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QUEEN'S COLLEGE
MATHEMATICAL COURSE.

JUNIOR ALGEBRA.

BY

N. F. DUPUIS, M. A.,
PROFESSOR OF MATHEMATICS.

PUBLISHED BY JOHN HENDERSON & Co.
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PREFACE.

The present work comprises the Junior Algebra of the course for the degree of B.A. in Queen's University.

Having been written for the use of students in class it is unlike the majority of works upon algebra offered to beginners. It contains no *very* elementary portions, as the students are supposed to be able to matriculate into the University before taking up the work. It contains no lists of exercises, since it is expected that the Teacher will select or frame such exercises as may best suit his immediate purpose.

The work deals mostly with principles, and the examples, which are fully worked, are introduced for the purpose of exemplifying these principles. For the fuller elucidation of principles a Teacher is supposed to be available.

In the establishing of theorems and formulæ the method followed is inductive rather than analytical, as the former method is believed to be fully as satisfactory as the other, and much more within the grasp of beginners.

The work consists of a small edition, and its production is somewhat of an experiment. If it is found to serve a useful purpose it may be followed by a similar work upon the Junior Geometry of the B.A. course.

N. F. D.

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ERRATA.

The following corrections are necessary:

Page 12, line 5,	for 3	read 2
" 12, Ex. 19, line 5	" x^3z	" x^2z
" 16, line 12	" $x-1$	" $x+1$
" 47, Art. 52, line 4,	" a	" a
" 60, Ex. 103, line 6,	" $2\beta+3\gamma$	" $2\beta-3\alpha$
" 63, Ex. 106′, line 3,	" $4x-a$	" $4b-a$
" 72, Ex. 115, line 1,	" $=1$	" $=11$
" 92, line 3,	" $n2r$	" $n\mathrm{P}2$
" 118, line 2,	" Σ^2	" Σn^2

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Ex.

MULTIPLICATION.

1. Wherever practicable multiplication should be performed in *one line*; by reducing the quantities to be operated upon to the form of binomial factors the multiplication can be readily effected by referring it to the results obtained by the multiplication of well known simple forms.

Fundamental Forms.

i. $(a \pm b)^2 = a^2 + b^2 \pm 2ab.$

ii. $(a+b)(a-b) = a^2 - b^2.$

iii. $(x+a)(x+b)(x+c) = x^3 + x^2 . \overline{a+b+c}$
$+ x . \overline{ab+bc+ca} + abc.$

iv. $(mx+a)(nx+b) = mnx^2 + \overline{mb+na} . x + ab.$

Ex. 1. $(a+b+c)^2 = a^2 + (b+c)^2 + 2a(b+c)$
$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca.$

2. $(x + \frac{1+\sqrt{3}}{2})(x + \frac{1-\sqrt{3}}{2}) = (x + \frac{1}{2} + \frac{1}{2}\sqrt{3})$
$(x + \frac{1}{2} - \frac{1}{2}\sqrt{3}) = (x+\frac{1}{2})^2 - \frac{3}{4} = x^2 + x - \frac{1}{2}.$

3. $(x^2 + ax - b)(x^2 - ax + b) = x^4 + (\overline{ax-b} + \overline{-ax+b})x^2$
$+ (ax-b)(-ax+b) = x^4 - a^2x^2 + b^2.$

4. $(a+b-c+d)(a-b+c-d) = (\overline{a+b} - \overline{c-d})(\overline{a-b} + \overline{c-d})$
$= (a+b)(a-b) + (a+b)(c-d) - (a-b)(c-d) - (c-d)^2$
$= a^2 - b^2 - c^2 - d^2 + 2bc - 2bd + 2cd.$

5. $3(p+q)(q+r)(r+p) = 3(p+q)(qr+qp+r^2+rp)$
$= 3(p^2q + pq^2 + p^2r + pr^2 + q^2r + qr^2 + 2pqr).$

and $(p+q+r)^3=(\overline{p+q}+r)^3=(p+q)^3+3(p+q)^2r$
$+3(p+q)r^2+r^3=p^3+q^3+r^3+3p^2q+3pq^2+3p^2r$
$+3pr^2+3q^2r+3qr^2+6pqr.$

$\therefore \quad 3(p+q)(q+r)(r+p)=(p+q+r)^3-p^3-q^3-r^3.$

---

2. *Symmetry.*—When an expression involves two or more letters in exactly the same way it is said to be symmetrical with respect to these letters. In writing such expressions we usually ignore the alphabetical order of these letters and give attention to their cyclic order. Thus,

$$a^2(b-c)+b^2(c-a)+c^2(a-b)$$

is symmetrical with respect to a, b and c, and in every term the letters follow the same order, $a\ b\ c\ a\ b$.

$$a^2(b+c-d)+b^2(c+d-a)+c^2(d+a-b)+d^2(a+b-c)$$

is symmetrical in a, b, c and d; but

$$(a+b+c+d)^2+(a+b-c-d)^2+(a-b+c-d)^2 \\ +(a-b-c+d)^2$$

is symmetrical in b, c, d, but not in a, since a is positive in every term, while the others are each positive in two terms and negative in two.

The study of symmetrical expressions is of very great importance for many reasons. The principal one at present is that having the expansion of one term of such an expression the expansions of the remaining terms may be written down at sight.

Ex. 6. To find the value of

$$s(s-2a)(s-2b)+s(s-2b)(s-2c)+s(s-2c)(s-2a) \\ -(s-2a)(s-2b)(s-2c) \text{ when } s=a+b+c.$$

This is evidently symmetrical in a, b and c.

$$s(s-2a)(s-2b)=s(s^2-2s.\overline{a+b}+4ab),$$
$$\therefore \quad s(s-2b)(s-2c)=s(s^2-2s.\overline{b+c}+4bc),$$
$$s(s-2c)(s-2a)=s(s^2-2s.\overline{c+a}+4ca);$$

and their sum is, $s(4.\overline{ab+bc+ca}-s^2)$.

But $(s-2a)(s-2b)(s-2c)=s^3-2s^2(a+b+c)$

$+4s(ab+bc+ca)-8abc=s(4.\overline{ab+bc+ca}-s^2)-8abc.$

$\therefore$ the whole expression becomes $8abc$.

7. To show that $8(a+b+c)^3-(a+b)^3-(b+c)^3-(c+a)^3$

$$=3(2a+b+c)(2b+c+a)(2c+a+b).$$

In cases of this kind we may either bring one of these expressions to the form of the other, or we may bring them both to the same third expression. The latter method is usually the simpler one, but the former is a better exercise of ingenuity.

$$8(a+b+c)^3=(2a+2b+2c)^3=(\overline{a+b}+\overline{b+c}+\overline{c+a})^3.$$

$\therefore$ Denoting $a+b$ by p, $b+c$ by q, and $c+a$ by r, we have (Art. 1, Ex. 5)

$$(p+q+r)^3-p^3-q^3-r^3=3(p+q)(q+r)(r+p)$$
$$=3(a+2b+c)(b+2c+a)(c+2a+b)$$
$$=3(2a+b+c)(2b+c+a)(2c+a+b).$$

---

3. *Multiplication by detached Coefficients.*—In multiplying together polynomials with one leading letter it is often advantageous to work upon the coefficients only, and to supply the leading letter after the completion of the work.

Ex. 8. To multiply x^3+3x^2-2x+1 by $2x^2-x+2$

$$\begin{array}{llllll}
1 & +3 & -2 & +1 & & \\
2 & -1 & +2 & & & \\
\hline
2 & +6 & -4 & +2 & & \\
 & -1 & -3 & +2 & -1 & \\
 & & +2 & +6 & -4 & +2 \\
\hline
\end{array}$$

Product : $2x^5+5x^4-5x^3+10x^2-5x+2$

8. To multiply $3x^3-x+2$ by x^3+2x^2-3.

Here we must supply zeros for the coefficients of the missing powers of x,

$$\begin{array}{c}
3+0-1+2 \\
1+2+0-3 \\
\hline
\text{\&c., \&c.}
\end{array}$$

10. To multiply ax^3+bx^2+cx+d by px^2+qx+r.

$$
\begin{array}{l}
a + b + c + d \\
\underline{p + q + r\qquad\quad} \\
ap \;+bp\; +cp\; +dp \\
\qquad +aq\; +bq\; +cq\; +dq \\
\qquad\qquad +ar\; +br\; +cr\; +dr \\
\hline
\end{array}
$$

$$
\begin{array}{r|r|r|r|r|l}
apx^5 + bp & x^4 + cp & x^3 + dp & x^2 + dq & x + dr. \\
aq & bq & cq & cr & \\
 & ar & br & &
\end{array}
$$

By observing the form which the product here assumes, and the manner in which its terms are made up, we may write it down at once in any similar case.

Thus,

$$
\begin{array}{l}
2ax^3+bx^2-cx+1 \\
\underline{\;cx^2-bx\;+\;2\qquad} \\
\end{array}
$$

$$
\begin{array}{r|r|r|r|l}
2acx^5 - 2ab & x^4 + 4a & x^3 + 2b & x^2 - 2c & x+2 \\
bc & -b^2 & +bc & -b & \\
 & -c^2 & +c & &
\end{array}
$$

11. To multiply $2x+3y+z-1$ by $x+y-2z+1$.

We readily see in this case that the product must contain the combinations of letters, x^2, y^2, z^2, xy, yz, zx, x, y, z, and a numerical term. Hence we may arrange as follows:

$$
\begin{array}{llcccccccccc}
2-3+1-1 & \quad & x^2 & y^2 & z^2 & xy & yz & zx & x & y & z & n \\
x \quad y \quad z & & 2 & -3 & -2 & -1 & +7 & -3 & +1 & -4 & +3 & -1 \\
\underline{1+1-2+1} & & & & & & & & & & &
\end{array}
$$

$\therefore$ product is $2x^2-3y^2-2z^2-xy+7yz-3zx+x-4y+3z-1$.

---

4. *Multiplication of Series.*—It often becomes necessary to square a series, or to multiply one series by another. In nearly all such cases the terms of the series are arranged according to the ascending powers of the leading letter. Multiplication of series finds its application in the algebraical development of functions, &c.

Ex. 12. To multiply $a+bx+cx^2+dx^3+\ldots.$
by $a'+b'x+c'x^2+d'x^3+\ldots.$

Product

$$
\begin{array}{r|r|r|l}
aa' + ab' & x + ac' & x^2 + ad' & x^3 + \ldots. \\
ba' & bb' & bc' & \\
 & ca' & cb' & \\
 & & da' &
\end{array}
$$

By observing the mode of formation of the various coefficients we are enabled to perform such multiplications with great facility. For example the coefficient of x^3 is formed by taking the terms,

$$\begin{array}{cccc} a & b & c & d \\ d' & c' & b' & a' \end{array}$$

multiplying each pair together and taking the sum of the products.

Ex. 13. To square the series $1 + ax + bx^2 + cx^3 + \dots$

$$\begin{array}{rl} & 1 + ax + bx^2 + cx^3 + \dots \\ \hline \text{Square} = & 1 + 2ax + \left.\begin{array}{c} 2b \\ a^2 \end{array}\right| x^2 + \left.\begin{array}{c} 2c \\ 2ab \end{array}\right| x^3 + \dots \end{array}$$

14. To show that the square of the series,

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

is formed by writing $2x$ in the place of x.

$$\begin{array}{rl} & 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \\ \hline \text{square} = & 1 + 2x + 2x^2 + \left.\begin{array}{c} \frac{2}{6} \\ 1 \end{array}\right| x^3 + \dots \end{array}$$

$$= 1 + (2x) + \frac{(2x)^2}{2} + \frac{(2x)^3}{6} + \dots$$

Ex. To multiply $1 + x + 2x^2 + 4x^3 + \dots$ by $1 - x - x^2 - x^3 - \dots$

Ex. Show to three terms that if the series $1 - \frac{1}{2}x^2 + \frac{1}{24}x^3 \dots$ and $x - \frac{1}{6}x^3 \dots$ be squared and added the sum is unity.

DIVISION.

5. Division, being the reverse of multiplication, may like that process be carried out upon the coefficients, and when only one letter is involved in the expressions under consideration, the process becomes in this way very much simplified.

If we multiply Ax^2+Bx+C by ax^2+bx+c, we obtain for coefficients,

$$\begin{array}{r|r|r|r} aA + aB & +aC & +bC & +cC \\ bA & bB & cB & \\ & cA & & \end{array}$$

By observing how the terms in this product are formed we may reverse the process and thus perform division. Thus, if $a+b+c$ be the coefficients of the divisor, we see that aA divided by a gives A, the first term of the quotient: then bA subtracted from the second term and the remainder divided by a gives B, the second term of the quotient; and lastly, $cA+bB$ subtracted from the third term and the remainder divided by a gives C, the third term of the quotient.

In this operation the only quantity by which we really divide is a, and hence if this be unity its presence may be quite ignored. Again, since algebraical subtraction is equivalent to addition with a changed sign, we make our subtractions additions by changing the signs of every term, except the first, in the divisor.

Ex. 15. To divide $4x^4-4x^3-5x^2+7x-2$ by $2x^2-3x+1$.

We may write the divisor in any position which is convenient,

as, $\begin{array}{r|l} 2 & 4-4-5+7-2 \\ +3 & +6+3-6+2 \\ -1 & -2-1 \\ \hline & 2+1-2\quad 0\quad 0 \end{array}$ or, $\begin{array}{r} \underline{2+3-1} \\ 4-4-5+7-2 \\ +6+3-6+2 \\ -2-1 \\ \hline 2+1-2\quad 0\quad 0 \end{array}$

and the quotient is $2x^3 - x - 2$.

The second position of the divisor is for some reasons the most convenient.

Ex. 16. To Divide $x^9 - 3x^5 + 6x - 4$ by $x^2 - 2x + 1$.

Here the coefficient of x^2 of the divisor being 1 we may ignore it altogether.

$$\begin{array}{r}
\overline{2-1} \\
1+0+0+0-3+0+0+0+6-4 \\
2\quad 4\quad 6\quad 8\quad 4\quad 0-4-8+4 \\
-1-2-3-4-2\quad 0+2 \\
\hline
1+2+3+4+2\quad 0-2-4\quad 0\quad 0
\end{array}$$

$\therefore$ quotient $= x^7 + 2x^6 + 3x^5 + 4x^4 + 2x^3 - 2x - 4$.

6. If the case is one of inexact division, we must stop the process at a certain point if we wish to obtain the correct remainder. This point is of course reached when the last obtained term of the quotient does not contain x.

In order to determine this point we draw a vertical line to the left of the divisor as usually written, *i. e.* between the first and second terms of the divisor as completely written; all the terms of the quotient proper are to the left of this line, and no term of the quotient line to the right of the vertical is to be used in forming a partial product in getting the remainder.

Ex. 17. Divide $x^7 - x^5 + 5x^3 + 10x^2 - 5x - 1$ by $x^4 - 2x^3 + x^2 - 2$.

$$\begin{array}{r|l}
 & \overline{2-\ 1+0+2} \\
1+0-1+0 & +5+10-5-1 \\
2\quad 4\quad 4 & 4-\ 2\quad 4\quad 4 \\
-1-2 & -2+4 \\
 & 2 \\
\hline
1+2+2+2 & +9+12-1+3
\end{array}$$

$\therefore$ quotient $= x^3 + 2x^2 + 2x + 2$,

and remainder $= 9x^3 + 12x^2 - x + 3$.

7. In cases of exact division the process may be carried out in a somewhat similar manner when several letters are involved.

Ex. 18. To divide $p^2 + pq + 2pr - 2q^2 + 7qr - 3r^2$ by $p - q + 3r$.

Since the dividend is of two dimensions and the divisor of one, the quotient must be of one. We may arrange as follows :

		p^2	q^2	r^2	pq	pr	qr
p	-1	1	-2	-3	1	3	7
q	1				-1	-2	-7
r	-3						
		1	2	-1	.	.	.
		p	q	r			

$\therefore$ quotient $= p+2q-r.$

Here the coefficient of p in the quotient must be 1 so that when multiplied by that of p in the divisor it may cancel that of p^2 in the dividend. Similarly we find that of q and of r. Then for pq we have $-1.2+1.1=-1$ which cancels that of pq in the dividend, &c.

Ex. 19. Divide $2x^3-3x^2y-3x^2z-3xy^2-3xz^2+12xyz+2y^3-3y^2z-3yz^2+2z^3$ by $x+y-2z$.

The quotient may obviously contain all possible terms of two dimensions, and can contain no others.

		x^3	y^3	z^3	x^2y	x^3z	xyz	y^2z	xy^2	xz^2	yz^2
x	-1	2	2	2	-3	-3	12	-3	-3	-3	-3
y	-1				-2	4	-1	4	-2	1	1
z	2						-1	-1	5	2	2
							-10				
		2	2	-1	-5	1	1	.	.	.	.
		x^2	y^2	z^2	xy	xz	yz				

.... possible terms of two dimensions.

$\therefore$ quotient $= 2x^2+2y^2-z^2-5xy+xz+yz.$

---

8. In a case of inexact division, as in example 17, if we neglect the vertical line and its indications the quotient will extend to an indefinite number of terms, which will follow a certain law of formation, and it will thus become an infinite series. This is very similar to cases of inexact division in arithmetic when the quotient is run out into a circulating decimal.

9. *Expansion by Division.*—Let it be required to divide 1 by $1+x$, running the quotient into a series ; we obtain,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - + \ldots .$$

similarly $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + + \ldots .$

By means of these forms we may effect the expansion of any expression which can be expanded by mere division.

Ex. 20. To expand $\dfrac{a}{b+x}$ into an infinite series.

$$\frac{a}{b+x} = \frac{a}{b} \cdot \frac{1}{1+\dfrac{x}{b}} = \frac{a}{b}\left(1 - \frac{x}{b} + \frac{x^2}{b^2} - \frac{x^3}{b^3} + - \ldots\right).$$

Ex. 21. To expand $\dfrac{a^2-1}{2a^2-1}$.

$$\frac{a^2-1}{2a^2-1} = (1-a^2)\cdot\frac{1}{1-2a^2} = (1-a^2)(1 + 2a^2 + \overline{2a^2}^2 + \ldots),$$
$$= 1 + a^2 + 2a^4 + 4a^6 + \ldots .$$

Ex. 22. To expand $\dfrac{1+z}{(1-z)^2}$.

$$\frac{1+z}{(1-z)^2} = (1+z)\cdot\frac{1}{1-2z+z^2} = (1+z)(1 + \overline{2z-z^2} + \overline{2z-z^2}^2 + \ldots)$$
$$= 1 + 3z + 5z^2 + 7z^3 + \ldots .$$

Ex. 23. To expand $\dfrac{x^2-2x-1}{1+x+x^2}$.

This becomes $(x^2-2x-1)(1 - \overline{x+x^2} + \overline{x+x^2}^2 - \ldots),$

$$= \ -1 - x + 3x^2 - 2x^3 - x^4 + 3x^5 \ldots .$$

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SUBSTITUTION.

10. Substitution is the writing of one quantity for another in an expression. Thus,

$Aa^3 + Ba^2 + Ca + D$ is obtained from

$Ax^3 + Bx^2 + Cx + D$, by substituting a for x.

