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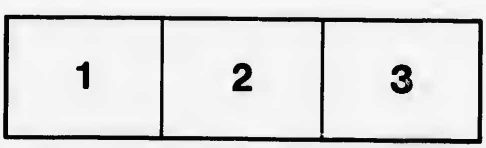
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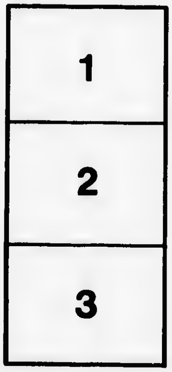
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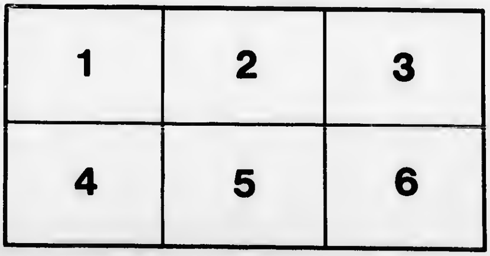
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PUBLIC SCHOOL

Euclid and Algebra

Authorized for use in the Public Schools of Ontario
by the Department of Education

PRICE 25 CENTS

TORONTO
THE HUNTER, ROSE COMPANY, Limited

PREFACE

This little book has been prepared to present the work in Geometry and Algebra laid down in the curriculum of the Public Schools of Ontario.

These subjects are taught during only one year of the course, and to pupils of whom the majority do not have opportunity of proceeding further. Consequently, completeness of treatment as well as simplicity has been considered.

The aim has been to present practical definite aspects of the subjects rather than to seek to lay a foundation for higher work in mathematics.

The problems and exercises are numerous, over fifteen hundred in all. Many of these are suitable for *viva voce* class work.

CONTENTS.

EUCLID'S ELEMENTS

BOOK 1.—PROPOSITIONS 1—26.

Euclid was the first teacher of mathematics in the first great university of the world—that founded at Alexandria about 300 B. C. He was the author of works on Geometry, Arithmetic, Astronomy, Optics and Music.

His work known as Euclid's Elements consisted of thirteen books, of which the first six and the last three treat of Geometry; the remaining books treat of Arithmetic.

The proofs of the propositions which are given in this edition of a part of the first book of the Elements are considered to be just about the same as those presented by Euclid to his classes. They have been used ever since his time as models of deductive reasoning, and their form, as well as the geometrical facts which they present, should be studied.

DEFINITIONS.

In the definitions, Euclid names the things with which he proposes to deal, and states the distinguishing marks by which these things are to be recognized.

The definitions should be carefully considered, and committed to memory as they are used.

1. **A point** is that which has position but has no magnitude.

A point is indicated by a dot with a letter attached, as the point A.

.A

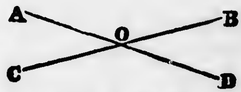
2. **A line** is that which has length, but has neither breadth nor thickness.

A line is indicated by a stroke, with a letter placed at each end, as the line *AB*.

The extremities of a line are points, and the intersection of two lines is a point.

The letters placed at the ends of a line are used also to indicate the extremities.

A letter placed at the intersection of two lines indicates the point of intersection, as the point O.



3. A **straight** line is one which lies evenly between its extreme points.

Only one straight line can lie between two given points.

4. **A surface** is that which has length and breadth, but no thickness.

5. A **plane** surface (or a **plane**) is one in which, if any two points be taken, the straight line between them lies wholly in that surface.

6. **An angle** is the inclination of two straight lines to one another which meet together, but are not in the same straight line.

The point at which the lines meet is called the **vertex**, and the lines are called the **arms** of the angle.

An angle is named by three letters, one placed at the vertex, and one on each of the arms; but these must be arranged so that the one denoting the vertex shall be the middle letter.

Thus the angle represented is called the angle *ABC*, or the angle *CBA*.

7. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a **right angle**, and the straight line which stands on the other is called a **perpendicular** to it.

8. An **obtuse angle** is one which is greater than a right angle.

9. An **acute angle** is one which is less than a right angle.

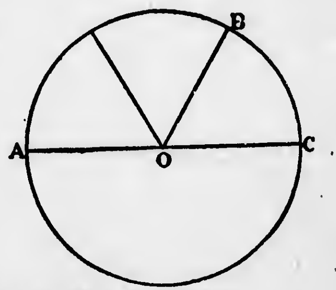
10. Any portion of a plane surface bounded (or contained) by one or more lines is called a **plane figure.**

11. **A circle** is a plane figure contained by one line which is called the **circumference**, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another. This point is called the **centre** of the circle.

A circle is usually named by three letters, each of which denotes a point on the circumference.

12. **A radius** of a circle is a straight line drawn from the centre to the circumference.

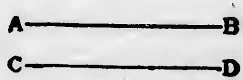
13. **A diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.



Thus *ABC* is a circle of which *O* is the centre; *OA*, *OB* and *OC* are radii, and *AC* is a diameter.

14. **Parallel** straight lines are such as are in the same plane, and being produced ever so far both ways, do not meet.

Thus *AB* and *CD* are parallel straight lines.



15. **A rectilineal figure** is one which is contained by straight lines.

These straight lines are called the **sides** of the figure, and the sum of them is called the **perimeter** of the figure.

A rectilineal figure is named by stating in order the letters which denote its angular points.

16. **A triangle** is a plane figure contained by three straight lines.

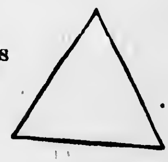
Any one of the angular points of a triangle may be called the **vertex** of the triangle: the side opposite to the vertex is called the **base.**

17. **A quadrilateral** is a plane figure contained by four straight lines.

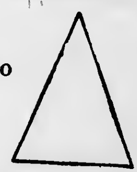
The straight line which joins opposite angular points of a quadrilateral is called a **diagonal.**

18. **A polygon** is a plane figure contained by more than four straight lines.

19. An **equilateral** triangle is one that has three equal sides.



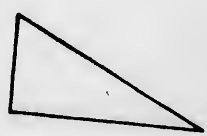
20. An **isosceles** triangle is one that has two equal sides.



21. A **scalene** triangle is one that has three unequal sides.

22. A **right-angled** triangle is one that has a right angle.

The side opposite to the right angle in a right-angled triangle is called the **hypotenuse.**

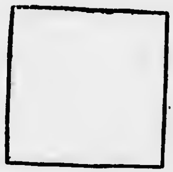


23. An **obtuse-angled** triangle is one that has an obtuse angle.

24. An **acute-angled** triangle is one that has three acute angles.

25. A **rhombus** is a quadrilateral that has all its sides equal.

26. A **square** is a quadrilateral that has all its sides equal, and all its angles right angles.



27. A **parallelogram** is a quadrilateral whose opposite sides are parallel.

28. A **rectangle** is a quadrilateral whose opposite sides are parallel, and whose angles are right angles.

29. A **trapezium** is a quadrilateral that has two sides parallel.

POSTULATES.

It is not possible, even with the best of instruments, to draw straight lines or circles. But since Euclid wishes to reason about figures made up of straight lines and circles, he requests that his attempts to draw these be considered successful. He makes these requests in three postulates, given below, which should be memorized, and referred to by number.

Let it be granted:

1. That a straight line may be drawn from any one point to any other point.

2. That a terminated straight line may be produced to any length either way.

3. That a circle may be described with any centre, and at any distance from that centre.

AXIOMS.

In the axioms, Euclid makes twelve simple statements, the truth of which is self-evident. They form the foundation on which the whole science of geometry is built. They should be committed to memory, and referred to by number.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals, the sums are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the sums are unequal, the greater sum being obtained from the greater unequal.

5. If equals be taken from unequals, the remainders are unequal, the greater remainder being obtained from the greater unequal.

6. Things which are doubles of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. Magnitudes which coincide with one another are equal to one another.

9. The whole is greater than its part, and equal to the sum of all its parts.

10. Two straight lines cannot enclose a space.

11. All right angles are equal to one another.

12. If a straight line meets two other straight lines, so as to make the interior angles on one side of it together less than two right angles, these two straight lines will meet if continually produced on the side on which are the angles which are together less than two right angles.

SYMBOLS AND ABBREVIATIONS.

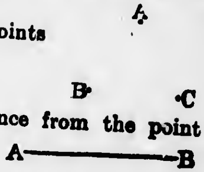
The following symbols and abbreviations are used in the propositions :—

$=$ stands for 'is equal to,' 'are equal to,' or 'be equal to.'
$\angle$ " 'angle.'
$\triangle$ " 'triangle.'
$\odot$ " 'circle.'
$\bigcirc^{ce}$ " 'circumference.'
$\therefore$ " 'therefore.'
Def. " 'Definition.'
Post. " 'Postulate.'
Ax. " 'Axiom.'
Prop. " 'Proposition.'
Hyp. " 'Hypothesis.'
Constr. " 'Construction.'

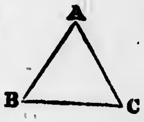
THE PROPOSITIONS

INTRODUCTION TO PROPOSITION 1.

1. State the definition of a circle, a triangle, an equilateral triangle.
2. What is a Postulate? An Axiom? State Post. 1, Post. 2 and Ax. 1.
3. Make a triangle whose angular points are the points *A*, *B* and *C*.

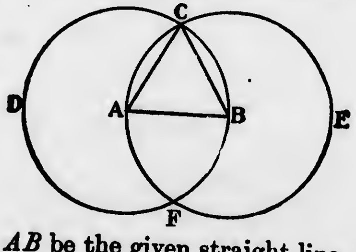


4. *(a)* Find a point which is at the same distance from the point *A* that the point *B* is.
   (b) In what line do all such points lie?
   (c) Draw a line every point of which will be at the same distance from *B* that *A* is.
   (d) Find a point that is equidistant from *A* and *B*.
5. *ABC* is an equilateral triangle. How does the distance of the point *A* from the point *B* compare with the distance of the point *A* from the point *C*?



PROPOSITION 1. PROBLEM.

To describe an equilateral triangle on a given straight line.



Let *AB* be the given straight line.

It is required to describe an equilateral triangle on *AB*.

With centre A and distance AB, describe ⊙ BCD. *Post.* 3

With centre B and distance BA, describe ⊙ ACE. *Post.* 3.

Let the circumferences intersect at the point C.

Join AC and CB. *Post* 1.

ABC shall be an equilateral △.

For, because A is the centre of ⊙ BCD,

$\therefore \quad AC = AB.$ *Def. of* ⊙.

And because B is the centre of ⊙ ACE,

$\therefore \quad BC = AB$ *Def. of* ⊙.

Now, since AC and BC are each equal to AB,

$\therefore \quad AC = BC$ *Ax.* 1.

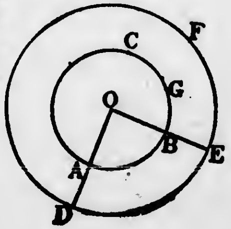
Thus AB, BC and AC are all equal, and an equilateral triangle ABC has been described on AB. *Def. of equilateral* △.

QUESTIONS ON PROPOSITION 1.

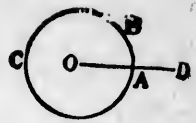
1. If the two circumferences intersect also at F, what kind of a triangle will be formed by joining AF and BF?
2. What kind of a quadrilateral is the figure $ACBF$?

INTRODUCTION TO PROPOSITION 2.

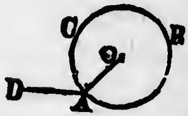
1. Define a circle. Is it possible to draw a circle on a plane surface with a pair of compasses?
2. State the postulates. State Ax. 2 and Ax. 3.
3. *(a)* Find a point equidistant from A and B. A· ·B
   (b) Show how to describe a circle that will pass through A and B.
4. ABC and DEF are concentric circles, having as common centre the point O. OD and OE are rad[illegible] the circle DEF which cut the circumference of the circle ABC at the points A and B respectively.
   (a) Show that AD equals BE.
   (b) Draw from G, a point on the circumference ABC, a straight line equal to AD.



5. A radius of the circle ABC is produced to D. Show how to draw from B a straight line equal to AD.

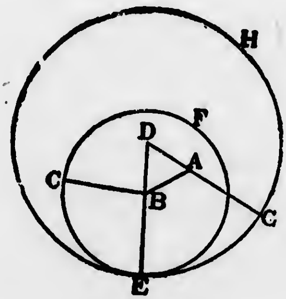


When AD is not in the same straight line as OA, show how to draw from B a straight line equal to AD.



PROPOSITION 2. PROBLEM.

From a given point to draw a straight line equal to a given straight line.



Let A be the given point, and BC the given straight line.
It is required to draw from A a straight line equal to BC.

Join AB. *Post.* 1.
On AB describe the equilateral △ DAB. *Prop.* 1.
With centre B and distance BC describe ⊙ CEF. *Post.* 3.
Produce DB to meet the O^{ce} CEF in E. *Post.* 2.
With centre D and distance DE describe ⊙ EGH. *Post.* 3.
Produce DA to meet the O^{ce} EGH in G. *Post.* 2.

Then will $AG = BC$.

For, because D is the centre of $\odot$ EGH,

$\therefore$ $DE = DG$. *Def. of* $\odot$.

Of these, the parts DB and DA are equal. *Def. of equilateral* $\triangle$.

$\therefore$ the remainders BE and AG are equal. *Ax.* 3.

And, because B is the centre of $\odot$ CEF,

$\therefore$ $BC = BE$. *Def. of* $\odot$.

But $AG = BE$.

$\therefore$ $AG = BC$. *Ax.* 1.

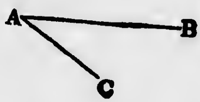
Thus from the point A a straight line AG has been drawn equal to BC.

QUESTIONS ON PROPOSITION 2.

1. Under what circumstances would the point D lie outside of the circle CEF?

   On the circumference of the circle CEF?
2. If D were without the circle CEF, would it be necessary to produce DB?
3. Could th: problem be solved by producing BD instead of DB?
4. Could the problem be solved by joining AC instead of AB?
5. Does the line AG always lie in the same direction, no matter which of the above methods of construction is used?

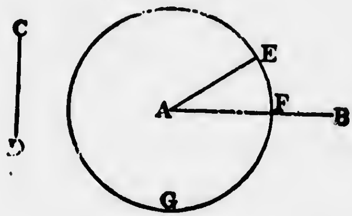
INTRODUCTION TO PROPOSITION 3.



AB and AC are two straight lines of which AB is the greater. Show how to cut off from AB a part equal to AC.

PROPOSITION 3. Problem.

From the greater of two given straight lines to cut off a part equal to the less.



Let AB and CD be the two given straight lines, of which AB is the greater.

It is required to cut off from AB a part equal to CD.

From A draw the straight line AE, equal to CD. *Prop.* 2.

With centre A and distance AE describe ⊙ EFG, cutting AB in F. *Post.* 3.

Then will $AF = CD$.

For, because A is centre of ⊙ EFG,

$\therefore \quad AF = AE.$ *Def. of* ⊙.

But $CD = AE.$ *Constr.*

$\therefore \quad AF = CD.$ *Ax.* 1.

Thus from AB a part AF has been cut off equal to CD.

QUESTIONS ON PROPOSITION 3.

1. Why not say "with centre A and distance CD describe ⊙ EFG, cutting AB in F"?
2. Can the required part be cut off from either end of the line AB?
3. Make the figure, putting in the construction necessary to draw AE equal to CD.

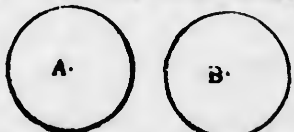
EXERCISES.

Find the solutions of the following exercises, and write the proof of each solution in a form similar to that used by Euclid in his propositions.

1. *AB* is a given straight line. Produce the line, making the whole length double that of *AB*. *A*———*B*
2. Describe an isosceles triangle on a given straight line, such that each of the equal sides shall be twice as long as the base.
3. On a given straight line describe an isosceles triangle having each of the equal sides equal to another given straight line.
4. Draw a straight line three times as long as a given straight line.
5. On a given straight line describe an isosceles triangle having each of the equal sides three times as long as the third side.
6. From a given point *C*, in a given straight line *AB*, draw a straight line equal to *AB*.
7. Produce the less of two given straight lines, making it equal to the greater.

INTRODUCTION TO PROPOSITION 4.

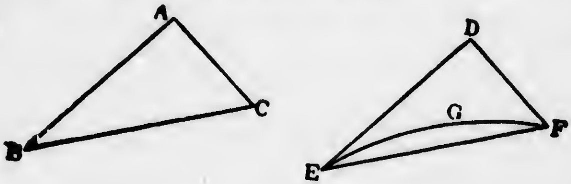
1. State Ax. 8, and Ax. 10.
2. What is the meaning of 'coincide'?
3. Are two angles necessarily equal, if the straight lines which form the angles are equal, each to each?
4. Two circles have equal radii: show that they have equal areas and equal circumferences.



5. Two squares have the sides of the one equal to the sides of the other. Show that they have equal areas and equal perimeters.

PROPOSITION 4. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to each other, they shall also have their third sides equal; and the two triangles shall be equal, and the other angles shall be equal, each to each, namely those to which the equal sides are opposite.



In the $\triangle$s ABC and DEF,

let $AB = DE$,

$AC = DF$,

and $\angle BAC = \angle EDF$.

It is required to prove that $BC = EF$,

$\triangle ABC = \triangle DEF$,

$\angle ABC = \angle DEF$,

and $\angle ACB = \angle DFE$.

If $\triangle ABC$ be applied to $\triangle DEF$, so that A falls on D, and AB falls on DE, then B will coincide with E,

because $AB = DE$. *Hyp.*

And because AB coincides with DE,

and $\angle BAC = \angle EDF$, *Hyp.*

$\therefore AC$ will fall on DF.

And because $AC = DF$, *Hyp.*

$\therefore C$ will coincide with F.

And because B coincides with E, and C with F,

$\therefore BC$ will coincide with EF.

For, if not, let it fall otherwise, as EGF

Then the two straight lines BC and EF will enclose a space, which is impossible. *Ax.* 10.

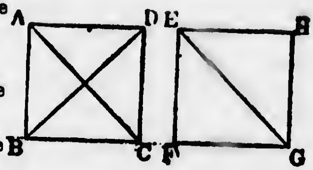
Hence BC coincides with and $\therefore = EF$,					*Ax.* 8.
$\triangle ABC$	"	"	"	$\therefore = \triangle DEF$,	
$\angle ABC$	"	"	"	$\therefore = \angle DEF$,	
and $\angle ACB$	"	"	"	$\therefore = \angle DFE$.	

Questions on Proposition 4.

1. State the axioms used in the proof.
2. Are the words, 'each to each,' necessary in the enunciation?
3. Does AC necessarily fall on DF, if AB coincides with DE?
4. Prove the proposition, beginning the superposition by applying B to E.

EXERCISES.

1. The sides of the square $ABCD$ are equal to the sides of the square $EFGH$.



Show that :

(*a*) The diagonals AC and EG are equal.

(*b*) The diagonals AC and BD are equal.

(*c*) The diagonal AC bisects, that is, divides into two equal parts, the angle BAD.

(*d*) The squares are equal in area.

2. A straight line AD bisects the vertical angle BAC of the isosceles triangle ABC, and meets the base at the point D. Show that D is the middle point of the base, and that AD is perpendicular to BC.
3. If two straight lines bisect each other at right angles, any point in either of them is equidistant from the extremities of the other.
4. The middle points of the sides of a square are joined in order. Show that the quadrilateral formed by these joining lines is equilateral.
5. $ABCD$ is a square, E is a point in AB, and F is a point in CD, such that AE is equal to CF; EF is joined. Show that the angle AEF is equal to the angle CFE.

Propositions are divided into two classes, **theorems** and **problems**.

A **theorem** is a truth that requires to be proved by means of other truths already known. The truths already known are either axioms or theorems.

A **problem** is a construction which is to be made by means of the principles of construction granted in the postulates or proved in previous problems.

The **enunciation**, or statement, of a theorem consists of two parts, the **hypothesis**, which states that which is assumed, and the **conclusion**, which states that which is asserted to follow from the hypothesis.

The hypothesis of Prop. 4 is "If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to each other;" the conclusion is "they shall have their third sides equal, and the two triangles shall be equal, and the other angles shall be equal, each to each, namely those to which the equal sides are opposite."

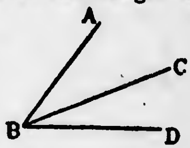
Figures which may be made to coincide, are said to be "equal in all respects."

The three sides and the three angles of a triangle are called the "parts of the triangle."

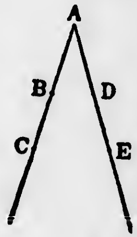
Thus, the triangles considered in Prop. 4, are proved to be equal in all respects, since the parts of the one are equal to the corresponding parts of the other.

Introduction to Proposition 5.

1. Define isosceles triangle. Which side of an isosceles triangle is called the base?
2. In the accompanying figure point out and name the angle which is
   (a) The sum of the angles ABC and CBD.
   (b) The difference of the angles ABD and CBD.

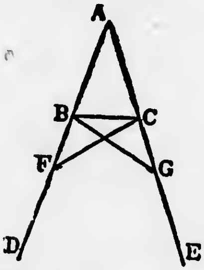


3. In the figure, AB is equal to AD, and AC is equal to AE.
   (a) Join BE and DC. Name the parts of the △s ACD and AEB, which are equal.
   (b) Prove that CD is equal to BE.
   (c) Join BD. Name all the parts of the △s BCD and DEB, which are equal.
   (d) What kind of a triangle is △ ABD?



PROPOSITION 5. THEOREM.

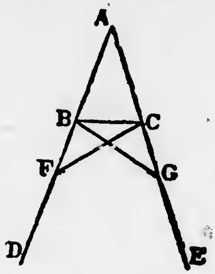
The angles at the base of an isosceles triangle are equal; and if the equal sides be produced, the angles on the other side of the base shall also be equal.



In $\triangle$ ABC, let $AB = AC$, and let AB and AC be produced to D and E.

It is required to prove that $\angle ABC = \angle ACB$,
and $\angle DBC = \angle ECB$.

In BD take any point F,
and from AE cut off AG, equal to AF. — *Prop. 3.*
Join BG and CF. — *Post. 1.*
In $\triangle$s AFC and AGB,
$AF = AG$, — *Constr.*
$AC = AB$, — *Hyp.*
and $\angle FAC = \angle GAB$,
$\therefore FC = GB$,
$\angle AFC = \angle AGB$,
and $\angle ACF = \angle ABG$. — *Prop. 4.*
Again, because $AF = AG$,
and $AB = AC$,
$\therefore BF = CG$. — *Ax. 3.*
And in $\triangle$s BFC and CGB,
$BF = CG$,
$FC = GB$,



and $\angle BFC = \angle CGB$,

$\therefore \angle BCF = \angle CBG$,

and $\angle FBC = \angle GCB$. *Prop. 4.*

Now, because $\angle ABG = \angle ACF$,

and $\angle CBG = \angle BCF$,

$\therefore \angle ABC = \angle ACB$, *Ax.* 3.

and these are the angles at the base.

It has also been proved that $\angle FBC = \angle GCB$, and these are the angles on the other side of the base.

It is evident that if a triangle has all its sides equal, it has all its angles equal; that is, an equilateral triangle is equiangular.

A truth, such as the above, which is easily and directly inferred from a proposition, is called a corollary of that proposition.

Questions on Proposition 5

1. **What is the hypothesis of Proposition 5?**
2. **What is the conclusion of Proposition 5?**
3. **Would it do equally well to say "In *AD* take any point *F*"?**

EXERCISES.

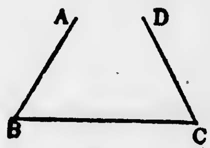
1. **Prove that the diagonal of a rhombus divides it into two isosceles triangles.**

2. Prove that the opposite angles of a rhombus are equal.
3. Prove that the diagonal of a rhombus bisects each of the angles through which it passes.
4. Two isosceles triangles, ABC and DBC, have the same base BC.
   (a) Prove that the angle ABD is equal to the angle ACD.
   (b) Prove that the angle BAD is equal to the angle CAD.
   (c) Prove that AD, or AD produced, bisects the base BC.
5. ABC is an isosceles triangle; and in the base BC two points D, E are taken such $BD = CE$; prove that ADE is an isosceles triangle.
6. Prove that the diagonals of a square divide the figure into four isosceles triangles.
7. Two equal circles, whose centres are A and B, intersect at the point C. Join CA and CB, and produce them to meet the circumferences at D and E respectively. Join DE. Prove that the angle CDE equals the angle CED.
8. ABC is an equilateral triangle: D, E and F are points in the sides AB, BC and CA, such that $AD=BE=CF$. Show that the triangle DEF is equilateral.

Introduction to Proposition 6.

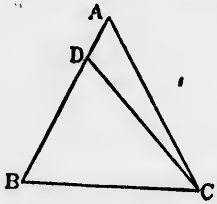
1. There are two straight lines, AB and CD.
   (a) If AB is not greater than CD, must AB be less than CD? Why?
   (b) If AB is not equal to CD, is AB necessarily greater than CD?
   (c) If AB is not greater than CD, nor less than CD, what relation must exist between AB and CD?

2. In the figure, $AB = CD$, and $\angle ABC = \angle DCB$, prove that $AC = BD$, and that $\triangle ABC = \triangle DCB$.



PROPOSITION 6. THEOREM.

If two angles of a triangle be equal, the sides opposite them shall also be equal.



In △ ABC let ∠ ABC = ∠ ACB.

It is required to prove that $AC = AB$.

If AC is not equal to AB, one of them must be the greater.

Let AB be the greater.

From BA cut off BD equal to AC. *Prop.* 3.

Join DC. *Post.* 1.

In △s DBC and ACB,

$DB = AC$, *Constr.*

$BC = CB$,

and ∠ DBC = ∠ ACB, *Hyp.*

∴ △ DBC = △ ACB. *Prop.* 4.

But this is impossible, since △ DBC is a part of △ ACB.

Therefore AB is not greater than AC.

Similarly it may be shown that AB is not less than AC.

∴ $AB = AC$.

Corollary. An equiangular triangle is equilateral.

QUESTIONS ON PROPOSITION 6.

1. How would you proceed to show "that AB is not less than AC"?
2. What is the hypothesis of Proposition 6?
3. What is the conclusion of Proposition 6?

4. What relation exists between the hypothesis of Prop. 5 and the conclusion of Prop. 6; and also between the first part of the conclusion of Prop. 5 and the hypothesis of Prop. 6?

Two propositions are said to be **converse**, when the hypothesis of each is the conclusion of the other.

EXERCISES.

1. The diagonals of the square *ABCD* intersect at *E*. Use Prop. 6 to prove that the triangle *EAB* is isosceles.
2. Prove that the diagonals of a rhombus bisect each other at right angles.
3. Show that the straight lines which bisect the angles at the base of an isosceles triangle, form with the base a triangle which is also isosceles.
4. In the figure of Prop. 1, if the straight line *AB* be produced both ways, to meet the one circumference at *D* and the other at *E*, show that the triangle *CDE* is isosceles.

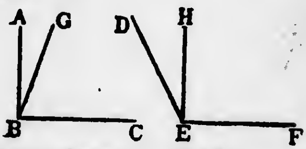
INTRODUCTION TO PROPOSITION 7.

1. *AB* and *CD* are two straight lines. *CE* = *AB*.

   (*a*) How does *AF* compare in length with *CE*?

   (*b*) How does *AF* compare in length with *CD*?

2. *ABC* and *DEF* are two angles, and ∠ *ABC* = ∠ *HEF*, which is a part of ∠ *DEF*.



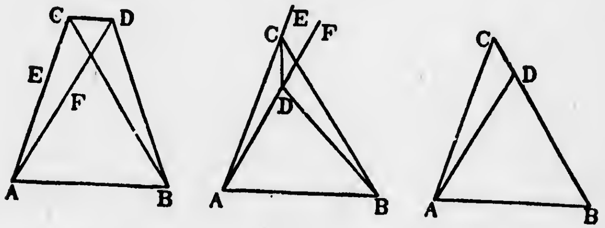
   (*a*) How does ∠ *GBC* compare in magnitude with ∠ *HEF*?

   (*b*) How does ∠ *GBC* compare in magnitude with ∠ *DEF*?

3. Show that two isosceles triangles cannot stand on the same base and on the same side of it, unless the vertex of the one triangle falls inside the other triangle.
4. Two triangles have the three sides of the one respectively equal to the three sides of the other. Can they be made to coincide?

PROPOSITION 7. THEOREM.

Two triangles on the same base and on the same side of it cannot have their conterminous sides equal.



If it is possible let the two △s *ABC*, *ABD* on the same base *AB*, and on the same side of it, have *AC* equal to *AD*, and *BC* equal to *BD*.

Three cases may occur.

(1) The vertex of each △ may be outside the other △.
(2) The vertex of one △ may be inside the other △.
(3) The vertex of one △ may be on a side of the other △.

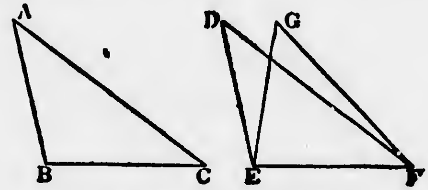
In the first case join *CD*; and in the second case join *CD* and produce *AC* and *AD* to *E* and *F*.

Because $AC = AD$,
$\therefore \angle ECD = \angle FDC$. *Prop.* 5.
But $\angle ECD$ is greater than $\angle BCD$. *Ax.* 9.
$\therefore \angle FDC$ is greater than $\angle BCD$.
Much more then is $\angle BDC$ greater than $\angle BCD$.
But because $BC = BD$,
$\therefore \angle BDC = \angle BCD$; *Prop.* 5.
that is, $\angle BDC$ is greater than and equal to $\angle BCD$,
which is impossible.

The third case needs no proof, because *BC* is not equal to *BD*. Hence two triangles on the same base and on the same side of it cannot have their conterminous sides equal.

PROPOSITION 8. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides, equal to them, of the other.



In the $\triangle$s ABC and DEF,

let $AB = DE$,

$AC = DF$,

and $BC = EF$.

It is required to prove that $\angle BAC = \angle EDF$.

Apply the $\triangle ABC$ to the $\triangle DEF$,

so that B is on E, and BC on EF

Then, because $BC = EF$,

C will coincide with F.

Then AB and AC will coincide with DE and DF.

For, if they do not, but fall otherwise, as GE and GF, then on the same base EF, and on the same side of it, there will be two $\triangle$s, DEF and GEF, having equal pairs of conterminous sides,

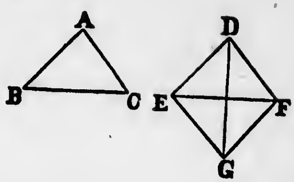
which is impossible. *Prop.* 7.

$\therefore$ BA coincides with ED, and AC with DF.

And hence $\angle BAC$ coincides with and $\therefore = \angle EDF$. *Ax.* 8.

Corollary. If two triangles have the three sides of the one respectively equal to the three sides of the other, the triangles are equal in all respects.

Questions on Proposition 8.



1. Apply the triangles so that they may fall on opposite sides of the common base *EF*. Join *DG*.

   (a) What kind of a triangle is *EDG*? *FDG*?

   (b) Prove that the $\angle EDF = \angle EGF$.

   (c) Prove the proposition in this way when *DG* does not pass between *E* and *F*.

   (d) Prove the proposition when *DG* passes through the point *F*.

2. What is meant by the 'parts of a triangle'?
3. (a) What parts were given equal in the two triangles considered in Prop. 4?

   (b) What parts were proved equal?
4. (a) What parts are given equal in Prop. 8?

   (b) What parts are proved equal?

   (c) Are the triangles equal in all respects?
5. Is it possible to make two triangles whose sides are respectively equal to three given straight lines, but which are not equal in all respects?

EXERCISES.

1. The opposite sides of a quadrilateral *ABCD* are equal. Prove that:

   (a) The opposite angles are equal.

   (b) The angle *ABD* is equal to the angle *CDB*.

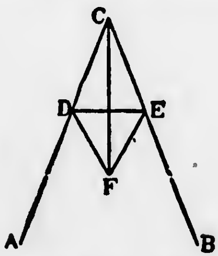
   (c) The middle point of *BD* is equidistant from *A* and *C*.
2. Show that two circumferences can cut each other in only one point on the same side of the line joining their centres.
3. Two isosceles triangles are on the same base, and on opposite sides of the base. Prove that the line joining their vertices bisects each of the vertical angles.
4. *ABC* is an isosceles triangle, of which *AB* and *AC* are the equal sides. Points *D* and *E* are taken in *AB*, and points *F* and *G* in *AC*, such that $AD = AF$, and $AE = AG$. *CD* and *BF* intersect in *H*; *CE* and *BG* intersect in *K*.

Prove that:

(a) AH bisects $\angle DAF$.
(b) $\angle BDH = \angle CFH$.
(c) $EH = HG$.
(d) A, H and K are in the same straight line.

PROPOSITION 9. PROBLEM.

To bisect a given rectilineal angle.



Let ACB be the given rectilineal angle.
It is required to bisect it.
In AC take any point D,
and from CB cut off CE equal to CD. *Prop.* 3.
Join DE, and on DE, on the side remote from C,
describe the equilateral $\triangle DEF$, *Prop.* 1.
Join CF.
CF shall bisect $\angle ACB$.
In $\triangle$s DCF and ECF,
$DC = EC$, *Constr.*
$CF = CF$,
and $DF = EF$, *Def. of equilateral* $\triangle$.
$\therefore \angle DCF = \angle ECF$. *Prop.* 8.
That is, CF bisects $\angle ACB$.

Questions on Proposition 9.

1. If the equilateral triangle were described on the same side of DE as C, what different cases would arise?

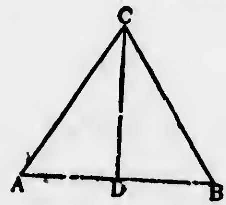
   Under what circumstances would the construction fail?

3. Show that CF bisects DE.

2. Show that CF also bisects the angle DFE.

PROPOSITION 10. Problem.

To bisect a given straight line.



Let AB be the given straight line.

It is required to bisect it.

On AB describe an equilateral $\triangle\ ABC$. *Prop.* 1.

Bisect $\angle\ ACB$ by CD, which meets AB at D. *Prop.* 9.

AB shall be bisected at D.

In the $\triangle$s ACD and BCD,

$AC = BC$,

$CD = CD$, *Def. of equilateral* $\triangle$.

and $\angle\ ACD = \angle\ BCD$, *Constr.*

$\therefore AD = BD$. *Prop.* 4.

That is, AB is bisected at D.

Questions on Proposition 10.

1. Is it necessary that the triangle described on AB should be equilateral?
2. Show that CD is at right angles to AB.
3. Every point equidistant from A and B lies in the line CD.

EXERCISES.

