. Binomial roots. The cube root of an expression the root of which consists of two terms, can be obtained directly by factoring the expression.

Thus

(1) $\sqrt[3]{a^3+3a^2b+3ab^2+b^3} = \sqrt[3]{(a+b)^3} = a+b$.

(2) $\sqrt[3]{8x^3-36x^2y+54xy^2-27y^3} = \sqrt[3]{(2x-3y)^3}$
$= 2x-3y$.

An expression which is the cube of a binomial may be readily recognized by comparing it with the known form

$$(a+b)^3 = a^3+b^3+3ab(a+b).$$

Consider the expression $8x^3+36x^2y+54xy^2+27y^3$. In it $8x^3=(2x)^3$ is in place of a^3; $27y^3=(3y)^3$ is in place of b^3.

If, then, the expression is a cube, it must be $(2x+3y)^3$.

Test it for the remaining terms by writing $2x$ for a and $3y$ for b.

We get $3ab(a+b)=18xy(2x+3y)=36x^2y+54xy^2$, which agrees with the given expression.

Therefore $8x^3+36x^2y+54xy^2+27y^3=(2x+3y)^3$.

Similarly $x^6-3x^5+3x^4-x^3=(x^2-x)^3$,

in which x^2 and $-x$ take the place of a and b as before.

The process, however, can be exhibited thus:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$= a^3 + (3a^2 + 3ab + b^2)b.$$

Hence $\sqrt[3]{a^3 + (3a^2 + 3ab + b^2)b} = a + b.$

Consider now the expression under the root sign.

(1) The cube root of the first term of the expression is a, the first term of the root.

(2) If a^3 be subtracted from the whole expression, the remainder is $b(3a^2 + 3ab + b^2)$.

(3) If the first term of the remainder be divided by $3a^2$, that is, 3 times the square of the part of the root already found, the quotient is b, the second term of the root.

(4) In order that there should be no remainder after the divisor is multiplied by b, the divisor must be $3a^2 + 3ab + b^2$, that is, 3 times the square of the first term + 3 times the product of the two terms, + the square of the last term.

Tabulated, it is as follows:

	$a^3 + 3a^2b + 3ab^2 + b^3$	$a + b$ *Ans.*
$3a^2$	a^3	
	$3a^2b + 3ab^2 + b^3$	
$3a^2 + 3ab + b^2$	$3a^2b + 3ab^2 + b^3$	
	No remainder.	

The above may be directly stated thus:

Take a, the cube root of a^3, the first term of the expression; cube it, and subtract from the whole expression. To get a new divisor, multiply a^2, the square of the term already found, by 3, making $3a^2$. Divide $3a^2$ into the first term of the remainder, obtaining b, the second term of the root. To the first term of the divisor, $3a^2$, add 3 times the product of the two terms of the root and also the square of the last term, making $3a^2 + 3ab + b^2$ for the new divisor.

Multiply this divisor by b, and subtract again.

If there is no remainder, the root required is $a + b$.

Ex. $\qquad 8x^3 - 36x^2y + 54xy^2 - 27y^3 \mid 2x - 3y$ Ans.

$\qquad\qquad 8x^3$

$3(2x)^2 = 12x^2 \qquad -36x^2y + 54xy^2 - 27y^3$

$\qquad\qquad -36x^2y + 54xy^2 - 27y^3$

$12x^2 - 3(2x)(-3y) + (-3y)^2$ $\qquad$ No remainder.

$12x^2 - 18xy + 9y^2.$

216. Polynomial roots. The mode of obtaining the cube root of the general polynomial can be seen by taking the case where the root consists of three terms.

$$(a+b+c)^3 = a^3 + (3a^2+3ab+b^2)b + \{3(a+b)^2+3(a+b)c+c^2\}c.$$

Hence

$$\sqrt[3]{a^3 + (3a^2+3ab+b^2)b + \{3(a+b)^2+3(a+b)c+c^2\}c} = a+b+c.$$

Consider the expression under the root sign.

1. The cube root of the first term is a, the first term of the root.

2. If a^3 be subtracted, the remainder is

$$(3a^2+3ab+b^2)b + \{3(a+b)^2+3(a+b)c+c^2\}c.$$

3. If $3a^2$, as before, be divided into the first term of the remainder, when expanded, b is obtained, the second term of the root.

4. Forming the divisor as in the last Art., we obtain $3a^2+3ab+b^2$, which, when multiplied by b and subtracted, leaves as a remainder

$$\{3(a+b)^2+3(a+b)c+c^2\}c.$$

5. If the first term of the remainder be divided by $3(a+b)^2$, that is, 3 times the square of the part of the root already found, the quotient is c, the third term of the root.

6. In order that there should be no remainder after the divisor is multiplied by c, the divisor must be

$$3(a+b)^2 + 3(a+b)c + c^2,$$

that is, 3 times the square of the first two terms + 3 times the sum of the first terms multiplied by the third + the square of the third term.

Tabulated, it is as follows:

$$3a^2 \quad a^3 + (3a^2 + 3ab + b^2)b + 3(a+b)^2 + 3(a+b)c + c^2 \;\; | \; \underline{a + b + c}.$$

$$a^3$$

$$\underline{3a^2 + 3ab + b^2} \quad 3a^2b + 3ab^2 + b^3 + \text{ etc.}$$

$$3a^2b + 3ab^2 + b^3$$

$$3(a+b)^2 \qquad + 3(a+b)^2c + 3(a+b)c^2 + c^3$$

$$+ 3(a+b)^2c + 3(a+b)c^2 + c^3$$

$$3(a+b)^2 + 3(a+b)c + c^2 \qquad \text{No remainder.}$$

It will be noticed that the terms are arranged in descending powers of a, and in the order a, b, c.

The student will readily see that the process is only an extension of that used in the preceding case.

E

$$a^6 - 3a^5 + 5a^3 - 3a - 1 \;|\; \underline{a^2 - a - 1} \text{ Ans.}$$

$$3(a^2)^2 = 3a^4 \quad | \; a^6$$

$$3a^4 - 3a^2(a) + a^2 \quad | \quad -3a^5 + 5a^3 - 3a - 1$$

$$3a^4 - 3a^3 + a^2 \quad | \quad -3a^5 + 3a^4 - a^3$$

$$3(a^2 - a)^2 = 3a^4 - 6a^3 + 3a^2 \quad | \quad -3a^4 + 6a^3 - 3a - 1$$

$$3(a^2 - a)^2 + 3(a^2 - a)(-1) + (-1)^2 \quad | \quad -3a^4 + 6a^3 - 3a - 1$$

$$= 3a^4 - 6a^3 + 3a + 1 \qquad \text{No remainder.}$$

217. Cube root of a fraction. The cube root of a fraction is found by finding the cube root of the numerator and the denominator separately, and dividing the former by the latter.

EXERCISE LVI

Find the cube roots of the following :

1. $64a^3b^6c^9$. **2.** $-125a^6b^{12}c^{15}$. **3.** $\frac{27a^{[illegible]}b^{[illegible]}c^{[illegible]}}{512d^{12}e^{15}}$.

4. $x^3+6x^2+12x+8$. **5.** $64x^3-48x^2+12x-1$.

6. $125a^3+75a^2b+15ab^2+b^3$. **7.** $x^3+12x^2+48x+64$.

8. $x^6+9x^5+30x^4+45x^3+30x^2+9x+1$.

9. $x^6-6x^5+40x^3-96x-64$.

10. $152x^3-27-63x^2+27x^6+63x^4-108x-108x^5$.

Arrange according to descending powers of x.

11. $a^3+\frac{1}{a^3}+3\left(a+\frac{1}{a}\right)$. **12.** $\frac{x^3}{y^3}-\frac{y^3}{x^3}-3\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)-5$.

CHAPTER XIV

THEORY OF INDICES

218. Definition. In Art. 13 we defined an index or exponent as a number used to show how many times another number is to be taken as a factor.

Thus $\quad a^m = a \times a \times a \ldots$ taken m times.

The student will see at once, therefore, that an exponent is used to count factors, just as a coefficient is used to count addends.

219. The Laws of Indices. The Laws of Indices, expressed in symbols, are:

(1) $a^m \times a^n = a^{m+n}$.

(2) $\dfrac{a^m}{a^n} = a^{m-n}$.

(3) $(a^m)^n = a^{mn}$.

(4) $(ab)^m = a^m b^m$.

(5) $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$.

Of these (1) is fundamental, the other four being derived from it, as will be seen below.

These Laws are proved as follows:

(1) $a^m \times a^n = a \times a \times a \ldots.$ taken m times $\times a \times a \times a \ldots.$ taken n times

$= a \times a \times a \times a \ldots.$ taken $m + n$ times

$= a^{m+n}$.

(2) $\frac{a^m}{a^n} = \frac{a \times a \times a \ldots \text{ taken } m \text{ times}}{a \times a \times a \ldots \text{ taken } n \text{ times}}$

$= a \times a \times a \ldots$ taken $m - n$ times

$= a^{m-n}$, where m is greater than n.

(3) $(a^m)^n = a^m \times a^m \times a^m \ldots$ taken n times

$= a^{m+m+m \ldots \text{ taken } n \text{ times}}$

$= a^{mn}$.

(4) $(ab)^m = ab \times ab \times ab \ldots$ taken m times

$= a \times a \times a \ldots$ taken m times $\times b \times b \times b \ldots$ taken m times

$= a^m \times b^m$.

(5) $\left(\frac{a}{b}\right)^m = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \ldots$ taken m times

$= \frac{a \times a \times a \ldots \text{ taken } m \text{ times}}{b \times b \times b \ldots \text{ taken } m \text{ times}}$

$= \frac{a^m}{b^m}$.

In the above, a may be either an integer or a fraction, positive or negative, without affecting in any way the reasoning.

220. General statement. So far we have considered only exponents, m and n, which may be considered positive integers. It is sometimes necessary, however, to employ exponents which are not positive numbers, viz., negative exponents, fractional exponents and zero exponents, just as we have used negative, fractional and zero coefficients. It is necessary, therefore, that we should determine how such exponents are to be interpreted. For this purpose we may either define these new exponents, and then show how operations with them are related to the fundamental law, or we may assume the law to hold for all values of m and n, positive, negative, fractional, or zero, and interpret on the basis of this assumption.

We shall follow the latter, assume the fundamental law and give to these exponents such meanings as this assumption makes necessary.

1. **Zero exponent.** Consider a^0.

$$a^m \times a^0 = a^{m+0} = a^m,$$

or

$$\frac{a^m \times a^0}{a^m} = \frac{a^m}{a^m}, \text{ dividing both sides by } a^m.$$

Therefore $$a^0 = 1. \qquad (6)$$

The meaning is quite clear when we consider that a means a to the first power or a^1, and if this factor be removed by dividing by itself we get 1 or a^0.

Thus $$\frac{a^1}{a^1} = a^{1-1} = a^0 = 1.$$

We must, therefore, interpret a^0 as being equal to 1, or, the zero index indicates the absence of any factor but unity.

2. **Negative exponent.** Consider a^{-3}.

$$a^{-3} \times a^3 = a^{-3+3} = a^0 = 1.$$

Therefore dividing by a^3,

we have $$\frac{a^{-3} \times a^3}{a^3} = \frac{1}{a^3},$$

or $$a^{-3} = \frac{1}{a^3}.$$

Or to consider the general case, a^{-m}, m being considered positive,

$$a^{-m} \times a^m = a^{-m+m} = a^0 = 1.$$

Therefore dividing by a^m,

we have $$a^{-m} = \frac{1}{a^m}.$$

Also $$a^m = \frac{1}{a^{-m}}. \quad (7)$$

We must, therefore, interpret a^{-m} as being equal to $\frac{1}{a^m}$ and vice versa. This shows that the negative index changes the factor from a multiplier to a divisor, and vice versa, after which the number of factors indicated by the index is to be counted in the ordinary way.

3. Fractional exponent. Consider $a^{\frac{2}{3}}$.

$$a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}}$$
$$= a^2.$$

Therefore $$(a^{\frac{2}{3}})^3 = a^2,$$

or $$a^{\frac{2}{3}} = \sqrt[3]{a^2}. \quad (8)$$

Or to consider the general case,

$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \text{ .... taken } n \text{ times} = a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n} \text{ taken } n \text{ times}}$$
$$= a^{\frac{m}{n} \times n} = a^m.$$

Therefore $$\left(a^{\frac{m}{n}}\right)^n = a^m,$$

or $$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

We must, therefore, interpret $a^{\frac{m}{n}}$ as being equal to $\sqrt[n]{a^m}$, that is, the nth root of a^m.

The fractional index may also be said to indicate that the number is to be divided into as many equal factors as are indicated by the denominator, and that the numerator shows how many of those factors are to be taken.

221. The following examples will illustrate the application of the laws:

Ex. 1. Find the value of $a^{x-y} \times a^{y-z} \times a^{z-x}$.

$$a^{x-y} \times a^{y-z} \times a^{z-x} = a^{x-y+y-z+z-x} \qquad \text{from (1)}$$
$$= a^0 = 1.$$

Ex. 2. Find the value of $\sqrt{a} \times \sqrt{b}$.

$$\sqrt{a} = a^{\frac{1}{2}}, \quad \sqrt{b} = b^{\frac{1}{2}}.$$

Therefore
$$\sqrt{a} \times \sqrt{b} = a^{\frac{1}{2}}b^{\frac{1}{2}}$$
$$= (ab)^{\frac{1}{2}} = \sqrt{ab} \qquad \text{from (4)}$$

Ex. 3. Find the value of $a^{\frac{2}{3}} \times a^{-\frac{3}{4}} \times (a^2)^{-\frac{1}{6}} \times \left(a^{\frac{1}{12}}\right)^5$.

$$a^{\frac{2}{3}} \times a^{-\frac{3}{4}}(a^2)^{-\frac{1}{6}} \times (a^{\frac{1}{12}})^5 = a^{\frac{2}{3}-\frac{3}{4}-\frac{2}{6}+\frac{5}{12}} \qquad \text{from (1) and (3)}$$
$$= a^0 = 1.$$

Ex. 4. Simplify $\sqrt{a} \times \sqrt[3]{b} \times \sqrt[4]{c}$.

$$\sqrt{a} \times \sqrt[3]{b} \times \sqrt[4]{c} = a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}$$
$$= a^{\frac{6}{12}} \times b^{\frac{4}{12}} \times c^{\frac{3}{12}}$$
$$= (a^6 \times b^4 \times c^3)^{\frac{1}{12}} \qquad \text{from (4)}$$
$$= \sqrt[12]{a^6b^4c^3}.$$

Ex. 5. Simplify $\sqrt[4]{x^2\sqrt[3]{x^6\sqrt{x^{-24}}}}$.

$$\sqrt[4]{x^2\sqrt[3]{x^6\sqrt{x^{-24}}}} = \left(x^2\sqrt[3]{x^6\sqrt{x^{-24}}}\right)^{\frac{1}{4}}$$
$$= x^{\frac{1}{2}}\left(x^6\sqrt{x^{-24}}\right)^{\frac{1}{12}}$$
$$= x^{\frac{1}{2}} \,.\, x^{\frac{1}{2}}(x^{-24})^{\frac{1}{24}}$$
$$= x \,.\, x^{-1}$$
$$= x^0 = 1.$$

EXERCISE LVIII

Find the value of

1. $8^{\frac{2}{3}}$. 2. $16^{\frac{1}{4}}$. 3. $16^{-\frac{1}{2}}$. 4. $27^{\frac{2}{3}}$.

5. $1000^{-\frac{1}{3}}$. 6. $\left(\frac{64}{27}\right)^{\frac{1}{3}}$. 7. $\left(\frac{64}{27}\right)^{-\frac{1}{3}}$. 8. $\left(\frac{1}{1000}\right)^{-\frac{1}{3}}$.

Write with fractional indices the following:

9. $\sqrt{a}$. 10. $\sqrt[10]{a}$. 11. $\sqrt[4]{(ab)^3}$. 12. $\sqrt{(ac)}$.

13. $3\sqrt[3]{a^{-2}}$. 14. $9\sqrt[4]{m}\sqrt[3]{n^{-2}}$. 15. $\sqrt[6]{a^2}\sqrt{a}$.

16. $25\sqrt[4]{a^2\sqrt{a^4}}$.

Write the following with radical signs:

17. $a^{\frac{1}{3}}$. 18. $a^{-\frac{1}{4}}$. 19. $a^{\frac{1}{3}}b^{\frac{1}{4}}$. 20. $m^{\frac{1}{n}}$.

21. $x^{\frac{m}{n}}y^{\frac{1}{m}}$. 22. $a^{-\frac{1}{4}}b^{\frac{1}{4}}c^{-\frac{1}{2}}$. 23. $x^{\frac{2}{3}}y^{\frac{4}{5}}$. 24. $3b^{-\frac{2}{3}}$.

Write the following with positive indices:

25. $a^{-\frac{2}{3}}$. 26. $b^{-\frac{1}{2}}$. 27. $\frac{1}{a^{-\frac{1}{2}}}$. 28. b^{-1}.

29. $\frac{a^{-\frac{1}{2}}}{b^{-\frac{1}{3}}}$. 30. $x^{-\frac{1}{2}}y^{-\frac{2}{3}}$. 31. $\frac{2x^{-1}a}{3^{-1}b^{-2}c}$.

32. $\left(\frac{a^{-1}+b^{-1}}{ab}\right)^{-1}$.

Write the following with negative indices:

33. $a^{\frac{1}{2}}$. 34. $\frac{1}{a^{\frac{1}{2}}}$. 35. $\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}$. 36. $x^{\frac{1}{2}}y$.

37. $\frac{x^{\frac{1}{2}}y^{\frac{3}{2}}}{a^{\frac{2}{3}}b^{-1}}$. 38. $\sqrt[3]{a^2b^4}$. 39. $\sqrt{\frac{a^4c^2}{x^2y^4}}$. 40. $\sqrt[2]{\frac{b}{x}}$.

In the following: (1) change all the denominators to numerators; (2) change all the numerators to denominators.

41. $\frac{2}{x^{-5}}$. **42.** $\frac{x^{-\frac{4}{5}}}{y^{-\frac{5}{3}}}$. **43.** $\frac{a^{-6}}{b^{-4}}$. **44.** $\frac{m^{-\frac{1}{2}}}{n^{-\frac{2}{3}}}$.

45. $\frac{a^{\frac{1}{3}}b^{\frac{1}{3}}c^{-\frac{1}{4}}}{x^{-\frac{1}{4}}y^{\frac{1}{3}}z^{-\frac{1}{4}}}$. **46.** $\frac{a^{\frac{1}{3}}b^{-2}}{c^{-\frac{1}{2}}d^{-\frac{2}{3}}}$. **47.** $\frac{8a^{-3}b^{-2}}{27c^{-4}d^{-5}}$. **48.** $\frac{27a^{3}m^{-2}}{8^{-1}b^{3}n}$.

Simplify the following:

49. $x^{3}\times x^{-3}$. **50.** $a^{\frac{2}{3}}\times a^{\frac{1}{4}}$.

51. $(a^{\frac{1}{3}}\times b^{\frac{3}{4}}\times c^{\frac{1}{4}})(a^{\frac{1}{2}}b^{-\frac{1}{4}}c^{-\frac{3}{4}})$. **52.** $(a^{0}\times a^{\frac{1}{2}})(b^{0}\times a^{-\frac{1}{2}})$.

53. $\frac{a^{-\frac{1}{2}}}{a^{\frac{1}{3}}}\times\frac{b^{\frac{1}{3}}}{b^{\frac{1}{2}}}\times\frac{c^{\frac{3}{4}}}{c^{\frac{1}{4}}}$. **54.** $\left(\frac{ab}{c}\right)^{\frac{1}{2}}\times\left(\frac{cx}{a}\right)^{\frac{1}{3}}\times\left(\frac{cy}{abc}\right)^{\frac{1}{4}}$.

55. $\left(\frac{a^{\frac{2}{3}}b^{-\frac{1}{2}}}{a^{-\frac{1}{2}}b^{\frac{1}{2}}}\right)^{2}$. **56.** $\frac{a^{m+n}\times a^{m-n}}{a^{2m}}$. **57.** $\frac{x^{\frac{1}{2}}y^{\frac{1}{3}}}{x^{\frac{1}{3}}y^{\frac{1}{2}}}\div\frac{x^{\frac{1}{6}}}{y^{\frac{1}{6}}}$.

58. $\left(\frac{16x^{-4}}{81y^{3}}\right)^{-\frac{3}{4}}$. **59.** $\sqrt[4]{\left(\frac{x^{3}y^{-5}}{a^{2}b^{-3}}\right)^{2}}$. **60.** $\sqrt[3]{\left(\frac{x^{3}y^{3}}{a^{2}b^{-3}}\right)^{-1}}$.

222. The following examples will illustrate how the ordinary operations of algebra — multiplication, division, factoring, square root, etc. — are performed with fractional, negative, and zero exponents.

Ex. 1. Multiply $a^{\frac{1}{3}}+a^{0}+a^{-\frac{1}{3}}$ by $a^{\frac{1}{3}}-a^{0}+a^{-\frac{1}{3}}$. *Ans.*

$$\begin{array}{l} a^{\frac{1}{3}}+a^{0}+a^{-\frac{1}{3}} \\ a^{\frac{1}{3}}-a^{0}+a^{-\frac{1}{3}} \\ \hline a^{\frac{2}{3}}+a^{\frac{1}{3}}+a^{0} \\ \quad -a^{\frac{1}{3}}-a^{0}-a^{-\frac{1}{3}} \\ \qquad\quad +a^{0}+a^{-\frac{1}{3}}+a^{-\frac{2}{3}} \\ \hline a^{\frac{2}{3}}\qquad +a^{0}+a^{-\frac{2}{3}} \text{ or } a^{\frac{2}{3}}+a^{-\frac{2}{3}}+1 \end{array}$$

Ex. 2. Divide $x - y$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.

$$\begin{array}{l|l} x^{\frac{1}{3}} - y^{\frac{1}{3}} & x - y \\ & x - x^{\frac{2}{3}}y^{\frac{1}{3}} \\ & \quad x^{\frac{2}{3}}y^{\frac{1}{3}} - y \\ & \quad x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}} \\ & \qquad x^{\frac{1}{3}}y^{\frac{2}{3}} - y \\ & \qquad x^{\frac{1}{3}}y^{\frac{2}{3}} - y \end{array} \qquad x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} \quad \textit{Ans.}$$

Ex. 3. Find the square root of $a^2 + 4a + 2 - 4a^{-1} + a^{-2}$.

$$\begin{array}{l|l} & a^2 + 4a + 2 - 4a^{-1} + a^{-2} \,|\, a + 2 - a^{-1} \quad \textit{Ans.} \\ & a^2 \\ 2a + 2 & \quad 4a + 2 \\ & \quad 4a + 4 \\ 2a + 4 - a^{-1} & \qquad -2 - 4a^{-1} + a^{-2} \\ & \qquad -2 - 4a^{-1} + a^{-2} \end{array}$$

Ex. 4. Resolve into factors $a^{\frac{2}{3}} - b^{\frac{2}{3}}$.

$$a^{\frac{2}{3}} - b^{\frac{2}{3}} = (a^{\frac{1}{3}})^2 - (b^{\frac{1}{3}})^2$$
$$= (a^{\frac{1}{3}} + b^{\frac{1}{3}})(a^{\frac{1}{3}} - b^{\frac{1}{3}}).$$

Similarly

$$ax - 8 + 12a^{-1}x^{-1} = (a^{\frac{1}{2}}x^{\frac{1}{2}} - 6a^{-\frac{1}{2}}x^{-\frac{1}{2}})(a^{\frac{1}{2}}x^{\frac{1}{2}} - 2a^{-\frac{1}{2}}x^{-\frac{1}{2}}).$$

EXERCISE LIX

Multiply

1. $x^{\frac{1}{3}} + y^{\frac{1}{3}}$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.
2. $x^{\frac{1}{2}} + a^{\frac{1}{2}}$ by $x^{\frac{1}{4}} + a^{\frac{1}{4}}$.
3. $x^{\frac{2}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{2}{3}} - y^{\frac{2}{3}}$.
4. $x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$ by $x^{\frac{1}{3}} - 1$.
5. $x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1$ by $x^{\frac{1}{3}} + 1$.
6. $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.

7. $x^n + x^{\frac{n}{2}} + 1$ by $x^n - x^{\frac{n}{2}} + 1$.

8. $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$.

Divide

9. $x + y$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.

10. $x - y$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.

11. $a + 125b$ by $a^{\frac{1}{3}} + 5b^{\frac{1}{3}}$.

12. $a - 64b$ by $a^{\frac{1}{3}} - 4b^{\frac{1}{3}}$.

13. $x^2 + y^2$ by $x^{\frac{2}{3}} + y^{\frac{2}{3}}$.

14. $x^{\frac{4}{3}} - y^{\frac{4}{3}}$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.

15. $x^{\frac{5}{2}} - 2x^{\frac{3}{2}}y + 2xy^{\frac{3}{2}} + x^{\frac{1}{2}}y^2 - y^{\frac{5}{2}}$ by $x + x^{\frac{1}{2}}y^{\frac{1}{2}} - y$.

16. $x^{-\frac{4}{3}} - 10x^{-\frac{2}{3}} + 9$ by $x^{-\frac{2}{3}} + 2x^{-\frac{1}{3}} - 3$.

Find the square root of

17. $x^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 1$.

18. $4a^{\frac{4}{3}} - 4a^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}$.

19. $a + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b + 2b^{-\frac{1}{2}} + 2a^{-\frac{1}{2}} + a^{-1}b^{-1}$.

20. $4a^{-2} - 12a^{-1} + 25 - 24a + 16a^2$.

21. $9x^{-4} + 24x^{-3} - 20x^{-2} - 48x^{-1} + 36$.

Factor

22. $x - y$, considering it the difference of two cubes.

23. $x + y$, considering it the sum of two cubes.

24. $x^{\frac{2}{3}} + 2a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}}$.

25. $x^{\frac{2}{3}} - 2a^{\frac{1}{3}}x^{\frac{1}{3}} + a^{\frac{2}{3}}$.

26. $x^{\frac{2}{3}} - a^{\frac{2}{3}}$.

27. $x^{\frac{3}{2}} - 64$.

28. $2x^{\frac{4}{3}} + 5x^{\frac{2}{3}}y^{\frac{2}{3}} + 2y^{\frac{4}{3}}$.

29. $3x^2 - 7xy^{-2} + 4y^{-4}$.

Simplify

30. $(a^2 - a^{-2})^2 - (a^2 - a^{-2})(a^2 + a^{-2})$.

31. $(a + b)(a^{-1} + b^{-1}) + (a - b)(a^{-1} - b^{-1})$.

32. $\dfrac{a^{\frac{1}{3}} - y^{\frac{1}{3}}}{a^{\frac{1}{3}} + y^{\frac{1}{3}}} - \dfrac{a - y}{a + y}$.

33. $\dfrac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}} + \dfrac{x^{-\frac{1}{2}} + y^{-\frac{1}{2}}}{x^{-\frac{1}{2}} - y^{-\frac{1}{2}}}$.

34. $\dfrac{1}{x^{\frac{1}{2}} - 1} - \dfrac{1}{x^{\frac{1}{2}} + 1} + \dfrac{1}{x - 1}$.

35. $\left(\dfrac{a^{x+y}}{a^y}\right)^x \div \left(\dfrac{a^y}{a^{y-x}}\right)^{x-y}$.

CHAPTER XV

SURDS

223. Definition. If a number be such that a specified root (Art. 24) cannot be accurately obtained, the root is called a **surd**.

Thus $\sqrt{3}$, $\sqrt[3]{4}$, $\sqrt[4]{5}$, $\sqrt[n]{a}$, $\sqrt{x+y}$ are surds.

Such expressions are also called **radicals,** the number indicating the root to be taken being called the **index,** and the number the root of which is to be taken, being called the **radicand.**

In $\sqrt[n]{a}$, n is the index and a the radicand.

224. Rational and Irrational Numbers. A number which can be expressed as an integer or a fraction, whether it be positive or negative, is called a **rational number;** a number which cannot be so expressed is called an **irrational number.**

Surds are irrational numbers.

225. It will be seen at once that surds are special cases of fractional indices, and hence, that all the Laws of Indices apply to them.

Thus $\sqrt{3} = 3^{\frac{1}{2}}$; $\sqrt[3]{4} = 4^{\frac{1}{3}}$; $\sqrt[n]{a} = a^{\frac{1}{n}}$.

226. Simplest Form of a Surd. A surd is said to be in its simplest form when the expression under the radical sign is the smallest possible integer, that is, when all the factors of the same power as the root index have been made rational factors.

Thus (1) $\sqrt{8} = \sqrt{2^2 \times 2} = 2\sqrt{2}$.

(2) $\sqrt{20} = \sqrt{2^2 \times 5} = 2\sqrt{5}$.

(3) $\sqrt[3]{a^4} = \sqrt[3]{a^3 \times a} = a\sqrt[3]{a}$.

227. Entire Surds and Mixed Surds. If a surd has no rational factor, it is called an **entire surd**; if it has a rational factor, it is called a **mixed surd.**

Thus $\sqrt{3}$ and $\sqrt[3]{a}$ are entire surds; $3\sqrt{5}$ and $m\sqrt{a}$ are mixed surds.

228. Rational Numbers in Surd Form. Any rational number may be expressed in surd form by raising it to the power indicated by the desired surd index.

Thus (1) $2 = \sqrt{2^2} = \sqrt{4}$.

(2) $2 = \sqrt[4]{2^4} = \sqrt[4]{16}$.

(3) $a = \sqrt[n]{a^n}$.

Similarly a mixed surd may be expressed as an entire surd by raising the rational factor to the power indicated by the surd index and making it a factor under the root.

