Technical and Bibliographic Notes/Notes techniques et bibliographiques

The Institute has attempted to obtain the best original copy available for filming. Features of this copy which may be bibliographically unique, which may alter any of the images in the reproduction, or which may significantly change the usual method of filming, are checked below.

L'Institut a microfilmé le meilleur exemplaire qu'il lui a été possible de se procurer. Les détails de cet exemplaire qui sont peut-être uniques du point de vue bibliographique, qui peuvent modifier une image reproduite, ou qui peuvent exiger une modification dans la méthode normale de filmage sont indiqués ci-dessous.

- [ ] Coloured covers/ Couverture de couleur
- [ ] Covers damaged/ Couverture endommagée
- [ ] Covers restored and/or laminated/ Couverture restaurée et/ou pelliculée
- [x] Cover title missing/ Le titre de couverture manque
- [ ] Coloured maps/ Cartes géographiques en couleur
- [ ] Coloured ink (i.e. other than blue or black)/ Encre de couleur (i.e. autre que bleue ou noire)
- [ ] Coloured plates and/or illustrations/ Planches et/ou illustrations en couleur
- [ ] Bound with other material/ Relié avec d'autres documents
- [ ] Tight binding may cause shadows or distortion along interior margin/ La reliure serrée peut causer de l'ombre ou de la distortion le long de la marge intérieure
- [ ] Blank leaves added during restoration may appear within the text. Whenever possible, these have been omitted from filming/ Il se peut que certaines pages blanches ajoutées lors d'une restauration apparaissent dans le texte, mais, lorsque cela était possible, ces pages n'ont pas été filmées.
- [ ] Additional comments:/ Commentaires supplémentaires:

- [ ] Coloured pages/ Pages de couleur
- [x] Pages damaged/ Pages endommagées
- [ ] Pages restored and/or laminated/ Pages restaurées et/ou pelliculées
- [ ] Pages discoloured, stained or foxed/ Pages décolorées, tachetées ou piquées
- [x] Pages detached/ Pages détachées
- [ ] Showthrough/ Transparence
- [ ] Quality of print varies/ Qualité inégale de l'impression
- [ ] Includes supplementary material/ Comprend du matériel supplémentaire
- [ ] Only edition available/ Seule édition disponible
- [ ] Pages wholly or partially obscured by errata slips, tissues, etc., have been refilmed to ensure the best possible image/ Les pages totalement ou partiellement obscurcies par un feuillet d'errata, une pelure, etc., ont été filmées à nouveau de façon à obtenir la meilleure image possible.

This item is filmed at the reduction ratio checked below/
Ce document est filmé au taux de réduction indiqué ci-dessous.

10X		12X		14X		16X		18X		20X		22X		24X		26X		28X		30X		32X
														X								

The copy filmed here has been reproduced thanks to the generosity of:

Library of Congress
Photoduplication Service

The images appearing here are the best quality possible considering the condition and legibility of the original copy and in keeping with the filming contract specifications.

Original copies in printed paper covers are filmed beginning with the front cover and ending on the last page with a printed or illustrated impression, or the back cover when appropriate. All other original copies are filmed beginning on the first page with a printed or illustrated impression, and ending on the last page with a printed or illustrated impression.

The last recorded frame on each microfiche shall contain the symbol → (meaning "CONTINUED"), or the symbol ▽ (meaning "END"), whichever applies.

Maps, plates, charts, etc., may be filmed at different reduction ratios. Those too large to be entirely included in one exposure are filmed beginning in the upper left hand corner, left to right and top to bottom, as many frames as required. The following diagrams illustrate the method:

L'exemplaire filmé fut reproduit grâce à la générosité de:

Library of Congress
Photoduplication Service

Les images suivantes ont été reproduites avec le plus grand soin, compte tenu de la condition et de la netteté de l'exemplaire filmé, et en conformité avec les conditions du contrat de filmage.

Les exemplaires originaux dont la couverture en papier est imprimée sont filmés en commençant par le premier plat et en terminant soit par la dernière page qui comporte une empreinte d'impression ou d'illustration, soit par le second plat, selon le cas. Tous les autres exemplaires originaux sont filmés en commençant par la première page qui comporte une empreinte d'impression ou d'illustration et en terminant par la dernière page qui comporte une telle empreinte.

Un des symboles suivants apparaîtra sur la dernière image de chaque microfiche, selon le cas: le symbole → signifie "A SUIVRE", le symbole ▽ signifie "FIN".