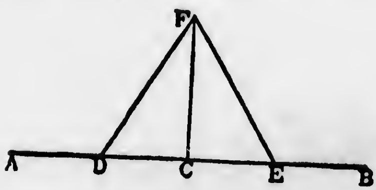
1. Divide a given angle into four equal parts.
2. Divide a given straight line into eight equal parts.
3. On a given base describe an isosceles triangle such that the sum of its equal sides may be equal to a given straight line.
4. Produce a given straight line so that the whole line may be five times as long as the part produced.
5. *D*, *E* and *F* are the middle points of the sides of an equiangular triangle, show that the triangle *DEF* is equiangular.
6. Two isosceles triangles, *ABC* and *DBC*, stand on the same base *BC*, but on opposite sides of it. *E* is the middle point of *AB*, and *F* the middle point of *AC*; and *BF* and *CE* intersect at *G*.
   (a) Prove that $DE = DF$.
   (b) Prove that $\angle EDB = \angle FDC$.
   (c) Prove that $CE = BF$.
   (d) Prove that $\angle DBF = \angle DCE$.
   (e) Prove that $\triangle GBC$ is isosceles.
   (f) Prove that $EG = GF$.
   (g) Prove that $\angle EGD = \angle FGD$.
   (h) Prove that *A*, *G* and *D* are in the same straight line.

INTRODUCTION TO PROPOSITION 11.

1. When is one straight line said to be perpendicular to another straight line?
2. *ABC* is an equilateral triangle, and *AD* is a straight line bisecting the vertical angle *BAC*, and meeting the base in *D*
   (a) Show that *AD* bisects the base.
   (b) Show that *AD* is perpendicular to the base.

PROPOSITION 11. Problem.

To draw a straight line perpendicular to a given straight line from a given point in the same.



Let AB be the given straight line, and C the given point in it.

It is required to draw from C a perpendicular to AB.

In AC take any point D.

From CB cut off CE equal to CD. *Prop.* 3.

On DE describe the equilateral $\triangle$ DEF. *Prop.* 1.

Join CF.

CF shall be perpendicular to AB.

In the $\triangle$s DCF and ECF,

$DC = EC$, *Constr.*

$CF = CF$,

and $DF = EF$, *Def. of equilateral* $\triangle$

$\therefore \angle DCF = \angle ECF$. *Prop.* 8

$\therefore$ CF is a perpendicular to AB. *Def. of perpendicular.*

Questions on Proposition 11.

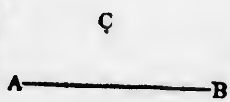
1. Is it necessary that the triangle described on DE should be equilateral?
2. If CD were greater than CB, what additional step in the construction would be necessary?
3. Show how to draw a perpendicular from the extremity of the line
4. Prove that $\angle FDC = \angle FEC$.
5. Can the proposition be proved without the use of Prop. 8?

EXERCISES.

1. In what line do all points lie, which are equidistant from a given point?
2. Find the line in which all points lie which are equidistant from two given points.
3. Find, if possible, a point which is equidistant from three given points.
4. Give a construction for finding the centre of the circle which passes through the three angular points of a triangle.
5. Find a point whose distance from the given point A is equal to a given straight line, and whose distance from a given point B is equal to another given straight line. Is this always possible?

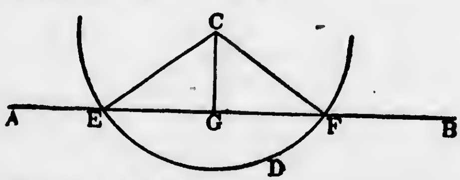
INTRODUCTION TO PROPOSITION 12.

1. Show that the straight line which joins the vertex of an isosceles triangle to the middle point of the base, is perpendicular to the base.
2. Make an isosceles triangle whose vertex is the point C, and whose base is a part of the straight line AB, which is not limited in length.



PROPOSITION 12. PROBLEM.

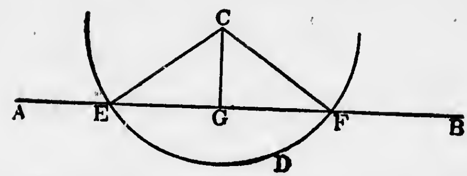
To draw a straight line perpendicular to a given straight line from a given point without it.



Let AB be the given straight line, and C the given point without it.

It is required to draw from C a perpendicular to AB.

Take any point D on the other side of AB.



With centre C and radius CD describe the ⊙ EDF, cutting AB, or AB produced, at E and F.

Bisect EF at G, *Prop.* 10.

and join CG.

CG shall be perpendicular to AB.

Join CE, CF.

In the △s CGE and CGF,

$EG = FG$, *Constr.*

$GC = GC$,

and $CE = CF$, *Def. of* ⊙

$\therefore \angle CGE = \angle CGF$. *Prop.* 8.

$\therefore$ CG is perpendicular to AB. *Def. of perpendicular.*

Questions on Proposition 12.

1. Prove the proposition without the use of Prop. 8.
2. Prove the proposition without the use of Prop. 8 or Prop. 10.
3. Is there any objection to taking the point D
   (*a*) In the line AB?
   (*b*) On the same side as C?

EXERCISES.

1. Describe an isosceles triangle, having given the base and the length of the perpendicular drawn from the vertex to the base.
2. In a given straight line find a point that is equidistant from two given points. Is this always possible?
3. From two given points, on opposite sides of a given straight line, draw straight lines to a point in the given line, making equal angles with it. Is this always possible?
4. ABC is a triangle, and D, E and F are the middle points of the sides, BC, CA and AB respectively. From D a straight line is drawn perpendicular to BC, and from E another straight line is drawn perpendicular to CA, meeting the former line in O. Show that OF is perpendicular to AB.

PROPOSITION 13. THEOREM.

The angles which one straight line makes with another on one side of it are together equal to two right angles.

Let AB make with CD on one side of it, the $\angle$s ABC, ABD.

It is required to prove $\angle\ ABC$ and $\angle\ ABD$ together = 2 right $\angle$s.

(1) If $\angle\ ABC = \angle\ ABD$,

each of them is a right angle; *Def. of right* $\angle$.

$\therefore \angle\ ABC$ and $\angle\ ABD$ together = 2 right $\angle$s.

(2) If $\angle\ ABC$ be not = $\angle\ ABD$,

from B draw BE perpendicular to CD. *Prop.* 11.

Then $\angle$s EBC, EBD are 2 right $\angle$s. *Constr.*

But $\angle\ ABC$ and $\angle\ ABD$ together = $\angle\ EBC$ and $\angle\ EBD$ together; *Ax.* 8.

$\therefore \angle\ ABC$ and $\angle\ ABD$ together = 2 right $\angle$s. *Ax.* 1.

Corollary 1. If two straight lines intersect, the four angles which they make at the point where they cut are together equal to four right angles.

Corollary 2. All the successive angles made by any number of straight lines meeting at one point are together equal to four right angles

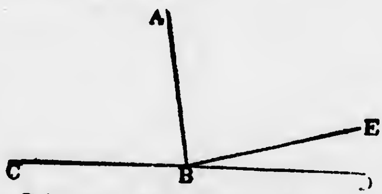
Corollary 3. Two straight lines cannot have a common segment.

EXERCISES.

1. The angles *ABC* and *ABD*, which are made by the straight line *AB* standing on the straight line *CD*, are bisected by the straight lines *BE* and *BF*. Show that the angle *EBF* is a right angle.
2. If the two exterior angles formed by producing a side of a triangle both ways are equal, show that the triangle is isosceles.
3. Show that the angles of a triangle formed by a diagonal and two of the sides of a square, together equal two right angles.
4. Construct an angle equal to half a right angle.
5. Make an isosceles triangle having each of its base angles equal to half a right augle, and each of the equal sides equal to a given straight line.
6. If one of the four angles which two intersecting straight lines make with each other be a right angle, all the others are right angles.

PROPOSITION 14. THEOREM.

If at a point in a straight line, two other straight lines on opposite sides of it make the adjacent angles together equal to two right angles, these two straight angles shall be in one and the same straight line.



At the point *B* in *AB*, let *BC* and *BD*, on opposite sides of *AB*, make ∠ *ABC* and ∠ *ABD* together = 2 right ∠s.

It is required to prove *BD* in the same straight line with *BC*.

If *BD* be not in the same straight line with *BC*,

produce *CB* to *E*; *Post.* 2.

then *BE* does not coincide with *BD*.

Now, since *CBE* is a straight line,

∴ ∠ *ABC* and ∠ *ABE* together = 2 right ∠ s. *Prop.* 13.

But ∠ ABC and ∠ ABD together = 2 right ∠ s; *Hyp.*

∴ ∠ ABC and ∠ ABE together = ∠ ABC and ∠ ABD together. *Ax.* 1.

Take away from these equals ∠ ABC, which is common;

∴ ∠ ABE = ∠ ABD, *Ax.* 3.

which is impossible.

∴ BE must coincide with BD;

that is, BD is in the same straight line with BC.

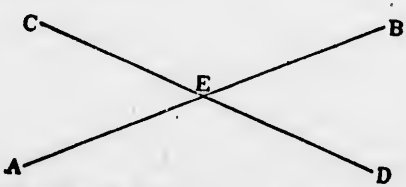
Questions on Proposition 14.

1. What relation does Prop. 14 bear to Prop. 13?
2. Show the necessity of the words 'on opposite sides' in the enunciation.

When two straight lines intersect each other, the opposite angles are called **vertically opposite angles.**

PROPOSITION 15. Theorem.

If two straight lines cut one another, the vertically opposite angles shall be equal.



Let AB and CD cut one another at E.

It is required to prove that ∠ AEC = ∠ BED,

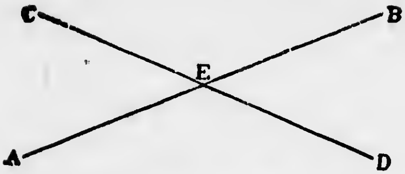
and ∠ BEC = ∠ AED.

Because CE stands on AB,

∴ ∠ AEC and ∠ BEC together = 2 right ∠ s. *Prop.* 13.

Because BE stands on CD,

∴ ∠ BEC and ∠ BED together = 2 right ∠ s. *Prop.* 13.



∴ ∠ *AEC* and ∠ *BEC* together = ∠ *BEC* and ∠ *BED* together. *Ax.* 1.

Take away from these equals ∠ *BEC*, which is common.

∴ ∠ *AEC* = ∠ *BED*. *Ax.* 3.

Similarly, ∠ *BEC* = ∠ *AED*.

EXERCISES.

1. If four straight lines meet at a point so that the opposite angles are equal, these straight lines are two and two in the same straight line.
2. What relation does the above theorem bear to Prop. 15?
3. Show that the bisectors of either pair of vertically opposite angles, in Prop. 15, are in the same straight line.
4. Show that, if *AB* is perpendicular to the straight line *CD*, which it meets at *B*, then if *AB* is produced to *E*, *BE* is also perpendicular to *CD*.
5. From two given points on the same side of a given straight line, show how to draw two straight lines which shall meet at a point in the given straight line and make equal angles with it.
6. In the figure of Prop. 15, made *EB* equal to *ED*, and *EC* equal to *EA*, and join *AD*, *DB* and *BC*. Then prove the angle *AED* equal to the angle *CEB*, without assuming any proposition after Prop. 5.
7. The side *AC* of the triangle *ABC* is bisected at *E*, and *BE* is drawn and produced to *F*, making *EF* equal to *EB*.

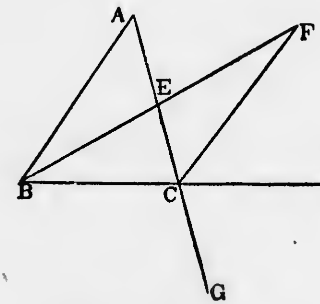
   Show that:

   (*a*) *FC* = *AB*.

   (*b*) ∠ *FCE* = ∠ *BAE*.

PROPOSITION 16. THEOREM.

If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.



Let ABC be a triangle, and let BC be produced to D.

It is required to prove $\angle\ ACD$ greater than $\angle\ BAC$,

and also greater than $\angle\ ABC$.

Bisect AC at E. *Prop.* 10.

Join BE, and produce it to F, making $EF = EB$. *Prop.* 3.

Join CF.

In $\triangle$s AEB and CEF,

$AE = CE$, *Constr.*

$EB = EF$, *Constr.*

and $\angle\ AEB = \angle\ CEF$, *Prop.* 15.

$\therefore \angle\ EAB = \angle\ ECF$. *Prop.* 4.

But $\angle\ ACD$ is greater than $\angle\ ECF$; *Ax.* 9.

$\therefore \angle\ ACD$ is greater than $\angle\ EAB$.

Also, if AC be produced to G,

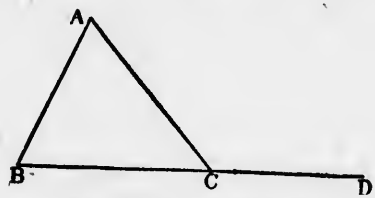
$\angle\ BCG$ is greater than $\angle\ ABC$.

But $\angle\ ACD = \angle\ BCG$; *Prop.* 15.

$\therefore \angle\ ACD$ is greater than $\angle\ ABC$.

PROPOSITION 17. THEOREM.

Any two angles of a triangle are together less than two right angles.



Let ABC be a triangle.

It is required to prove the sum of any two of its angles less than two right angles.

Produce BC to D.

Then $\angle ABC$ is less than $\angle ACD$. *Prop.* 16.

$\therefore \angle ABC$ and $\angle ACB$ are together less than $\angle ACD$ and $\angle ACB$ together.

But $\angle ACD$ and $\angle ACB$ together = two right $\angle$s. *Prop.* 13.

$\therefore \angle ABC$ and $\angle ACB$ are together less than two right $\angle$s.

Now $\angle ABC$ and $\angle ACB$ are any two angles of the triangle;

$\therefore$ any two angles of a triangle are together less than two right angles.

QUESTIONS ON PROPOSITION 17.

1. Show that the proposition can be proved by joining the vertex to a point in the opposite side.
2. State Axiom 12.
3. Enunciate Prop. 17 and Axiom 12, so as to show that they are converse theorems.

EXERCISES.

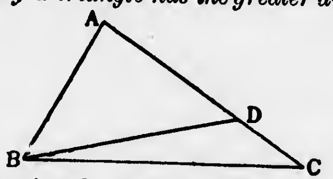
1. A is a given point and BC a given straight line.
   (*a*) Find a point in BC, whose distance from A is equal to the length of another straight line DE.
   (*b*) Show that two, and not more than two, such straight lines can be drawn.

(c) Show that only one perpendicular can be drawn from A to BC.

3. P is any point within the triangle ABC, and PA and PB are joined. Show that the angle APB is greater than the angle ACB.
4. Show that two angles of every triangle must be acute angles.
5. Show that two exterior angles of every triangle must be obtuse angles. Of what triangle will the three exterior angles be obtuse?
6. In the figure of Prop. 16, show that the area of the triangle ABC is equal to the area of the triangle FBC.

PROPOSITION 18. THEOREM.

The greater side of a triangle has the greater angle opposite to it.



Let ABC be a triangle, having AC greater than AB. It is required to prove $\angle ABC$ greater than $\angle ACB$.

From AC cut off AD equal to AB, *Prop.* 3.
and join BD.

Because $\angle ADB$ is an exterior angle of $\triangle BCD$,
$\therefore \angle ADB$ is greater than $\angle ACB$. *Prop.* 16.
But $\angle ADB = \angle ABD$, since $AB = AD$; *Prop.* 5.
$\therefore \angle ABD$ is greater than $\angle ACB$.
Much more, then, is $\angle ABC$ greater than $\angle ACB$.

QUESTIONS ON PROPOSITION 18.

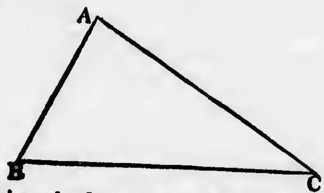
1. State the hypothesis and the conclusion of Prop. 18.
2. Is Prop. 18 equivalent to the theorem, "the greatest side of a triangle has the greatest angle opposite to it"?

EXERCISES.

1. Prove Prop. 18 by producing the shorter side.
2. $ABCD$ is a quadrilateral of which AD is the longest side, and BC the shortest; show that the angle ABC is greater than the angle ADC, and the angle BCD greater than the angle BAD.

PROPOSITION 19. THEOREM.

The greater angle of a triangle has the greater side opposite to it.



Let ABC be a triangle having $\angle ABC$ greater than $\angle ACB$.

It is required to prove AC greater than AB.

If AC be not greater than AB,

then AC must be equal to AB, or less than AB.

If $AC = AB$,

then $\angle ABC = \angle ACB$. *Prop.* 5.

But it is not. *Hyp.*

$\therefore AC$ is not equal to AB.

If AC be less than AB,

then $\angle ABC$ is less than $\angle ACB$. *Prop.* 18.

But it is not. *Hyp.*

$\therefore AC$ is not less than AB.

Hence AC must be greater than AB.

QUESTIONS ON PROPOSITION 19.

1. Enunciate the converse of Prop. 19.
2. Is Prop. 19 equivalent to the theorem "the greatest angle of a triangle has the greatest side opposite to it?"

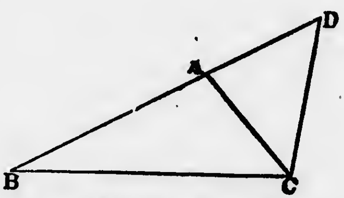
EXERCISES.

1. In an obtuse-angled triangle the greatest side is opposite the obtuse angle; and in a right-angled triangle the greatest side is opposite the right angle.
2. Show that three equal straight lines cannot be drawn from a given point to a given straight line.

3. The perpendicular is the shortest line that can be drawn from a given point to a given straight line: and of the others that which is nearer the perpendicular is less than the one more remote.
4. Any straight line drawn from the vertex of an isosceles triangle to a point in the base is less than either of the equal sides.
5. Enunciate and prove a theorem similar to Ex. 4, when the point is taken in the base produced.
6. The vertical angle ABC of the triangle ABC is bisected by the straight line BD, which meets the base in D. Show that AB is greater than AD, and CB is greater than CD.

PROPOSITION 20. THEOREM.

Any two sides of a triangle are together greater than the third side.



Let ABC be a triangle.

It is required to prove that any two of its sides are together greater than the third side.

Produce BA to D, making AD equal to AC. *Prop.* 3.

Join CD.

Then $\angle ACD = \angle ADC$, since $AD = AC$. *Prop.* 5.

But $\angle BCD$ is greater than $\angle ACD$;

$\therefore \angle BCD$ is greater than $\angle ADC$.

$\therefore BD$ is greater than BC. *Prop.* 19.

But $BD = BA$ and AC together;

$\therefore BA$ and AC are together greater than BC.

Now BA and AC are any two sides;

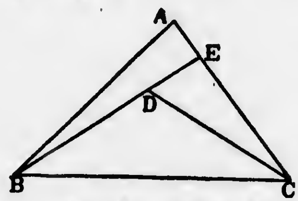
$\therefore$ any two sides of a triangle are together greater than the third side.

EXERCISES.

1. Prove Prop. 20, by bisecting the vertical angle by a straight line which meets the base.
2. Prove Prop. 20, by drawing a perpendicular from the vertex to the base.
3. *(a)* In the figure of Prop. 16, prove that *CF* is equal to *AB*.
   (b) Hence prove that the sum of any two sides of a triangle is greater than twice the straight line drawn from the middle point of the third side to the opposite vertex.
4. *(a)* Prove that the sum of the sides of any quadrilateral is greater than twice either diagonal.
   (b) Hence prove that the sum of the sides is greater than the sum of the diagonals.
5. Take any point *O*, and join to the angular points of the triangle *ABC*.
   (a) Prove that the sum of *OA* and *OB* is greater than *AB*.
   (b) Prove that twice the sum of *OA*, *OB* and *OC* is greater than the sum of the sides.
6. If a point be taken within a quadrilateral, and joined to each of the angular points, show that the sum of these joining lines is the least possible, when the point taken is the point of intersection of the diagonals.
7. Four points lie in a plane, no one of them being within the triangle formed by joining the other three. Find the point, the sum of whose distances from these four points is the least possible.
8. A point *P* is taken within the triangle *ABC*. Show that the sum of the sides *AB* and *AC* is greater than the sum of *PB* and *PC*.
9. In any triangle, the difference between any two sides is less than the third side.

PROPOSITION 21. THEOREM.

If, from the ends of a side of a triangle, there be drawn two straight lines to a point within the triangle, these will be together less than the other sides of the triangle, but will contain a greater angle.



Let ABC be a $\triangle$, and from B and C let BD and CD be drawn to any point D within the $\triangle$.

It is required to prove that BD and DC are together less than BA and AC, but that $\angle BDC$ is greater than $\angle BAC$.

Produce BD to meet AC in E.

Then BA and AE are together greater than BE. *Prop.* 20.

Add to each EC.

Then BA and AC are together greater than BE and EC.

Again, DE and EC are together greater than DC. *Prop.* 20.

Add to each BD.

Then BE and EC are together greater than BD and DC.

And it has been shown that BA and AC are together greater than BE and EC;

$\therefore$ BA and AC are together greater than BD and DC.

Next, because $\angle BDC$ is greater than $\angle DEC$, *Prop.* 16.

and $\angle DEC$ is greater than $\angle BAC$, *Prop.* 16.

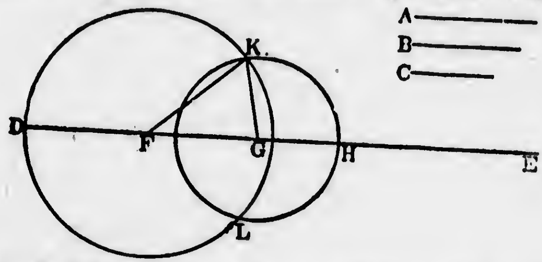
$\therefore$ $\angle BDC$ is greater than $\angle BAC$.

EXERCISES.

1. In the figure of Prop. 21, join DA, and show that the sum of DA, DB and DC is less than the sum of the sides of the triangle ABC, but greater than half the sum.
2. In the figure of Prop. 21, show that the angle BDC is greater than the angle BAC, by joining AD and producing it towards the base.

PROPOSITION 22. PROBLEM.

To make a triangle the sides of which shall be equal to three given straight lines, any two of which are greater than the third.



Let A, B, C be the three given straight lines, any two of which are greater than the third.

It is required to make a triangle the sides of which shall be respectively equal to A, B and C.

Take a straight line DE terminated at D, but unlimited towards E.

From it cut off $DF = A$, $FG = B$, $GH = C$. *Prop.* 3.

With centre F and radius FD, describe the ⊙ DKL.

With centre G and radius GH, describe the ⊙ HKL, cutting the other circle at K.

Join KF, KG.

KFG is the triangle required.

Because $FK = FD$, *Def. of* ⊙.

$\therefore\ FK = A$.

Because $GK = GH$, *Def. of* ⊙.

$\therefore\ GK = C$.

And FG was made equal to B.

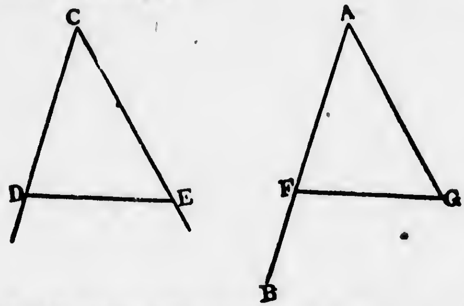
$\therefore$ Δ KFG has its sides respectively equal to A, B and C.

QUESTIONS ON PROP. 22.

1. Why does the enunciation state that any two of the given lines are together greater than the third?
2. Will the circumferences of two circles intersect, if the sum of their radii is greater than the distance between their centres?

PROPOSITION 23. PROBLEM.

At a given point in a given straight line, to make an angle equal to a given angle.



Let AB be the given straight line, A the given point in it, and $\angle\ DCE$ the given angle.

It is required to make at A an angle equal to $\angle\ DCE$.

In CD, CE take any points D, E.

Join DE.

Make $\triangle\ AFG$, such that $AF = CD$, $FG = DE$, and $GA = EC$ — *Prop.* 22.

BAG is the required angle.

In the $\triangle$s AFG and CDE,

$AF = CD$,

$AG = CE$,

and $FG = DE$,

$\therefore \angle\ FAG = \angle\ DCE$. — *Prop.* 8.

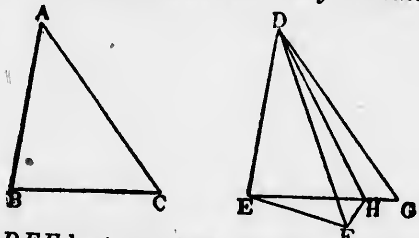
EXERCISES.

1. Prove Prop. 23, giving all the construction, instead of assuming Prop. 22.
2. Construct a triangle, having given two sides and the angle between them.
3. Construct a triangle, having given the base, and having the angles adjacent to the base equal to two given angles.

Is this always possible?

PROPOSITION 24. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the contained angles unequal, the base of the triangle which has the greater contained angle shall be greater than the base of the other.



Let *ABC*, *DEF* be two triangles, having *AB* = *DE*, *AC* = *DF*, but ∠ *BAC* greater than ∠ *EDF*.

It is required to prove *BC* greater than *EF*.

At *D* make ∠ *EDG* equal to ∠ *BAC*. *Prop.* 23.

Cut off *DG* equal to *AC* or *DF*. *Prop.* 3.

Join *EG*.

Bisect ∠ *FDG* by *DH*, meeting *EG* at *H*; *Prop.* 9.

and if *F* does not lie on *EG*, join *FH*.

In the △s *ABC* and *DEG*,

BA = *ED*, *Hyp.*

AC = *DG*, *Constr.*

and ∠ *BAC* = ∠ *EDG*, *Constr.*

∴ *BC* = *EG*. *Prop.* 4.

Again, in the △s *FDH* and *GDH*,

FD = *GD*, *Constr.*

DH = *DH*,

and ∠ *FDH* = ∠ *GDH*, *Constr.*

∴ *FH* = *GH*. *Prop.* 4.

Hence *EH* and *FH* together = *EH* and *GH* together = *EG*.

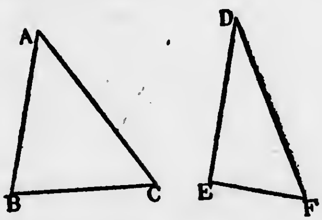
But *EH* and *FH* are together greater than *EF*; *Prop.* 20

∴ *EG* is greater than *EF*;

∴ *BC* is greater than *EF*.

PROPOSITION 25. THEOREM.

If two triangles have two sides of the one respectively equal to two sides of the other, but their bases unequal, the angle contained by the two sides of the triangle which has the greater base shall be greater than the angle contained by the two sides of the other.



Let ABC, DEF be two triangles, of which $AB = DE$, $AC = DF$, but base BC is greater than base EF.

It is required to prove $\angle BAC$ greater than $\angle EDF$.

If $\angle BAC$ be not greater than $\angle EDF$, it must be either equal to it or less than it.

But $\angle BAC$ is not $= \angle EDF$,

for then base BC would be equal to base EF. *Prop.* 4

But it is not. *Hyp.*

And $\angle BAC$ is not less than $\angle EDF$,

for then base BC would be less than base EF. *Prop.* 24.

But it is not. *Hyp.*

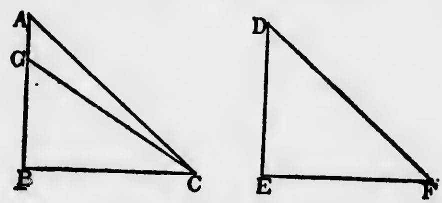
$\therefore \angle BAC$ must be greater than $\angle EDF$.

EXERCISES.

1. Show that Prop. 24 and Prop. 25 are converse propositions.
2. Assuming the truth of Prop. 25, deduce the truth of Prop. 24.
3. D is the middle point of the side BC of the triangle ABC. Prove that the angle ADB is greater or less than the angle ADC, according as AB is greater or less than AC.
4. State and prove the converse of the preceding theorem.

PROPOSITION 26. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely either the sides adjacent to the equal angles, or the sides opposite to equal angles in each; then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.



In $\triangle$s *ABC* and *DEF*,

let $\angle$ *ABC* = $\angle$ *DEF*, and $\angle$ *ACB* = $\angle$ *DFE*.

First, let the sides adjacent to the equal $\angle$s in each be equal, that is, let *BC* = *EF*.

It is required to prove that *AB* = *DE*, *AC* = *DF*, and $\angle$ *BAC* = $\angle$ *EDF*.

For if *AB* be not equal to *DE*, let *AB* be the greater, and make *GB* = *DE*, and join *GC*.

Then in $\triangle$s *GBC* and *DEF*,

GB = *DE*,

BC = *EF*,

and $\angle$ *GBC* = $\angle$ *DEF*,

$\therefore$ $\angle$ *GCB* = $\angle$ *DFE*. *Prop.* 4.

But $\angle$ *ACB* = $\angle$ *DFE*, *Hyp.*

$\therefore$ $\angle$ *GCB* = $\angle$ *ACB*;

which is impossible.

$\therefore$ *AB* is not greater than *DE*.

Similarly it may be shown that *AB* is not less than *DE*.

$\therefore$ *AB* = *DE*.

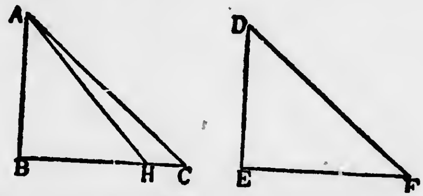
Then in △s *ABC* and *DEF*,

because *AB* = *DE*, *BC* = *EF*, and ∠ *ABC* = ∠ *DEF*,

∴ *AC* = *DF*, and ∠ *BAC* = ∠ *EDF*. *Prop.* 4.

Next, let the sides which are opposite to equal angles in each triangle be equal, that is, let *AB* = *DE*.

It is required to prove that *AC* = *DF*, *BC* = *EF*, and ∠ *BAC* = ∠ *EDF*.



For if *BC* be not equal to *EF*, let *BC* be the greater, and make *BH* = *EF*, and join *AH*.

Then in △s *ABH* and *DEF*,

AB = *DE*, and *BH* = *EF*, and ∠ *ABH* = ∠ *DEF*,

∴ ∠ *AHB* = ∠ *DFE*. *Prop.* 4.

But ∠ *ACB* = ∠ *DFE*, *Hyp.*

∴ ∠ *AHB* = ∠ *ACB*;

that is, the exterior ∠ of △ *AHC* is equal to the interior and opposite ∠ *ACB*, which is impossible. *Prop.* 16.

∴ *BC* is not greater than *EF*.

Similarly it may be shown that *BC* is not less than *EF*.

∴ *BC* = *EF*.

Then in △s *ABC* and *DEF*,

because *AB* = *DE*, *BC* = *EF*, and ∠ *ABC* = ∠ *DEF*,

∴ *AC* = *DF*, and ∠ *BAC* = ∠ *EDF*. *Prop.* 4.

EXERCISES.

1. The angle BAC is bisected by the line AD. From D the lines DB and DC are drawn making the angles ADB and ADC equal. Prove that DB, DC are equal.
2. The bisector AD of the angle BAC of the triangle ABC meets BC in D. Prove that if the angles ADB, ADC are equal, the triangle is isosceles.
3. The equal angles of an isosceles triangle ABC are bisected by the lines BD, CE, which meet the opposite sides in D and E respectively; prove that BD and CE are equal.
4. Any point in the bisector of an angle is equidistant from the arms of the angle.
5. Find the point in the base of a triangle which is equidistant from the sides.
6. Prove Prop. 26 by superposition.
7. If two sides of a triangle be produced, prove that the two bisectors of the angles so formed meet at a point equidistant from the three sides of the triangle.

EXERCISES ON PROPOSITIONS 1-26.

1. The vertical angle BAC of an isosceles triangle ABC is bisected by AD. If any point E be taken in AD, prove that EB equals EC.
2. ABC is a triangle such that AB equals AC. A line CD is drawn to cut AB in D; the point E is taken in AC such that AE equals AD. Prove that the angle AEB is equal to the angle ADC.
3. If the opposite sides of a quadrilateral are equal, the opposite angles are equal.
4. AB is a given straight line and C is a given point; draw through A and C a straight line AD whose length is four times AB.
5. Describe an equilateral triangle having a given point A for the middle point of one side, and having each of the sides equal in length to a given straight line BC.
6. In a given circle BCD place a straight line BC equal to the straight line AE, and having its extremities B and C in the circumference. Is this always possible?

7 Upon the base BC describe a triangle such that the perpendicular from the vertex on the base equals a given straight line EF, and cuts the base in D.

8. Describe a quadrilateral having given the length of each side, and of one diagonal.

9. Construct a quadrilateral having each of its sides equal to a given straight line, and having one of its angles equal to a given angle.

10. If the diagonals of a quadrilateral bisect each other, the opposite sides and angles are equal.

11. Any three sides of a quadrilateral are together greater than the fourth side.

12. If a quadrilateral has four equal sides and equal diagonals, it is a square.

13. If two right angled triangles have equal hypotenuses, and a side of one equal to a side of the other, they are equal in every respect.

14. AB, AC are the equal sides of an isosceles triangle; BD and CD are drawn perpendicular respectively to AB and AC to meet in D. Prove that BD equals CD, and that AD bisects the angle BAC.

15. ABC is a triangle, BC is the base, and AC is greater than AB. A is joined to D, the middle point, of BC. Prove that the angle ADC is obtuse.

16. If E is any point in AD, of Ex. 15, and E be joined to B and C, prove that EB is less than EC.

17. P is any point in the plane of the angle BAC; show how to draw through P a straight line, which will make an isosceles triangle with the arms of the angle BAC.

18. The base of a triangle is produced both ways, and the exterior angles are bisected. Prove that the point of intersection of these lines is equidistant from the sides of the triangle; also, that the line joining this point of intersection to the opposite angle of the triangle bisects the angle.

19. If one side of a triangle be less than another, the angle opposite the less side must be acute.

20. Prove that the sum of the perpendiculars, drawn from the vertices of a triangle to the opposite sides, is less than the perimeter of the triangle.

21. Prove that the sum of the three lines joining the middle points of the three sides of a triangle to the opposite vertices is less than the perimeter of the triangle.

22. In any triangle ABC, D is any point in the base BC. Prove that AD is less than the greater of the two sides, AB and AC.

23. Two sides, AB and AC, of the triangle ABC, are produced to D and E respectively, and the angle DBC is equal to the angle ECB. Show that AB is equal to AC.

24. E is the point of intersection of the diagonals of the square $ABCD$, F is the middle point of AB, and G the middle point of BC. EF is produced, making FH equal to EF; and EG is produced, making GK equal to EG. Show that H, B and K lie in the same straight line.

25. Construct a right-angled triangle, having given the hypotenuse and the sum of the sides.

26. Construct a triangle, having given each of the angles at the base and the perimeter of the triangle.

27. Construct a triangle, having given two sides and an angle opposite to one of them.

How many solutions are generally possible?