Thus (1) $3\sqrt[3]{a} = \sqrt[3]{3^3a} = \sqrt[3]{27a}$.

(2) $b\sqrt{a} = \sqrt{ab^2}$.

EXERCISE LX

Reduce to their simplest form the following:

1. $\sqrt{50}$.
2. $\sqrt{128}$.
3. $\sqrt{96}$.
4. $\sqrt{75}$.
5. $\sqrt{a^2b^3c}$.
6. $\sqrt[3]{56}$.
7. $\sqrt[3]{81}$.
8. $4\sqrt[4]{32}$.
9. $\sqrt[3]{432}$.
10. $\sqrt{343a^3b^4c^5}$.
11. $\sqrt[4]{x^5y^4z^9}$.
12. $\sqrt[3]{160x^4y^5z^2}$.
13. $\sqrt{50x^4y^3 - 75x^5y^7}$.
14. $\sqrt{50x^2 - 100xy + 50y^2}$.

Express as Entire Surds:

15. $3\sqrt{2}$. **16.** $2\sqrt{3}$. **17.** $3\sqrt[3]{2}$.

18. $2\sqrt[3]{3}$. **19.** $2\sqrt[4]{5}$. **20.** $a\sqrt{b}$.

21. $\frac{a}{b}\sqrt[n]{\frac{b^{n+1}}{a^{n+1}}}$. **22.** $\frac{a+b}{a-b}\sqrt{\frac{a^2-b^2}{a^2+2ab+b^2}}$.

23. $\frac{a+b}{a-b}\sqrt[3]{a^2-b^2}$. **24.** $\frac{b}{a^n}\sqrt{\frac{a^{2n+1}}{b}}$.

229. Order of Surds. Surds are of the same order when the same root is required.

Thus $\sqrt{2}$ and $\sqrt{10}$ are of the same order and are called **quadratic surds,** or second order surds; $\sqrt[3]{2}$ and $\sqrt[3]{4^2}$ are of the same order and are called **cubic surds,** or third order surds; $\sqrt[n]{a}$ is an *n*th order surd.

230. Reduction of Surds to the same order. Surds of a different order can be reduced to the same order by bringing their fractional indices to a common denominator.

Reduce to the same order the following:

(1) $\sqrt{3}$ and $\sqrt[3]{2}$.

$$\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{3}{6}} = \sqrt[6]{3^3} = \sqrt[6]{27}.$$

$$\sqrt[3]{2} = 2^{\frac{1}{3}} = 2^{\frac{2}{6}} = \sqrt[6]{2^2} = \sqrt[6]{4}.$$

(2) $\sqrt[n]{a}$ and $\sqrt[m]{b}$.

$$\sqrt[n]{a} = a^{\frac{1}{n}} = a^{\frac{m}{mn}} = \sqrt[mn]{a^m}.$$

$$\sqrt[m]{b} = b^{\frac{1}{m}} = b^{\frac{n}{mn}} = \sqrt[mn]{b^n}.$$

231. Comparison of Surds. Surds of a different order can be compared with each other by reducing them to the same order. Thus in the last article

$$\sqrt{3} = \sqrt[6]{27} \text{ and } \sqrt[3]{2} = \sqrt[6]{4}.$$

hence $\sqrt{3}$ is greater than $\sqrt[3]{2}$.

232. Similar Surds. Surds are said to be similar to one another when they have the same **surd factor**.

Thus $\sqrt{3}$, $\sqrt{27}$ and $\sqrt{3a^2}$ are similar surds for $\sqrt{27} = 3\sqrt{3}$, and $\sqrt{3a^2} = a\sqrt{3}$.

233. Fractional Surd Factors. A fractional surd factor may be made integral by multiplying both the numerator and denominator of the fraction by a number which will make the denominator an exact root.

Thus (1) $\sqrt{\frac{3}{2}} = \sqrt{\frac{3}{2} \times \frac{2}{2}} = \frac{1}{2}\sqrt{6}.$

(2) $\sqrt[4]{\frac{3}{8}} = \sqrt[4]{\frac{3}{8} \times \frac{2}{2}} = \frac{1}{2}\sqrt[4]{6}.$

EXERCISE LXI

Reduce the following to the same order, and where possible compare them:

1. $\sqrt{5}$, $\sqrt[3]{4}$. 2. $\sqrt[3]{6}$, $\sqrt{7}$. 3. $\sqrt[3]{2}$, $\sqrt[4]{3}$.

4. $\sqrt[4]{5}$, $\sqrt[3]{6}$, $\sqrt[6]{25}$. 5. $\sqrt[3]{x^2 + y^2}$, $\sqrt{x + y}$.

6. $\sqrt[3]{\frac{x}{y}}$, $\sqrt[4]{\frac{y}{x}}$.

Show that the following are similar surds:

7. $\sqrt{20}$, $\sqrt{45}$. 8. $\sqrt{50}$, $\sqrt{72}$. 9. $\sqrt{75}$, $\sqrt{343}$.

Express with an integral surd factor, and reduce to the simplest form the following:

10. $\sqrt[3]{\frac{1}{9}}$. 11. $\sqrt[4]{\frac{1}{343}}$. 12. $\sqrt[4]{\frac{2}{27}}$.

13. $\sqrt{\frac{7}{5a}}$. 14. $\sqrt{\frac{7}{5a^3}}$. 15. $(a - b)\sqrt{\frac{a + b}{a - b}}$.

234. Addition and Subtraction of Surds. To add or subtract similar surds, add or subtract the rational factors, and multiply by the irrational factor.

Ex. 1. $\sqrt{50} + \sqrt{72} = 5\sqrt{2} + 6\sqrt{2}$
$= 11\sqrt{2}.$

Ex. 2. $\sqrt[3]{16} + \sqrt[3]{128} - \sqrt[3]{250} = 2\sqrt[3]{2} + 4\sqrt[3]{2} - 5\sqrt[3]{2}$
$= \sqrt[3]{2}.$

When the surds are not similar, the addition or subtraction can only be indicated.

EXERCISE LXII

Simplify the following:

1. $3\sqrt{5} + 4\sqrt{5}.$
2. $5\sqrt[3]{4} + 2\sqrt[3]{4} - 6\sqrt[3]{4}.$
3. $2\sqrt[4]{3} + 5\sqrt[4]{3} - 7\sqrt[4]{3}.$
4. $7\sqrt{18} + 4\sqrt{32} - 7\sqrt{8}.$
5. $2\sqrt[3]{4} + 5\sqrt[3]{32} - 9\sqrt[3]{108}.$
6. $\sqrt[3]{56} + \sqrt[3]{189} + \sqrt{162}.$
7. $\sqrt{2} - 3\sqrt{8} + 4\sqrt{18} - \sqrt{50}.$
8. $\sqrt[3]{24} + \sqrt[3]{81} - \sqrt[3]{192} - \frac{1}{2}\sqrt[3]{375}.$
10. $\sqrt[4]{a^4b} + \sqrt[4]{b^5} + \sqrt[4]{c^4b}.$
11. $\sqrt[3]{128} + \sqrt[3]{250} - \sqrt[3]{432} - \sqrt[3]{456}.$
12. $6x\sqrt{x^3y^3} + 4y\sqrt{x^5y^5}.$
13. $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{8}} + \sqrt{\frac{1}{32}} - \sqrt{2}.$
14. $\sqrt[3]{\frac{1}{9}} + \sqrt[3]{\frac{1}{243}} + \sqrt[3]{\frac{1}{6561}}.$
15. $2\sqrt{50} + 3\sqrt{72} - \sqrt{27}.$
16. $\sqrt{a+b} + \sqrt{a^3 + a^2b} + \sqrt{(a+b)^3} + \sqrt{ab^2 + b^3}.$
17. $(a-b)\sqrt{a^2-b^2} + (a^2-b^2)\sqrt{\frac{a-b}{a+b}} - 2b\sqrt{a^2-b^2}.$

235. Multiplication of Surds. The process of multiplication of surds follows from the Laws of Indices. If the surds are of the same order, the irrational factors are multiplied under a common root, and the product of the rational coefficients is the rational coefficient of the product. If the surds are of a different order, they must first be reduced to the same order.

Ex. 1. $\sqrt{3} \times \sqrt{2} = 3^{\frac{1}{2}} \times 2^{\frac{1}{2}}$

$$= (2 \times 3)^{\frac{1}{2}} = \sqrt{6}.$$

Ex. 2. $3\sqrt{3} \times 4\sqrt{5} = 3 \times 3^{\frac{1}{2}} \times 4 \times 5^{\frac{1}{2}}$

$$= 12 \times (3 \times 5)^{\frac{1}{2}}$$
$$= 12\sqrt{15}.$$

Ex. 3. $\sqrt{2} \times \sqrt[3]{3} = 2^{\frac{1}{2}} \times 3^{\frac{1}{3}} = 2^{\frac{3}{6}} \times 3^{\frac{2}{6}}$

$$= (2^3 \times 3^2)^{\frac{1}{6}}$$
$$= \sqrt[6]{72}.$$

Ex. 4. Multiply $3\sqrt{2} + 2\sqrt{3}$ by $4\sqrt{2} - \sqrt{3}$.

$$\begin{array}{l} 3\sqrt{2} + 2\sqrt{3} \\ 4\sqrt{2} - \sqrt{3} \\ \hline 12\sqrt{4} + 8\sqrt{6} \\ \qquad - 3\sqrt{6} - 2\sqrt{9} \\ \hline 24 + 5\sqrt{6} - 6 = 18 + 5\sqrt{6} \end{array}$$

Ex. 5. Multiply $2\sqrt{a+b} - 3\sqrt{a}$ by $\sqrt{a+b} + \sqrt{a}$.

$$\begin{array}{l} 2\sqrt{a+b} - 3\sqrt{a} \\ \sqrt{a+b} + \sqrt{a} \\ \hline 2(a+b) - 3\sqrt{a(a+b)} \\ \qquad + 2\sqrt{a(a+b)} - 3a \\ \hline \end{array}$$

Collecting terms, $2b - a - \sqrt{a(a+b)}$.

236. Division of Surds. The same principles that apply in multiplication of surds apply in division. Where the surd factors are of the same order, they are divided under the same root sign; if they are not of the same order, they must be reduced to the same order.

Ex. 1. $\frac{\sqrt{8}}{\sqrt{4}} = \sqrt{\frac{8}{4}} = \sqrt{2}.$

Ex. 2. $\frac{\sqrt{3}}{\sqrt[3]{2}} = \frac{3^{\frac{1}{2}}}{2^{\frac{1}{3}}} = \frac{3^{\frac{3}{6}}}{2^{\frac{2}{6}}} = \left(\frac{3^3}{2^2}\right)^{\frac{1}{6}}$

$= \sqrt[6]{\frac{27}{4}} = \frac{1}{4}\sqrt[6]{27 \times 4^5}.$

Ex. 3. $\frac{8\sqrt{10}}{4\sqrt{5}} = 2\sqrt{\frac{10}{5}} = 2\sqrt{2}.$

EXERCISE LXIII

Simplify

1. $\sqrt{5} \times \sqrt{15}.$
2. $\sqrt{5} \times \sqrt{20}.$
3. $\sqrt{5} \times \sqrt{30}.$
4. $3\sqrt{10} \times 4\sqrt{15}.$
5. $\sqrt[3]{5} \times \sqrt[3]{75}.$
6. $\sqrt[3]{7} \times \sqrt[3]{49}.$
7. $7\sqrt[4]{27} \times 8\sqrt[4]{18}.$
8. $\sqrt[4]{32} \times \sqrt[4]{1296}.$
9. $\sqrt[3]{4} \times \sqrt[4]{5}.$
10. $\sqrt{2} \times \sqrt[3]{3} \times \sqrt[4]{4}.$
11. $\sqrt[3]{16} \times \sqrt{12} \times \sqrt[6]{250}.$
12. $\sqrt{15} \div \sqrt{5}.$
13. $\sqrt{20} \div \sqrt{5}.$
14. $\sqrt[4]{75} \div \sqrt[4]{120}.$
15. $\frac{1}{4}\sqrt{9} \times \frac{3}{4}\sqrt[3]{21}.$
16. $\frac{3}{4}\sqrt[3]{21} \div \frac{1}{4}\sqrt[3]{9}.$
17. $\sqrt{\frac{3}{5}} \times \sqrt[6]{\frac{1}{45}}.$
18. $\sqrt{4} \times \sqrt[3]{6} \times \sqrt[6]{8}.$
19. $\sqrt[3]{144} \div \sqrt[3]{9}.$
20. $\frac{2}{5}\sqrt{6} \div \frac{3}{4}\sqrt[3]{9}.$
21. $(\sqrt{12} + 2\sqrt{15} - 2\sqrt{25})\sqrt{15}.$
22. $(\sqrt[3]{185} + \sqrt[3]{40} - \sqrt[3]{600})\sqrt[3]{10}.$
23. Multiply $2\sqrt{3} - 3\sqrt{2}$ by $\sqrt{3} + \sqrt{2}.$

24. Multiply $4\sqrt{18} - 3\sqrt{12}$ by $\sqrt{2} - \sqrt{3}$.

25. " $4 - \sqrt{10}$ by $5 - 2\sqrt{10}$.

26. " $\sqrt[3]{9} - \sqrt[3]{12}$ by $\sqrt[3]{3} + \sqrt[3]{12}$.

27. Divide $\sqrt[3]{9} - \sqrt[3]{12} - \sqrt[3]{45}$ by $\sqrt[3]{3}$.

28. " $84\sqrt{6} - 21\sqrt{8} + 14\sqrt{20}$ by $7\sqrt{2}$.

237. The Square of a Polynomial involving Surds. The methods employed in Art. 136 may be applied to polynomial expressions involving surds.

Ex. 1. Find the value of $(\sqrt{6} - \sqrt{3})^2$.

$$(\sqrt{6} - \sqrt{3})^2 = 6 + 3 - 2\sqrt{3 \times 6} = 9 - 6\sqrt{2}.$$

Ex. 2. Find the value of $(\sqrt{3} - \sqrt{6} + \sqrt{12})^2$.

$$(\sqrt{3} - \sqrt{6} + \sqrt{12})^2 = 3 + 6 + 12 - 2\sqrt{18} + 2\sqrt{36} - 2\sqrt{72}$$
$$= 21 - 18\sqrt{2} + 12 = 9(1 - 2\sqrt{2}).$$

238. Rationalizing Denominators. Where a surd occurs in the denominator of a fraction it is usual to get rid of it by multiplying both the numerator and denominator of the fraction by an expression which makes the denominator integral. The process is called rationalizing the denominator. As all problems in division may be expressed in fractional form, the same principle may be applied in effecting simplification in such cases.

Ex. 1. Rationalize the denominator of $\dfrac{2\sqrt{3} + \sqrt{2}}{\sqrt{2} - \sqrt{3}}$.

$$\frac{2\sqrt{3} + \sqrt{2}}{\sqrt{2} - \sqrt{3}} = \frac{2\sqrt{3} + \sqrt{2}}{\sqrt{2} - \sqrt{3}} \times \frac{\sqrt{2} + \sqrt{3}}{\sqrt{2} + \sqrt{3}} =$$
$$= \frac{2\sqrt{6} + 8 + \sqrt{6}}{2 - 3} = -(3\sqrt{6} + 8).$$

Ex. 2. Rationalize the denominator of $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\times\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}=$$

$$=\frac{(\sqrt{a}+\sqrt{b})^2}{a-b}=\frac{a+b+2\sqrt{ab}}{a-b}.$$

Expressions of the form $\sqrt{a}+\sqrt{b}$, $\sqrt{a}-\sqrt{b}$ are called conjugate surds. The product of two conjugate surds is always rational.

EXERCISE LXIV

Find the value of

1. $(\sqrt{3}+\sqrt{2})^2$.
2. $(4\sqrt{2}+6\sqrt{3})^2$.
3. $(\sqrt{2}-2\sqrt{5}-3\sqrt{3})^2$.
4. $(\sqrt{x+y}+\sqrt{x-y})+\sqrt{x^2+y^2}$.

Rationalize the denominator of

5. $\dfrac{\sqrt{3+1}}{2\sqrt{2}}$.
6. $\dfrac{\sqrt{3}-\sqrt{2}}{2\sqrt{3}}$.
7. $\dfrac{2\sqrt{3}}{\sqrt{3}-2\sqrt{2}}$
8. $\dfrac{3\sqrt{2}}{2\sqrt{2}-\sqrt{3}}$.
9. $\dfrac{\sqrt{2}-1}{\sqrt{3}+\sqrt{2}}$.
10. $\dfrac{\sqrt{10}+\sqrt{5}}{\sqrt{10}-\sqrt{5}}$
11. $\dfrac{\sqrt{10}-\sqrt{12}}{\sqrt{10}+\sqrt{12}}$.
12. $\dfrac{2-3\sqrt{2}}{3\sqrt{2}+2}$.
13. $\dfrac{7+\sqrt{3}}{7-\sqrt{3}}$.
14. $\dfrac{4\sqrt{2}+6\sqrt{3}}{3\sqrt{3}-2\sqrt{2}}$.
15. $\dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$.
16. $\dfrac{\sqrt{x+1}-1}{\sqrt{x+1}+1}$
17. $\dfrac{\sqrt{x+y}+\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}}$.
18. $\dfrac{\sqrt{x^2+y^2}+\sqrt{x^2-y^2}}{\sqrt{x^2+y^2}-\sqrt{x^2-y^2}}$.
19. $\dfrac{x\sqrt{y}+y\sqrt{x}}{\sqrt{x}+\sqrt{y}}$.
20. $\dfrac{\sqrt{x+2y}+\sqrt{x-2y}}{\sqrt{x+2y}-\sqrt{x-2y}}$.

21. $\dfrac{x+\sqrt{4a^2-x^2}}{x-\sqrt{4a^2-x^2}}$.

22. $\dfrac{x^2+x\sqrt{x^2+y^2}}{x^2+y^2+x\sqrt{x^2+y^2}}$.

23. $\dfrac{a\sqrt{a}-\sqrt{b+1}}{\sqrt{a^3}+\sqrt{b+1}}$.

24. $\dfrac{\sqrt{2}-\sqrt{5}-\sqrt{7}}{\sqrt{2}+\sqrt{5}+\sqrt{7}}$.

239. Special Theorems. The following theorems in quadratic surds, based upon the fact that a rational quantity and a surd cannot be equal, are of sufficient importance to be specially noted.

I. A rational quantity and a surd cannot together be equal to a surd.

For, if possible, let

$$\sqrt{x}=y+\sqrt{z}\qquad\text{where } y \text{ is rational.}$$

Squaring both sides,

$$x=y^2+z+2y\sqrt{z},$$

or

$$2y\sqrt{z}=x-y^2-z,$$

or

$$\sqrt{z}=\frac{x-y^2-z}{2y}.$$

That is, a surd is equal to a rational quantity, which is impossible.

II. If $a+\sqrt{b}=x+\sqrt{y}$, then $a=x$ and $b=y$. For if a does not equal x, let it equal $x+d$.

Substituting this value for a, we have

$$x+d+\sqrt{b}=x+\sqrt{y}$$

or

$$d+\sqrt{b}=\sqrt{y}, \text{ which is impossible by I.}$$

III. If $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, then $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

Let

$$\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}.$$

Squaring,

$$a+\sqrt{b}=x+y+2\sqrt{xy},$$

therefore

$$a=x+y$$

and

$$\sqrt{b}=2\sqrt{xy}.$$

Subtracting, $a - \sqrt{b} = x + y - 2\sqrt{xy}$

$= (\sqrt{x} - \sqrt{y})^2.$

Therefore $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}.$

240. The Square Root of a Binomial Surd. The special theorems of Art. 239 may be applied to determine the square root of a binomial quadratic surd when such a root exists.

Ex. 1. Extract the square root of $6 + \sqrt{20}$.

Assume $\sqrt{6 + \sqrt{20}} = \sqrt{x} + \sqrt{y}.$ (1)

Squaring, $6 + \sqrt{20} = x + y + 2\sqrt{xy},$

therefore $x + y = 6.$ (2)

Also by Art. 239, III. $\sqrt{6 - \sqrt{20}} = \sqrt{x} - \sqrt{y}.$ (3)

Multiplying (1) by (3), we have

$$\sqrt{36 - 20} = x - y,$$

or $x - y = 4.$ (4)

From (2) $x + y = 6.$

Solving (4) and (2), $x = 5$ and $y = 1.$

Therefore $\sqrt{6 + \sqrt{20}} = 1 + \sqrt{5}.$

In simple examples such as the preceding the result may frequently be written from inspection.

Let $(\sqrt{x} + \sqrt{y})^2 = x + y + 2\sqrt{xy}$

$= 6 + 2\sqrt{5}.$

Now find the two numbers, 5 and 1, whose sum is 6 and product 5; $\sqrt{5} + \sqrt{1}$, or $1 + \sqrt{5}$ is the root required.

Similarly $2a + \sqrt{4(a^2 - 1)} = 2a + 2\sqrt{(a+1)(a-1)};$

from which $a + 1$ and $a - 1$ are the required numbers and consequently $\sqrt{a+1} + \sqrt{a-1}$ is the root of the given expression.

EXERCISE LXV

Extract the square root of

1. $6 - \sqrt{20}$.
2. $4 + 2\sqrt{3}$.
3. $4 - 2\sqrt{3}$
4. $8 + 2\sqrt{7}$.
5. $8 - 2\sqrt{15}$.
6. $8 + 2\sqrt{15}$.
7. $11 + 6\sqrt{2}$.
8. $11 - 6\sqrt{2}$.
9. $10 + 4\sqrt{6}$.
10. $73 + 40\sqrt{3}$.
11. $12 - 2\sqrt{35}$.
12. $23 + 6\sqrt{10}$.
13. $2a + 2\sqrt{a^2 - x^2}$.
14. $2x - \sqrt{4(x^2 - 1)}$.
15. $x^2 + 2y\sqrt{x^2 - y^2}$.

241. Equations involving Surds. Equations involving surd terms can sometimes be solved by rationalizing the surd terms. If the equation contains only one surd term, this can be rationalized by bringing that term to one side of the equation and all the other terms to the other side, and then raising both sides to the power indicated by the radical index.

If there be two surd terms, then it is necessary to put them on opposite sides of the equation and proceed as before. If a surd term still remains, the operation must be repeated. The following examples will serve to illustrate the process:

Ex. 1. Solve the equation $\sqrt{x - 12} = 4$.

Squaring, $x - 12 = 16$.

Therefore $x = 4$.

Ex. 2. Solve the equation $\sqrt{x - 5} - \sqrt{x - 4} = 1$.

Transposing, $\sqrt{x - 5} = 1 + \sqrt{x - 4}$.

Squaring, $x - 5 = 1 + x - 4 + 2\sqrt{x - 4}$.

Transposing, $2\sqrt{x - 4} = -2$.

Simplifying and squaring, $x - 4 = 1$, or $x = 5$.

Ex. 3. Solve the equation $\sqrt{x} + \sqrt{x-4} = \dfrac{8}{\sqrt{x-4}}$.

Multiply through by the denominator, $\sqrt{x-4}$,

and $\sqrt{x(x-4)} + x - 4 = 8$,

or $\sqrt{x(x-4)} = 12 - x$.

Squaring, $x(x-4) = 144 + x^2 - 24x$.

Hence $20x = 144$,

and $x = 7\frac{1}{5}$.

Ex. 4. Solve the equation $\sqrt{7 + 3\sqrt{5x - 16}} = 4$.

Squaring, $7 + 3\sqrt{5x-16} = 16$.

Hence $3\sqrt{5x-16} = 9$.

Therefore $\sqrt{5x-16} = 3$.

Squaring again, $5x - 16 = 9$.

Therefore $x = 5$.

Ex. 5. Solve the equation $\sqrt{x+2} + \sqrt{x+4} = \sqrt{4x+10}$.

Squaring, $x + 2 + x + 4 + 2\sqrt{(x+2)(x+4)} = 4x + 10$.

Therefore $2\sqrt{(x+2)(x+4)} = 2x + 4$.

Dividing by 2 and squaring, $(x+2)(x+4) = x^2 + 4x + 4$.

Clearing and collecting, $2x = -4$

$x = -2$.

EXERCISE LXVI

Solve the following equations:

1. $\sqrt{x-16} = 3$.

2. $\sqrt{x-a^2} = b$.

3. $\sqrt{x^2+9x-1} = x - 4$.

4. $6 + 4\sqrt{x} = 10$.

5. $1 + 3\sqrt{x} = y$.

6. $\sqrt{x-12} = 3\sqrt{x+3}$.

7. $\sqrt{2x} - \sqrt{2x-15} = 1$.

8. $\sqrt{x+5} - \sqrt{x} = 5$.

9. $\sqrt{x+10} - \sqrt{x-5} = 3.$

10. $\sqrt{4x} - \sqrt{4x-11} = \dfrac{5}{\sqrt{4x-11}}.$

11. $\sqrt{x+7} + \sqrt{x} = \dfrac{28}{\sqrt{x+7}}.$

12. $\dfrac{\sqrt{2x}+9}{\sqrt{2x}-7} = \dfrac{\sqrt{2x}+17}{\sqrt{2x}-1}.$

13. $\dfrac{\sqrt{x}+3}{\sqrt{x}+\sqrt{3}} = \dfrac{\sqrt{x}-\sqrt{3}}{\sqrt{x}-3}.$

14. $\dfrac{\sqrt{x-1}}{\sqrt{x+5}} = \dfrac{\sqrt{x-3}}{\sqrt{x-1}}.$

15. $\dfrac{\sqrt{x}-3}{\sqrt{x}+1} = \dfrac{\sqrt{x}+4}{\sqrt{x}-2}.$

16. $\sqrt{4x+5} = \sqrt{x} + \sqrt{x+3}.$

17. $\sqrt{a+x} + \sqrt{a-x} = \sqrt{a-2x}.$

18. $(x+\sqrt{5})^2 - (x-\sqrt{5})^2 = 4(5-\sqrt{5}.$

19. $\dfrac{3x-1}{\sqrt{3x}+1} = 1 + \dfrac{\sqrt{3x}-1}{2}.$

20. $\sqrt{x} - \sqrt{\dfrac{2}{x}} = \sqrt{x+2}.$

21. $\dfrac{1}{\sqrt{3}-x} + \dfrac{1}{\sqrt{3}+x} = \dfrac{2}{x\sqrt{3}}.$

22. $x^{\frac{1}{2}} + \left(\dfrac{a}{x}\right)^{\frac{1}{2}} = (x-a)^{\frac{1}{2}}.$

23. $\dfrac{1}{1+\sqrt{x}} + \dfrac{1}{1-\sqrt{x}} = \dfrac{3}{1+x}.$

24. $\dfrac{1}{1+\sqrt{1-x}} + \dfrac{1}{1-\sqrt{1-x}} = \dfrac{\sqrt{3}}{2-x}.$

CHAPTER XVI

QUADRATIC EQUATIONS

242. Definition. A quadratic equation is one which contains the second power of the unknown quantity, and no higher power.

Thus $x^2 - 4 = 0$ and $x^2 + 4x + 4 = 0$ are quadratic equations.

243. Pure Quadratics. A quadratic equation which contains only the square of the unknown quantity is called a **pure quadratic.**

Thus $x^2 - 9 = 0$ is a pure quadratic.

Such equations are sometimes called **incomplete quadratics,** because they lack the first power of the unknown quantity.

244. Solution of pure quadratics. To solve a pure quadratic, equate the square of the unknown quantity to the constant term, with the proper sign, and take the square root of both sides. We have seen (Art. 208) that each number has two square roots equal in absolute magnitude but of different sign. Hence all such equations have two roots.

Ex. 1. Solve the equation $x^2 - 9 = 0$.

Transposing, $x^2 = 9$.

Therefore $x = \pm 3$.

Ex. 2. Solve the equation

$$(x + a)(x + b) + (x - a)(x - b) = x^2 + a^2 + b^2.$$

Multiplying out,

$$x^2 + (a+b)x + ab + x^2 - (a+b)x + ab = x^2 + a^2 + b^2.$$

Collecting, $\qquad x^2 = a^2 + b^2 - 2ab = (a-b)^2.$

Therefore $\qquad x = \pm(a-b).$

245. Adfected Quadratics. A quadratic equation which contains both the first and second powers of the unknown quantity is called an **adfected quadratic**.

Thus $x^2 + 9x + 20 = 0$ is an adfected quadratic.

Adfected quadratics are also called complete quadratics.

246. Solution of adfected quadratics by factoring. We have seen in Chapter VII that the solution of adfected quadratics can be effected by transposing all the terms of the equation to one side and resolving it into factors, and then equating each factor separately to zero.

If there is difficulty in finding the factors of the expression it can be readily changed into an equivalent expression which is the difference of two squares, the factors of which can be obtained at once. The process depends upon the fact that the expression $x^2 + 2ax$ can be made a perfect square by adding a^2, the square of half the coefficient of x, since $x^2 + 2ax + a^2 = (x+a)^2$.

Thus if to $x^2 + 6x$ we add $\left(\frac{6}{2}\right)^2$ or 3^2 we get $x^2 + 6x + 9$, which is equal to $(x+3)^2$, a perfect square. The following examples show how this fact may be applied to effect the solution of equations:

Ex. 1. Solve the equation $x^2 - 3x - 10 = 0$.

$x^2 - 3x$ is made a perfect square by adding $\left(\frac{3}{2}\right)^2$ to it. If $\left(\frac{3}{2}\right)^2$ be added it must also be subtracted if the value of the expression is to remain unchanged.