Let X stand for a general expression of the form

$$Ax^n + Bx^{n-1} + \dots Sx + T,$$

where n is an integer greater than unity.

If we divide this by $x - a$ we will obtain another expression one dimension lower, which we may denote by X_1 and, in the case of inexact division, a remainder R which does not contain x.

Hence we may write,

$$Ax^n + Bx^{n-1} + \dots Sx + T = X_1\,(x - a) + R.$$

Then,

i. If we substitute a for x throughout, we get

$$Aa^n + Ba^{n-1} + \dots Sa + T = R.$$

Hence we conclude that if we divide an expression containing only positive integral powers of x by $x - a$ the remainder will be the original expression with a substituted for x.

ii. If $x - a$ divides X exactly, R is nothing; and substituting a for x we have,

$$Aa^n + Ba^{n-1} + \dots Sa + T = 0.$$

Hence if $x - a$ is an exact divisor of an expression containing only positive integral powers of x, the substitution of a for x in the expression causes it to become zero; and conversely, if this substitution renders the expression zero it is exactly divisible by $x - a$.

11. Applications of i. and ii.

Ex. 24. To find the remainder when $x^7 - 3x^3 + 2$ is divided by $x - 1$.

$$\text{Remainder} = 1^7 - 3.1^3 + 2 = 0.$$

$$\therefore\ x^7 - 3x^3 + 2 \text{ is exactly divisible by } x - 1.$$

In a large number of cases the substitution is most readily effected by means of the division itself.

Ex. 25. To find the remainder when $p^5 - 2p^4 + 3p - 10$ is divided by $p - 4$.

$$\begin{array}{rrrrrr|l}
 & & & & & \underline{\ 4\ } & \\
1 & -2 & 0 & 0 & 3 & -10 & \\
 & 4 & 8 & 32 & 128 & 524 & \\
\hline
 & 2 & 8 & 32 & 131 & 514 & = \text{Remainder.}
\end{array}$$

Hence to substitute a for x in X divide the expression X by $x - a$ and take the remainder.

Ex. 26. To find the value of $n^5 - 3n^2 + 2n + 10$ when -3 is substituted for n. Divide by $n + 3$.

$$\begin{array}{rrrrrr|l}
 & & & & & \underline{-3} & \\
1 & 0 & 0 & -3 & 2 & 10 & \\
 & -3 & 9 & -27 & 90 & -276 & \\
\hline
 & -3 & 9 & -30 & 92 & -266 & \therefore\ \text{Result} = -266.
\end{array}$$

Ex. 27. Is $x - 2$ a divisor of $x^4 - 3x^3 - 4x^2 + 2x + 20$?

If we substitute 2 for x we obtain

$$2^4 - 3.2^3 - 4.2^2 + 2.2 + 20 = 0,$$

$$\therefore\ x - 2 \text{ is a divisor.}$$

Ex. 28. Is $a - b$ a divisor of $ab(b - a) + bc(c - b) + ca(a - c)$?

Substitute b for a, and we obtain,

$$b^2.0 + bc(c - b) + cb(b - c) = 0,$$

$$\therefore\ a - b \text{ is a divisor.}$$

12. In transforming equations it frequently becomes necessary to substitute a binomial expression for x in the general expression $Ax^n + Bx^{n-1} + \ldots\ Sx + T$. This may obviously be done by writing the binomial in the place of x, and then expanding, as follows :

Ex. 29. To substitute $y-1$ for x in x^3-3x^2+2x+1.

We have, $(y-1)^3-3(y-1)^2+2(y-1)+1$,

$=y^3-6y^2+11y-5$, by expansion.

The following will enable us to perform this important substitution more readily :

Since $x=y-1$, $\therefore y=x+1$, and the given expression is to be put under the form

$$(x+1)^3+R_2(x+1)^2+R_1(x+1)+R\,;$$

Where we have to determime the remainders R, R_1, R_2.

If we divide the original expression by $x+1$ the remainder is R. If we now set aside this remainder and divide what is left by $x-1$ the remainder is R_1. Proceeding in this way we obtain all the remainders. The whole operation is as follows:

$$\begin{array}{cccc}
 & & & \underline{-1} \\
1 & -3 & +2 & +1 \\
 & -1 & +4 & -6 \\
\hline
 & -4 & 6 \;| & -5 \\
 & -1 & 5 \;| & \\
\hline
 & -5 \;| & 11 & \\
 & -1 \;| & & \\
\hline
 & -6 & &
\end{array}$$

$\therefore$ $(x+1)^3-6(x+1)^2+11(x+1)-5$ is the expression ;

or, $y^3-6y^2+11y-5$.

Ex. 30. Express $3p^5-p^3+4p^2+5p-8$ in terms of $p-2$.

$$\begin{array}{ccccccc}
3 & 0 & -1 & 4 & 5 & -8 & \underline{|\,2} \\
 & 6 & 12 & 22 & 52 & 114 & \\
\hline
 & 6 & 11 & 26 & 57\;| & 106 & \\
 & 6 & 24 & 70 & 192\;| & & \\
\hline
 & 12 & 35 & 96\;| & 249 & & \\
 & 6 & 36 & 142\;| & & & \\
\hline
 & 18 & 71\;| & 238 & & & \\
 & 6 & 48\;| & & & & \\
\hline
 & 24\;| & 119 & & & & \\
 & 6\;| & & & & & \\
\hline
 & 30 & & & & &
\end{array}$$

$\therefore\ 3(p-2)^5+30(p-2)^4+119(p-2)^3+238(p-2)^2$
$+249(p-2)+106$, is the required expression.

Ex. 30′. In $m^3-3m^2n+2mn^2-3n^3$ substitute $m-n$ for m.

1	−3	2	−3	1
	1	−2	0	
	−2	0	−3	
	1	−1		
	−1	−1		
	1			
	0			

$\therefore\ (m-n)^3-n^2(m-n)-3n^3$ is the expression.

13. The following form of substitution is of importance in many operations.

Ex. 31. What does x^7-4x^3+2x-1 become when $x^2+x-1=0$?

This may be solved by division directly as follows:

						−1	1
1	0	0	0	−4	0	2	−1
	−1	1	−2	3	−1	4	−4
		1	−1	2	−3	1	
1	−1	2	−3	1	−4	7	−5

$\therefore\ 7x-5$ is the result.

Or it may be done thus:

$$\because\ x^2+x-1=0,\quad \therefore\ x^2=1-x$$
$$x^4=1+x^2-2x=2-3x.$$
$$\therefore\ x^3=x-x^2=2x-1\ ;$$
$$\text{and } x^7=x^4.x^3=(2-3x)(2x-1)=13x-8.$$
$$\therefore\ x^7-4x^3+2x-1=7x-5.$$

14. We will now extend Art. 11 to the case where the number to be substituted is partly a whole number and partly a decimal.

Ex. 31′. Find the value of x^3-3x^2+2x+1, when $x=2.85$.

In this case we work through for each figure separately, as follows:

	I.	II.	III.	
1	−3	+2	+1	2.85
	2	−2	0	
	−1	0	1 . . .	
	2	2		
	1	2 . .		
	2			
	3 .			
	.8	3.04	4.032	
	3.8	5.04	5.032 . . .	
	8	3.68		
	4.6	8.72 . .		
	8			
	5.4 .			
	05	.2725	.449625	
	5.45	8.9925	5.481625, result.	

The above work is fully expanded in order to show the various steps. We first work through for 2, as in former examples. Then we work through for the 8, remembering that as it is in the tenths place the figures in column I. will be moved one place to the right, two places in column II., and three places in column III. And in like manner for 5.

The work may be very much condensed, as follows :

Ex. 32. Find the value of $y^4 - 4y^2 - 1$ when $y = 2.13$.

1	0	−4	0	−1	2.13
	2	0	0	−1	
	4	8	16	.8081	
	6	20	18081	1.43586161, result.	
	81	2081	20244		
	82	2163	20925387		
	83	2246			
	843	227129			

15. If we have an expression $y^3 - a$ and we put for y any particular value we have seen (Art. 14) how to find the value of the expression. If that value is zero, then $y^3 - a = 0$, and therefore $y = \sqrt[3]{a}$. Hence, if we can discover a quantity which when put for y makes the expression $y^3 - a$ zero, that quantity is a cube root of a. And similar reasoning would apply in the

case of any other root. The process of the last article supplies the means of approximating to this value and thus becomes an elegant means for the arithmetical extraction of roots.

Ex. 33. To find the cube root of 2299968.

1	0	0	−2299968	132
	1	1	+1	
	2	3	−1299	
	33	399	+1197	
	36	507	− 102968	
	392	51484	+ 102968	

Ex. 34. To extract the cube root of 3.

1	0	0	−3	1.44
	1	1	1	
	2	3	−2000	
	34	436	1744	
	38	588	−0256000	
	424	60496	0241984	
	428	62208	..14016	

We know that 1 is the first figure of the root; we, therefore, work through for one. We then find the next figure of the root by employing 4 in the second column as a trial divisor, and 20 in the third column as a dividend; but as the 4 will be increased by the subsequent operation we make a proper allowance in the quotient figure. The principal points to be attended to are that the number carried to the third column must always be less than the number above it from which it is to be subtracted, and that the remainder after subtraction must not be greater than the last completed number in the second column.*

After obtaining 3 or 4 figures the number of figures may be doubled by employing the last completed number in the second column as a divisor and the last remainder in the third as a dividend. Thus dividing 14016 by 6221 we obtain 225; hence the cube root of 3 is 1.44225 true to the last figure. In a precisely similar manner we may extract fifth and seventh roots, &c.

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* In some special questions it may be greater by a small amount.

OF FACTORS AND FACTORING.

16. In the expression

$$a(b-c)(a+b+c)(a^2+2a-1)(ab+bc)$$

a is a monomial factor ; $b-c$ is a binomial factor, and $a+b+c$ is a trinomial factor. These are linear factors, containing terms of only one dimension, while a^2+2a-1 and $ab+bc$ are quadratic factors, inasmuch as they contain terms of two dimensions.

An expression may have real quadratic factors when it has no real linear factors, *e.g.*

$$x^4+3x^2+2=(x^2+1)(x^2+2)$$

in which neither of the quadratic factors has any real linear factor.

17. Theoretically any expression of the form

$$Ax^n+Bx^{n-1}+\ldots\ldots Sx+T$$

may be written as the product of n linear factors containing x, as

$$A(x-\alpha)(x-\beta)(x-\gamma)\ldots\ldots(x-\varsigma),$$

in which the values of α, β, &c., depend upon those of A, B, C, &c.; but practically the discovery of the values of α, β, γ, &c., cannot always be effected by any means at our command, so that the actual process of factoring can be carried out only in special cases. These are, however, frequently of great importance. Only the simpler processes of factoring will be dealt with here.

18. Factoring by reference to known formulæ.

The formulæ more generally useful are :

i. $a^2-b^2=(a+b)(a-b)$.

ii. $a^2+b^2\pm 2ab=(a\pm b)^2$.

iii. $x^2+(a+b)x+ab=(a+x)(b+x)$.

iv. $a^2+b^2+c^2+2(ab+bc+ca)=(a+b+c)^2$.

v. $a^3+b^3+c^3-3abc=(a+b+c)(a.\overline{a-b}+b.\overline{b-c}+c.\overline{c-a})$.

vi. $(a+b+c)^3-a^3-b^3-c^3=3(a+b)(b+c)(c+a)$.

Ex. 35. $x^2-2x-3=x^2-2x+1-4=\overline{x-1}^2-2^2=(x-3)(x+1)$.

or $x^2-2x-3=x^2+(1-3)x+1\times-3=(x+1)(x-3)$.

Ex. 36. $m^2+4m-6=m^2+4m+4-10=(m+2)^2-\sqrt{10}^2$,

$=(m+2+\sqrt{10})(m+2-\sqrt{10})$.

Ex. 37. $a^2+2ab+b^2-a-b-6=(a+b)^2-(a+b)-6$

$=(a+b)^2-(a+b)+\frac{1}{4}-\frac{25}{4}=(a+b-3)(a+b+2)$.

Ex. 37′. $a^3+2b^3-3ab^2=a^3+b^3+b^3-3abb$

$=(a+b+b)(a.\overline{a-b}+b.\overline{b-b}-b.\overline{a-b})$

$=(a+2b)(a-b)^2$.

Ex. 38. $6a^2b+3a^2+12ab^2+12ab+3a+12b^2+6b$

$=3\{2ab(a+2b)+a(a+2b)+2b(a+2b)+a+2b\}$

$=3\{(a+2b)(2ab+a+2b+1)\}$

$=3(a+2b)(2b+1)(a+1)$.

19. If the quantity $pqrs=0$, then one of the factors must be zero, and all may be zero. Conversely if one of the factors be zero, then the product is zero, provided that none of the other factors be infinite.

This principle furnishes a ready means of finding factors when they are rational and not too complex.

Ex. 39. To factor $a(b+bc-c)+b(c+ca-a)+c(a+ab-b)$.

This is symmetrical in a, b, c, therefore if a be a factor b and c will also be factors.

To know if a be a factor put $a=0$. Then it reduces to $bc-bc=0$ $\therefore$ a is a factor, and the literal factors are abc.

Since the highest term in the given expression is of three dimensions, there can be but three literal factors; but there may be a numerical factor. Denote it by n, then,

$$a(b+bc-c)+b(c+ca-a)+c(a+ab-b)=n.abc$$

must be true quite independently of any values which we may give to a, b and c, that is it must be identically true. Make then, $a=1$, $b=2$, $c=3$, and we obtain $18=6n$, $\therefore\ n=3$.

Hence,

$$a(b+bc-c)+b(c+ca-a)+c(a+ab-b)=3abc.$$

Ex. 40. To factor $ab(a-b)+bc(b-c)+ca(c-a)$.

Putting $a=0$, we find no monomial factors. Putting $a-b=0$, or $b=a$ we find $a-b$, and hence from symmetry $b-c$ and $c-a$ to be factors. $\therefore$ the expression is equivalent to $n.(a-b)(b-c)(c-a)$, and we readily find $n=-1$.

$$\therefore\ ab(a-b)+bc(b-c)+ca(c-a)=-(a-b)(b-c)(c-a).$$

Ex. 41. To factor $2ac(2a-c)+2cb(2c-b)+2ba(2b-a)-7abc$.

We readily find this to be equivalent to

$$-(2a-c)(2c-b)(2b-a).$$

Ex. 42. To factor $ab(b^2-a^2)+bc(c^2-b^2)+ca(a^2-c^2)$.

This is symmetrical in a, b, c, and is of four dimensions; hence there are four literal factors.

We readily find that $a-b$, $b-c$, $c-a$ are factors. And since the expression is symmetrical in a, b, c, and can contain only one more factor, it also must be symmetrical in a, b, c. Therefore it can only be $a+b+c$. And the expression is equal to

$$(a-b)(b-c)(c-a)(a+b+c).$$

Ex. 43. To factor $ab(c-d)+bc(d-a)+cd(a-b)+da(b-c)$.

Putting $a-b=0$, or $b=a$,

$$a^2(c-d)+ac(d-a)+da(a-c)=0$$

$\therefore\ a-b$ is a factor, and from symmetry $b-c$, $c-d$, and $d-a$ are factors. But being of only three dimensions it cannot have four literal factors. Therefore, it can have none or it must be identically zero.

20. Since $(x+a)(x+b)(x+c)=x^3+x^2(a+b+c)+x(ab+bc+ca)+abc$,

we see that the independent term abc is the product of the three quantities a, b, c, which with x make up the linear factors $x+a$, $x+b$, and $x+c$. A like relation will be found to exist for any number of factors. Hence in finding a rational linear factor of a rational integral expression involving x in consecutive powers it is necessary to try only the factors of the independent term.

Ex. 44. To find linear factors of $x^4-3x^3-3x^2+7x+6$.

The factors of 6 are ± 1, ± 2, ± 3, ± 6.

Substitute 1 for x; value $=8$ $\therefore$ $x-1$ is not a factor.
" -1 for x; " $=0$ $\therefore$ $x+1$ is a factor.
" 2 for x; " $=0$ $\therefore$ $x-2$ is a factor.

And dividing by $(x+1)(x-2)$ we reduce the expression to x^2-2x-3, whose factors are $(x+1)(x-3)$.

$\therefore$ The whole expression is equivalent to

$$(x+1)^2(x-2)(x-3).$$

Ex. 45. To factor a^4-6a^2-7a-6.

Substituting the various factors of 6 for a we find two linear factors, $(a+2)(a-3)$; and dividing by these we obtain,

$$(a+2)(a-3)(a^2+a+1),$$

the third factor being a quadratic factor which cannot be further reduced.

HIGHEST COMMON MEASURE.

21. If from the expression *acef* and *abcd* we take out the factors common to both, viz., a and c, the product of these is common to both, and is the highest factor which they have in common. Hence it is called the *highest common factor* or *highest common measure* of the quantities, and is usually denoted by H.C.F. or H.C.M.

Hence to find the H.C.M. of two quantities resolve them into factors and take the product of all the factors common to both.

Ex. 46. H.C.M. of $a^3 - ab^2 + a^2b - b^3$ and $a^3 + 3a^2b + 3ab^2 + b^3$.

$$a^3 - ab^2 + a^2b - b^3 = (a+b)(a+b)(a-b),$$

$$a^3 + 3a^2b + 3ab^2 + b^3 = (a+b)(a+b)(a+b),$$

$$\therefore \text{ H.C.M.} = (a+b)^2.$$

22. If two expressions na, nb have a common factor n, their sum, their difference and the sum and the difference of any multiples of the expressions will have the same common factor.

For, $na \pm nb = n(a \pm b)$;

and $na.p \pm nb.q = n(ap \pm bq)$.

This lies at the basis of the common method of finding the H.C.M. of given expressions.

Ex. 47. To find the H.C.M. of $6x^3 - 7x^2 - 9x - 2$ and $2x^3 + 3x^2 - 11x - 6$.

Taking coefficients only,

α . . .	$6 - 7 - 9 - 2$	β . . .	$2 + 3 - 11 - 6$
3β . . .	$6 + 9 - 33 - 18$	3α . . .	$18 - 21 - 27 - 6$
$3\beta - \alpha$ . . .	$16 - 24 - 16$	$3\alpha - \beta$ . . .	$16 - 24 - 16$
Divide by 8,	$2 - 3 - 2$	$\div 8x$ . . .	$2 - 3 - 2$

$$\therefore\ 2x^2 - 3x - 2 = \text{H.C.M.}$$

Ex. 48. H.C.M. of $6x^3+15x^2-6x+9$ and $9x^3+6x^2-51+36$.

Here we can divide through by 3, which will be a factor of the H.C.M.

$$\begin{array}{llr}
\alpha \ldots & & 2+\ 5-\ 2+\ 3 \\
3\alpha \ldots & & 6+15-\ 6+\ 9 \\
2\beta \ldots & & \underline{6+\ 4-34+24} \\
\gamma & & 11+28-15 \\
5\gamma \ldots & & 55+140-75 \\
11\delta \ldots & & \underline{55+198+99} \\
 & & 58+174 \\
\div 58 & & 1+3
\end{array}
\qquad
\begin{array}{llr}
\beta \ldots & & 3+\ 2-17+12 \\
4\alpha \ldots & & \underline{8+20-\ 8+12} \\
\text{diff. } \div x=\delta & & 5+18+\ 9 \\
5\delta \ldots & & 25+90+45 \\
3\gamma \ldots & & \underline{33+84-45} \\
 & & 58+174 \\
\div 58x & & 1+3
\end{array}$$

$\therefore$ H.C.M. $=3(x+3)$.