28. Construct a triangle, having given the base, one of the angles at the base, and the sum of the sides.

29. Construct a triangle, having given the base, one of the angles at the base, and the difference of the sides.

30. If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

31. Given three sides of a quadrilateral, and the angles adjacent to one of them; construct the quadrilateral.

32. The vertex A of the triangle ABC is joined to D, the middle point of the base. AD is produced to E, making DE equal to AD, and B and E joined. Prove that the angle EBD is equal to the angle BCA.

33. D is the middle point of the hypotenuse AB of the right-angled triangle ABC. CD is drawn and produced to E, making DE equal to CD. Show that the angle AEB is a right angle.

34. Construct a rhombus having its diagonals equal to two given straight lines.

35. $ABCD$, $EFGH$ are two squares. If they be placed so that F falls on C, and FE along CD, show that FG will either fall along CB, or in the same straight line with it.

36. The three exterior angles of a triangle, made by producing the sides in succession, are together greater than three right angles.

37. Through a given point draw a straight line, such that the per-

pendiculars drawn from two given points to it may be equal, and on opposite sides of it.

38. In a given straight line find a point the difference of whose distances from two given points on the same side of the line may be the greatest possible.

Examine the case where the given points are situated on opposite sides of the given line.

39. Three equal straight lines, *OA*, *OB* and *OC* are equally inclined to each other; if their extremities, *A*, *B* and *C* be joined, prove that *ABC* is an equilateral triangle.

40. If the angles, *AOB*, *BOC* and *COA*, in Ex. 39, be bisected by the lines *OD*, *OE* and *OF* respectively, which meet *AB*, *BC* and *CA* in *D*, *E* and *F*; prove that *OD*, *OE* and *OF* are equal and equally inclined to one another.

41. If through the extremities of the lines in Ex. 39, perpendiculars be drawn, prove that the triangle formed by these perpendiculars is equilateral, and that *A*, *B* and *C* bisect the sides.

42. On the sides of a square, and remote from it, equilateral triangles are described, and their vertices joined in order. Prove that the quadrilateral so formed is equilateral, and that its diagonals and those of the square intersect in the same point.

43. Will Ex. 42 be true, if, instead of describing equilateral triangles, we describe equal isosceles triangles?

NOTE.—In the following deductions, assume as the definition of a **rectangle**, that it is "a quadrilateral, with its opposite sides equal, and all its angles right angles."

44. In a rectangle, prove:
   (*a*) That the diagonals are equal.
   (*b*) That the angles made by a diagonal and a pair of opposite sides are equal, and that either diagonal bisects the rectangle.
   (*c*) That the diagonals bisect each other.
   (*d*) That the diagonals and the sides make four isosceles triangles.

45. Let *ABCD* be a rectangle, *E* and *F* the middle points of *AB* and *CD* respectively. Join *ED*, *EC* and *EF*, and prove:
   (*a*) That *ED* equals *EC*.
   (*b*) That *EF* is perpendicular to *CD*.
   Infer also that *EF* is perpendicular to *AB*.
   (*c*) That *AEFD* and *EBCF* are equal rectangles.

46. $ABCD$ is a rectangle, E the middle point of AB, and G any point in AD. Join GE and produce it to meet CB produced in H. Prove GE equal to EH and AG equal to BH.

47. If F is the middle point of CD in Ex. 46: join EF, and prove that $2\,EF = HC + GD$.

48. In 47, let K be any point in GD, and F the middle point of CD. Join KF and produce it to meet BC, produced in L. Prove that $2\,EF = HL + GK$.

49. Prove in Ex. 48 that the area of the trapezium $GHLK$ equals the area of the rectangle $ABCD$.

Hence show that the measure of the area of this trapezium is equal to one half the product of the measures of its perpendicular height and the sum of its parallel sides.

50. Show how to cut a rhombus so as to form out of the parts a rectangle.

51. $ABCD$ is a rectangle. In AB take any point E, and from DC cut off DF equal to AE. Join EF, and prove that the quadrilaterals $AEFD$ and $EBCF$ are rectangles.

52. A triangle and a rectangle stand on the same base, and the vertex of the triangle is in the side of the rectangle which is opposite to the common base; prove that the area of the rectangle is double the area of the triangle.

Hence show that the measure of the area of a triangle is equal to one-half the product of the measures of its base and height.

ELEMENTARY ALGEBRA.

CHAPTER I.

Arithmetical Numbers.

We measure and compare concrete quantities of the same kind by stating how many times a certain definite quantity of that same kind, called the *unit of measurement,* must be repeated to make up each of the given quantities. We indicate the number of times by the symbols—

1, 2, 3, 4, 5, 6, 7, 8, 9,

and combinations of these, according to the decimal system of notation.

When there is a remainder, and we cannot arrive at the quantity to be measured by repetitions of the unit of measurement, we divide the unit into halves or thirds or tenths, etc., and then measure the remainder in terms of these subdivisions of the unit. We indicate the result by such symbols as—

$\frac{1}{2}$, $\frac{2}{3}$, $\frac{7}{10}$ or $\cdot 7$, etc.

These symbols, used to represent numbers, are dealt with in arithmetic.

Algebraical Numbers.

There exist pairs of quantities of such a nature that the one counted in with the other annuls the other, in whole or in part. Such as—

loss *and* gain,
distance up *and* distance down,
addend *and* subtrahend.

These quantities can be measured and compared in terms of related units, such that one unit of the one kind destroys or cancels one unit of the other kind.

Such related units are—

one dollar loss *and* one dollar gain,
one yard up *and* one yard down,
1 to be added *and* 1 to be subtracted.

It has been found convenient to denote these related units, called respectively the **positive unit** and the **negative unit**, by the symbols—

$$+1 \quad \textit{and} \quad -1.$$

$+1$ represents the unit first chosen, and consequently is the fundamental unit of algebraical numbers.

Repetitions of these units (and of subdivisions of them) are indicated by the symbols—

$$+1,\ +2,\ +3,\ +4,\ \text{etc.},\ +\tfrac{1}{2},\ +\tfrac{3}{4},\ \text{etc.},$$
$$-1,\ -2,\ -3,\ -4,\ \text{etc.},\ -\tfrac{1}{2},\ -\tfrac{3}{4},\ \text{etc.}$$

These are the symbols used to denote the numbers of elementary algebra.

The mark $+$ is called the positive sign, and is read **plus**. The mark $-$ is called the negative sign, and is read **minus.**

Algebraical numbers which differ only in sign are called **complementary** numbers.

The sign $+$ is often omitted, and when neither $+$ nor $-$ is placed before an arithmetical number, the sign $+$ is to be understood.

In dealing with these algebraical numbers, arithmetical results are included, for either of the above series of numbers includes all the numbers of arithmetic. The value of an algebraical number, considered independently of its sign, is called its **absolute**, or **arithmetical**, value.

+1 indicates that the positive unit is taken once, and is read *plus one*.

+2 indicates that the positive unit is taken twice, and is read *plus two*.

−1 indicates that the negative unit is taken once, and is read *minus one*.

−2 indicates that the negative unit is taken twice, and is read *minus two*.

+1 and +1 counted together give +2.
−1 and −1 counted together give −2.
+1 and −1 counted together give 0.

In arithmetic, when *one yard* is the unit, the number 3 indicates the measurement of the quantity *three yards*.

In algebra, when *one yard up* is the positive unit, the number +3 indicates the measurement of the quantity *three yards up*, and −3 the measurement of the quantity *three yards down*.

EXERCISE 1.

Find the algebraical numbers which represent the following measurements:

1. Thirty feet up when the positive unit is one foot up.
2. Thirty feet down when the positive unit is one foot up.
3. Thirty feet up when the positive unit is one foot down.
4. Thirty feet down when the positive unit is one foot down.
5. A gain of 25 dollars, the positive unit being a gain of 5 dollars.
6. A loss of 40 cents, the positive unit being a gain of 10 cents.
7. Ten miles east, when the positive unit is one hundred yards west.

---

8. If a debt of 5 dollars is represented by + 5, what will + 3 represent? What will − 4 represent?

9. What is the positive unit when a debt of 10 dollars is represented by + 2?

10. What is the negative unit when an expansion of 3 inches is denoted by − 2?

ADDITION.

In arithmetic we add numbers by counting together the units indicated.

Likewise in algebra we add numbers by counting together the units indicated; but we must remember that complementary numbers, such as +1 and −1, or +2 and −2, or +3 and −3, etc., when added cancel each other.

Thus, +2 and −2	when added give	0.
+2 and +3	"	+5.
−2 and −3	"	−5.
+2 and −3	"	−1.
−2 and +3	"	+1.

EXERCISE 2.

Add the numbers—

1. +2, +3, +5.
2. −2, −3, −4.
3. +1, −2, +3, −4, +5, —6.
4. −12, −15, +14, +20, −36, +50.
5. $+\frac{1}{2}, -\frac{2}{3}, +\frac{4}{5}, -\frac{5}{6}$.
6. $+1\frac{1}{2}, -2\frac{2}{3}, +6\frac{1}{6}, -2\frac{5}{12}$.
7. 32, −16, 14, −28, +30, −50, 60.
8. 1500, −18·27, +131·4, −1415, ·1513.
9. State a rule for the addition of two numbers which have like signs.
10. State a rule for the addition of two numbers which have unlike signs.
11. Add the algebraical numbers which represent the quantities, 15 dollars loss, 18 dollars gain and 20 dollars gain, when one dollar loss is the positive unit.

SUBTRACTION.

By the subtraction of one number from another we mean the taking away, or counting out, from the latter the units indicated by the former.

If we wish to subtract $+3$ from a given number we can do so by counting in -3 with the given number, since this will have the effect of counting out $+3$.

To subtract $+4$ from $+7$, we add $+7$ and -4.

To subtract -4 from $+7$, we add $+7$ and $+4$.

To subtract -4 from -7, we add -7 and $+4$.

To subtract $+4$ from -7, we add -7 and -4.

That is, to subtract an algebraical number we add the complementary number.

Exercise 3.

1. Subtract $+3$ from $+5$.
2. Subtract -3 from $+5$.
3. Subtract $+3$ from -5.
4. Subtract -3 from -5.
5. From -36 take -20, and add -30 to the result.
6. From the sum of -28 and -32 subtract the sum of -60 and $+30$.
7. Add -13, -12, $+18$, and subtract the result from the sum of 8, -5, -20 and $+30$.
8. From the sum of -12, $+15$, -13 and 10 take the sum of -18 and $+27$.
9. What must be added to the sum of -4, -5 -6 and $+8$ to give $+13$?
10. What number must be subtracted from the sum of -19, -20 and -5 to give a remainder $+17$?

MULTIPLICATION.

In arithmetic we multiply 7 by 3 by counting together 7 and 7 and 7, and obtain 21; that is, we do with 7 that which we did with the unit to obtain 3. Also, when we multiply $\frac{2}{3}$ by $\frac{4}{5}$, we divide $\frac{2}{3}$ into five equal parts and take four of these parts—that is, we do with $\frac{2}{3}$ that which we did with the unit to obtain $\frac{4}{5}$.

Similarly, in algebra, we multiply $+7$ by $+3$, by doing with $+7$ just that which we did with $+1$ to get $+3$; that is, we count together $+7$ and $+7$ and $+7$ and obtain $+21$. Also, we multiply $+7$ by -3, by counting together -7 and -7 and -7, and obtain -21; for we obtain -3 from $+1$ by changing its sign, that is, taking the complementary number, and counting it three times.

To multiply -7 by $+3$ we count together -7 and -7 and -7, and obtain -21.

To multiply -7 by -3 we count together $+7$ and $+7$ and $+7$, which gives $+21$.

Exercise 4.

1. Multiply $+4$ by $+2$.
2. Multiply -4 by $+2$.
3. Multiply $+4$ by -2.
4. Multiply -4 by -2.
5. Multiply the sum of $+10$, -8 and -16 by the sum of -4, -6 and $+8$.
6. Subtract the sum of $+60$, -30, -32 and $+24$ from the product of -24 and $+13$.
7. From the product of -12 and -13 subtract the sum of -4, 5, -6, $+7$ and -8.
8. What is the sign of the product of two positive numbers?
9. What is the sign of the product of two negative numbers?
10. What is the sign of the product of two numbers which have different signs?

DIVISION.

The division of one number by another means the finding of that number which multiplied by the latter number gives the former.

We wish to divide $+6$ by $+2$. Now we counted $+1$ twice to obtain $+2$; hence to divide $+6$ by $+2$ we must find that number which counted twice will give $+6$: therefore $+3$ is the result.

To divide $+6$ by -2 we must find that number which, after having its sign changed, and being counted twice, will give $+6$: therefore -3 is the result.

Similarly, -6 divided by $+2$ gives -3,
and -6 divided by -2 gives $+3$.

Exercise 5.

1. Divide $+12$ by $+4$.
2. Divide -12 by $+4$.
3. Divide $+12$ by -4.
4. Divide -12 by -4.
5. Divide the sum of -20 and -40 by the sum of $+30$ and -20.
6. From the product of -40 and -40 subtract the product of -30 and -30, and divide the result by the sum of $+40$ and -30.
7. What is the sign of the result of dividing a positive number by a positive number?
8. What is the sign of the result of dividing a positive number by a negative number?
9. What is the sign of the result of dividing a number by another of different sign?
10. What is the sign of the result of dividing a number by another of the same sign?

Besides their uses to distinguish the two series of algebraical numbers, the signs + and − are used to indicate the *operations* of addition and subtraction respectively, as in arithmetic.

The signs × and ÷ are also used to indicate multiplication and division respectively.

The bar — also indicates division.

An algebraical number is sometimes enclosed in brackets to distinguish the two uses of + and −.

Thus: $(+5) + (-3) = +2.$

$(-7) - (+3) = -10.$

$(+5) \times (-3) = -15.$

$(+12) \div (-6) = -2.$

$$\frac{+12}{-6} = -2.$$

Exercise 6.

Find the value of:

1. $(+5) + (-6) + (+7) + (-8).$
2. $(+4) - (-4) + (-4) - (+4).$
3. $(+5) \times (-6) + (+3) \times (-2).$
4. $(-10) \div (+2) - (+12) \div (-3).$
5. $(+5) \times (-5) + (+6) \div (-2).$
6. $(+12) \div (-4) + (-3) \times (-2) - (+6) \times (-1).$
7. $\frac{-60}{-12} + \frac{-30}{-5} + \frac{-40}{+4}.$
8. $\frac{-20}{+4} + \frac{40}{-4} - \frac{60}{6}.$
9. $(-15) \div (-3) + (-40) \div (+4) - (-20) \div (-5).$
10. $\frac{-52}{-4} + \frac{-8}{-4} - \frac{-60}{4} - \frac{-10}{10}.$

CHAPTER II.

Definitions, Substitutions, Combining Like Terms.

In algebra we deal not only with **definite** numbers, such as those of the preceding chapter, but we reason also about **general** numbers, which we represent by letters.

Thus, no matter what number is denoted by either a or b, we have

$$a + b - b = a.$$

When we use letters to represent numbers, we indicate the operation of multiplication by writing one letter immediately after the other. Thus ab means the product of the number a and the number b.

We also indicate the product of a definite number and a general number by writing the general number after the other with no sign between. Thus $-2a$ means the product of -2 and a.

When a number is expressed as the product of two numbers, each of these numbers is called the **coefficient** of the other, and each is also called a **factor** of the product.

A coefficient is called a **numerical** coefficient when it is a definite number, and a **literal** coefficient when it is a general number. In the product $-4x$, -4 is the numerical coefficient of x; in the product of ax, a is the literal coefficient of x.

Exercise 7.

1. What is meant by mn? $-6m$? $+4n$? $3m$? $-4mn$?

2. If m is $+6$ and n is -3, what is the value of mn? $-6n$? $+4n$? $3m$? $-4mn$?

3. State the meaning of each of the following sets of symbols: abc, $ab + c$, $a + bc$, $-2 + bc$, $-2b + c$, $-2bc$.

4. If a is $+2$, b is ∓ 3, and c is -4, find the value of abc $ab + c$, $a + bc$, $-2 + bc$, $-2b + c$, $-2bc$.

5. Find the value of each of the following expressions, in which $a = +2$, $b = -3$, $c = -2$, $d = +4$, $x = +1$, $y = -5$, $z = -10$:—

(1) $a+b+c+d$. (2) $a+b+cx+dy$.
(3) $ab-cd+ax-yz$. (4) $ac-bd+2x-3y+4z$.
(5) $abc+abd+bcd+xyz$. (6) $abcx-abdy+acdx-bcdy$.
(7) $abcdxyz$.

6. If the value of $+3x$ is $+30$, what must be the value of x?

7. When $-3x$ is $+30$, what is the value of x?

8. When $-3x$ is -30, what is the value of x?

9. If the value of ab is $+40$, and the value of a is -10, what is the value of b?

10. If $ab = -50$, and $b = +5$, what is the value of a?

11. If $abc = +60$, and $a = +2$, and $b = -3$, what is the value of c?

12. If $abc = -90$, and $a = -9$, what is the value of bc?

13. $abcd$ is equal to 40, when $a = -1$, $b = -2$, and $c = +5$, what is the value of d?

14. If yz is equal to -10, what is the value of $abyz$ when $a = -3$ and $b = +2$?

15. If $a+b$ is equal to $+20$, and b is equal to $+6$, what is the value of a?

16. If $ab+cd$ is equal to 100, and ab is equal to $+40$, what is the value of cd?

17. If $ab+cd = 60$, and $ab = 20$, and $c = -4$, what is the value of d?

18. If $ax+by = -50$, and $a = -4$, $b = -10$, and $x = -5$, what is the value of y?

19. $ab-cd = 60$, $a = +2$, $c = +3$, $d = +4$, what must be the value of b?

20. $-3x+40 = 70$, what is the value of x?

A collection of algebraical symbols, that is, of letters, figures and signs, which, when all the operations indicated are carried out, results in a number, is called an **algebraical expression,** or, shortly, an **expression.**

The parts of an algebraical expression which are to be used as addends in finding the final value of the expression are called **terms.**

Thus $ax - by + 2z - 36$ is an expression of which the terms are ax, $-by$, $+2z$, -36.

Those terms which are written with the sign + (or without the sign) are called **positive terms.**

Those written with the sign − are called **negative terms.**

Terms which differ only in the sign are called **complementary** terms.

Thus in the expression $a + 3x - by + by - 3x$, $+3x$ and $-3x$ are complementary terms. Such also are $-by$ and $+by$.

An expression consisting of one term is called a **monomial**: as $4ab$.

An expression of two terms is called a **binomial**: as $2a + 3c$.

An expression of three terms is called a **trinomial**: as $a + b + c$.

All expressions of more than one term are called **multinomials,** or **polynomials.**

Combining Like Terms.

Two terms which contain the same letters involved in the same way are called **like terms.**

Thus, $+5m$, $+3m$, $-6m$ are like terms.

abc, $4abc$, $-5abc$ are also like terms.

Consider the like terms $+5m$ and $+3m$. $+5m$ means m repeated five times, and $+3m$ means m repeated three times. Hence, their sum is m repeated eight times, that is $+8m$.

Consider the like terms $+8m$ and $-6m$. $+8m$ means m repeated eight times, and $-6m$ means that the sign of m is changed, and the resulting number repeated six times. Hence their sum is m repeated twice, that is $+2m$.

From these examples we see that two like terms may be combined into a like term, whose numerical coefficient is the sum of the numerical coefficients of the given terms.

The sum of $-6abc$ and $+9abc$ is $+3abc$, since the sum of -6 and $+9$ is $+3$.

EXERCISE 8.

Simplify the following expressions by combining like terms:

1. $+8a+6a-9a$.
2. $+8abc+6abc-9abc$.
3. $+13x-12y+4y-5x$.
4. $2y-16y+13y-14x$.
5. $\frac{1}{2}x-\frac{1}{3}x+\frac{1}{4}x-\frac{1}{5}x+\frac{1}{6}x$.
6. $\frac{1}{3}abc+\frac{2}{5}abc-\frac{3}{4}abc+\frac{5}{6}abc$.
7. $-3pq+10pq-7pq+x+y+z$.
8. $pq-4pq+6pq+3p+2q$.
9. $ap+bq+13pq-24pq-16pq$.
10. $3x-4x+8x+12xy-4xy-10xy$.
11. $40abcd-38abcd-16abcd+32abcd$.
12. $12x+13y+14z+6x-5y-8z-3x+4y-5z$.
13. $7x-3z+4x+14y-20z-32x-4y$.
14. $27mn-36nl+40lm+42nl-32mn$.

    What does mn mean? What does nm mean?
15. $+3mn+4nm+5mn$.
16. $4abc-5acb+6bca-17cba$.
17. $13xyz+12ayz-20yzx-17zay+50zyx+100azy$.
18. $8X+13Y-16Y+14X-4Y$.

CHAPTER III.

ADDITION.

To add $+3x$ and $-4y$, we merely write the terms, one after the other, with sign unchanged; thus, $+3x-4y$.

Also, to add $+3x+2y$ and $+6x-5y$, we write $+3x+2y+6x-5y$.

This last result may be simplified by combining the like terms giving $+9x-3y$.

It is sometimes more convenient to write the expressions to be added, one under the other, having like terms in vertical columns. Thus, the above sum would be written—

$$\begin{array}{r} +3x+2y \\ +6x-5y \\ \hline +9x-3y \end{array}$$

Exercise 9.

Add together—

1. $+5a+6b;\ +7a-2b.$
2. $14a+3b;\ -6a+2b;\ -3a-5b.$
3. $6a+7b;\ 9b-13a;\ 14a+6b+c.$
4. $-3a-2b+4c;\ -6a-7b-13b;\ 17a+18b+19a.$
5. $22x-y+6z;\ 12y-13x;\ 40x-30z+25y.$
6. $x+y+z;\ y+z-x;\ z+x-y;\ x+y-z;\ -x-y-z.$
7. $x+2y+3z;\ 4y+5z-6x;\ 7z+8x-9y;\ 10x+11y-12z.$
8. $2a+b-c-d;\ 3a-6b+14c+d;\ 12a-30b;\ 16b+13c.$
9. $a+b+c+d;\ b+c+d-a;\ c+d-a-b;\ d+a-b-c.$
10. $10xy+13yz;\ 10yz+13xy-20zx;\ 27xy+12yz+zx.$
11. $7+13ab-16bc;\ 22bc-16ab+13;\ -17+ab.$
12. $pqr+abc-100;\ 45abc-16pqr;\ 17-pqr-abc.$
13. $14ab+\frac{1}{3}bc+\frac{1}{4}cd;\ -\frac{1}{2}ab+\frac{5}{6}bc-10cd.$
14. $\cdot25ab-\cdot63bc+\cdot9cd;\ \cdot5bc-\cdot4ab+\cdot75cd.$
15. $\frac{1}{2}x+\frac{1}{3}y-\frac{1}{4}z+\frac{1}{6}xyz;\ x-y+z-xyz;\ \frac{1}{3}x-\frac{1}{2}y-\frac{3}{4}z+xyz.$
16. $13abcd+14abde-15bcde;\ 12bcda-15bade-10becd.$

17. $17pq + 15qr - 13rp + 18rq$; $22qr - 16rp + 13pq$; $-22qr - 40pr - 13pq$; $12pq + 13rq + 14qp$.

18. $270xy - 1320xyz + 1405xyzw$; $190xyz - 300yx - xzyw$; $xy + xyz - xyzw$.

19. $x+y+z+w+a+b$; $x-y+z-w+a-b$; $a-b-x-y-z-w$; $z+w-x-y+a+b$; $-a-b-x-y-z-w$; $+a-b$; $-x+y-z$; $-w$.

20. $b+\frac{1}{2}c+\frac{1}{3}a$; $c+\frac{1}{3}a-\frac{1}{2}b$; $a+\frac{2}{3}b-\frac{3}{4}c$; $-2a-2b-2c$.

SUBTRACTION.

We have already shown that to subtract an algebraical number is the same as to add the complementary number. Now, the complementary of any expression may be obtained by changing the sign of each of its terms: for, the sum of any expression, and the expression obtained by changing the sign of each of its terms, is zero.

Hence, to subtract any algebraical expression from another, we add to the latter the expression obtained by changing the sign of each term of the former.

Thus, to subtract $6a + 4b$ from $13a - 12b$, we must add $13a - 12b$ and $-6a - 4b$.

This may be done by writing the terms in succession, as

$$13a - 12b - 6a - 4b,$$

and combining like terms.

Or, as in the case of addition, by writing the like terms in vertical columns; thus:—

$$\begin{array}{r} 13a - 12b \\ -6a - 4b \\ \hline 7a - 16b. \end{array}$$

When the subtraction is performed in the latter way, it is usual to leave the signs of the expression to be subtracted unchanged when writing it under the other. Then the signs must be changed mentally when performing the addition.

Thus, to subtract $8ab - 6bc + 4de$ from $13ab - 8bc - 6de$ we write

$$\begin{array}{r} 13ab - 8bc - 6de \\ 8ab - 6bc + 4de \\ \hline 5ab - 2bc - 10de. \end{array}$$

and obtain $5ab - 2bc - 10de$.

EXERCISE 10.

Subtract :—

1. $2a + 3b + 4c$ from $6a + 5b + 6c$.
2. $a + b + c$ from $2a - b + c$.
3. $a - b - c$ from $a + b + c$.
4. $2x - 3y + 4z + 6w$ from $10x - y - z - 10w$.
5. $a + 2b - 3c - 4d$ from $b + c - d - a$.
6. $17x - 14y + 20z$ from $40x - y - z - w$.
7. $m + n + p$ from $2m$.
8. $2m + 3n - 4p$ from $-p + 5m$.
9. $2ab + 3bc + 5ca$ from $4ab + 4bc + 4ca$.
10. $10xyz + 12yzw - 13zwx$ from $-5yzw$.
11. Subtract $2xy - 3yz + 4zx$ from $3xy - 4yz + zx$ and add $xy - yz + zx$ to the remainder.
12. Subtract the sum of $10l - 7m + 3n - p$ and $3m - n + p - 3l$ from $12l - 11m + 10n$.
13. Subtract from $x + y + z$ the sum of $2x + y + z$, $x + 2y - 3z$ and $-x - y + z$.
14. Subtract from $ax - 3bx + 4cx$, the sum of $-2ax + 4bx - 5cx$, $ax - 3bx + 6cx$ and $2ax - 2bx - 2cx$.
15. Subtract the sum of $ax - a + 1$, $2ax - 2a - 2$, and $-ax - a - 4$ from $ax + a + 1$.

BRACKETS.

We indicate that an algebraical expression is to be added as a whole by placing it in brackets with a + sign prefixed.

Thus, $a + (b + c)$ means that the sum of b and c is to be added to a. But this is the same as writing down the terms of the expression to be added with their signs unchanged. That is,

$$a + (b + c) = a + b + c.$$

From this we see that when a pair of brackets is preceded by the sign +, the brackets may be removed.

Again, we indicate that an algebraical expression is to be subtracted as a whole by placing it in brackets with a − sign prefixed.

Thus, $a - (b + c)$ means that the expression $b + c$ is to be subtracted from a. But this is the same as writing down the terms of the expression to be subtracted, with their signs changed. That is,

$$a - (b + c) = a - b - c.$$

Hence we see that when a pair of brackets is preceded by the sign −, the brackets may be removed if we change the sign of each term which was enclosed by the brackets.

Sometimes several pairs of brackets are used, one pair being wholly enclosed by another pair. In such cases the pairs of brackets are made of different shapes :—

{ } [ ]

In removing these brackets, it is best to begin with the innermost pair, and remove only one pair at a time.

Thus,

$$\begin{aligned} & x - \{x - (2x - y) + 2y.\} \\ = & x - \{x - 2x + y + 2y.\} \\ = & x - x + 2x - y - 2y. \\ = & 2x - 3y. \end{aligned}$$

EXERCISE 11.

Remove the brackets from the following expressions and combine the like terms :

1. $x + (y - z)$.
2. $x - (y - z)$.
3. $2x - (y - 3x) + (4x - 4y)$.
4. $(3x - 3y + 4z) - (2x - 2y + 3z) - (x - y + z)$.

5. $(x - y) + (y - z) + (z - x) + (x + y + z)$.

6. $x - (y - z) - (z - x) - (x - y)$.

7. $(x - y + z) + 3x - (2y - 2x) - (x - y + z)$.

8. $20 + (13 - a) + 4 - \{5 - (6 - 2a)\}$.

9. $x - \{(y - z) - (z - x)\} - (x - y + z)$.

10. $2p - (3q + 4r) - \{2p - (3q + 4r)\} + p + q + r$.

11. $a - \{2b - 3a - (4a - 2b)\} + 2b - 3a - (4a - 2b)$.

12. $3a + [7b - \{5c - (2a - b) + 3c\}]$.

13. $a + b + c + d - [a - \{b - (c - d)\}]$.

14. $[2x - y - \{3x + 2y - (y - x)\}] - [\{(x - y) - x\} + y]$.

15. $2 - [2 - \{2 - (2 - a) - 2\} - 2]$.

A bar is sometimes placed over the terms which are to be considered as one quantity. Thus, $a - \overline{b - c}$ means the same as $a - (b - c)$.

16. $2x - [2y - \{2x - y - (2x - y - \overline{x - y}) + 2y\} - y]$.

17. $a + b + c + d - [a - b + \{c + d - (b - \overline{c - d} - b - d)\}]$.

18. $\overline{2x - 3y + 4z + 3w} - \overline{x - 2y - 3z - 4w}$.

19. $a - [-2a - \{2b - (2c - \overline{3a - 4b} - 4c) - 3a - b\} - c]$.

20. $1 - [\{2 - \overline{3 - 4x}\}] - 2 - [3 - \{4 - (5 - \overline{6 - x})\}]$.

Evidently any number of terms of an expression may be enclosed within a pair of brackets, preceded by the sign + without changing the signs of the terms; and within a pair of brackets preceded by the sign −, if the signs of the terms are changed.

Exercise 12.

Enclose the third, fourth and fifth terms of each of the following expressions in a pair of brackets, preceded (1) by the sign +, (2) by the sign − :

1. $2a - b - 3c + 4d - e.$
2. $a - b - c - d - e + f.$
3. $2x - 3y + 4z - 3x - 3y + 3z.$
4. $2x - y - 3x - 4y - 4x + y.$
5. $a + b - c - a + b - c.$
6. $2x - 3y + 4z - 5w + 6u.$
7. $3a - 3b - 4c - 4d - e.$
8. $2b - 4 - 2c - y + 3z.$
9. $-x + y - z - w + 6u.$
10. $-\frac{1}{2}x + \frac{1}{3}y - \frac{1}{3}z + \frac{1}{4}w - \frac{2}{3}u.$

Enclose the third and fourth terms of the following in a pair of brackets preceded by the sign −, and then all of the expression after the first term in another pair of brackets preceded by the sign − :

11. $a-b-c+d+e.$
12. $2x-3y-4z+5w.$
13. $x-y+z-3.$
14. $ab+bc+ca-abc.$
15. $-x+y-z+w.$
16. $-\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}c+\frac{1}{2}d.$
17. $(a-b)-(c-d)+(a+d).$
18. $x-(y-2z)-(z-2x)-(x-2y)+w.$

CHAPTER IV.

MULTIPLICATION.

We know that the product of two arithmetical numbers is the same, no matter which number is taken as multiplier.

Thus 3 multiplied by 4 gives the same result as 4 multiplied by 3; and $\frac{2}{3}$ multiplied by $\frac{4}{5}$ gives the same result as $\frac{4}{5}$ multiplied by $\frac{2}{3}$.

Evidently the same will be true of algebraical numbers: for the absolute value of the product is independent of the signs, and the sign of the product is independent of the order of the factors.

MULTIPLICATION OF MONOMIALS.

We have already learned that the product of two general numbers is indicated by writing the letters one immediately after the other. In finding the product of two monomials, we have then only to multiply the numerical factors of the terms and annex the literal factors.

Thus $(+\ 4x) \times (+\ 3y)$
$= +\ 12xy.$

And $-\ 3xy \times -\ 4ab$
$= +\ 12xyab.$

When a literal factor is repeated in a product, the number of times the factor occurs is indicated by a small figure written a little above and to the right of the letter.

Thus the product of $-\ 4a$ and $-\ 5a$ is written $+\ 20a^2$.

Instead of aa we write a^2,
" aaa " a^3,
" $aaaa$ " a^4,

and so on.

$(a + b)$ multiplied by $(a + b)$, indicated by $(a+b)(a+b)$, is also written $(a + b)^2$.

$(a + b)(a + b)(a + b)$, is also written $(a + b)^3$.

$7 \times 7 \times 7 \times 7 \times 7$, is also written 7^5.

The products, $a^2, a^3, a^4, \ldots\ldots$ are called the second, third, fourth, ... **powers** of a.

The number which indicates the power is called the **index** or **exponent**. When a letter has no index, 1 is to be understood; thus, $a = a^1$.

The product of a^2 and a^3 is a^{2+3}:

$$\begin{aligned} \text{for } a^2 &= a\,a, \\ \text{and } a^3 &= a\,a\,a, \\ \therefore a^2 \times a^3 &= a\,a\,a\,a\,a \\ &= a^5. \end{aligned}$$

$$\begin{aligned} \text{Similarly, } x^3 \times x^5 &= x^8: \\ \text{for } x^3 &= x\,x\,x, \\ \text{and } x^5 &= x\,x\,x\,x\,x, \\ \therefore x^3 \times x^5 &= x\,x\,x\,x\,x\,x\,x\,x \\ &= x^8. \end{aligned}$$

Hence we see that the index of the product of two powers of the same letter is equal to the sum of the indices of the factors.

Exercise 13.

Multiply:

1. -3 by $+4$.
2. -5 by -3.
3. -2 by -5.
4. $-3b$ by $+4a$.
5. $-5x$ by $-3y$.
6. $-2m$ by $-5n$.

7. $-3b$ by $+4b$.
8. $-5xy$ by $-3x$.
9. $-2mn$ by $-5mn$.
10. $-3b^2$ by $+4b^3$.
11. $-5x^3$ by $-3x^4$.
12. $-2m^{10}$ by $-5m^6$.
13. $-4ab^2c^3$ by $-5a^3b^2c$.
15. $-\frac{2}{3}a^5b^3c^4d^2$ by $+\frac{1}{2}ab^2cd^3e$.
15. $-3xyz^2$, $8x^3y^2$ and $-\frac{5}{12}x^3z^2$.
16. $-2m^2n^3p^2$, $3mnpq$ and pq.
17. $-x$, $-x$ and $-x$.

Find the value of :

18. $(-x)^4$.
19. $(-2ab)^3$.
20. $(-2a^2b)^3$.
21. $(-3)^5$.
22. $(-2a^2b)^4$.
23. $(-ab^2c^3)^2$.
24. $(-3x^2y)^2 \times (-2xy^3)^2 \times (-xyz)^3$.
25. $-(a^2bc)^2 \times (-a^2bc)^3$.