Hence if $$x^2 - 3x - 10 = 0,$$

then $$x^2 - 3x + \left(\frac{3}{2}\right)^2 - 10 - \left(\frac{3}{2}\right)^2 = 0.$$

That is, $$\left(x - \frac{3}{2}\right)^2 - \frac{49}{4} = 0,$$ simplifying last two terms together,

or $$\left(x - \frac{3}{2}\right)^2 - \left(\frac{7}{2}\right)^2 = 0.$$

Factoring, $$\left(x - \frac{3}{2} + \frac{7}{2}\right)\left(x - \frac{3}{2} - \frac{7}{2}\right) = 0,$$

or $$(x + 2)(x - 5) = 0.$$

Hence $$x + 2 = 0 \text{ or } x - 5 = 0,$$

and therefore $$x = -2 \text{ or } x = 5.$$

If the coefficient of the square be not unity, then it is necessary to divide through by that coefficient before solving.

Ex. 2. Solve the equation $4x^2 + 21x - 18 = 0$.

Dividing through by 4, $$x^2 + \frac{21x}{4} - \frac{9}{2} = 0.$$

Adding and subtracting, $\left(\frac{1}{2} \text{ of } \frac{21}{4}\right)^2$

then $$x^2 + \frac{21}{4}x + \left(\frac{21}{8}\right)^2 - \frac{9}{2} - \left(\frac{21}{8}\right)^2 = 0.$$

Collecting, $$\left(x + \frac{21}{8}\right)^2 - \frac{729}{64} = 0.$$

Factoring, $$\left(x + \frac{21}{8} + \frac{27}{8}\right)\left(x + \frac{21}{8} - \frac{27}{8}\right) = 0,$$

or $$(x + 6)\left(x - \frac{6}{8}\right) = 0.$$

Hence $$x + 6 = 0 \text{ or } x - \frac{6}{8} = 0,$$

and therefore $$x = -6 \text{ or } x = \frac{6}{8} = \frac{3}{4}.$$

247. Solution by Completing the Square. The above solution may also be effected directly without resolving the expression into factors, by transposing the constant quantity to the right-hand side of the equation, adding to each side the quantity necessary to make the left-hand side a perfect square, and then extracting the square root of both sides. The method is practically identical with that of Art. 246, the essential process in each case being called **completing the square.**

Ex. 1. Solve the equation $x^2 + 9x + 20 = 0$.

Transposing, $$x^2 + 9x = -20.$$

Completing the square, $$x^2 + 9x + \left(\frac{9}{2}\right)^2 = -20 + \left(\frac{9}{2}\right)^2.$$

Hence $$\left(x + \frac{9}{2}\right)^2 = \frac{1}{4}.$$

Extracting square root, $$x + \frac{9}{2} = \pm\frac{1}{2}.$$

Hence $$x = +\frac{1}{2} - \frac{9}{2} \text{ or } -\frac{1}{2} - \frac{9}{2}$$
$$= -4 \text{ or } -5.$$

Ex. 2. Solve the equation $\frac{3x-2}{2x-3} = \frac{5x}{x+4} - 2$.

Multiplying through by the common denominator,
$$(3x - 2)(x + 4) = 5x(2x - 3) - 2(x + 4)(2x - 3).$$

Clearing and collecting, $$3x^2 - 35x = -32.$$

Dividing by 3, $$x^2 - \frac{35}{3}x = -\frac{32}{3}.$$

Completing the square, $x^2 - \frac{35}{3}x + \left(\frac{35}{6}\right)^2 = -\frac{32}{3} + \left(\frac{35}{6}\right)^2.$

Hence $\left(x - \frac{35}{6}\right)^2 = \frac{841}{36}.$

Extracting the square root, $x - \frac{35}{6} = \pm\frac{29}{6}.$

Hence $x = +\frac{29}{6} + \frac{35}{6}$ or $-\frac{29}{6} + \frac{35}{6}$

$= 10\frac{2}{3}$ or 1.

Ex. 3. Solve the equation $\frac{x}{x-2} + \frac{4}{x+1} = 3.$

Multiplying through by the common denominator,

$$x(x+1) + 4(x-2) = 3(x+1)(x-2).$$

Simplifying, $-2x^2 + 8x - 2 = 0$;

therefore $x^2 - 4x = -1.$

Completing the square, $x^2 - 4x + 4 = -1 + 4 = 3.$

Taking the square root, $x - 2 = \pm\sqrt{3}.$

Therefore $x = 2 + \sqrt{3}$ or $2 - \sqrt{3}.$

248. Should the equation be without a constant term, the solution is directly effected by factoring.

Ex. Solve equation $x^2 - ax = 0.$

Factoring, $x(x-a) = 0.$

Hence $x = 0$ or $x - a = 0.$

Therefore $x = 0$ or $x = a.$

The student will observe that in every quadratic equation there are **two** roots. The zero root in the last example must not be overlooked.

EXERCISE LXVII

Solve the following equations, working 1 to 16 by factoring and the remainder by completing the square only. Verify the result.

1. $x^2 - 9 = 0$.
2. $x^2 - 49 = 0$.
3. $4x^2 - 40 = 0$.
4. $\frac{x}{2} + \frac{2}{x} = \frac{x}{3} + \frac{3}{x}$.
5. $2x^2 - 3x = 0$.
6. $4x^2 - 9x = 0$.
7. $2x^2 - 5x = 0$.
8. $3x^2 = 5x$.
9. $5x^2 = -6x$.
10. $ax^2 - bx = 0$.
11. $x^2 - 5x + 6 = 0$.
12. $x^2 - 10x + 16 = 0$.
13. $x^2 - 10x + 21 = 0$.
14. $x^2 - 20x + 51 = 0$.
15. $x^2 + 24x + 80 = 0$.
16. $x^2 - 16x + 48 = 0$.
17. $x^2 - 2x - 48 = 0$.
18. $x^2 - 13x - 140 = 0$.
19. $4x^2 - 2x - 12 = 0$.
20. $6x^2 - x - 12 = 0$.
21. $2x^2 - 27x = 14$.
22. $12x^2 - 37x - 21 = 0$.
23. $3x^2 - 13x = 10$.
24. $7x^2 + 9x = 10$.
25. $\frac{x^2}{4} + \frac{3x}{2} = 27\frac{1}{2}$.
26. $\frac{4x}{x-1} - \frac{x+3}{x} = 4$.
27. $\frac{x+2}{x-1} - \frac{4-x}{2x} = \frac{7}{3}$.
28. $\frac{x+2}{x-7} - \frac{x+5}{x-5} = 1$.
29. $\frac{6x+5}{4x-3} = \frac{4x+4}{x-3}$.
30. $\frac{3}{x-1} + \frac{4}{x-3} = \frac{15}{x+3}$.

249. The general equation—Solution by a formula. The most general expression for the quadratic equation is $ax^2 + bx + c = 0$, since every equation involving fractions can be reduced to this form by multiplying by the common denominator and collecting the terms. The general equation can be solved exactly as in Art. 248.

Solve the equation $ax^2 + bx + c = 0$.

Dividing through by a, $x^2 + \frac{b}{a}x = -\frac{c}{a}$.

Completing the square. $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$

Hence $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$.

Extracting the square root, $x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$.

Hence $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

This result may be used as a formula for the solution of equations, as shown in the following examples:

Ex. 1. Solve the equation $6x^2 + 17x + 12 = 0$.

Comparing the equation with the general equation,

$$ax^2 + bx + c = 0,$$
$$a = 6, \quad b = 17, \quad c = 12.$$

Substituting in formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

we have
$$x = \frac{-17 \pm \sqrt{17^2 - 4 \times 6 \times 12}}{2 \times 6}$$
$$= \frac{-17 \pm \sqrt{1}}{12}$$
$$= \frac{-17 + 1}{12} \text{ or } \frac{-17 - 1}{12}$$
$$= -1\tfrac{1}{3} \text{ or } -1\tfrac{1}{2}.$$

Ex. 2. Solve the equation $x^2 - 3x - 88 = 0$.

Comparing the equation with the general equation,

$$a = 1, \quad b = -3, \quad c = -88.$$

Therefore $$x = \frac{3 \pm \sqrt{9 + 4 \times 88}}{2}$$

$$= \frac{3 \pm 19}{2} = 11 \text{ or } -8.$$

Ex. 3. $$\frac{1}{x+a} + \frac{1}{x-a} = \frac{1}{x-2a}.$$

Multiplying by the common denominator,

$$(x-a)(x-2a) + (x+a)(x-2a) = x^2 - a^2.$$

Collecting terms, $x^2 - 4ax - 2a^2 = 0.$

Comparing with the general equation,

$$a = 1, \quad b = -4a, \quad c = -2a^2.$$

Therefore $$x = \frac{4a \pm \sqrt{16a^2 + 8a^2}}{2}$$

$$= 2a \pm a\sqrt{6} = a(2 \pm \sqrt{6}).$$

Ex. 4. Solve the equation $3x^2 - 8x + 10 = 0.$

Here $a = 3, \quad b = -8, \quad c = 10.$

Therefore $$x = \frac{8 \pm \sqrt{64 - 120}}{6}$$

$$= \frac{8 \pm \sqrt{-56}}{6}.$$

250. Imaginaries. Expressions of the form $\sqrt{-56}$ (Ex. 4) are called **imaginary expressions,** as their values cannot be even approximately expressed, there being no numbers positive or negative whose squares are negative. It is convenient, however, to take from under the root any square positive factor, leaving the remaining negative factor under the root sign.

For example, $\sqrt{-56} = \sqrt{-2^2 \times 14} = 2\sqrt{-14}.$

Similarly $\sqrt{-a^2} = a\sqrt{-1}.$

When such a solution is obtained it means that there is no real number which will make the left-hand side of the equation equal to zero.

The meaning of such expressions will be discussed in a later chapter. In the following exercise the student should practice **completing the square** and the use of the formula, checking the results of one method by the other.

EXERCISE LXVIII

Solve the following equations:

1. $x^2 - 5x + 3 = 0.$
2. $x^2 - 15x = 154.$
3. $3x^2 + x - 200 = 0.$
4. $4x^2 - 26x - 14 = 0.$
5. $8x^2 - 10x - 12 = 0.$
6. $72x^2 - 172x - 20 = 0.$
7. $5x^2 - 8x = 21.$
8. $9x^2 - 12x + 4 = 0.$
9. $72 - 9x^2 = 18x.$
10. $12x^2 - 17ax + 6a^2 = 0$
11. $28x^2 = 3ax + 18a^2.$
12. $25x^2 = 5x + 6.$
13. $x^2 - \frac{5}{6}x = -\frac{1}{6}.$
14. $\frac{7}{x+5} - \frac{8}{x-6} = \frac{3}{x-1}.$
15. $\frac{1}{x^2-1} + \frac{1}{x-1} = \frac{7}{8} - \frac{1}{x+1}.$
16. $\frac{10}{x-3} + \frac{12}{x+4} = 3.$
17. $\frac{3}{2x+1} + \frac{6}{2x-1} - 5 = 0.$
18. $\frac{4}{x+1} + \frac{5}{x+2} = \frac{12}{x+3}.$
19. $\frac{x}{x^2-2x-15} - \frac{7\frac{1}{2}}{x^2+2x-35} = \frac{1}{x^2+10x+21}.$
20. $\frac{4x-5}{x+2} + \frac{2x-3}{x+4} = 10.$
21. $\frac{x+3}{x-3} + \frac{4-x}{x-2} = -3.$
22. $\frac{x-7}{x+2} + \frac{x-5}{x+5} = 1.$
23. $\frac{x}{a} + \frac{b}{x} = \frac{a}{x} + \frac{x}{b}.$
24. $x^2 - ax + a^2 = 0.$
25. $x^2 - 4ax + 2a^2 = 0.$
26. $\frac{1}{x+a+b} = \frac{1}{x} + \frac{1}{a} + \frac{1}{b}.$
27. $\frac{1}{a+x} + \frac{1}{b+x} = \frac{a+b}{ab}.$

28. $a^2x^2 - b^2x^2 - 4abx - a^2 + b^2 = 0.$

29. $\dfrac{x}{x+1} + \dfrac{x+1}{x} = \dfrac{13}{6}.$

30. $\dfrac{x+4}{x-4} + \dfrac{x-2}{x-3} = 6\tfrac{1}{2}.$

31. $\dfrac{x+3}{x+2} + \dfrac{x-3}{x-2} = \dfrac{2x-3}{x-1}.$

32. $\dfrac{x-2}{x-3} - \dfrac{x-3}{x-4} = 2\left\{\dfrac{x-5}{x-6} - \dfrac{x-6}{x-7}\right\}.$

33. $\dfrac{x+2}{x} + \dfrac{x}{x-2} = \dfrac{x+3}{x} + \dfrac{x}{x-3}.$

34. $\dfrac{x}{x-1} + \dfrac{x+2}{x+1} = \dfrac{8x-13}{4(x-2)}.$

35. $\dfrac{a^2}{b+x} + \dfrac{a^2}{b-x} = c.$

251. Higher Degree Equation in Quadratic form. An equation in which the unknown quantity occurs in two terms only, the unknown factor in one being the square of the unknown factor in the other, is said to be in the **quadratic form.**

Thus $x^4 - 7x^2 + 10 = 0$ and $x^{2n} + x^n + a = 0$, are in quadratic form.

Equations in the quadratic form can be solved like quadratic equations.

Ex. 1. Solve $x^4 - 10x^2 + 21 = 0.$

Factoring, $(x^2 - 7)(x^2 - 3) = 0.$

Hence $x^2 = 7$ or $x^2 = 3.$

Therefore $x = \pm\sqrt{7}$ or $x = \pm\sqrt{3}.$

The result could be immediately written down by the formula of Art. 249.

Thus
$$x^2 = \frac{10 \pm \sqrt{100 - 84}}{2}$$
$$= \frac{10 \pm 4}{2} = 7 \text{ or } 3.$$

That is, $x = \pm\sqrt{7}$ or $\pm\sqrt{3}$.

Ex. 2. Solve $x^4 + 10x^2 - 24 = 0$.

By the formula
$$x^2 = \frac{-10 \pm \sqrt{100 + 96}}{2}$$
$$= \frac{-10 \pm 14}{2} = 2 \text{ or } -12.$$

Hence $x = \pm\sqrt{2}$,

or $x = \pm\sqrt{-12} = \pm 2\sqrt{-3}$.

Ex. 3. Solve $x^6 - 17x^3 + 72 = 0$.

By the formula
$$x^3 = \frac{17 \pm \sqrt{289 - 288}}{2}$$
$$= \frac{17 \pm 1}{2} = 8 \text{ or } 9.$$

Extracting cube root, $x = \sqrt[3]{8}$ or $\sqrt[3]{9}$
$= 2$ or $\sqrt[3]{9}$.

Ex. 4. Solve $x^{\frac{2}{3}} - 7x^{\frac{1}{3}} - 30 = 0$.

By the formula
$$x^{\frac{1}{3}} = \frac{7 \pm \sqrt{49 + 120}}{2}$$
$$= \frac{7 \pm 13}{2} = 10 \text{ or } -3.$$

Cubing, $x = 1000$ or -27.

252. Solution by Substitution. In more complex examples the solution can sometimes be simplified by substituting a letter for an expression involving x. The following three cases will serve to illustrate the principle:

CASE I. When the first and second powers of an expression involving x are found in the equation.

Ex. Solve $(3x^2 + 2x)^2 - 3(3x^2 + 2x) - 4 = 0$.

If y be substituted for $3x^2 + 2x$, the equation becomes
$$y^2 - 3y - 4 = 0.$$

Factoring. $(y-4)(y+1)=0.$

Hence $y=4$ or $y=-1.$

We must now solve the two equations,

$$3x^2+2x-4=0 \text{ and } 3x^2+2x+1=0.$$

$$x=\frac{-2\pm\sqrt{4+48}}{6} \qquad x=\frac{-2\pm\sqrt{4-12}}{6}$$

$$=\frac{-1\pm\sqrt{13}}{3}. \qquad =\frac{-1\pm\sqrt{-2}}{3}.$$

CASE II. When an expression involving x and its reciprocal occur in the equation.

Ex. Solve $$\frac{x^2}{x+1}+3\left(\frac{x+1}{x^2}\right)-4=0.$$

If y be substituted for $\frac{x^2}{x+1}$, the equation becomes

$$y+\frac{3}{y}-4=0,$$

or $$y^2-4y+3=0,$$

or $$y=\frac{4\pm\sqrt{16-12}}{2}$$

$$=3 \text{ or } 1.$$

The two equations $\frac{x^2}{x+1}=3$ and $\frac{x^2}{x+1}=1$ must now be solved, from which $x=\frac{3\pm\sqrt{21}}{2}$ or $\frac{1\pm\sqrt{5}}{2}$.

CASE III. When an expression involving x occurs in the equation free from, and also under, a radical.

Ex. Solve $x^2+6x+7+\sqrt{x^2+6x+10}=3.$

Adding 3 to each side, the equation becomes

$$x^2+6x+10+\sqrt{x^2+6x+10}=6.$$

If now y be substituted for $x^2+6x+10$, the equation becomes

$$y+\sqrt{y}=6.$$

or

$$y^{\frac{1}{2}}=\frac{-1\pm\sqrt{1+24}}{2}=\frac{-1\pm 5}{2}$$
$$=2 \text{ or } -3.$$

The two equations $x^2+6x+10=2$,
and $x^2+6x+10=-3$ must now be solved to get the value of x.

EXERCISE LXIX

Solve the following :

1. $5x^4+6x^2-11=0$.
2. $x^6-26x^3-27=0$.
3. $12x^6-25x+12=0$.
4. $4x^4+23x^2-72=0$.
5. $x^{\frac{2}{3}}-x^{\frac{1}{3}}-6=0$.
6. $4x^3-13x^{\frac{3}{2}}+36=0$.
7. $2\sqrt{x}-3\sqrt[4]{x}+1=0$.
8. $x^{-6}+19x^{-3}=216$.
9. $\dfrac{x^2-1}{9}+\dfrac{1}{x^2}-1=0$.
10. $x^2+\dfrac{1}{x^2}=a^2+\dfrac{1}{a^2}$.
11. $x^2-\dfrac{1}{x^2}=a^2-\dfrac{1}{a^2}$
12. $27x^3-\dfrac{16}{x^3}+46=0$.
13. $\left(\dfrac{x+3}{x}\right)^2+3\left(\dfrac{x+3}{x}\right)=10$.
14. $\left(x+\dfrac{1}{x}\right)^2+4\left(x+\dfrac{1}{x}\right)=12$.
15. $\left(\dfrac{7}{x^2-3x}\right)^2+\dfrac{14}{x^2-3x}-9=0$.
16. $(x^{-2}+1)-4(x^{-2}+1)+3=0$.
17. $(x^2-x)^2-(x^2-x)=132$.
18. $\left(\dfrac{1+x^2}{x}\right)^2+2\left(\dfrac{1+x^2}{x}\right)=8$.
19. $\left(\dfrac{1-x}{x}\right)^2-3\left(\dfrac{1-x}{x}\right)^2-10=0$.

20. $\left(\frac{6}{x^2}-\frac{x}{2}\right)^2-3\left(\frac{6}{x^2}-\frac{x}{2}\right)=4.$

21. $\left(\frac{x^2}{x-1}\right)+3\left(\frac{x-1}{x^2}\right)-4=0.$

22. $\frac{x}{x^2+1}+12\left(\frac{x^2+1}{x}\right)=7.$

23. $\frac{x}{x+2}-\frac{x+2}{2x}=-\frac{1}{2}.$

24. $\frac{x^2+1}{3}+\frac{3}{x^2+1}=4.$

25. $2x^2+4x+\frac{8}{2x^2+4x}+6=0.$

26. $x^2+4x+\sqrt{x^2+4x}=12.$

27. $x^2-x-4\sqrt{x^2-x+2}=10.$

28. $x+2\sqrt{x+1}=24.$

29. $\frac{x^2}{\sqrt{x^2-7}}-\frac{16\sqrt{x^2-7}}{x}=6.$

30. $x^2+(x+2)(x+3)+\sqrt{2x^2+5x+6}=30.$

253. Character of the Roots of the Quadratic. In the solution of the general equation $ax^2+bx+c=0$ (Art. 249), we obtained

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

As there are only two square roots of the expression b^2-4ac, there are only two values of x that will satisfy the equation; that is, only two roots of the equation, viz.,

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

254. The expression under the radical b^2-4ac is called the **discriminant** of the equation; since, as we shall see, the character of the roots depends upon it.

Clearly, b^2-4ac may be either positive or negative, or zero.

1. If b^2-4ac be positive, then $\sqrt{b^2-4ac}$ is a real number, and the roots of the equation are real roots, and unequal. If b^2-4ac be also a perfect square, the roots are rational, if not they are irrational.

2. If b^2-4ac be equal to zero, then the term under the radical vanishes, and the roots are equal, viz.,

$$\frac{-b}{2a} \text{ and } \frac{-b}{2a}.$$

3. If b^2-4ac be negative, the roots of the equation are imaginary, since the term under the radical has a negative sign.

This cannot occur if the sign of c be negative, unless the sign of the first term of the equation be also negative.

Whether the roots of a particular equation are real, equal, or imaginary, can be at once determined by examining the expression b^2-4ac for the equation.

Ex. 1. Determine the nature of the roots of the equation $4x^2-6x-4=0$.

$$a=4,\ b=-6,\ c=-4.$$

$$b^2-4ac=36+64=100.$$

Hence the roots are real, rational, and unequal.

Ex. 2. Determine the nature of the roots of the equation $3x^2-9x+20=0$.

$$b^2-4ac=81-240=-159.$$

Hence $$\sqrt{b^2-4ac}=\sqrt{-159}.$$

The roots are therefore imaginary.

Ex. 3. Determine the nature of the roots of the equation $16x^2+16x+4=0$.

$$b^2-4ac=256-256=0.$$

The roots are equal.

255. Relation between the roots and coefficients. We have seen that an equation can be solved by resolving it into factors, and solving the factors separately.

Thus $x^2 - 9x + 20 = 0$ becomes when factored

$(x-4)(x-5) = 0$, giving the roots $x = 4$ and $x = 5$.

In this case the relation between the roots and the [illegible]ents appears at once from Art. 130, viz.: the [illegible]duct of [illegible] roots is equal to the constant term, and the [illegible] equal to the coefficient of x with the [illegible]

[illegible] be shown to be general, as follows:

[illegible] denote the roots of the equation [illegible]

Then
$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Adding,
$$r_1 + r_2 = \frac{-b}{2a} + \frac{-b}{2a} = -\frac{b}{a}.$$

Multiplying,
$$r_1 \times r_2 = \frac{(-b)^2 - (b^2 - 4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}.$$

If, therefore, the equation be written

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0, \text{ we have}$$

(1) The product of the roots is equal to the constant term.

(2) The sum of the roots is equal to the coefficient of x with its sign changed.

256. Formation of equation from the roots. The roots of an equation being given, the equation can at once be found.

1. Since the roots result from the solution of the factors separately, the factors can be determined from the roots. Thus if the roots are 4 and 5, we can write

$$x = 4 \text{ and } x = 5,$$

or

$$x - 4 = 0 \text{ and } x - 5 = 0,$$

and hence $(x - 4)(x - 5) = 0$ is the desired equation. The roots with their signs changed become the second terms of the factors.

In general, if the known roots are a and b the equation is $(x - a)(x - b) = 0$.

2. By the last Article $r_1 \times r_2 =$ the constant term of the equation, and $r_1 + r_2 =$ the coefficient of x with its sign changed.

Hence $x^2 - (r_1 + r_2)x + r_1 r_2 = 0$ is the equation whose roots are r_1 and r_2.

Ex. 1. Form the equation whose roots are 2 and -5.

By Method 1. $(x - 2)(x + 5) = 0$ is the equation.

By Method 2. $x^2 - (2 - 5)x + 2(-5) = 0$ is the equation.

These, of course, give the same result,

$$x^2 + 3x - 10 = 0.$$

Ex. 2. Form the equation whose roots are

By Method 1. $\{x - (1 + \sqrt{3})\}\{x - (1 - \sqrt{3})\} = 0$ is the equation.

Clearing of brackets, $x^2 - 2x - 2 = 0$.

By Method 2. The sum of the roots $= 2$,

The product $= 1 - 3 = -2$.

Hence equation is $x^2 - 2x - 2 = 0$

EXERCISE LXX

Determine, without solving the equations, the character of the roots of the following:

1. $x^2 - 7x + 12 = 0$.
2. $x^2 + 4x - 10 = 0$.
3. $2x^2 + 6x - 8 = 0$.
4. $36x^2 - 35x + 6 = 0$.
5. $x^2 + 4x + 16 = 0$.
6. $x^2 + 8x - 16 = 0$.
7. $3x^2 + 7x + 6 = 0$.
8. $2x^2 - 13x + 15 = 0$.
9. $2x^2 - 3x + 8 = 0$.
10. $x^2 + x + 1 = 0$

Form the equations whose roots are

11. $2, 3$.
12. $-2, -3$.
13. $-\frac{1}{2}, +\frac{1}{2}$.
14. $-a, +a$.
15. $2 + \sqrt{3}, 2 - \sqrt{3}$.
16. $3 + \sqrt{2}, 3 - \sqrt{2}$.
17. $1 + \sqrt{-2}, 1 - \sqrt{-2}$.
18. $a + \sqrt{-b}, a - \sqrt{-b}$.

19. Find the sum and the product of the roots of

$$(x-1)(x-2) = 3(x-5); \quad (a-x)(x-b) = x(x-c).$$

20. Form the equation the sum of whose roots is $2a$, and product $a^2 - 1$. Obtain the roots separately.

21. Find the value of n for which the equation $2x^2 - nx + 18 = 0$ has equal roots, and solve the equation.

22. Find the greatest value of n for which the roots of $2x^2 - 7x + n = 0$ will not be imaginary. Solve the equation for this value of n.

23. If the sum of the roots of $nx^2 - 5x + 20n = 0$ equals their product, find n and solve the equation.

24. Find the values of n for which the equation $3nx^2 - (6n + 12)x + 25 = 0$ will have equal roots, and solve the resulting equation.

CHAPTER XVII

SIMULTANEOUS EQUATIONS OF THE SECOND DEGREE

257. Simultaneous equations of the second degree involving two unknown quantities, cannot always be solved by the methods given for quadratic equations, as they lead in general to equations of the fourth degree, when the process of eliminating one of the unknown quantities (Art. 116) is performed. In certain cases, however the solution may be effected and rules can be obtained for so doing.

258. CASE I. **By Substitution.** When one of the equations is of the first degree, the solution can be effected by substitution.

Ex. Solve $x^2 + y^2 = 34.$ (1)

$x + y = 8.$ (2)

From (2) $x = 8 - y.$

Substituting in (1) $(8 - y)^2 + y^2 = 34.$

or $2y^2 - 16y + 30 = 0.$

Hence $y = \frac{16 \pm \sqrt{256 - 240}}{4}$

$= 5$ or $3.$

Substituting these values for y in (2) to obtain x, we have $x = 3$ or 5.

The student will observe that the values of x and y go in pairs, and that the equations are satisfied only when these pairs are properly selected, thus when $x = 3$, $y = 5$ and when $x = 5$, $y = 3$.

259. CASE II. **Homogeneous Equations.** When the equations are homogeneous they can always be solved by first obtaining one of the unknown quantities in terms of the other.

Ex. Solve

$$3x^2 + 4xy = 20, \qquad (1)$$

$$5xy + 2y^2 = 12. \qquad (2)$$

By multiplying (1) by 3 and (2) by 5, we can eliminate the constant quantities.

Multiplying, (1) becomes $9x^2 + 12xy = 60, \qquad (3)$

and (2) " $25xy + 10xy^2 = 60.$

Subtracting, $9x^2 - 13xy - 10y^2 = 0.$

Solving for x in terms of y, considering $-13y$ and $-10y^2$ as constants,

$$x = \frac{13y \pm \sqrt{169y^2 + 360y^2}}{18}$$

$$= \left(\frac{13 \pm 23}{18}\right)y = 2y \text{ or } -\frac{5}{9}y.$$

Substituting these values for x in equation (1), we have two equations to solve for y, viz.,

$$3(2y)^2 + 8y^2 = 20 \text{ and } 3\left(-\frac{5}{9}y\right)^2 + 4y\left(-\frac{5}{9}y\right) = 20.$$

or $12y^2 + 8y^2 = 20.$ $\qquad 75y^2 - 180y^2 = 1620.$

Whence $y^2 = 1$ $\qquad y^2 = -\frac{108}{7}$

or $y = \pm 1,$ $\qquad y = \pm\frac{6}{7}\sqrt{-21},$

then $x = 2y$ $\qquad x = -\frac{5}{9}y$

$= \pm 2.$ $\qquad = \mp\frac{10}{21}\sqrt{-21}.$

260. CASE III. **Symmetrical Equations.** When the equations are symmetrical with respect to x and y, that is, when x and y can be interchanged without changing the equation, a solution may be effected by first obtaining the values of $x+y$ and $x-y$ separately, and then from these the values of x and y may be readily found. This also applies to certain equations of a degree higher than the second.

Ex. Solve $\quad x-y=5,$ (1)

$\quad xy=14.$ (2)

Squaring (1), $\quad x^2+y^2-2xy=25.$

Multiplying (2) by 4, $\quad 4xy=56.$

Adding, $\quad x^2+y^2+2xy=81.$

Therefore $\quad x+y=\pm 9.$

The two pairs of equations must now be solved, viz

$$x-y=5 \text{ and } x-y=5,$$
$$x+y=9 \qquad x+y=-9.$$

Adding, $\quad 2x=14 \qquad 2x=-4,$

or $\quad x=7 \qquad x=-2.$

Subtracting, $\quad 2y=4 \qquad 2y=-14,$

or $\quad y=2 \qquad y=-7.$

Therefore when $x=7$, $y=2$, when $x=-2$, $y=-7$.

This problem of course can also be solved by substitution, as one of the equations is of the first degree.