Ex. 49. H.C.M. of $10y^3+y^2-9y+24$ and $20y^4-17y^2+48y-3$

$$\begin{array}{llr}
\alpha \ldots & & 10+1-9+24 \\
5\delta \ldots & & \underline{10-5+0+15} \\
\epsilon & & 6-9+9 \\
8\delta-\alpha & & 6-9+9
\end{array}
\qquad
\begin{array}{llr}
\beta \ldots & & 20+0-17+48-3 \\
2ay \ldots & & \underline{20+2-18+48} \\
\delta & & 2-\ 1+\ 0+3
\end{array}$$

$\therefore$ H.C.M. $=2y^2-3y+3$.

LEAST COMMON MULTIPLE.

23. The least number of which two given expressions are factors is their least common multiple.

If *acef* and *abed* be two expressions, their L.C.M. is *abcdef* since this is the lowest expression which contains both.

To find the L.C.M. of two quantities we take the factors which are common to both and those which are peculiar to each and multiply them together. Thus a, e, are common to both the foregoing expressions, c, f, are peculiar to one, and b, d, to the other.

Ex. 50. L.C.M. of $x^2-(a-b)x-ab$ and $x^2-2ax+a^2$.

By factoring these become $(x-a)(x+b)$ and $(x-a)^2$,

$\therefore$ $(x-a)$ is common to both, $(x+b)$ is peculiar to the first, and the second $(x-a)$to to the second.

$$\therefore \text{ L.C.M.} = (x-a)^2(x+b).$$

24. If there be more than two quantities we proceed in a similar manner.

Ex. 51. L.C.M. of n^2-3n+2, n^2+2n-3, n^3-2n^2-n+2.

Factoring these become, $(n-1)(n-2)$, $(n-1)(n+3)$, $(n-1)(n+1)(n-2)$,

$$\therefore \text{ L.C.M.} = (n-1)(n-2)(n+1)(n+3).$$

25. The product of any two quantities is equal to the product of their H.C.M. into their L.C.M.

For if $A = abcpq$ be one quantity,

and $B = acprs$ be the other;

their H.C.M. $= a.c.p$,

their L.C.M. $= a.c.p.b.q.r.s$,

$$\therefore \text{ H.C.M.} \times \text{L.C.M.} = a^2c^2p^2bqrs = AB.$$

Hence knowing the H.C.M. of two quantities we find their L.C.M. by dividing their product by their H.C.M.

FRACTIONS.

26. The same principles of operation apply to algebraical as to arithmetical fractions.

i. To add or substract fractions. Bring them to a common denominator, and then add or substract the numerators, as the case may be, and write the common denominator beneath.

ii. To multiply fractions together. Multiply together the numerators for a new numerator, and the denominators for a new denominator.

iii. To divide one fraction by another. Invert the divisor and perform multiplication.

Fractions have such a multiplicity of forms that no general method of working can be laid down. It is frequently advantageous to factor the parts when possible.

A number of examples is here given.

Ex. 52. $$\frac{3x^2+x-2}{2x^2-x-3}=\frac{(3x-2)(x+1)}{(2x-3)(x+1)}=\frac{3x-2}{2x-3}.$$

Ex. 53. $$\frac{a^3-3a^2+3a-2}{3a^3-4a^2+4a-1}=\frac{(a^2-a+1)(a-2)}{(a^2-a+1)(3a-1)}=\frac{a-2}{3a-1}.$$

Ex. 54. $$\frac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}}=\frac{(x+1)^2+(x-1)^2}{(x+1)^2-(x-1)^2}=\frac{2x^2+2}{4x}=\frac{x^2+1}{2x}.$$

Ex. 55. $$\frac{x^2+\left(\dfrac{a}{b}+\dfrac{b}{a}\right)xy+y^2}{x^2+\left(\dfrac{a}{b}-\dfrac{b}{a}\right)xy-y^2}=\frac{x^2+\left(\dfrac{a}{b}+\dfrac{b}{a}\right)xy+\dfrac{a}{b}.\dfrac{b}{a}y^2}{x^2+\left(\dfrac{a}{b}-\dfrac{b}{a}\right)xy-\dfrac{a}{b}.\dfrac{b}{a}y^2}$$

$$=\frac{(x+\frac{a}{b}y)(x+\frac{b}{a}y)}{(x+\frac{a}{b}y)(x-\frac{b}{a}y)}=\frac{ax+by}{ax-by}.$$

Ex. 56. $\dfrac{y}{(x-y)(x-z)}+\dfrac{x}{(y-x)(y-z)}+\dfrac{x+y}{(z-x)(z-y)}$.

By putting in one factor in each denominator and arranging in cyclic order we have,

$$\frac{-y(y-z)}{(x-y)(y-z)(z-x)}+\frac{-x(z-x)}{(x-y)(y-z)(z-x)}+\frac{-(x+y)(x-y)}{(x-y)(y-z)(z-x)}$$

$$=\frac{z(y-x)}{(x-y)(y-z)(z-x)}=-\frac{z}{(y-z)(z-x)}=\frac{z}{(y-z)(x-z)}.$$

Ex. 57. $\dfrac{a}{(c-a)(a-b)}+\dfrac{b}{(a-b)(b-c)}+\dfrac{c}{(b-c)(c-a)}$

$$=\frac{a(b-c)+b(c-a)+c(a-b)}{(a-b)(b-c)(c-a)}=0$$

since the numerator is zero identically.

Ex. 58. $$\frac{a}{(a-b)(b-c)(x-a)}+\frac{b}{(b-a)(b-c)(x-b)}$$

$$+\frac{c}{(c-a)(c-b)(x-c)}.$$

The common denominator is $(a-b)(b-c)(c-a)(x-a)(x-b)(x-c)$.

The first numerator $=$ $-a(b-c)(x-b)(x-c)$
and by symmetry the others are $\begin{cases}-b(c-a)(x-c)(x-a)\\ -c(a-b)(x-a)(x-b)\end{cases}$

$\therefore$ whole numerator $=-\{a(b-c)(x-b)(x-c)+b(c-a)(x-c)(x-a)+c(a-b)(x-a)(x-b)\}$.

Now $a-b$ is a factor of this, and therefore $b-c$ and $c-a$ are factors. To find the fourth factor which probably contains x, let the factors be,

$$(m+nx)(a-b)(b-c)(c-a)$$

where n is numerical and m may be so. To find them put $a=2$, $b=1$, $c=0$, and we obtain, $m=0$, $n=1$,

$\therefore$ the factors are, $x(a-b)(b-c)(c-a)$,

which reduces the whole fraction to

$$\frac{x}{(x-a)(x-b)(x-c)}.$$

Ex. 59.

$$\frac{bc}{(a-b)(b-c)(c-d)}+\frac{cd}{(b-c)(c-d)(d-a)}+\frac{da}{(c-d)(d-a)(a-b)}$$
$$+\frac{ab}{(d-a)(a-b)(b-c)}.$$

The numerator in this case becomes,

$$ab(c-d)+bc(d-a)+cd(a-b)+da(b-c).$$

$a-b$ is a factor of this, and from symmetry $b-c$, $c-d$, and $d-a$ are factors.

But it cannot have four literal factors, therefore it must be zero identically ; hence the sum of the factions is zero.

27. The following relations among the terms of fractions can often be employed with great advantage in algebraical transformations. They are useful in reducing fractional expressions, and they lie at the basis of the relations employed in proportion.

I. If $\dfrac{a}{b}=\dfrac{c}{d}$;

1. $\dfrac{a}{c}=\dfrac{b}{d}$, for $ad=bc$ in both cases by mere cross multiplication.

2. $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$. For $\dfrac{a}{b}+1=\dfrac{c}{d}+1$, and $\dfrac{a}{b}-1=\dfrac{c}{d}-1$;

$$\therefore \quad \frac{a+b}{b}=\frac{c+d}{d} \text{ and } \frac{a-b}{b}=\frac{c-d}{d}.$$

$$\therefore \quad \frac{a+b}{b}\cdot\frac{b}{a-b}=\frac{c+d}{d}\cdot\frac{d}{c-d},$$

$$\text{or} \quad \frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

II. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$,

3. $\dfrac{a}{b}=\dfrac{ma+nc+pe}{mb+nd+pf}$.

Let $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = z,$

then $a = bz$, $c = dz$, $e = fz$;

$$\therefore\ ma + nc + pe = z(mb + nd + pf),$$

$$\therefore\ z = \frac{a}{b} = \frac{ma + nc + pe}{mb + nd + pf}.$$

Cor. If $m = n = p$, $\dfrac{a}{b} = \dfrac{a + c + e}{b + d + f}$;

and this is true for any number of terms.

III. If $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{a}$

4. $\dfrac{a}{c} = \dfrac{a}{b} \cdot \dfrac{b}{c} = \dfrac{a}{b} \cdot \dfrac{a}{b} = \dfrac{a^2}{b^2}.$

5. $\dfrac{a}{d} = \dfrac{a}{b} \cdot \dfrac{b}{c} \cdot \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{a}{b} \cdot \dfrac{a}{b} = \dfrac{a^3}{b^3}.$

Ex. 60. If $\dfrac{x}{2y - z} = \dfrac{y}{2z - x} = \dfrac{z}{2x - y}$, then each fraction $= 1$.

For $\dfrac{x}{2y - z} = \dfrac{x + y + z}{(2y - z) + (2z - x) + (2x - y)} = \dfrac{x + y + z}{x + y + z} = 1.$

Ex. 61. If $\dfrac{x}{a(y + z)} = \dfrac{y}{b(z + x)} = \dfrac{z}{c(x + y)}$,

then $\dfrac{x}{a}(y - z) + \dfrac{y}{b}(z - x) + \dfrac{z}{c}(x - y) = 0.$

Multiply the first fraction, both numerator and denominator, by $\dfrac{y - z}{a}$, and it becomes $\dfrac{\dfrac{x}{a}(y - z)}{y^2 - z^2}$;

and similarly for the other fractions.

Then $$\frac{x}{a(y + z)} = \frac{\dfrac{x}{a}(y - z) + \dfrac{y}{b}(z - x) + \dfrac{z}{c}(x - y)}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2};$$

And since the denominator is necessarily zero, the numerator must also be zero in order that the first fraction may be finite.

Ex. 62. If $\frac{x^2-yz}{a^2}=\frac{y^2-zx}{b^2}=\frac{z^2-xy}{c^2}=1$, to show that

$$(x+y+z)(a^2+b^2+c^2)=a^2x+b^2y+c^2z$$

Multiply the numerator and denominator of the first fraction by x, of the second by y and of the third by z. Then summing numerators and denominators,

$$\frac{x^3+y^3+z^3-3xyz}{a^2x+b^2y+c^2z}=1\ ;$$

or factoring, $$\frac{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}{a^2x+b^2y+c^2z}=1.$$

But from the original fractions,

$$x^2+y^2+z^2-xy-yz-zx=a^2+b^2+c^2\ ;$$

$$\therefore\ (x+y+z)(a^2+b^2+c^2)=a^2x+b^2y+c^2z.$$

RATIO.

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28. The *ratio* of one quantity to another is the numerical quotient which arises from dividing the one quantity by the other, or it is the number which expresses how often the one quantity is contained in the other. Hence a ratio is an abstract quantity, and in order that magnitudes may have a ratio the one to the other, they must be of the same kind. Thus there can be no ratio between miles and years although there is between numbers expressing aggregates of miles and years.

If a, b, c, denote certain lengths, a has a certain ratio to b and to c, but a has no ratio to bc, since bc denotes an area. And thus in geometrical applications of Algebra the terms of a ratio must be homogeneous. But if a, b, c, denote numbers, any combination of them may be employed as terms of a ratio.

The ratio of a to b may be expressed either as $\frac{a}{b}$, or $a:b$. In any case a is the antecedent and b the consequent. If a is greater than b it is a ratio of *greater inequality*, if less than b of *less inequality*, and if equal, a ratio of *equality*. The ratio $b:a$

is the *inverse ratio* of $a:b$. The ratio $a^2:b^2$ is the *duplicate ratio* of $a:b$; $a^3:b^3$ the *triplicate*, and $a^{\frac{3}{2}}:b^{\frac{3}{2}}$ is sometimes called the *sesquiplicate* ratio of $a:b$.

The doctrine of ratio is extremely important in modern mathematics, for it frequently happens that the terms of a ratio are of little or no importance while the ratio itself is all-important.

We have examples of ratio in the circular measure of angles, in sines, cosines, tangents, specific weight, &c.

The propositions of Art. 27 apply directly to ratios as fractions. From these it is evident that a ratio is not changed when both terms are multiplied or divided by the same quantity.

29. Let $a:b$ be a given ratio; then dividing both terms by x, $\dfrac{a}{x}:\dfrac{b}{x}$ is the same as $a:b$; but when x becomes infinitely great each term becomes infinitely small. Hence quantities which become infinitely small, and are thence called vanishing quantities, may have a definite and finite ratio. This principle lies at the foundation of the Differential Calculus.

Ex. 63. What is the ratio of $(a+x)^2-a^2$ to x when x becomes infinitely small?

$$(a+x)^2-a^2=2ax+x^2$$

$\therefore$ ratio $=\dfrac{2ax+x^2}{x}=2a+x=2a$ when x becomes infinitely small.

Ex. 64. To find the ratio of $3x^2-2x+2$ to x^2+x-1 when x becomes infinitely great.

$$\text{Ratio }=\frac{3x^2-2x+2}{x^2+x-1}=\frac{3-\dfrac{2}{x}+\dfrac{2}{x^2}}{1+\dfrac{1}{x}-\dfrac{1}{x^2}}=3 \text{ when } x=\infty.$$

30. Prop. The addition of the same quantity to both terms of a ratio of inequality brings it nearer to a ratio of equality.

Let $\frac{a}{b}$ be the ratio, which suppose greater than 1.

$$\text{then } \frac{a}{b} - 1 = \frac{a-b}{b}.$$

Now add x to each side, and, $\frac{a+x}{b+x} - 1 = \frac{a-b}{b+x}$.

But $\frac{a-b}{b+x}$ is less than $\frac{a-b}{b}$, $\therefore$ $\frac{a+x}{b+x}$ is nearer unity than $\frac{a}{b}$ is.

A similar proof applies when $a:b$ is less than 1.

31. Ratios are *compounded* by taking the product of the antecedents for a new antecedent, and the product of the consequent for a new consequent.

PROPORTION.

32. When two ratios are equal the terms taken in order are said to be in *proportion*, or to form a proportion.

Thus, if $\frac{a}{b} = \frac{c}{d}$, then a, b, c, d are the consecutive terms of a proportion, which is often expressed as

$$a:b::c:d.$$

a and d are the *extremes*, and b and c the *means*. The terms a, b, as also c, d, constitute a couplet; and the proportion is read,

a is to b as c is to d.

If the terms of the last couplet be divided by c d, we have,

$$a:b::\frac{1}{d}:\frac{1}{c},$$

or a is to b *inversely* as d is to c.

33. The following variations in a given proportion are directly derived from Art. 27; some of them have been distinguished by special names.

If $a:b::c:d$,

1. $a:c::b:d$, ... Alternando.
2. $b:a::d:c$, ... Invertendo.
3. $a+b:b::c+d:d$, ... Componendo.
4. $a-b:b::c-d:d$, ... Dividendo.
5. $a:a+b::c:c+d$, ... Convertendo.
6. $a+b:a-b::c+d:c-d$. sum of first & 2 is to difference
7. $a^n:b^n::c^n:d^n$.
8. $a:b::a+c:b+d$.

If $a:b::c:d::e:f$,

9. $a:b::a+c+e:b+d+f$.
10. $a:b::ma+nc+pe:mb+nd+pf$.
11. $a^n:b^n::a^n+c^n+e^n:b^n+d^n+f^n$.

If $a:b::b:c::c:d$,

12. $a:c::a^2:b^2$.
13. $a:d::a^3:b^3$.

If $a:b::b:c$, then $ac=b^2$, and b is said to be a mean proportfonal between a and c.

Proportions are most readily worked as fractions.

Ex. 65. If $ax+cy:ay+cx::bx-cy:by-cx$, then each ratio is that of x to y.

$$\text{For, } \frac{ax+cy}{ay+cx}=\frac{bx-cy}{by-cx}=\frac{ax+bx}{ay+by}=\frac{(a+b)x}{(a+b)y}=\frac{x}{y}.$$

Ex. 66. If $a:b::b:c$, then, $a+b+c:a-b+c::(a+b+c)^2:a^2+b^2+c^2$.

For, $b^2=ac$ $\therefore$ $a^2+b^2+c^2=(a+c)^2-b^2=(a+b+c)(a-b+c)$; and $(a+b+c)(a^2+b^2+c^2)=(a+b+c)^2(a-b+c)$;

$$\therefore\ a+b+c:a-b+c::(a+b+c)^2:a^2+b^2+c^2.$$

EQUATIONS.

34. When an expression is put equal to another expression or to zero, the result is called an equation. Thus, $ax = b$, $ax + b = c$, $ax + by - c = 0$, are equations.

Fundamentally, equations are of two kinds. Thus, $3x - 6 = 0$ is true only under the *condition* that x is 2, and it is consequently called an equation of condition; but, $2(a - x) - (a - 3x) = a + x$ is true for all values of x, and is said to be identically true. Such equations are *identical* equations, or *identities*. Identities are often distinguished by the sign $\equiv$.

35. The solution of an equation of condition consists in finding such a value for the unknown quantity as will render the equation an identity. Thus to solve

$$3x - 2(x + 1) = 2(1 - 2x) + 6,$$

we must find a number which when put for x will make the expression an identity. We readily find 2 to be such a value.

In solving an equation we consider it as an identity and then proceed upon the selfevident principle that if two equal quantities be modified similarly and simultaneously. they must remain equal throughout all modifications.

36. Equations are divided into degrees measured by the highest dimension of the literal symbol taken as the unknown quantity, in the case of one unknown.

Thus, $ax + y^2 - z^3 = 0$ is of the first degree in x, of the second in y, and of the third in z.

Equations of the first degree are *linears*, of the second degree *quadratics*, of the third *cubics*, of the fourth *quartics*, &c.

LINEAR EQUATIONS OF ONE UNKNOWN QUANTITY.

37. These, being the simplest of all equations, usually offer no special difficulties in their solution.

Ex. 67. $$3(x-2)=\frac{2-3x}{4}+\frac{x-3}{2}$$

$$\therefore\ 3x-6=\frac{2-3x+2x-6}{4}=-\frac{x+4}{4}$$

$$\therefore\ 12x-24+x=-4$$

$$\therefore\ 13x=20 \text{ and } x=\frac{20}{13}.$$

38. It sometimes happens that equations which are not strictly linear can be solved as such, but these are usually *made* for the occasion. Examples of modes of reduction and solution follow.