MULTIPLICATION OF A MULTINOMIAL AND A MONOMIAL.

We indicate the product of $a+b$ and c by placing $a+b$ in brackets and writing c either immediately after it or before it.

Thus $$(a+b)\,c, \text{ or } c\,(a+b).$$

To multiply c by $a+b$, we must use the quantity c as the unit in forming the expression $a+b$.

That is, we must do with c that which we did with $+1$ to obtain a; and also we must do with c that which we did with $+1$ to obtain b; and then add the two results.

Hence to find the product of $(a+b)$ and c, we must add the product of a and c and the product of b and c.

Thus $$(a+b)\,c = ac + bc.$$

This result is true, no matter what the values of a and b may be.

Similarly it may be shown that

$$(a+b+c+d+\ldots\ldots)\,m$$
$$=am+bm+cm+dm+\ldots\ldots$$

Now, every expression may be written in the form

$$a+b+c+\ldots\ldots$$

by enclosing each term in a bracket and placing the sign + before it.

Thus
$$2x - 3y + 4xz^3 - 6w^2$$
$$=(2x)+(-3y)+(+4xz^3)+(-6w^2).$$

And the product of this expression and a monomial, say $-2xy$, will be obtained by adding the products of each of the quantities enclosed in the brackets and $-2xy$.

The product is

$$-4x^2y+6xy^2-8x^2yz^3+12xyw^2.$$

Hence the product of a multinomial expression and a monomial may be obtained by multiplying each term of the multinomial by the monomial.

Exercise 14.

Multiply:

1. $a+b$ by x. 2. $c+d$ by m. 3. $2a+3b$ by x.
4. $3c+5d$ by m. 5. $2a+3b$ by $4x$. 6. $3c+5d$ by $10m$.
7. $2a+3b$ by $-2x$. 8. $3c+5d$ by $-3m$. 9. $3a-4b$ by $-2c$.
10. $2c-4d$ by -4. 11. $a+b$ by a. 12. $c+d$ by d.
13. x^2+x by $2x$. 14. $2x^2-3x$ by $3x$.
15. x^2+2x+1 by x. 16. x^2-2x-2 by $3x^2$.
17. x^2+xy+y^2 by $x.^2$ 18. x^2+xy+y^2 by $-xy$.
19. x^2+xy+y^2 by y^2. 20. $2a+3b-4c$ by $-2abc$.
21. x^2 by x^2-y^2. 22. y^2 by x^2-y^2.
23. $-2xyz$ by $x^3+x^2y+x^2z+xyz$.

24. Multiply $a^2+b^2+c^2-bc-ca-ab$ by a; also by b; also by c. Then add the three results.

25. Multiply x^2-xy+y^2 by x^2; also by $+xy$; also by y^2. Then add the three results.

MULTIPLICATION OF MULTINOMIALS.

To find the product of $a + b$ and $c + d$. Enclosing each expression in brackets, we have

$$(a+b)(c+d) = a(c+d)+b(c+d).$$
$$\text{But } a(c+d) = ac+ad,$$
$$\text{and } b(c+d) = bc+bd.$$
$$\therefore (a+b)(c+d) = ac+ad+bc+bd.$$

Again,

$$(a+b+c)(d+e+f)$$
$$= a(d+e+f)+b(d+e+f)+c(d+e+f)$$
$$= ad+ae+af+bd+be+bf+cd+ce+cf.$$

From these examples we conclude that the product of two multinomials may be found by taking the sum of the products formed by multiplying each term of the one expression by each term of the other.

The product of

$$2x+3y \text{ and } 3x+4y$$

is obtained thus:

$$(2x+3y)(3x+4y)$$
$$=2x(3x+4y)+3y(3x+4y)$$
$$=6x^2+8xy+9xy+12y^2$$
$$=6x^2+17xy+12y^2.$$

And the product of $a-3b$ and $2a+4b$:

$$(a-3b)(2a+4b)$$
$$=a(2a+4b)-3b(2a+4b)$$
$$=2a^2+4ab-6ab-12b^2$$
$$=2a^2-2ab-12b^2.$$

The work is commonly arranged thus

$$\begin{array}{l} 2a+4b \\ \underline{a-3b} \\ 2a^2+4ab \text{..........product of } 2a+4b \text{ and } a. \\ \underline{\quad\;\; -6ab-12b^2} \text{....product of } 2a+4b \text{ and } -3b. \\ 2a^2-2ab-12b^2 \end{array}$$

Exercise 15.

Find the product of :

1. $2x+3y$ and $x+2y$.
2. $3a+b$ and $2a+5b$.
3. $2a+3b$ and $a+2b$.
4. $3x+y$ and $2x+5y$.
5. $2m+3n$ and $m+2n$.
6. $3p+q$ and $2p+5q$.
7. $2a-b$ and $a+2b$.
8. $3x-2y$ and $2x-3y$.
9. $2m-n$ and $m+2n$.
10. $3m-2n$ and $3m-3n$.
11. $a+6$ and $2a+5$.
12. $7x-4$ and $3x+5$.
13. $x+12$ and $x+5$.
14. $x+6$ and $x+10$.
15. $x+4$ and $x+15$.
16. $x+3$ and $x+20$.
17. $x+2$ and $x+30$.
18. $x+1$ and $x+60$.
19. $x-12$ and $x-5$.
20. $x-6$ and $x-10$.
21. $x-4$ and $x-15$.
22. $x-3$ and $x-20$.
23. $x-2$ and $x-30$.
24. $x-1$ and $x-60$.
25. $x-12$ and $x+5$.
26. $x+12$ and $x-5$.
27. $x-4$ and $x+15$.
28. $x+4$ and $x-15$.
29. $x-6$ and $x+10$.
30. $x+6$ and $x-10$.
31. $x-3$ and $x+20$.
32. $x+3$ and $x-20$.
33. $x-2$ and $x+30$.
34. $x+2$ and $x-30$.
35. $x-1$ and $x+60$.
36. $x+1$ and $x-60$.
37. $3x+24$ and $3x+1$.
38. $3x-24$ and $3x+1$.
39. $3x+12$ and $3x+2$.
40. $3x+12$ and $3x-2$.
41. $3x+8$ and $3x+3$.
42. $3x-8$ and $3x+3$.
43. $3x-8$ and $3x-3$.
44. $3x-10$ and $3x-2$.
45. $x+y$ and $x-y$.
46. $a+b$ and $a-b$.
47. $m+n$ and $m-n$.
48. $p+q$ and $p-q$.
49. $2x+y$ and $2x-y$.
50. $2x+3y$ and $2x-3y$.
51. $ab+c$ and $ab-c$.
52. $pq+bm$ and $pq-bm$.
53. a^2+b^2 and a^2-b^2.
54. a^2+bc and a^2-bc.
55. $2a^2+3b^2$ and $2a^2-3b^2$.
56. $lmn+pqr$ and $lmn-pqr$.

Exercise 16.

Find the product of:

1. $2x + y + 3z$ and $3x$.
2. $2x + y + 3z$ and $2y$.
3. $2x + y + 3z$ and $3x + 2y$.
4. $bc + ca + ab$ and $a + b + c$.
5. $x + y + z$ and $x + y + z$
6. $a + b + c$ and $a + b + c$.
7. $2x + 3y + 4z$ and $2x + 3y + 4z$.
8. $x^2 + y^2 + z^2 - yz - zx - xy$ and $x + y + z$.
9. $x^2 + y^2 + z^2 + yz - zx + xy$ and $x - y + z$.
10. $x^2 + y^2 + z^2 - yz + zx + xy$ and $x - y - z$.
11. $2a^2 - 4 + 6a + 3a^3$ and $2a + 3$.
12. $3x - 4x^2 + x^3 - 6$ and $2x - 3$.
13. $x^3 + x^2 y + xy^2 + y^3$ and $x - y$.
14. $3m^2 + 4 + 2m + 5m^3$ and $m^2 - 2 + m$.
15. $1 + 2z + 3z^2 + 4z^3$ and $3z^2 + 2z + 1$.

An expression frequently consists of terms which contain different powers of the same letter.

It will usually be found convenient to arrange all such expressions according to descending or ascending powers of that letter. And in multiplying two such expressions containing powers of the same letter, it is advisable to arrange both expressions in the same order.

Arranging according to descending powers of m, example 14 of the previous exercise would be worked in the following way:

$$\begin{array}{rrrrrr}
5m^3 & + 3m^2 & + 2m & + 4 & & \\
m^2 & + m & - 2 & & & \\
\hline
5m^5 & + 3m^4 & + 2m^3 & + 4m^2 & & \\
 & + 5m^4 & + 3m^3 & + 2m^2 & + 4m & \\
 & & - 10m^3 & - 6m^2 & - 4m & - 8 \\
\hline
5m^5 & + 8m^4 & - 5m^3 & & & - 8
\end{array}$$

This arrangement of the terms makes easy the placing of like terms in vertical columns.

Exercise 17.

Find the product of:

1. $2x^2 + 3xy - 4y^2$ and $x^2 - 2xy + y^2$.
2. $x^4 + x^3 + x^2 + x + 1$ and $x^5 - x^2 + x - 1$.
3. $4a^2 + 9b^2 + c^2 - 3bc - 2ca - 6ab$ and $2a + 3b + c$.
4. $x^2 - xy + y^2 + x + y + 1$ and $x + y - 1$.
5. $a^3 + 3a^2b + 3ab^2 + b^3$ and $a^2 - 2ab + b^2$.
6. $2x^4 - 3x^3 + x^2 - 2x + 1$ and $x^3 + 2x^2 + 2x - 1$.
7. $x^3 + 3x^2y + 3xy^2 + y^3$ and $x^3 - 3x^2y + 3xy^2 - y^3$.
8. $(x + y)$, $(y + z)$ and $(z + x)$.
9. $(a + b)$, $(b + c)$ and $(c + a)$.
10. $(x + 2)$, $(x + 3)$ and $(x + 4)$.
11. $(x - 2)$, $(x - 3)$ and $(x - 4)$.
12. $(x + 2y)$, $(x + 3y)$ and $(x + 4y)$.
13. $(x - 2y)$, $(x - 3y)$ and $(x - 4y)$.
14. $x^2 + x + 1$ and $x^2 - x + 1$.
15. $x^2 + xy + y^2$ and $x^2 - xy + y^2$.

Simplify, by removing brackets and combining like terms:

16. $x(x^2 + x + 1) - (x^2 + x + 1)$.
17. $x^2(x^2 + xy + y^2) + y^2(x^2 + xy + y^2) - xy(x^2 + xy + y^2)$.
18. $3(2a^2 - 3a + 1) + 4a(a - 2) + 2a^2 + 9a - 3$.
19. $(3a + b)(2a + 3b) + (a + 2b)(2a - 5b) + 7(a^2 + b^2)$.
20. $2\left[2a - 3\left\{a - 2(a + 7) - 1\right\} + 1\right]$.
21. $(a + b)\left\{2a - (a - b) - b\right\} - a(a + b)$.
22. $a\left[a + a\left\{a - (a + 2) - 2\right\} + 2\right]$.
23. $3(a + 2b + 3c) - 4(2a + 3b + c) + 2(3a + b + 2c)$.
24. $2\left\{3(a - b)x + 4(b - c)y\right\} - \left\{2(a + b)x - 3(b + c)y\right\}$.
25. $4\left\{(a - b)(x + y) - (a + b)(x - y)\right\} - 2\left\{(a + b)(x + y) - (a - b)(x - y)\right\}$.

When a factor is common to all the terms within a pair of brackets, that factor can be removed from each term and placed outside the brackets.

Thus:

$$(ax + bx) = (a + b)x,$$

and $4m^2n - 6mn^2 + 3mn = (4m - 6n + 3)\, mn.$

Exercise 18.

Place in brackets the terms which contain like powers of x, and then remove that power of x outside of the brackets:

1. $4mx^2 + 5nx^2$.
2. $2ax + 5bx + cx$.
3. $ax + bx + cx^2 + dx^2$.
4. $2x + 3x^2 + 5cx + 4mx^2$.
5. $mx^3 + 4mnx + nx^3 + pqx$.
6. $2x + cx^3 + dx + ex^2 + fx^3 + 4x^2$.
7. $a^3bx + a^2bx^2 + a^2x^3 + a^2b^2x + ab^2x^2$.
8. $2x - (3x^2 + ax) + (bx^3 - cx^2) - cx^3$.
9. $ax^2 - (bx + cx^3) - (ax + bx^2) + (ax^3 - cx)$.
10. $3x - a\,(2x^2 + 3x^3) + b\,(x^3 - x)$.

Use brackets to indicate:

11. The product of $x + y$ and z.
12. The product of $x + y$ and $a + b$.
13. That the product of $a + b$ and $c + d$ is to be added to x.
14. That the product of $m - n$ and $m + n$ is to be subtracted from m^2.
15. The third power of the sum of a and b.
16. That the product of the sum and difference of a and b is equal to the difference of their squares.

DIVISION.

DIVISION OF A MONOMIAL BY A MONOMIAL.

We formed the product of one monomial by another by multiplying their numerical coefficients, and writing after this product the literal factors. Hence it is evident that to divide one monomial by another we must divide the numerical coefficient of the dividend by the numerical coefficient of the divisor, and write after this the literal factors of the dividend which remain after removing the literal factors of the divisor.

To divide $-20ab$ by $+5b$.

Divide -20 by $+5$, and we obtain -4.

Remove b from ab, and we obtain a.

Hence the quotient is $-4a$.

To divide $-16abc$ by $-4ac$.

-16 divided by -4 by gives $+4$,

and ac removed from abc gives b.

Hence the quotient is $+4b$.

To divide a^5 by a^3.

Since $a^5 = aaaaa$,

and $a^3 = aaa$,

$\therefore$ the quotient obtained by dividing a^5 by a^3

$$= aa = a^2.$$

From this we see that if one power of a letter be divided by another power of the same letter, the index of the quotient is obtained by subtracting the index of the divisor from the index of the dividend.

To divide $-20a^3x^2y$ by $+5a^2x$.

-20 divided by $+5$ gives -4,

and a^3x^2y divided by a^2x gives axy

Hence the quotient is $-4axy$

EXERCISE 19

Divide .

1. ab by a. 2. ab by b. 3. ab by ab.
4. $3ab$ by a. 5. $+4ab$ by b. 6. $-5ab$ by ab.
7. a^8 by a^3. 8. $x^3 y^4$ by $x^2 y^2$. 9. $x^2 y^3 z^4$ by xyz^2.
10. $-14a^8$ by $-2a^3$. 11. $20x^3 y^4$ by $-5x^2 y^2$.
12. $-40x^2 y^3 z^4$ by $-4xyz^2$. 13. $-4a^3b^2cd^3$ by $-2a^3bcd^3$
14. $-x^2yz^3$ by xz^3. 15. $8p^{10}q^{14}r$ by $-4pq^4r$.
16. $72\,a^2b^3c^4$ by $-9ab^2c$. 17. $abcpq$ by $-apq$.
18. $-a^3b^2c^2d^2$ by $-abcd$. 19. $72xy^5$ by $-xy^5$.
20. $-45x^{10}y^5z^8$ by $-9x^{10}y^5z^7$.

DIVISION OF A MULTINOMIAL BY A MONOMIAL.

We found the product of a multinomial and a monomial by multiplying each term of the multinomial by the monomial. Hence it is evident that we can find the quotient of a multinomial by a monomial by dividing each term of the dividend by the divisor.

To divide $2ab - 6a^2c$ by $-2a$.

$+2ab$ divided by $-2a$ gives $-b$,

and $-6a^2c$ divided by $-2a$ gives $+3ac$;

$\therefore$ the quotient is $-b + 3ac$.

EXERCISE 20.

Divide :

1. $ax + bx + cx$ by x. 2, $ax^2 + bx^2 + cx^2$ by x^2.
3. $2ax + 3bx + 4cx$ by x. 4. $ax^2 + 2bx^2 + 3cx^2$ by x^2.
5. $-3ax + 5bx - 46cx$ by $-x$.
6. $-4ax^2 + 8bx^2 - 6cx^2$ by $-2x^2$.
7. $-10abxy + 5a^2bx - 15ab^2x^3y^2$ by $-5ax$.
8. $-8a^3x^2y^4 + 12a^2x^3y^4 - 10a^4x^4y^2$ by $-2ax^2y^2$.
9. $-5abc - 10abd - 15bcd$ by $-5b$.
10. $16a^3b^2cd - 4a^2b^3c^2d - 8abc^3d^3 + 4abcd$ by $4abcd$.

DIVISION OF A MULTINOMIAL BY A MULTINOMIAL.

We found the product of two multinomials by multiplying each term of the one by each term of the other and taking the sum of the partial products. Hence it is evident that in finding the quotient of one multinomial by another, for every term of the quotient found we must remove from the dividend the product of that term and each term of the divisor.

Thus, to divide $x^2 + 7x + 12$ by $x + 3$.

It is evident that x is one term of the quotient.

Hence we must subtract $x^2 + 3x$ from $x^2 + 7x + 12$: this leaves $4x + 12$.

The next term of the quotient is $+4$.
The product of $x + 3$ and $+4$ is $4x + 12$,
$\therefore$ the quotient is $x + 4$.

The work is conveniently arranged thus :

$$
\begin{array}{r|l}
x+3\,) & x^2+7x+12\,(\,x+4 \\
 & \underline{x^2+3x} \\
 & 4x+12 \\
 & \underline{4x+12}
\end{array}
$$

A term of the quotient may easily be found if the terms of both the divisor and the dividend are arranged in descending or ascending powers of some common letter. Then the first term of the dividend divided by the first term of the divisor will give a term of the quotient.

Hence it is that to divide one multinomial by another we usually proceed thus :

1. Arrange divisor and dividend in descending or ascending powers of some common letter, and keep the remainder after each subtraction in this same order.
2. The first term of the dividend divided by the first term of the divisor will give a term of the quotient.

3. Multiply each term of the divisor by this term of the quotient, and subtract the product from the dividend.
4. Repeat these operations, taking the remainder as a new dividend, until there is no remainder, or one in which the highest power of the common letter is lower than that in the divisor.

To divide $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + \dots$ by $x^2 - 2xy + y^2$.

$$x^2 - 2xy + y^2)\,x^4 - 4x^3y + 6x^2y^2 - \dots + y^4\,(x^2 - 2xy + y^2$$
$$\underline{x^4 - 2x^3y + x^2y^2}$$
$$- 2x^3y + 5x^2y^2 - 4xy^3$$
$$\underline{- 2x^3y + 4x^2y^2 - 2xy^3}$$
$$x^2y^2 - 2xy^3 + y^4$$
$$\underline{x^2y^2 - 2xy^3 + y^4}$$

The quotient is $x^2 - 2xy + y^2$.

It will be noticed that the term $+y^4$ was not brought down with the remainder after the first subtraction, as there was no like term in the expression to be subtracted.

Exercise 21.

Divide:

1. $x^2 + 3x + 2$ by $x + 1$.
2. $x^2 + 4x + 4$ by $x + 2$.
3. $x^2 + 5x + 6$ by $x + 2$.
4. $a^2 + 5a + 4$ by $a + 4$.
5. $x^2 + 7x + 10$ by $x + 5$.
6. $x^2 + 7x + 6$ by $x + 1$.
7. $a^2 - 3a + 2$ by $a - 1$.
8. $x^2 - 5x - 14$ by $x + 2$.
9. $a^2 - 7a + 12$ by $a - 3$.
10. $x^2 - 7x - 18$ by $x - 9$.
11. $m^2 - m - 2$ by $m + 1$.
12. $c^2 - 3c - 18$ by $c + 3$.
13. $x^2 + 2xy + y^2$ by $x + y$.
14. $x^2 + 3xy + 2y^2$ by $x + 2y$.
15. $x^2 + 5xy + 6y^2$ by $x + 3y$.
16. $x^2 - 5xy - 14y^2$ by $x - 7y$.
17. $a^2 - 11ab - 12b^2$ by $a + b$.
18. $m^2 - 12mn + 35n^2$ by $m - 5n$.
19. $3x^2y + 8xy + 4y$ by $xy + 2y$.
20. $x^2 + ax + bx + ab$ by $x + a$.
21. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ by $x^2 + 2xy + y^2$.

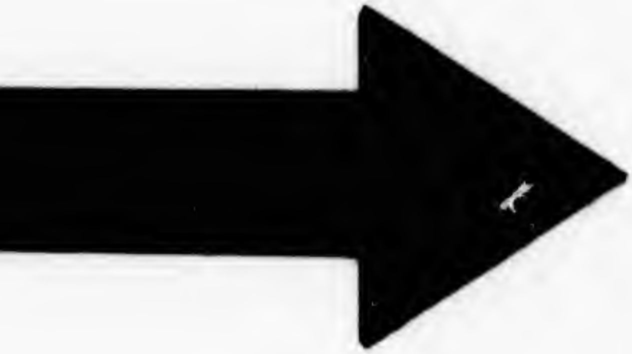


IMAGE EVALUATION TEST TARGET (MT-3)

150mm

6"

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2.8
2.5
3.2
3.6
2.2
4.0
2.0
1.8
1.1
1.0

22. $m^6+2m^5-4m^4-2m^3+12m^2-2m-1$ by m^2+2m-1.
23. $x^4-2x^3-7x^2+8x+12$ by x^2-x-6.
24. $4y^8-16y^6+16y^4-1$ by $2y^4-4y^2-1$
25. $x^4+x^2y^2+y^4$ by x^2+xy+y^2.
26. $2x^4y^4-3x^3y^3-9x^2y^2+9xy-2$ by $2x^2y^2+3xy-2$.
27. $2x^4y^4+7x^3y^3z+3x^2y^2z^2-17xyz^3-15z^4$ by $x^2y^2+4xyz+5z^2$.
28. $2+7x-8x^2-16x^3+x^4+4x^5$ by $1+3x-5x^2-4x^3$.
29. $2a^6-3a^5bc+5a^4b^2c^2-5a^3b^3c^3+5a^2b^4c^4-3ab^5c^5+2b^6c^6$ by $2a^2-abc+2b^2c^2$.
30. $m^5-m^4-11m^3+4m^2+4m+3$ by m^3-3m^2-2m-1.
31. $x^2+y^2+2xy+2x+2y+1$ by $x+y+1$.
32. $a^2+b^2-2ab+2a-2b+1$ by $a-b+1$.
33. $a^2-ab-2b^2-3bc-c^2$ by $a-2b-c$.
34. $2x^2+5xy-3y^2-3x-9y$ by $2x-y-3$.
35. $l^3+6l^2m+12lm^2+8m^3$ by $l^2+4lm+4m^2$
36. $x^2+y^2+z^2+2yz-2zx-2xy$ by $x-y-z$.

Miscellaneous Examples.

Exercise 22.

1. Add $+10$, -7, -6, -13 and $+9$.
2. Multiply the sum of -10 and $+6$, by the sum of -6 and $+10$, and subtract -30 from the product.
3. Subtract the sum of $a-2b-c$ and $2a+b-2c$ from the sum of $b+2c-a$ and $c-a-2b$.
4. When $x=-3$ and $y=+2$, find the value of $x^2+2xy+y^2$.
5. Multiply $x^6+x^5+x^4+x^3+x^2+x+1$ by x^2-2x+1.

---

6. Add $2a+3b+4c$, $-3a+b-5c$, $2a-2b+2c$.
7. Multiply the sum of a^2+ab+b^2 and a^2-ab+b^2 by $a+b$; then multiply the result by $a-b$.
8. Divide x^4+4x^2+16 by x^2+2x+4.
9. Simplify $16-[5-\{3-1-(2-4)-6\}-5]$.
10. Find the value of $(3x+2y)^3+(2x+3y)^3$ when $x=-y$.

11. Find the expression which, added to $2x^2 - 3x - 4$, gives the sum $-6 - 4x - x^2$.

12. Multiply $\{x - (2x - 3y) + 4y\}$ by $x + y$.

13. Divide $1 - x^5$ by $1 - x$.

14. Find the value of

$$\frac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{(a-b)(b-c)(c-a)}$$

when $a = 1$, $b = 2$, $c = 3$.

15. If $ax + by = -10$, when $a = +2$, $b = -2$, and $x = +3$, find the value of y.

---

16. Subtract $(a+b)(c-d)$ from $(a-b)(c+d)$.

17. Find the product of $x - 2y$, $x - y$, $x + y$ and $x + 2y$.

18. Simplify $a\left[1 - 2\{2 - 3(1 - 4b)\}\right]$.

19. Divide $x^5 - 5xy^4 + 4y^5$ by $x^2 - 2xy + y^2$.

20. If $\dfrac{abc}{a+b} = -60$, when $a = 4$ and $b = -2$, find the value of c.

---

21. Divide $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ by $x^2 + 2xy + y^2$.

22. Multiply $2a - 3b - 3(a+b) + 4\{2a - (a-b)\}$ by $a - b\{2 - (4 - 6)\}$.

23. Find the value of the product of $x^2 + xy + y^2$ and $x^2 - xy + y^2$, when $x = -1$, and $y = +2$.

24. Find the value of $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a}$ when $a = +2$, $b = -2$ $c = +1$.

25. If $ab = 24$, and $bc = 36$ and $abc = 144$, find the values of a, b and c.

CHAPTER V.

IMPORTANT RESULTS IN MULTIPLICATION AND DIVISION. FACTORS AND MULTIPLES.

To find the **square** of a **binomial**:

(1)

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \qquad \\ +ab+b^2 \\ \hline a^2+2ab+b^2. \end{array}$$

(2)

$$\begin{array}{r} a-b \\ a-b \\ \hline a^2-ab \qquad \\ -ab+b^2 \\ \hline a^2-2ab+b^2. \end{array}$$

From the first example we see that the square of the sum of two numbers is equal to the sum of the squares of the numbers, increased by twice their product.

From the second example we see that the square of the difference of two numbers is equal to the sum of the squares of the numbers, diminished by twice their product.

Or, rememberin at the sign forms a part of the term, we may say in each case, that the square of a binomial is equal the sum of the square of each term, and twice the product of the terms.

Exercise 23.

Find, without ordinary multiplication, the square of each of the following binomials:

1. $x+y$.
2. $y+z$.
3. $m+n$.
4. $2x+y$.
5. $2y+3z$.
6. $3m+4n$.
7. $x-y$.
8. $y-z$.
9. $m-n$.
10. $2x-y$.

11. $2y - 3z$ 12. $3m - 4n$.
13. $2p + q$. 14. $ab + c$.
15. $ab + 2c$. 16. $7ab + 3$.
17. $2xy + 5$. 18. $3mn - 4a$.
19. $x + \frac{3}{2}$. 20. $x + \frac{2}{3}$.
21. $2x + \frac{3}{5}$. 22. $2x - \frac{3}{5}$.
23. $2a + \frac{3}{2}b$. 24. $3m + \frac{1}{5}$.
25. $4xy + \dfrac{z}{2}$. 26. $7x - \frac{3}{4}$.
27. $2x^2 - \frac{1}{2}$. 28. $2 - \dfrac{x^2}{2}$.
29. $20 + 3$. 30. $30 + 2$.
31. $60 + 1$. 32. $100 + 3$.
33. 102. 34. 105.
35. $100\frac{1}{2}$. 36. $99\frac{1}{2}$.
37. 1003. 38. $2000\frac{1}{2}$.

---

39. What term must be added to $x^2 + y^2$ to form the square of $x + y$?

40. What term must be added to $x^2 + y^2$ to form the square of $x - y$?

41. What term must be added to $x^2 + 2xy$ to form the square of $x + y$?

42. What term must be added to $x^2 + 4xy$ to form the square of $x + 2y$?

43. Form a complete square by adding a term to $x^2 - 10x$.

44. Form a complete square by adding a term to $4x^2 + 12x$.

Exercise 24.

Express each of the following as the square of a binomial:

1. $x^2 + 2xy + y^2$. 2. $x^2 - 2xy + y^2$.
3. $a^2 + 2ab + b^2$. 4. $c^2 - 2cd + d^2$.
5. $x^2 + 4xy + 4y^2$. 6. $x^2 - 4xy + 4y^2$.

7. $4x^2 + 4x + 1.$
8. $a^2 + 4ab + 4b^2.$
9. $a^2 + 2abc + b^2c^2.$
10. $9p^2 - 12pq + 4q^2.$
11. $16y^2 - 8y + 1.$
12. $x^2 + x + \frac{1}{4}.$
13. $\frac{x^2}{4} + x + 1.$
14. $\frac{4m^2}{9} + \frac{4m}{3} + 1.$
15. $4a^2b^2 - 4abc + c^2.$
16. $\frac{9x^2}{16} - 2 + \frac{16}{9x^2}.$
17. $\frac{x^2}{y^2} + 2 + \frac{y^2}{x^2}.$
18. $16 - 8x + x^2.$
19. $25a^2 - 2a + \frac{1}{25}.$
20. $(a+b)^2 + 2(a+b)c + c^2.$
21. $(x+y)^2 + 2(x+y)z + z^2.$
22. $(a+b)^2 + 2(a+b)(c+d) + (c+d)^2.$

Exercise 25.

Make a square by adding a term to each of the following, and state the expression of which each is then the square.

1. $m^2 + 2mn.$
2. $p^2 + 2pq.$
3. $4x^2 + 4xy.$
4. $m^2 + 6m.$
5. $p^2 + 8p.$
6. $4x^2 + 12x.$
7. $m^2 - 2mn.$
8. $p^2 - 2pq.$
9. $4x^2 - 4xy.$
10. $m^2 - 6m.$
11. $p^2 - 8p.$
12. $4x^2 - 12x.$
13. $x^2 - 16x.$
14. $y^2 + 2y.$
15. $z^2 - 4z.$
16. $x^2 + 3x.$
17. $y^2 - 5y.$
18. $4z^2 + 5z.$
19. $x^2y^2 + 4xy.$
20. $4a^2b^2 + 8ab.$
21. $p^2q^2 - 5pq.$
22. $16x^2y^2 - 2xy.$
23. $\frac{x^2}{4} + x.$
24. $\frac{y^2}{9} + \frac{2y}{3}.$
25. $m^2 + n^2.$
26. $p^2 + q^2.$
27. $1 + a^2.$
28. $1 - 2a.$
29. $9 + 6m.$
30. $64a^2 + 16a.$
31. $4mn + 1.$
32. $6m^2n + m^2n^2.$
33. $x^2 + ax.$
34. $x^2 + bx.$
35. $x^2 + 2cx.$
36. $4x^2 + ax.$
37. $4x^2 + bx.$
38. $4x^2 + 1.$
39. $\frac{1}{4} + x.$
40. $\frac{1}{4} + 4x^2.$
41. $-2 + x^2.$
42. $z + \frac{1}{9}.$

To find the **product** of the **sum** and **difference** of two numbers.

$$\begin{array}{ll} a+b & \ldots\ldots \text{ sum of } a \text{ and } b. \\ \underline{a-b} & \ldots\ldots \text{ difference of } a \text{ and } b. \\ a^2+ab & \\ \underline{\quad -ab-b^2} & \\ a^2 \qquad -b^2 & \ldots\ldots \text{ difference of squares of } a \text{ and } b. \end{array}$$

From the above example we see that the product of the sum and difference of two numbers is equal to the difference of the squares of the numbers.

Exercise 26.

Write the product of:

1. $x+y$ and $x-y$.
2. $m+n$ and $m-n$.
3. $b+c$ and $b-c$.
4. $b+2c$ and $b-2c$.
5. $2m+n$ and $2m-n$.
6. $2m+3n$ and $2m-3n$.
7. $x+7$ and $x-7$.
8. $x+4$ and $x-4$.
9. $x+1$ and $x-1$.
10. $1+x$ and $1-x$.
11. $1+2x$ and $1-2x$.
12. $3-2x$ and $3+2x$.
13. $2ab+c$ and $2ab-c$.
14. $ab+2c$ and $ab-2c$.
15. $4hm+5p$ and $4hm-5p$.
16. $2ab+\frac{1}{2}$ and $2ab-\frac{1}{2}$.
17. a^2+b^2 and a^2-b^2.
18. $2a^2-bc$ and $2a^2+bc$.
19. $(a+b)+c$ and $(a+b)-c$.
20. $a+b+c$ and $a+b-c$.
21. $(2a+b)+2c$ and $(2a+b)-2c$.
22. $2a+3b+2c$ and $2a+3b-2c$.
23. $l+m+n$ and $l-m+n$.
24. $2a-b+3c$ and $2a+b+3c$.
25. $a-b+c$ and $a+b-c$.
26. $2a-b+3c$ and $2a+b-3c$.
27. x^2+x+1 and x^2-x+1.
28. a^2x^2+ax+1 and a^2x^2-ax+1.
29. x^2+xy+y^2 and x^2-xy+y^2.
30. $(1+x+x^2)$, $(1-x+x^2)$ and $(1-x^2+x^4)$.

Conversely, we can always find the two factors which give a product of the form $a^2 - b^2$. That is, if an expression can be written as the difference of the squares of two numbers, the expression is the product of the sum and difference of those two numbers.

Thus,
$$x^2 - y^2 = (x+y)(x-y);$$
and
$$a^2x^2 - b^2y^2 = (ax)^2 - (by)^2$$
$$= (ax+by)(ax-by).$$
Also
$$a^2 + 2ab + b^2 - c^2 = (a+b)^2 - c^2$$
$$= (a+b+c)(a+b-c).$$
And
$$a^4 - b^4 = (a^2+b^2)(a^2-b^2)$$
$$= (a^2+b^2)(a+b)(a-b).$$

Exercise 27.

Find the factors of:

1. $c^2 - d^2$.
2. $m^2 - n^2$
3. $q^2 - r^2$.
4. a^2[illegible]$^2 - y^2$.
5. $x^2 - y^2z^2$.
6. $x^2y^2 - z^2w^2$.
7. $4m^2 - n^2$.
8. $9p^2 - 4q^2$.
9. $16x^2 - 9y^2$.
10. $4 - x^2$.
11. $9 - 4y^2$.
12. $1 - \frac{x^2}{4}$.
13. $4a^2b^2 - x^2y^2$.
14. $x^4 - 9$.
15. $16 - y^6$.
16. $49a^2b^2c^2 - 9$.
17. $a^2b^4c^2 - d^4$.
18. $1 - 16a^2b^2c^2$.
19. $25 - 16p^2q^4$.
20. $1 - 25p^2q^2$.
21. $81a^2 - 25b^4$.
22. $a^6 - 25b^2$.
23. $a^4 - b^4$.
24. $a^8 - b^8$.
25. $16 - a^4$.
26. $625x^4 - y^4$.
27. $\frac{1}{25} - x^2y^2$.
28. $b^8c^8 - 256$.
29. $x^8 - y^8$.
30. $a^8b^8 - c^8$.
31. $27^2 - 23^2$.
32. $103^2 - 97^2$.
33. $220^2 - 25$.
34. $(a+b)^2 - c^2$.
35. $a^2 + 2ab + b^2 - c^2$.
36. $4a^2 + 4ab + b^2 - c^2$.
37. $a^2 - 4ab + 4b^2 - 4c^2$.
38. $a^2 - (b+c)^2$.
39. $a^2 - b^2 - 2bc - c^2$.
40. $a^2 - b^2 + 2bc - c^2$.
41. $9x^2 - 12xy + 4y^2 - 25z^2$.
42. $(x+y)^2 - (x-y)^2$.
43. $(a+b)^2 - (c+d)^2$.