Ex. 2. Solve $\quad x^2+y^2=25,$ (1)

$\quad xy=12.$ (2)

Multiplying (2) by 2, $\quad 2xy=24.$ (3)

Adding (1) and (3), $\quad x^2+y^2+2xy=49.$

Therefore $\quad x+y=\pm 7.$

Subtracting (2) from (1), $x^2 + y^2 - 2xy = 1$.

Therefore $$x - y = \pm 1.$$

The following pairs of equations must now be solved, viz.:

$$\begin{array}{llll} x + y = 7 & x + y = 7 & x + y = -7 & x + y = -7 \\ x - y = 1 & x - y = -1 & x - y = 1 & x - y = -1 \end{array}$$

$$\begin{array}{llll} \text{then } x = 4 & x = 3 & x = -3 & x = -4 \\ \quad\;\; y = 3 & y = 4 & y = -4 & y = -3 \end{array}$$

Ex. 3. Solve
$$x^3 + y^3 = 91. \qquad (1)$$
$$x + y = 7. \qquad (2)$$

Dividing (1) by (2),
$$x^2 - xy + y^2 = 13. \qquad (3)$$

Squaring (2), $$x^2 + 2xy + y^2 = 49. \qquad (4)$$

Subtracting, $$3xy = 36,$$
$$xy = 12. \qquad (5)$$

Subtracting (5) from (3),
$$x^2 - 2xy + y^2 = 1,$$

and therefore, $$x - y = \pm 1,$$

from (2) $$x + y = 7.$$

Solving now the pairs of equations,

$$\begin{array}{llllll} x - y = 1 & x - y = -1, & \text{we have} & x = 4 & x = 3 \\ x + y = 7 & x + y = 7 & & y = 3 & y = 4 \end{array}$$

EXERCISE LXXI

Solve the following equations:

1. $x^2 + y^2 = 148,$
   $x - y = 10.$

2. $x^2 + y^2 = 80,$
   $x + y = 12.$

3. $x^2 - y^2 = 20,$
   $x - y = 2.$

4. $x^2 - y^2 = 36,$
   $x + y = 12.$

5. $x^2 + 3xy - 2y^2 = 32,$
$2x - 3y = 2.$

6. $7x^2 - 8xy = 159,$
$5x + 2y = 7.$

7. $2x^2 - 3xy = 8,$
$4xy - y^2 = 28.$

8. $4x^2 + 6xy = 72,$
$3xy + 4y^2 = 34.$

9. $x^2 + 6xy - 9y^2 = 7,$
$x^2 - xy + y^2 = 13.$

10. $x^2 + y^2 = 5,$
$2xy - y^2 = 3.$

11. $x^2 + 3xy = -5,$
$2xy - y^2 = -24.$

12. $x^2 + 2xy - 4y^2 = 4,$
$x^2 - 5xy + 8y^2 = 2.$

13. $4x^2 - 2xy - 7y^2 = 20,$
$3x^2 + xy - 5y^2 = 1.$

14. $x^2 + 2xy + 3y^2 = 17$
$2x^2 + 3xy + 5y^2 = 2$[illegible]

15. $x^2 + xy + y^2 = 39,$
$3y^2 - 5xy = 25.$

16. $x^2 + y^2 = 130,$
$xy = 63.$

17. $x^2 - y^2 = 48,$
$xy = 32.$

18. $x + y = 14,$
$xy = 48.$

19. $x + y = 26,$
$xy = 160.$

20. $x - y = 20,$
$xy = 300.$

21. $x^3 + y^3 = 133,$
$x + y = 7.$

22. $x^3 - y^3 = 152,$
$x - y = 2.$

23. $x^3 + y^3 = 28,$
$x + y = 4.$

24. $x^3 - y^3 = 63,$
$x - y = 3.$

25. $x^2 + y^2 = 50,$
$xy = 7.$

26. $x + y = 5,$
$x^2 + y^2 = 13.$

27. $x^3 + y^3 = 280,$
$x^2 - xy + y^2 = 28.$

28. $x^3 - 27y^3 = 271,$
$x - 3y = 1.$

29. $x^2 - 3xy + y^2 = 1,$
$3x^2 - xy + 3y^2 = 13.$

30. $\frac{x+y}{x-y} + \frac{x-y}{x+y} = \frac{5}{2}$
$2x + 3y = 9$

CHAPTER XVIII

PROBLEMS IN QUADRATIC EQUATIONS

261. The statement in algebraic symbols of the conditions of a problem frequently gives rise to a quadratic equation, and since every quadratic equation has two roots it would at first appear that such problems have always two solutions. But the solution of an equation does not always furnish a solution of the problem from which the equation was formed. The roots of an equation may be fractional, negative or imaginary, and such numbers may not be applicable to the quantity sought in the problem. Some of these peculiarities are illustrated in the following examples:

Ex. 1. The number of feet in the perimeter of a square exceeds the number of square yards in its area by 27. Find its side in yards.

Let x = the number of yards in a side,

then $12x$ = " " " feet " the perimeter.

Therefore $$12x - x^2 = 27,$$

or $$x^2 - 12x + 27 = 0,$$

from which $$x = 3 \text{ or } 9.$$

Each of these roots is equally appropriate to express the length of the side of a square. The problem has, therefore, two distinct solutions.

Ex. 2. The hypotenuse of a right-angled triangle is 13, and its area is 30. Find its sides.

Let x and y represent the sides.

Then $$x^2 + y^2 = 169,$$

and $$xy = 60.$$

Solving these equations in the usual way, we obtain

$$x = 5, \text{ or } 12,$$
$$y = 12, \text{ or } 5.$$

We have thus two solutions of the equations but only one solution of the given problem, the double values of x and y being a mere interchange between the lengths of two sides of the same triangle.

Ex. 3. For each of a number of books was paid twice as many cents as there were books. Had 19 cents been paid for each, the total cost would have been 35 cents greater. How many books were in the set?

Let x = the number of books,

then $2x$ = " " " cents paid for each.

Therefore $$2x^2 + 35 = 19x,$$

or $$2x^2 - 19x + 35 = 0,$$

from which $$x = 7 \text{ or } 2\tfrac{1}{2}.$$

We have again two solutions of the equation but only one solution of the problem, since $2\frac{1}{2}$ is not a number used in counting books. If, however, we replace "books" by "yards of cloth" we shall obtain the same equation with the same roots and we shall then have two solutions, since $2\frac{1}{2}$ is perfectly intelligible when used in counting yards.

Ex. 4. If the circumference of a wheel were 4 feet greater it would make one revolution less in a distance of 80 feet. Find its circumference.

Let x = the circumference of the wheel in feet,

then $\frac{80}{x}$ = the number of revolutions in 80 feet,

and $\frac{80}{x+4}$ = " " after the increase.

Therefore $$\frac{80}{x} - \frac{80}{x+4} = 1.$$

Simplifying, $80(x+4) - 80x = x^2 + 4x$,

or $$x^2 + 4x - 320 = 0,$$

from which $$x = 16 \text{ or } -20.$$

Now -20, taken as the number of feet in the circumference of a wheel, gives no intelligible meaning. There is, therefore, but one solution to the problem.

Ex. 5. Divide a line 10 inches long so that the square on one segment may be double the square on the other segment.

Let x = the number of inches in the smaller segment,

then $10 - x$ = " " " larger "

Therefore $$(10 - x)^2 = 2x^2,$$

or $$x^2 + 20x - 100 = 0,$$

from which $$x = -10 \pm 10\sqrt{2}.$$

The positive root $10(\sqrt{2} - 1)$ gives the ordinary solution, but the negative root, $-10(\sqrt{2} + 1)$ may also be used to count inches along a given line. If, then, we count $10(\sqrt{2} + 1)$ inches in the *opposite direction*, we shall obtain an external point of section which also satisfies the given condition.

Ex. 6. Divide a line 10 inches long into two parts such that the sum of the squares on the two parts may be 40.

Let x and $10 - x$ denote the two parts.

Then $$x^2 + (10 - x)^2 = 40.$$

Simplifying, $$x^2 - 10x + 30 = 0,$$

from which $$x = 5 \pm \sqrt{-5}.$$

Both roots in this case are imaginary. Now imaginary expressions have a definite meaning, and are of great use in mathematics, but they are *not* used in counting units of measurement along a given line. The problem is, therefore, entirely impossible.

EXERCISE LXXII

PROBLEMS

1. The product of two consecutive numbers exceeds their sum by 89. Find the numbers.

2. The difference of two numbers is 5, and the square of their sum exceeds the sum of their squares by 300. Find the numbers.

3. The sides of a rectangle are as 3 : 4, and the diagonal is 75 inches. Find the sides.

4. Bought a number of apples for 50 cents. Had I received 5 more for the same money the price of each apple would have been one-half a cent less. How many did I receive?

5. If a train travelled 5 miles an hour faster it would complete a journey of 105 miles in 30 minutes less time. Find its rate in miles per hour.

6. Bought cloth for \$35, and by selling it for \$[illegible] a yard the gain was equal to the cost of one yard. Find the number of yards bought.

7. A brick is $3\frac{1}{2}$ inches longer than wide, and 14 [illegible] are

required to pave a rectangular area 20 feet long and 18 feet broad. Find the length of a brick.

8. A field of grain is 60 rods long and 40 rods wide. What uniform width must be cut from each side to leave $\frac{3}{4}$ of the grain standing?

9. The length and breadth of a room are as 3 : 2, and if 3 feet be added to each, the new area of the floor is to the former area as 35 : 24. Find the dimensions of the room.

10. Divide a line 10 inches long into two parts such that the square on one part may be double the square on the other part. Give the result correct to 3 decimal places.

11. Divide a line 20 inches long into two parts such that the square on one part may equal the rectangle contained by the whole line and the other part.

12. Produce a line 6 inches long so that the rectangle contained by the whole line thus produced and the part produced equals the square on the original line.

13. A piece of cloth on being washed shrinks in length one-sixteenth, and in width by one-tenth. Its area is thus reduced by 600 square inches, and its perimeter by 22 inches. Find its original length and width.

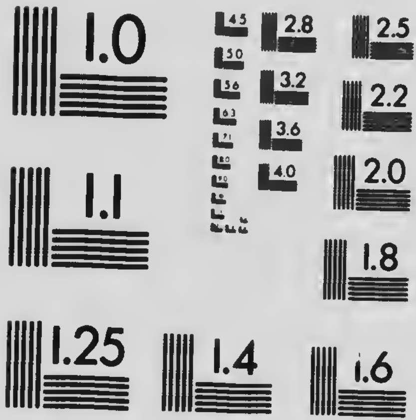
14. A vessel having two pipes can be filled in 2 hours' less time by one than by the other, and by both together in 1 hour $52\frac{1}{2}$ minutes. How long would each pipe alone require to fill the vessel?

15. Find the price of eggs per score when 10 more for 75 cents lowers the price $37\frac{1}{2}$ cents per 100.

16. The colored part of a square picture is surrounded by a white border $1\frac{1}{2}$ inches wide, and the whole by a frame 1 inch wide. The colored portion and the frame together contain $42\frac{1}{4}$ square inches. Find the area of the frame.

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



17. The area of a right-angled triangle is $\frac{3}{25}$ of the area on the square of its hypotenuse. Find the ratio of the sides containing the right angle.

18. A rectangle whose area is 168 square inches is inscribed in a circle whose radius is $12\frac{1}{2}$ inches. Find its sides.

19. Divide 40 into two parts such that the sum of the fractions obtained by dividing each part by the other may be $2\frac{1}{12}$.

20. The rate of an automobile is 10 miles per hour greater than that of a bicycle rider. If it takes the automobile 6 hours longer to run 255 miles than it takes the cyclist to ride 63 miles, find the rate of the automobile.

21. If the length and width of a rectangle be each increased by 3 inches, the diagonal will be increased from 13 inches to 17 inches. Find its sides.

22. A boatman can row $7\frac{1}{2}$ miles down a river and back again in 3 hours and 45 minutes. If the rate of the stream is $1\frac{1}{2}$ miles per hour, find his rate of rowing in still water.

23. Sold goods for $24, losing a rate per cent equal to the cost of the goods. Find their cost.

24. Find two numbers whose sum, product and difference of squares are all equal.

25. The sum of two numbers divided by their difference gives the same quotient as the greater number divided by the lesser. Find the quotient.

26. The hypotenuse of a right-angled triangle is $7\sqrt{2}$ and its area is $11\frac{1}{2}$. Find its sides.

27. The perimeter of a right-angled triangle is 40 feet and the difference of the sides containing the right angle is 7 feet. Find the area.

28. The fore wheel of a carriage turns 22 times more than the hind wheel in $\frac{3}{8}$ of a mile; if the circumference of each were increased by 1 foot it would turn only 18 times more. Find the circumference of each.

29. Two partners together invest \$2000 in business. The one receives \$1800 for stock and profit at the end of 2 months and the other \$900 at the end of 8 months. Find the amount invested by each, assuming that each man's profit is proportional to his capital multiplied by the number of months it remained in business.

30. An increase of 1 second in the time required to run 110 yards decreases the rate of running by $\frac{15}{16}$ mile per hour. Find the rate of running.

31. A stone dropped into a well is heard to strike the water in $3\frac{27}{40}$ seconds. Find the depth of the well, assuming that in t seconds the stone falls $16t^2$ feet and that sound travels 1120 feet per second.

32. Two trains start at the same time, one from A to B, the other from B to A. When they meet the former has travelled 12 miles further than the latter and reaches B $4\frac{2}{3}$ hours after meeting. The latter reaches A in $7\frac{5}{7}$ hours after meeting. How many miles from A to B?

33. An increase of $\frac{10}{11}$ mile per hour in the rate of a carriage would cause a wheel 16 feet in circumference to revolve in $\frac{1}{11}$ seconds less time. Find the rate of the carriage.

34. A rectangle has a strip 2 inches wide taken from one side and one end, and the area is thereby reduced by two-fifths. If a strip of the same width were taken off all around, the area would be reduced by seven-tenths. Find the original length and width of the rectangle.

EXAMINATION PAPERS

I

1. Find the value of $x+\frac{1}{x}$ when $x=\frac{3+\sqrt{5}}{2}$ and when $x=\frac{3-\sqrt{5}}{2}$. Why are the two results alike?

2. What values of n will make $3^n=9^5$, $2^{n+5}=4^n$?

3. Simplify $\frac{3+\sqrt{3}}{1+\sqrt{3}}+\frac{3+2\sqrt{3}}{2+\sqrt{3}}$, and $\frac{1}{2-\sqrt{2}}\div\frac{\sqrt{2}+1}{\sqrt{2}-1}$.

4. If $x+y+z=xyz$, prove that $(1-x)^2=(1-xy)(1-xz)$.

5. Solve the equations $x+\sqrt{x}=20$ and $x-\sqrt{x}=20$. Why do both equations give the same pair of roots?

6. Solve equations $\frac{1}{x}+\frac{1}{y}=\frac{7}{20}$, $xy=40$.

7. The perimeter of a right-angled triangle is 60 feet, and the area is 120 square feet. Find its sides.

II

1. Find the value of x^2+xy+y^2 when $x=3+\sqrt{5}$, $y=3-\sqrt{5}$.

2. Simplify $\frac{1}{4}\{(x^{n-1}+x)^2-(x^{n-1}-x)^2\}$.

3. Simplify $\frac{\sqrt{3}-1}{2\sqrt{2}}\left\{6-\frac{(\sqrt{3}-1)^2}{2}\right\}$.

4. Find the square root of $x(x+1)(x+2)(x+3)+1$. What property of numbers is proved by this example?

5. Solve equation $(4+2\sqrt{3})x^2+(\sqrt{3}+1)x=2$, and verify the result.

6. Solve equations $\frac{1}{x}+\frac{1}{y}=\frac{x+y}{10}=\frac{7}{x+y+3}$.

7. A boatman who can row 6 miles per hour in still water, requires half an hour to row 1 mile down a river and back again. Find the rate of the stream in miles per hour correct to three places of decimals.

III

1. Find the value of $3x + 3x^2 - x^3$ when $x = 2 - \sqrt{3}$.

2. If $a = 25$, $b = 16$, write down the value of

$$a^{\frac{1}{2}} - b^{\frac{1}{2}},\ (a-b)^{\frac{1}{2}},\ (a^{-1} - b^{-1})^{-1},\ (a^{-\frac{1}{2}} + b^{-\frac{1}{2}})^{-\frac{1}{2}}.$$

3. Find the square root of

$$a^2(a^2 + b^2 + c^2) + b^2c^2 + 2a(b + c)(bc - a^2).$$

4. If $\frac{1}{a}(bx^2 + a^2y) = \frac{1}{b}(ay^2 + b^2x)$, then either $ay = bx$ or $bx + ay = ab$.

5. Solve the equation $x^2 + 6\sqrt{x^2 - 2x + 5} = 11 + 2x$, and verify the result.

6. The perimeter of a rectangle is 50 feet 6 inches, and its area is 85 square feet. Find its sides. With the same perimeter, could its area be 160 square feet?

7. *A* starts from P to Q at the rate of 8 miles per hour. Three hours later, *B* starts from Q to P at such a rate as to reach P in 19 hours. They meet when *B* has travelled as many hours as he travels miles in one hour. How many miles from P to Q?

IV

1. If $x = \frac{1}{2}(1 \pm \sqrt{5})$ find the value of $\frac{1}{2}(x^2 + 1)(x^2 - 1)$

2. Multiply

$$(a+b)^{\frac{2}{3}} - (a^2 - b^2)^{\frac{1}{3}} + (a-b)^{\frac{2}{3}} \text{ by } (a+b)^{\frac{1}{3}} + (a-b)^{\frac{1}{3}}.$$

3. Extract the square root of

$$\left(x^3 - 2 + \frac{1}{x^3}\right) \div \left(x - 2 + \frac{1}{x}\right).$$

4. If p be the difference between any number and its reciprocal, q^2 the difference between the square of the same number and the square of its reciprocal, show that $q^4 - p^4 = 4p^2$.

5. Solve the equation $2x^2 - 3\sqrt{x^2 + 2x + 11} + 4x = 49$.

6. Solve the equations $x^3 + y^3 = 133$, $xy(x + y) = 70$.

7. Find, correct to 3 places of decimals, the sides of a rectangle whose length exceeds its breadth by half an inch, and whose area is one square inch.

V

1. Simplify $\frac{1}{2}\sqrt{2 + \sqrt{2}} \times \sqrt{4 - 2\sqrt{2}}$ and $(7 - 4\sqrt{3})^{\frac{1}{2}}$.

2. Simplify $2^n + 2^n$, $2^n - 2^{n-1}$, $4(2^{n-1})$, $\frac{1}{4}(8^n)$, and verify the results when $n = 3$, 4, and 0.

3. If $x = a^2 - bc$, $y = b^2 - ca$, $z = c^2 - ab$, show that

$$\frac{x^2 - yz}{a} = \frac{y^2 - zx}{b} = \frac{z^2 - xy}{c} = ax + by + cz.$$

4. The sum of the squares of two numbers is m times their product, and the difference of their squares is n times their product. Find the value of $m^2 - n^2$.

5. Solve the equation $p + x + \sqrt{2px + x^2} = q$, and verify the result.

6. Solve equations $\frac{x}{y} - \frac{y}{x} = \frac{x + y}{xy} = \frac{9}{20}$.

7. A rectangle is one inch long. Find its side, providing that when a square whose side equals the width of the rectangle is removed the area of the remainder may be one-half a square inch.

VI

1. Simplify
$$\frac{2+\sqrt{3}}{3\sqrt{12}-5\sqrt{3}+5\sqrt{2}-\sqrt{32}},$$
and express its value to 4 places of decimals.

2. Divide $a^2 - a^2b^{-2} - 1 + b^{-2}$ by $a + ab^{-1} + 1 + b^{-1}$.

3. Extract the square root of
$$2a^2(b-c)^2 - b^2(c-a)^2 + 2c^2(a-b)^2.$$

4. Solve the equation $x + 4\sqrt{2x-3} + 6 = 0$, and also $x - 4\sqrt{2x-3} + 6 = 0$. Why have two apparently different equations the same roots?

5. Solve the equations $x^3 - y^3 = 63$, $xy(x-y) = 12$.

6. If $\sqrt{2} - 1$ is a root of $nx^2 - 2x\sqrt{2} + 1 = 0$, find n and the remaining root.

7. Two cyclists set out at the same time, on the same road, the one from A to B, and the other from B to A. They reached their destinations in 2 hours 40 minutes and 1 hour 30 minutes, respectively, after meeting. How long after starting before they met? If the difference in their rates was 2 miles per hour, find the distance from A to B.

VII

1. Simplify
$$\frac{\sqrt{6-2\sqrt{5}}}{2+\sqrt{14-6\sqrt{5}}}.$$
and express its value to 4 places of decimals.

2. Divide $x^3 + x^{-3} + 3(x + x^{-1})$ by $x + x^{-1}$, and express the quotient as a square.

3. Simplify
$$(x-1+\sqrt{2})(x-1-\sqrt{2})(x+1+\sqrt{2})(x+1-\sqrt{2}).$$

4. Solve the equation $x + \sqrt{7x - 19} = 1$, and verify the result.

5. Solve $x^2y^2 - 6xy - 72 = 0$, $\frac{x-1}{5} + \frac{y+1}{3} = 2$.

6. If $\frac{1+\sqrt{3}}{2}$ is one root of $2x^2 - nx - 1 = 0$, find n and the remaining root.

7. The difference in the circumferences of the front and the hind wheels of a carriage is 3 feet. When the carriage moves at the rate of 10 miles per hour the front wheel makes one-half a revolution per second more than the hind wheel. Find the circumference of each wheel.

CHAPTER XIX

GRAPHIC SOLUTIONS OF SECOND DEGREE EQUATIONS

262. The graph of a second degree expression. In the chapter on Elementary Graphs we showed how the solution of first degree equations of one and two unknown quantities could be effected. We shall show in this chapter how the same principles can be applied to the solution of equations of the second degree.

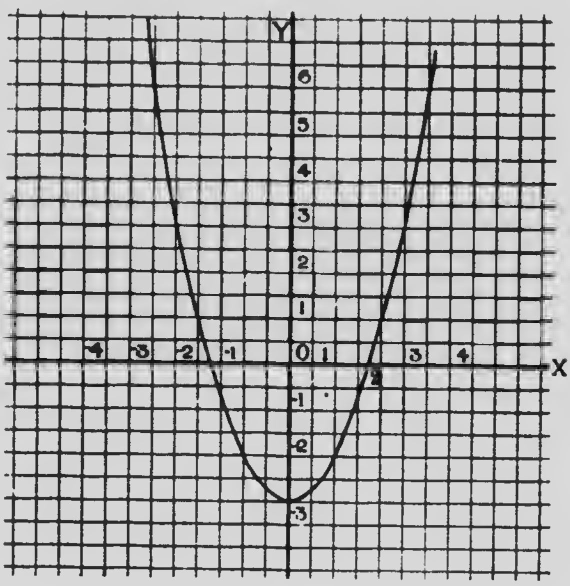


Fig. 1.

Ex. 1. Plot the graph of $x^2 - 3$.

Let $$x^2 - 3 = y.$$

Assigning values to x, we get the corresponding values of y by calculation.

Thus when $x = -3, \ -2, \ -1, \ 0, \ 1, \ 2, \ 3,$

$y = \ 6, \ 1, \ -2, \ -3, \ -2, \ 1, \ 6.$

These values plotted give the curve shown in Fig. 1

In plotting the graph, **two** of the small divisions are taken as the **unit** of length.

The points where the curve cuts the axis of x give the solution for the equation $x^2 - 3 = 0$.

These values, of course, are at once seen from the equation to be $\pm\sqrt{3} = \pm 1.7$. It will be noticed that there are only two roots possible.

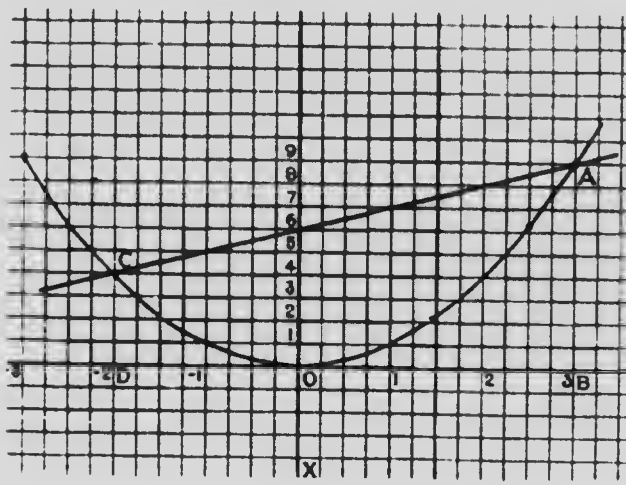


Fig. 2.

Ex. 2. Plot the graph of $y = x^2$.

Assigning values to x, and calculating the corresponding values of y, we obtain the following set of values:

$x = -3, \ -2\frac{1}{2}, \ -2, \ -1\frac{1}{2}, \ -1, \ -\frac{1}{2}, \ 0, \ \frac{1}{2}, \ 1, \ 1\frac{1}{2}, \ 2, \ 2\frac{1}{2}, \ 3.$

$y = \ 9, \ 6\frac{1}{4}, \ 4, \ 2\frac{1}{4}, \ 1, \ \frac{1}{4}, \ 0, \ \frac{1}{4}, \ 1, \ 2\frac{1}{4}, \ 4, \ 6\frac{1}{4}, \ 9.$

As the values of y increase much more rapidly than those of x, take four of the small divisions as the unit of measurement along the axis of x, and one along the axis of y. This scale must also be observed in taking measures from the curve.

The curve is shown in Fig. 2.

It will be observed that the graph is symmetrical with regard to the y axis, and that it touches the x axis at the origin.

This graph is of special interest because of its assistance in solving graphically quadratic equations of one unknown quantity.

263. Graphic solution of equations of one unknown quantity. The solution of an equation of one unknown quantity may be performed in two ways:

(1) As in Ex. 1, by equating the whole expression to y, plotting the graph and finding the points where the y axis is cut by the graph.

(2) When the expression contains terms of both the first and second degree, the work can be greatly lessened by using the graph plotted in Ex. 2.

This solves graphically the equation

$$x^2 - x - 6 = 0 \text{ or } x^2 = x + 6.$$

If we draw the graphs $y = x^2$ and $y = x + 6$ separately, the particular values of y common to the two graphs are the values for which $x^2 = x + 6$, and hence the corresponding values of x are the roots of the equation.

The graph of $y = x + 6$ is plotted on the same diagram as $y = x^2$, and on the same scale. It is the line AC, Fig. 2.

Hence $x = OB$ or 3, and $x = OD$ or -2 are the two values of x corresponding to the common values of y. Hence the roots are 3, and -2.

The student will notice that this method does not give t graph of $x^2 - x - 6$, but it solves the equation $x^2 - x - 6$ The method is applicable to any quadratic of one unkno quantity, and has the advantage that the general eq $y = x^2$ is common to them all, and its graph need not redrawn. It is only necessary to plot a line which can done by means of two points.

264. Graphic solution of simultaneous equations. T is effected by plotting the graph of each and taking points of intersection.

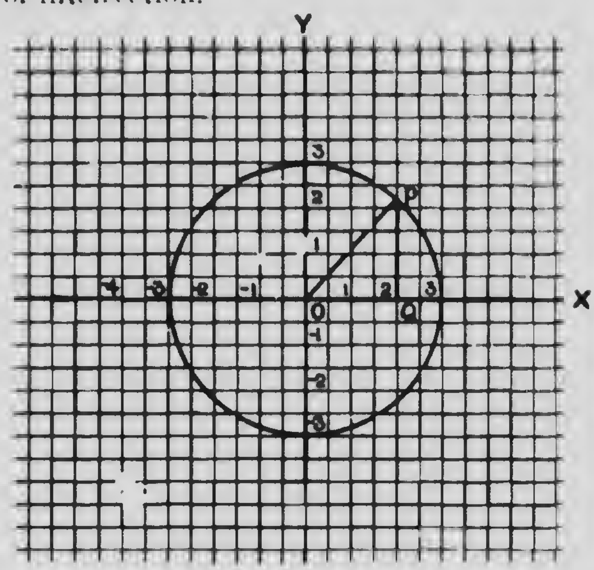


Fig. 3.

Ex. 1. Plot the graph of $x^2 + y^2 = 9$.

Solving for y, $y = \pm\sqrt{9 - x^2}$.

Then when

$x =$	-4,	-3,	-2,	-1,	0,	1,	2,	3,	4
$y =$	$\pm\sqrt{-7}$,	0,	± 2.2,	± 1.8,	± 3,	± 1.8,	± 2.2,	0,	$\pm$

Making the unit of length two divisions of the section paper. The graph is shown in Fig. 3.

The student will observe that the graph is a circle, the radius of which is 3. The graph can therefore be plotted at once with a pair of dividers.

All second degree equations of the form $x^2 + y^2 = r^2$ are circles and can be so plotted. This fact can be deduced at once from the graph of the circle. For if any ordinate PQ be taken, OP being the radius, then geometrically

$$\overline{OQ}^2 + \overline{PQ}^2 = \overline{OP}^2,$$

or

$$x^2 + y^2 = r^2.$$

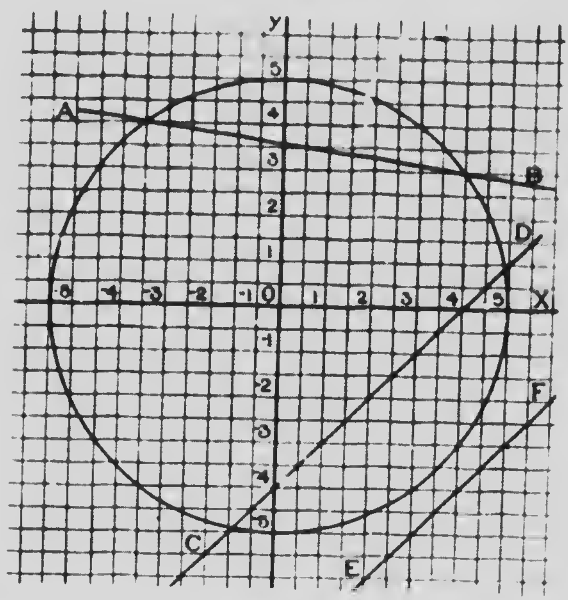


Fig. 4.