Ex. 67. $$\frac{3+2x}{2x}-\frac{2x-3}{2x-1}+\frac{2x-5}{x-2}=\frac{4x+7}{2x+2}.$$

$$\therefore\ \frac{3}{2x}+1-\frac{2x-1-2}{2x-1}+\frac{2x-4-1}{x-2}=\frac{4x+4+3}{2x+2},$$

$$\therefore\ \frac{3}{2x}+1-1+\frac{2}{2x-1}+2-\frac{1}{x-2}=2+\frac{3}{2x+2},$$

$$\therefore\ \frac{3}{2x}-\frac{1}{x-2}=\frac{3}{2x+2}-\frac{2}{2x-1};$$

whence $14x^3-22x^2+28x=14x^3-22x^2-14x+12$,

$$\therefore\ 42x=12,$$

and $x=\frac{2}{7}$.

Ex. 69. $\dfrac{z-1}{z-2}+\dfrac{z-5}{z-6}=\dfrac{z-2}{z-3}+\dfrac{z-4}{z-5}$.

Hence, $1+\dfrac{1}{z-2}+1+\dfrac{1}{z-6}=1+\dfrac{1}{z-3}+1+\dfrac{1}{z-5}$,

$$\therefore \frac{1}{z-2}-\frac{1}{z-3}=\frac{1}{z-5}-\frac{1}{z-6},$$

$$\therefore \frac{1}{(z-2)(z-3)}=\frac{1}{(z-5)(z-6)};$$

and the numerators being the same the denominators must be equal;

$$\therefore z^2-5z+6=z^2-11z+30;$$

whence, $z=4$.

The principles of Art. 27 may sometimes be employed.

Ex. 70. $\left(\dfrac{a+x}{a-x}\right)^2=1+\dfrac{cx}{ab}$.

expanding, $\dfrac{a^2+2ax+x^2}{a^2-2ax+x^2}=\dfrac{ab+cx}{ab}$,

$$\therefore \frac{a^2+x^2}{2ax}=\frac{2ab+cx}{cx},$$

num. – denom. $\dfrac{(a-x)^2}{2ax}=\dfrac{2ab}{cx}$,

$$\therefore a-x=2a\sqrt{\frac{b}{c}},$$

$$\therefore x=a-2a\sqrt{\frac{b}{c}}.$$

Ex. 71. $\dfrac{8x+12}{x^2+6x+5}=\dfrac{8x+44}{x^2+10x+21}$.

Hence, $\dfrac{x^2+6x+5}{x^2+10x+21}=\dfrac{8x+12}{8x+44}$,

$$\therefore \frac{x^2+6x+5}{4x+16}=\frac{8x+12}{32};$$

whence $x=2$.

OF INDICES AND SURDS.

39. It being understood as an elementary principle that a^n means $a.a.a.$ &c., to n factors, we propose here to extend this notation to negative and fractional indices and to examine principles of working with such.

i. Since $a^n = a.a.a.$ &c., to n factors,

and $a^m = a.a.a.$ &c., to m factors,

$\therefore \quad a^n.a^m = a.a.a.$ &c., to $n+m$ factors.

But $a^{n+m} = a.a.a.$ &c., to $n+m$ factors.

$\therefore \quad a^n.a^m = a^{n+m}$; . . . . . (A).

And, to multiply powers we add their indices.

This in short is the rule which has been assumed throughout multiplication.

ii. Since $\dfrac{a^m}{a^n} = \dfrac{a.a.a.\text{ \&c., to } m \text{ factors}}{a.a.a.\text{ \&c., to } n \text{ factors}}$,

by dividing both numerator and denominator by $a.a.a.$ &c., to n factors we obtain,

$$\frac{a^m}{a^n} = a.a.a.\text{ \&c., to } m-n \text{ factors.}$$

But $a^{m-n} = a.a.a.$ &c., to $m-n$ factors,

$\therefore \quad \dfrac{a^m}{a^n} = a^{m-n}$; . . . . . (B).

And to divide one power by another, we subtract the index of the divisor from the index of the dividend; and this is the rule assumed throughout division.

iii. If in ii, (B) we make $m=n$ we have

$$1 = a^0.$$

$\therefore$ any quantity raised to the power indicated by zero is equivalent to unity.

iv. If in ii, (B) we make $m=0$ we have

$$\frac{a^0}{a^n}=\frac{1}{a^n}=a^{-n} \quad \ldots\ldots (C).$$

Hence a negative index is to be interpreted as expressing the reciprocal of the quantity expressed by the same index when positive.

v. Since $(a^m)^n=(a.a.\ \&c., \ldots\ldots m \text{ factors})^n$

$=(a.a.\ \&c, \ldots m \text{ fact.})(a.a.\ \&c, \ldots m \text{ fact.})\ \&c \ldots n$ brackets,

$=a.a.a.\ \&c$, to mn factors.

$\therefore \quad (a^m)^n=a^{mn} \quad \ldots\ldots (D).$

Hence, to raise a power to any given power we multiply the index of the first power by the index of the power to which it is to be raised.

vi. In v. (D) if we divide the indices on both sides by n or multiply by $\frac{1}{n}$ we obtain,

$$a^m=a^{mn\cdot\frac{1}{n}}=a^m.$$

But a^m is the n^{th} root of $(a^m)^n$; therefore multiplying an index by $\frac{1}{n}$ is equivalent to extracting the n^{th} root.

Hence, $a^{\frac{1}{2}}$ means the square root of a

$a^{\frac{1}{3}}$ " cube " a, &c.

vii. If we have $a^{\frac{1}{n}}.a^{\frac{1}{n}}.a^{\frac{1}{n}} \ldots$ to m factors, the result must be $a^{\frac{1}{n}\cdot m}=a^{\frac{m}{n}}$.

Hence, $a^{\frac{m}{n}}$ means that the n^{th} root of a is to be raised to the m^{th} power.

Ex. 72. If $\frac{a^n}{a^{1-n}}.a^{1+n-m}=\frac{a^{1+n+m}}{a^{1-n-m}}$, to find a relation between m and n.

Reducing to one line by iv.,

we have $a^{3n-m} = a^{2m+2n}$;

hence $3n - m = 2m + 2n$ and $n = 3m$.

Ex. 73. To simplify, $\dfrac{2^{2n}.4^{1-n}}{3^{n-1}} = \dfrac{8^{4-n}.2^{-1}}{9^{n-2}}$.

Reducing to one line,

$$2^{2n}.2^{2(1-n)}.2^{-3(4-n)}.2^{1} = 3^{n-1}.3^{-2(n-2)}.$$

$$\therefore\ 2^{3n-9} = 3^{3-n}.$$

$n = 3$ evidently satisfies this condition, since we then have $2^0 = 3^0$ or $1 = 1$.

Ex. 74. To simplify $\dfrac{8^{\frac{n}{3}}.2^{n+1}}{4^{\frac{3}{n}}} = 2^n$.

In this case, $(2^3)^{\frac{n}{3}}.2^{n+1} = 2^n.(2^2)^{\frac{3}{n}}$,

or, $2^n.2^{n+1} = 2^n.2^{\frac{6}{n}}$,

$$\therefore\ n + 1 = \frac{6}{n},$$

and n evidently is 2.

40. An expression denoting a root which cannot be exactly obtained, as $\sqrt{2}$, $\sqrt[3]{5}$, &c., is called a *surd*, or *irrational quantity*. Surds are divided into orders depending upon the index of the root to be obtained; if it be a square root we have a *quadratic* surd, if a cube root a *cubic* surd, &c.,

The product of a rational quantity with a surd is known as a *mixed* surd, but when all the factors are under the surd sign it is termed an *entire* surd.

Surds may be indicated by indices as $2^{\frac{1}{2}}$, $5^{\frac{1}{3}}$, &c., and in many cases their properties are best studied in this way; but in the case of quadratic surds, at least, and frequently in the case of other surds, it is more customary to employ the sign $\sqrt{\ }$.

The following principles establish the rules for the working of surds. Let n denote any quantity whatever, integral or fractional; then,

i. Since $a^n.b^n = (ab)^n$, $\therefore\ \sqrt[n]{a}.\sqrt[n]{b} = \sqrt[n]{ab}$.

ii. Since $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$, $\therefore$ $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.

iii. $a\sqrt{b} = \sqrt{a^2}\sqrt{b} = \sqrt{a^2 b}$.

similarly $a\sqrt[3]{b} = \sqrt[3]{a^3}.\sqrt[3]{b} = \sqrt[3]{a^3 b}$.

iv. $\sqrt{a}.\sqrt[3]{a} = a^{\frac{1}{2}}.a^{\frac{1}{3}} = a^{\frac{1}{2}+\frac{1}{3}} = a^{\frac{5}{6}}$.

v. $\sqrt{p^2 q} = \sqrt{p^2}.\sqrt{q} = p\sqrt{q}$.

Ex. 75. $\sqrt{3}.\sqrt{2}.\sqrt{5} = \sqrt{30}$.

Ex. 76. $\sqrt{184} = \sqrt{2^2.46} = 2\sqrt{46}$.

Ex. 77. $\frac{\sqrt{24}}{3\sqrt{2}} = \frac{2\sqrt{6}}{3\sqrt{2}} = \frac{2}{3}\sqrt{3}$.

Ex. 78. $\sqrt[3]{54a^4x^2y^3} = \sqrt[3]{27a^3y^3.2ax^2} = 3ay\sqrt[3]{2ax^2}$.

42. Fractional expressions with a compound surd in the denominator are simplified by rendering the denominator rational. The methods of doing this are shown in the following examples :

Ex. 83. $\frac{1}{2-\sqrt{2}} = \frac{1}{2-\sqrt{2}} \times \frac{2+\sqrt{2}}{2+\sqrt{2}} = \frac{2+\sqrt{2}}{4-2} = 1 + \frac{1}{2}\sqrt{2}$.

Ex. 84. $\frac{3-\sqrt{2}}{\sqrt{2}-1} = \frac{(3-\sqrt{2})(\sqrt{2}+1)}{2-1} = 1 + 2\sqrt{2}$.

Ex. 85. $\frac{2}{1+\sqrt{2}-\sqrt{3}} = \frac{2(1-\sqrt{2}+\sqrt{3})}{2\sqrt{6}-4}$

$$= \frac{2(1-\sqrt{2}+\sqrt{3})(2\sqrt{6}+4)}{24-16} = \frac{(1-\sqrt{2}+\sqrt{3})(\sqrt{6}+2)}{2}$$

$$= \frac{\sqrt{6}+\sqrt{2}+2}{2}.$$

43. The following propositions with respect to quadratic surds in particular are important :

i. *The product of dissimilar quadratic surds cannot be rational.*

For, let $\sqrt{p}$ and $\sqrt{q}$ be their simplest surd factors ; then neither p nor q contains square factors, and being dissimilar they are not made up of the same factors ; therefore, their product cannot be made up of square factors, and consequently $\sqrt{pq}$ is not rational.

ii. *A surd cannot be made up by combining rational quantities and surds by addition and subtraction.*

For if possible, let $\sqrt{p} = m \pm \sqrt{n}$;

squaring, $p = m^2 + n \pm 2m\sqrt{n}$;

$$\therefore \quad \sqrt{n} = \pm \frac{p - m^2 - n}{2m} = \text{a rational quantity.}$$

iii. *A surd cannot be made up by combining two dissimilar surds by addition and subtraction.*

For if possible let $\sqrt{p} = \sqrt{q} \pm \sqrt{r}$

squaring, $p = q + r \pm 2\sqrt{qr}$,

$\therefore \ \sqrt{qr} = \pm \frac{1}{2}(p - q - r) =$ a rational quantitity.

But since q and r are dissimilar, $\sqrt{qr}$ cannot be rational.

iv. *If* $x + \sqrt{y} = a + \sqrt{b}$, *then* $x - \sqrt{y} = a - \sqrt{b}$.

For $x - a = \sqrt{b} - \sqrt{y}$. But since $x - a$ is rational it cannot be equal to the difference between two surds.

Hence $x - a = 0$, and $\sqrt{b} - \sqrt{y} = 0$;

$\therefore \ x = a$ and $\sqrt{b} = \sqrt{y}$;

and $x - \sqrt{y} = a - \sqrt{b}$.

44. To find the square root of a binomial quadratic surd.

Let $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$.

squaring, $a + \sqrt{b} = x + y + 2\sqrt{xy}$;

$\therefore$ Art. 43, iv., $x + y = a$, and $4xy = b$.

Hence $(x+y)^2 - b = (x-y)^2 = a^2 - b$,

and $\therefore \ x - y = \sqrt{a^2 - b}$.

But $x - y = a$,

$\therefore \ x = \frac{1}{2}(a + \sqrt{a^2 - b})$,

and $y = \frac{1}{2}(a - \sqrt{a^2 - b})$.

$$\therefore \ \sqrt{a + \sqrt{b}} = \sqrt{\left\{\tfrac{1}{2}(a + \sqrt{a^2 - b})\right\}} + \sqrt{\left\{\tfrac{1}{2}(a - \sqrt{a^2 - b})\right\}}.$$

Ex. 86. To find the square root of $3 + 2\sqrt{2}$.

Here $x + y = 3$, and $xy = 2$.

$\therefore\ (x-y)^2=3^2-4=1,$

and $x-y=1.$

Hence, $x=2$, and $y=1$,

$\therefore\ \sqrt{3+2\sqrt{2}}=1+\sqrt{2}.$

Ex. 87. To find the square root of $23-4\sqrt{15}$.

$x+y=23$, and $4xy=240$;

$\therefore\ x-y=\sqrt{(23^2-240)}=17$;

$\therefore\ x=20,\ y=3,$

and $\sqrt{(23-4\sqrt{15})}=2\sqrt{5}+\sqrt{3}.$

45. In the case of trinomial quadratic surds which are complete squares we may proceed as follows :

Let $\sqrt{p}+\sqrt{q}+\sqrt{r}$ be the root.

Then $(\sqrt{p}+\sqrt{q}+\sqrt{r})^2=p+q+r+2\sqrt{pq}+2\sqrt{qr}+2\sqrt{rp}.$

But $p=\dfrac{2\sqrt{pq}.2\sqrt{rp}}{2\times 2\sqrt{qr}}.$

Hence if P, Q, R, denote the surd terms, taken in order,

$$p=\frac{PR}{2Q},\ q=\frac{QP}{2R},\ r=\frac{RQ}{2P}.$$

Ex. 88. To find the square root of $54-4\sqrt{2}+6\sqrt{5}-12\sqrt{10}$.

$$p=\frac{4\sqrt{2}.12\sqrt{10}}{2\times 6\sqrt{5}}=8,\ \therefore\ \sqrt{p}=2\sqrt{2}.$$

$$q=\frac{4\sqrt{2}.6\sqrt{5}}{2\times 12\sqrt{10}}=1,\ \therefore\ \sqrt{q}=1.$$

$$r=\frac{12\sqrt{10}.6\sqrt{5}}{2\times 4\sqrt{2}}=45,\ \therefore\ \sqrt{r}=3\sqrt{5}.$$

$\therefore\ 1\pm 2\sqrt{2}\pm 3\sqrt{5}$, form the terms in the root, and a little inspection shows us that the signs must be

$$1-2\sqrt{2}+3\sqrt{5}.$$

In cases of this kind the subsequent squaring of the root is the only sure test of correctness.

SURD EQUATIONS SOLVED AS LINEARS.

46. Equations containing surds can sometimes be solved as linears, but in all cases they involve certain peculiarities which will be more fully comprehended hereafter.

Ex. 89. Given $\sqrt{a^2+x^2}+\sqrt{a^2-x^2}=b$ to find x.

$$\text{Squaring, } 2a^2+2\sqrt{a^4-x^4}=b^2:$$

$$\text{transposing and squaring, } 4a^4-4x^4=(b^2-2a^2)^2;$$

$$\therefore\ x^4=a^4-\tfrac{1}{4}(b^2-2a^2)^2;$$

$$\therefore\ x=\sqrt[4]{\left\{a^4-\tfrac{1}{4}(b^2-2a^2)^2\right\}}.$$

Since the fourth root of a quantity has four values, x has four values which will satisfy the equation; and thus the equation although apparently solved as a linear, is in fact a quartic.

Ex. 90. Given $\sqrt{a}+\sqrt{x}=\sqrt{ax}$ to find x.

Here we reduce the number of surds containing x by dividing by $\sqrt{x}$,

$$\text{and}\quad \sqrt{\frac{a}{x}}=\sqrt{a}-1.$$

$$\therefore\quad \frac{x}{a}=\frac{1}{(\sqrt{a}-1)^2};$$

$$\text{and}\quad x=\frac{a}{(\sqrt{a}-1)^2}.$$

Ex. 91. Given, $\dfrac{\sqrt{a+x}}{a}+\dfrac{\sqrt{a+x}}{x}=\dfrac{\sqrt{x}}{b}$ to find x.

$$\frac{x\sqrt{a+x}+a\sqrt{a+x}}{ax}=\frac{\cancel{ax}\sqrt{x}}{b},$$

$$\therefore\quad b(a+x)^{\frac{3}{2}}=ax^{\frac{3}{2}}.$$

Squaring and extracting cube root,

$$b^{\frac{2}{3}}(a+x)=a^{\frac{2}{3}}x,$$

$$\text{whence, } x=\frac{ab^{\frac{2}{3}}}{a^{\frac{2}{3}}-b^{\frac{2}{3}}}.$$

OF THE QUADRATIC EQUATION.

47. A quadratic equation contains the second power of the unknown quantity. If it contains that power only it is called a *pure* quadratic, but if it contains the first power also it is a *mixed* or *adfected* quadratic. This distinction is, however, of little importance.

48. *Origin of a Quadratic.* If two linear equations, with the same unknown quantity, be multiplied together the product is a quadratic.

Thus, $(x-a=0)(x-b=0)$ gives $x^2-(a+b)x+ab=0$.

And conversely, every quadratic can be formulated as the product of two linears.

Thus, if $x^2+px+q=0$ denote any quadratic,

$$x^2+px+\frac{p^2}{4}+q-\frac{p^2}{4}=0,$$

$= x = \frac{-p \pm \sqrt{p^2-4q}}{2}$

$$\therefore\ \left(x+\frac{p}{2}\right)^2-\sqrt{\left(\frac{p^2}{4}-q\right)}^2=0,$$

$$\therefore\ \left\{x+\frac{p}{2}+\sqrt{\left(\frac{p^2}{4}-q\right)}\right\}\cdot\left\{x+\frac{p}{2}-\sqrt{\left(\frac{p^2}{4}-q\right)}\right\}=0.$$

In which the quantities within the $\}\{$ are linear equations.

Hence every quadratic may be considered as the product of two linears.

49. *Roots of a Quadratic.* A quantity which, when put for the unknown quantity in an equation, satisfies it, or renders it true, is called a *root* of the equation.

A linear has but one root; but a quadratic, being the product of two linears, is satisfied by the root of each linear; every quadratic has accordingly two roots.*

Thus, if $x^2+px+q=(x-a)(x-b)=0$, where $p=-(a+b)$ and $q=ab$, then $x=a$, or $x=b$ satisfies the equation since either substitution renders the expression zero, hence a and b are the roots of the quadratic.