Les cartes, planches, tableaux, etc., peuvent être filmés à des taux de réduction différents. Lorsque le document est trop grand pour être reproduit en un seul cliché, il est filmé à partir de l'angle supérieur gauche, de gauche à droite, et de haut en bas, en prenant le nombre d'images nécessaire. Les diagrammes suivants illustrent la méthode.

1	2	3

1
2
3

1	2	3
4	5	6

Hon J. A. Garfield
with the author's compliments

Elementary theorems relating to the geometry of a space of three dimensions and of uniform positive curvature in the fourth dimension.

5

(By *Simon Newcomb*, Washington U. S. of North America.)

The following theorems are founded on the ideas of *Riemann*, as set forth in his celebrated dissertation „Ueber die Hypothesen, welche der Geometrie zu Grunde liegen", though they may not be entirely accordant with his remarks on the result of his theory. It appears not uninteresting to consider the subject from the stand point of elementary geometry instead of following the analytic method which has been commonly employed by writers on the non-Euclidian geometry. The system here set forth is founded on the following three postulates.

1. I assume that space is triply extended, unbounded, without properties dependent either upon position or direction, and possessing such planeness in its smallest parts that both the postulates of the Euclidian geometry, and our common conceptions of the relations of the parts of space are true for every indefinitely small region in space.

2. I assume that this space is affected with such curvature that a right line shall always return into itself at the end of a finite and real distance $2D$ without losing, in any part of its course, that symmetry with respect to space on all sides of it which constitutes the fundamental property of our conception of it.

3. I assume that if two right lines emanate from the same point, making the indefinitely small angle α with each other, their distance apart at the distance r from the point of intersection will be given by the equation

$$s = \frac{2\alpha D}{\pi} \sin \frac{r\pi}{2D}.$$

The right line thus has this property in common with the Euclidian right line that two such lines intersect in only a single point. It may be that the number of points in which two such lines can intersect admits of being determined from the laws of curvature, but not being able so to de-

[Washington, 1877].

termine it, I assume as a postulate the fundamental property of the Euclidian right line. It remains to be seen whether this assumption leads to any conclusions either inconsistent with themselves or to the Euclidian geometry in any small region of space.

The following nomenclature may be used.

A ***complete right line*** is one returning into itself as supposed in postulate 2. Any small portion of it is to be conceived of as a Euclidian right line. It may be called a right line simply when no ambiguity will result therefrom.

The locus of all complete right lines passing through the same point, and lying in the same Euclidian plane containing that point will be called a ***complete plane***.

A ***region*** will mean any indefinitely small portion of space, in which we are to conceive of the Euclidian geometry as holding true. Within any region whatever figures may be designated as Euclidian in order to avoid confusing them with the more complicated relations which have place in the geometry of curved space.

The following propositions are for the most part, presented without demonstration as being either too obvious to require it or obtainable by processes which leave no doubt of their validity. A few will need at least the outlines of a demonstration.

I. From postulates 1. and 2. it follows that ***all complete right lines are of the same length*** $2D$. Hence D is the greatest possible distance at which any two points in space can be situated, it being supposed that the distance is measured on the line of least absolute length. If two moving points start out in opposite directions from a point A on a right line α, they will meet at the distance D in a point which we may designate as $A'\alpha$.

II. ***The complete plane is a Euclidian plane in every region of its extent.*** For, let α, α', and α'' be three successive positions of the generating right line, and let r, r', and r'' be three points each at any distance r from the common point of intersection of the lines α, α' and α''. Then, considering the Euclidian plane containing the line α and the point r', there can, owing to the symmetry of space on each side (postulate 1.) be no reason why the line α' should intersect this plane in one direction rather than in another, it will therefore wholly lie in it. And, from the same postulate, there is no reason why the line α'' should pass on one side of the plane $\alpha r'$ rather than on the other; it will therefore lie in it. Therefore, in every region, the consecutive positions of the generating line lie in the same Euclidian plane.

III. ***Every system of right lines, passing through a common point A and making an indefinitely small angle with each other, are parallel to each other in the region A' at distance D.*** From postulate 3. it follows that in this region we have $\frac{ds}{dr} = 0$, while, by proposition II., every pair lie in the same plane. Conversely, since two points completely determine a right line, it follows that ***all lines which are parallel in the same region intersect in a common point at the distance D from that region.***

IV. ***If a system of right lines pass in the same plane through A, the locus of their most distant points will be a complete right line.*** It is obvious that this locus will be everywhere perpendicular to the generating line, because there is no reason why the angle on one side should be different from that on the other. Moreover, there is no reason why the locus, at any point should deviate to one side of the Euclidian plane containing two consecutive positions of the generating line rather than to the other. It will therefore, in every region, be a Euclidian right line. And, when the generating line has turned through 180°, the most distant point will have returned to its original position: it will therefore have described a complete right line.