It will also be observed that for any value of x greater than 3, the value of y is **an imaginary quantity**, no point for which x is greater than 3 being on the circle.

Thus when $x = 4, \; y = \pm\sqrt{-7}.$

Ex. 2. Solve the equation

$$x^2 + y^2 = 25, \quad (1)$$

$$x + 7y = 25. \quad (2)$$

The graph of (1) can be plotted at once by the dividers, since it is a circle of radius 5.

In (2), when $x = 4, \ -3,$

$y = 3, \ \ 4.$

Plotting for the two points on the same scale as the circle, we have the line AB. The points where the line and circle cut each other give the common values of x and y, and hence the solution of the equations

$$x = 4, \ -3, \quad y = 3, \ 4.$$

265. Draw on the same diagram the graphs of the equations

$$x - y = 4, \quad (1)$$

$$x - y = 8. \quad (2)$$

The graph of (1) is the line CD.

The " (2) " EF.

The solution of the equations

$$x^2 + y^2 = 25,$$

and $x - y = 4,$ as shown by the common points, is $x = 4.9, \ -.9,$

$$y = .9, \ -4.9.$$

The graph EF does not cut the circle, and there are no points in common. Hence no solution can be obtained graphically.

Solving the equation, however, we obtain

$$x = \frac{8 \pm \sqrt{-14}}{2}, \quad y = \frac{-8 \pm \sqrt{-14}}{2}.$$

Here the roots are found to be **imaginary** quantities.

It can be shown to be a general proposition, that when the graphs of two equations do not cut, their roots are imaginary.

Ex. 3. Solve the equations

$$xy = 12, \qquad (1)$$

$$2x - 3y = 14. \qquad (2)$$

In (1), $y = \frac{12}{x}$.

Hence when

$x = -12, -8, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6$, etc.

$y = -1, -1\frac{1}{3}, -2, -3, -4, -6, -12, 12, 6, 4, 3, 2$, etc.

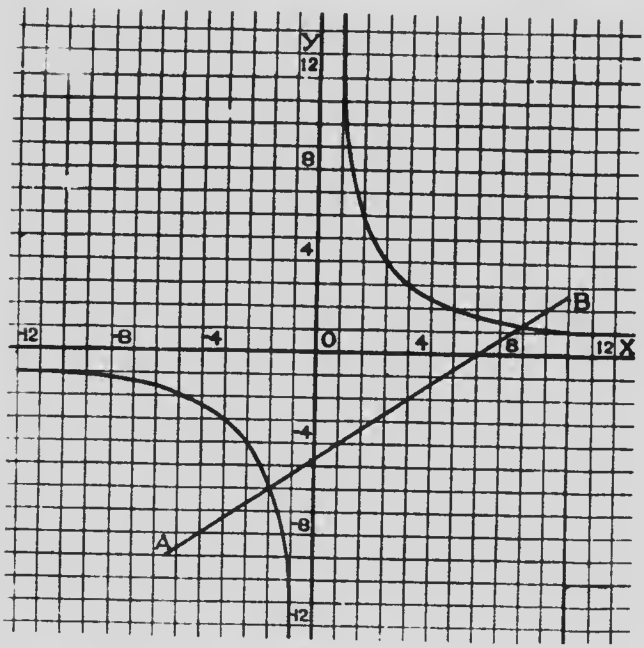


Fig. 5.

Plotting these points and drawing a graph through them, we have the two-branched curve shown in Fig. 5.

These branches are completely separated—all the values of x and y in one branch being positive, and in the other negative.

Equation (2) plotted on the same diagram, gives the line AB.

The two roots are $x = -2$, and 9,

$y = -6$, and $1\frac{1}{3}$,

the scale being one square per unit.

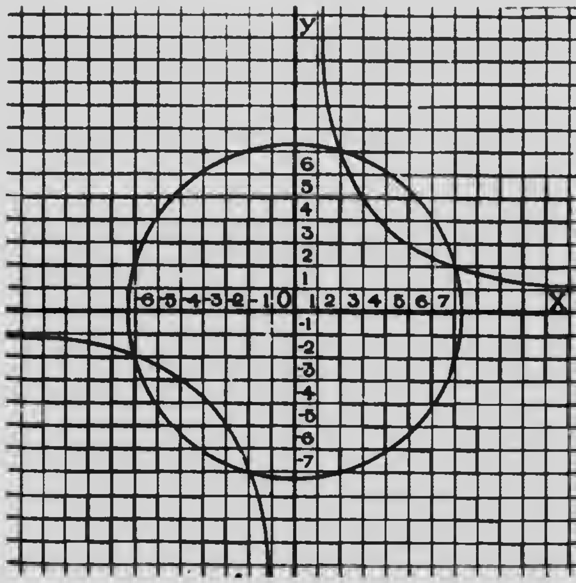


Fig. 6.

The student will observe that a line may be drawn to cut either branch in two points, or both in one, but that only two points can be common; that is, there are only two solutions.

It will be observed also that the smaller value we give x, the larger value do we get for y, and vice versa.

When x is very small, y is very large.
When x is very large, y is very small.

This shows that the branches reach out continuously in the four directions.

Ex. 4. Solve the equations

$$x^2 + y^2 = 53, \qquad (1)$$

$$xy = 14. \qquad (2)$$

The radius of the circle (1) is $\sqrt{53} = 7.28$. The circle can be described at once (Fig. 6).

Equation (2) can be plotted as before, from the following values derived from $y = \frac{14}{x}$.

$x = -7$,	-6,	-5,	-4,	-3,	-2,	1,	2,	3,	4,	5,	6,	7.
$y = -2$,	-2.3,	-2.8,	$-3\frac{1}{2}$,	$-4\frac{2}{3}$,	7,	14,	7,	$4\frac{2}{3}$,	$3\frac{1}{2}$,	2.8,	2.3,	2.

These values plotted give the two-branched curve as in the last case.

These equations have four solutions, two on each branch, produced from equation (2). The solutions are

$$x = \pm 7, \quad \pm 2,$$

$$y \pm 2, \quad \pm 7.$$

The graph shows how it is that two second degree equations of two unknown quantities have four solutions.

EXERCISE LXXIII

Plot the equations

1. $y = 2x$. 2. $y = 3x$. 3. $y = \frac{x}{3}$.

4. $y^2 = 4x$. This curve is a parabola.

5. $\frac{x^2}{9} + \frac{y^2}{4} = 1.$ This curve is an ellipse.

6. $\frac{x^2}{9} - \frac{y^2}{4} = 1.$ This curve is an hyperbola.

7. Solve the equations

$$y = 2x,$$
$$y^2 = 4x, \text{ graphically.}$$

8. Solve the equations

$$\frac{x^2}{9} + \frac{y^2}{4} = 1, \quad y = \frac{x+1}{2}.$$

(Get an approximate result.)

9. Solve graphically

$$\frac{x^2}{9} - \frac{y^2}{4} = 1, \text{ and } y = \frac{x+1}{3}.$$

(Get an approximate result.)

10. Plot $x^2 + x - 2$, and solve equation $x^2 + x - 2 = 0$

11. Plot $4x^2 - 3x = 10$, and solve the equation

$$4x^2 - 3x - 10 = 0.$$

12. Using the graph of $y = x^2$, solve the equation

$$x^2 - x = 20.$$

ANSWERS AND RESULTS

Exercise I. Page 15

1. 8, 16, 12, 64, 48, 144.
2. 15, 45, 75, 225, 34, 64.
3. 6, 5, 7, 12, 15, 20.
4. 7, 5, 9, 5, 6.
5. 6.
6. 8.
7. 11.
8. 20.
9. 9.
10. 90.
11. 26.
12. 16.
13. 25.
14. 88.
15. 5.
16. 0.
17. $\frac{1}{6}$.
18. 10.
19. 8.
20. $1\frac{5}{6}$.
21. 16.
22. $7\frac{1}{2}$.

Exercise II. Page 17

2. 3, 5, 7, 8.
3. 7, 11, 6, 9, 12.
4. 5, 4, 6, 5.
6. 4^3, 2^6.
7. 3^4, 9^2; 3^6, 9^3.
8. 5^4, 4^4.
9. 32, 25, 16, 27, 1, 15625.
10. 8, 4.
11. 32, 8, 9, 3.
12. 4, 3.
13. 4, 3, 27, 4.
14. 36.
15. 64.
16. 248, 7.
17. 6, 30.
18. 12.

Exercise III. Page 19

1. $x+5$, $5x$.
2. $10-7$, $10-x$.
3. $n+2$, $n+x$.
4. $n+5$, $n+y$.
5. $2x+2$.
6. $2a+2b$, $a-b$.
7. $m-p-q$, $m-(p+q)$.
8. \$$3n$, \$$nx$.
9. $2n+3p$.
10. $100x$, $\dfrac{x}{100}$.
11. $100b-nx$, $b-\dfrac{nx}{100}$.
12. $12x+y$, $36x+12y$.
13. $5m$, mx, $\dfrac{mp}{60}$.
14. $\dfrac{m}{4}$ hrs., $\dfrac{m}{x}$ hrs.
15. $nq+mp$, $\dfrac{(nq+mp)x}{100}$.
16. $\dfrac{x+y}{6}$, xy, $\dfrac{xy}{144}$.
17. $4x+2$, $\dfrac{2x+1}{6}$.
18. $x+y+10$, $x-y$

19. $7(10) + 5$; $10x + y$.
20. 4, 11, 5.
21. $35x + 50$.
22. 10.
23. $6x^2$, 3.
24. xyz, $2(xy + yz + xz)$, $4(x + y + z)$.
25. $2n + 1$, $2n - 1$
26. $xy + x$.
27. $(a + b)^2 = a^2 + b^2 + 2ab$.
28. $(a - b)^2 = a^2 + b^2 - 2ab$.
29. $\frac{a^3 - b^3}{a - b} = a^2 + b^2 + ab$.
30. $(a + b)(a - b) = a^2 - b^2$.

Exercise IV. Page 24

1. $+18$, -90.
2. 30 ft. north, 15 ft. south, $7\frac{1}{2}$ ft. south, 1 ft. north.
3. 5 ft. up, 10 ft. down.
4. \$2.50 gain; \$3.25 loss.
5. $-2\frac{3}{4}$, $+3\frac{7}{8}$, $+4\frac{1}{8}$, $-2\frac{1}{2}$.
6. -1, $+2$.
7. $-2a$, $3a$.
8. -10, $+3$, -4.
9. \$12 cash $= +12$, \$5 debt $= -5$.
10. $+17$, $-30\frac{1}{2}$
11. -2 oz., 14 oz., 2 oz.
12. Halved, doubled, sign changed.

Exercise V. Page 30

1. $+8$, -8, -2, $+2$, -3, $+3$.
2. $+50$, -50, $+18$, -18, $+10$, -10.
3. a, $9b$, $-4a^2$, $-10ab$, $24xy$, $7m$.
4. $4x^2$.
5. $-7x^2 + 3y^2$.
6. $-6m^2 - 10n^2$.
7. $-3ab$.
8. $-a^2b - 4ab^2$.
9. $-6(a + b)$.
10. 6, -8, -12.
11. $10 - 25 + 5 = -10$ 10 miles west, the end of the journey. $10 + 25 + 5 = 40$ miles, the distance travelled.

Exercise VI. Page 32

1. $6a + 4b - 4c$.
2. $2a + 4b + 11c$.
3. $9a - 5b + 2c$.
4. $8a - 2b - 4c - 4x$.
5. $5ab + 2ac + 2bc$.
6. $-ax - 3bx + cx$
7. 0.
8. $7(a + b)$.
9. $a + b - c$.
10. $9(a^2 + b)$.
11. $6a(b + c) + e + x$.
12. $9a - 4b - 17c - 12d + 1$[illegible].
13. $3(a^2 + b^2) + 13ab$.
14. $-a^3 + 12a^2b - 2ab^2$.
15. $2a^4 - a^3b - 5a^2b^2 + 5ab^3 - 7b^4$.
16. $15x - 7y + 3z$.
17. $-4x - 13y + 23z$.
18. $-4a + 6b + 6c$.
19. $-11a + 13b + 10c$.

Exercise VII. Page 36

1. $8,\ -8,\ -2,\ +2,\ 3,\ -3.$ **2.** $-9,\ 14,\ 3,\ -7,\ -4,\ +7.$
3. $10x,\ +y,\ 19a^2,\ -9ab,\ 3a-5b,\ 4x+3y.$
4. $4ax,\ 7by,\ 4a.$ **5.** $-2x^2,\ -10y^2,\ 2xy.$ **6.** $-10a^2,\ 0.$
7. $-9a+5b.$ **8.** $5,\ -9,\ -15.$ **9.** $-2,\ 11,\ -9,\ -9.$
10. $-10(x-y)^2.$ **11.** $+44-(-23)=67.$

Exercise VIII. Page 37

1. $2a-4b+4c.$ **2.** $-a+4b+d.$ **3.** $-2b+3c-x.$
4. $4x^3-7x^2y+3xy^2.$ **5.** $2+3x-6x^3.$
6. $2a^2-2b^2+2bc-2ac.$ **7.** $3x^2-4xy+8y^2+2xz-yz.$
8. $2x^2+xy.$ **9.** $-x^2y-4y^3.$
10. $-a^2-b^2+c^2-ab+2bc+2ac.$ **11.** $-a+2b-c+ab^2.$
12. $-x^3+15x^2y+7y^3.$ **13.** $3(a-b)+13(x-y).$
14. $2(a+b)+(c+d)+2(x-y)+p+q.$ **15.** $11a-4b-3c.$
16. $a-4b-2c-d-6e.$ **17.** $a-16c-5d+21e.$
18. $3a-3b-c.$

Exercise IX. Page 40

1. $2a.$ **2.** $2b.$ **3.** $2a-2c.$
4. $-b+2c.$ **5.** $6a-3c.$ **6.** $3a-4b+4c.$
7. $2a-4b-c.$ **8.** $x+2a+3.$ **9.** $x-4a.$
10. $-2y-2.$ **11.** $2a-b.$ **12.** $x-y+3z.$
13. $-2b+2c.$ **14.** $7a-6b.$ **15.** $5a-4b.$
16. $(a-b)+(c+d)-(e+f);\ (a-b+c)+(d-e-f).$
17. $a-(b-c-d)-(e+f;\ a+(-b+c+d)+(-e-f).$
18. $a-\{(b-c)-(d-e)+f\}.$ **20.** $0.$

Exercise X. Page 46

1. $-15,\ -15,\ 15,\ -77,\ 40.$
2. $-15x^2,\ -6a^2bc,\ -20x^2y,\ 6a^3b^3,\ 5a^3bx.$
3. $-x^3y^2,\ 91x^2y^3z^2.\ x^3y^4,\ -8m^2n^2x,\ -35a^3b^3c.$
4. $-30a^4b^4c^4.$ **5.** $120a^3b^3x^3$ **6** $25,\ 8,\ 54.$ **7.** $0,\ 32.$
8. $-1,\ 5.$ **9.** $9a^6,\ 8a^6,\ 16a^4b^8,\ -243a^{10}b^{15}.$
10. $-3,\ 7,\ -10,\ -70,\ 29.$ **11.** $120.$

Exercise XI. Page 49

1. $2x^3 - 4x^2 + 6x$.
2. $-9x^3 - 12x^2 + 6x$.
3. $2x^2y - 4xy^2 + 2y^3$.
4. $-8a^3b + 12a^2b^2 - 4ab^3$.
5. $-x + 2x^2 - 3x^3$.
6. $x^2y^2z + xy^2z^2 - x^2yz^2$.
7. $-21a^3bx^2 + 6a^2b^2xy + 9abx^2y^2 - 12ab^3xy$.
8. $-6a^3bc + 9ab^3c + 3abc^3 + 6a^2bc^2 + 12ab^2c^2 - 6a^2b^2c$.
9. $a^2b - a^3b + a^2b^2 - a^3bc + a^2b^2c - a^3b^2c$.
10. $8x^3 - 11x^2 + 12x$.
11. $-3x^3 - 16x^2 + 11x - 4$.
12. $6ab - 13b^2$.
13. $2a^3 - 11a^2b + 10ab^2 - 4b^3$
14. $14a - 4b - 14c$.
15. $-2bc + c^2$.
16. $2x^2 + 10xy - 12y^2$.
17. $2px$.
18. $2bx + 2by$.
19. 0.
20. $cx + ay + bz$.

Exercise XII. Page 52

1. $6x^3 - 7x^2 + 11x - 6$.
2. $2x^3 + 3x^2 - 8x + 3$.
3. $20x^3 - 18x^2 - 25x - 6$.
4. $-3x^3 + 5x^2 + 7x - 10$.
5. $x^3 - 8$.
6. $a^3 + 1$.
7. $a^3 - b^3$.
8. $a^3 + b^3$
9. $a^4 - 4a^2 + 12a - 9$.
10. $4a^4 - 13a^2b^2 + 9b^4$.
11. $6x^5 - x^4 + 4x^3 + 2x^2 - 7x + 2$.
12. $6x^5 - 5x^4 - 32x^3 + 5x^2 + 19x - 5$.
13. $1 - 4x^2 + 12x^3 - 5x^4 + 8x^5 - 12x^6$.
14. $2 - 8x + 13x^2 - 11x^3 + 6x^4 - 3x^5 + x^6$.
15. $-3x^5 + 8x^4 + x^3 + 3x^2 + 10x - 3$.
16. $x^6 - 1$
17. $a^8 + 2a^6 + 3a^4 + 2a^2 + 1$.
18. $x^8 - 3x^7 - 2x^6 + 11x^5 - 14x^4 + 3x^3 + 16x^2 - 29x + 21$.
19. $x^3 + 3xy - y^3 + 1$.
20. $a^3 + b^3 + c^3 - 3abc$.
21. $2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4$.
22. $x^3 - 8y^3 + z^3 + 6xyz$.
23. $a^8 - 4a^6b^2 + 6a^4b^4 - 4a^2b^6 + b^8$.
24. $x^{12} - x^9y^3 + x^6y^6 - x^3y^9 + y^{12}$.
25. $x^2 + 19x + 4$.
26. $5 + x$.
27. -12.
28. $2a^2 - 2b^2$.
29. 0.
30. $a^3 - b^3$.
31. $b^4 + 2a^2b^2$.
32. $2b^4 - 32$.

Exercise XIII. Page 56

1. $-4, \ -4, \ 4, \ 4.$
2. $-54, \ -30, \ 5, \ -21.$
3. $4ab^2, \ -4a^2b, \ 7x, \ -6n^4.$
4. $-21x^4y^5, \ -4ax^2, \ -5b^3, \ -6a^3b^2c^2.$
5. $3ab, \ ab.$
6. $ab, \ -15yz^2.$
7. $2x^2 - 3x + 4.$
8. $-3y^2 - 4y + 2.$
9. $a^2 - 2ab + 3b^2.$
10. $-a^2 + 2ab + 4c.$
11. $3x^3 - 5x^2y + 6xy^2 + y^3.$
12. $7x^2z^2 - 9xy + 8y^2z.$
13. $3 - 4(a+b) + 5(a+b)^2.$
14. $x^2 - z(x-y) + y^2.$
15. $(3b - a)(a + b).$

Exercise XIV. Page 60

1. $x + 7.$
2. $x - 3.$
3. $x - 8.$
4. $x - 9.$
5. $2x + 3.$
6. $x^2 + 2x - 3.$
7. $a + b.$
8. $a^2 + ab + b^2.$
9. $a^2 - ab + b^2.$
10. $a - b.$
11. $x^2 + 3x + 2.$
12. $2x^2 - 3x + 7.$
13. $x + 5.$
14. $a^3 - 3a^2 + 3a + 1.$
15. $3a^2 + 4ab + b^2.$
16. $2x^3 - 5x^2 - 2x + 9$, Rem. $9x - 24.$
17. $3x^4 - 5x^3 + 2x + 3$, Rem. 1.
18. $a^2 - ab + b^2.$
19. $a^4 + a^2b^2 + b^4.$
20. $a^{10} - a^5b^5 + b^{10}, \ a^{12} - a^9b^3 + a^6b^6 - a^3b^9 + b^{12}.$
21. $a^5 - b^5, \ a^6 - b^3.$
22. $x^4 + 2x^3 + 3x^2 + 2x + 1, \ x^2 + x + 1.$
23. $x^4 + x^2 + 1, \ x^3 + 2x^2 + 2x + 1.$
24. $a^2 + 2ab + b^2.$
25. $x^3 - 2x^2y + 2xy^2 - y^3.$
26. $-8a^4 + 8a^2b^2 + 3ab^3 + b^4.$
27. $x^3 + px.$
28. $x^2 + mx - n.$
29. $a^2x^2 + 2abxy + b^2y^2.$
30. $3a^2x - 4ax^2 + 16x^3.$
31. $x^2 - 3x + 2.$
32. $x^5 - 2x^4 + 8x^2 - 16x.$
33. $x^3 + 2x^2 + 3x - 1.$
34. $a^2 + 2ab + b^2.$
35. $a + b + 3.$
36. $a^2 + b^2.$
37. $a^2 - ab + b^2 + a + b + 1.$
38. $a^2 - 2a + 1.$
39. $x^3 + 4x^2 + 8x + 6$, Rem. $13x - 10.$
40. $x^3 - 5x^2 + 7x.$

Exercise XV. Page 63

1. $(a + m + p)x + (b + n + q)y.$
2. $(4a + c)x + (7 - 3b)y.$
3. $(m - 1)x - (x - 1)y.$
4. $(a^2 + ab + b^2)x + (a^2 - 2ab - b^2)y.$

5. $(b+1)x+(a+4)y$. 6. $nx-my$. 7. $-nx$...
8. $2(a-c)x-2(a-b)y$. 9. $(a-p)x^2+(b+q)xy-(c+r)y^2$
10. $(c-b)x^2+(a+b)xy-(c-b)y^2$.
11. $(a-b-1)x+(b-1)y-(a-1)z$.
12. $x^2+(a+b)x+ab$. 13. $x^2-(a+b)x+ab$.
14. $x^2+(a-b)x-ab$. 15. $x^2-(a-b)x-ab$.
16. $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$.
17. $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$.
18. $x^3+(a-b+c)x^2-(ab+bc-ac)x-abc$.
19. $x^3-(a-b+c)x^2-(ab-ac+bc)x+abc$.
20. $3x^2-(ab+bc+ca)$. 21. $a+b-c$.
22. $a^2+b^2+c^2-ab-ac-bc$. 23. $4a^2+b^2+c^2+2ab-2ac+bc$.
24. $x+2y-z$. 25. $x-b$. 26. $x-a$.
27. $x^2-mx+mn$. 28. $x^2+(b+c)x+bc$, $x^2+(a+c)x+ac$.
29. $x^2-(a+c)x+ac$, $x^2-(a+b)x+ab$. 30. $x+b$.
31. $a(b+c)-2bc$. 32. $a^2+2ab+b^2-c^2$, $-a^2+b^2+2bc+c^2$.

Exercise XVI. Page 69

1. $x=12$. 2. $x=9$. 3. $x=27$. 4. $x=47$.
5. $x=2$. 6. $x=2$. 7. $x=-16$. 8. $x=-1$
9. $x=-3$. 10. $x=2$. 11. $x=0$. 12. $x=6$.
13. $x=-8$. 14. $x=-2\frac{1}{4}$. 15. $x=31$. 16. $x=98$.
17. $x=3$. 18. $x=\frac{3}{10}$. 19. $x=15$. 20. $x=3$.
21. $x=4\frac{1}{3}$. 22. $x=2$. 23. $x=8a$. 24. $x=a+b$.
25. $x=0$. 26. $x=a+b$. 27. $x=\frac{a}{2}$. 28. $x=\frac{b}{2}$.
29. $x=1$. 30. $x=b-a$. 31. $x=a+b$. 32. $x=b-a$.
33. $x=c-a-b$. 34. $x=a+b$. 35. $x=c$. 36. $x=a+b-c$.

Exercise XVII. Page 72

1. 12. 2. 46, 29. 3. 10. 4. 71 cts., 54 cts.
5. 57 cts. 6. 11 ft. 7. 19 ft. 8. \$13.
9. 30 yrs. 10. 40. 11. 25, 17, 14. 12. 39, 3…

13. 452, 423. 14. 24, 29, 36. 15. \$1.88, \$1.93, \$3.83.
16. \$100. 17. 55, 35. 18. 10 19. 28, 5.
20. 21. 21. \$560, \$460. 22. 19, 24.
23. 54 cts., 40 cts. 24. 20 cts. 25. 40 cts. 26. 40 cts.
27. 85, 51, 33. 28. \$3.20. 29. 15. 30. \$3.80.
31. 60. 32. \$67. 33. 23. 34. \$5.00. 35. $41\frac{2}{3}$.
36. 60 gals. 37. 400 sq. in. 38. 324 sq. in.
39. 12800 sq. yds.

Exercise XVIII. Page 78

1. $x = 6$. 2. $x = 7\frac{3}{8}$. 3. $x = 12$. 4. $x = 3$.
5. $x = 1$. 6. $x = 3\frac{1}{8}$. 7. $x = \frac{1}{3}$. 8. $x = 3\frac{1}{4}$.
9. $x = 2$. 10. $x = 9$. 11. $x = 1$. 12. $x = \frac{3}{2}$.
13. $x = \frac{9}{16}$. 14. $x = -5$. 15. $x = 0$. 16. $x = 31$.
17. $x = 2$. 18. $x = 2\frac{1}{4}$. 19. $x = \frac{8}{11}$. 20. $x = 4$.
21. $x = 2$.

Exercise XIX. Page 81

1. 17. 2. 21. 3. 24. 4. 9, 8. 5. 33.
6. $1\frac{1}{4}$ mi. 7. 40. 8. 39. 9. 36 ft. 10. 75 cts.
11. 40. 12. 149. 13. 16, $10\frac{1}{2}$. 14. 18.
15. \$350, \$650. 16. \$480, \$300. 17. 21. 18. 54.
19. 19, 24. 20. 10 yrs. 21. 108 sq. ft. 22. 15.
23. \$600. 24. \$1200. 25. \$160. 26. \$48.
27. \$12. 28. \$800.

Exercise XX. Page 88

1. $x = 12$, $y = 8$. 2. $x = 22$, $y = 3$. 3. $x = 11$, $y = 13$. 4. $x = 13$, $y = 23$.
5. $x = 3$, $y = -5$. 6. $x = 2$, $y = 7$. 7. $x = 5$, $y = 4$. 8. $x = 8$, $y = 3$.
9. $x = 4$, $y = -2$. 10. $x = -4$, $y = 3$. 11. $x = 6$, $y = 10$. 12. $x = 15$, $y = 10$.

13. $x = 3$, $y = 5$. 14. $x = 11$, $y = 2\frac{1}{2}$. 15. $x = 5$, $y = -1$. 16. $x = 22$, $y = 24$.
17. $x = -3$, $y = 2$. 18. $x = \frac{1}{3}$, $y = \frac{1}{4}$. 19. $x = 2$, $y = \frac{1}{2}$. 20. $x = 16$, $y = -3$.
21. $x = 3$, $y = 1$. 22. $x = 2$, $y = -5$.

Exercise XXI. Page 91

1. 19, 28 2. 12, 8. 3. 40 cts., 21 cts
4. $2.25, $1.20. 5. 30 cts., 40 cts. 6. 10 lbs.
7. $22, $26. 8. $12, $15. 9. $1.20, $0.60.
10. 180, 200. 11. $3.50, $2.25. 12. $12 + 20 = 32$.
13. 125. 14. 36. 15. 36. 16. 26.
17. The sum of the digits; the difference of the digits.
18. When the large digit is in the tens' place, the number is 7 times the sum of its digits.
19. $\frac{3}{8}$. 20. $\frac{4}{15}$. 21. 28. 22. 24.
23. 20 ft., 12 ft. 24. $2\frac{3}{4}$ miles. 25. 27 miles.

EXAMINATION PAPERS

I. Page 94

1. 16. 2. $x + y + x + y$, $2x + 2y$, $2(x + y)$; xy, $144xy$.
3. $-4a + 7b - 7c$. 4. $a^3 + a^2b$. 5. $a^2 - ab + b^2$.
6. $3\frac{1}{2}$. 7. 30 lbs.

II. Page 95

1. $-2\frac{1}{3}$. 2. 12, 64, 81. 3. $-2\frac{1}{2}ab + \frac{1}{3}b^2$.
4. $2x^2 - 15x - 8$. 5. $a - 10b$. 6. $x = 3$.
7. $7\frac{1}{2}$ weeks, $75, $82.50, $90.

III. Page 95

1. $\frac{1}{6}$. 2. 9 sq. ft. 3. Twice the larger number; twice the smaller number. 4. 0, $2\frac{1}{3}$. 5. $x - a$.
6. 20. 7. 30, 50.

IV. Page 96

1. $3\frac{1}{8}$.
2. $8x - 4$, $4x^2 - 4x - 15$, $4x^2 - 4x + 1$.
3. $2a - 5b + 3b^2$.
4. $2a^3 + 2b^3 + 2c^3 - 6abc$.
5. $a^2 - ab + b^2$.
6. $x = 15$.
7. 10 years.

V. Page 97

1. 120.
2. $a^2 + 3b^2$.
3. -1.
4. 0.
5. [illegible]
6. $x = 5$, $y = 2$.
7. 18, 6.