44. $a^2 - 2ab + b^2 - c^2 - 2cd - d^2$.

45. $a^2 - 4ab + 4b^2 - 4c^2 + 4cd - d^2$.

46. $9 - 4x^2 + 4xy - y^2$.

47. $a^2 - b^2 + c^2 - d^2 - 2ac - 2bd$.

48. $4a^2 - 9b^2 + c^2 - 25d^2 + 4ac + 30bd$.

49. $(x^2 + y^2)^2 - x^2y^2$.

50. $x^4 + x^2y^2 + y^4$.

51. $x^4 + x^2 + 1$.

To find the **product of two binomials** which have a **common term.**

Consider the following products:

$$\begin{array}{l} x + 5 \\ \underline{x + 7} \\ x^2 + 5x \\ \underline{\quad\; + 7x + 35} \\ x^2 + 12x + 35. \end{array} \qquad \begin{array}{l} x - 5 \\ \underline{x - 7} \\ x^2 - 5x \\ \underline{\quad\; - 7x + 35} \\ x^2 - 12x + 35. \end{array}$$

$$\begin{array}{l} x - 5 \\ \underline{x + 7} \\ x^2 - 5x \\ \underline{\quad\; + 7x - 35} \\ x^2 + 2x - 35. \end{array} \qquad \begin{array}{l} x + 5 \\ \underline{x - 7} \\ x^2 + 5x \\ \underline{\quad\; - 7x - 35} \\ x^2 - 2x - 35. \end{array}$$

We see that the product consists of three terms:

(1) The square of the common term.

(2) The product of the common term and the sum of the unlike terms.

(3) The product of the unlike terms.

Exercise 28.

Find, without ordinary multiplication, the product of:

1. $x + 2$ and $x + 3$.
2. $x + 3$ and $x + 5$.
3. $x + 7$ and $x + 11$.
4. $x - 2$ and $x - 3$.
5. $x - 3$ and $x - 5$.
6. $x - 7$ and $x - 11$.

7. $x+2$ and $x-3$.
8. $x+3$ and $x-5$.
9. $x+7$ and $x-11$.
10. $x-2$ and $x+3$.
11. $x-3$ and $x+5$.
12. $x-7$ and $x+11$.
13. $m+9$ and $m-7$.
14. $d+9$ and $d-7$.
15. $y+9$ and $y-7$.
16. $mn+9$ and $mn-7$.
17. $de-9$ and $de+7$.
18. $yz+9$ and $yz-7$.
19. $2x+3$ and $2x-5$.
20. $3y+11$ and $3y-7$.
21. $3z-6$ and $3z-8$.
22. $4mn-6$ and $4mn+12$.
23. $3abc+7$ and $3abc-9$.
24. $4x^2y+13$ and $4x^2y-5$.
25. $3a^2y+7$ and $3a^2y+17$.
26. $x+3$ and $x+3$.
27. $m+11$ and $m+11$.
28. $2x+5$ and $2x+5$.
29. $2m+7$ and $2m+7$.

When the unlike terms are not definite numbers, their sum may be indicated and used as one quantity by placing in brackets.

Thus:

$$\begin{array}{l} x+a \\ \underline{x+b} \\ x^2+ax \\ \underline{\quad +bx+ab} \\ x^2+(a+b)x+ab \end{array}$$

$$\begin{array}{l} x-a \\ \underline{x-b} \\ x^2-ax \\ \underline{\quad -bx+ab} \\ x^3+(-a-b)x+ab \end{array}$$

or, $x^2-(a+b)x+ab$

$$\begin{array}{l} x+a \\ \underline{x-b} \\ x^2+ax \\ \underline{\quad -bx-ab} \\ x^2+(a-b)x-ab \end{array}$$

$$\begin{array}{l} x-a \\ \underline{x+b} \\ x^2-ax \\ \underline{\quad +bx-ab} \\ x^2+(-a+b)x-ab \end{array}$$

or, $x^2-(a-b)x-ab$

Exercise 29.

Write the product of:

1. $x+a$ and $x+2a$.
2. $x+3b$ and $x+4b$.
3. $x+2a$ and $x-4a$.
4. $x-3b$ and $x+4b$.
5. $2x+a$ and $2x+3a$.
6. $3x-4b$ and $3x+7b$.

7. $4mn+p$ and $4mn-2p$.
8. $abc-2x$ and $abc+5x$.
9. $x+2y$ and $x-5y$.
10. $3x+2y$ and $3x-5y$.
11. $2x^2-9a$ and $2x^2-7a$.
12. $3a^3+10b$ and $3a^3-10b$.
13. $10lmn+5x$ and $10lmn-9x$.
14. $xy+z$ and $xy+w$.
15. $(a+b)+2$ and $(a+b)+3$.
16. $(m+n)+5$ and $(m+n)-6$.
17. $(2m+n)+10$ and $(2m+n)-15$.
18. $a+b+3$ and $a+b+4$.
19. $m+n+10$ and $m+n-7$.
20. $-x+7$ and $-x-9$.
21. $-4mn+11$ and $-4mn-5$.
22. $-a+b+3$ and $-a+b-10$.
23. $x+2y+3z$ and $x+2y-8z$.
24. $2x+2z-3y$ and $2x-5z-3y$.
25. $2mn+4+3p$ and $2mn-7+3p$.

Conversely, we can find the two factors which give a product of the form $x^2+9x+14$, if we can find two terms which (1) added together give the co-efficient of x, (2) multiplied together give the third term.

Thus, to find the factor of $x^2+9x+14$, we must find two numbers which, added together, give $+9$, and multiplied together give $+14$.

By trial, we find the numbers to be $+2$ and $+7$.

Hence, $x^2+9x+14 = (x+2)(x+7)$.

Again, to find the factors of $x^2-9x-22$, we must find two numbers whose sum is -9, and whose product is -22.

Since the product is a negative number, the required numbers have different signs.

And since their sum is -9, the absolute value of the negative one is greater than that of the other by 9.

Hence the numbers are -11 and $+2$.

Therefore, $x^2-9x-22 = (x-11)(x+2)$.

To find the factors of $x^2 - 5ax - 36a^2$.

The sum of the two unlike terms is $-5a$, and their product is $-36a^2$.

Hence the two terms are $-9a$ and $+4a$.

$$\therefore x^2 - 5ax - 36a^2 = (x - 9a)(x + 4a).$$

When simple factors of an expression of the form $x^2 + 8x + 12$ can be found, we may always proceed as in the following examples, where we write the expression to be factored as the difference of two squares.

$$\begin{aligned} &x^2 + 8x + 12 \\ &\quad = x^2 + 8x + 16 + 12 - 16 \\ &\quad = (x + 4)^2 - 2^2 \\ &\quad = (x + 4 + 2)(x + 4 - 2) \\ &\quad = (x + 6)(x + 2). \end{aligned}$$

$$\begin{aligned} &x^2 - 17x - 60 \\ &\quad = x^2 - 17x + (\tfrac{17}{2})^2 - 60 - (\tfrac{17}{2})^2 \\ &\quad = (x - \tfrac{17}{2})^2 - \tfrac{529}{4} \\ &\quad = (x - \tfrac{17}{2} + \tfrac{23}{2})(x - \tfrac{17}{2} - \tfrac{23}{2}) \\ &\quad = (x + 3)(x - 20). \end{aligned}$$

Exercise 30.

Find factors of:

1. $x^2 + 4x + 3$.
2. $x^2 + 6x + 5$.
3. $x^2 + 10x + 16$.
4. $x^2 + 7x + 12$.
5. $x^2 + 8x + 15$.
6. $x^2 + 6x + 8$.
7. $y^2 + 10y + 21$.
8. $y^2 + 8y + 7$.
9. $y^2 + 12y + 20$.
10. $y^2 + 14y + 24$.
11. $y^2 - 6y + 8$.
12. $y^2 - 4y + 3$.
13. $m^2 - 8m + 16$.
14. $m^2 - 10x + 24$.
15. $m^2 - 10m + 21$.
16. $m^2 - 10m + 16$.
17. $m^2 - 10m + 25$.
18. $m^2 - 10m + 9$.
19. $a^2 - 14a + 24$.
20. $a^2 - 10a + 24$.
21. $a^2 + 14a + 24$.
22. $a^2 + 10a + 24$.

23. $m^2n^2+6mn+8.$
24. $p^2q^2-4pq+3.$
25. $a^2m^2-8am+12.$
26. $a^2m^2+12am+11.$
27. $x^2y^2z^2-20xyz+51.$
28. $a^2b^2-20ab+19.$
29. $a^2+4a-12.$
30. $b^2-4b-12.$
31. $m^2-2m-63.$
32. $m^2+2m-63.$
33. $x^2-4x-21.$
34. $x^2+4x-21.$
35. $y^2+6y-27.$
36. $y^2-6y-27.$
37. $a^2+4a-32.$
38. $b^2-8b-9.$
39. $x^2-5x+6.$
40. $x^2+5x-14.$
41. $b^2-7b+10.$
42. $y^2+9y-22.$
43. $a^2x^2+3ax-70.$
44. $a^2x^2-9ax-70.$
45. $a^2+3ab+2b^2.$
46. $x^2+5xy+6y^2.$
47. $c^2-15cd-16d^2.$
48. $n^2+4np+4p^2.$
49. $x^2-17xy-60y^2$
50. $x^2+17xy-60y^2.$
51. $4x^2+12x+5.$
52. $9x^2+36x+35.$
53. $(a+b)^2+8(a+b)+12.$
54. $(m+n)^2+11(m+n)-12.$
55. $(a+b)^2-5(a+b)c-14c^2.$
56. $(a+b)^2-5c(a+b)+6c^2.$

To find the **square** of a **trinomial**.

Consider the expression $a+b+c$. We can put it in the form of a binomial expression, if we place two of the terms within a pair of brackets. Hence we can find its square thus:

$$\begin{aligned}\{a+b+c\}^2 &= \{(a+b)+c\}^2\\ &= (a+b)^2+2(a+b)c+c^2\\ &= a^2+2ab+b^2+2ac+2bc+c^2\\ &= a^2+b^2+c^2+2ab+2ac+2bc\end{aligned}$$

Similarly,

$$\begin{aligned}\{x^2-2x+3\} &= \{(x^2-2x)+3\}^2\\ &= (x^2-2x)^2+2(x^2-2x)3+3^2\\ &= x^4-4x^3+4x^2+6x^2-12x+9\\ &= x^4-4x^3+10x^2-12x+9\end{aligned}$$

Or, since in finding the square of *any* expression we multiply the whole expression by each term, and take the sum, it is evident that in forming the square of any expression we take the square of each term, and twice the product of every two terms.

To make sure that we take twice the product of every pair of terms, we may perform the operation in this way: take twice the product of each term and all terms which follow it.

Exercise 31.

Write the square of:

1. $x+y+z$. 2. $x+2y+z$. 3. $x+2y+3z$.
4. $x-y+z$. 5. $x+2y-z$. 6. $x-2y-3z$.
7. $a+b+c+d$. 8. $a-b+2c-3d$. 9. $2x-3y+4z-5w$.
10. x^2+x+1. 11. x^2+xy+y^2. 12. $x^2+2xy+y^2$.
13. x^3+x^2+x+1. 14. x^3-x^2+x-1. 15. $a^3+3a^2b+3ab^2+b^3$.

Find the expressions of which the following are the squares:

16. $a^2+2ab+b^2-2ac-2bc+c^2$.
17. $x^2-2xy+y^2-2yz+2xz+z^2$.
18. $x^2+4xy+4y^2+6xz+12yz+9z^2$.
19. $4x^2-4xy+y^2-12xz+6yz+9z^2$.
20. $a^2-2ab+b^2-2a+2b+1$.

To find the **cube** of a **binomial**:

By actual multiplication we find that $(a+b)^3=a^3+3a^2b+3ab^2+b^3$.

Since every binomial may be put in the form $a+b$, we are able to write out its cube without the ordinary process of multiplication.

Thus:

$$\begin{aligned}(a-b)^3 &= a^3+3a^2(-b)+3a(-b)^2+(-b)^3.\\ &= a^3-3a^2b+3ab^2-b^3.\end{aligned}$$

And $(2x+3y)^3=(2x)^3+3(2x)^2(+3y)+3(+2x)(+3y)^2+(+3y)^3$

$$= 8x^3+36x^2y+54xy^2+27y^3.$$

Also $(2x-3y)^3 = (2x)^3 + 3\,(+2x)^2\,(-3y) + 3\,(+2x)\,(-3y)^2 + (-3y)^3$

$$= 8x^3 - 36x^2y + 54xy^2 - 27y^3.$$

Exercise 32.

Expand:

1. $(x+y)^3$. 2. $(x-y)^3$. 3. $(2x+y)^3$.
4. $(x-2y)^3$. 5. $(3x+2y)^3$. 6. $(3a-2b)^3$.
7. $(x+4)^3$. 8. $(2x+1)^3$. 9. $(1-2x)^3$.
10. $(2x+3)^3$. 11. $(2-3x)^3$. 12. $(2-\frac{m}{2})^3$.
13. $\{(a+b)+c\}^3$. 14. $(a+b+c)^3$. 15. $(a+b-c)^3$.
16. $(x+2y+3z)^3$. 17. $(2x-y+z)^3$. 18. $(1-x+x^2)^3$.

Consider the following multiplications:

(1)

$$\begin{array}{l} a^2+ab+b^2 \\ \underline{a\ \ -b} \\ a^3+a^2b+ab^2 \\ \underline{\quad\ -a^2b-ab^2-b^3} \\ a^3 \qquad\qquad\quad -b^3. \end{array}$$

(2)

$$\begin{array}{l} a^2-ab+b^2 \\ \underline{a\ \ +b} \\ a^3-a^2b+ab \\ \underline{\quad\ +a^2b-ab^2+b^3} \\ a^3 \qquad\qquad\quad +b^3. \end{array}$$

We may write the results thus:

$$(a-b)\,(a^2+ab+b^2) = a^3-b^3.$$
$$(a+b)\,(a^2-ab+b^3) = a^3+b^3$$

Exercise 33.

Multiply:

1. $(x+y)\,(x^2-xy+y^2)$. 2. $(x-y)\,(x^2+xy+y^2)$.
3. $(c+d)\,(c^2-cd+d^2)$. 4. $(a-c)\,(a^2+ac+c^2)$.
5. $(2x+a)\,(4x^2-2xa+a^2)$. 6. $(2a-c)\,(4a^2+2ac+c^2)$.
7. $(2a+b)\,(4a^2-2ab+b^2)$. 8. $(2x-3y)\,(4x^2+6xy+9y^2)$.
9. $(2a+3b)(4a^2-6ab+9b^2)$. 10. $(x-\frac{1}{3})\,(x^2+\frac{x}{3}+\frac{1}{9})$.
11. $(x+\frac{1}{2})\,(x^2-\frac{x}{2}+\frac{1}{4})$. 12. $(a^2-b^2)\,(a^4+a^2b^2+b^4)$.

13. $(ac+b^2)(a^2c^2-ab^2c+b^4)$. 14. $(x^2+yz)(x^4-x^2yz+y^2z^2)$.
15. $(4x+1)(16x^2-4x+1)$. 16. $(2+xy)(4-2xy+x^2y^2)$.
17. $(2m+3n)(4m^2-6mn+9n^2)$.
18. $(a^2-1)(a^4+a^2+1)$. 19. $(1+a^2)(1-a^2+a^4)$.
20. $(2x^2+\frac{1}{2})(4x^4-x^2+\frac{1}{4})$.

Complete the following statements by writing the necessary factors:

21. $(x+y)(\quad) = x^3+y^3$.
22. $(x-y)(\quad) = x^3-y^3$.
23. $(a+2b)(\quad) = a^3+8b^3$.
24. $(2x-3y)(\quad) = 8x^3-27y^3$.
25. $(1-2a^2)(\quad) = 1-8a^6$.
26. $(2y+\frac{1}{2})(\quad) = 8y^3+\frac{1}{8}$.
27. $(5m-3n)(\quad) = 125m^3-27n^3$.
28. $(m^2+nm+n^2)(\quad) = m^3-n^3$.
29. $(a^2-ab+b^2)(\quad) = a^3+b^3$.
30. $(4x^2+10xy+25y^2)(\quad) = 8x^3-125y^3$.
31. $(9m^2-3m+1)(\quad) = 27m^3+1$.
32. $(1+4m+16m^2)(\quad) = 1-64m^3$.

If an expression can be written as the **sum** or **difference** of the cubes of two numbers, it can readily be shown to be the product of two factors.

Thus,
$$8x^3+27y^3$$
$$=(2x)^3+(3y)^3$$
$$=(2x+3y)(4x^2-6xy+9y^2).$$

And,
$$125m^3n^3-1$$
$$=(5mn)^3-1^3$$
$$=(5mn-1)(25m^2n^2+5mn+1).$$

Exercise 34.

Find factors of:

1. p^3+q^3. 2. p^3-q^3. 3. $8p^3+q^3$.
4. p^3-8q^3. 5. $8p^3+27q^3$. 6. $27m^3-1$.
7. $8-125z^3$. 8. $1000x^6-y^6$. 9. $x^3y^3+z^3$.

10. $a^3 + b^3c^3$. 11. $(a+b)^3 + c^3$. 12. $a^3 + (b+c)^3$.

13. $(a+b)^3 - c^3$. 14. $a^3 - (b+c)^3$. 15. $a^3 + (b-c)^3$.

16. $a^3 - (b-c)^3$. 17. $a^3 + 3a^2b + 3ab^2 + b^3 - c^3$.

18. $a^3 - b^3 - 3b^2c - 3bc^2 - c^3$.

19. $27 - a^3 - 3a^2b - 3ab^2 - b^3$.

20. Show that $x+y$ is a factor of $(x-3)^3 + (y+3)^3$.

21. Show that $x+y$ is a factor of $(6x+5y)^3 + (4x+5y)^3$.

22. Show that $x-y$ is a factor of $\{(a+1)x+by\}^3 - \{(ax+(1+b)y\}^3$.

23. Show that $x+y$ is a factor of $\{(1-m)x+py\}^3 + \{mx+(1-p)y\}^3$.

24. Express $x^6 - 1$ as the product of four factors.

25. Express $64 - y^6$ as the product of four factors.

26. Show that $x+y$ is a factor of $\{(1-m)x+py+qz\}^3 + \{mx+(1-p)y-qz\}^3$.

We have shown how to find the factors of a trinomial of the form $x^2 + mx + n$, by arranging the expression as the difference of two squares.

The same method may be applied in finding the factors of a trinomial, such as $8x^2 + 22x + 15$.

For,

$$
\begin{aligned}
& 8x^2 + 22x + 15 \\
&= 8\left(x^2 + \tfrac{11}{4}x + \tfrac{15}{8}\right) \\
&= 8\left\{\left(x + \tfrac{11}{8}\right)^2 - \tfrac{121}{64} + \tfrac{15}{8}\right\} \\
&= 8\left\{\left(x + \tfrac{11}{8}\right)^2 - \tfrac{1}{64}\right\} \\
&= 8\left(x + \tfrac{11}{8} + \tfrac{1}{8}\right)\left(x + \tfrac{11}{8} - \tfrac{1}{8}\right) \\
&= 8\left(x + \tfrac{3}{2}\right)\left(x + \tfrac{5}{4}\right) \\
&= (2x+3)(4x+5).
\end{aligned}
$$

Also,

$$
\begin{aligned}
& 5a^2 - 12a + 4 \\
&= 5\left\{a^2 - \tfrac{12}{5}a + \tfrac{4}{5}\right\} \\
&= 5\left\{\left(a - \tfrac{6}{5}\right)^2 - \tfrac{36}{25} + \tfrac{4}{5}\right\} \\
&= 5\left\{\left(a - \tfrac{6}{5}\right)^2 - \left(\tfrac{4}{5}\right)^2\right\} \\
&= 5\left(a - \tfrac{2}{5}\right)(a-2) \\
&= (5a-2)(a-2).
\end{aligned}
$$

EXERCISE 35.

Factor:

1. $6x^2+5x+1$.
2. $6x^2+7x+2$.
3. $6x^2-x-2$.
4. $6x^2+x-2$.
5. $5x^2+12xy+4y^2$.
6. $5b^2-12bc+4c^2$.
7. $5c^2-8cd-4d^2$.
8. $5x^2+12xy+4y^2$.
9. $12a^2+a-20$.
10. $12a^2-a-20$.
11. $12a^2-31a+20$.
12. $12a^2+53a+20$.
13. $12a^2-43a-20$.
14. $12a^2-53ab+20b^2$.
15. $8x^2+22xy+15y^2$.
16. $8x^2-22xy+15y^2$.
17. $8a^2-2ab-15b^2$.
18. $8a^2+2ab-15b^2$.
19. $50x^2+51x+1$.
20. $48x^2+90xy+27y^2$

A **simple factor** is one that cannot be resolved. Such as, 3, -5, a, $x-y$.

A **common factor** of two or more algebraical expressions is an expression which will exactly divide each of them, and the **highest common factor** is the product of all the common simple factors.

Highest common factor is denoted by the letters H.C.F.

The common simple factors of $4a^2bc^2$ and $6ab^2c^3d$, are 2, a, b, c and c.

Hence the H. C. F. of $4a^2bc^2$ and $6ab^2c^3d$ is $2abc^2$.

Also the common factors of $6(a^2-b^2)$ and $8(a^2-2ab+b^2)$ are 2 and $a-b$.

Hence their H. C. F. is $2(a-b)$.

EXERCISE 36.

Find the H. C. F. of:

1. ab^2 and a^2b.
2. abc^2 and a^2bc.
3. m^3n and m^2
4. $ab^2c^3d^4$ and $a^4b^3c^2d$.
5. $6a^2b^3c$ and $9ab^3c^3$.
6. $24x^2y$ and $36x^3y^3$.

7. $15x^2y^2z^3$ and $25\,y^3z^2$.
8. $14a^3b^3c^5$ and $7b^5c^3$.
9. $4ab$, $6a^2b^2c$ and $12b^3c^2$.
10. a^3x^4, $20a^4x^3$ and $10x^5$.
11. x^2+5x+6 and x^2+4x+3.
12. x^2+5x+6 and x^2+3x+2.
13. $6\,(x_2-17x+70)$ and $8\,(x^2-3x-28)$.
14. $a^2+6ab+8b^2$ and $a^2+4ab+4b^2$.
15. $x^2+10xy-24y^2$ and x^2-4y^2.
16. $x^2+3xy+2y^2$ and $x^2+6xy+8y^2$.
17. $3a^2-4ab+b^2$ and $4a^2-5ab+b^2$.
18. $(a+b)^2-c^2$ and $(a+c)^2-b^2$.
19. x^2+5x+4, x^2-1 and x^2+2x-3.
20. x^3+y^3, $x^2+2xy+y^2$ and x^2-y^2.

A **common multiple** of two or more algebraical expressions is an expression which is exactly divisible by each of them, and the **lowest common multiple** of the expressions is that common multiple which has the least number of simple factors.

Lowest common multiple is denoted by the letters L. C. M.

The L.C.M. of $4a^2bc^2$ and $6ab^2c^3d$

is $12a^2b^2c^3d$.

Also, the L. C. M. of

$6\,(a^2-b^2)$ and $8\,(a^2-2ab+b^2)$,

that is, of $6\,(a-b)\,(a+b)$ and $8\,(a-b)\,(a-b)$

is $24\,(a-b)^2\,(a+b)$.

Exercise 37.

Find the L. C. M. of:

1. a^2bc and ab^2.
2. $3m^3n$ and $4m^2n^2$.
3. $8xy^2z^3$ and $6yz^2w^3$.
4. $5a^2b^2c^2$ $10ab^3c^5$ and $15a^5b$.
5. a^2-b^2 and $(a+b)^2$.
6. (a^3+b^3) and $(a+b)^2$.

7. $4x^2-1$ and $4x^2+4x+1$. 8. x^2-4 and x^3+8.
9. $b(b^2-1)$ and $b(b+1)$. 10. x^3+y^3 and $(x+y)^3$.
11. x^2+5x+6 and $x^2+7x+12$.
12. $x^2+9x+20$ and x^2+x-20.
13. x^2-3x-4 and x^2-1.
14. $2x^2-x-3$ and $4x^2-12x+9$.
15. $a^2+6ab+9b^2$ and $a^2+7ab+12b^2$.
16. $6x^2+x-12$ and $4x^2+4x-3$.
17. $4b^2+11b-3$ and b^2+6b+9.
18. $(a+b+c)^2$ and $(a+b)^2-c^2$.
19. $a^2-b^2+c^2-2ac$ and $a^2-b^2-c^2-2bc$.
20. x^2+4x+4, x^2+5x+6 and x^2+6x+9.

CHAPTER VI.

SIMPLE EQUATIONS.

An **equation** is a statement that two expressions are equal. Thus $2x + 3 = 3x$; or, $x(x - 4) = x^2 - 4x$.

The two expressions which are stated to be equal are called the **members** or **sides** of the equation.

Equations are commonly divided into two classes: identical equations and conditional equations.

An **identical equation,** or an **identity,** is one of which the two sides are equal whatever numbers the letters stand for. Thus, $x(x - 4) = x^2 - 4x$.

A **conditional equation** is one of which the two sides are equal only when the letters stand for particular numbers. Thus, $2x + 3 = 3x$, is true only when x stands for 3.

The term equation when used without any qualifying word usually means a conditional equation.

A letter which must have a particular value, in order that the statement of equality may be true, is called the **unknown quantity.** And the value of this unknown quantity is the number which, when substituted for it, will satisfy the equation. This value is called a **root** of the equation.

To **solve** an equation is to find the root, that is, the value of the unknown quantity.

In the equation $2x + 3 = 9$,

$2x + 3$ is the left-hand member, or side,
9 is the right-hand member, or side,
x is the unknown quantity,
3 is the value of x, which satisfies the statement, and therefore 3 is the root of the equation.

A **simple equation** is one which contains only the first power of the unknown quantity.

The unknown quantity is usually denoted by the letter x, although any other letter would do equally well.

In solving equations, the following axioms are assumed:

If equals be added to equals the sums are equal.

If equals be taken from equals the remainders are equal.

If equals be multiplied by equals the products are equal.

If equals be divided by equals the quotients are equal.

Consider the following equations:

(1)
$$2x = 6.$$

If we divide each of the equal members of this equation by 2, we obtain equal quotients.

$$\therefore x = 3.$$

(2)
$$5x - 6 = 3x.$$

If we add 6 to each of the equal members we obtain

$$5x - 6 + 6 = 3x + 6.$$

That is, $5x = 3x + 6.$

If now we subtract $3x$ from each side, we obtain

$$5x - 3x = 3x - 3x + 6$$

or, $5x - 3x = 6,$

or, $2x = 6.$

$\therefore$ $x = 3.$

(3)
$$4x - 7 = 2x + 5.$$

If we add 7 to each side, and also subtract $2x$ from each side, we have

$$4x - 7 + 7 - 2x = 2x - 2x + 7 + 5,$$

or, $4x - 2x = 7 + 5,$

that is, $2x = 12.$

$\therefore$ $x = 6.$

From these examples we see that any term may be transferred from one side of an equation to the other, provided we change its sign; for this is equivalent to adding the complementary term to each side.

This transposition of terms enables us to place all the terms containing the unknown quantity on one side of the equation, and all the other terms on the other side.

And by combining terms the equation is reduced to the form

$$ax = b.$$

Dividing both sides by a, we have

$$x = \frac{b}{a}.$$

(4) $$4(x-5)+3(x-2)=2(2x+4)-15.$$

Removing the brackets, we have:

$$4x-20+3x-6=4x+8-15.$$

Transposing terms, we have:

$$4x+3x-4x=20+6+8-15.$$

That is, $$3x=19.$$

$$x=6\tfrac{1}{3}.$$

Exercise 38.

Solve the equations:

1. $2x=8.$ 2. $2x=-8.$ 3. $-3x=9.$
4. $13x=-39.$ 5. $mx=4m.$ 6. $ax=-a^2.$
7. $2x+5=17.$ 8. $3x-7=17.$ 9. $5x+3=-22.$
10. $11x=45-4x.$ 11. $12x+5=7x-30.$
12. $13x-15+2x=60.$ 13. $2x-27-14x+5=100-3x.$
14. $20x-17=18+13x.$ 15. $14-x=x-6.$
16. $3-4x-5+6x=7-8x.$
17. $5-10x-8-3x=2-6x+30.$
18. $2(x-5)+3(x-14)=8.$

19. $4x - (x + 3) = 12.$
20. $2(x - 5) + 3(4 - 2x) = 6(2x - 4).$
21. $x = 4(2x - 3) + 3(x - 7).$
22. $5(x - 6) + 6(x - 5) = 0.$
23. $(x - 3) + 2(x - 4) + 3(x - 5) = 4(x - 6) - (10 - x).$
24. $5 - 3(2 - 3x) = 4 + (x - 17).$
25. $14 - 2\{x - 3(x - 4) - 4\} = 24.$
26. $5 - 3\{2 - x - 4(5 - x) - 3\} = -50.$
27. $3\{x - 2(3 - 2x)\} = 10 - 2(x - 7).$
28. $16 + 3(x - 7) = 4\{2 - 3(2 - x) + x\}.$
29. $2 - [2 - \{2 - (2 - 2x)\}] = x.$
30. $(x - 2)(x + 3) = x^2 + 2x + 20.$
31. $(x + 4)^2 = (x - 3)(x + 7).$
32. $(x - 4)(x - 5) + 17 = x(x + 6) - 100.$
33. $2(x - 6) + (x - 4)^2 = x(x + 10).$
34. $(x - 4)(x - 5) + (x - 6)(x - 2) = (x + 1)(x - 3) + x^2.$
35. $4(x - 2)^2 + 3(x + 3)^2 = 7(x - 1)(x + 1).$
36. $(2x + 3)(2x - 5) = 4(x - 1)^2.$
37. $(x + 1)^2 + (x + 2)^2 = (x - 1)^2 + (x + 3)^2.$
38. $(2x + 3)(x - 5) + (x + 2)(x + 7) = 3(x - 1)(x - 4).$
39. $x + (x - 7)^2 - (x - 9)^2 = (x - 7)(x - 9) - x^2.$
40. $(x + 1)(x + 3)(x + 5) - (x - 2)(x + 4)(x + 7) = 100.$

When some of the coefficients are fractions, if we multiply both sides of the equation by a number which is a multiple of each of the denominators, we get rid of the fractions.

Evidently the L. C. M. of the denominators will be a suitable multiplier.

To solve:

(1)
$$\tfrac{1}{3}x + 14\tfrac{1}{6} = 15 + \tfrac{1}{4}x.$$

Multiplying both sides by 12, we have

$$4x + 170 = 180 + 3x.$$
$$\therefore\ 4x - 3x = 180 - 170.$$
$$\therefore\ x = 10.$$

To solve:

(2) $\frac{5}{6}(x-2) - \frac{1}{3}(x-4) = 10.$

Multiplying both sides by 6, we have

$$5(x-2) - 2(x-4) = 60.$$
$$\therefore 5x - 10 - 2x + 8 = 60.$$
$$\therefore 5x - 2x = 60 + 10 - 8.$$
$$\therefore 3x = 62.$$
$$\therefore x = 20\tfrac{2}{3}.$$

Note that $\frac{1}{3}(x-4)$ is the same as $\frac{x-4}{3}$.

Exercise 39.

Solve:

1. $\frac{2}{3}x = 8.$
2. $\frac{3}{4}x = 10.$
3. $\frac{1}{5}x = 3\frac{1}{2}.$
4. $\frac{2x}{3} = 12.$
5. $\frac{3x}{4} = 15.$
6. $\frac{2x}{5} = 6.$
7. $\frac{2}{3}x + 10 = \frac{4}{5}x + 2\frac{1}{2}.$
8. $\frac{3x}{4} + \frac{2x}{5} = 10\frac{1}{2} + x.$
9. $10 - 3\frac{1}{2}x = x + 60 - \frac{x}{3}.$
10. $\frac{1}{4}(x-6) = \frac{1}{3}(x+12).$
11. $\frac{x+8}{4} = \frac{x+2}{6}.$
12. $\frac{x}{4} + \frac{x}{6} + \frac{5x}{12} = \frac{3}{2}.$
13. $\frac{1}{4}(x-3) + \frac{1}{5}(2x+4) = \frac{3}{10}(x-5).$
14. $\frac{3}{4}(x-2)(2x-6) = \frac{2}{3}(x-5)(3x-2) - \frac{x^2}{2}.$
15. $\frac{2x-3}{4} - \frac{5-6x}{3} = 0.$
16. $\frac{1}{7}(5x-9) + \frac{2}{5}(7x+8) = 23.$
17. $\frac{3x-17}{5} - \frac{2x-9}{9} = \frac{x+40}{49}.$

18. $\frac{2x+1}{3} - 1 = \frac{5x+5}{11} + 1.$

19. $\frac{5x+2}{7} + \frac{3x+4}{4} = \frac{25x}{2} - 87.$

20. $\frac{3x+5}{4} - \frac{x+15}{5} = 5x - 24.$

21. $\frac{3}{4}(x+2) - \frac{2}{3}(x+4) = \frac{4}{5}(x-5) - 2.$

22. $(x-2)(x-3) - \frac{1}{4}(x-6)(x+12) = \frac{3}{4}x(x+10).$

23. $1 - \frac{x-1}{4} = \frac{10x-45}{7} - 1.$

24. $\frac{5}{12}(3x-7) - \frac{1}{6}(2x-4) = \frac{1}{12}(5x+8).$

25. $\frac{4x}{7} - \frac{2x+3}{5} = \frac{3x}{8} - 12.$

26. $\frac{5x}{3} - \frac{5x-1}{11} = 5x - 9.$

27. $\frac{3x+2}{5} = \frac{2x-3}{2} - \frac{x-1}{10}.$

28. $\frac{5}{9} - \frac{x+2}{2} = \frac{5x-3}{4}.$

29. $\frac{5x}{6} - \frac{x}{4} - \frac{x-2}{10} = \frac{x}{60}.$

30. $\frac{2x+9}{4} - \frac{6}{5}(x-1) = 3 - \frac{4x}{7}.$

CHAPTER VII.

SOLUTION OF PROBLEMS.