* This statement is not without exceptions, to some of which reference will be made hereafter.

50. *General solution of a Quadratic.* The most general form in which a quadratic can be written is

$$ax^2 + bx + c = 0\ ;$$

when a, b, c, denote any quantities whatever, and this we are to resolve into linear factors.

Multiply throughout by $4a$, and add and subtract b^2, and we obtain

$$4a^2x^2 + 4abx + b^2 - b^2 + 4ac = 0.$$

$$\therefore\ (2ax + b)^2 - (b^2 - 4ac) = 0,$$

$$\text{or,}\ (2ax + b + \sqrt{b^2 - 4ac})(2ax + b - \sqrt{b^2 - 4ac}) = 0.$$

Whence if x_1, x_2, denote the two roots,

$$\left.\begin{aligned} x_1 &= \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \\ x_2 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a}. \end{aligned}\right\} \quad \ldots (A).$$

These may be combined in one formula by using the double sign $\pm$, and we get,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \ldots\ldots (B).$$

A study of this form serves for the solution of all quadratics.

Ex. 92. Let $3x^2 - 2x + 4 = 0$;

$$\text{then, } x = \frac{2 \pm \sqrt{2^2 - 48}}{6} = \frac{2 \pm \sqrt{-44}}{6},$$

$$= \tfrac{1}{3}(1 \pm \sqrt{-11}).$$

Ex. 93. Let $(a - b)x^2 + ax + b = 0.$

$$\text{then, } x = \frac{-a \pm \sqrt{a^2 - 4b(a - b)}}{2(a - b)},$$

$$= \frac{-a \pm (a - 2b)}{2(a - b)},$$

$$= \frac{b}{b - a},\ \text{ or } -1.$$

51. *Sum and product of the roots.*

Adding the values of the roots in Art. 50, (A), we obtain,

$$x_1 + x_2 = -\frac{b}{a};$$

and multiplying, we obtain,

$$x_1 x_2 = \frac{c}{a}.$$

Hence in a quadratic the sum of the roots is equal to the quotient arising from dividing the coefficient of x by that of x^2 taken with a changed sign ; and the product of the roots is equal to the quotient arising from dividing the constant term by the coefficient of x^2.

Ex. 94. Given $(sl+\beta)^2+(cl+a)^2=r^2$, and $s^2+c^2=1$, a quadratic in l to find the sum and product of the roots.

Squaring and arranging in powers of l,

$$l^2+2l(ca+s\beta)+a^2+\beta^2-r^2=0,$$

$$\therefore\ l_1+l_2=-2(ca+s\beta),$$

and $\quad l_1 l_2 = a^2+\beta^2-r^2.$

52. *Nature of the roots.* In the formula Art. 50 (B), since b^2 is essentially positive, and since a may be rendered positive by change of signs, the character of the quantity under the surd will depend upon the sign and value of c, a being positive

i. If c is negative, then b^2-4ac is positive, and has a square root either rational or irrational. Hence in this case the roots are always *real* quantities.

Thus, if $x^2+4x-n=0$, x has two real values for every positive value of n.

ii. If c is positive and less than $\frac{b^2}{4a}$, the quantity b^2-4ac is positive, and the roots are real.

iii. If c, being positive and less than $\frac{b^2}{4a}$, gradually increases in value, then the two values of x, *i.e.* the two roots become more and more nearly equal as b^2-4ac becomes smaller ; and finally the roots meet and become equal in value when b^2-4ac

becomes zero. The surd part then disappears and the quadratic has *equal* roots which may be both positive or both negative, but which are always rational.

The very important condition then that a quadratic may have equal roots is that $4ac = b^2$.

Ex. 95. If $r^2 - 2drc + d^2 - a^2 = 0$ be a quadratic in r, find the condition that r may have two equal values.

$$\text{Condition, } 4(d^2 - a^2) = 4c^2d^2,$$

$$\text{or, } \frac{a^2}{d^2} = 1 - c^2.$$

iv. If c be positive and greater than $\frac{b^2}{4a}$, $b^2 - 4ac$ is negative, and as the square root of a negative quantity has no real existence but is wholly imaginary, the roots of the equation will be *imaginary* or *impossible*. These imaginary results are not to be dismissed as of no consequence, as they are frequently of very great importance. Let it be required for example to divide 10 into parts such that their product may be 30. If x be one part, $10 - x$ will be the other, and

$$x(10 - x) = 30 = 10x - x^2 ;$$

$$\text{whence } x = \frac{10 \pm \sqrt{-20}}{2} = 5 \pm \sqrt{-5} ;$$

where the imaginary result $\sqrt{-5}$ shows that there is some impossibility involved in the question. Upon examination we find that the largest product which it is possible to obtain from the two parts of 10 is 25.

v. If b be zero, the value of x reduces to $\pm\sqrt{\frac{-c}{a}}$.

In this case the roots are equal in value, but of opposite signs. The condition that this should take place is, then, that the coefficient of x in the first power shall be zero. If c be positive the roots are imaginary, but if negative they are real.

Ex. 93. Given $(rs - a)^2 + (rc - b)^2 - 1 = 0$ to find the conditions under which r will have values equal in magnitude but opposite in sign.

$$\text{Expanding, } r^2(s^2 + c^2) - 2r(as + bc) + a^2 + b^2 - 1 = 0.$$

$$\therefore \text{ Condition is } as + bc = 0,$$

$$\text{or,} \quad \frac{s}{c} = -\frac{b}{a}.$$

The relations developed in the present article are of the highest importance in coordinate geometry, and in the application of algebra to geometry.

53. *Limits of positive and negative values of quadratic expressions.*

Let ax^2+bx+c be a general quadratic expression. Resolving it into linear factors we find the expression to be equivalent to

$$a\left(x+\frac{b+\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b-\sqrt{b^2-4ac}}{2a}\right).$$

Disregarding the factor a for the present, when the two factors within brackets have the same sign their product will be positive, but when they have different signs it will be negative; and the only effect of a change in sign of a is to reverse these results.

But the bracketed factors can have different signs only when one is greater than zero and the other less.

Suppose the first factor to be greater than zero and the second one less; then we must have

$$x > -\frac{b+\sqrt{b^2-4ac}}{2a}, \text{ and } < \frac{\sqrt{b^2-4ac}-b}{2a}.$$

Between these limits for the value of x the expression is negative for positive values of a and positive for negative values; and for all values of x beyond these limits the sign of the expression is the opposite to that which it has when the value of x is taken between the limits.

Ex. 94. What are the limits of negative values for the expression $3x^2+2x-5$?

$$\text{This is equivalent to, } 3\left(x+\frac{2+\sqrt{4+60}}{6}\right)\left(x+\frac{2-\sqrt{4+60}}{6}\right),$$

$$\text{or } 3(x+\tfrac{5}{3})(x-1).$$

$\therefore$ x must be less than 1 and greater than $-\frac{5}{3}$, and if any quantity between these limits be substituted for x in the given expression the result will be negative.

Ex. 95. Under what conditions will $7x-3x^2-2$ be negative?

This is equivalent to $-(3x^2-7x+2)=-3(x-\frac{1}{3})(x-2)$.

Hence the expression will give positive results for all values of x between 2 and $\frac{1}{3}$, and negative results for all other values.

54. *Of maximum and minimum solutions of quadratic expressions.*

By dividing by the coefficient of x^2 any quadratic may be put into the form,

$$x^2+px+q=0.$$

We know, Art. 49, that there are two quantities real or imaginary which when substituted for x in this expression will render it true. These are the roots. If, however, we put any other quantity whatever for x the expression will not be equal to zero, but to some finite quantity which we may denote by y. The value of y will depend upon that of the quantity substituted for x. If among all the quantities which can be substituted for x there be one which will make y *greater* than it can be made by substituting any other value for x, that value of x furnishes us the *maximum* solution, and y or the quadratic expression is said to attain its maximum. If on the other hand the particular substitution renders y *less* than any other substitution does, we have a *minimum* solution and y or the quadratic expression attains its minimum.

55. *To find the maximum or minimum solution of a quadratic.*

Let $x^2+px+q=y$;

then, $x=\dfrac{-p\pm\sqrt{p^2-4q+4y}}{2}$.

Now, in order that x may be a *real* quantity the expression under the surd sign must not be negative. It is readily seen that increasing the value of y has no tendency to make $p^2-4q+4y$ negative, and hence that y has no maximum. By diminishing y howexer the value of the whole surd expression will be gradually diminished until it passes through zero and becomes negative. Hence y has a *minimum* limit; that value which makes the surd expression zero.

Again, let the expression be,

$$-x^2+px+q=y;$$

Changing signs, $x^2-px-q=-y$;

$$\text{whence } x=\frac{p\pm\sqrt{p^2+4q-4y}}{2}.$$

In this case, since y is negative, increasing the numerical value of y diminishes the expression under the surd; hence y has a *maximum* limit when this expression becomes zero, but it has no minimum limit.

We infer then,

i. That every quadratic admits of a minimum or a maximum solution according as the coefficient of x^2 is respectively positive or negative.

ii. That the maximum or minimum solution is obtained by solving the equation for x and then equating to zero the quantity under the surd sign.

In the general equation $x^2+px-q=y$, the minimum value of y is $\frac{4q-p^2}{4}$, and the corresponding value of x, or the value of x which renders the expression a minimum is $\frac{p}{2}$.

Ex. 95. It is required to divide a number a into two parts such that their product may be a maximum.

Let x be one part, and $a-x$ the other.

Then $x(a-x)=y$ a maximum;

$\therefore\ ax-x^2=y$

or $x=\frac{a\pm\sqrt{a^2-4y}}{2}$.

Hence, for a maximum, $a^2-4y=0$, or $y=\left(\frac{a}{2}\right)^2$,

and $x=\frac{a}{2}$. And the number must be divided into equal parts.

If a denotes a line, we see from this that for a given perimeter the square contains a greater are than any other rectangle.

Ex. 96. To divide a given numberinto two parts such that the sum of their squares divided by their product may give a maximum or a minimum, and to determine it.

Let a be the given quantity ;

then $x^2 + (a-x)^2 = 2x^2 - 2ax + a^2 =$ sum of squares,

and $x(a-x) = ax - x^2 =$ product.

$$\therefore \quad \frac{2x^2 - 2ax + a^2}{ax - x^2} = y = \text{a max. or a min.}$$

From this we obtain,

$$x = \frac{a}{2} \pm \frac{a}{2}\sqrt{1 - \frac{4}{2+y}}.$$

Whence we readily see that y can have a minimum value, but no maximum.

Put $1 - \dfrac{4}{2+y} = 0 \quad \therefore y = 2 =$ the min. value ; and $x = \dfrac{a}{2}$.

Hence the number must be divided into two equal parts ; and the sum of the squares of the parts divided by their product cannot be less than two.

—— This article is of particular importance in the application of algebra to geometry.

56. *Graphic representation of the quadratic.*

All the prominent properties of the quadratic may be exhibited graphically by means of a curve.

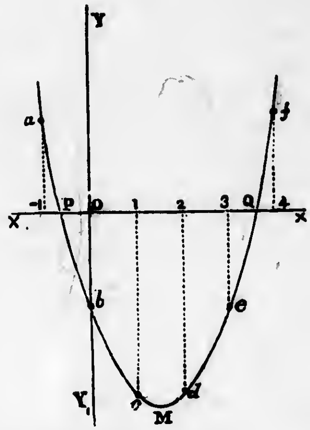
Take for illustration the quadratic expression $x^2 - 3x - 2$. We know that for two particular values of x, (the roots), the expression will be zero, but that it will have some finite value when any other quantity is substituted for x. Let y denote that value ; then

$$x^2 - 3x - 2 = y.$$

Substitute different values for x, integers for convenience, and we obtain corresponding values of y as follows :

if $x =$	-1	0	1	2	3	4	....
$y =$	2	-2	-4	-4	-2	2	....

Draw two lines XX_1, YY_1, intersecting at right angles in O. Let the different values substituted for x be denoted by distances measured from O along XX_1, the positive values to the right of O and the negative to the left.



Also, let the corresponding values of y be measured from the line XX_1 parallel to the line YY_1, the positive values upwards and the negative downwards. We thus get a series of points a, b, c, d, e, f . . . . The curve which passes through these points and through all points similarly obtained by substituting all possible quantities for x represents the quadratic expression x^2-3x-2.

i. Consider the points P and Q where the curve cuts the line XX_1. For these points y is zero, and they accordingly represent the equation, $x^2-3x+2=0$. And the values of x for these points *i.e.* OP and OQ, or the distances of these points from O represent the roots of the equation. We thus see that one root is positive and has a value between 3 and 4, and the other root is negative with a value between 0 and 1. If both points, P and Q, were upon the same side of O the roots would have the same sign, positive if upon the right side and negative if upon the left.

ii. Since the curve actually cuts the line XX_1 the points P and Q are not imaginary but real, and the equation has consequently real roots.

If the curve after approaching the line XX_1 turned and receded from it without meeting it, the roots would be imaginary.

iii. Suppose that the curve merely touches the line XX_1 at its lowest extremity M. This might be brought about by moving the curve bodily upwards : but in so doing the points P and Q would gradually approach one another and finally meet at the point of contact, and the distances OP and OQ would be one and the same. Hence this denotes equal roots. If the curve were still more elevated the points P and Q would become imaginary.

Hence we see that if a quadratic changes its form continuously so as to pass from real to imaginary roots or vice versa, it must pass through the condition of equal roots. Compare Art. 52, iii.

iv. As the curve lies wholly below the line XX_1 from P to Q, the quadratic expression x^2-3x-2 is negative for all values of x between these limits, and positive for all other values.

v. Since the curve sweeps downward to a lowest point and then begins to ascend, the quadratic has a minimum value. If the curve were reversed and the apex turned upwards, it would denote the existence of a maximum value for the corresponding quadratic.

vi. If YY_1 passed through M so that the curve was symmetrical with reference to the line YY_1, OP would be equal to OQ in magnitude, but would differ from it in sign. Hence the roots would be equal in magnitude, but opposite in sign. Art. 52, v.

Ex. The quadratic $6+x-x^2$ has equal roots, one positive and the other negative. It is positive for all values of x between the roots, and negative for all values beyond them. It admits of a maximum but not of a minimum.

The curve described as above is known in Geometry as the *Parabola*.

Of the double solution furnished by the quadratic equation.

57. When the statement of a problem involves a quadratic equation, the two roots indicate in general two possible solutions to the problem; the double solution being sometimes directly applicable and sometimes not.

In purely arithmetical questions it usually happens that only one of the solutions is directly applicable, the other becoming so only after some changes in the wording of the problem.

Ex. 97. A man died in a year A.D. which was $33\frac{1}{4}$ times his age: 13 years before the year was the square of his age. To find his age at death.

Let x=his age, then $33\frac{1}{4}\,x$=the year A.D.

and $33\frac{1}{4}x-13=(x-13)^2$

whence $x=56$ or $3\frac{1}{4}$.

Here 56 is evidently the answer to the problem, but what does the $3\frac{1}{4}$ mean ?

Since $33\frac{1}{4}\times 3\frac{1}{4} - 13 = \frac{1521}{16} = (3\frac{1}{4} - 13)^2$

$\therefore$ $3\frac{1}{4}$ satisfies the algebraical condition, but $3\frac{1}{4} - 13 = -7\frac{3}{4}$, a negative quantity.

Hence we may interpret the two solutions as follows :

13 years ago the year A.D. was the square of the years
- since (after) the man was born . . . . . $x=56$.
- before the man was born . . . . $x=3\frac{1}{4}$.

58. It sometimes happens in even arithmetical questions that both solutions are applicable.

Ex. 98. A man buys a horse and sells him for \$24, thus losing as much per cent as the horse cost in dollars : To find the cost.

$$\text{If } x = \text{the cost}, \frac{x}{100} \cdot x = \text{ loss } = x - 24$$

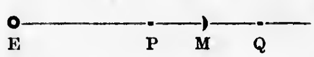
$$\text{Whence } x = 60 \text{ or } 40.$$

And $\because$ both solutions satisfy the condition, the problem is to a certain extent indeterminate.

59. In geometrical problems and problems involving geometrical magnitudes, the double solution is frequently of the highest importance, and it should not be neglected, inasmuch as it often increases materially our knowledge of the problem in hand.

Ex. 99. The attraction of a planet is directly proportional to its mass and inversely proportional to the square of its distance. The mass of the earth is 75 times that of the moon. and their distance apart is 240,000 miles. It is required to find a point in the line joining them where their attractions are equal.

Let P be the point and denote EP by x. Then $PM = 240000 - x$.



Attraction of $\circ = 75 \times \frac{1}{x^2}$; of $) = 1 \times \frac{1}{(240000 - x)^2}$;

and these are to be equal ;

hence $75(240000 - x)^2 = x^2$,

whence, $x = 215160$ or 271330 miles.

We thus see that there are two points of equal attraction, the latter of which lies beyond the moon at the point Q; a result which, when once obtained, recommends itself to our judgment as true.

60. When a quadratic equation so involves a surd as to necessitate the process of squaring in the course of the solution, it sometimes happens that the roots obtained are not those of the equation proposed, but of an equation differing in sign only from the original.

Ex. 100. Given $3x + \sqrt{30x - 71} = 5$ to find x.

By the regular mode of solution we here obtain the values 4 and $2\frac{2}{9}$ for x, *neither* of which will satisfy the given equation, they being in fact roots of the equation,

$$3x - \sqrt{30x - 71} = 5.$$

In cases of this kind it is only by verification that we can determine whether we have a correct solution of the proposed equation or not.

Again from the equation $3x + \sqrt{2x - 2} = 7$, we obtain $x = 3$ and $x = 1\frac{8}{9}$, of which $x = 1\frac{8}{9}$ only will satisfy the given equation, while $x = 3$ satisfies the equation $3x - \sqrt{2x - 2} = 7$.

The difficulty in these cases seems to arise from the fact that when we square a quantity we lose all trace of its original sign, and we have afterwards no means of determining algebraically what sign it was at first affected by.

Thus: $\sqrt{2x - 2} = 7 - 3x$ and $-\sqrt{2x - 2} = 7 - 3x$, evidently become identical upon squaring, whereas they cannot possibly be satisfied by the same quantities; so that any solution must give us either both roots belonging to only one of these equations, or one root belonging to each.

Whether any value of x can satisfy the equation $3x + \sqrt{30x - 71} = 5$ or not we do not know, but if there be such a value it cannot be found by the usual mode of solving a quadratic.

TWO OR MORE UNKNOWN QUANTITIES.

61. It frequently happens that the conditions of a problem require the introduction of more than one unknown quantity in its statement.

In such cases we require for the complete determination of the unknowns as many equations as there are unknown quantities, and these equations must moreover be *independent*, that is, they must be such that any one of them cannot be obtained from the others by any legitimate process. The equations in such a set are termed *simultaneous equations*. Thus :

$$3x+y-z=2$$
$$x-y+2z=3$$
$$x+2y+z=8$$

is a set of three simultaneous equations involving the three unknown quantities x, y and z; and they are thus named because the values obtained for x, y and z must satisfy all the equations *at the same time*. This takes place when $x=1, y=2$ and $z=3$.