VI. Page 97

1. $-\frac{1}{3}$.
2. $a^4 + a^2b^2$.
3. 1.
4. $ab + bc + ac$.
5. $x = 2$.
6. $x = 4$, $y = 9$.
7. 19, 47.

VII. Page 98

1. $z = 5$.
2. 2.
3. $2(a^2b + a^2c + ac^2 + bc^2 + abc - b^3)$
4. $1 - b + ab$.
5. $x = a^2 + ab + b^2$.
6. 12 ft., 10 ft.
7. 24, 19.

VIII. Page 99

1. $3a^2 + ab + 4b^2$.
2. $x^2 - a^2$.
3. $ax + by - 1$.
4. $9(ab + bc - 2ac - 2c^2)$.
5. $x = 8$.
6. 37.
7. \$200, \$30.

Exercise XXII. Page 102

1. $3(a - b + c)$.
2. $b(a + b - c)$.
3. $5c(a - 2b - 3c)$.
4. $3x(2x^2 - 3x + 5)$.
5. $7p(p^2 - 5p + 9)$.
6. $11(2m^2 - 3mn + 7n^2)$.
7. $5ab(a - 3b)$.
8. $2ab(3ax - 4by + 3xy)$.
9. $12a^2b(a^2 - 2ab + 3b^2)$.
10. $13xy^2(2ax - 3by)$.
11. $(a + b)(x + y)$.
12. $(a - b)(x - y)$.
13. $(a + b)(c - d)$.
14. $(a - d)(b - c)$.
15. $(m + n)(a + 1)$.
16. $(m - 1)(a - b)$.
17. $(a + b)(a + c)$.
18. $(b - c)(a + b)$.
19. $(a - x)(x + b)$.
20. $(b - a)(a + c)$.
21. $(x^2 + 2)(x - 3)$.
22. $(a^2 - b)(a - 2)$.
23. $(2a - 1)(a^2 + 1)$
24. $(1 + a)(1 + b)$.
25. $(1 - a)(1 - b)$.
26. $(a^2 + 1)(a + 1)$.

27. $(a^4 - a^2 + 1)(a + 1)$. 28. $(ab - cd)(x + y)$.
29. $(2ax - 3by)(x - y)$. 30. $(a - b)(ax - by)$.
31. $(a - b)(2x + 3y - 5)$. 32 $x(a + b)(ac - by)$.
33. $(a^2 + bc)(b^2 - 3ac)$. 34. $(x - c)(x + a - b)$.

Exercise XXIII. Page 104

1. $x^2 + 2xy + y^2$. 2. $x^2 - 2xy + y^2$. 3. $x^2 - y^2$.
4. $a^2 + 6ab + 9b^2$. 5. $a^2 - 6ab + 9b^2$. 6. $a^2 - 9b^2$.
7. $4a^2 + 12ab + 9b^2$. 8. $4a^2 + 12ab + 9b^2$.
9. $4a^2 - 9b^2$. 10. $9x^2 - 30x + 25$. 11. $4x^2 + 20x + 25$
12. $25x^2 - 9$. 13. $1 + 6a + 9a^2$. 14. $25b^2 - 10b + 1$
15. $1 - 16a^2$. 16. $a^4 + 2a^2b^2 + b^4$. 17. $a^4 - 2a^2b^2 + b^4$.
18. $a^4 - b^4$. 19. $a^2x^2 - 2abxy + b^2y^2$.
20. $4a^4 - 12a^3b + 9a^2b^2$. 21. $a^4 + 2a^3b + a^2b^2$.
22. $1 - 4x + 4x^2$. 23. $1 + 4x + 4x^2$. 24. $1 - 4x^2$.
25. $m^4n^2 - 2m^3n^3 + m^2n^4$ 26. $4p^4 - 20p^3q^2 + 25p^2q^4$.
27. $9y^2 - 4x^2$. 28. 6561. 29. 8091.
30. 6396. 31. $2(a^2 + b^2)$. 32. $4ab$.
33. $a^4 - 16b^4$. 34. $16x^4 - 81y^4$. 35. $a^{16}b^{16}$.

Exercise XXIV. Page 106

1. $(x + y)(x - y)$. 2. $(x - y)^2$. 3. $(x + 1)^2$.
4. $(2x + 3y)(2x - 3y)$. 5. $(2a + 3)^2$. 6. $(1 + 3a)^2$
7. $(4x + 1)(4x - 1)$. 8. $(2a - 1)^2$. 9. $(2a + 5b)^2$
10. $(4x^2 + 9y^2)(2x + 3y)(2x - 3y)$. 11. $(x^3 - 9)^2$
12. $(3x - 5y)^2$. 13. $5(a^2 + 2b^2)(a^2 - 2b)^2$. 14. $3(x - 2)$
15. $6(2x + 3y)^2$. 16. $(ab^2 + 10c^3)(ab^2 - 10c^3)$. 17. $(ab - 5)$
18. $(a + b)^2$. 19. $2(9x^2 + 16y^2)(3x + 4y)(3x - 4y)$.
20. $(5a^2x - 3b^2y)^2$. 21. $x^2(3a - 7b)^2$. 22. $3y^2(x - 3ay)^2$
23. $a^2(9x^2 + 1)(3x + 1)(3x - 1)$. 24. $3xy^2(4x^2 + 1)(2x + 1$
25. $(2a - 3b)^2$. 26. $(a + 2b)(a - 2b)$.
27. $(a + b)(a - b)(c + d)(c - d)$ 28. $(1 + a)(1 - a)(1 + b)(1 - b)$
29. $(b + c)(b - c)(a + 1)(a - 1)$. 30. $(a + b)^2$.

Exercise XXV. Page 107

1. $x^2 + 8x + 15$.
2. $x^2 - 8x + 15$.
3. $x^2 - 2x - 15$.
4. $y^2 - y - 30$.
5. $y^2 - 7y + 6$.
6. $y^2 + 2y - 99$.
7. $a^2 - 11a + 24$.
8. $b^2 + 10b - 75$.
9. $c^2 + 10c - 200$.
10. $x^2 - xy - 6y^2$.
11. $x^2 - 19xy - 20y^2$.
12. $x^2 + 10xy - 39y^2$.
13. $x^4 - 9x^2 + 14$.
14. $a^6 - 8a^3 + 15$.
15. $b^8 + 20b^4 - 125$.
16. $x^2 - 2abx - 3a^2b^2$.
17. $1 + 7x - 30x^2$.
18. $a^2 - 8ab^2 + 7b^4$.
19. $x^4 + 10x^2yz - 24y^2z^2$.
20. $a^4 + 15a^3b - 100a^2b^2$.
21. $4x^2 - 30xy + 50y^2$.
22. $9a^2 + 21ab - 44b^2$.
23. $x^2 + (a+b)x + ab$.
24. $9x^2 + 3(a-b)x - ab$.
25. $(a+b)^2 + 2(a+b) - 15 = a^2 + 2ab + b^2 + 2a + 2b - 15$.
26. $x^4 + 2x^3 - x^2 - 2x - 24$.
27. $(x^2 - 5x + 4)(x^2 - 5x + 6)$
$= (x^2 - 5x)^2 + 10(x^2 - 5x) + 24$
$= x^4 - 10x^3 + 35x^2 - 50x + 24$.
28. $x^4 - 10x^3 + 25x^2 - 36$.
29. $x^4 - 25x^2 + 144$.
30. $a^4 - 26a^2b^2 + 25b^4$.

Exercise XXVI. Page 109

1. $(x+3)(x+5)$.
2. $(x+2)(x+6)$.
3. $(x+1)(x+7)$.
4. $(y-3)(y-4)$.
5. $(y-2)(y-6)$.
6. $(y-1)(y-12)$.
7. $(a+5)(a-3)$.
8. $(a-5)(a+3)$.
9. $(a-7)(a+5)$.
10. $(b+7)(b-5)$.
11. $(b-12)(b+1)$
12. $(b+12)(b-1)$.
13. $(m-3)(m-17)$.
14. $(m-30)(m+10)$.
15. $(m+20)(m-1)$.
16. $(x^2-7)(x^2+3)$.
17. $(x+1)(x-1)(x+2)(x-2)$.
18. $(x+1)(x-1)(x+3)(x-3)$.
19. $(x^3-4)(x^3+3)$.
20. $(x^2+4)(x+2)(x-2)(x^4+2)$.
21. $(x+5)(x-5)(x+4)(x-4)$.
22. $(xy-16)(xy+2)$.
23. $(x^2y+20)(x^2y-5)$.
24. $(x+5y)(x+15y)$.
25. $(m+12np)(m-7np)$.
26. $(ab-4c)(ab-14c)$.
27. $(1+6x^2)(1+x)(1-x)$.
28. $4(x+3)(x-2)$.
29. $3(a-9)(a+8)$.
30. $5(x^2+10)(x+2)(x-2)$.
31. $2a(x^3-7a)(x^3+2a)$.
32. $a^2b(1+19x)(1-x)$.

33. $x(1-6x)(1+x)$.
34. $(x+2a)(x+b)$.
35. $(x-2a)(x-3b)$.
36. $(x^2+a^2)(x+b)(x-b)$.
37. $(x+2a)(x-2a)(x+3b)(x-3b)$.
38. $3(3a-5)(a+1)$.
39. $(2a+4)(2a-4)(2a+1)(2a-1)$.
40. $(x-3)(x+1)(x-1)^2$.
41. $(x+1)(x-1)(x+3)(x+5)$.

Exercise XXVII. Page 111

1. $2x^2+5xy+2y^2$.
2. $2x^2-5xy+2y^2$.
3. $3x^2+5xy+2y^2$.
4. $3x^2+xy-2y^2$.
5. $4x^2-11xy+6y^2$.
6. $6x^2-xy-15y^2$.
7. $14x^2+83xy-6y^2$.
8. $6x^4-13x^2y^2+2y^4$.
9. $72x^2+55xy-8y^2$.
10. $5x^4-26x^2y+5y^2$.
11. $12x^2-10xy-8y^2$.
12. $6ax^2-35axy-6ayz$.
13. $(2x-1)(x-2)$.
14. $(2x+3y)(x+y)$.
15. $(2x-y)(x-3y)$.
16. $(2x-3)(x-4)$.
17. $(2x-3)(x+4)$.
18. $(3x-4)(4x+5)$.
19. $(4x-3y)(3x-4y)$.
20. $a(x-6y)(6x-y)$.
21. $a^2(5x-11y)(2x+3y)$.
22. $(ax+y)(x+ay)$.
23. $(2x+y)(x+2y)(2x-y)(x-2y)$.
24. $(3x+y)(x+3y)(3x-y)(x-3y)$.

Exercise XXVIII. Page 113

1. $x^3+3x^2y+3xy^2+y^3$.
2. $x^3-3x^2y+3xy^2-y^3$.
3. x^3+3x^2+3x+1.
4. x^3-3x^2+3x-1
5. $a^3+6a^2b+12ab^2+8b^3$.
6. $a^3-6a^2+12a-8$.
7. $8a^3+12a^2+6a+1$.
8. $8a^3-12a^2b+6ab^2-b^3$.
9. $a^3+9a^2b+27ab^2+27b^3$.
10. $27a^3-27a^2b+9ab^2-b^3$.
11. $8a^3+36a^2b+54ab^2+27b^3$.
12. $125a^3-150a^2b+60ab^2-8b^3$.
13. $2(a^3+3ab^2)$.
14 $2(3a^2b+b^3)$.
15. a^2+ab+b^2.
16. a^2-ab+b^2.
17. $a^2+2ab-ac+b^2-bc+c^2$.
18. $a^2+ab-ac+b^2-2bc+c^2$.
19. $a^2+5ab+7b^2$.
20. $a^2+4ab+7b^2$.
21. $a+b$.
22. $a-b$.
23. 0.

Exercise XXIX. Page 115

1. $(x+y)(x^2-xy+y^2)$.
2. $(x+2)(x^2-2x+4)$.
3. $(x+1)(x^2-x+1)$.
4. $(x+4)(x^2-4x+16)$.
5. $(x-y)(x^2+xy+y^2)$.
6. $(1-y)(1+y+y^2)$.
7. $(1-2y)(1+2y+4y^2)$.
8. $(2x+5y)(4x^2-10xy+25y^2)$.
9. $(4x-10y)(16x^2-40xy+100y^2)$.
10. $(x^2+y^2)(x^4-x^2y^2+y^4)$.
11. $(x^2+1)(x^4-x^2+1)$.
12. $(3+y^2)(9-3y^2+y^4)$.
13. $(x+2)(x-2)(x^2-2x+4)(x^2+2x+4)$.
14. $(2a+3bc)(4a^2-6abc+9b^2c^2)$.
15. $(5xy-7z)(25x^2y^2+35xyz+49z^2)$.
16. $a(3a-4)(9a^2+12a+16)$.
17. $2a(5-2a)(25+10a+4a^2)$.
18. $(x^2+2y)(x^4-2x^2y+4y^2)$.
19. $(a+b)(a^2-ab+b^2)(a^6-a^3b^3+b^6)$.
20. $(2a^2+3b^3)(4a^4-6a^2b^3+9b^6)$.
21. $(2ab+1)(4a^2b^2-2ab+1)(2ab-1)(4a^2b^2+2ab+1)$.
22. $(a^4+b^4)(a^8-a^4b^4+b^8)$.
23. $(a^2+b^2)(a^4-a^2b^2+b^4)(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$.
24. $(a-b-1)(a^2-2ab+b^2+a-b+1)$.
25. $(a+b)(a^2-4ab+7b^2)$.
26. $(a-b)(a^2+4ab+7b^2)$.
27. $(a-b)(a^2+7ab+19b^2)$.
28. $(x+1)(x-2)(x^4-2x^3+3x^2-2x+4)$.
29. $(x-1)(x+2)(x^4+2x^3+3x^2+2x+4)$.
30. $(1+a)(1+b)(1-a+a^2)(1-b+b^2)$.
31. $(1-a)(1-b)(1+a+a^2)(1+b+b^2)$.

Exercise XXX. Page 116

1. $x^2+y^2+z^2+2xy+2xz+2yz$.
2. $x^2+y^2+z^2+2xy-2xz-2yz$.
3. $x^2+y^2+z^2-2xy+2xz-2yz$.
4. $x^2+y^2+z^2-2xy-2xz+2yz$.
5. $x^2+y^2+z^2-2xy+2xz-2yz$.

6. $a^2 + 4b^2 + c^2 - 4ab + 2ac - 4bc.$
7. $4a^2 + b^2 + 9c^2 - 4ab - 12ac + 6bc.$
8. $a^2 + 4b^2 + 25c^2 - 4ab + 10ac - 20bc.$
9. $a^2 + b^2 + c^2 + d^2 - 2ab + 2ac - 2ad - 2bc + 2bd - 2cd.$
10. $a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4.$
11. $a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4.$
12. $1 - 2x + 3x^2 - 2x^3 + x^4.$ 13. $x^4 + 2x^3 - x^2 - 2x + 1.$
14. $x^6 - 2x^5 + 3x^4 - 4x^3 + 3x^2 - 2x + 1.$
15. $a^2b^2 + a^2c^2 + b^2c^2 + 2abc(a + b + c).$
16. $(a + b + c)^2.$ 17. $(a - b + c)^2.$ 18. $(a - b)^2.$
19. $(a + b)^2.$ 20. $b^2.$ 21. $b^2.$
22. $(1 + 2a - b)^2.$ 23. $(a + b - c)^2.$ 24. $(a - 2b - 3c)^2.$

Exercise XXXI. Page 118

1. $a^2 + 2ab + b^2 - c^2.$ 2. $a^2 - 2ac + c^2 - b^2.$
3. $4a^2 + b^2 - c^2 - 4ab.$ 4. $a^2 - 4b^2 - c^2 - 4bc.$
5. $9z^2 - x^2 + 4xy - 4y^2.$ 6. $x^4 + x^2 + 1.$
7. $x^4 + x^2y^2 + y^4.$ 8. $x^4 + 9x^2 + 81.$
9. $4x^4 + 3x^2 + 9.$ 10. $x^4 + 4a^4.$
11. $4a^4 + 1.$ 12. $a^8 + a^4 + 1.$
13. $a^2 + 2ab + b^2 - c^2 - 2cd - d^2.$
14. $a^2 - b^2 - c^2 - d^2 + 2bc - 2bd + 2cd.$
15. $a^4 - 2a^2b^2 + b^4.$ 16. $2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4.$
17. $2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4.$

Exercise XXXII. Page 119

1. $(a + b + c)(a + b - c).$ 2. $(a + b - c)(a - b + c).$
3. $(a + b + c)(a - b - c).$ 4. $(a + 2b + c)(a + 2b - c$
5. $(3a + b)(a - b).$ 6. $(4a - b)b.$
7. $(x + 1)(x - 1)(x + 3)(x - 3).$ 8. $(x + 2)(x - 2)(x + 4)(x - 4).$
9. $(x^2 - x + 6)(x - 3)(x + 2).$ 10. $(3a + b)(a + 5b).$
11. $8(a - b)(a - 3b).$ 12. $(a + b + 3c)(a + b - 3c$
13. $(a + b - 2c)(a - b + 2c).$ 14. $(b - c + a)(b - c - a).$

15. $(a+b-c)(a-b+c)$.
16. $(a+b+2)(a+b-2)$.
17. $(ax+by+1)(ax+by-1)$.
18. $(1+3a-5b)(1-3a+5b)$.
19. $(5c+2a-b)(5c-2a+b)$.
20. $(a+b+c+d)(a+b-c-d)$.
21. $(a+b+c-d)(a-b+c+d)$.
22. $(a-b+c-d)(a-b-c+d)$.
23. $(a+b-c+d)(a-b-c-d)$.
24. $(x-3)(x+1)(x+3)(x-5)$.
25. $(x-1)(x-2)(x-3)(x-4)$.
26. $(x-3)(x-7)(x+5)^2$.
27. $3(x-5)(x+3)(x+5)(x+7)$.

Exercise XXXIII. Page 120

1. $(x^2+xy+y^2)(x^2-xy+y^2)$.
2. $(x^2+2x+3)(x^2-2x+3)$.
3. $(3a^2+2ab+b^2)(3a^2-2ab+b^2)$.
4. $(a^2+ab+3b^2)(a^2-ab+3b^2)$.
5. $(2a^2+3a+5)(2a^2-3a+5)$.
6. $(a^2+ab-b^2)(a^2-ab-b^2)$.
7. $(a^2+2ab+2b^2)(a^2-2ab+2b^2)$.
8. $(2a^2+2a+1)(2a^2-2a+1)$.
9. $(2x^2+3xy+3y^2)(2x^2-3xy+3y^2)$.
10. $(a^2+4ab+8b^2)(a^2-4ab+8b^2)$.
11. $(2x^2+4x+5)(2x^2-4x+5)$.
12. $(a+b)(a-b)(a+3b(a-3b)$.
13. $(7x^2+4xy-2y^2)(7x^2-4xy-2y^2)$.
14. $(5a^2+3ax+8x^2)(5a^2-3ax+8x^2)$.
15. $(2x+y)(2x-y)(x+3y)(x-3y)$.
16. $(5a^2-2ab+b^2)(a^2-2ab+5b^2)$.
17. $(a^2+3b^2)(b^2+3a^2)$.
18. $(a^2+1)(a^2-4a+5)$.
19. $(a+b+c)(a+b-c)(a-b+c)(a-b-c)$.

Exercise XXXIV. Page 122

1. $(a-b)(a+b-c)$.
2. $(a-b)(a+b-1)$.
3. $(x+1)(x+a-1)$.
4. $(a+2b)(a-2b+3)$.
5. $(2a^2-1)(2a^2+3a+1)$.
6. $(a+b)(a-b)^3$.
7. $(a-1)(a+1)^2$.
8. $(a+1)(a-1)^2$.
9. $(y+1)(y^2-y+1)$.
10. $(y^2-y+1)(y+a+1)$.
11. $(a+1)(a-1)(b+1)(b-1)$.
12. $(a+1)(a-1)(b+1)(b-1)$.

13. $(x+2)(x-2)(x^2+1)$. **14.** $(x^2+x-1)(x^2-x-1)$.
15. $(a^2+3ab+b^2)(a^2-3ab+b^2)$.
16. $(a^2+3ab-b^2)(a^2-3ab-b^2)$.
17. $(x^2+9y^2)(x^2+y^2)$. **18.** $(x^2+2xy+7y^2)(x^2-2xy+7y^2)$.
19. $2a(a-1)(4a+3)$. **20.** $3x(1+x)(4-3x)$.
21. $(a^2+\frac{1}{2})^2$. **22.** $(a^2+a+\frac{1}{2})(a^2-a+\frac{1}{2})$.
23. $(a^2+1)(a^2-4a+5)$. **24.** $(a-1)(a+3)(a^2+2a+5)$.
25. $ab(a^3+b^2)(a^3-b^2)$. **26.** $3a(a^2+b)(a^4-a^2b+b^2)$.
27. $(m-n)^2(m^2+mn+n^2)$. **28.** $(m+n)^2(m^2-mn+n^2)$.
29. $(a^2+ab+2b^2)(a^2+ab-2b^2)$. **30.** $(x^2+xy+y^2)(x^2+xy-y^2)$.
31. $(a-c)(ac+1)$. **32.** $2(a-c)(1-ac)$.
33. $(x^2+ax+b^2)(x^2-ax+b^2)$. **34** $(x^2+ax+b^2)(x^2+ax-b^2)$.
35. $(x-1)(y-1)(x+1)(y^2+y+1)$.
36. $(a-1)(b-2)(a^2+a+1)(b+2)$.
37. $(1-x)(1+x+y-xy)$. **38.** $(1+x)(1-x+y+xy)$.
39. $(x+3)(x+5)(x-5)(x-7)$. **40.** $x^2(x^2+2ab)(x^2+4ab)$.
41. $(a-b)^2(a+b)^2(a^2+b^2)$. **42.** $(a+b)^2(a-b)^2(a^2+b^2)$.
43. $(x-y)(x^2+xy+y^2)$. **44.** $(x-y)^3(x+y)$.
45. $(x-y)^3(x-y)$ **46.** $(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$.
47. $(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$.
48. $(ax+ay-bx+by)(ax-ay+bx+by)$.
49. $(ax+ay+bx-by)(ax-ay-bx-by)$.
50. $(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d)$
51. $(a+b+c+d)(a-b-c+d)(a+b-c-d)(a-b+c-d)$.
52. $(a+b)(a-b)(a^2+b^2-1)$. **53.** $(ax-by)(bx+ay)$.
54. $(a+b-c)(a^2+b^2+c^2+2ab+ac+bc)$.
55. $(a-b+c)(a^2+b^2+c^2-2ab-ac+bc)$.
56. $(a+b+c)(a+b-4c)$. **57.** $(a-b+3c)(a-b-4c)$.
58. $(2x+3y+4z)(3x+y-2z)$. **59.** $(a-b)(b-c)(c-a)$.
60. $(a-b)(b-c)(c-a)(a+b+c)$. **61.** $\{1+y-x^2(1-y)$
62. $(1+x)(1-x)(1+x+y-xy)(1-x+y+xy)$.
63. $(x+y)(x-y)(x+y+a)(x+y-a)$.
64. $8a(b+c)(c-a)(a-b)$. **65.** $4(a-c)(a+d)(b+c)(b-d)$.

Exercise XXXV. Page 125

1. $x = 2, \ -3$. 2. $x = 4, \ -3$. 3. $x = 10, \ -4$.
4. $x = 5, \ -7$. 5. $x = 6, \ -10$. 6. $x = 8, \ -4$
7. $x = 3, \ -1$. 8. $x = -2, \ 5$. 9. $x = 2, \ 5$.
10. $x = 4, \ 9$. 11 $x = 4, \ -4$. 12. $x = 3, \ 7$.
13. $x = \frac{5}{3}, \ -\frac{5}{3}$. 14. $x = 0, \ 3, \ -3$. 15. $x = -\frac{7}{4}$.
16. $x = 0, \ -4$. 17. $x = 1, \ -1, \ 3, \ -3$.
18. $x = 2, \ -2, \ 5, \ -5$. 19. $x = 0, \ 1, \ -1$.
20. $x = 1, \ -1, \ -1$. 21. $x = 1, \ -1, \ 2$.
22. $x = 2a, \ -a$. 23. $x = a, \ -b$. 24. $x = a$.
25. $x = a + b, \ a - b$. 26. $x = a - b, \ b - a$.

Exercise XXXVI. Page 127

1. 10, 15. 2. 6, 9. 3. 10, 12. 4. 5, 7.
5. 11, 17 : 1, 7. 6. 144 sq. ft. 7. 12 ft.
8. 14 ft. 9. 20 ft. 10. 1 inch. 11. 12, 21.
12. 60 sq. ft. 13. 7ft., 24 ft. 14. 10 rods.
15. 10, 11. 16. 5 ft. 18. 25 sq. ft.
19. 2 inches. 20. 2 inches.

Exercise XXXVII. Page 131

1. bc^2. 2. $3abc$. 3. $5a^2b$. 4. $14a^2b^3$.
5. $12x^2y^3$. 6. $5xy$. 7. $5(a - b)$. 8. $4ab$.
9. $7m$. 10. $mx + ny$. 11. $a^2 + b^2$. 12. $ab(a - b)$.
13. $a - b$. 14. $a - 7$. 15. $a - 2$. 16. $x^2 + x + 1$.
17. $x^2 - y^2$. 18. $x^2 - xy + y^2$. 19. $x^4 - x^2y^2 + y^4$.
20. $x^2 - 2x + 4$. 21. $2x + 1$. 22. $x(x^2 + 1)$.
23. $x + 1$. 24. $a^2 + ab + b^2$. 25. $25a^2 - 20ab + 16b^2$.
26. $a(a + 1)$. 27. $p - q$. 28. $2(a^2 - b^2)$. 29 $a + b + c$.
30. $b(a - c)$. 31. $2a^2 + b^2$. 32. $x - y$. 33. $x^2 - 5$.

Exercise XXXVIII. Page 134

1. abc. 2. $6a^2b^2c^2d$. 3. $60m^3n^3p^3$. 4. $168l^3m^3n^4$
5. $ab(a^2 - b^2)$. 6. $ax(a - x)$. 7. $x(x + 1)(x - 1)$.
8. $12(x^2 - 1)$. 9. $a^2 - 4b^2$. 10. $ab(a + 1)^2(b + 1)$.
11. $(1 - x)^2(1 + x)$. 12. $(x - y)^2(x + 2y)$. 13. $30x(x + 2)$.
14. $4x^2 - 1$. 15. $ab(a - b)^2$ 16. $a^3 - b$.
17. $60(x^2 - y^2)^2$. 18. $12x^2(x - 1)^2$ 19. $x^2y^2(x^2 - y^2)$.
20. $(x - 1)(x - 2)(x - 3)$. 21. $6ab^2(x - 1)^2(x + 1)(x + 2)$
22. $(x - a)(x - b)(x - c)$. 23. $a^6 - b^6$.
24. $(x^2 - 4)(x^2 - 9)$. 25. $a^2b^2(a^2 - b^2)^2$.
26. $(4x + 5)(3x - 4)(4x - 3)$. 27. $4a^2b^2 - (a^2 + b^2 - c^2)^2$.
28. $(x^2 - 1)(2x - 1)(x - 2)$.

Exercise XXXIX. Page 140

1. $2x + 3$. 2. $x^2 - 2x + 5$. 3. $x + 3$.
4. $x^2 + 3x + 4$. 5. $x^2 + 2x - 3$. 6. $x + 4$.
7. $3x - 2$. 8. $x^2 + 4x - 5$. 9. $2x - 3$. 10. $x - 7$.
11. $x^3 - 1$. 12. $x - 3$. 13. $x^2 + 2x + 3$
14. $x^2 - 4x + 4$. 15. $3x^2 + 2x + 1$. 16. $x^2 - 3x + 1$.
17. $x^2 + 5x + 1$. 18. $a(x - y)$. 19. $3(2x - 3y)$.
20. $ax + b$. 21. $x^2 - 2x + 3$. 22. $x - 3$.
23. $x^2 - 1$. 24. $x - a$.

Exercise XL. Page 142

1. $(x + 1)(x - 2)(x^2 + x + 1)$. 2. $(x + 2)(2x - 1)(x^2 + 2x - $ [illegible]
3. $(2x + 3)(3x - 4)(x^2 + 3x - 1)$.
4. $(x - 4)(3x - 2)(3x^2 + 2x + 1)$.
5. $(x - y)(9x^2 - 4y^2)(4x^2 - 9y^2)$. 6. $(a^2 - b^2)(a + 2b)(a - 2b)$
7. $(a^3 + b^3)(a^2 - ab - 6b^2)$. 8. $(x^2 - 1)(x^2 - 4)(x^2 - 9)$.
9. $(2a - 3b)(a + 2b)(3a + 2b)$ 10. $(a^2 - 4b^2)(a^4 + 2a^2b^2 + 9$ [illegible]
11. $ax(a + x)(a - 2x)(a + 3x)$. 12. $(x^3 + 1)(x - 3)(2x^2 - x -$ [illegible]
13. $(x^2 - 1)(x^2 - 4)(4x^2 - 1)(x + 3)$.

Exercise XLI. Page 146

1. $\frac{a}{2b}$. 2. $\frac{3ac}{5bd}$. 3. $\frac{3ay}{5}$. 4. $\frac{3c^2x^2}{4a}$.

5. $\frac{2}{3x}$. 6. $\frac{4l^3m^2n}{7}$. 7. $\frac{a}{a-b}$. 8. $\frac{a}{2b-3c}$.

9. $\frac{2}{a-2b}$. 10. $\frac{a}{m}$. 11. $\frac{b}{ac}$. 12. $\frac{ab}{xy}$.