V. ***The locus of all the points at distance D from a fixed point A, is a complete plane, and, indeed, a double plane if we consider as distinct the coincident surfaces in which the two opposite lines meet.*** For, let us imagine a series of right lines passing in one plane through a common point A. The locus of their most distant points will then, by the last proposition be a complete right line β. Then, suppose this plane to revolve round a Euclidian right line lying in it at the point A. The locus β will then revolve round the point A' in a plane, and will therefore describe a complete plane.

We have here a partially independent proof of proposition II., since the locus in question must be alike in all its parts. The basis of this second proof is proposition IV. which rests on the basis that the most distant region of a revolving line describes a Euclidian plane.

VI. ***Conversely, all right lines perpendicular to the same complete plane meet in a point at the distance D on each side of the plane.*** This point may be called the pole of the plane, and the plane itself may be called the polar plane of the point. The position of a complete plane in space is completely determined by that of its pole, and *vice versa*. The poles of all planes passing through a point lie in the polar plane of that point.

VII. *For every complete right line, there is a conjugate complete right line such that every point of the one is at distance D from every point of the other.* A line may be changed into its conjugate by two rotations of 90° each around a pair of opposite points.

The three last propositions may be combined as follows. If we call one locus polar to another when every point of the one is at distance D from every point of the other, then, the polar of a point will be a plane, that of a right line will be another right line, and that of a plane will be a point. And, every locus will be completely determined by its polar.

VIII. *Any two planes in space have, as a common perpendicular, the right line joining their poles, and intersect each other in the conjugate to that right line.*

IX. *If a system of right lines pass through a point, their conjugates will be in the polar plane of that point. If they also be in the same plane, the conjugates will all pass through the pole of that plane.*

X. From postulate 3. it may be deduced that the relation between the sides, a, b, and c of a plane triangle in curved space, and their opposite angles A, B, C, will be the same as in a Euclidian spherical triangle of which the corresponding sides are $\frac{a\pi}{2D}$, $\frac{b\pi}{2D}$ and $\frac{c\pi}{2D}$ *). That is, the relation

*) To prove this, in a rectilineal triangle of which a, b and c are the sides, and A, B and C the opposite angles let us consider b and C as constant, and a, c and B as functions of A. To find the differential variations of a, c and B, we substitute dA for α in Postulate III: we then find

$$\frac{da}{dA} = \frac{2D}{\pi \sin B} \sin \frac{c\pi}{2D},$$

$$\frac{dc}{dA} = \frac{2D}{\pi \operatorname{tang} B} \sin \frac{c\pi}{2D},$$

$$\frac{dB}{dA} = -\cos \frac{c\pi}{2D}.$$

The integrals of these equations may be expressed in the form

$$\sin \frac{c\pi}{2D} \sin B = C_1,$$

$$\cos \frac{c\pi}{2D} = \sqrt{1-C_1^2} \cos \frac{\pi}{2D}(a - C_2),$$

$$\cos B = \sqrt{1-C_1^2} \sin (A - C_3),$$

in question is expressed by the formulae

$$\sin\frac{a\pi}{2D}:\sin\frac{b\pi}{2D}:\sin\frac{c\pi}{2D} = \sin A:\sin B:\sin C.$$

From this it follows that the right line is a minimum distance between any two points whether we follow it in one direction or the other, that is, whether we consider it as greater or less than D. For, let A and B be the two points, and P the middle point of a line joining AB, lying near the straight line AB. Since $AP = PB < D$, it is evident that the shortest line joining AB and passing through P is composed of the two right lines $AP+PB$. But, by the formulae of spherical trigonometry we have $AB < AP+PB$, so that AB is a minimum line so long as its product by $\frac{\pi}{2D}$ is less than π, that is, so long as it is less than $2D$.

XI. ***Space is finite, and its total volume admits of being definitely expressed by a number of Euclidian solid units which is a function of D.*** We may conceive space as filled in the following way. If a crowd of beings should proceed to form a sphere of matter by building out on all sides from a common centre, they themselves living on the constantly growing surface, then, just before the sphere attained the radius D each being would see those who were diametrically opposite directly above him, so that, in each region, the only vacant space left would be contained between two Euclidian planes separated by the distance $2(D-r)$, r being the radius of the solid sphere. If the building should be continual until these two surfaces met at every point, all space would be filled.