62. If the number of independent equations be less than that of the unknown quantities, the equation can be satisfied by an infinite number of *sets* of values for the unknown quantities, and the problem is said to be indeterminate. Thus if we have one equation with two unknowns, as $2x-3y=10$, it is evident that if we put *any value whatever* for x we can find a corresponding value for y. This species of equation is extensively employed in co-ordinate geometry, where x denotes an abscissa of some locus and y the corresponding ordinate.

63. If the number of equations be greater than that of the unknown quantities, then some of the equations must be incompatible with the others, or else they are *dependent*, and hence redundant.

Thus, if
$$\begin{aligned} 3x + 2y &= 8 \\ 2x - y &= 3 \\ x + y &= 1 \end{aligned}$$

be a set of three equations with the two unknowns x and y, the values which satisfy the first two cannot possibly satisfy the third, or those which satisfy the second and third cannot satisfy the first, &c.; i.e., one of the equations is incompatible with the other two.

If the third equation were $x+3y=5$, then since this may be derived from the other two, or any one of them from the remaining two, one equation is dependent, and, thus giving no new relation, is redundant.

But if the equations are literal and are to be also compatible, some relation must exist among the literal co-efficients.
Art. 72.

LINEAR SIMULTANEOUS EQUATIONS—ELIMINATION.

64. When we so combine two or more equations as to get rid of a quantity we are said to *eliminate* that quantity between the equations; and the process of *solving* a set of simultaneous equations consists in eliminating the unknown quantities, one after another, until we finally have a single equation containing only one of the unknowns.

The methods of elimination will be considered under the following heads:

1. By comparison.
2. By substitution.
3. By cross-multiplication and addition and subtraction.
4. By indeterminate or arbitrary multipliers.
5. By determinant forms.

These modes are all applicable in any case, but they are not all equally convenient. Thus 1 and 2 are not often convenient with more than two unknowns; 3 may be applied to any number, and is one of the most practical; 4 applies with greatest advantage to three unknowns; and 5 applies most profitably to three or more.

65. *Elimination by comparison.* This method consists in finding the value of the same unknown quantity or some function of it in terms of the other, from each equation, and then equating these values.

Ex. 99. Let $\frac{x}{2}+\frac{y}{3}=4$; $\frac{5x}{6}-\frac{2y}{3}=3$,

be the equations.

Then from the first, $x=8-\frac{2y}{3}$;

and from the second, $x=\frac{18}{5}+\frac{4y}{5}$.

$$\therefore\ 8-\frac{2y}{3}=\frac{18}{5}+\frac{4y}{5}.$$

Whence $y=3$, and thence $x=6$.

Ex. 100. Given $\frac{m}{x}+\frac{n}{y}=a$, $\frac{n}{x}+\frac{m}{y}=b$ to find x and y.

Here, $\frac{mn}{x}=na-\frac{n^2}{y}=mb-\frac{m^2}{y}$.

$\therefore \frac{1}{y}(m^2-n^2)=mb-na$, and $y=\frac{m^2-n^2}{mb-na}$,

and from symmetry $x=\frac{m^2-n^2}{ma-nb}$.

66. *Elimination by substitution.* In this method we substitute for one of the unknown quantities in one of the equations its value drawn from another equation.

Ex. 101. Given $\frac{4x+5y}{40}=x-y$, and $\frac{2x-y}{3}=\frac{1}{2}-2y$ to determine x and y.

From the first, $4x+5y=40x-40y$; whence, $x=\frac{5y}{4}$. And substituting this for x in the second.

$$\frac{2.\frac{5y}{4}-y}{3}=\frac{1}{2}-2y$$

Whence $y=\frac{1}{5}$,

and hence $x=\frac{1}{4}$.

67. *Elimination by cross-multiplication and addition and subtraction.* The following examples will illustrate this very important method:

Ex. 102. Let $ax - by = 0$, $x + y = c$ be the equations.

Multiplying the second equation by b and adding to the first we eliminate y and obtain

$$ax + bx = bc\text{; whence } x = \frac{bc}{a+b};$$

$$\text{and thence, } y = \frac{ac}{a+b}.$$

Ex. 103. Let the equations be,

$$\begin{aligned} 2x + 4y + 5z &= 49, \ldots\ldots \alpha \\ 3x + 5y + 6z &= 64, \ldots\ldots \beta \\ 4x + 3y + 4z &= 55 \ldots\ldots \gamma \end{aligned}$$

$$\begin{aligned} 2\alpha - \gamma \ldots\ldots \quad 5y + 6z &= 43, \ldots\ldots \delta \\ 2\beta + 3\gamma \ldots\ldots - 2y - 3z &= -19, \ldots\ldots \varepsilon \\ 2\varepsilon + \gamma \ldots\ldots \quad y &= 5; \end{aligned}$$

whence $x = 7$, $z = 3$.

Ex. 104. Given,

$$\begin{aligned} 9x - 2z + u &= 41 \ldots\ldots \alpha \\ 7y - 5z - t &= 12 \ldots\ldots \beta \\ 4y - 3x + 2u &= 5 \ldots\ldots \gamma \\ 3y - 4u + 3t &= 7 \ldots\ldots \delta \\ 7z - 5u &= 11 \ldots\ldots \varepsilon \end{aligned}$$

Since t occurs the least often eliminate it first.

$$\delta + 3\beta \ldots\ldots 24y - 15z - 4u = 43, \ldots\ldots \zeta$$

β and δ cannot hereafter be employed.

Next eliminate u.

$$\begin{aligned} \zeta + 2\gamma \ldots\ldots 32y - 15z - 6x &= 53, \ldots\ldots \eta \\ 2\alpha - \gamma \ldots\ldots 21x - 4y - 4z &= 77, \ldots\ldots \theta \\ 5\alpha + \varepsilon \ldots\ldots 15x - z &= 72 \ldots\ldots \kappa \end{aligned}$$

To eliminate y;

$$\eta + 8\theta \ldots\ldots 162x - 47z = 669 \ldots\ldots \lambda$$

And finally,

$$47\kappa - \lambda \ldots\ldots 543x = 2715;$$

whence $x = 5$, $y = 4$, $z = 3$, $u = 2$, $t = 1$.

Ex. 105. Given $ax+by=c$, and $a'x+b'y=c'$ to find x and y.

Multiply the first equation by b' and the second by b and subtract one product from the other, and we get,

$$x(ab'-a'b)=b'c-bc'.$$

$$\therefore\ x=\frac{b'c-bc'}{ab'-a'b};$$

and from symmetry, $y=\dfrac{a'c-ac'}{ab'-a'b}$.

We notice here that in order to eliminate y we multiply the first equation by the coefficient of y in the second, and the second equation by the coefficient of y in the first; and similarly to eliminate x; hence the term *cross-multiplication.*

68. *Elimination by indeterminate or arbitrary multipliers.* This method may be readily applied to the case of two equations, or to the case of three.

Ex. 105′. Given $3x+y=7$, and $10x-2y=2$ to find x and y.

Multiply one of the equations, the first for example, by the indeterminate multiplier λ and add the product to the other, and we have,

$$x(3\lambda+10)+y(\lambda-2)=7\lambda+2.$$

Now this is necessarily true whatever value may be given to λ. But if $\lambda=2$, y will disappear from the equation and we obtain $16x=16$, or $x=1$.

Similarly if $3\lambda+10=0$, x disappears from the equation and there results, $y\left(-\frac{10}{3}-2\right)=2-\frac{70}{3}$; whence $y=4$.

69. If we have a set of three equations, for example:

$$\begin{aligned} 2x - 3y + z &= 2 \\ x + y - 2z &= 1 \\ 3x + 2y - 3z &= 5, \end{aligned}$$

it is possible to multiply them by such multipliers that when the products are added the coefficients of two letters may both become zero at the same time, and thus we may eliminate both letters at one operation. In the example given if we multiply the first equation by 1, the second by -7, and the third by 5 and add, we obtain $10x=20$.

To investigate a rule for finding the proper multipliers.

$$\begin{aligned} \text{Let, } ax &+ by + cz = d \\ a'x &+ b'y + c'z = d' \\ a''x &+ b''y + c''z = d''. \end{aligned}$$

Then multiplying the first by l, the second by m, and the third by n and adding, we have,

$$x(la+ma'+na'')+y(lb+mb'+nb'')+z(lc+mc'+nc'') = ld+md'+nd''$$

Now if y and z are both to disappear their coefficients in this equation must be zero. We must accordingly have

$$lb+mb'+nb''=0$$
$$lc+mc'+nc''=0$$

Eliminating n between these, we obtain

$$\frac{l}{b'c''-b''c'}=\frac{m}{b''c-bc''},$$

and from symmetry each $=\dfrac{n}{bc'-b'c}$,

And having three equal fractions the numerators must be proportional to the denominators.

Hence l, m, n may be *any* quantities proportional to

$$b'c''-b''c',\quad b''c-bc'',\quad bc'-b'c$$

respectively ; and these quantities themselves are usually taken as the multipliers.

To apply this, notice, i. That the multipliers are made up solely from the coefficients of the letters to be eliminated.

ii. That the multiplier for any line involves only coefficients belonging to the remaining lines.

iii. That each multiplier is the difference of two products, these being formed of terms taken always in the same order.

Ex. 106. Given,
$$\begin{aligned} x-y-2z&=3\\ 2x+y-3z&=11\\ 3x-2y+z&=4. \end{aligned}$$

We find for l, m and n, in order to eliminate y and z, the values -5, 5 and 5 respectively. Then multiplying and adding, we get $20x=60$ and hence $x=3$.

Ex. 106′. Given,

$$\begin{aligned} ax+by-az&=b(a+b)\\ bx-ay+z&=a-b\\ x+2y-2z&=4x-a. \end{aligned}$$

To eliminate y and z the multipliers are,

$$l=2a-2,\ m=2b-2a,\ n=a-b,$$

whence we obtain, after reduction, $x=a$; and similarly $y=b$ and $z=a-b$.

70. *Elimination by determinant forms.*

If from the simultaneous equations,

$$\begin{aligned} a_1x+b_1y+c_1z&=d_1\\ a_2x+b_2y+c_2z&=d_2\\ a_3x+b_3y+c_3z&=d_3, \end{aligned}$$

We eliminate y and z by Art. 68, or by any other means we obtain for the value of x,

$$x=\frac{d_1b_2c_3+d_2b_3c_1+d_3b_1c_2-d_1b_3c_2-d_2b_1c_3-d_3b_2c_1}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_1b_3c_2-a_2b_1c_3-a_3b_2c_1}.$$

The complex expressions forming the numerator and denominator of this fraction are *determinants*; and as we see they occur in the common process of elimination. The numerator may evidently be obtained from the denominator by substituting d for a throughout; and hence from the principle of symmetry in order to obtain equivalent expressions for y and z we must substitute d for b and c respectively in the above form.

Taking the denominator then as the *type* form the numerators may all be derived from it by substitution.

In the case of three simultaneous equations involving three unknowns as above, each term in the denominator is the product of three *elements* or is of three dimensions. With four equations each term will be of four dimensions, and so on; and determinants are thus divided into *orders* according to the dimensions of the terms.

A determinant of the third order contains six terms, while one of the fourth order contains no less than twenty-four terms.

For the purpose of denoting these expressions without writing them in full the following notation is commonly employed:

$\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix}$ denotes $a_1b_2 - a_2b_1$ which is a determinant of the second order.

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ denotes $a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$,

which is of the third order and is the same as

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1).$$

From this we see that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

In like manner the determinant of the fourth order,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix}$$

$$- a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

These relations between determinants of different orders enable us to expand a given determinant, or to find its value.

Ex. 107. To find the value of $\begin{vmatrix} 3 & 4 & 2 \\ 1 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$

We have, $\begin{vmatrix} 3 & 4 & 2 \\ 1 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 3 \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} + 2 \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix}$

$$= 3(1-3) - (4-2) + 2(12-2) = 6$$

Ex. 108.

$$\begin{vmatrix} 3 & 1 & 2 & 3 \\ 4 & 0 & 2 & 1 \\ 6 & 4 & 1 & 2 \\ 7 & 3 & 0 & 1 \end{vmatrix} = 3 \begin{vmatrix} 0 & 2 & 1 \\ 4 & 1 & 2 \\ 3 & 0 & 1 \end{vmatrix} - 4 \begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 3 & 0 & 1 \end{vmatrix} + 6 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 0 & 1 \end{vmatrix} - 7 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 4 & 1 & 2 \end{vmatrix}$$

$$= 3(-8+9) - 4(1-8+3) + 6(2-12) - 7(3-16) = 50$$

The following principles established in works on determinants assist us in the evaluation.

i. If a column or row contains a common factor that factor may be placed outside and each element in the column or row divided by it.

ii. Any column may be added to or subtracted from another column, or any row may be added to or subtracted from another row without changing the value of the determinant.

iii. If two columns or two rows be exchanged the sign of the determinant is changed.

iv. If two columns or two rows be the same the determinant is zero.

Applying these in evaluating the last determinant, we have,

$$\begin{vmatrix} 3 & 1 & 2 & 3 \\ 4 & 0 & 2 & 1 \\ 6 & 4 & 1 & 2 \\ 7 & 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 1 & 3 \\ 2 & 4 & 0 & 1 \\ 1 & 6 & 4 & 2 \\ 0 & 7 & 3 & 1 \end{vmatrix}$$ by bringing the third column first, which does not change the sign, it being a double exchange;

$$= \tfrac{1}{2}\begin{vmatrix} 2 & 3 & 1 & 3 \\ 2 & 4 & 0 & 1 \\ 2 & 12 & 8 & 4 \\ 0 & 7 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 8 & 8 & 3 \\ 0 & 7 & 3 & 1 \end{vmatrix}$$ by dividing the first column by 2 and then subtracting the first row from the second and the second from the third;

$$= \begin{vmatrix} 1 & -1 & -2 \\ 8 & 8 & 3 \\ 7 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -2 \\ 0 & 5 & 3 \\ 4 & 2 & 1 \end{vmatrix}$$ by subtracting the second column from the first;

$$= \tfrac{1}{2}\begin{vmatrix} 4 & 2 & -4 \\ 0 & 5 & 3 \\ 4 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & -4 \\ 0 & 5 & 3 \\ 0 & 0 & 5 \end{vmatrix}$$ by subtracting the first row from the last and dividing by 4.

$$= 2(5\times 5 - 0\times 3) = 50.$$

Ex. 109. Given

$$\begin{aligned} x \quad &+y &+z \quad &= 0 \\ (b+c)x &+(c+a)y &+(a+b)z &= 0 \\ bcx \quad &+cay &+abz \quad &= 1. \end{aligned}$$

$$x = \begin{vmatrix} 0 & 1 & 1 \\ 0 & c+a & a+b \\ 1 & ca & ab \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ bc & ca & ab \end{vmatrix}$$

Now if in the second of these determinants we put $b=c$ we obtain two columns alike and the determinant becomes zero; hence $b-c$ is a factor, and from symmetry $a-b$ and $c-a$ are factors.

$\therefore$ the second determinant $= - (a-b)(b-c)(c-a)$.
But the first $= b-c$;

$$\therefore x = -\frac{1}{(a-b)(c-a)} = \frac{1}{(a-b)(a-c)};$$

$$\text{Similarly, } y = \frac{1}{(b-c)(b-a)},$$

$$z = \frac{1}{(c-a)(c-b)}.$$

71. If we have a set of equations which do not contain a constant term, we can determine only the ratios of the unknown quantities to one another and not the unknowns themselves.

$$\text{Let } a_1x + b_1y + c_1z = 0,$$
$$a_2x + b_2y + c_2z = 0,$$

be a set of two such equations.

Put $\dfrac{x}{z} = m$, $\dfrac{y}{z} = n$, and they become,

$$a_1m + b_1n + c_1 = 0$$
$$a_2m + b_2n + c_2 = 0;$$

and we see that the unknown quantities to be determined are m and n, i.e., the ratios of $x : z$ and $y : z$, or any other two ratios which we chose to fix upon.

$$\text{Now, } m = \frac{x}{z} = -\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\begin{vmatrix} b_1 & c_1 \\ b^2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\therefore \quad \frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} \text{ by symmetry.}$$

Hence x, y, z may be any quantities respectively proportional to the denominators. This result is practically identical with that of Art. 68.

Ex. 110. To find the ratios $a : b : c$ when $x : y : z = mb + nc - la : nc + la - mb : la + mb - nc$.

Denote the ratio $\dfrac{x}{mb+nc-la} = \dfrac{y}{nc+la-mb} = \dfrac{z}{la+mb-nc}$ by $\dfrac{1}{v}$.

$$\text{Then,}\quad \begin{aligned} -la + mb + nc - vx &= 0 \\ la - mb + nc - vy &= 0 \\ la + mb - nc - vz &= 0 \end{aligned}$$

and considering a, b, c, v, as unknowns, we have

$$\frac{a}{\begin{vmatrix} m & n & -x \\ -m & n & -y \\ m & -n & -z \end{vmatrix}} = \frac{b}{\begin{vmatrix} -l & n & -x \\ l & n & -y \\ l & -n & -z \end{vmatrix}} = \frac{c}{\begin{vmatrix} -l & m & -x \\ l & -m & -y \\ l & m & -z \end{vmatrix}} = \frac{v}{\begin{vmatrix} -l & m & n \\ l & -m & n \\ l & m & -n \end{vmatrix}}.$$

and by expanding the determinants, we obtain

$$\frac{a}{2mn(y+z)} = \frac{b}{2nl(z+x)} = \frac{c}{2lm(x+y)} = \frac{v}{-4lmn}$$

$$\therefore a : b : c = mn(y+z) : nl(z+x) : lm(x+y).$$

72. *Of sets in which the number of equations is greater than that of the unknown quantities.*

In order that such equations may coexist there must necessarily be some relation among the coefficients. Thus if we are to have,

$$\begin{aligned} ax + by &= c \\ bx + ay &= 2c \\ x + y &= a + b + c, \end{aligned}$$

we must also have $(a+b)(a+b+c) = 3c$;

and unless this relation exists the given equations cannot possibly coexist.

Let $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ be three equations involving the two unknowns x and y.

Eliminating y between the first and second, and then between the first and third, we obtain,

$$x = -\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = -\frac{\begin{vmatrix} c_1 & b_1 \\ c_3 & b_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}}$$

or, $(c_1b_2 - c_2b_1)(a_1b_3 - a_3b_1) = (a_1b_2 - a_2b_1)(c_1b_3 - c_3b_1)$,

And multiplying out, rejecting terms which cancel each other, dividing through by b and arranging, we have,

$$a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = 0,$$

whence, from Art. 70, we have,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \quad \text{as the required condition of coexistence.}$$

Ex. 111. If the equations $y = mx + h$, $y = m_1x + h_1$, $y = m_2x + h_2$, are to exist together, determine the condition.

Here,
$$\begin{vmatrix} 1 & m & h \\ 1 & m_1 & h_1 \\ 1 & m_2 & h_2 \end{vmatrix} = 0$$

or, $m(h_1 - h_2) + m_1(h_2 - h) + m_2(h - h_1) = 0$.

Read Over

INDETERMINATE ANALYSIS OF THE FIRST DEGREE.

73. As stated in Article 62, if the number of equations be less than that of the unknown quantities an indefinite number of sets of values may be found to satisfy the equations.