13. $\frac{l}{ma}$. 14. $\frac{-1}{1+x}$. 15. $\frac{a}{a+b}$. 16. $\frac{a}{c(a+b)}$.

17. $\frac{1}{x+y}$. 18. $\frac{3x}{5y}$. 19. $\frac{3a}{2b(2a-3b)}$. 20. $\frac{a^2}{c^2}$.

21. $\frac{a^2-ab+b^2}{a^2}$. 22. $\frac{a^2+b^2}{a^4+a^2b^2+b^4}$. 23. $\frac{1}{a^2+b^2}$.

24. a^2-2a+2. 25. $\frac{a+b}{a+3b}$. 26. $\frac{a+b-c}{a+b+c}$.

27. $\frac{a}{a-b-c}$. 28. $\frac{a-b+c}{a+b-c}$. 29. $\frac{x+1}{x-1}$. 30. $\frac{1}{1-x^2}$.

31. $\frac{x^2+b^2}{x^2-b^2}$. 32. $\frac{2}{a-b}$. 33. $\frac{2}{a-2b}$. 34. $\frac{1}{1+a}$.

35. $\frac{a}{b}$. 36. $\frac{x-2}{2x-3}$. 37. $\frac{x^2-x+1}{x^2+x+1}$. 38. $\frac{x+1}{x-1}$.

39. $\frac{3x-4}{3x+4}$. 40. $\frac{1}{b+c-a}$. 41. $\frac{a+b-c}{c-a+b}$.

Exercise XLII. Page 149

1. $2x-1+\frac{1}{x-2}$. 2. $a-b+\frac{2b^2}{a+b}$.

3. $x^2-xy+y^2-\frac{y^3}{x+y}$. 4. $a+2b+\frac{3b^2}{a-b}$.

5. $2x-3-\frac{x}{x^2-x+1}$. 6. $x-\frac{3x+1}{x^2+1}$.

7. $x-1-\frac{x-7}{x^2+x+1}$. 8. $x+1+\frac{2x-3}{5x^2-5x+3}$.

9. $x + a + \dfrac{bx + ab}{x^2 - ax - b}$.

10. $\dfrac{1}{1 + x}$.

11. $-\dfrac{1}{1 + a}$.

12. $\dfrac{x^2}{x - y}$.

13. $\dfrac{2a}{a + b}$.

14. $\dfrac{2b}{a + b}$.

15. $\dfrac{2x^2 - xy - y^2}{x + y}$.

16. $\dfrac{2(x^2 + y^2)}{x + y}$.

17. $\dfrac{a^2 - 32}{a - 5}$

18. $\dfrac{1 - a^2}{a + 3}$.

19. $\dfrac{1}{1 - ab}$.

20. $\dfrac{a^3 + b^3}{a - b}$.

21. $\dfrac{1 + x^2}{1 + x + x^2}$.

22. $\dfrac{a^2 - 2ay - x^2}{x + a}$.

23. $\dfrac{x - 3}{x + 1}$.

24. $\dfrac{2(x^2 + x - 2)}{x + 1}$.

25. $\dfrac{1}{1 - x}$.

26. $\dfrac{a}{1 + b}$.

27. $\dfrac{1}{c} + \dfrac{1}{a} + \dfrac{1}{b}$.

28. $\dfrac{x^2}{6y^2} - \dfrac{2}{3} + \dfrac{4y^2}{3x^2}$.

29. $\dfrac{x}{8} - \dfrac{1}{5} - \dfrac{3}{8x} + \dfrac{1}{2x^2}$.

30. $\dfrac{1}{b^4} + \dfrac{1}{a^2b^2} + \dfrac{1}{a^4}$.

31. $\dfrac{x}{x + y} - \dfrac{y}{a + b}$.

32. $\dfrac{x}{a + 2b} + \dfrac{y}{a - b}$.

Exercise XLIII. Page 152

1. $\dfrac{5a}{12}$.

2. $\dfrac{5a}{12}$.

3. $\dfrac{3x - 5}{6}$.

4. $\dfrac{b}{2}$.

5. $\dfrac{2a - 3b}{2}$.

6. $2a$.

7. $\dfrac{2}{a}$.

8. 1.

9. $\dfrac{x - 9}{3x}$.

10. $\dfrac{2}{3x}$.

11. 0.

12. $\dfrac{a^2 + b^2 + c^2}{abc}$.

13. $\dfrac{2x}{a}$.

14. $\dfrac{x^3 + y^3}{x^2y^3}$.

15. $\dfrac{a^2 + b^2}{a^2 - b^2}$.

16. $\dfrac{a + y}{a^2}$.

17. $\dfrac{1}{a + b}$.

18. $\dfrac{a}{a - b}$.

19. $\dfrac{4xy}{x^2 - y^2}$.

20. $\dfrac{x}{x + y}$.

21. $\dfrac{a + b}{ab}$.

22. $\dfrac{ab}{a^2 - b^2}$.

23. $\dfrac{m + n}{mn}$.

24. $\dfrac{m + n}{m^2}$.

25. $\dfrac{a+b}{a^2-ab+b^2}$. 26. $\dfrac{a-b}{a+b}$. 27. $\dfrac{a+b}{ab}$.

28. $\dfrac{a-x}{x}$. 29. $\dfrac{2a}{a-b}$. 30. $\dfrac{2}{x-5}$. 31. $\dfrac{3}{2x+1}$.

32. $(a+b)^2$. 33. $\dfrac{2b}{a^2-b^2}$. 34. $\dfrac{2x^3}{1+x^2+x^4}$. 35. $\dfrac{2y^3}{y^2-x^2}$.

36. $2x$. 37. $\dfrac{4x(x^2+1)}{x^4+x^2+1}$. 38. 2. 39. $\dfrac{3}{x-7}$.

40. $\dfrac{2}{(x-4)(x-6)}$. 41. $\dfrac{2}{(x-2)(x-3)(x-4)}$.

42. $\dfrac{2}{(x-1)(2x+1)(2x+3)}$. 43. $\dfrac{5}{(2x-1)(2x+1)(x-2)}$.

44. $\dfrac{2}{x+3}$. 45. $\dfrac{2(x^2+y^2)}{x^3+y^3}$. 46. $\dfrac{2(x^3+y^3)}{x^3-y^3}$.

47. $\dfrac{2(x+y)}{x^2+xy+y^2}$. 48. $\dfrac{1}{(x^3+1)(x^2+1)}$. 49. $\dfrac{1}{(1+x^2)(1-x^3)}$.

50. $\dfrac{8x^4-x^3-24x^2+5x-4}{(x+1)(x-1)(x+3)(x-2)}$. 51. $\dfrac{8x^4+x^3-24x^2-5x-4}{(x+1)(x-1)(x-3)(x+2)}$.

Exercise XLIV. Page 155

1. $-\dfrac{2a}{a+1}$. 2. $\dfrac{4}{1-a^4}$. 3. $\dfrac{x+2}{x^2-1}$. 4. $\dfrac{2x}{1-x^4}$.

5. $\dfrac{2}{x+2}$. 6. 0. 7. $\dfrac{4y^3}{x^4-y^4}$. 8. $\dfrac{4x^2y^2}{x^4-y^4}$.

9. $\dfrac{4a}{a+x}$. 10. $\dfrac{2x^2}{x^2-a^2}$. 11. $\dfrac{3x-2}{x+2}$. 12. 2. 13. $\dfrac{y}{2(x+y)}$.

14. $\dfrac{y}{x-y}$. 15. $\dfrac{1}{x}$. 16. $\dfrac{1}{x}$. 17. $\dfrac{-x}{2(x^2+1)}$. 18. 0.

19. 0. 20. $\dfrac{-7}{(x-2)(x-3)(x+5)}$. 21. $\dfrac{b(a+c)}{ac(a-b)(b-c)}$.

22. $\dfrac{2a}{a^2-(b-c)^2}$. 23. $\dfrac{2a}{(a+b)(a+3b)}$. 24. $\dfrac{2}{(x-2)(x-3)}$.

25. 2. 26. 1. 27. $\dfrac{16x^{15}}{1-x^{16}}$. 28. $\dfrac{84x^3-186x^2+93x-6}{4(3x+2)(3x-2)^2}$.

Exercise XLV. Page 160

1. $\frac{2h}{7xy}$. 2. 1. 3. $\frac{8acd^2}{5b^2}$. 4. $\frac{3abz}{5cy}$.

5 $\frac{6by}{7ax}$. 6. $\frac{pq}{22x^2y^2}$. 7. $\frac{b+c}{4}$. 8. $\frac{5ax}{2c}$.

9. $\frac{1}{2}$. 10. $\frac{(a-2)(a-5)}{a^2}$. 11. $\frac{a^2}{b^2}$. 12. $\frac{(a-b)(2a-3b)}{ab}$.

13. $\frac{(a+b)x}{2}$. 14. $\frac{2(a-b)}{a}$. 15. $\frac{1-x}{x+7}$. 16. $\frac{a-b}{a-2b}$.

17 $(a+b)^3$. 18. $\frac{x}{2a+x}$. 19. $\frac{2(x+7)^2}{x^2-x-30}$. 20. 1.

21. 1. 22. $\frac{ab-a-b}{b^2(a^2-b^2)}$. 23. $\frac{a}{a+b}$. 24. 1.

25. $\frac{x+y}{xy}$. 26. $\frac{1}{a+x}$. 27. $\frac{1-a^2}{1-x^2}$. 28. $2(a-b)$.

29. 1. 30. $\frac{(a+y)(x-y)}{(a-x)(x+y)}$.

Exercise XLVI. Page 168

1. $x^2+2+\frac{1}{x^2}$. 2. $x^2-2+\frac{1}{x^2}$. 3. $x^2-\frac{1}{x^2}$.

4. $\frac{a^2}{b^2}-\frac{x^2}{y^2}$. 5. $\frac{1}{x^3}-1$. 6 $\frac{1}{x^3}+x^3$. 7. $\frac{a^3}{b^3}+1$.

8. $x+\frac{1}{x}$ 9. $1+\frac{1}{x}+\frac{1}{x^2}$. 10. $\frac{a^2}{b^2}+1+\frac{b^2}{a^2}$.

11. $x-\frac{1}{x}$. 12. $\frac{1}{x(x+1)}$. 13. $x+1$. 14. $\frac{b}{a}-\frac{a}{b}$.

15. $\frac{-1}{b+1}$ 16. $\frac{a-x}{ax}$. 17. $\frac{a(a+x)}{x}$. 18. $\frac{1}{1-x^2}$.

19. $\frac{-1}{(x^2+x+1)}$. 20. $\frac{2}{3}x^3+\frac{2}{3}x^2-\frac{5}{18}x+\frac{1}{3}$. 21. $\frac{x^2}{2}-\frac{x}{3}+\frac{1}{4}$

22. $\frac{x^2}{y^2}+\frac{y}{x}$. 23. $\frac{x^2+xy+y^2}{x^2-xy+y^2}$. 24. $x^4+1+\frac{1}{x^4}$.

25. $\frac{x^2}{y^2} - 1 + \frac{y^2}{x^2}$. 26. $\frac{a^4}{b^4} + \frac{b^4}{a^4} - 2$. 27. $\frac{a^2}{b^2} - \frac{b^2}{c^2} - \frac{c}{a^2} + \frac{2b}{a}$.

28. $\frac{a}{b} + 1$. 29. $a^4 - 1$. 30. $a^4 + \frac{4}{a^4}$.

31. $\frac{a^2}{2} - 1 + \frac{1}{a^2}$. 32. $x^3 - 2 + \frac{1}{x^3}$. 33. $\frac{x^3}{8} + 2 + \frac{8}{x^3}$.

a^2. 35. $\frac{x^3}{3} - \frac{3x^2}{4} + \frac{5x}{2} - \frac{2}{3}$.

Exercise XLVII. Page 165

1. $\frac{3x-18}{x^4}$. 2. $\frac{3x+9}{4x-8}$. 3. $-\frac{1}{5}$. 4. $\frac{14x-2}{19-6x}$.

5. $\frac{3x}{6x+10}$. 6. $x^2 - 2x + 2$. 7. $\frac{a+1}{a-1}$. 8. $a + b$.

9. $\frac{1}{a+b+c}$. 10. $\frac{ab}{4}$. 11. b. 12. $-\frac{2x}{1+x^2}$.

13. $\frac{2b}{a}$. 14. $\frac{2a^2-5a+2}{2(2a^2+5a+2)}$. 15. $\frac{c+a}{ac(b-c-a)}$.

16. 2. 17. $\frac{1}{a-b}$. 18. $\frac{4}{3x}$. 19. $\frac{4}{x+1}$.

20. $\frac{x^2}{1+x+x^2}$. 21. $\frac{x^2-xy+y^2}{x^2+xy+y^2}$. 22. $\frac{2}{x}$.

Exercise XLVIII. Page 168

1. $x = 5$. 2. $x = 11$. 3. $x = -3$. 4. $x = 4$.
5. $x = 40$. 6. $x = 37\frac{1}{2}$. 7. $x = 350$. 8. $x = 480$.
9. $x = 42$. 10. $x = 16$. 11. $x = 1\frac{3}{4}$. 12. $x = \frac{1}{3}$.
13. $x = 5$. 14. $x = 3$. 15. $x = 2$. 16. $x = \frac{4}{13}$.
17. $x = 7$. 18. $x = 2$. 19. $x = 3\frac{1}{2}$. 20. $x = 3\frac{3}{4}$.
21. $x = 0, \frac{4}{13}$. 22. $x = 1$. 23. $x = -1$. 24. $x = 5\frac{1}{3}$.
25. $x = 2$. 26. $x = \frac{1}{2}$. 27. $x = 2\frac{1}{2}$. 28. $x = \frac{3}{4}$.
29. $x = 1\frac{1}{4}$. 30. $x = 12$. 31. $x = 12$. 32. $x = \frac{3}{2}$.
33. $x = 0, 3$. 34. $x = -1$. 35. $x = 8\frac{2}{3}$. 36. $x = \frac{a}{a-1}$.

37. $x = \frac{2ab}{a+b}$. 38. $x = \frac{a+b}{2}$. 39. $x = \frac{ab}{b-a}$. 40. $x = a - b$.

41. $x = \frac{1}{a-b}$. 42. $x = b - a$. 43. $x = \frac{a^2}{b-a}$. 44. $x = \frac{a}{a-b-1}$

45. $x = \frac{7ab}{8a+3b}$. 46. $x = \frac{2ab}{a+b}$. 47. $x = \frac{a+b}{2ab}$. 48. $x = \frac{1}{2a}$.

Exercise XLIX. Page 172

1. $x = 2$. 2. $x = \frac{3}{2}$. 3. $x = \frac{1}{2}$. 4. $x = 2$.

5. $x = 0$. 6. $x = 1\frac{3}{4}$. 7. $x = \frac{-ab}{a+2b}$. 8. $x = a + b$.

9. $x = 7$. 10. $x = 9$. 11. $x = 2\frac{1}{2}$. 12. $x = 4$.

13. $x = 4\frac{1}{2}$. 14. $x = \frac{4}{5}$. 15. $x = 4$. 16. $x = 2$.

17. $x = 6$. 18. $x = -7$. 19. $x = 17\frac{1}{2}$. 20. $x = 6$.

21. $x = 75\frac{1}{2}$. 22. $x = 3$. 23. $x = 2$. 24. $x = 2a$.

25. $x = \frac{bc}{b+c}$. 26. $x = \frac{a+b}{ab}$. 27. $x = \frac{ab}{a+b}$.

28. $x = \frac{abc}{ab+bc+ca}$. 29. $x = \frac{am-bn}{m-n}$. 30. $x = \frac{a^2m+b^2n}{am+bn}$.

Exercise L. Page 176

1. $x = 4\frac{1}{3}$. 2. $x = \frac{2ab}{a-b}$. 3. $x = 2\frac{2}{7}$. 4. $x = 1$.

5. $x = \frac{a}{a+b-1}$. 6. $x = \frac{ab}{a+b-c}$. 7. $x = \frac{1}{2}$. 8. $x = -$

9. $x = 0$. 10. $x = \frac{ab-ac-bc}{c}$. 11. $x = \frac{a^2+b^2}{a+b}$.

12. $x = \frac{-ab(a+b)}{a^2+b^2}$. 13. $x = \frac{c-a}{2}$. 14. $x = -2(a+b+c$

15. $x = \frac{br-cq}{cp-ar}$. 16. $x = \frac{b}{a}$. 17. $x = 2,\ 2\frac{1}{2}$.

18. $x = a+b+c$. 19. $x = -(a+b+c)$. 20. $x = \frac{a-b}{2}$

21. $x = 2(b-a)$. 22. $x = a+b+c$. 23. $x = a+b+c$.

Exercise LI. Page 181

1. $\frac{6}{15}$. 2. $\frac{12}{15}$. 3. $\frac{1}{5}$. 4. [illegible]
5. \$2400, \$4000. 6. \$1501. 7. 45 seconds.
8. 320 yds. 9. $2\frac{1}{3}$ hrs., $93\frac{1}{3}$ mi. 10. [illegible] mi. per hr.
11. $6\frac{1}{2}$ mi. 12. $3\frac{1}{3}$ days. 13. [illegible] days.
14. 30 days. 15. $37\frac{1}{2}$ min. 16. 40 days.
17. $24\frac{1}{2}$ mi. ; 1 mi. per hr. 18. 18 mi. per hr. ; 78 mi.
19. 1 hr. ; double. 20. $2\frac{2}{5}$ mi. per hr.
21. $16\frac{4}{11}$ min., $49\frac{1}{11}$ min. after 6.
22. 3[illegible]$\frac{8}{11}$ min. ; 42 min. after 3. 23. 5:30 ; 5:42.
24. $7:5\frac{5}{11}$; $7:21\frac{9}{11}$; $7:38\frac{2}{11}$. 25. 3:30.
26. 49, 15. 27. 6. 28. $10\frac{6}{25}$. 29. 16 mi.
30. 24, 40. 31. $1\frac{2}{3}$ hrs. 32. \$22200. 33. \$15250.
34. $10\frac{10}{11}$. 35. \$4575, \$2910. 36. 60 days.
37. 550 yds. : 6 min. 38. 9.18 a.m. ; $\frac{3}{5}$.
39. $10.37\frac{1}{2}$ a.m. 40. \$1250. 41. 20 mi.

Exercise LII. Page 190

1. $x = 16$, $y = 21$.
2. $x = 11$, $y = 7$.
3. $x = 6$, $y = 9$.
4. $x = 3$, $y = 5$.
5. $x = 2$, $y = -1$.
6. $x = 4$, $y = -1$.
7. $x = 5$, $y = 9$.
8. $x = 13$, $y = 5$.
9. $x = 9$, $y = -1$.
10. $x = 3$, $y = 5$.
11. $x = 5$, $y = 0$.
12. $x = 3$, $y = 1$.
13. $x = 19$, $y = 3$.
14. $x = 4$, $y = 3$.
15. $x = 17$, $y = 5$.
16. $x = 17$, $y = 3$.
17. $x = 7$, $y = -2$.
18. $x = -1$, $y = 1$.
19. $x = 11$, $y = 5$.
20. $x = 4$, $y = 7$.
21. $x = 1$, $y = 3$.
22. $x = 6$, $y = 8$.
23. $x = \frac{3}{2}$, $y = -3$.
24. $x = 9$, $y = 12$.
25. $x = 1$, $y = \frac{1}{5}$.
26. $x = 3\frac{1}{2}$, $y = 7$.
27. $x = \frac{1}{63}$, $y = 36$.

28. $x = \frac{3}{7}$, $y = -\frac{2}{3}$.

29. $x = -20$, $y = 6\frac{2}{3}$.

30. $x = a + b$, $y = a - b$.

31. $x = \frac{a - b^2}{1 - ab}$, $y = \frac{a^2 - b}{1 - ab}$.

32. $x = \frac{ab(a + b)}{a^2 + b^2}$, $y = \frac{ab(a - b)}{a^2 + b^2}$.

33. $x = \frac{a}{a - b}$, $y = \frac{b}{a + b}$.

34. $x = \frac{2}{a + b}$, $y = \frac{2}{a - b}$.

35. $x = \frac{a + b}{ab}$, $y = \frac{a - b}{ab}$.

36. $x = a(b + 1)$, $y = b(a + 1)$.

37. $x = a(b - c)$, $y = b(c - a)$.

38. $x = ac$, $y = bd$.

39. $x = \frac{a - b}{a + b}$, $y = \frac{a + b}{a - b}$.

40. $x = \frac{(a + b)^2}{a^2 + b^2}$, $y = \frac{a^2 - b^2}{a^2 + b^2}$.

41. $x = \frac{1}{b}$, $y = \frac{1}{a}$.

42. $x = \frac{a + b}{ab}$, $y = \frac{a - b}{ab}$

43. $x = \frac{1 + a}{1 + b}$, $y = \frac{a + b}{1 + b}$.

44. $x = y = \frac{a^2 - b^2}{a}$.

Exercise LIII. Page 196

1. $x = 4$, $y = 2$, $z = 5$.

2. $x = -1$, $y = 4$, $z = 3$.

3. $x = 3$, $y = 2$, $z = 5$.

4. $x = 5$, $y = 6$, $z = 1$

5. $x = 3$, $y = 2$, $z = 1$.

6. $x = 2\frac{1}{2}$, $y = 3\frac{1}{3}$, $z = 0$.

7. $x = 4$, $y = -2$, $z = 3$.

8. $x = 1$, $y = 2$, $z = 3$

9 $x = 9$, $y = 3$, $z = 11$.

10. $x = 4$, $y = 1$, $z = -1$.

11. $x = 7$, $y = -2$, $z = 3$.

12. $x = 2$, $y = 1$, $z = 2$.

13. $x = 2$, $y = -3$, $z = 4$.

14. $x = 3$, $y = 5$, $z = -\frac{1}{2}$.

15. $x = 5$, $y = 7$, $z = -3$.

16. $x =$ [illegible], $y =$ [illegible], $z =$ [illegible]

17. $x = 26,\ y = 7,\ z = -2.$

18. $x = 15,\ y = 0,\ z = 18.$

19. $x = 2,\ y = 1,\ z = \frac{2}{3}.$

20. $x = \frac{2}{3},\ y = -\frac{1}{3},\ z = 1.$

21. $x = \frac{3}{2},\ y = -1,\ z = \frac{1}{2}.$

22. $x = \frac{1}{2},\ y = \frac{1}{3},\ z = \frac{1}{4}$

23. $x = 3,\ y = 5,\ z = 7.$

24. $x = b + c - a,\ y = c + a - b,\ z = a + b - c.$

25. $x = \frac{b^2 + c^2 - a^2}{2bc},\ y = \frac{c^2 + a^2 - b^2}{2ca},\ z = \frac{a^2 + b^2 - c^2}{2ab}.$

26. $x = b - c,\ y = c - a,\ z = a - b.$

27. $x = \frac{1}{(a-b)(a-c)},\ y = \frac{1}{(b-c)(b-a)},\ z = \frac{1}{(c-a)(c-b)}.$

28. $x = abc,\ y = ab + bc + ca,\ z = a + b + c.$

29. $x = 4,\ y = 7,\ z = 10.$

30. $x = n + p - m,\ y = p + m - n,\ z = m + n - p.$

31. $x = b + c - a,\ y = c + a - b,\ z = a + b - c.$

32. $x = a,\ y = b,\ z = c.$

Exercise LIV. Page 198

1. 20, 10. 2. 15, 3. 3. 11. 4. 37. 5. 49.
6. \$42, \$48. 7. \$3.20, \$2.40. 8. 90 cts., 64 cts.
9. \$12.40, \$1.60. 10. 48, 45, 27. 11. \$45, \$27.
12. 24 days, 30 days. 13. 60 days, $8\frac{4}{7}$ days.
14. $1\frac{1}{2}$ hrs., 1 hr., $\frac{3}{4}$ hr. 15. 6 mi.; 2 mi. per hr.
16. $58\frac{1}{2}$ min. 17. 8, $7\frac{1}{2}$. 18. 396 ft.; 12 mi. per hr.
19. 8 ft., 11 ft. 20. 10 days, 30 days, 30 days. 21. 5, 7.
22. 14, 10, 8 gal. 23. \$13, \$8, \$3. 24. 1000, 600, 100.
25. \$1000, \$1200. 26. 30, 20, 60, 10 days. 27. 10, 65.
28. 30 mi. 29. \$3000, \$2000; \$2500, \$1500.
30. 267. 31. 600 sq. ft. 32. $\frac{5}{11}$. 33. 12, 11. 34. 5, $4\frac{8}{9}$.
35. 24, 60. 36. $4\frac{1}{3}$ ft., 12 ft. 37. \$2200, \$6000.

EXAMINATION PAPERS

I. Page 203

1. 2. 2. $a^3 + b^3 - 3b + \frac{1}{a^3}$. 3. $x^2 + ax - a$.
4. $2x + 1$, $(2x + 1)(x - 4)(x^2 - 4x + 2)$. 5. $x = \frac{3}{2}$, $y = 1$
6. 7. 7. $16\frac{2}{3}$.

II. Page 204

1. $\frac{x^2 + 7x + 14}{(x^2 + 5)(x + 2)}$, $x^2 - 2x + 2$. 2. $x - 2y$.
3. $x = \frac{c - a}{2}$, $\frac{2a}{a + c}$. 4. 2. 5. $x^2 - 3x + 2$, $x^2 - 4x + 3$
6. \$90, 3 persons. 7. 104 inches.

III. Page 205

1. ac. 3. $\frac{b}{b - a}$. 4. $x = \frac{a^2 - b^2}{b}$, $y = \frac{a^2 - b^2}{a}$
5. $60a^2b^2(a^2 - b^2)^2$. 6. $3\frac{1}{3}$ days. 7. 7 mi. of hills, 4 mi. level.

IV. Page 205

1. $a(x + 2)$, $a(x^2 + 2x + 3)$. 2. $x + 3$. 3. $\frac{x(x - 1)}{x^2 - x + 1}$.
4. $\frac{a - c}{ac}$. 5. $x = \frac{32}{33}$, $y = \frac{28}{11}$, $z = \frac{86}{33}$. 6. 4. 7. $5\frac{1}{2}$ lbs.

V. Page 206

1. $\frac{a + 1}{b} + 1$. 2. $-\frac{2(a + b)}{b}$. 3. $2\frac{4}{7}$.
5. $x = a - 1$, $y = b + 1$. 6. 12 days, 24 days.
7. 35, $9\frac{1}{2}$, $23\frac{1}{2}$.

VI. Page 207

1. $\frac{1}{a} + \frac{2}{a^2}$, $\frac{1}{a} + \frac{2}{a^2} + \frac{3}{a^3}$. 2. Each expression $= (x + y + a +$
3. $x^3 + 3x^2 - x - 3$, $x^3 - 3x^2 - x + 3$. 4. 1. 5. $x = m -$
6. $\frac{m}{7t}$. 7. 38 lbs., 60 lbs.

VII. Page 208

1. 6, 12, 12 ; 28, 56, 56 ; 120, 240, 240. 2. 2.

3. $x^2 - ax + b$, $(x^2 - a^2)(x^2 - ax + b)$. 4. $\dfrac{48}{(x^2 - 1)(x^2 - 9)}$.

5. $x = 2a + b$, $y = a$. 6. 50 cents. 7. 5, $2\frac{1}{2}$ mi. per hr.

Exercise LV. Page 215

2. (1) $(-1, -4)$; (2) $(\frac{33}{13}, \frac{36}{13}) = (2\frac{1}{2}, 2\frac{5}{8})$ nearly ; (3) (1, 3).

5. (1) $x = 5$, $y = 4$. (2) $x = 2\frac{5}{7}$, $y = -1\frac{1}{7}$. (3) $x = -2$, $y = 4$.
(4) $x = 4$, $y = -3$. (5) $x = 37\frac{1}{5}$, $y = -37\frac{4}{5}$. (6) $x = 3$, $y = 4$.

Exercise LVI. Page 221

1. $2abc$. 2. $5a^3b^2c^4$. 3. $8a^8b^6c$. 4. $\dfrac{7a^2b}{9c^2}$.

5. $\dfrac{10c^4ba^5}{3c^2d}$. 6. $\dfrac{6ab}{4cd}$. 7. $x + 6$. 8. $x - 5$.

9. $x - 4$. 10. $x - 18$. 11. $9a - b$. 12. $5x - 7y$.

13. $a^2 - 5ax + 4x^2$. 14. $1 + 2x + 3x^2$. 15. $17 - 11x + x^3$.

16. $x + 2y + 3z$. 17 $5x^2y - 3xy^2 + 2y^3$. 18. $3x^2 + xy - 8y^2$.

19. $\dfrac{x^2}{4} + \dfrac{x}{2y} - \dfrac{1}{5y^2}$. 20. $a^3 - 6 + \dfrac{1}{a^3}$. 21. $x^2 - xy + y^2$.

22. $ab + bc - ca$. 23. $a^2 - b^2 + c^2 - d^2$.

Exercise LVII. Page 226

1. $4ab^2c^3$. 2. $-5a^2b^4c^5$. 3. $\dfrac{3ab^2c^3}{8d^4e^5}$. 4. $x + 2$.

5. $4x - 1$. 6. $5a + b$. 7. $x + 4$. 8. $x^2 + 3x + 1$.

9. $x^2 - 2x - 4$. 10. $3x^2 - 4x - 3$. 11. $a + \dfrac{1}{a}$. 12. $\dfrac{x}{y} - 1 - \dfrac{y}{x}$.