EXERCISE 40.

If x is an algebraical number, state the meaning of each of the following expressions:

1. $2x$. 2. $-2x$. 3. $x+2$. 4. $x-2$.
5. $x+5$. 6. $2x+5$. 7. $2x-5$. 8. $7-2x$.
9. $2(x+3)$. 10. $(x+6)^2$. 11. $(3x-2)^2$. 12. $14-4(x+6)$.
13. $2x=7$. 14. $(x+2)^2=x(x+3)$.
15. $20-3(x+4)=5$.

EXERCISE 41

Using x to represent the unknown number, write an algebraical expression which represents:

1. Double the number.
2. Five times the number.
3. The sum of the number and 2.
4. The result of subtracting 4 from the number
5. The sum of double the number and 20.
6. The square of the sum of the number and 10.
7. The product of the number and the sum of the number and 7.
8. The amount by which the number exceeds 60.
9. The excess of the number over 50.
10. The excess of 100 over the number
11. The amount which must be added to the number to make 40.

12. The amount which must be subtracted from double the number to make 40.
13. That the excess of double the number over 10 is equal to 30.
14. That the square of the sum of twice the number and three is equal to four times the square of the number.
15. That if 20 be added to three times the number the sum will be 50.
16. The number greater by one.
17. The three following consecutive numbers.
18. That the product of the two following consecutive numbers is equal to 56.
19. That the square of the sum of the number and six, is greater by 20 than the square of the sum of the number and five.
20. That the product of the next two consecutive numbers is greater than the product of the two preceding consecutive numbers by 42.

The principal application of elementary algebra is the solution of problems.

Consider the following problems:

(1) Find a number such that if 10 be added to double the number the sum will be 50.

Let x represent the number.

Then $2x$ represents double the number.

And $2x+10$ represents the sum of double the number and 10.

But this sum is 50.

$$\therefore \quad 2x+10=50.$$
$$\therefore \quad 2x=40.$$
$$\therefore \quad x=20.$$

Therefore, the required number is 20.

(2) A has \$100 and B has \$40 how much must A give B in order that B may have as much as A?

Let x represent the *number* of dollars which A must give B.

Then $100 - x$ represents the number of dollars which A will have left.

And $40 + x$ represents the number of dollars which B will then have.

$$\therefore\ 100 - x = 40 + x.$$
$$\therefore\ -2x = -60.$$
$$\therefore\ x = 30.$$

Therefore, A must give B \$30.

Hence we see that we can solve a problem by representing the unknown number by x, and then expressing the conditions of the problem in the form of an equation. The solution of this equation gives us the unknown number.

Exercise 42.

Solve the following problems:

1. The sum of two numbers is 45; if x represents one of the numbers, what will represent the other?
2. If x and 76 represent two numbers whose sum is 120; find x.
3. A person has x dollars, a second person has 10 dollars more than the first, together they have 40 dollars; how many dollars has each?
4. The sum of two numbers is 73, and one of the numbers is 44; find the other.
5. The difference of the ages of two persons is 14 years, and the age of the elder is 42 years; find the age of the younger.
6. The sum of two numbers is 54 and their difference 22; find the numbers.

7 If x and $2x + 6$ represents two numbers whose difference is 22, find the numbers.

8. One boy has 22 marbles more than another boy; if he also has 6 more than double as many, find how many each has.

9. The result of adding 5 to double a certain number is the same as subtracting 15 from four times the number. Find the number.

10. One number is greater than another by 2: their sum is 50. Find the greater number.

11. One number is less than another by 3; their sum is 39. Find the numbers.

12. The sum of two numbers is 50; one is greater than the other by 10. What are the numbers?

13. Divide 100 into two numbers whose difference is 10.

14. One number is double another; if 40 be added to the less and 1 be taken from the greater number, the former result is double the latter. Find the numbers.

15. Find the number which exceeds its sixth part by 35.

16. The sum of \$93.05 was raised by A and B together, B contributed \$8.79 more than A, how much did each contribute?

17. Add 30 to a certain number, and the sum will be as much above 31 as the original number is below 31. What is the number?

18. A drover bought a certain number of calves for \$4.50 each, and 8 less cows at \$30 each, and paid altogether, \$174. How many cows did he buy?

19. How much chicory at 10 cents a pound must be mixed with 8 pounds of coffee at 40 cents a pound, to make a mixture worth 22 cents a pound?

20. The sum of \$225 was raised by A, B and C together; B contributed \$19 more than A, and C \$58 more than B. How much did each contribute?

21. A father is 52 years old and his son is 4 years old; in how many years will the father be exactly 7 times as old as his son?

22. If 42 be added to a certain number, the result is 4 times that number; find the number.

23. A is three times as old as B, and 7 years ago their united ages amounted to as many years as now represent the age of A; find their ages.

24. A person has $630; part of it he loans at the rate of 4 per cent., and the remainder at the rate of 5 per cent., and he received equal sums as interest from the two parts; how much did he loan at each rate?

25. John and Charles play a game of marbles; John has 22 marbles, and Charles 13 before they begin, and at the end of the game John has 4 times as many as Charles How many did John win?

26. Find a number such that its fifth part may exceed its seventh part by 12.

27. A father's age is six times as great as that of his son, but 4 years ago it was 11 times as great. Find the age of each.

28. How much tea worth 30 cents a pound, must be mixed with 12 pounds at 50 cents a pound, to make a mixture worth 36 cents a pound?

29. Divide $300 among three persons, A, B and C, so that B may receive twice as much and C three times as much as A.

30. Divide $480 into two parts, so that the first part put out at interest for a year at 5 per cent., may exceed the interest on the other part at 6 per cent., by $20.70.

31. Find a number such that if 20 be added to it, the sum will be three times the remainder when 20 is subtracted from it.

32. A can earn $2 and B $1.75 a day. How long will it take B to earn as much as A can earn in 21 days?

33. At an election where 743 votes were polled, the successful candidate had a majority of 61. How many ballots were cast for each?

34. *A* had 4 times as many apples as *B*. *A* gave *B* 12, and he had twice as many left as *B* then had. How many had each at first?

35. If a man walks $49\frac{1}{2}$ miles in ploughing a field 40 rods long and 20 rods wide, find the average width of each furrow, no allowance being made, for turning at each end..

36. A farmer, at a sale, disposed of a certain number of horses at $100 each; 5 times as many cows as horses at $30 each, and as many sheep as horses and cows together at $5 each. The total proceeds of the sale amounted to $840. How many of each did he sell?

37. When, after 8 o'clock, will the two hands of a clock first be in a straight line with each other?

38. The sum of two numbers is 89 and their difference is 31. Find the numbers.

39. A father divides $51.00 among his three children, so that every time the first receives one dollar the second receives two dollars; and as often as the second receives three dollars the third receives four dollars. How much does each child receive?

40. Two casks contain equal quantities of water. From the first, 42 gallons are drawn; from the second, 6 gallons. The quantity remaining in one vessel is now three times that in the other. How much did each cask contain at first?

41. *A* has twice as many marbles as *B*. *A* loses 32 and *B* wins 17, and then *B* has twice as many as *A*. How many had each at first?

42. *A* and *B* have 128 apples between them. *A* gives *B* a sufficient number to double his quantity, and they

then have equal quantities. How many had each at first ?

43. A man has 5 times as many half dollars as he has dollars, and 3 times as many quarters as he has halves. The whole sum is \$58. How many of each has he ?

44. *A's* earnings for the past year are \$75 less than twice *B's* ; *B's* are \$50 more than one-half *C's*, and *C's* are \$25 more than one third of *A's* and *B's* together. What are the earnings of each ?

45. Two boys, John and James, have 74 marbles between them, and John has 46 more than James. How many marbles have each ?

46. A sixth part of the sum of two numbers is 17, and one-quarter of their difference is 11. Find the numbers.

47. Two travellers, *A* and *B*, agree to share their expenses equally. At the end of the journey they find their total outlay to be \$48.50, aud that *B* must pay to *A* \$6.25 to settle according to agreement. Find the actual outlay of each before settling.

48. In a family of three persons the average age is 21 years. The father's age is five times one-quarter the combined ages of mother and child, and the mother's age is 3 years more than four times the child's age. Find the age of each.

49. A rectangular field contains $2\frac{1}{2}$ acres. If it is 25 rods in length, what is its width ?

50. A cask contains a certain quantity of brandy. If half the brandy be drawn off and 45 gallons of water poured into the cask, and one-quarter of the mixture be brandy, find the number of gallons of brandy originally in the cask.

51. Find when, after 3 o'clock, the minute hand of a clock first coincides with the hour hand.

52. Find the distance from A to B, if 4 miles more than one-seventh the distance is 16 miles less than one-half the distance.

53. A workman was employed for 30 days on condition that for every day he worked he should receive \$1.25, and for every day he was idle he should forfeit 50 cents. At the end of the time he received \$28.75. Find the number of days he worked.

54. A boy engaged with a farmer for one year for \$128 and a suit of clothes. He left at the end of nine months and was entitled to \$92 and the suit. Find the price of the suit.

55. A bill of \$67.50 was paid in dollars, fifty-cent and twenty-five cent pieces. There were 3 times as many fifty-cent pieces as dollars, and 10 more twenty-five than fifty-cent pieces. How many of each were used?

56. A man has 11 hours at his disposal; how far may he ride in a coach which travels 8 miles an hour, so as to return in time, walking back at the rate of 3 miles an hour?

57. The rate of a man rowing in still water is double that of a stream. If it takes the man 40 minutes to row 5 miles down the stream, find the rate of the stream.

58. A starts upon a walk at the rate of 4 miles an hour, and after 30 minutes B starts at the rate of $4\frac{3}{4}$ miles an hour; when and where will he overtake A?

59. A garrison of 700 men had provisions to last for 40 days, but 12 days afterward 300 men were killed. How long will the provisions last the remainder of the garrison?

60. A boy is one-third the age of his father, and has a brother one-sixth of his own age; the ages of all three amount to 75 years. Find the age of each.

61. Two boys have equal sums of money; but if one had 15 cents more and the other 9 cents less, the one would

have three times as much as the other. What sum had each?

62. A man sells 50 acres more than one-half his farm, and there yet remains 30 acres less than one-third of it. How many acres were there in the farm?

63. A bag contains a certain number of sovereigns, twice as many shillings, and three times as many pence; and the whole sum is £267. Find the number of sovereigns, shillings and pence in it.

64. A huckster bought a certain number of apples at the rate of 5 for 2 cents, and sold one-half of them at the rate of 3 for 1 cent, and the other half at the rate of 2 for 1 cent, gaining altogether 4 cents. How many apples did he buy?

65. A certain number of men and one-half as many women were employed on a work. Each man received $1.25, and each woman 75 cents; their total wages being $45.50. How many of each were employed?

66. A hare is 80 of her own leaps before a greyhound; she takes 3 leaps for every 2 he takes, but he covers as much ground in 1 leap as she does in 2. How many leaps does the hare take before she is caught?

MISCELLANEOUS REVIEW QUESTIONS.

Exercise 43.

1. Divide $8x^3 + y^3$ by $2x + y$.
2. Simplify $2\left[2 - 2\{2 - 2(2 - a)\} - a\right]$.
3. Factor $x^2 + 8x - 560$ and $x^2 - 8x - 560$.
4. Solve $20x - 13 - 56x = 3(x - 7)$.
5. Find a number which is as much greater than 20 as three times the number is greater than 70.

6. Divide $256x^4+16x^2+1$ by $1-4x+16x^2$
7. Expand $(a-2b+c-3d)^2$.
8. Find the product of $(a-2b+c-3d)$ and $(a+2b+c+3d)$.
9. Solve $4(x-2)-5(x-3)=6(2-x)$.
10. Divide 16 into two such parts that if four times one part be added to five times the other part, the result may be 76.

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11. Divide x^5+y^5 by $x+y$ and x^5-y^5 by $x-y$.
12. Divide the difference of the squares of x^2+x+1 and x^2-x+1 by their sum.
13. Simplify $a-2\left[a-2\left\{a-2(2-a)\right\}\right]$.
14. Solve $3x-2(2x-4)=x$.
15. Express $x^{16}-y^{16}$ as the product of five factors.

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16. Divide x^6-y^6 by $x-y$.
17. Solve $(2x-3)(3x-2)=6(x-1)^2$.
18. Find the factors of $64x^2-z^2$
19. Find a number such that double the number is as much greater than 40 as three times the number is less than 85.
20. Show that $(a+b)(b+c)(c+a)=a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.

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21. Divide x^6-y^6 by $x+y$.
22. Find factors of $x^2-17x-60$ and x^2+x-20.
23. State in words the meaning of $(x+y)^3=x^3+y^3+3xy(x+y)$ and prove the equality.
24. Solve $2x-\left\{3-4(x-2)+x\right\}=60$.
25. Show that $(a-b)(b-c)(c-a)=a^2(c-b)+b^2(a-c)+c^2(b-a)$.

26. Multiply $b^2 + bc + c^2$ by $b^2 - bc + c^2$.
27. Find the factors of $15x^2 + 34xy + 15y^2$ and of $15x^2 - 16xy - 15y^2$.
28. Prove that the product of the sum of the squares of two numbers, increased by their product, and the difference of the numbers, is equal to the difference of the cubes of the numbers.
29. Solve $(x+2)(x-9)-(x-6)(x-3)=20-13x$.
30. Prove that $(a+b+c)(a^2+b^2+c^2-bc-ca-ab)=a^3+b^3+c^3-3abc$.

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31. Multiply $x^4 - x^3 + x^2 - x + 1$ by $x^4 + x^3 + x^2 + x + 1$.
32. Prove that $(x+y+z)(x^2+y^2+z^2-yz-zx-xy)=x^3+y^3+z^3-3xyz$.
33. From the preceding write the product of $(x+2y+3z)$ and $(x^2+4y^2+9z^2-6yz-3zx-2xy)$.
34. Find two numbers whose sum is 156 and whose difference is 40.
35. Find the value of $x^3+y^3+z^3-3xyz$ when $x=a+1$, $y=a-1$ and $z=-2a$.

---

36. Find the product of $l^2 - lm + m^2$ and $l^2 + lm + m$.
37. Write the quotient of $x^3+y^3+z^3-3xyz$ by $x+y+z$.
38. Write the quotient of $x^2+y^3+8z^3-6xyz$ by $x+y+2z$.
39. The difference of two numbers is 6, and half their sum is 12, what are the numbers?
40. Solve $(x+2)^3=(x+1)^2(x+4)$.

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41. Show that $(b-c)^2+(c-a)^2+(a-b)^2=2(a^2+b^2+c^2-bc-ca-ab)$.
42. Write the quotient of $x^3-y^3+z^3+3xyz$ by $x-y+z$.
43. Simplify $2\left[x-3\left\{x-4(x-a)-a\right\}-a\right]-a$.

44. Two numbers differ by 1; show the difference of their squares is equal to the sum of the numbers.

45. Solve $(x+7)^2+(5-x)(x+5)=36x$.

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46. Write the square of $3x-y-z$.

47. Find two factors of $a^2-b^2+c^2-2ac$.

48. Show that $(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(bc+ca+ab)x+abc$. From this expand the product $(x+2)(x+3)(x+4)$.

49. Divide $x^3-y^3-27z^3-9xyz$ by $x-y-3z$.

50. If a is 2, find the value of x which will make $(x-a)^2=x(x-14)$.

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51. Find the continued product of $(x+1)$, $(x-1)$, (x^2+x+1) and (x^2-x+1).

52. Show that $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2$. State in words the meaning of this identity.

53. Factor $x^2+3xy+2y^2+x+y$.

54. Divide 72 into two parts, such that three times one part is equal to five times the other part.

55. Solve $2(x+1)^3+5(x-2)^3=7x^3-24(x+3)(x-9)$.

---

56. Show that $(b-c)^3+(c-a)^3+(a-b)^3-3(b-c)(c-a)(a-b)=0$.

57. If $x=2a-b-c$, $y=2b-c-a$, $z=2c-a-b$, show that $x^3+y^3+z^3=3xyz$.

58. Factor $12a^2+ab-20b^2$.

59. Simplify $x^2-\left[(x-z)^2-\left\{z^2-(x+z)^2\right\}\right]$.

60. Solve $(\cdot 3x+2)-(7-\cdot 4x)=14\cdot 6$.

---

61. Show that $(x+y+z)^2=(y+z)^2+(z+x)^2+(x+y)^2-x^2-y^2-z^2$.

62. Divide $x^3 - 3x^2 + 3x + y^3 - 1$ by $x + y - 1$.

63. Find the value of $(99x + y)^3 + (x + 99y)^3$, when $x = 49$ and $y = -49$.

64. What is the price of bread per loaf if an increase of 25 per cent. in the price would reduce the number of loaves that could be purchased for one dollar by 2?

65. Solve $\frac{1}{5}(2x + 3) - \frac{1}{6}(3x + 4) = 12$.

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66. Divide $(x^2 + 5x - 14)(x^2 + 8x - 9)$ by $x^2 + 6x - 7$.

67. Solve $(x - 6)(x + 4) - (x - 8)(x + 3) = 0$.

68. A father has three sons: the father's age is 36, and the joint ages of the sons is 30. In how many years will the joint ages of the sons be equal to that of the father?

69. Expand $(m + 2n - p + 2q)^2$.

70. Find the value of $A^3 + B^3$, when $A = (2a + 2b - c)$, $B = (2c - b - a)$ and $a + b + c = 0$.

CHAPTER VIII.

Fractions.

We commonly indicate the division of one expression by another by placing the first expression above the other and separated from it by a bar or stroke.

Thus, $\frac{a}{b}$ is the quotient obtained by dividing a by b.

Such an indicated quotient is called a **fraction.**

The expression placed above the line is called the **numerator,** and that below the line is called the **denominator.**

The numerator and denominator are called the **terms** of the fraction.

Since the quotient is not altered, if we multiply both divisor and dividend by the same quantity, it is evident that the value of a fraction is not altered by multiplying each of its terms by the same quantity.

That is,

$$\frac{a}{b} = \frac{ax}{bx}.$$

And since the quotient is not altered, if we divide both divisor and dividend by the same quantity, it is evident that the value of a fraction is not altered by dividing each of its terms by the same quantity.

Thus,

$$\frac{a^2}{ab} = \frac{a}{b},$$

and

$$\frac{a^2 - x^2}{(a+x)^2} = \frac{a-x}{a+x}.$$

A fraction is said to be in its **lowest terms** when the numerator and denominator have no common factor. We may reduce a fraction to its lowest terms by dividing numerator and denominator by their H. C. F.

Exercise 44.

Reduce to their lowest terms :

1. $\dfrac{4x}{3x^2}$. 2. $\dfrac{5a}{10}$. 3. $\dfrac{6a^2}{9a^3}$.

4. $\dfrac{15a^2b}{20a^3b^2}$. 5. $\dfrac{24x^2yz^2}{30xy^2z^2}$. 6. $\dfrac{25l^2m^3p^2}{30lm^2p^4}$.

7. $\dfrac{ax+ay}{bx+by}$. 8. $\dfrac{xy+y^2}{x^2+xy}$. 9. $\dfrac{a^2-b^2}{(a+b)^2}$.

10. $\dfrac{c^2-d^2}{c^2+2cd+d^2}$. 11. $\dfrac{x^2-y^2}{x^2-2xy+y^2}$. 12. $\dfrac{a^2-b^2}{(a-b)^3}$.

13. $\dfrac{x^2+3xy+2y^2}{(x+y)^2}$. 14. $\dfrac{x^2+2xy+y^2}{(x+y)^3}$. 15. $\dfrac{m^2-2mn+n^2}{m^2-3mn+2n^2}$.

16. $\dfrac{a^2+5a+6}{a^2+a-6}$. 17. $\dfrac{p^2-10pq-24q^2}{p^2+4pq+4q^2}$.

18. $\dfrac{(x-1)^3}{x^3-1}$. 19. $\dfrac{x^3+1}{x^3+3x^2+3x+1}$.

20. $\dfrac{b^2+b+1}{b^3-1}$. 21. $\dfrac{x^2-y^2}{x^4-y^4}$.

22. $\dfrac{a^3+3a^2+2a}{a^3+3a^2+3a+1}$. 23. $\dfrac{6x^2+13xy+6y^2}{6x^2+7xy+2y^2}$.

24. $\dfrac{x^6-1}{x^6+2x^3+1}$. 25. $\dfrac{(a+b+c)^2}{a^2-b^2-2bc-c^2}$.

26. $\dfrac{(x+y)^2-z^2}{x^2-(y+z)^2}$. 27. $\dfrac{a^2+b^2-c^2-d^2-2ab-2cd}{a^2-(b-c-d)^2}$.

Exercise 45.

Complete the following identities by writing the required term :

1. $\dfrac{a}{b}=\dfrac{\quad}{bc}$. 2. $\dfrac{x}{y}=\dfrac{\quad}{yz}$. 3. $\dfrac{a}{b}=\dfrac{\quad}{b^2}$

4. $\dfrac{m}{n}=\dfrac{mp}{\quad}$ 5. $\dfrac{x}{y}=\dfrac{xp}{\quad}$. 6. $\dfrac{a}{b}=\dfrac{a^2}{\quad}$.

7. $\frac{4xy}{z^2} = \frac{\quad}{2z^3}$.
8. $\frac{5a^2b}{3c^3} = \frac{10a^3b^2}{\quad}$.
9. $\frac{abc}{x^3} = \frac{4a^2bc}{\quad}$.

10. $\frac{xc}{bc} = \frac{\quad}{bx}$.
11. $\frac{mx}{nx} = \frac{am}{\quad}$.
12. $\frac{4a}{12ab} = \frac{3}{\quad}$.

13. $\frac{a-x}{a+x} = \frac{a^2-x^2}{\quad}$.
14. $\frac{x+2}{x+3} = \frac{\quad}{x^2+6x+9}$.

15. $\frac{x+5}{x-7} = \frac{x^2-2x-35}{\quad}$.
16. $\frac{a+2b}{a+4b} = \frac{\quad}{a^2+4ab}$.

17. $\frac{a^2-9a-10}{a^2-12a+20} = \frac{\quad}{a^2-4a+4}$.

18. $\frac{b^2+8b+15}{b^2+6b+9} = \frac{\quad}{b^2-9}$.
19. $\frac{1}{a+b} = \frac{\quad}{a^2-b^2}$.

20. $\frac{x-y}{1} = \frac{\quad}{x-y}$.
21. $b-c = \frac{\quad}{b+c}$.

22. $b-c = \frac{b^2+4bc-5c^2}{\quad}$.
23. $1 = \frac{\quad}{x+y}$.
24. $a = \frac{\quad}{a+b}$

Exercise 46.

Change into equivalent fractions, having the same denominator, the common denominator being the L. C. M. of the denominators:

1. $\frac{a}{x}, \frac{b}{2x}$.
2. $\frac{a}{x}, \frac{b}{x^2}$.

3. $\frac{x}{bc}, \frac{x}{ca}, \frac{x}{ab}$.
4. $\frac{a}{bc}, \frac{b}{ca}, \frac{c}{ab}$.

5. $\frac{2}{a}, \frac{3}{b}, \frac{4}{c}$.
6. $\frac{a}{bx^2}, \frac{b}{a^2x}, \frac{1}{a^2b^2}$.

7. $\frac{a}{a+b}, \frac{b}{a-b}$.
8. $\frac{x}{x^2-y^2}, \frac{1}{x-y}$.

9. $\frac{3}{x+3}, \frac{2}{x+4}$.
10. $\frac{x+3}{a^2+6a+8}, \frac{4}{a+4}$.

11. $\frac{a}{2}$, $\frac{a+1}{a-1}$, $\frac{a-1}{a+1}$. 12. $\frac{1}{x^2+6x+5}$, $\frac{1}{x^2+7x+10}$.

13. $\frac{2}{x+1}$, $\frac{2x}{(x+1)^2}$, $\frac{2x^2}{(x+1)^3}$.

14. $\frac{1}{b-c}$, $\frac{1}{b+c}$, $\frac{b}{b^2-c^2}$, $\frac{c}{(b+c^2)}$.

15. $\frac{x+y}{x-y}$, $\frac{x-y}{x+y}$, $\frac{x^2}{x^2-y^2}$. 16. a, $\frac{1}{b}$, $\frac{c}{d}$, $\frac{1}{b-d}$.

17. $\frac{1}{(b+c)(c+a)}$, $\frac{1}{(c+a)(a+b)}$, $\frac{1}{(a+b)(b+c)}$.

18. $\frac{a+b}{(b-c)(c-a)}$, $\frac{b+c}{(c-a)(a-b)}$, $\frac{c+a}{(a-b)(b-c)}$.

The sum of the quotients obtained by dividing several numbers by the same divisor is equal to the quotient obtained by dividing the sum of the several numbers by the same divisor.

That is,

$$\frac{a}{x}+\frac{b}{x}+\frac{c}{x}=\frac{a+b+c}{x}.$$

Also the difference of the quotients obtained by dividing two numbers by the same divisor is equal to the quotient obtained by dividing the difference of the numbers by the same divisor.

That is,

$$\frac{x}{d}-\frac{y}{d}=\frac{x-y}{d}.$$

Hence we see that any number of fractions which have a common denominator may be combined into a single fraction.

Thus,

$$\frac{x}{a}-\frac{y}{a}+\frac{z}{a}=\frac{x-y+z}{a},$$

and

$$\frac{x}{b-c}-\frac{y}{b-c}-\frac{z}{b-c}=\frac{x-y-z}{b-c}.$$

To combine $\frac{5}{2x-1} - \frac{4}{2x+1} + \frac{5}{4x^2-1}$ into a single fraction.

The L. C. M. of the denominators is $4x^2 - 1$.

$$\frac{5}{2x-1} - \frac{4}{2x+1} + \frac{5}{4x^2-1}$$

$$= \frac{5(2x+1)}{4x^2-1} - \frac{4(2x-1)}{4x^2-1} + \frac{5}{4x^2-1}$$

$$= \frac{10x+5-8x+4+5}{4x^2-1}$$

$$= \frac{2x+14}{4x^2-1}.$$

To combine $\frac{a-b}{ab} + \frac{b-c}{bc} + \frac{c-a}{ab}$.

The L. C. M. of the denominators is abc.

$$\frac{a-b}{ab} + \frac{b-c}{bc} + \frac{c-a}{ca}$$

$$= \frac{ac-bc}{abc} + \frac{ab-ac}{abc} + \frac{bc-ab}{abc}$$

$$= \frac{ac-bc+ab-ac+bc-ab}{abc}$$

$$= \frac{o}{abc} = o$$

Exercise 47.

Combine into a single fraction:

1. $\frac{x}{2} + \frac{x}{3}$.
2. $\frac{2x}{3} + \frac{3x}{4}$.
3. $\frac{1}{2x} + \frac{1}{3x}$.
4. $\frac{2}{3x} + \frac{3}{4x}$.
5. $\frac{x}{x+y} + \frac{y}{x+y}$.
6. $\frac{x}{x-y} - \frac{y}{x-y}$.
7. $\frac{4}{x+y} - \frac{3}{x-y}$.
8. $\frac{x}{x+y} - \frac{y}{x-y}$.

9. $\frac{1}{x-2}-\frac{1}{x+2}$.

10. $\frac{1}{a-4}+\frac{1}{a+4}$.

11. $\frac{1}{x(x+y)}+\frac{1}{x(x-y)}$.

12. $\frac{x+y}{x-y}+\frac{x-y}{x+y}$.

13. $\frac{1}{a+2}+\frac{1}{a-2}-\frac{a}{a^2-4}$.

14. $\frac{1}{x+y}-\frac{1}{x-y}+\frac{y}{x^2-y^2}$.

15. $\frac{1}{x^2+2x+1}+\frac{1}{x^2+4x+3}$.

16. $\frac{2}{x^2-3x+2}-\frac{3}{x^2-1}$.

17. $\frac{x+4}{x^2+5x+6}-\frac{x+2}{x^2+4x+3}$.

18. $1+\frac{a-b}{a+b}+\frac{b}{a+b}$.

19. $x+\frac{x^2+y^2}{x-y}-\frac{y^2}{x+y}$.

20. $\frac{x}{x-y}+\frac{y}{y-x}$.

21. $\frac{b}{a-b}+\frac{a}{a+b}+\frac{2b^2}{b^2-a^2}$.

22. $\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}$.

23. $\frac{1}{a-1}-\frac{2}{a-2}+\frac{2}{a+2}-\frac{1}{a+1}$.

24. $\frac{3}{x-1}-\frac{x+1}{x-1}-\frac{x^2}{x^2-1}+1$.

To multiply $\frac{a}{b}$ and $\frac{c}{d}$.

Since the product of quotient and divisor is the dividend,

$$\therefore \frac{a}{b}\times b=a.$$

$$\text{Also } \frac{c}{d}\times d=c.$$

$$\therefore \frac{a}{b}\times\frac{c}{d}\times bd=ac.$$

$$\therefore \frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}.$$

Hence the product of two fractions may be written as one fraction, whose numerator is the product of the numerators of the two fractions, and whose denominator is the product of the denominators.

Evidently we may find the product of any number of fractions, by multiplying together the numerators for a new numerator, and the denominators for a new denominator.

To multiply $\frac{2a}{5c}$, $\frac{3bx^2}{4ad}$ and $\frac{10c^2d}{7ax^3}$.

The product is $\frac{2a \times 3bx^2 \times 10c^2d}{5c \times 4ad \times 7ax^3}$,

that is, $\frac{60abc^2dx^2}{140a^2cdx^3}$.

This result, when reduced to lowest terms, is

$$\frac{3bc}{7ax}$$

We may shorten the work by cancelling like factors in numerator and denominator, just as is done in arithmetic.

Thus:

$$\frac{a^2-b^2}{x^2-y^2} \times \frac{x+y}{a-b} \times \frac{c}{(a+b)^2}$$

$$= \frac{(a-b)(a+b)}{(x-y)(x+y)} \times \frac{x+y}{a-b} \times \frac{c}{(a+b)^2}$$

$$= \frac{c}{(x-y)(a+b)}.$$

Exercise 48.

Find the following products in their lowest terms:

1. $\frac{2x}{5y} \times \frac{3xy}{4x^2}$.

2. $\frac{2a^2}{6bc} \times \frac{3b^2c}{4ad}$.

3. $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{a}$.

4. $\frac{x^2}{yz} \times \frac{y^2}{zx} \times \frac{z^2}{xy}$.

5. $\dfrac{4xy}{5zw} \times \dfrac{3xz}{4yw} \times \dfrac{5xw}{3yz}$. 6. $\dfrac{a^3}{b^4} \times \dfrac{b^3}{c^5} \times \dfrac{c^4}{a}$.

7. $\dfrac{a+b}{(a-b)^2} \times \dfrac{a^2-b^2}{a-b}$. 8. $\dfrac{(x+y)^2}{x^2-y^2} \times \dfrac{x-y}{x}$.

9. $\dfrac{a+b}{a^2-ab} \times \dfrac{a-b}{ab+b^2}$. 10. $\dfrac{x}{x+1} \times \dfrac{x+1}{x+2} \times \dfrac{x+2}{x}$.

11. $\dfrac{x^2-1}{x^2-4} \times \dfrac{x+2}{x-1} \times \dfrac{x}{x+1}$.

12. $\dfrac{(x+y)^2}{(x-y)^2} \times \dfrac{x^2-y^2}{x^2+2xy+y^2}$.

13. $\dfrac{x^2+3x+2}{x^2+4x+4} \times \dfrac{x^2+5x+6}{x^2+4x+3}$.

14. $\dfrac{a^2-b^2}{a^2+ab-2b^2} \times \dfrac{a+2b}{a+b}$.

15. $\dfrac{m^2+4mn+3n^2}{m^2-4mn-5n^2} \times \dfrac{m}{m+3n}$.

16. $\dfrac{x^3-1}{(x+1)^3} \times \dfrac{x^2-1}{x^2+x+1}$.

17. $\dfrac{a^2-(b+c)^2}{a^2-(b-c)^2} \times \dfrac{(a+c)^2-b^2}{(a-c)^2-b^2}$.

18. $\dfrac{x^2+5x+6}{(x+1)^2} \times \dfrac{x^2+5x+4}{(x+2)^2} \times \dfrac{x+1}{x+4}$.

19. $\dfrac{a+b}{(a-b)^2} \times \dfrac{a^2-b^2}{a^2+b^2} \times \dfrac{a^4-b^4}{(a+b)^3}$.

20. $\dfrac{(a-b)^2-c^2}{(a+b)^2-c^2} \times \dfrac{c^2-(a+b)^2}{(c-b)^2-a^2} \times \dfrac{c^2-(b-c)^2}{a^2-(b+c)^2} \times \dfrac{a+b+c}{a-b+c}$.

To divide $\frac{a}{b}$ by $\frac{c}{d}$.

$$\text{Since } \frac{ad}{bc} \times \frac{c}{d} = \frac{a}{b},$$

$$\therefore \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

$$\text{But } \frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}.$$

$$\therefore \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

Hence we see that to divide by a fraction we invert the fraction and multiply by it.

$$\text{Thus } \frac{ax}{by} \div \frac{ab}{xy}$$

$$= \frac{ax}{by} \times \frac{xy}{ab}$$

$$= \frac{x^2}{b^2}.$$

$$\text{And } \frac{x^2-y^2}{x^2} \div \frac{x+y}{x}$$

$$= \frac{x^2-y^2}{x^2} \times \frac{x}{x+y}$$

$$= \frac{x-y}{x}.$$

Exercise 49.

Simplify:

1. $\frac{ax}{by} \div \frac{bx}{ay}$.
2. $\frac{x^2}{y^2} \div \frac{ax}{by}$.
3. $\frac{4ab}{5c^2} \div \frac{2ac}{b^2}$.
4. $\frac{mnp}{q^2} \div \frac{np^2}{mq}$.
5. $\frac{a^2-b^2}{(a-b)^2} \div \frac{a+b}{a-b}$.
6. $\frac{x^2+5xy+6y^2}{x^2+7xy+10y^2} \div \frac{x^2-9y^2}{(x+5y)^2}$.

7. $\dfrac{x^2-4}{x^2-9} \div \dfrac{(x+2)^2}{(x+3)^2}$.

8. $\dfrac{x^2+6x+5}{x^2+7x+12} \div \dfrac{x+5}{x+4}$.

9. $\dfrac{c-2d}{c+3d} \div \dfrac{c^2-4d^2}{c^2+5d+6d^2}$

10. $\dfrac{a^3-1}{a^3+1} \div \dfrac{a^2+a+1}{a^2-a+1}$.

11. $\dfrac{a^2xy}{b^2z} \times \dfrac{bxz^2}{y^2} \div \dfrac{ax^2}{by}$.

12. $\dfrac{a^2-4}{a^2-25} \times \dfrac{a^2-8a+15}{a^2-a-6} \div \dfrac{a-2}{a+5}$.