Thus, if $ax + by = c$ be the given equation involving the two quantities x and y we may evidently put any quantity whatever for x and find a corresponding value for y.

In practice the number of solutions is restricted by the condition that the values of x and y must be positive whole numbers.

Ex. 112. It is require dto pay three dollars in 11-cent pieces and 7-cent pieces.

Let x denote the number of 11-cent pieces and y that of the 7-cent pieces.

Then, $11x + 7y = 300$ is the equation.

From this, $x = \dfrac{300 - 7y}{11} = 27 + \dfrac{3 - 7y}{11}$.

As x is to be a whole number, the expression $\frac{3-7y}{11}$, and its multiple by a whole number, must be a whole number.

We now endeavor to multiply by such an integer that the coefficient of y may be greater or less by unity than some multiple of 11. 8 is such a number, since $8\times 7=56=5\times 11+1$.

Hence, $\frac{24-56y}{11}=2-5y+\frac{2-y}{11}$ must be a wh. no.

and $\therefore$ $\frac{2-y}{11}=$ a wh. no. $=p$ say.

Then $y=2-11p$, and putting this value in the original equation we obtain, $x=26+7p$.

Hence, $x=26+7p$, $y=2-11p$ is the required solution, where p may be any integer, positive or negative, which will give positive values for x and y.

If	p	=	0	−1	−2	−3
	x	=	26	19	12	5
	y	=	2	13	25	37

which four sets are all the possible positive integral solutions. Any other integral values for p would make either x or y negative, which is not consistent with the original condition.

Ex. 113. It is required to find a number which when divided by 3 leaves a remainder 2, divided by 5 leaves 3, and divided by 7 leaves 5.

Let x be the number ; then,

$\frac{x-2}{3}$, $\frac{x-3}{5}$, $\frac{x-5}{7}$ must all be whole numbers.

Put $\frac{x-2}{3}=p$ $\therefore$ $x=3p+2$;

and writing this for x in the second fraction,

$\frac{3p-1}{5}$ must be a whole number,

$\therefore$ $\frac{p-2}{5}=q$ must be a whole number,

$\therefore$ $p=5q+2$ and $x=15q+8$;

and this in the third gives,

$$\frac{15q+3}{7}, \text{ or } \frac{q+3}{7} = \text{whole number} = r;$$

$$\therefore \quad q = 7r - 3 \text{ and } x = 105r - 37,$$

where r may be any positive integer whatever. Making $r=1$ gives 68 for the smallest number satisfying the required conditions.

74. If we have $ax+by=c$ an indeterminate equation of the first degree, it is readily seen that by increasing x, y may be made to pass through zero, and conversely by increasing y, x may be made to pass through zero. If then negative values of x and y are to be excluded, x cannot be greater than $\frac{c}{a}$ nor less than zero, and hence the number of solutions is necessarily limited.

But if $ax-by=c$ be the equation, an increase in the value of x must be accompanied by an increase in that of y, and as both may be indefinitely increased the number of solutions is quite unlimited.

75. In the equation $ax \pm by = c$, a, b and c cannot have a common factor, for we may divide throughout by such factor and thus get rid of it.

Again, a and b must be prime to each other, for if they have a common factor, it must also be a factor of $ax \pm by$; but as it is not a factor of c, the equation $ax \pm by = c$ is impossible. Thus $2x+10y=31$ cannot have an integral solution.

76. In Ex. 112 we found for values of x and y,

$$x = 26 + 7p, \quad y = 2 - 11p.$$

Now it will be noticed that the coefficient of p in the value of x is the coefficient of y in the original equation; and similarly the coefficient of p in the value of y is that of x in the original equation. This may be proved to be always the case.* Hence if $ax+by=c$ be the original equation, the values of x and y may be written, $x = \alpha \pm bp$, $y = \beta \mp ap$, where α and β are fixed quantities, which solve the equation when $p=0$.

* Demonstrations of this kind belong to an advanced course of Algebra.

If then one solution can be determined by any means, all the other solutions may be obtained at once.

Thus, if we find one solution of Ex. 112 to be $x=12$ and $y=24$, we have $x=12+7p$, $y=24-11p$ as general formulæ, and by making $p=-1, 0, 1, 2$ successively we get all the possible solutions.

If the equation be $ax-by=c$, we have only to change the sign of b in what proceeds.

77. In Ex. 113, the coefficient of r in the value of x is the L.C.M. of the three denominators, 3, 5 and 7. Hence if l denote this quantity the value of x may be written,

$$x=\gamma+lr\ ;$$

and if one solution (γ) can in any way be found, others will be obtained by adding on multiples of l.

SIMULTANEOUS QUADRATICS.

78. If an equation contains two unknowns, its degree is measured by the term of highest dimensions in these unknowns.

Thus, $2x+3xy+4=0$ is a quadratic since the second term is of two dimensions. In like manner if x, y, z, be unknowns, $x^2+yz=0$ is a quadratic, $x^2y+z^2+yz=0$ is a cubic, $xy^2z+z^2-2xy=0$ is a quartic, &c.

The most general quadratic in x and y that can be written is,
$$ax^2+bxy+cy^2+dx+ey+f=0\ ;$$
and the most general in x, y and z, is

$$ax^2+by^2+cz^2+dxy+exz+fyz+gx+hy+kz+l=0.$$

79. In general the elimination of an unknown between two quadratics produces an equation of a higher order; but if one of the equations be linear the resulting equation will be still a quadratic.

In any case elimination between two quadratics cannot produce an equation of a degree higher than the fourth. As a consequence the solution of simultaneous quadratics may require finally the solution of a quadratic only, or of a cubic, or of a quartic. The problem may, therefore, admit of two, three or four solutions depending upon conditions.

Solution of simultaneous quadratics is often effected by ingenious combinations and artifices rather than by any fixed principles of elimination. These artifices are best learned by observation and practice.

TWO EQUATIONS WHEREOF ONE IS A QUADRATIC AND THE OTHER A LINEAR.

80. The solution of these is effected by substituting in the quadratic the value of one of the unknowns as derived from the linear equation.

Ex. 114. Given, $ax^2+by^2+cxy+dx+ey+f=0$,
and $mx+ny+p=0$.

From the second equation, $x = -\dfrac{p+ny}{m}$.

And this value in the first gives, after reduction,

$$y^2(an^2+b^2m^2-cmn)+y(2apn-cpm-dmn+em^2)+ap^2-dpm+fm^2$$
$=0$; a quadratic in y.

Ex. 115. Given $3x^2-2y^2+xy-y=1$,
and $2x-3y=1$.

Here, $x = \dfrac{3y+1}{2}$, which in the first gives,

$$\tfrac{3}{4}(3y+1)^2-2y^2+\frac{y}{2}(3y+1)-y=11.$$

From which we obtain, $y = 1$ or $-\dfrac{41}{45}$;

and thence, $x = 2$ or $-\dfrac{49}{25}$.

81. If the quadratic equation be divisible by the linear the equations are equivalent to a pair of linears only, and x and y have but one value each.

Ex. 116. Given $3x^2 - 5xy - 2y^2 = 17$,

$$x - 2y = 1.$$

The first equation is $(x - 2y)(3x + y) = 17$.

But $x - 2y = 1$; $\therefore\ 3x + y = 17$.

Whence, $x = 5,\ y = 2$.

If we solve this by substituting from the second equation in the first we obtain,

$$x = 1 + 2y;\ \therefore\ 3 + 12y^2 + 12y - 5y - 12y^2 = 17$$

or $y = 2$, one value only.

82. Sometimes equations may be solved by combining them in some simple manner.

Ex. 117. Given
$$x^2 + y^2 = 13$$
$$x + y = 5$$

Subtracting the first from the square of the second we have,

$$2xy = 12;$$

and subtracting this from the first, we get

$$(x - y)^2 = 1, \text{ or } x - y = 1;$$

$$\therefore\ x = 3,\ y = 2.$$

SIMULTANEOUS EQUATIONS CONTAINING TWO QUADRATICS.

83. It is not always possible to solve these as quadratics, and experience is usually the only guide as to whether it is possible or not.

Ex. 118. Given
$$2x^2 + 3xy = 26,$$
$$3y^2 + 2xy = 39.$$

Here, $2x^2 + 3xy = x(2x + 3y) = 26$,

and $3y^2 + 2xy = y(2x + 3y) = 39$;

$\therefore$ dividing $\dfrac{x}{y} = \dfrac{2}{3}$, and $x = \dfrac{2y}{3}$.

Putting this value for x in the first,

$$\frac{8y^2}{9} + 2y^2 = 26.$$

Whence, $y = \pm 3$ and $x = \pm 2$.

84. If the terms involving the unknowns be homogeneous, we may advantageously obtain a third equation in which the unknown quantity is the ratio of one of the original unknowns to the other.

Ex. 119. Given, $x^2 + xy + 4y^2 = 6,$
$$3x^2 + 8y^2 = 14.$$

Let $\dfrac{x}{y} = v \quad \therefore\ x = vy.$

$$\text{Then, } \frac{x^2+xy+4y^2}{3x^2+8y^2} = \frac{v^2y^2+vy^2+4y^2}{3v^2y^2+8y^2}$$
$$= \frac{v^2+v+4}{3v^2+8} = \frac{3}{7},$$

whence we find, $v = 4$ or $-\frac{1}{2}$;

and writing $x = 4y$ in the second equation gives,

$$y = \pm\tfrac{1}{2}, \text{ and } \therefore\ x = \pm 2.$$

If we take the other value of v and write $y = -2x$ we obtain

$$x = \pm\frac{\sqrt{10}}{5}, \quad y = \mp\frac{2\sqrt{10}}{5},$$

Hence x and y have each four values all of which satisfy the equations.

85. If the equations be each symmetrical with respect to the unknowns, it is frequently of advantage to employ two new unknowns, one of which is the sum and the other the difference of the original unknowns.

Ex. 120. Given $x^2 + y^2 + x + y = 8,$
$$x + y + xy = 5.$$

Put $x = u + v$, $y = u - v$; then the equations become,

$$u^2 + v^2 + u = 4,$$
$$u^2 - v^2 + 2u = 5.$$

Adding, $2u^2 + 3u = 9$;

whence, $u = \frac{3}{2}$ or -3.

With these values of u we find,

when $u=\frac{3}{2}$, $v=\pm\frac{1}{2}$, $x=2$ or 1, $y=1$ or 2.

when $u=-3$, $v=\pm\sqrt{-2}$, $x=-3\pm\sqrt{-2}$, $y=-3\mp\sqrt{-2}$.

Hence x and y have each four values, which give four pairs satisfying the given equations.

In the present example, as in all cases where x and y are symmetrically involved, their values are interchangable.

86. The substitution of the last article may sometimes be employed where the equations are not strictly symmetrical in x and y.

Ex. 121. Given, $x+\sqrt{x^2-y^2} = \frac{8}{y}\left(\sqrt{x+y}+\sqrt{x-y}\right)$.

$$(x+y)^{\frac{3}{2}} - (x-y)^{\frac{3}{2}} = 26.$$

Put $x+y=2s^2$, $x-y=2t^2$.

The equations become,

$$(s+t)^2 = \frac{8}{s^2-t^2}(s+t)\sqrt{2} \;\ldots\ldots\; \alpha$$

$$\sqrt{8}(s^3-t^3)=26 \;\ldots\ldots\ldots\ldots\; \beta$$

From α we get at once,

$$s+t=\frac{8\sqrt{2}}{s^2-t^2}$$

or $\quad s^3-t^3+st(s-t)=8\sqrt{2} \;\ldots\ldots\; \gamma$

Substituting for s^3-t^3 from β in γ, we get

$$st(s-t)=\frac{3}{\sqrt{2}} \;\ldots\ldots\ldots\ldots\; \delta$$

$\beta\div\delta$ gives, $\dfrac{s^2+st+t^2}{st}=\dfrac{13}{3}$,

$\therefore$ Art. 27, $\dfrac{(s+t)^2}{st}=\dfrac{16}{3}$, $\dfrac{(s-t)^2}{st}=\dfrac{4}{3}$;

$\therefore \quad \dfrac{s+t}{s-t}=2$, and $s=3t$.

Whence we readily find, $s=\dfrac{3}{\sqrt{2}}$, $t=\dfrac{1}{\sqrt{2}}$, and

hence $x=5$, and $y=4$.

INEQUALITIES.

87. An equation declares that there is equality between its two members, but a non-equation or *inequality* declares that one of its members is greater or less than the other; and the problems which present themselves in inequalities usually require us to prove that one expression is greater or less than another.

Since the square of a quantity is always positive, $(x-y)^2$ or x^2+y^2-2xy is a positive quantity whether x be greater or less than y.

Hence, x^2+y^2 is greater than $2xy$; or expressed symbollically,

$$x^2+y^2 > 2xy.$$

The proof of a large number of inequalities depends upon this principle.

If $x=y$ the inequality becomes an equality

The following principles are important:

If $a > b$,

Then, 1. $na > nb$, and $\frac{a}{n} > \frac{b}{n}$;

but $\frac{n}{a} < \frac{n}{b}$.

2. $a+c > b+c$, and $a-c > b-c$;

but, $c-a < c-b$.

3. If a and b be both positive,

$\sqrt{a} > \sqrt{b}$ and $a^m > b^m$,

but $a^{-m} < b^{-m}$.

4. If both sides are divided by a negative quantity the character of the inequality is reversed.

Ex. 122. $a^2+b^2+c^2+\ldots > ab+bc+cd+\ldots$

For, $a^2+b^2 > 2ab$, $b^2+c^2 > 2bc$ &c.

$$\therefore\ 2a^2+2b^2+\ \ldots\ > 2ab+2bc+\ \ldots$$

$$\therefore\ a^2+b^2+\ \ldots\ > ab+bc+cd+\ \ldots$$

Ex. 123. For the same base and perimeter the area of an isosceles triangle is greater than that of a scalene one.

Let $s=\frac{1}{2}$ perimeter of each, and $b=$ the common base. Also, let a, c be the sides of the scalene triangle and e the side of the isosceles one.

Then, $A_1=$ area of isosceles $=\sqrt{s(s-e)^2(s-b)}$,

and $A_2=$ " scalene $=\sqrt{s(s-a)(s-b)(s-c)}$.

$$\therefore\ A_1^2 \gtrless A_2^2 \text{ as } (s-e)^2 \gtrless (s-a)(s-c),$$

$$\text{as } e^2-2se \gtrless ac-s(a+c),$$

$$\text{as } e^2 \gtrless ac,$$

since $a+c=2e$;

$$\text{as, } \left(\frac{a+c}{2}\right)^2 \gtrless ac,$$

$$\text{as, } a^2+c^2 \gtrless 2ac.$$

But $a^2+c^2 > 2ac$

$$\therefore\ A_1 > A_2.$$

Ex. 124. $x^6+y^6 > x^5y+xy^5$.

$$x^6+y^6 \gtrless x^5y+xy^5.$$

$$\text{as } (x^5-y^5)(x-y) \gtrless 0.$$

But if $x>y$, both factors are positive and their product is positive and therefore >0.

And if $x<y$, both factors are negative and their product is positive and therefore >0.

$$\therefore\ x^6+y^6 > x^5y+xy^5.$$

SERIES.

88. A succession of terms formed according to some regular law is called a series. If the number of terms be limited the series is finite, but if unlimited it is infinite. Series may be formed or developed in a number of different ways, one of which is given in Art. 9. Their study is important inasmuch as in many cases we are compelled to employ them. We have examples of what are the sums of the first few terms of well known series in logarithms, sines, &c. The law of formation of the terms of a series, or the "law of the series," may be very simple or very complex.

The simplest series is one in which each term differs from the one before it by a constant quantity. Such a series is termed an *equi-difference series*, an *arithmetic series*, or an *arithmetic progression*.

OF ARITHMETIC SERIES.

89. The quantities with which we have normally to deal in an arithmetic series are a, the first term; n, the number of terms; d, the common difference between consecutive terms; z, the last or n^{th} term; and s the sum of n terms.

Having any three of these we can find the remaining two by means of the relations which we proceed to develope.

Let a, $a+d$, $a+2d$, $a+3d$, &c., be the consecutive terms of the series. Then it is readily seen that the n^{th} term is $a+(n-1)d$;

$$\therefore\ z=a+(n-1)d.\ .\ .\ .\ (A)$$

To find S.

$$S=a+(a+d)+(a+2d)+\ .\ .\ .\ .(a+\overline{n-1}.d).$$

and reversing the order of the terms,

$$S=(a+\overline{n-1}.d)+(a+\overline{n-2}.d)+\ .\ .\ .\ .\ +a$$

adding, $2S=(2a+\overline{n-1}.d)+(2a+\overline{n-1}.d)+\ .\ .\ .\ .$ to n terms,

$$= n(2a + \overline{n - 1}.d)\,;$$

$$\therefore\; S = \frac{n}{2}(2a + \overline{n - 1}.d)\,. \;.\;.\;.\; (B)$$

Formulæ (A) and (B) involve all possible relations among the five quantities given above.

Ex. 125. Given $z = 13$, $d = 3$, $n = 5$, to find S.

From (A), $13 = a + 12 \;\therefore\; a = 1$;

then from (B), $S = \frac{5}{2}(2 + 12) = 35$.

Ex. 126. A falling body descends $\frac{f}{2}$ feet in the first second, $\frac{3f}{2}$ in the second second, $\frac{5f}{2}$ in the third and so on; how far will it fall in the n^{th} second? How far in t seconds?

Here, $a = \frac{f}{2}$, $d = f$,

$$\therefore\; z = \frac{f}{2} + (n - 1)f = nf - \frac{f}{2} = f(n - \tfrac{1}{2})$$

= the distance in the n^{th} second.

and $s = \frac{t}{2}(2.\frac{f}{2} + \overline{t - 1}.f) = \frac{1}{2}ft^2$.

= distance fallen from rest in t seconds.

90. Multiplying out Art. 89, B, we have,

$$S = n\left(a - \frac{d}{2}\right) + n^2 \,.\, \frac{d}{2}.$$

Hence, unless d be zero, an expression giving generally the sum of an arithmetic series must involve the *square of the number of terms*; and unless $d = 2a$ it will involve also the first power of that number.

Thus, $\frac{n^2}{2} - \frac{n}{3}$ expresses the sum of n terms of *some* arithmetic series. To find it;

$S = \frac{n}{2}(2a + \overline{n - 1}\,.\,d) = \frac{n^2}{2} - \frac{n}{3}$, must be true for all values of n since each is a general expression for the sum.

Let $n=1$; $\therefore\ a=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$,

" $n=2$; $\therefore\ 2a+d=2-\frac{2}{3}=\frac{4}{3}$ $\therefore\ d=1$,

and the series is,

$$\tfrac{1}{6}+\tfrac{7}{6}+\tfrac{13}{6}+\tfrac{19}{6}+\ \ldots\,.$$

Ex. 127. The sums of two A.P.[s] are as $11-5n$ to $11+3n$, to find the ratio of their sixth terms.

Let a, d, s denote the 1[st] term, the common diff., and the sum of n terms in the first series; and a_1, d_1, s_1 denote like quantities in the second.