Exercise LVIII. Page 232

1. 4.
2. 4.
3. $\frac{1}{4}$.
4. $\frac{1}{81}$.
5. $\frac{1}{10^4}$
6. $\frac{256}{81}$.
7. $\frac{81}{256}$.
8. 10.
9. $a^{\frac{1}{2}}$.
10. $a^{\frac{1}{3}}$.
11. $(ab)^{\frac{3}{4}}$.
12. $(ab)^{\frac{1}{4}}$.
13. $3a^{-\frac{2}{3}}$.
14. $9m^{\frac{1}{4}}n^{\frac{7}{8}}$.
15. a
16. $25a$.
17. $\sqrt[3]{a^2}$.
18. $\sqrt[3]{a^{-2}}$.
19. $\sqrt[3]{a^2}\sqrt[4]{b^3}$.
20. $\sqrt[n]{m}$.
21. $\sqrt[n]{x^m}\sqrt[m]{n^r}$.
22. $\sqrt[4]{a^{-3}b^3c^{-2}}$.
23. $\sqrt[7]{x^6}\sqrt[9]{y^4}$.
24. $3\sqrt[6]{b^4c^{-2}}$.
25. $\frac{1}{a^{\frac{3}{4}}}$.
26. $\frac{1}{b^{\frac{1}{2}}}$.
27. $a^{\frac{1}{2}}$.
28. $\frac{1}{b^{\frac{3}{4}}}$.
29. $\frac{b^{\frac{3}{4}}}{a^{\frac{1}{2}}}$.
30. $\frac{1}{x^{\frac{1}{2}}y^{\frac{2}{3}}}$.
31. $\frac{6c^3}{a^2b^2x}$.
32. $\frac{ab}{\frac{1}{a}+\frac{1}{b}}$.
33. $\frac{1}{a^{-\frac{1}{2}}}$.
34. $a^{-\frac{1}{2}}$.
35. $\frac{y^{-\frac{1}{2}}}{x^{-\frac{1}{2}}}$.
36. $\frac{1}{x^{-\frac{1}{2}}y^{-\frac{2}{3}}}$.
37. $\frac{a^{-\frac{2}{3}}b^{-\frac{1}{2}}}{x^{-\frac{1}{2}}y^{-\frac{3}{2}}}$.
38. $\frac{1}{a^{-\frac{2}{3}}b^{-\frac{1}{3}}}$.
39. $\frac{x^{-\frac{1}{2}}y^{-1}}{a^{-1}c^{-\frac{1}{2}}}$.
40. $\frac{b^{-\frac{3}{2}}y^{-2}}{c^{-2}x^{-1}}$.
41. $2x^{-5}$, $\frac{1}{2^{-1}x^5}$.
42. $x^{\frac{1}{3}}y^{-\frac{2}{3}}$, $\frac{1}{x^{-\frac{1}{3}}y^{\frac{2}{3}}}$.
43. $a^{\frac{3}{4}}b^{\frac{3}{4}}$, $\frac{1}{a^{-\frac{3}{4}}b^{-\frac{3}{4}}}$.
44. $m^{-\frac{1}{n}}n^{\frac{1}{m}}$, $\frac{1}{m^{\frac{1}{n}}n^{-\frac{1}{m}}}$.
45. $a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{3}{4}}x^{-\frac{1}{2}}y^{-\frac{1}{3}}z^{-\frac{3}{4}}$, $\frac{1}{a^{-\frac{1}{2}}b^{-\frac{1}{3}}c^{-\frac{3}{4}}x^{\frac{1}{2}}y^{\frac{1}{3}}z^{\frac{3}{4}}}$.
46. $a^{\frac{1}{3}}b^{-\frac{3}{4}}c^{-\frac{1}{7}}d^{\frac{2}{3}}$, $\frac{1}{a^{-\frac{1}{3}}b^{\frac{3}{4}}c^{\frac{1}{7}}d^{-\frac{2}{3}}}$.
47. $8(27)^{-1}a^2b^3c^{-3}d^{-2}$, $\frac{1}{27(8)^{-1}a^{-2}b^{-3}c^3d^2}$.
48. $216a^3mb^{-3}n^{-1}$, $\frac{1}{(216)^{-1}a^{-3}m^{-1}b^3n}$.

49. 1. 50. $a^{\frac{17}{12}}$. 51. $a^{\frac{1}{6}}b^{\frac{1}{6}}c^{1\frac{1}{2}}$. 52. 1.
53. $a^{\frac{1}{6}}b^{-\frac{1}{6}}c^{-1\frac{1}{2}}$. 54. $c^{-\frac{1}{6}}x^{\frac{1}{3}}y^{\frac{1}{6}}$. 55. $a^{2}b^{-2}$. 56. 1.
57. 1. 58. $\frac{27x^3y^{\frac{9}{4}}}{8}$. 59. $\frac{b^4x^4}{a^{\frac{4}{3}}y}$. 60. $\frac{a^{\frac{3}{5}}}{b^4x^4y^4}$.

Exercise LIX. Page 234

1. $x^{\frac{2}{3}} - y^{\frac{2}{3}}$.
2. $x^{\frac{5}{6}} + x^{\frac{1}{2}}a^{\frac{1}{3}} + x^{\frac{1}{3}}a^{\frac{1}{2}} + a^{\frac{5}{6}}$.
3. $x^{\frac{4}{3}} - y^{\frac{4}{3}}$.
4. $x^{\frac{5}{3}} - x^{\frac{4}{3}} + x - x^{\frac{2}{3}} + x^{\frac{1}{3}} - 1$.
5. $x^{\frac{5}{3}} + x^{\frac{4}{3}} - x - x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$.
6. $a^{\frac{4}{3}} + a^{\frac{2}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$.
7. $x^n + 1 + x^{-n}$.
8. $x^{\frac{5}{2}} + x^{\frac{3}{2}}y - xy^{\frac{3}{2}} - y^{\frac{5}{2}}$.
9. $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$.
10. $x^{\frac{5}{6}} + x^{\frac{4}{6}}y^{\frac{1}{6}} + x^{\frac{3}{6}}y^{\frac{2}{6}} + x^{\frac{2}{6}}y^{\frac{3}{6}} + x^{\frac{1}{6}}y^{\frac{4}{6}} + y^{\frac{5}{6}}$.
11. $a^{\frac{2}{3}} - 5a^{\frac{1}{3}}b^{\frac{1}{3}} + 25b^{\frac{2}{3}}$.
12. $a^{\frac{2}{3}} + 4a^{\frac{1}{3}}b^{\frac{1}{3}} + 16b^{\frac{2}{3}}$.
13. $x^{\frac{4}{3}} - x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}$.
14. $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.
15. $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + y^{\frac{3}{2}}$.
16. $x^{-\frac{2}{3}} - 2x^{-\frac{1}{3}} - 3$.
17. $x^{\frac{1}{3}} + 1$.
18. $2a^{\frac{1}{3}} - y^{\frac{1}{3}}$.
19. $a^{\frac{1}{2}} + b^{\frac{1}{2}} + a^{-\frac{1}{2}}b^{-\frac{1}{2}}$.
20. $2a^{-1} - 3 + 4a$.
21. $3x^{-2} + 4x^{-1} - 6$.
22. $(x^{\frac{1}{3}} - y^{\frac{1}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})$.
23. $(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})$.
24. $(x^{\frac{1}{3}} + a^{\frac{1}{3}})^2$.
25. $(x^{\frac{1}{3}} - a^{\frac{1}{3}})^2$.
26. $(x^{\frac{1}{3}} + a^{\frac{1}{3}})(x^{\frac{1}{3}} - a^{\frac{1}{3}})$.
27. $(x^{\frac{1}{4}} - 4)(x^{\frac{1}{2}} + 4x^{\frac{1}{4}} + 16)$.
28. $(2x^{\frac{2}{3}} + y^{\frac{2}{3}})(x^{\frac{2}{3}} + 2y^{\frac{2}{3}})$.
29. $(3x - 4y^{-2})(x - y^{-2})$.
30. $2a^{-4} - 2$.
31. 4.
32. $\frac{2(a^{\frac{1}{3}}y^{\frac{2}{3}} - a^{\frac{2}{3}}y^{\frac{1}{3}})}{a + y}$.
33. 0.
34. $\frac{3}{x - 1}$.
35. a^{xy}.

Exercise LX. Page 237

1. $5\sqrt{2}$. 2. $8\sqrt{2}$. 3. $4\sqrt{6}$. 4. $5\sqrt{3}$.
5. $ab\sqrt{c}$. 6. $2\sqrt[3]{7}$. 7. $3\sqrt[3]{3}$. 8. $8\sqrt[4]{2}$.
9. $6\sqrt[3]{2}$. 10. $7abc^2\sqrt[3]{b^2c}$. 11. $xyz\sqrt[4]{xz^2}$.

12. $2xyz\sqrt[3]{5z^3}$. 13. $5x^2y\sqrt{2y} - 3x^2y^3$. 14. $5(x-y)\sqrt{2}$
15. $\sqrt{18}$. 16. $\sqrt{12}$. 17. $\sqrt[3]{54}$. 18. $\sqrt[3]{24}$.
19. $\sqrt[4]{80}$. 20. $\sqrt{a^2b}$. 21. $\sqrt[n]{\frac{b}{a}}$. 22. $\sqrt{\frac{a+b}{a-b}}$
23. $\sqrt[3]{\frac{(a+b)^4}{(a-b)^2}}$. 24. $\sqrt{ab}$.

Exercise LXI. Page 239

1. $\sqrt[6]{125}$, $\sqrt[6]{16}$. 2. $\sqrt[6]{36}$, $\sqrt[6]{343}$. 3. $\sqrt[12]{16}$, $\sqrt[12]{27}$
4. $\sqrt[12]{125}$, $\sqrt[12]{1296}$, $\sqrt[12]{625}$. 5. $\sqrt[6]{(x^2+y^2)^2}$, $\sqrt[6]{(x+y)}$
6. $\sqrt[12]{\frac{x^4}{y^4}}$, $\sqrt[12]{\frac{y^3}{x^3}}$. 7. $2\sqrt{5}$, $3\sqrt{5}$. 8. $5\sqrt{2}$, $6\sqrt{2}$.
9. $5\sqrt{3}$, $9\sqrt{3}$. 10. $\frac{1}{3}\sqrt[3]{3}$. 11 $\frac{1}{4}\sqrt[4]{7}$. 12. $\frac{1}{3}\sqrt[4]{6}$.
13. $\frac{1}{5a}\sqrt{35a}$. 14. $\frac{1}{5a^3}\sqrt{35a}$. 15. $\sqrt{a^2-b^2}$.

Exercise LXII. Page 240

1. $7\sqrt{5}$. 2. $\sqrt[3]{4}$. 3. 0. 4. $23\sqrt{2}$.
5. $-15\sqrt[3]{4}$. 6. $5\sqrt[3]{7} + 9\sqrt{2}$. 7. $2\sqrt{2}$.
8. $-\frac{3}{2}\sqrt[3]{3}$. 10. $(a+b+c)\sqrt[4]{b}$. 11. $3\sqrt[3]{2} - 2\sqrt[3]{57}$
12. $2x^2y(3+2y^2)\sqrt{xy}$. 13. $-\frac{1}{8}\sqrt{2}$. 14. $\frac{13}{24}\sqrt[3]{3}$.
15. $28\sqrt{2} - 3\sqrt{3}$. 16. $(2a+2b+1)\sqrt{a+b}$.
17. $(2a-4b)\sqrt{a^2-b^2}$.

Exercise LXIII. Page 242

1. $5\sqrt{3}$. 2. 10. 3. $5\sqrt{6}$. 4. $60\sqrt{6}$.
5. $5\sqrt[3]{3}$. 6. 7. 7. $168\sqrt[4]{6}$. 8. $12\sqrt[4]{2}$.
9. $\sqrt[12]{32000}$. 10. $2\sqrt[3]{3}$. 11. $4\sqrt{30}$. 12. $\sqrt{3}$.
13. 2. 14. $\frac{1}{2}\sqrt[4]{10}$. 15. $\frac{9}{16}\sqrt[3]{21}$. 16. $\sqrt[3]{63}$.
17. $\frac{1}{5}\sqrt[6]{75}$. 18. $2\sqrt[6]{288}$. 19. $2\sqrt[3]{2}$. 20. $\frac{5}{27}\sqrt[6]{194}$

21. $30+6\sqrt{5}-10\sqrt{15}$. 22. $\sqrt[3]{1850}+2\sqrt[3]{50}-10\sqrt[3]{6}$.
23. $-\sqrt{6}$. 24. $42-18\sqrt{6}$. 25. $40-13\sqrt{10}$.
26. $3-\sqrt[3]{36}+3\sqrt[3]{4}-2\sqrt[3]{18}$. 27. $\sqrt[3]{3}-\sqrt[3]{4}-\sqrt[3]{15}$.
28. $12\sqrt{3}-6+2\sqrt{10}$.

Exercise LXIV. Page 244

1. $5+2\sqrt{6}$. 2. $140+48\sqrt{6}$. 3. $49-4\sqrt{10}-6\sqrt{6}+12\sqrt{15}$.
4. $2x+3\sqrt{x^2-y^2}$. 5. $\frac{\sqrt{6}+\sqrt{2}}{4}$. 6. $\frac{3-\sqrt{6}}{6}$.
7. $\frac{-2(3+2\sqrt{6})}{5}$. 8. $\frac{3(4+\sqrt{6})}{5}$. 9. $\sqrt{6}-\sqrt{3}+\sqrt{2}-2$.
10. $3+2\sqrt{2}$. 11. $2\sqrt{30}-11$. 12. $\frac{6\sqrt{2}-11}{7}$.
13. $\frac{26+7\sqrt{3}}{23}$. 14. $\frac{70+24\sqrt{6}}{19}$. 15. $2x^2-1-2x\sqrt{x^2-1}$.
16. $\frac{x+2-2\sqrt{x+1}}{x}$. 17. $\frac{x+\sqrt{x^2-y^2}}{y}$.
18. $\frac{x^2+\sqrt{x^4-y^4}}{y^2}$. 19. $\sqrt{xy}$. 20. $\frac{x+\sqrt{x^2-4y^2}}{2y}$.
21. $\frac{2a^2+x\sqrt{4a^2-x^2}}{x^2-2a^2}$. 22. $\frac{x\sqrt{x^2+y^2}}{x^2+y^2}$.
23. $\frac{a^3+b+1-2a\sqrt{ab+a}}{a^3-b-1}$. 24. $\frac{\sqrt{10}-\sqrt{35}}{5}$.

Exercise LXV. Page 247

1. $\sqrt{5}-1$. 2. $1+\sqrt{3}$. 3. $\sqrt{3}-1$. 4. $1+\sqrt{7}$.
5. $\sqrt{5}-\sqrt{3}$. 6. $\sqrt{5}+\sqrt{3}$. 7. $3+\sqrt{2}$. 8. $3-\sqrt{2}$.
9. $2+\sqrt{6}$. 10. $5+4\sqrt{3}$. 11. $\sqrt{7}-\sqrt{5}$. 12. $\sqrt{5}+3\sqrt{2}$.
13. $\sqrt{a+x}+\sqrt{a-x}$. 14. $\sqrt{x+1}-\sqrt{x-1}$. 15. $y+\sqrt{x^2-y^2}$.

Exercise LXVI. Page 248

1. $x=25$. 2. $x=a^2+b^2$. 3. $x=1$. 4. $x=1$.
5. $\frac{1}{9}(y-1)^2$. 6. $x=-4\frac{7}{8}$. 7. $x=32$. 8. $x=4$.

9. $x=5$. 10. $x=9$. 11. $x=9$ 12. $x=1562\frac{1}{2}$.
13. $x=3$. 14. $x=4$. 15. $x=\frac{1}{5}$. 16. $x=1$.
17. $x=\frac{2a}{\sqrt{5}}$. 18. $x=\sqrt{5}-1$. 19. $x=3$. 20. $x=\sqrt{2}-1$.
21. $x=\frac{3}{4}$. 22. $x=\frac{-\sqrt{a}}{\sqrt{a}+2}$. 23. $x=\frac{1}{5}$. 24. $x=4(2-\sqrt{3})$

Exercise LXVII. Page 255

1. $x=\pm 3$. 2. $x=\pm 7$. 3. $x=\pm\sqrt{10}$
4. $x=\pm\sqrt{6}$. 5. $x=0,\ \frac{3}{2}$. 6. $x=0,\ \frac{3}{4}$.
7. $x=0,\ \frac{5}{2}$. 8. $x=0,\ \frac{5}{3}$. 9. $x=0,\ -\frac{4}{5}$
10. $x=0,\ \frac{b}{a}$. 11. $x=2,\ 3$. 12. $x=2,\ 8$.
13. $x=3,\ 7$. 14. $x=3,\ 17$. 15. $x=4,\ 20$.
16. $x=4,\ 12$. 17. $x=8,\ -6$. 18. $x=20,\ 7$
19. $x=2,\ -\frac{3}{2}$. 20. $x=3,\ -\frac{4}{3}$. 21. $x=14,$ [illegible]
22. $x=\frac{7}{3},\ \frac{3}{4}$. 23. $x=0,\ \frac{13}{3}$. 24. $x=\frac{9}{7},\ -2$.
25. $x=-3\pm\sqrt{119}$. 26. $x=3,\ -1$. 27. $x=1,\ -\frac{1}{5}$.
28. $x=10,\ 1$. 29. $x=-\frac{1}{5},\ -\frac{3}{2}$. 30. $x=7,\ \frac{3}{2}$.

Exercise LXVIII. Page 258

1. $x=\frac{5\pm\sqrt{13}}{2}$. 2. $x=22,\ -7$. 3. $x=8,\ -$[illegible]
4. $x=7,\ -\frac{1}{2}$. 5. $x=2,\ -\frac{3}{4}$. 6. $x=\frac{5}{2},\ -\frac{1}{2}$
7. $x=3,\ -\frac{7}{5}$. 8. $x=\frac{2}{3},\ \frac{2}{3}$. 9. $x=2,\ -4$
10. $x=\frac{3a}{4},\ \frac{2a}{3}$. 11. $x=\frac{6a}{7},\ -\frac{3a}{4}$. 12. $x=\frac{3}{5},\ -\frac{5}{3}$.
13. $x=\frac{1}{2},\ \frac{1}{3}$. 14. $x=2,\ -\frac{13}{2}$. 15. $x=3,\ -$[illegible]
16. $x=8,\ -\frac{5}{3}$. 17. $x=\frac{9\pm\sqrt{241}}{20}$. 18. $x=3,\ -$[illegible]
19. $x=-3\frac{1}{2}$. 20. $x=\frac{-12\pm\sqrt{38}}{2}$. 21. $x=0,\ \frac{7}{2}$.

22. $x = 6 \pm \sqrt{91}$. 23. $x = \pm\sqrt{-ab}$. 24. $x = \frac{a}{2}(1 \pm \sqrt{-3})$

25. $x = a(2 \pm \sqrt{2})$. 26. $x = -a, -b$. 27. $x = 0, -\frac{a^2+b^2}{a+b}$.

28. $x = \frac{a+b}{a-b}, \frac{b-a}{b+a}$. 29. $x = 2, -3$.

30. $x = \frac{81 \pm \sqrt{657}}{18}$. 31. $x = 0, 4$. 32. $x = \frac{1 \pm \sqrt{73}}{2}$.

33. $x = 0, \frac{5 \pm \sqrt{-23}}{4}$. 34. $x = 3, \frac{1}{3}$.

35. $x = \frac{1}{c}\sqrt{bc(bc - 2a^2)}$.

Exercise LXIX. Page 262

1. $x = \pm 1, \pm\frac{1}{3}\sqrt{-55}$. 2. $x = 3, -1$. 3. $x = \sqrt[3]{\frac{3}{4}}, \sqrt[3]{\frac{4}{3}}$.
4. $x = \pm\frac{2}{3}, \pm 2\sqrt{-2}$. 5. $x = 27, -8$. 6. $x = 2\sqrt[3]{2}, 3\sqrt[3]{3}$.
7. $x = 1, \frac{1}{16}$. 8. $x = \frac{1}{2}, -\frac{1}{3}$. 9. $x = \pm 1, \pm 3$.

10. $x = \pm a, \pm\frac{1}{a}$. 11. $x = \pm a, \pm\frac{1}{a}\sqrt{-1}$.

12. $x = \frac{2}{3}, \sqrt[3]{2}$. 13. $x = 3, -\frac{1}{2}$.

14. $x = 1, 1, -3 \pm 2\sqrt{2}$. 15. $x = \frac{3 \pm \sqrt{2}}{2}, \frac{3 \pm \sqrt{23}}{2}$.

16. $x = \pm 2, \pm\frac{1}{\sqrt{2}}$. 17. $x = 4, -3, \frac{1 \pm \sqrt{-43}}{2}$.

18. $x = 1, 1, -2 \pm \sqrt{3}$. 19. $x = \frac{1}{8}, -1$.

20. $3, -2, \frac{-1 \pm \sqrt{97}}{8}$. 21. $x = \frac{3 \pm \sqrt{-3}}{2}, \frac{1 \pm \sqrt{-3}}{2}$.

22. $x = \frac{1 \pm \sqrt{-35}}{6}, \frac{1 \pm \sqrt{-63}}{8}$. 23. $x = 2, -1$.

24. $x = \pm\sqrt{2}, \pm 2\sqrt{2}$. 25. $x = -1, -1, 1 \pm \sqrt{-1}$.

26. $x = -2 \pm 2\sqrt{5}, -2 \pm \sqrt{7}$. 27. $x = 2, -1, 1 \pm \sqrt{137}$.

28. $x = 26 - 2\sqrt{26}$. 29. $x = \pm\frac{5}{3}, \pm\frac{2}{3}\sqrt{21}$.

30. $x = 0, -\frac{5}{2}, \frac{-5 \pm \sqrt{-63}}{4}$.

Exercise LXX. Page 267

11. $x^2 - 5x + 6 = 0$.
12. $x^2 + 5x + 6 = 0$.
13. $4x^2 - 1 = 0$
14. $x^2 - a^2 = 0$.
15. $x^2 - 4x + 1 = 0$.
16. $x^2 - 6x + 7 = 0$
17. $x^2 - 2x + 3 = 0$.
18. $x^2 - 2ax + a^2 + b$.
19. $6, 17$; $\frac{a+b+c}{2}$, $\frac{ab}{2}$.
20. $x^2 - 2ax + a^2 - 1 = 0$; $a + 1$, $a - 1$.
21. $n = 12$, $x = 3$, 3; $n = -12$, $x = -3$, -3.
22. $6\frac{1}{4}$; $x = \frac{7}{4}$, $\frac{7}{4}$.
23. $n = \frac{1}{4}$, $x = 10 \pm 4\sqrt{5}$ [illegible]
24. $n = 3$, $x = \frac{5}{3}$, $\frac{5}{3}$; $n = \frac{1}{3}$, $x = \frac{5}{2}$, $-\frac{5}{2}$.

Exercise LXXI. Page 271

1. $x = 12$, -2,
   $y = 2$, -12.
2. $x = 8$, 4,
   $y = 4$, 8.
3. $x = 6$,
   $y = 4$.
4. $x = 7\frac{1}{2}$,
   $x = 4\frac{1}{2}$.
5. $x = 4$, $-\frac{14}{5}$ (?),
   $y = 2$, $-\frac{68}{15}$ (?).
6. $x = 3$, $-\frac{33}{27}$ (?),
   $y = -2$, [illegible]
7. $x = \pm 4$, $\pm \frac{1}{3}\sqrt{-5}$,
   $y = \pm 2$, $\pm \frac{14}{5}\sqrt{-5}$.
8. $x = \pm 3$, $\pm 12\sqrt{-2}$,
   $y = \pm 2$, $\mp \frac{17}{2}\sqrt{-2}$.
9. $x = \pm 4$, $\pm \frac{31}{\sqrt{79}}$,
   $y = \pm 3$, $\mp \frac{2}{\sqrt{79}}$.
10. $x = \pm 2$, $\pm \frac{4}{5}\sqrt{5}$,
    $y = \pm 1$, $\pm \frac{3}{5}\sqrt{5}$.
11. $x = \pm 5$, $\pm \frac{1}{\sqrt{7}}$,
    $y = \mp 2$, $\mp \frac{12}{\sqrt{7}}$.
12. $x = \pm 2$, $\pm \frac{10}{\sqrt{29}}$,
    $y = \pm 1$, $\pm \frac{1}{\sqrt{29}}$.
13. $x = \pm 3$, $\pm \frac{31}{13}\sqrt{-1}$,
    $y = \mp 2$, $\pm \frac{28}{13}\sqrt{-1}$.
14. $x = \pm 1$, $\pm \frac{1}{\sqrt{2}}$,
    $y = \pm 2$, $\pm \frac{3}{\sqrt{2}}$.
15. $x = \pm 2$, $\pm \frac{46}{7}$,
    $y = \pm 5$, $\mp \frac{5}{7}$.
16. $x = \pm 9$, ± 7,
    $y = \pm 7$, ± 9.

17. $x = \pm 8, \ \pm 4\sqrt{-1}$, $y = \pm 4, \ \pm 8\sqrt{-1}$.
18. $x = 8, \ 6$, $y = 6, \ 8$.
19. $x = 10, \ -16$, $y = 16, \ -10$.
20. $x = 30, \ -10$, $y = 10, \ -30$.
21. $x = 2, \ 5$, $y = 5, \ 2$.
22. $x = 6, \ -4$, $y = 4, \ -6$.
23. $x = 3, \ 1$, $y = 1, \ 3$.
24. $x = 4, \ -1$, $y = 1, \ -4$.
25. $x = \pm 7, \ \pm 1$, $y = \pm 1, \ \pm 7$.
26. $x = 2, \ 3$, $y = 3, \ 2$.
27. $x = 6, \ 4$, $y = 4, \ 6$.
28. $x = 10, \ -9$, $y = 3, \ -3\frac{1}{3}$.
29. $x = \pm 1, \ \pm 2$, $y = \pm 2, \ \pm 1$.
30. $x = 3, \ 9$, $y = 1, \ -3$.

Exercise LXXII. Page 276

1. 10, 11; −8, −9.
2. 10, 15; −10, −15.
3. 45, 60.
4. 20.
5. 30.
6. 10.
7. 8 in.
8. 5 rods.
9. 18 ft., 12 ft.
10. 4.142, 5.857.
11. $10(3 - \sqrt{5})$, $10(\sqrt{5} - 1)$.
12. 3.708.
13. 128 in., 30 in.
14. 5 hrs., 3 hrs.
15. $37\frac{1}{2}$ cents.
16. 30 sq. in.
17. $\frac{1}{3}$.
18. 7, 24.
19. 16, 24.
20. 17, or 25 mi. per hour.
21. 5 in., 12 in.
22. $4\frac{1}{2}$ mi. per hour.
23. \$60, or \$40.
24. $\frac{1}{2}(3 \pm \sqrt{5})$, $\frac{1}{2}(1 \pm \sqrt{5})$.
25. $1 \pm \sqrt{2}$.
26. $2\sqrt{6} + 1$, $2\sqrt{6} - 1$.
27. 60 sq. ft.
28. 9 ft., 10 ft.
29. \$1500, \$500.
30. 15 mi. per hour.
31. 196 ft.
32. 96.
33. 10 mi. per hour.
34. 10 in., 8 in.

EXAMINATION PAPERS

I. Page 280

1. 3, 3.
2. 10, 5.
3. $2\sqrt{3}$; $\frac{1}{2}(2 - \sqrt{2})$.
5. 9, 16; 9, 16.
6. $x = 4, \ 10$, $y = 10, \ 4$.
7. 10, 24, 26.

II. Page 280

1. 32.
2. x^n.
3. $\frac{1}{\sqrt{2}}$.
4. $x^2 + 3x + 1$.
5. $1 - \sqrt{3}$, $\frac{1}{2}(\sqrt{3} - 1)$.
6. $x = 2, 5, -5 \pm \sqrt{15}$,
   $y = 5, 2, -5 \mp \sqrt{15}$.
7. 3.464.

III. Page 281

1. 1.
2. 1, 3, $14\frac{4}{5}$, $\frac{2}{3}\sqrt{5}$.
3. $a^2 - ab - ac - bc$.
5. $x = 1, 1, 1 \pm 2\sqrt{15}$.
6. $21\frac{1}{4}$ ft., 4 ft.; No.
7 57, or 152.

IV. Page 281

1. $\frac{1}{2}(5 \pm 3\sqrt{5})$.
2. $2a$.
3. $x + 1 + \frac{1}{x}$.
5. $x = 5, -7, -\frac{1}{2}(-2 \pm \sqrt{69})$.
6. $x = 2, 5$,
   $y = 5, 2$.
7. 1.281, .781.

V. Page 282

1. 1, $2 + \sqrt{3}$.
2. 2^{n+1}, 2^{n-1}, 2^{n+1}, 2^{3n-}[illegible].
4. 4.
5. $\frac{(p-q)^2}{2q}$.
6. $x = 5, \frac{4}{9}$.
   $y = 4, -\frac{5}{9}$.

VI. Page 283

1. $(2 + \sqrt{3})(\sqrt{3} - \sqrt{2}) = 2.1862$.
2. $(a + 1)(1 + b^{-1})$.
3. $ab + bc - 2ac$.
4. $x = 6$, or 14.
5. $x = 4, -1$,
   $y = 1, -4$.
6. 1, $\sqrt{2} + 1$.
7. 2 hrs., 28 mi.

VII. Page 283

1. $\frac{\sqrt{5}}{5} = .4472$.
2. $(x + x^{-1})^2$.
3. $x^4 - 6x^2 + 1$.
4. $x = 4, 5$, but $\sqrt{7x - 19} = -3$, or -4.
5. $x = 6, \frac{10}{3}, \frac{14 \pm \sqrt{286}}{3}$,
   $y = 2, \frac{18}{5}, \frac{14 \mp \sqrt{286}}{5}$.
6. 2, $\frac{1}{2}(1 - \sqrt{3})$.
7. 8 ft., 11 ft.