13. $\dfrac{x^2-y^2}{x^2+y^2} \div \dfrac{(x^2+y^2)^2}{x^4-y^4} \times \dfrac{x^2+y^2}{(x-y)^2}$.

14. $\dfrac{a^4-b^4}{a^2-2ab+b^2} \div \dfrac{a^2+ab}{a-b} \times \dfrac{a-b}{b(a+b)}$.

15. $\left(\dfrac{x+2}{x-2} - \dfrac{x-1}{x+2}\right) \div \dfrac{x}{x+2}$.

16. $1 \div \left(\dfrac{a-b}{a+b} + \dfrac{a+b}{a-b}\right)$.

17. $\left(\dfrac{a}{b} + \dfrac{b}{a}\right) \div \left(\dfrac{1}{a} + \dfrac{1}{b}\right)$.

CHAPTER IX.

EQUATIONS AND PROBLEMS INVOLVING FRACTIONS.

When an equation involves fractions, we have seen that we may get rid of the fractions by multiplying both sides of the equation by the L.C.M. of the denominators of the fractions.

To solve $$\frac{3}{x-2}=\frac{4}{x-1}.$$

Multiplying both sides by $(x-2)(x-1)$ we get

$$3(x-1)=4(x-2).$$

That is, $3x-3=4x-8.$

$\therefore$ $3x-4x=3-8$

and $x=5.$

To solve $$\frac{x-1}{x-2}=\frac{x+4}{x+2}.$$

Multiplying both sides by $(x-2)(x+2)$ we get

$$(x-1)(x+2) = (x-2)(x+4),$$

or $x^2+x-2 = x^2+2x-8$

$\therefore$ $-x = -6,$

and $x = 6.$

Exercise 50.

Solve the equations :

1. $\frac{2}{x}+\frac{3}{4x}=\frac{11}{8}.$

2. $4+\frac{5}{6x}=\frac{3}{2x}.$

3. $\frac{12}{x}+\frac{9}{x}+\frac{4}{3x}=7\frac{4}{9}.$

4. $\frac{1}{1+x}=\frac{2}{3x}.$

5. $\frac{20}{x}-\frac{39}{x+2}=0.$

6. $\frac{x-4}{2} = \frac{x-2}{3}$.   7. $\frac{2}{x-4} = \frac{3}{x-2}$.

8. $1 + \frac{2}{x-4} = 1 + \frac{3}{x-2}$.

9. $\frac{x-2}{x-4} = \frac{x+1}{x-2}$.   10. $\frac{3x-6}{x-4} = \frac{3x+3}{x-2}$.

11. $\frac{x-2}{4x-16} = \frac{x+1}{4x-8}$.   12. $\frac{6x-18}{3x-8} = \frac{6x-15}{3x-7}$.

13. $\frac{2x-6}{3x-8} = \frac{2x-5}{3x-7}$.   14. $\frac{x-3}{x-4} - \frac{x-5}{x+1} = 0$.

15. $\frac{3x-5}{x-1} - \frac{2x-7}{x-3} = 1$.   16. $\frac{18x-6}{2x-1} - \frac{24x-12}{3x-2} = 1$.

17. $\frac{3x-1}{2x-1} - \frac{4x-2}{3x-2} = \frac{1}{6}$.

18. $\frac{x-4}{x-5} - \frac{x-5}{x-6} = \frac{2x}{(x-5)(x-6)}$.

19. $\frac{x-7}{x-8} - \frac{x-8}{x-9} = \frac{x-1}{x^2-17x+72}$.

20. $\frac{x-4}{x-5} - \frac{x-5}{x-6} = \frac{x-7}{x-8} - \frac{x-8}{x-9}$.

21. $\frac{x+1}{x-1} + \frac{x-8}{x-6} = \frac{x+2}{x} + \frac{x-7}{x-5}$.

22. $\frac{3+x}{3-x} - \frac{2+x}{2-x} - \frac{1+x}{1-x} = 1$.

22. $\frac{x^2-x+1}{x-1} + \frac{x^2+x+1}{x+1} = 2x$.

24. $\frac{4z+17}{z+3} + \frac{3z-10}{z-4} = 7$.

25. $\frac{4}{z+2} + \frac{7}{z+3} = \frac{37}{z^2+5z+6}$.

26. $\frac{1}{2}(x-\frac{1}{3}) - \frac{1}{3}(x-\frac{1}{4}) + \frac{1}{4}(x-\frac{1}{5}) = 0$

27. $\frac{2}{2y-5} + \frac{1}{y-3} = \frac{6}{3y-1}$.

28. $\dfrac{6x+1}{15} - \dfrac{2x-4}{7x-16} = \dfrac{2x-1}{5}$.

29. $\dfrac{x+2}{2} + \dfrac{3x-1}{5} + \dfrac{1}{8} = \dfrac{6x+13}{8}$.

30. $\dfrac{4}{3} - \dfrac{7}{3z} - \dfrac{3}{4} = 1 - \dfrac{5}{2z}$.

31. $\dfrac{y}{3} - \dfrac{y^2-5y}{3y-7} = \dfrac{2}{3}$.

32. $\dfrac{x}{2} + \dfrac{6-x}{6} - \dfrac{3}{x-1} = \dfrac{x-1}{3}$.

33. $\dfrac{3}{x-1} - \dfrac{x+1}{x-1} = \dfrac{x^2}{1-x^2}$.

34. $x(x-4)(x-6)(x-8) = (x-1)(x-3)(x-5)(x-9)$.

35. $\dfrac{3}{(3x-2)(2x-5)} = \dfrac{5}{(5x-9)(2x-3)}$.

36. $\dfrac{2y^2-3y+1}{y^2-2y+2} = \dfrac{2y-3}{y-2}$.

37. $\dfrac{x+3}{7} + (x+2)^2 = (x+1)^2$.

38. $\dfrac{x}{4} - \dfrac{x-4}{32} + \dfrac{13-2x}{40} = \dfrac{8-x}{2} - \dfrac{7}{8}$.

39. $\dfrac{5}{6}\left(x - \dfrac{1}{3}\right) + \dfrac{7}{6}\left(\dfrac{x}{5} - \dfrac{1}{7}\right) = 4\dfrac{8}{9}$.

EXERCISE 51.

1. Divide 108 into two parts, so that 25% of one part may equal 20% of the other part.
2. What number subtracted from the denominator of $\frac{3}{8}$ will make the fraction equal to $\frac{5}{6}$?
3. What number must be subtracted from the denominator of $\frac{5}{12}$ and added to the denominator of $\frac{7}{9}$ to make the resulting fractions equal?

4. What number must I add to the numerator and subtract from the denominator of $\frac{3}{14}$ to make the fraction equal to $\frac{2}{3}$?

5. Eight times a number, consisting of two digits, is equal to three times the number composed of the digits reversed. If the units digit is 5 greater than the tens digit in the former number, find the number.

6. Find the number whose one-half and one-fifth exceeds its one-third and two-sevenths by 34.

7. What number subtracted from both numerator and denominator of $\frac{3}{4}$ will reduce the fraction to $\frac{1}{2}$?

8. What number added to the numerator and denominator of $\frac{15}{7}$ will make the fraction equal to 2?

9. What number added to the numerator and subtracted from the denominator of $\frac{5}{8}$ will make a fraction equal to $\frac{9}{10}$?

10. What number added to the numerator and subtracted from the denominator of $\frac{7}{8}$ will make the same fraction as when twice the number is added to the numerator and also subtracted from the denominator of $\frac{3}{4}$?

11. A company took a risk at 4%, and reinsured $\frac{3}{4}$ of it at 3%. The premium received exceeded the premium paid by $27. Find the amount of the risk.

12. A man lends $375 at a certain rate of interest, and $412 at a rate 2% higher. If the interest for one year from both investments is $47.59, find the rate at which each was lent.

13. A man sold a horse at 20% profit. If the horse had cost him $40 more, and had sold for the same amount as before, he would have lost 5%. What was the cost?

14. A man divided a farm among three sons. To the first he gave 110 acres; to the second $\frac{2}{9}$ of the whole, and to the third $1\frac{2}{3}$ as much as to both the others. How many acres did the farm contain?

15. A man has a certain sum of money invested at 4°/ₒ, and 3 times that amount at 6°/ₒ. From both investments he obtains an income of $382.14. What is the total amount invested?

16. A quantity of goods was sold at 25°/ₒ gain; but, had they cost $40 less, the gain at the same selling price would have been 35%. What did the goods cost?

17. A person gave 5 cents each to a number of beggars, and had 14 cents left. He found that he would have required 22 cents more to enable him to give the beggars 8 cents each. How many beggars were there?

18. Divide 78 into two parts, so that 9% of one part may equal 17% of the other.

19. In building a house the owner pays twice as much for material as for labor. Had he paid 5% more for material and 7% more for labor, the house would have cost $10,144. What was the cost?

20. A merchant bought 100 barrels of flour, part at $7 per barrel, and the remainder at $5 a barrel. By selling the former at 15% gain, and the latter at 14% loss, he just cleared himself on the transaction. How many barrels of each did he buy?

21. A grocer spent equal sums in tea, coffee and sugar, making 12% on the tea, 8% on the coffee, and losing 15% on the sugar. His total gain being $63.50, find the cost of each commodity.

22. Divide $500 into two parts, such that the simple interest on one part for 4 years, at 6°/ₒ per annum, may be $12 more than that on the other part for 6 years, at 5°/ₒ per annum.

23. Find a number consisting of two digits, whose units digit exceeds its tens digit by 5, and when the number is divided by the sum of the digits the quotient is 3.

24 What number divided into 367 will give a quotient 21 , and a remainder 10?

25. A can do a piece of work in m days, and B in n days. In what time can they do it working together?

26. A and B working together can do a piece of work in p days, and A can do it alone in q days. In what time can B do the work himself?

27. A can do a piece of work in x days, and B in y days. How long will it take B to finish the work if A has worked at it z days?

28. A can do a piece of work in 10 days, and B in m days. If both working together could do it in 6 days, find m.

29. A can earn \$$m$ a day, and B \$$n$ a day. If they work together, how much can they earn in t days?

30. If they earn together \$$p$, how many days do they work, and how should the money be divided?

MISCELLANEOUS REVIEW QUESTIONS.

Exercise 52.

1. Multiply $a+b+c$ by $a+b-c$.
2. Show that $(a+b)^2+2(a+b)c+c^2=(a+b+c)^2$.
3. Factor $1-2x+x^2-y^2-z^2+2yz$.
4. A man receives \$5 a day for his work, and forfeits \$3 a day for each working day he is idle; at the end of 20 days he receives \$28. How many days has he worked?
5. Divide $(x^2+3x)^2-7(x^2+3x)-18$ by x^2+3x+2.

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6. Show that $(x+\frac{1}{2})^2=x(x+1)+\frac{1}{4}$.
   Infer an easy rule for squaring such a number as $20\frac{1}{2}$.
7. Find the continued product $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.
8. Divide $a^2-b^2-c^2+d^2+2bc-2ad$ by $a+b-c-d$.
9. Solve $(x-4)^2-(x-6)^2=4(x-5)$.

10. Ten apples and six pears cost 22 cents, and one apple cost half as much as one pear. Find the cost of each.

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11. Find the continued product of $(x+2)$, $(x-2)$, (x^2+2x+4) and (x^2-2x+4).
12. There are two numbers whose sum is 10; their product is $24\frac{3}{4}$. Find the sum of the cubes of the numbers.
13. Find factors of $50x^2-151x+3$.
14. A, B and C are three houses, in order, along a road. The distance from A to C is one mile, and the number of feet in the distance from A to B is the same as the number of yards in the distance from B to C. Find the distance from B to C.
15. Solve $2(x-3)(x-11)-(2x-1)(x-7)=100$.

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16. Find the value of $(a-b)^3+(b-c)^3+(c-a)^3-3(b-c)(c-a)(a-b)$; when $a=1$, $b=2$ and $c=3$.
17. Find the continued product of $(a+b)$, $(a+c)$ and $(a+d)$. From the result infer the product of $(a+1)$, $(a-2)$ and $(a+3)$.
18. The sum of two numbers is 20; the difference of their squares, 20. What is the difference of the numbers?
19. Find two numbers whose difference is 10, and whose sum diminished by 10 is 10.
20. Solve $x(x-3)(x-7)=(x-1)(x-4)(x-5)$.

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21. Find factors of $(x^2+4x)^2-9(x^2+4x)-36$.
22. Find the expression which multiplied by $1-y-z$ will give $1-y^3-3yz-z^3$.
23. Simplify $a-[2b+\{3c-3a-(a+b)\}]+2a-(b+c)$.
24. Divide $(a+b+c)(bc+ca+ab)-abc$ by $a+b$.
25. Solve $(x-6)(x+2)+(5+x)(x-5)-(7+x)(2x-3)=0$.

26. If $2s = a + b + c$, show that $16s(s-a)(s-b)(s-c) = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$.

27. Put $a+3$ for x in the expression $x^3 - 6x^2 + 12x - 8$, and arrange the result in descending powers of a.

28. Show that the sum of the squares of two consecutive numbers is greater than twice their product by 1.

29. What value of x will make $(x+a)(x+b)$ greater than $(x-2a)(x-3b)$ by 32, where $a = 1$ and $b = 2$?

30. Show that the sum of the cubes of any three consecutive whole numbers is divisible by three times the middle number.

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31. Factor $30x^2 - 73xy - 5y^2$.

32. Show that $2(a^3 + b^3 + c^3 - 3abc) = \{(a-b)^2 + (b-c)^2 + (c-a)^2\}(a+b+c)$.

33. Expand $(a+b+c)^3$; from the result infer the expansion of $(x+2y-z)^3$.

34. Add a term to $16a^2b^2 + 2abc^2$, which will make the expression a square, and state of what expression it is then the square.

35. Solve $(x-5)^2 + (2x-4)^2 = (3x-7)^2 - (2x-3)^2$.

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36. Find the L. C. M. of $x^2 - 1$, $x^3 + 1$ and $x^2 - x + 1$.

37. Simplify $\dfrac{a-b}{a+b} + \dfrac{a+b}{a-b}$.

38. If $x = 2$, $y = 3$ and $z = 5$, find the value of

$$\frac{y-z}{x} + \frac{z-x}{y} + \frac{x-y}{z}.$$

39. Show that the product of two consecutive even integers is 1 less than a square integer.

40. Solve $\dfrac{3}{x} + \dfrac{5}{2x} = 2 + \dfrac{3}{2x}$.

41. Prove that $x(x+1)(x+2)(x+3)+1=(x^2+3x+1)^2$.

42. Infer from Ex. 41, the square root of $12 \times 13 \times 14 \times 15+1$.

43. Solve $\frac{x+4}{x-3}=\frac{2x-5}{2x-8}$.

44. The sum of two numbers is 15. The fraction formed by dividing the less by the greater is equal to $\frac{2}{3}$. Find the numbers.

45. Show that

$$1+\frac{a^2+b^2-c^2}{2ab}=\frac{(a+b+c)(a+b-c)}{2ab}.$$

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46. Simplify

$$\frac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2} \times \frac{(a-b)^2-(c-d)^2}{(a-c)^2-(b-d)^2}.$$

47. A sum of money is to be divided among a number of boys. If 8 cents is given to each there will be 5 cents over; if 9 cents is given to each there will be 5 cents short. Find the number of boys.

48. Solve $\frac{(x+3)(2x+4)}{(x-1)(x+8)}-2=0$.

49. Prove that

$$1+n+\tfrac{1}{2}n(n+1)+\tfrac{1}{6}n(n+1)(n+2)$$
$$=\tfrac{1}{6}(n+1)(n+2)(n+3).$$

50. Show that the square of the sum of two numbers is greater than the sum of their squares by just as much as the sum of their squares is greater than the square of their difference.

ANSWERS.

Ex. 1. (1) $+30$. (2) -30. (3) -30. (4) $+30$. (5) $+5$. (6) -4. (7) -176. (8) A debt of 3 dollars; an asset of 4 dollars. (9) A debt of 5 dollars. (10) An expansion of $1\frac{1}{2}$ inches.

Ex. 2. (1) $+10$. (2) -9. (3) -3. (4) $+21$. (5) $-\frac{1}{3}$. (6) $+2\frac{7}{12}$. (7) $+42$. (8) $198{\cdot}2813$.

Ex. 3. (1) $+2$. (2) $+8$. (3) -8. (4) -2. (5) -46. (6) -30. (7) $+20$. (8) -9. (9) $+20$. (10) -61.

Ex. 4. (1) $+8$. (2) -8. (3) -8. (4) $+8$. (5) $+28$. (6) -334. (7) $+162$. (8) $+$. (9) $+$. (10) $-$.

Ex. 5. (1) $+3$. (2) -3. (3) -3. (4) $+3$. (5) -6. (6) $+70$. (7) $+$. (8) $-$. (9) $-$. (10) $+$.

Ex. 6. (1) -2. (2) 0. (3) -36. (4) -1. (5) -28. (6) $+9$. (7) $+1$. (8) -25. (9) -9. (10) $+31$.

Ex. 7. (2) -18; $+18$; -12; $+18$; $+72$. (4) -24; $+2$; -10; -14; -10; $+24$. (5) $+1$; -23; -46; -15; $+62$; -4; $+2400$. (6) $+10$. (7) -10. (8) $+10$. (9) -4. (10) -10. (11) -10. (12) $+10$. (13) $+4$. (14) $+60$. (15) $+14$. (16) $+60$. (17) -10. (18) $+7$. (19) $+36$. (20) -10.

Ex. 8. (1) $+5a$. (2) $+5abc$. (3) $+8x-8y$. (4) $-y-14x$. (5) $+\frac{23}{60}x$. (6) $+\frac{49}{60}abc$. (7) $+x+y+z$. (8) $+3pq+3p+2q$. (9) $ap+bq-27pq$. (10) $+7x-2xy$. (11) $+18abcd$. (12) $15x+12y+z$. (13) $-21x-23z+10y$. (14) $-5mn+6nl+40lm$. (15) $+12mn$. (16) $-12abc$. (17) $43xyz+95ayz$. (18) $22X-7Y$

Ex. 9. (1) $12a+4b$. (2) $5a$. (3) $7a+22b+c$. (4) $8a-4b+23c$. (5) $49x+36y-24z$. (6) $x+y+z$. (7) $13x+8y+3z$. (8) $17a-19b+26c$. (9) $2c+4d$. (10) $50xy+35yz-19zx$. (11) $3-2ab+6bc$. (12) $-16pqr+45abc-83$. (13) $\frac{27}{2}ab+\frac{7}{6}bc-\frac{39}{4}cd$. (14) $-\cdot 15ab-\cdot 13bc+1\cdot 65cd$. (15) $\frac{11}{6}x-\frac{7}{6}y+\frac{1}{6}xyz$. (16) $25abcd-abde-25bcde$. (17) $43pq+46qr-69pr$. (18) $-29xy-1129xyz+1403xyzw$. (19) $-2x-2y-2w+4a-2b$. (20) $-\frac{5}{6}b-\frac{5}{4}c-\frac{7}{3}a$.

Ex. 10 (1) $4a+2b+2c$. (2) $a-2b$. (3) $2b+2c$. (4) $8x+2y-5z-16w$. (5) $-2a-b+4c+3d$. (6) $23x+13y-21z-w$. (7) $m-n-p$. (8) $3m-3n+3p$. (9) $2ab+bc-ca$. (10) $-10xyz-17yzw+13zwx$. (11) $2xy-2yz-4zx$. (12) $5l-7m+8n$. (13) $-x-y+2z$. (14) $-2bx+5cx$. (15) $-ax+5a+6$.

Ex. 11. (1) $x+y-z$. (2) $x-y+z$. (3) $9x-5y$. (4) o. (5) $x+y+z$. (6) x. (7) $5x-2y$. (8) $38-3a$. (9) $z-x$. (10) $p+q+r$. (11) a. (12) $5a+6b-8c$. (13) $2b+2d$. (14) $-2x-2y$. (15) a. (16) $3x$. (17) $2b-c$. (18) $x-y+7z+7w$. (19) $3a-3b+3c$. (20) $2-5x$.

Ex. 12. (1) $2a-b+(-3c+4d-e)$; $2a-b-(3c-4d+e)$. (2) $a-b+(-c-d-e)+f$; $a-b-(c+d+e)+f$. (3) $2x-3y+(4z-3x-3y)+3z$; $2x-3y-(-4z+3x+3y)+3z$. (4) $2x-y+(-3x-4y-4x)+y$; $2x-y-(3x+4y+4x)+y$. (5) $a+b+(-c-a+b)-c$; $a+b-(c+a-b)-c$. (6) $2x-3y+(4z-5w+6u)$; $2x-3y-(-4z+5w-6u)$. (7) $3a-3b+(-4c-4d-e)$; $3a-3b-(4c+4d+e)$. (8) $2b-4+(-2c-y+3z)$; $2b-4-(2c+y-3z)$. (9) $-x+y+(-z-w+6u)$; $-x+y-(z+w-6u)$. (10) $-\frac{1}{2}x+\frac{1}{3}y+(-\frac{1}{3}z+\frac{1}{4}w-\frac{2}{3}u)$; $-\frac{1}{2}x+\frac{1}{3}y-(\frac{1}{3}z-\frac{1}{4}w+\frac{2}{3}u)$. (11) $a-\{b+(c-d)-e\}$ (12) $2x-\{3y+(4z-5w)\}$. (13) $x-\{y+(z+3)\}$. (14) $ab-\{-bc+(-ca+abc)\}$. (15) $-x-\{-y+(z-w)\}$. (16) $-\frac{1}{2}a-\{-\frac{1}{2}b+(\frac{1}{2}c-\frac{1}{2}d)\}$. (17) $a-\ b+(c-d)-a-d\}$. (18). $x-\{y+(-2z+z)-2x+x-2y-w\}$.

Ex. 13. (1) -12. (2) $+15$. (3) $+10$. (4) $-12ab$. (5) $+15xy$. (6) $+10mn$. (7) $-12b^2$. (8) $+15x^2y$. (9) $+10m^2n^2$. (10) $-12b^5$. (11) $+15x^7$. (12) $+10m^{16}$. (13) $+20a^4b^4c^4$. (14) $-\frac{1}{3}a^6b^5c^5d^5e$. (15) $+10x^7y^3z^4$. (16) $-6m^3n^4p^4q^2$. (17) $-x^3$. (18) $+x^4$. (19) $-8a^3b^3$. (20) $-8a^6b^3$. (21) -243. (22) $+16a^8b^4$. (23) $+a^2b^4c^6$. (24) $-36x^9y^{11}z^3$. (25) $+a^{10}b^5c^5$.

Ex. 14. (1) $ax+bx$. (2) $cm+dm$. (3) $2ax+3bx$. (4) $3cm+5dm$. (5) $8ax+12bx$. (6) $30cm+50dm$. (7) $-4ax-6bx$. (8) $-9cm-15dm$. (9) $-6ac+8bc$. (10) $-8c+16d$. (11) a^2+ab. (12) $cd+d^2$. (13) $2x^3+2x^2$. (14) $6x^3-9x^2$. (15) x^3+2x^2+x. (16) $3x^4-6x^3-6x^2$. (17) $x^4+x^3y+x^2y^2$. (18) $-x^3y-x^2y^2-xy^3$. (19) $x^2y^2+xy^3+y^4$. (20) $-4a^2bc-6ab^2c+8abc^2$. (21) $x^4-x^2y^2$. (22) $x^2y^2-y^4$. (23) $-2x^4yz-2x^3y^2z-2x^3yz^2-2x^2y^2z^2$. (24) $a^3+b^3+c^3-3abc$. (25) $x^4+x^2y^2+y^4$.

Ex. 15. (1) $2x^2+7xy+6y^2$. (2) $6a^2+17ab+5b^2$. (3) $2a^2+7ab+6b^2$. (4) $6x^2+17xy+5y^2$. (5) $2m^2+7mn+6n^2$. (6) $6p^2+17pq+5q^2$. (7) $2a^2+3ab-2b^2$. (8) $6x^2-13xy+6y^2$. (9) $2m^2+3mn-2n^2$. (10) $9m^2-15mn+6n^2$. (11) $2a^2+17a+30$. (12) $21x^2+23x-20$. (13) $x^2+17x+60$. (14) $x^2+16x+60$. (15) $x^2+19x+60$. (16) $x^2+23x+60$. (17) $x^2+32x+60$. (18) $x^2+61x+60$. (19) $x^2-17x+60$. (20) $x^2-16x+60$. (21) $x^2-19x+60$. (22) $x^2-23x+60$. (23) $x^2-32x+60$. (24) $x^2-61x+60$. (25) $x^2-7x-60$. (26) $x^2+7x-60$. (27) $x^2+11x-60$. (28) $x^2-11x-60$. (29) $x^2+4x-60$. (30) $x^2-4x-60$. (31) $x^2+17x-60$. (32) $x^2-17x-60$. (33) $x^2+28x-60$. (34) $x^2-28x-60$. (35) $x^2+59x-60$. (36) $x^2-59x-60$. (37) $9x^2+75x+24$. (38) $9x^2-69x-24$. (39) $9x^2+42x+24$. (40) $9x^2+30x-24$. (41) $9x^2+33x+24$. (42) $9x^2-15x-24$. (43) $9x^2-33x+24$. (44) $9x^2-36x+20$. (45) x^2-y^2. (46) a^2-b^2. (47) m^2-n^2. (48) p^2-q^2.

(49) $4x^2-y^2$. (50) $4x^2-9y^2$. (51) $a^2b^2-c^2$. (52) $p^2q^2-b^2m^2$. (53) a^4-b^4. (54) $a^4-b^2c^2$. (55) $4a^4-9b^4$. (56) $l^2m^2n^2-p^2q^2r^2$.

Ex. 16. (1) $6x^2+3xy+9xz$. (2) $4xy+2y^2+6yz$. (3) $6x^2+7xy+9xz+2y^2+6yz$. (4) $a^2b+a^2c+ab^2+ac^2+b^2c+bc^2+3abc$. (5) $x^2+y^2+z^2+2xy+2yz+2zx$. (6) $a^2+b^2+c^2+2ab+2bc+2ca$. (7) $4x^2+9y^2+16z^2+12xy+24yz+16zx$. (8) $x^3+y^3+z^3-3xyz$. (9) $x^3-y^3+z^3+3xyz$. (10) $x^3-y^3-z^3-3xyz$. (11) $6a^4+13a^3+18a^2+10a-12$. (12) $2x^4-11x^3+18x^2-21x+18$. (13) x^4-y^4. (14) $5m^5+8m^4-5m^3-8$. (15) $12z^5+17z^4+16z^3+10z^2+4z+1$.

Ex. 17. (1) $2x^4-xy^3-8x^2y^2+11xy^3-4y^4$. (2) $x^9+x^8+x^7+x^5-x^4-x^3-x^2-1$. (3) $8a^3+27b^3+c^3-18abc$. (4) $x^3+y^3+3xy-1$. (5) $a^5+a^4b-2a^3b^2-2a^2b^3+ab^4+b^5$. (6) $2x^7+x^6-x^5-8x^4+2x^3-3x^2+4x-1$. (7) $x^6-3x^4y^2+3x^2y^4-y^6$. (8) $x^2y+x^2z+xy^2+xz^2+y^2z+yz^2+2xyz$. (9) $a^2b+a^2c+ab^2+ac^2+b^2c+bc^2+2abc$. (10) $x^3+9x^2+26x+24$. (11) $x^3-9x^2+26x-24$. (12) $x^3+9x^2y+26xy^2+24y^3$. (13) $x^3-9x^2y+26xy^2-24y^3$. (14) x^4+x^2+1. (15) $x^4+x^2y^2+y^4$. (16) x^3-1. (17) $x^4+x^2y^2+y^4$. (18) $12a^2-8a$. (19) $15a^2+10ab$. (20) $10a+92$. (21) 0. (22) $-3a^2+2a$. (23) $a-4b+9c$. (24) $(4a-3b)\,x+(11b-5c)\,y$. (25) $-12bx+4ay$.

Ex. 18. (1) $(4m+5n)\,x^2$. (2) $(2a+5b+c)\,x$. (3) $(a+b)\,x+(c+d)x^2$. (4) $(2+5c)x+(3+4m)x^2$. (5) $(m+n)x^3+(4mn+pq)\,x$. (6) $(2+d)\,x+(e+4)\,x^2+(c+f)\,x^3$ (7) $(a^3b+a^2b^2)x+(a^2b+ab^2)x^2+a^2x^3$. (8) $(2-a)x+(-c-3)x^2+(b-c)\,x^3$. (9) $(-a-b-c)\,x+(a-b)\,x^2+(a-c)\,x^3$. (10) $(3-b)\,x-2ax^2+(b-3a)\,x^3$. (11) $(x+y)\,z$. (12) $(x+y)\,(a+b)$. (13) $x+(a+b)\,(c+d)$. (14) $m^2-(m-n)\,(m+n)$. (15) $(a+b)^3$. (16) $(a+b)\,(a-b)=a^2-b^2$.

Ex. 19. (1) b. (2). a. (3) 1. (4) $3b$. (5) $+4a$. (6) -5. (7) a^5. (8) xy^2. (9) xy^2z^2. (10) $7a^5$. (11) $-4xy^2$. (12)

$10xy^2z^2$. (13) $2b$. (14) $-xy$. (15) $-2p^9q^{10}$. (16) $-8abc^3$. (17) $-bc$. (18) a^2bcd. (19) -72. (20) $5z$.

Ex. 20. (1) $a+b+c$. (2) $a+b+c$. (3) $2a+3b+4c$. (4) $a+2b+3c$. (5) $3a-5b+46c$. (6) $2a-4b+3c$. (7) $2by-ab+3b^2x^2y^2$. (8) $4a^2y^2-6axy^2+5a^3x^2$. (9) $ac+2ad+3cd$. (10) $4a^2b-ab^2c-2c^2d^2+1$.

Ex. 21. (1) $x+2$. (2) $x+2$. (3) $x+3$. (4) $a+1$. (5) $x+2$. (6) $x+6$. (7) $a-2$. (8) $x-7$. (9) $a-4$. (10) $x+2$. (11) $m-2$. (12) $c-6$. (13) $x+y$. (14) $x+y$. (15) $x+2y$. (16) $x+2y$. (17) $a-12b$. (18) $m-7n$. (19) $3x+2$. (20) $x+b$. (21) $x^3+3x^2y+3xy^2+y^3$. (22) m^4-3m^2+4m+1. (23) x^2-x-2. (24) $2y^4-4y^2+1$. (25) x^2-xy+y^2. (26) $x^2y^2-3xy+1$. (27) $2x^2y^2-xyz-3z^2$. (28) $2+x-x^2$. (29) $a^4-a^3bc+a^2b^2c^2-ab^3c^3+b^4c^4$. (30) m^2+2m-3. (31) $x+y+1$. (32) $a-b+1$. (33) $a+b+c$. (34) $x+3y$. (35) $l+2m$. (36) $x-y-z$.

Ex. 22. (1) -7. (2) 14. (3) $-5a+6c$. (4) 1. (5) x^8-x^7-x+1. (6) $a+2b+c$. (7) $2a^4-2b^4$. (8) x^2-2x+4. (9) 17. (10) 0. (11) $-2-x-3x^2$. (12) $7y^2+6xy-x^2$. (13) $1+x+x^2+x^3+x^4$. (14) -6. (15) 8. (16) $-2bc+2ad$. (17) $x^4-5x^2y^2+4y^4$. (18) $3a-24ab$. (19) $x^3+2x^2y+3xy^2+4y^3$. (20) 15. (21) $x^3+3x^2y+3xy^2+y^3$. (22) $3a^2-14ab+8b^2$. (23) 21. (24) $-2\frac{1}{2}$. (25) $a=4$, $b=6$, $c=6$.

Ex. 23. (1) $x^2+2xy+y^2$. (2) $y^2+2yz+z^2$. (3) $m^2+2mn+n^2$. (4) $4x^2+4xy+y^2$. (5) $4y^2+12yz+9z$. (6) $9m^2+24mn+16n^2$. (7) $x^2-2xy+y^2$. (8) $y^2-2yz+z^2$. (9) $m^2-2mn+n^2$. (10) $4x^2-4xy+y^2$. (11) $4y^2-12yz+9z^2$. (12) $9m^2-24mn+16n^2$. (13) $4p^2+4pq+q^2$. (14) $a^2b^2+2abc+c^2$. (15) $a^2b^2+4abc+4c^2$. (16) $49a^2b^2+42ab+9$. (17) $4x^2y^2+20xy+25$. (18) $9m^2n^2-24mna+16a^2$. (19) $x^2+3x+2\frac{1}{4}$. (20) $x^2+\frac{4x}{3}+\frac{4}{9}$. (21) $4x^2+\frac{12x}{5}+\frac{9}{52}$.

(22) $4x^2 - \frac{12x}{5} + \frac{9}{25}$. (23) $4a^2 + 6ab + \frac{9b^2}{4}$. (24) $9m^2 + \frac{6m}{5} + \frac{1}{25}$. (25) $16x^2y^2 + 4xyz + \frac{z^2}{4}$. (26) $49x^2 - \frac{21x}{2} + \frac{9}{16}$. (27) $4x^4 - 2x^2 + \frac{1}{4}$. (28) $4 - 2x^2 + \frac{x^4}{4}$. (29) $20^2 + 2 \times 20 \times 3 + 3^2 = 529$. (30) 1024. (31) 3721. (32) 10609. (33) 10404. (34) 11025. (35) $10100\frac{1}{4}$. (36) $9900\frac{1}{4}$. (37) 1006009. (38) $4002000\frac{1}{4}$. (39) $2xy$. (40) $-2xy$. (41) y^2. (42) $4y^2$. (43) 25. (44) 9.

Ex. 24. (1) $(x+y)^2$. (2) $(x-y)^2$. (3) $(a+b)^2$. (4) $(c-d)^2$. (5) $(x+2y)^2$. (6) $(x-2y)^2$. (7) $(2x+1)^2$. (8) $(a+2b)^2$. (9) $(a+bc)^2$. (10) $(3p-2q)^2$. (11) $(4y-1)^2$. (12) $(x+\frac{1}{2})^2$. (13) $\left(\frac{x}{2}+1\right)^2$. (14) $\left(\frac{2m}{3}+1\right)^2$. (15) $(2ab-c)^2$. (16) $\left(\frac{3x}{4}-\frac{4}{3x}\right)^2$. (17) $\left(\frac{x}{y}+\frac{y}{x}\right)^2$. (18) $(4-x)^2$. (19) $(5a-\frac{1}{5})^2$. (20) $\{(a+b)+c\}^2$. (21) $\{(x+y)+z\}^2$. (22) $\{(a+b)+(c+d)\}^2$.