XII. The third postulate affords us the means of readily determining the elementary relations of circular and spherical figures in space. We

C_1, C_2 and C_3 being arbitrary constants. These constants must be so taken that we shall have simultaneously

$$\begin{aligned} A &= 0, \\ c &= b, \\ a &= 0, \\ B &= 2\pi - C \end{aligned}$$

which conditions give

$$\begin{aligned} C_1 &= \sin C \sin\frac{b\pi}{2D}, \\ \sqrt{1-C_1^2}\cos\frac{\pi C_2}{2D} &= \cos\frac{b\pi}{2D}, \\ \sqrt{1-C_1^2}\sin C_3 &= \cos C. \end{aligned}$$

When the values of the arbitrary constants derived from these equations are substituted in the integrals, we have the fundamental equations of spherical trigonometry.

see at once by the equation

$$ds = \frac{2D}{\pi} \sin \frac{r\pi}{2D} d\alpha$$

that the circumference of a circle of radius r is $4D \sin \frac{r\pi}{2D}$. When $r = D$ the circle will form two complete right lines, as it should, because the two ends of every diameter then meet. For the area of a circle of radius r we readily find the expression

$$\text{Area} = \frac{16D^2}{\pi} \sin^2 \frac{r\pi}{4D}.$$

The area of a complete plane, counting both sides of the surface, is found by putting $r = 2D$, and is therefore $\frac{16D^2}{\pi}$. This is, in fact, easily found to be the surface generated by a complete right line revolving through 360°. The reason for considering the complete plane as a double surface will be seen presently.

The surface of a sphere of radius r is

$$\text{Surface} = \frac{16D^2}{\pi} \sin^2 \frac{r\pi}{2D},$$

and its volume

$$\frac{8D^3}{\pi}\left(r - \frac{D}{\pi} \sin \frac{r\pi}{D}\right).$$

The total volume of space will be found by putting $r = D$, which will give

$$\text{Total volume} = \frac{8D^3}{\pi}.$$

XIII. ***The two sides of a complete plane are not distinct, as in a Euclidian surface.*** If we draw a complete straight line on one side of a plane, it will, at the point of completion, be found on the other side, and must be completed a second time if it is to be closed without intersecting the plane. It will, in fact, be a complete circle of radius D. (prop. XII.) If, in the case supposed in XI., just before space is filled, a being should travel to distance $2D$, he would, on his return, find himself on the opposite surface to that on which he started, and would have to repeat his journey in order to return to his original position without leaving the surface. In this property we find a certain amount of reason for considering the complete plane as a double surface.

XIV. The following proposition is intimately connected with the preceding one. If, moving along a right line, we erect an indefinite series of perpendiculars, each in the same Euclidian plane with the one which precedes it, then, on completing the line and returning to our starting point, the perpendiculars will be found pointing in a direction the opposite of that with which we started.

It may be remarked that the law of curvature here supposed does not seem to coincide with one of the conclusions of ***Riemann***. The latter says: „Man würde, wenn man die in einem Flächenelement liegenden Anfangsrichtungen zu kürzesten Linien verlängert, eine unbegrenzte Fläche mit constantem positiven Krümmungsmaass, also eine Fläche erhalten, welche in einer ebenen dreifach ausgedehnten Mannigfaltigkeit die Gestalt einer Kugelfläche annehmen würde und welche folglich endlich ist." If, by this is meant that if the triply extended curved space became plane space, the complete plane would become a sphere, a discussion of the proposition would be too long to be entered upon here. I cite it only to remark that the complete plane described in the present paper must by no means be confounded with a sphere from which it differs in several very essential characteristics.

α. It has no diameter; a straight line, whether normal to it or not, only intersects it in a single point.

β. The shortest line connecting any two points of it lies upon it.

γ. The locus of the most distant point upon it is not a point, but a right line.

In the same way, the complete right line does not possess the properties of a circle. It does not intersect its normal plane at more than a single point; the most distant point upon it is, on the contrary, at greatest possible distance from the normal plane.

It may be also remarked that there is nothing within our experience which will justify a denial of the possibility that the space in which we find ourselves may be curved in the manner here supposed. It might be claimed that the distance of the farthest visible star is but a small fraction of the greatest distance D, but nothing more. The subjective impossibility of conceiving of the relation of the most distant points in such a space does not render its existence incredible. In fact our difficulty is not unlike that which must have been felt by the first man to whom the idea of the sphericity of the earth was suggested in conceiving how, by travelling in a constant direction, he could return to the point from which he started without, during his journey, finding any sensible change in the direction of gravity.

Washington, 1877.

Abdruck aus dem „Journal für die reine und angewandte Mathematik“ Bd. 88
Druck von G. Reimer in Berlin.