Ex. 25. (1) $(m+n)^2$. (2) $(p+q)^2$. (3) $(2x+y)^2$. (4) $(m+3)^2$. (5) $(p+4)^2$. (6) $(2x+3)^2$. (7) $(m-n)^2$. (8) $(p-q)^2$. (9) $(2x-y)^2$. (10) $(m-3)^2$. (11) $(p-4)^2$. (12) $(2x-3)^2$. (13) $(x-8)^2$. (14) $(y+1)^2$. (15) $(z-2)^2$. (16) $\left(x+\frac{3}{2}\right)^2$. (17) $\left(y-\frac{5}{2}\right)^2$. (18) $\left(2z+\frac{5}{4}\right)^2$. (19) $(xy+2)^2$. (20) $(2ab+2)^2$. (21) $\left(pq-\frac{5}{2}\right)^2$. (22) $\left(4xy-\frac{1}{4}\right)^2$. (23) $\left(\frac{x}{2}+1\right)^2$. (24) $\left(\frac{y}{3}+1\right)^2$. (25) $(m+n)^2$ or $(m-n)^2$. (26) $(p+q)^2$ or $(p-q)^2$. (27) $(1+a)^2$ or $(1-a)^2$ (28) $(1-a)^2$. (29) $(3+m)^2$. (30) $(8a+1)^2$. (31) $(2mn+1)^2$. (32) $(3m+mn)^2$. (33) $\left(x+\frac{a}{2}\right)^2$. (34) $\left(x+\frac{b}{2}\right)^2$. (35) $(x+c)^2$. (36) $\left(2x+\right.$

$\frac{a}{4})^2$. (37) $\left(2x+\frac{b}{4}\right)^2$. (38) $(2x+1)^2$. (39) $\left(\frac{1}{2}+x\right)^2$. (40) $\left(\frac{1}{2}+2x\right)^2$ or $(\frac{1}{2}-2x)^2$. (41) $\left(\frac{1}{x}-x\right)^2$. (42) $\left(\frac{3}{2}z+\frac{1}{3}\right)^2$.

Ex. 26. (1) x^2-y^2. (2) m^2-n^2. (3) b^2-c^2. (4) b^2-4c^2. (5) $4m^2-n^2$. (6) $4m^2-9n^2$. (7) x^2-49. (8) x^2-16. (9) x^2-1. (10) $1-x^2$. (11) $1-4x^2$. (12) $9-4x^2$. (13) $4a^2b^2-c^2$. (14) $a^2b^2-4c^2$. (15) $16h^2m^2-25p^2$. (16) $4a^2b^2-\frac{1}{4}$. (17) a^4-b^4. (18) $4a^4-b^2c^2$. (19) $a^2+2ab+b^2-c^2$. (20) $a^2+2ab+b^2-c^2$. (21) $4a^2+4ab+b^2-4c^2$. (22) $4a^2+12ab+9b^2-4c^2$. (23) $l^2+2ln+n^2-m^2$. (24) $4a^2+12ac+9c^2-b^2$. (25) $a^2-b^2+2bc-c^2$. (26) $4a^2-b^2+6bc-9c^2$. (27) x^4+x^2+1. (28) $a^4x^4+a^2x^2+1$. (29) $x^4+x^2y^2+y^4$. (30) $1+x^4+x^8$.

Ex. 27. (1) $(c+d)(c-d)$. (2) $(m+n)(m-n)$. (3) $(q+r)(q-r)$. (4) $(ax+y)(ax-y)$. (5) $(x+yz)(x-yz)$. (6) $(xy+zw)(xy-zw)$. (7) $(2m+n)(2m-n)$. (8) $(3p+2q)(3p-2q)$. (9) $(4x+3y)(4x-3y)$. (10) $(2+x)(2-x)$. (11) $(3+2y)(3-2y)$. (12) $\left(1+\frac{x}{2}\right)\left(1-\frac{x}{2}\right)$. (13) $(2ab+xy)(2ab-xy)$. (14) $(x^2+3)(x^2-3)$. (15) $(4+y^3)(4-y^3)$. (16) $(7abc+3)(7abc-3)$. (17) $(ab^2c+d^2)(ab^2c-d^2)$. (18) $(1+4abc)(1-4abc)$. (19) $(5+4pq^2)(5-4pq^2)$. (20) $(1+5pq)(1-5pq)$. (21) $(9a+5b)(9a-5b)$. (22) $(a^3+5b)(a^3-5b)$. (23) $(a^2+b^2)(a+b)(a-b)$. (24) $(a^4+b^4)(a^2+b^2)(a+b)(a-b)$. (25) $(4+a^2)(2+a)(2-a)$. (26) $(25x^2+y^2)(5x+y)(5x-y)$. (27) $\left(\frac{1}{5}+xy\right)\left(\frac{1}{5}-xy\right)$. (28) $(b^4c^4+16)(b^2c^2+4)(bc+2)(bc-2)$. (29) $(x^4+y^4)(x^2+y^2)(x+y)(x-y)$. (30) $(a^4b^4+c^4)(a^2b^2+c^2)(ab+c)(ab-c)$.

(31) $(27+23)(27-23) = 50\times 4 = 2^4\cdot 5^2$. (32) $2^4\cdot 5^2\cdot 3$. (33) $5^3\cdot 3^2\cdot 43$. (34) $(a+b+c)(a+b-c)$. (35) $(a+b+c)(a+b-c)$. (36) $(2a+b+c)(2a+b-c)$. (37) $(a-2b+2c)(a-2b-2c)$. (38) $(a+b+c)(a-b-c)$. (39) $(a+b+c)(a-b-c)$. (40) $(a+b-c)(a-b+c)$. (41) $(3x-2y+5z)(3x-2y-5z)$. (42) $4xy$. (43) $(a+b+c+d)(a+b-c-d)$. (44) $(a-b+c+d)(a-b-c-d)$. (45) $(a-2b+2c-d)(a-2b-2c+d)$. (46) $(3+2x-y)(3-2x+y)$. (47) $(a+b-c+d)(a-b-c-d)$. (48) $(2a+3b+c-5d)(2a-3b+c+5d)$. (49) $(x^2+xy+y^2)(x^2-xy+y^2)$. (50) $(x^2+xy+y^2)(x^2-xy+y^2)$. (51) $(x^2+x+1)(x^2-x+1)$.

Ex. 28. (1) x^2+5x+6. (2) $x^2+8x+15$. (3) $x^2+18x+77$. (4) x^2-5x+6. (5) $x^2-8x+15$. (6) $x^2-18x+77$. (7) x^2-x-6. (8) $x^2-2x-15$. (9) $x^2-4x-77$. (10) x^2+x-6. (11) $x^2+2x-15$. (12) $x^2+4x-77$. (13) $m^2+2m-63$. (14) $d^2+2d-63$. (15) $y^2+2y-63$. (16) $m^2n^2+2mn-63$. (17) $d^2e^2-2de-63$. (18) $y^2z^2+2yz-63$. (19) $4x^2-4x-15$. (20) $9y^2+12y-77$. (21) $9z^2-42z+48$. (22) $4m^2n^2+24mn-72$. (23) $9a^2b^2c^2-6abc-63$. (24) $16x^4y^2+32x^2y-65$. (25) $9a^4y^2+72a^2y+119$. (26) x^2+6x+9. (27) $m^2+22m+121$. (28) $4x^2+20x+25$. (29) $4m^2+28m+49$.

Ex. 29. (1) $x^2+3ax+2a^2$. (2) $x^2+7bx+12b^2$. (3) $x^2-2ax-8a^2$. (4) $x^2+bx-12b^2$. (5) $4x^2+8ax+3a^2$. (6) $9x^2+9bx-28b^2$. (7) $16m^2n^2-4mnp-2p^2$. (8) $a^2b^2c^2+3abcx-10x^2$. (9) $x^2-3xy-10y^2$. (10) $9x^2-9xy-10y^2$. (11) $4x^2-32ax^2+63a^2$. (12) $9a^6-100b^2$. (13) $100l^2m^2n^2-40lmnx-45x^2$. (14) $x^2y^2+xyz+xyw+zw$. (15) $(a+b)^2+5(a+b)+6=a^2+2ab+b^2+5a+5b+6$. (16) $m^2+2mn+n^2-m-n-30$. (17) $4m^2+4mn+n^2-10m-5n-150$. (18) $a^2+2ab+b^2+7a+7b+12$. (19) $m^2+2mn+n^2+3m+3n-70$. (20) $x^2+2x-63$. (21) $16m^2n^2-24mn-55$.

(22) $a^2-2ab+b^2+7a-7b-30$. (23) $x^2+4xy+4y^2-5xz-10yz-24z^2$. (24) $4x^2-12xy+9y^2-6xz+9yz-10z^2$. (25) $4m^2n^2+12mnp+9p^2-6mn-9p-28$.

Ex. 30. (1) $(x+3)(x+1)$. (2) $(x+5)(x+1)$. (3) $(x+8)(x+2)$. (4) $(x+4)(x+3)$. (5) $(x+5)(x+3)$. (6) $(x+4)(x+2)$. (7) $(y+7)(y+3)$. (8) $(y+7)(y+1)$. (9) $(y+10)(y+2)$. (10) $(y+12)(y+2)$. (11) $(y-4)(y-2)$. (12) $(y-3)(y-1)$. (13) $(m-4)(m-4)$. (14) $(m-6)(m-4)$. (15) $(m-7)(m-3)$. (16) $(m-8)(m-2)$. (17) $(m-5)(m-5)$. (18) $(m-9)(m-1)$. (19) $(a-12)(a-2)$. (20) $(a-6)(a-4)$. (21) $(a+12)(a+2)$. (22) $(a+6)(a+4)$. (23) $(mn+4)(mn+2)$. (24) $(pq-3)(pq-1)$. (25) $(am-6)(am-2)$. (26) $(am+11)(am+1)$. (27) $(xyz-17)(xyz-3)$. (28) $(ab-19)(ab-1)$. (29) $(a+6)(a-2)$. (30) $(b-6)(b+2)$. (31) $(m-9)(m+7)$. (32) $(m+9)(m-7)$. (33) $(x-7)(x+3)$. (34) $(x+7)(x-3)$. (35) $(y+9)(y-3)$. (36) $(y-9)(y+3)$. (37) $(a+8)(a-4)$. (38) $(b-9)(b+1)$. (39) $(x-2)(x-3)$. (40) $(x+7)(x-2)$. (41) $(b-2)(b-5)$. (42) $(y+11)(y-2)$. (43) $(ax+10)(ax-7)$. (44) $(ax-14)(ax+5)$. (45) $(a+2b)(a+b)$. (46) $(x+3y)(x+2y)$. (47) $(c-16d)(c+d)$. (48) $(n+2p)(n+2p)$. (49) $(x-20y)(x+3y)$. (50) $(x+20y)(x-3y)$. (51) $(2x+5)(2x+1)$. (52) $(3x+5)(3x+7)$. (53) $(a+b+6)(a+b+2)$. (54) $(m+n+12)(m+n-1)$. (55) $(a+b-7c)(a+b+2c)$. (56) $(a+b-3c)(a+b-2c)$.

Ex. 31. (1) $x^2+y^2+z^2+2xy+2yz+2zx$. (2) $x^2+4y^2+z^2+4xy+4yz+2zx$. (3) $x^2+4y^2+9z^2+4xy+12yz+6zx$. (4) $x^2+y^2+z^2-2xy-2yz+2zx$. (5) $x^2+4y^2+z^2+4xy-4yz-2zx$. (6) $x^2+4y^2+9z^2-4xy+12yz-6zx$. (7) $a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$. (8) $a^2+b^2+4c^2+9d^2-2ab+4ac-6ad-4bc+6bd-12cd$. (9) $4x^2+9y^2+16z^2+25w^2-12xy+16xz-20xw-24yz+30yw-40zw$.

(10) $x^4+2x^3+3x^2+2x+1$. (11) $x^4+2x^3y+3x^2y^2+2xy^3+y^4$. (12) $x^4+4x^3y+6x^2y^2+4xy^3+y^4$. (13) $x^6+2x^5+3x^4+4x^3+3x^2+2x+1$. (14) $x^6-2x^5+3x^4-4x^3+3x^2-2x+1$. (15) $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$. (16) $(a+b-c)$. (17) $(x-y+z)$. (18) $(x+2y+3z)$. (19) $(2x-y-3z)$. (20) $(a-b-1)$.

Ex. 32. (1) $x^3+3x^2y+3xy^2+y^3$. (2) $x^3-3x^2y+3xy^2-y^3$. (3) $8x^3+12x^2y+6xy^2+y^3$. (4) $x^3-6x^2y+12xy^2-8y^3$. (5) $27x^3+54x^2y+36xy^2+8y^3$. (6) $27a^3-54a^2b+36ab^2-8b^3$. (7) $x^3+12x^2+48x+64$. (8) $8x^3+12x^2+6x+1$. (9) $1-6x+12x^2-8x^3$. (10) $8x^3+36x^2+54x+27$. (11) $8-36x+54x^2-27x^3$. (12) $8-6m+\frac{3}{2}m^2-\frac{m^3}{8}$. (13) $a^3+b^3+c^3+3a^2b+3a^2c+3ab^2+3ac^2+3b^2c+3bc^2+6abc$. (14) See 13. (15) $a^3+b^3-c^3+3a^2b-3a^2c+3ab^2+3ac^2-3b^2c+3bc^2-6abc$. (16) $x^3+8y^3+27z^3+6x^2y+9x^2z+12xy^2+27xz^2+36y^2z+54yz^2+36xyz$. (17) $8x^3-y^3+z^3-12x^2y+12x^2z+6xy^2+6xz^2+3y^2z-3yz^2-12xyz$. (18) $1-3x+6x^2-7x^3+6x^4-3x^5+x^6$.

Ex. 33. (1) x^3+y^3. (2) x^3-y^3. (3) c^3+d^3. (4) a^3-c^3. (5) $8x^3+a^3$. (6) $8a^3-c^3$. (7) $8a^3+b^3$. (8) $8x^3-27y^3$. (9) $8a^3+27b^3$. (10) $x^3-\frac{1}{27}$. (11) $x^3+\frac{1}{8}$. (12) a^6-b^6. (13) $a^3c^3+b^6$. (14) $x^6+y^3z^3$. (15) $64x^3+1$. (16) $8+x^3y^3$. (17) $8m^3+27n^3$. (18) a^6-1. (19) $1+a^6$. (20) $8x^3+\frac{1}{8}$. (21) x^2-xy+y^2. (22) x^2+xy+y^2. (23) $a^2-2ab+4b^2$. (24) $4x^2+6xy+9y^2$. (25) $1+2a^2+4a^4$. (26) $4y^2-y+\frac{1}{4}$. (27) $25m^2+15mn+9n^2$. (28) $m-n$. (29) $a+b$. (30) $2x-5y$. (31) $3m+1$. (32) $1-4m$.

Ex. 34. (1) $(p+q)(p^2-pq+q^2)$. (2) $(p-q)(p^2+pq+q^2)$. (3) $(2p+q)(4p^2-2pq+q^2)$. (4) $(p-2q)(p^2+2pq+4q^2)$. (5) $(2p+3q)(4p^2-6pq+9q^2)$. (6) $(3m-1)(9m^2+3m+1)$. (7) $(2-5z)(4+10z+25z^2)$. (8) $(10x^2-y)(100x^2+10x^2y+y^2)$. (9) $(xy+z)(x^2y^2-xyz+z^2)$. (10) $(a+bc)(a^2-abc$

$+ b^2c^2)$. (11) $(a + b + c)(a^2 + 2ab + b^2 - ac - bc + c^2)$. (12) $(a+b+c)(a^2 - ab - ac + b^2 + 2bc + c^2)$. (13) $(a+b-c)(a^2 + 2ab + b^2 + ac + bc + c^2)$. (14) $(a - b - c)(a^2 + ab + ac + b^2 + 2bc + c^2)$. (15) $(a + b - c)(a^2 - ab + ac + b^2 - 2bc + c^2)$. (16) $(a - b + c)(a^2 + ab - ac + b^2 - 2bc + c^2.)$ (17) $(a + b - c)(a^2 + 2ab + b^2 + ac + bc + c^2)$. (18) $(a - b - c)(a^2 + ab + ac + b^2 + 2bc + c^2)$. (19) $(3 - a - b)(9 + 3a + 3b + a^2 + 2ab + b^2)$. (24) $(x - 1)(x + 1)(x^2 - x + 1)(x^2 + x + 1)$. (25) $(2 - y)(2 + y)(4 - 2y + y^2)(4 + 2y + y^2)$.

Ex. 35. (1) $(3x+1)(2x+1)$. (2) $(3x+2)(2x+1)$. (3) $(3x-2)(2x+1)$. (4) $(3x+2)(2x-1)$. (5) $(5x+2y)(x+2y)$. (6) $(5b-2c)(b-2c)$. (7) $(5c+2d)(c-2d)$. (8) $(5x+2y)(x+2y)$. (9) $(4a-5)(3a+4)$. (10) $(3a-4)(4a+5)$. (11) $(3a-4)(4a-5)$. (12) $(12a+5)(a+4)$. (13) $(12a+5)(a-4)$. (14) $(12a-5b)(a-4b)$. (15) $(2x+3y)(4x+5y)$. (16) $(2x-3y)(4x-5y)$. (17) $(2a-3b)(4a+5b)$. (18) $(2a+3b)(4a-5b)$. (19) $(50x+1)(x+1)$. (20) $(8x+3y)(6x+9y)$.

Ex. 36. (1) ab. (2) abc. (3) m^2. (4) ab^2c^2d. (5) $3ab^3c$. (6) $12x^2y$. (7) $5y^2z^2$. (8) $7b^3c^3$. (9) $2b$. (10) x^3. (11) $x+3$. (12) $x+2$. (13) $2(x-7)$. (14) $a+2b$. (15) $x-2y$. (16) $x+2y$. (17) $a-b$. (18) $a+b+c$. (19) $x+1$. (20). $x+y$.

Ex. 37. (1) a^2b^2c. (2) $12m^3n^2$. (3) $24xy^2z^3w^3$. (4) $30a^5b^3c^5$. (5) $(a-b)(a+b)^2$. (6) $(a+b)(a^3+b^3)$. (7) $(2x-1)(2x+1)^2$. (8) $(x-2)(x^3+8)$. (9) $b(b^2-1)$. (10) $(x^3+y^3)(x+y)^2$. (11) $(x+2)(x+3)(x+4)$. (12) $(x+4)(x-4)(x+5)$. (13) $(x+1)(x-1)(x-4)$. (14) $(x+1)(2x-3)^2$. (15) $(a+4b)(a+3b)^2$. (16) $(2x-1)(2x+3)(3x-4)$. (17) $(4b-1)(b+3)^2$. (18) $(a+b-c)(a+b+c)^2$. (19) $(a+b+c)(a+b-c)(a-b-c)$. (20) $(x+2)^2(x+3)^2$.

Ex. 38. (1) 4. (2) -4. (3) -3. (4) -3. (5) 4. (6) $-a$. (7) 6. (8) 8. (9) -5. (10) 3. (11) -7. (12) 5.

(13) $-13\frac{5}{9}$. (14) 5. (15) 10. (16) $\frac{9}{10}$. (17) -5. (18) 12. (19) 5. (20) $1\frac{5}{8}$. (21) $\frac{33}{10}$. (22) $5\frac{5}{11}$. (23) -8. (24) $-1\frac{1}{2}$. (25) $6\frac{1}{2}$. (26) $13\frac{1}{9}$. (27) $2\frac{8}{17}$. (28) $\frac{11}{13}$. (29) 0. (30) -26. (31) $-9\frac{1}{4}$. (32) $9\frac{2}{15}$. (33) $\frac{1}{4}$. (34) $2\frac{1}{3}$. (35) -25. (36) $4\frac{3}{4}$. (37) $2\frac{1}{2}$. (38) $\frac{13}{17}$. (39) $4\frac{11}{21}$. (40) $1\frac{12}{17}$.

Ex. 39. (1) 12. (2) $13\frac{1}{3}$. (3) $17\frac{1}{2}$. (4) 18. (5) 20. (6) 15. (7) $56\frac{1}{4}$. (8) 70. (9) -12. (10) -66. (11) -20. (12) $1\frac{4}{5}$. (13) -7. (14) $-\frac{14}{23}$. (15) $\frac{29}{30}$. (16) 6. (17) 9. (18) 10. (19) 8. (20) 5. (21) $6\frac{32}{43}$. (22) $1\frac{5}{7}$. (23) 5. (24) $5\frac{5}{6}$. (25) 56. (26) $2\frac{2}{5}$. (27) 6. (28) $\frac{11}{63}$. (29) $-\frac{3}{7}$. (30) $3\frac{1}{2}$.

Ex. 41. (1) $2x$. (2) $5x$. (3) $x+2$. (4) $x-4$. (5) $2x+20$. (6) $(x+10)^2$. (7) $x(x+7)$. (8) $x-60$. (9) $x-50$. (10) $100-x$. (11) $40-x$. (12) $2x-40$. (13) $2x-10=30$. (14) $(2x+3)^2=4x^2$. (15) $3x+20=50$. (16) $x+1$. (17) $x+1, x+2, x+3$. (18) $(x+1)(x+2)=56$. (19) $(x+6)^2-20=(x+5)^2$. (20) $(x+1)(x+2)-(x-1)(x-2)=42$.

Ex. 42. (1) $45-x$. (2) 44. (3) \$15. (4) 29. (5) 28. (6) 38, 16. (7) 16, 38. (8) 16. (9) 10. (10) 26. (11) 18. (12) 20. (13) 45. (14) 14. (15) 42. (16) *A* \$42.13. (17) 16. (18) 4. (19) 12. (20) *A* \$43. (21) 4. (22) 14. (23) *A* 42. (24) \$350. (25) 6. (26) 210. (27) Son 8. (28) 28. (29) *A* \$50. (30) \$450. (31) 40. (32) 24. (33) 341. (34) *A* 72. (35) 10 in. (36) 3 horses. (37) $10\frac{10}{11}$ minutes. (38) 29. (39) \$9. (40) 60. (41) *A* 54, (42) *A* 96. (43) 8 dollars. (44) *A* \$125. (45) James 14. (46) 29. (47) *B* \$18. (48) Child 5. (49) 16. (50) 30. (51) $16\frac{4}{11}$. (52) 56. (53) 25. (54) \$16. (55) \$20. (56) 24. (57) 2_{2}. (58) $2\frac{2}{3}$ hrs. (59) 49. (60) Father 54. (61) 21. (62) 120. (63) 240 sov. (64) 240. (65) 28 men. (66) 320.

Ex. 43. (1) $4x^2 - 2xy + y^2$. (2) $12 - 10a$. (3) $(x - 20)(x+28)$, $(x+20)(x-28)$. (4) $\frac{8}{39}$. (5) 25. (6) $16x^2 + 4x + 1$. (7) $a^2+4b^2+c^2+9d^2-4ab-4bc+2ac-6ad+12bd-6cd$. (8) $a^2-4b^2+c^2-9d^2+2ac-12bd$. (9) 1. (10) 12, 4. (11) $x^4-x^3y+x^2y^2-xy^3+y^4$, $x^4+x^3y+x^2y^2+xy^3+y^4$. (12) $2x$. (13) $11a-16$. (14) 4. (15) $(x-y)(x+y)(x^2+y^2)(x^4+y^4)(x^8+y^8)$. (16) $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$. (17) 0. (18) $(8x+z)(8x-z)$. (19) 25. (21) $x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5$. (22) $(x-20)(x+3)$, $(x+5)(x-4)$. (24) $14\frac{1}{5}$. (26) $b^4+b^2c^2+c^4$. (27) $(3x+5y)(5x+3y)$, $(3x-5y)(5x+3y)$. (29) $3\frac{11}{15}$. (31) $x^8+x^6+x^4+x^2+1$. (33) $x^3+8y^3+27z^3-18xyz$. (34) 58, 98. (35) 0. (36) $l^4+l^2m^2+m^4$. (37) $x^2+y^2+z^2-xy-yz-zx$. (38) $x^2+y^2+4z^2-xy-2yz-2zx$. (39) 9, 15. (40) $-1\frac{1}{3}$. (42) $x^2+y^2+z^2+xy+yz-zx$. (43) $20x-21a$. (45) $3\frac{4}{11}$. (46) $9x^2+y^2+z^2-6xy-6xz+2yz$. (47) $(a-b-c)(a+b-c)$. (48) $x^3+9x^2+26x+24$. (49) $x^2+y^2+9z^2+xy+3xz-3yz$. (50) $-\frac{2}{5}$. (51) x^6-1. (53) $(x+y)(x+2y+1)$. (54) 45, 27. (55) $-8\frac{21}{68}$. (58) $(3a+4b)(4a-5b)$. (59) $-(x^2+z^2)$. (60) 28. (62) $x^2-xy+y^2-2x+y+1$. (63) 0. (64) 10. (65) $-120\frac{2}{3}$. (66) $x^2+7x-18$. (67) 0. (68) 3. (69) $m^2+4n^2+p^2+4q^2+4mn-2mp+4mq-4np+8nq-4pq$. (70) 0.

Ex. 44. (1) $\frac{4}{3x}$. (2) $\frac{a}{2}$. (3) $\frac{2}{3a}$. (4) $\frac{3}{4ab}$. (5) $\frac{4x}{5y}$. (6) $\frac{5lm}{6p^2}$. (7) $\frac{a}{b}$. (8) $\frac{y}{x}$. (9) $\frac{a-b}{a+b}$. (10) $\frac{c-d}{c+d}$. (11) $\frac{x+y}{x-y}$. (12) $\frac{a+b}{(a-b)^2}$. (13) $\frac{x+2y}{x+y}$. (14) $\frac{1}{x+y}$. (15) $\frac{m-n}{m-2n}$. (16) $\frac{a+2}{a-2}$. (17) $\frac{p-12q}{p+2q}$. (18) $\frac{(x-1)^2}{x^2+x+1}$. (19) $\frac{x^2-x+1}{x^2+2x+1}$. (20) $\frac{1}{b-1}$. (21) $\frac{1}{x^2+y^2}$. (22) $\frac{a^2+2a}{a^2+2a+1}$. (23) $\frac{2x+3y}{2x+y}$.

(24) $\frac{x^3-1}{x^3+1}$. (25) $\frac{a+b+c}{a-b-c}$. (26) $\frac{x+y-z}{x-y-z}$. (27) $\frac{a-b-c-d}{a+b-c-d}$.

Ex. 45. (1) ac. (2) xz. (3) ab. (4) np. (5) yp. (6) ab. (7) $8xyz$. (8) $12abc^3$. (9) $4ax^3$. (10) ax. (11) an. (12) $9b$. (13) $(a+x)^2$. (14) x^2+5x+6. (15) $(x-7)^2$. 16. a^2+2ab. (17) a^2-a-2. (18) $b^2+2b-15$. (19) $a-b$ (20) $(x-y)^2$. (21) b^2-c^2. (22) $b+5c$. (23) $x+y$. (24) a^2+ab.

Ex. 46. (1) $\frac{2a}{2x}, \frac{b}{2x}$. (2) $\frac{ax}{x^2}, \ldots.$ (3) $\frac{ax}{abc}, \ldots.$ (4) $\frac{a^2}{abc}, \ldots.$ (5) $\frac{2bc}{abc}, \ldots.$ (6) $\frac{a^2b}{a^2b^2x^2}, \ldots.$ (7) $\frac{a^2-ab}{a^2-b^2}, \ldots.$ (8) $\frac{x^2+xy}{x^2-y^2}, \ldots$ (9) $\frac{3x+12}{x^2+7n+12}, \ldots$ (10) $\frac{4a+8}{a^2+6a+8}, \ldots.$ (11) $\frac{a^3-a}{2(a^2-1)}, \ldots.$ (12) $\frac{x+2}{(x+1)(x+2)(x+5)}, \ldots.$ (13) $\frac{2(x+1)^2}{(x+1)^3}, \ldots$ (14) $\frac{(b+c)^2}{(b-c)(b+c)^2}, \ldots.$ (15) $\frac{(x+y)^2}{x^2y^2}, \ldots.$ (16) $\frac{abd(b-d)}{bd(b-d)}, \ldots.$ (17) $\frac{a+b}{(a+b)(b+c)(c+a)}, \ldots.$ (18) $\frac{a^2-b^2}{(a-b)(b-c)(c-a)}, \ldots.$

Ex. 47. (1) $\frac{5x}{6}$. (2) $\frac{17x}{12}$. (3) $\frac{5}{6x}$. (4) $\frac{17}{12x}$. (5) 1. (6) 1. (7) $\frac{x-7y}{x^2-y^2}$. (8) $\frac{x^2-2xy-y^2}{x^2-y^2}$. (9) $\frac{4}{x^2-4}$. (10) $\frac{2a}{a^2-16}$. (11) $\frac{2}{x^2-y^2}$. (12) $\frac{2(x^2+y^2)}{x^2-y^2}$. (13) $\frac{a}{a^2-4}$ (14) $\frac{-y}{x^2-y^2}$. (15) $\frac{2x+4}{(x+1)^2(x+3)}$. (16) $\frac{8-x}{(x^2-1)(x-2)}$. (17) $\frac{x}{(x+1)(x+2)(x+3)}$. (18) $\frac{2a+b}{a+b}$. (19) $\frac{x(2x-y)}{x-y}$. (20) 1. (21) 1. (22) 0. (23) $\frac{-6a^2}{(a^2-1)(a^2-4)}$

(24) $\frac{1 + x - x^2}{x^2 - 1}$.

Ex. 48. (1) $\frac{3}{10}$. (2) $\frac{ab}{4d}$. (3) 1. (4) 1. (5) $\frac{x^3}{yzw}$. (6) $\frac{a^2}{bc}$. (7) $\frac{(a+b)^2}{(a-b)^2}$. (8) $\frac{x+y}{x}$. (9) $\frac{1}{ab}$. (10) 1. (11) $\frac{x}{x-2}$. (12) $\frac{x+y}{x-y}$. (13) 1. (14) 1. (15) $\frac{m}{m-5n}$. (16) $\frac{(x-1)^2}{(x+1)^2}$. (17) $\frac{(a+b+c)^2}{(a+b-c)^2}$. (18) $\frac{x+3}{x+2}$. (19) 1. (20) 1.

Ex. 49. (1) $\frac{a^2}{b^2}$. (2) $\frac{bx}{ay}$. (3) $\frac{2b^3}{5c^3}$. (4) $\frac{m^2}{pq}$. (5) 1. (6) $\frac{x+5y}{x-3y}$. (7) $\frac{(x-2)\ (x+5)}{(x+2)\ (x-3)}$. (8) $\frac{x+1}{x+3}$. (9) 1. (10) $\frac{a-1}{a+1}$. (11) az. (12) 1. (13) $\frac{(x+y)^2}{x^2+y^2}$. (14) $\frac{(a-b)}{ab}\frac{(a^2+b^2)}{(a+b)}$. (15) $\frac{7x+2}{x(x-2)}$. (16) $\frac{a^2-b^2}{2(a^2+b^2)}$. (17) $\frac{a^2+b^2}{a+b}$.

Ex. 50. (1) 2. (2) $\frac{1}{6}$. (3) 3. (4) 2. (5) $2\frac{2}{19}$. (6) 8. (7) 8. (8) 8. (9) 8. (10) 8. (11) 8. (12) 2. (13) 2. (14) $3\frac{2}{7}$. (15) 5. (16) 2. (17) 2. (18) $-\frac{1}{2}$. (19) 0. (20) 7. (21) 3. (22) $1\frac{1}{2}$. (23) 0. (24) 2. (25) 1. (26) $\frac{8}{23}$. (27) $2\frac{21}{29}$. (28) -2. (29) 2. (30) $\frac{2}{5}$. (31) -7. (32) $3\frac{1}{4}$. (33) -2. (34) $4\frac{1}{2}$. (35) $7\frac{3}{4}$. (36) $1\frac{1}{3}$. (37) $-1\frac{3}{5}$. (38) 4. (39) 5.

Ex. 51. (1) 48. (2) $4\frac{2}{5}$. (3) $3\frac{1}{4}$. (4) $3\frac{4}{5}$. (5) 27. (6) 420. (7) 2. (8) 1. (9) $1\frac{3}{19}$. (10) $\frac{4}{23}$. (11) \$$1542\frac{6}{7}$. (12) 5, 7. (13) \$152. (14) 720. (15) \$6948. (16) \$540. (17) 12. (18) 51, 27. (19) \$9600. (20) 40, 60. (21) \$1270. (22) \$300. (23) 27. (24) 17. (25) $\frac{mn}{m+n}$.

(26) $\frac{pq}{q-p}$. (27) $\frac{(x-z)\,y}{x}$. (28) 15. (29) $(m+n)\,t$. (30) $\frac{p}{m+n}$, $\frac{mp}{m+n}$, $\frac{np}{m+n}$.

Ex. 52. (1) $a^2+2ab+b^2-c^2$. (3) $(x-y+z-1)\,(x+y-z-1)$. (4) 11. (5) x^2+3x-9. (7) $2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$. (8) $a-b+c-d$. (9) Identity. (10) 1; 2. (11) x^6-64. (12) $257\frac{1}{2}$. (13) $(50x-1)\,(x-3)$. (14) 1320. (15) $-3\frac{2}{13}$. (16) 0. (17) $a^3+a^2b+a^2c+a^2d+abc+abd+acd+bcd$, a^3+2a^2-5a-6. (18) 1. (19) 5, 15. (20) $2\frac{1}{2}$. (21) $(x-2)\;(x+1)\;(x+3)\;(x+6)$. (22) $1+y+z+y^2-yz+z^2$. (23) $7a-2b-4c$. (24) $ab+bc+ca+c^2$. (25) $-1\frac{1}{15}$. (27) a^3+3a^2+3a+1, (29) $3\frac{9}{11}$. (31) $(15x+y)\,(2x-5y)$. (33) $a^3+b^3+c^3+3a^2b+3a^2c+3ab^2+3ac^2+3b^2c+3bc^2+6abc$, $x^3+8y^3+z^3+6x^2y-3x^2z+12xy^2+3xz^2-12y^2z+6yz^2-12xyz$. (34) $\frac{c^4}{16}$, $4ab+\frac{c^2}{4}$. (35) $-\frac{1}{4}$. (36) $(x-1)\,(x^3+1)$. (37) $\frac{2\,(a^2+b^2)}{a^2-b^2}$. (38) $-\frac{1}{8}$. (40) 2. (42) 181. (43) $4\frac{3}{11}$. (44) 6, 9. (46) 1. (47) 10. (48) 7.

$+n)\ t.$ (30)

$-y-z-1).$
$2c^2a^2-a^4-$
10) 1 ; 2.
$-3).$ (14)
a^2c+a^2d+
. (19) 5,
) $(x+6).$
(24) $ab+$
(29) $3\frac{9}{11}$.
$^2b+3a^2c+$
$^2y-3x^2z+$
$4ab+\frac{c^2}{4}.$
$\frac{b^2)}{2}$. (38)
, 9. (46)