Then, s and s_1 may be expressed by $n(11-5n)$ and $n(11+3n)$ respectively.

$$\therefore\ \frac{n}{2}(2a+\overline{n-1}\,.\,d)=n(11-5n)\text{ and }\frac{n}{2}(2a_1+\overline{n-1}\,.\,d_1)$$
$$=n(11+3n).$$

Hence, by giving values to n as above we obtain,

$$a=6,\ d=-10,\text{ and }a_1=14,\ d_1=6\,;$$

and the ratios of their sixth term is,

$$\frac{a+5d}{a_1+5d_1}=\frac{6-50}{14+30}=\frac{-44}{44}=-1.$$

91. If the number of terms be the unknown quantity we may have a quadratic in n, and the problem then admits of a double solution. In some cases both values of n are equally applicable.

Ex. 128. In an A.P. $a=7$, and $d=-2$, to find how many terms will make 12 when summed.

$$s=n^2\,.\,\frac{d}{2}+n\left(a-\frac{d}{2}\right)=-n^2+8n=12\,;$$

$$\therefore\ n=\frac{8\pm\sqrt{64-48}}{2}=6\text{ or }2.$$

Ex. 128′. In the A.P.[s] $6,\ 7\frac{1}{2},\ 9\ \ldots$ and $-3,\ -1,\ 1\ \ldots,$

(1) discover if there be a common term, and if so its value; (2) if there be a common number of terms for which the sum of the terms in each series is the same, and if so find its value.

(1) Taking the expression for the n^{th} term, we have, if n is a common term,

$$a+\overline{n-1}\,.\,d=a_1+\overline{n-1}\,.\,d_1,$$

$$\text{or } 6+(n-1)\tfrac{3}{2}=-3+(n-1)2.$$

whence, $n=19$; and the 19^{th} term is common.

Its value is $6+18\times\frac{3}{2}=33$.

(2) Taking the expression for the sum, since s and n are to be common,

$$s=\frac{n}{2}(2a+\overline{n-1}\,.\,d)=\frac{n}{2}(2a_1+\overline{n-1}\,.\,d_1),$$

$$\therefore\ 2a+\overline{n-1}\,.\,d=2a_1+\overline{n-1}\,.\,d_1,$$

$$\text{or } 12+(n-1)\tfrac{3}{2}=-6+(n-1)2.$$

Whence $n=37$; and the sum of the first 37 terms is the same for each series.

The value is, $\frac{37}{2}(12+36\times\frac{3}{2})=1221$.

If in the above n were fractional there could be no common term, as the number of terms must necessarily be integral.

Since we divide by n, $n=0$ is one solution: but this would be excluded by the nature of the problem.

92. When three quantities form three consecutive terms of an A.P. the middle one is said to be an arithmetic mean between the extreme ones.

If then A be an arithmetic mean between a and b we have,

$$A-a=b-A, \text{ and } \therefore\ A=\frac{a+b}{2}.$$

Hence the arithmetic mean between two quantities is the half sum of the quantities.

93. The sum of n consecutive natural numbers counting from unity is, $\dfrac{n(n+1)}{2}$.

The sum of n consecutive odd numbers 1, 3, 5, &c., is, n^2.

The sum of n consecutive even numbers 2, 4, 6, &c., is $n(n+1)$.

94. *A* sum of *P* dollars is put to simple interest for *t* years at *r* per unit per annum.

The interest at the end of the first year is Pr			
"	"	"	2nd year is $2Pr$
"	"	"	3rd year is $3Pr$
...	...	...	...
"	"	"	t^{th} year is tPr,

$\therefore$ the whole amount at the end of t years is,

$$A = P + Prt = P(1 + rt).$$

Ex. 129. A sum of *P* dollars is deposited yearly with a banker to be left for *t* years from the date of the first deposit. To find the accumulated amount at the end of the period.

The first payment draws interest for t years and $= P(1 + rt)$
The second " " " $t - 1$ years and $= P(1 + r.\overline{t - 1})$
... ...
Payment before the last draws interest for 1 year $= P(1 + r)$
Last payment draws interest for 0 years $= P$

$$\begin{aligned}\therefore \text{ Amount } &= P + P(1 + r) + \dots + P(1 + rt),\\ &= P(1 + t) + Pr(1 + 2 + \dots t),\\ &= P(1 + t) + Pr.\frac{t(1 + t)}{2},\\ &= \frac{P}{2}(1 + t)\,(2 + rt).\end{aligned}$$

GEOMETRIC SERIES.

95. When the ratio of any term in a series to the preceding term is a constant quantity, the series is called an *equimultiple series*, a *geometric series*, or a *geometric progression*

The quantities with which we have normally to [illegible] are *a* the first term, *r* the common ratio, *n* the number of terms, *z* the n^{th} term, and *S* the sum of *n* terms. Any three of these being given the remaining two may be found by the relations now to be developed.

Let a, ar, ar^2, ar^3, &c., be consecutive terms of the series. Then it is readily seen that the n^{th} term is ar^{n-1}.

$$\therefore\ z = ar^{n-1} \ldots . \ (A)$$ Formula

To find S.

$$S = a + ar + ar^2 + \ \ldots . \ ar^{n-} + ar^{n-1}$$

Multiply by r, $rS = \quad ar + ar^2 + \ \ldots . \ ar^{n-} + ar^{n-1} + ar^n$

Subtract, $S(1-r) = a - ar^n$

$$\therefore\ S = a . \frac{1-r^n}{1-r} \ldots . \ (B)$$

Otherwise as follows:

By division, $\dfrac{a}{1-r} = a + ar + ar^2 + \ \ldots . \ ar^{n-1} + \dfrac{ar^n}{1-r}$.

$$\therefore\ S = a + ar + \ \ldots . \ ar^{n-1} = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

$$= a . \frac{1-r^n}{1-r}$$

Formulæ (A) and (B) involve all possible relations amongst the five quantities given above.

Ex. 130. The population of a city increases at the rate of 5 per cent per annum, and it is now 20000. What was it 10 years ago?

In this case, since the series is a decreasing one r is a fraction, viz.: $\dfrac{1}{1.05}$, $a = 20000$ and $n = 11$, as there are 11 terms to find z.

From (A), $z = ar^{n-1} = \dfrac{20000}{(1.05)^{10}} = 12422$ nearly.

Problems in Geometric series involving r or n as unknown quanties cannot in general be conveniently solved without logarithms.

96. If in (B) r is less than unity, r^n may be made as small as we please by taking n sufficiently great. The *limit* then to which s approaches as n becomes indefinitely increased is, $\dfrac{a}{1-r}$, and this expression is usually taken as the *sum of the infinite series* in which r is less than one. It must be borne in

mind, however, that no number of terms which we could ever take would by summation be as great as $\frac{a}{1-r}$, for as the number of terms is infinite there must always be a remainder; but by taking a sufficient number of terms we may make their sum approach the value of $\frac{a}{1-r}$ as near as we please while we can never make that sum surpass it.

Ex. 131. To find the value of the repeater $.\dot{3}\dot{6}$.

This is equal to $\frac{36}{100}+\frac{36}{10000}+\ldots$ ad infinitum.

$\therefore\ a=\frac{36}{100},\ r=\frac{1}{100},$

and $\therefore\ s=\frac{a}{1-r}=\frac{36}{100}\div(1-\frac{1}{100})=\frac{36}{100}\times\frac{100}{99}=\frac{36}{99}.$

Ex. 132. The series, $1+\frac{n}{n+1}+\frac{n^2}{(n+1)^2}+\ldots$ ad infin. $=n+1$

For $s=\dfrac{1}{1-\dfrac{n}{n+1}}=\dfrac{n+1}{n+1-n}=n+1.$

If $n=1,\ 1+\frac{1}{2}+\frac{1}{4}+\ \ldots\ldots=2,$

$n=2,\ 1+\frac{2}{3}+\frac{4}{9}+\ \ldots\ldots=3,$

$n=3,\ 1+\frac{3}{4}+\frac{9}{16}+\ \ldots\ldots=4,$

&c., &c., &c.

97. In any three consecutive terms of a geometric series the middle term is called a *geometric mean* between the extreme terms.

Prob. To insert a geometric mean between two given terms.

Let a and b be the given terms, and g the geometric mean required. Then, since a, g, b are to form three terms of a geometric series, we must have

$$\frac{g}{a}=\frac{b}{g} \text{ and } \therefore\ g=\sqrt{ab}$$

Hence the geometric mean between two quantities is the square root of their product. (Compare Art. 33 where it is called a *mean proportional.*)

The side of a square is a geometric mean between the sides of the equal rectangle. For if a, b be the sides of the rectangle, and s that of the square, area $= ab = s^2$.

98. Prob. To insert n terms between two given terms so as to form a geometric series.

Let a and b be the terms, and let the completed series be,

$$a\ t_1,\ t_2,\ t_3,\ \ldots\ldots t_n,\ b.$$

Then, $$\frac{t_1}{a} = \frac{t^2}{t_1} = \frac{t_3}{t_2} = \ldots\ldots = \frac{b}{t^n} = r.$$

But $$\frac{b}{a} = \frac{t_1}{a} \cdot \frac{t_2}{t_1} \ldots\ldots \frac{b}{t_n} = r^{n+1}$$ there being $n+1$ factors.

$$\therefore\ r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

And $$t_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}} = (a^n b)^{\frac{1}{n+1}}$$

$$t_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}} = (a^{n-1} b^2)^{\frac{1}{n+1}}$$

&c., &c.,

99. If a sum of P dollars be put at interest for one year it amounts to $P(1+r)$ dollars. If this be now taken as a new principal and be put at interest for another year it amounts to $P(1+r)(1+r)$ or $P(1+r)^2$. Similarly in three years it will amount to $P(1+r)^3$; and in t years to $P(1+r)^t$ dollars.

Therefore if A denotes the amount we have

$$A = P(1 + r)^t.$$

which is the fudamental formula in compound interest.

It is evident that the amounts at the ends of successive years form the geometric series,

$$P(1+r),\ P(1+r)^2,\ P(1+r)^3,\ \ldots\ P(1+r)^t.$$

Ex. 133. n annual payments of P dollars each are made into a bank to remain at compound interest. To find the total amount due at the date of the last payment.

Let R denote $1+r$.

The 1st payment remains $n-1$ yrs. $\therefore$ its amount is PR^{n-1}.
" 2nd " " $n-2$ " $\therefore$ " PR^{n-2}.
... ...
" last " " 0 " $\therefore$ " P.

$\therefore$ The total amount is $P(1+R+ \ldots . R^{n-1})$.

$$\therefore\ A = P \cdot \frac{R^n - 1}{R-1};$$

$$\text{or}\quad A = P \cdot \frac{(1+r)^n - 1}{r}.$$

This gives the amount of an annuity which has been foreborne or left unpaid for a period of n years.

To find the present value of such an annuity, or the sum which when put to interest will produce its equivalent, we have,

$$V = \frac{A}{R^n} = \frac{P}{R}\left(1 - \frac{1}{(1+r)^n}\right).$$

Ex. 134. A corporation borrows P dollars to be paid in n equal annual instalments, each instalment to include all interest due at the time of its payment. To find the value of the instalment.

Let P denote the instalment and a, b, c, &c., the sums paid in successive years upon the principal.

Then, 1st payment $= p = a + Pr$,
amount unpaid $= P - a$;
2nd payment $= p = b + (P-a)r$, whence $b = aR$,
amount unpaid $= P - a - b = P - a - aR$;
3rd payment $= p = c + (P - a - aR)r \ \therefore\ c = aR^2$,
amount unpaid $= P - a - b - c$, &c.
... ...
Similarly, n^{th} payment $= p = aR^{n-1}$,
amount unpaid $= P - a - b - c -$ &c.,
$= P - a - aR - aR^2 - \ldots - aR^{n-1}$.

But the amount unpaid after the last payment must be zero; hence,

$$P - a(1 + R + R^2 + \ldots . R^{n-1}) = 0,$$

$$\therefore\ P=\frac{a(R^n-1)}{R-1} \text{ and } a=\frac{P(R-1)}{R^n-1}=\frac{Pr}{R^n-1}.$$

Hence, $p=\dfrac{Pr}{R^n-1}+Pr=P.\dfrac{rR^n}{R^n-1}.$

HARMONIC SERIES.

100. A number of terms is said to form a Harmonic series when the reciprocals of the terms form an Arithmetic series; so that if the reciprocals of the terms be taken in any arithmetic series we have a Harmonic series.

Thus $1,\ 3,\ 5,\ 7,\ 9,$ is an Arithmetic series,
and $1,\ \frac{1}{3},\ \frac{1}{5},\ \frac{1}{7},\ \frac{1}{9},$ is a Harmonic series.

Let a, b, c be three terms in Harmonic Progression;

then $\dfrac{1}{a},\ \dfrac{1}{b},\ \dfrac{1}{c}$ are in A.P., and consequently,

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}= \text{the common difference.}$$

$$\therefore\ \frac{a-b}{a}=\frac{b-c}{c};$$

or $a:c::a-b:b-c.$

And three terms are in Harmonic progression or series, or they form a Harmonic proportion when *the first is to the third as the difference between the first and second is to the difference between the second and third.*

This is frequently taken as the definition of Harmonic Proportion; and a series of terms in which any three taken consecutively form a Harmonic Proportion is a Harmonic series.

Problems in H. P. are best solved as problems of A. P. by means of the relation given in the first definition of a Harmonic series.

Ex. 135. To find a Harmonic mean between A and B.

Let H be the mean. Then,

$\dfrac{1}{A},\ \dfrac{1}{H},\ \dfrac{1}{B}$ are to be in Arithmetic proportion,

$$\therefore \frac{1}{H} - \frac{1}{A} = \frac{1}{B} - \frac{1}{H}, \text{ and } \frac{2}{H} = \frac{1}{B} + \frac{1}{A},$$

$$\therefore H = \frac{2AB}{A+B}.$$

a : c :: a-b : b-c

101. Harmonic proportion is so named on account of the similarity which exists between its terms and the relative lengths of a string which sound the harmonics in music. Its chief application, however, is in Geometry.

Let A, X, B, Y be four points in a line. Then AX, AB, AY form three magnitudes which may be taken as terms of a harmonic proportion, if AX is to AY as the difference between AB and AX is to the difference between AY and AB; *i.e.*, if AX : AY :: BX : YB.

A ——a—— X ——b—— B ——c—— Y

The points A, X, B, Y are then said to form a harmonic range, and the line AB is said to be harmonically divided in X and Y. The properties of harmonically divided lines is an important one in modern geometry.

VARIATION.

102. When two quantities are so connected that a change of value in one is accompanied by a change of value in the other, in such a way that their ratio remains constant, one of the quantities is said to *vary* as the other. Variation is usually denoted by the mark $\propto$, and is only a kind of generalized proportion.

If $A \propto B$, then $\frac{A}{B}$ = constant $= n$ suppose

$$\therefore A = nB.$$

Hence when one quantity varies as another they are connected by a constant factor.

i. If $A = nB$, A varies directly as B.

ii. If $A = \frac{n}{B}$, A varies inversely as B.

iii. If $A = mBC$, A varies jointly as B and C.

iv. If $A = m\dfrac{B}{C}$, A varies directly as B and inversely as C.

Ex. 136. The space passed over by a body falling from rest varies as the square of the time, and experiment has shown that it descends 64 feet in 2 seconds. Find the relation between the space and the time.

$$S \backsim t^2 \quad \text{we may write } S = nt^2.$$

But when $t = 2$, $S = 64$.

$$\therefore\ 64 = 4n \text{ and } n = 16.$$

$$\therefore\ S = 16t^2.$$

Ex. 137. The earth's radius is 4,000 miles and its attraction upon a body without it varies inversely as the square of the distance from its centre. The number of beats which a pendulum makes in a day varies as the square root of the earth's attraction upon it. How much would a clock with a seconds pendulum lose daily if taken one mile high?

Let g = the earth's attraction at its surface, and r = the earth's radius. Then,

$$g \backsim \frac{1}{r^2}.$$

But if n = the number of beats per day at the earth's surface, and n_1 at the height of one mile,

$$n \backsim \sqrt{g} \backsim \frac{1}{r}\ \therefore\ n = \frac{a}{r}, \text{ where } a \text{ is a constant;}$$

$$\therefore\ a = rn;\ \text{and } n_1 = \frac{a}{r_1}, \text{ where } r_1 = 4001;$$

$$\therefore\ n = n.\frac{r}{r_1};$$

$$\text{and the loss} = n - n_1 = n.\frac{r_1 - r}{r_1} = 86400 \times \frac{1}{4001}$$

$$= 21.59 \text{ seconds.}$$

103. Let C vary as $\dfrac{A}{B}$; then we may write $C = \dfrac{nA}{B}$.

Now if C is constant, A must vary as B; and if B is constant A must vary as C. But multiplying by B, $BC = nA$; and therefore A varies as BC.

Hence if A varies as B when C is constant, and varies as C when B is constant, it varies as BC when both are allowed to change.

Ex. 138. It is proved in Euc. VI. 1, that the area of a triangle varies as its base when its altitude is unchanged; and similarly it varies as the altitude when the base is unchanged; hence it varies as the product of the base and altitude.

If then $\triangle$ denote the area, b the base and p the altitude, we have $\triangle \backsim bp$; and hence $\triangle = nbp$, where n is an unknown constant. Now the right-angled triangle whose sides are each 1 is one-half the square of which its hypothenuse is a diagonal, and therefore its area is $\frac{1}{2}$; $\therefore n = \frac{1}{2}$, and $\triangle = \frac{1}{2}bp$.

If the three sides of a triangle vary so as to keep all their ratios constant, the triangle remains always similar to a given triangle.

In this case $p \backsim b$ and hence we may write $p = mb$, and therefore $\triangle = \frac{1}{2}mb^2$; *i.e.* the area of a triangle varies as the square of one of its sides when the triangle remains similar to a given triangle.

PERMUTATIONS—VARIATIONS—COMBINATIONS.

104. If a number of objects be taken and formed into groups such that the relative positions of *all* the objects are not the same in two groups; then, if each group contains all the objects concerned it is called a *permutation*; but if it contains only a certain number of objects, less than the whole, it is a *variation*.

If the groups are such that no two groups contain the same assemblage of objects, each group is called a *combination*.

Frequently no distinction is drawn between variations and permutations, and it is readily seen that the permutations are only the variations in a particular case.

For this reason, and because the word variation has already been used in a different sense, we shall employ the word permutation for both.

PERMUTATIONS.

105. Take two letters a and b; the permutations which can be made out of these are ab and ba, *i.e.* two.

Take three letters and we have, abc, acb, bac, bca, cab, cba, or six permutations.

Similarly four letters will give us 24 permutations.

But $2 = 1.2$, $6 = 1.2.3$, $24 = 1.2.3.4$;

From analogy we infer that with n letters the number of permutations is expressed by $1.2.3 \ldots . n$.

106. Let there be 4 letters a, b, c, d, and let us take only two at a time; then we have, ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd, dc, or 12 in all. But $12 = 4.3$.

In like manner if three letters out of the four be taken at a time we would find the number of permutations expressed by $4 . 3 . 2$. And if we employ 5 letters, taking three at a time, we have for the number of permutations, $5 . 4 . 3$ or 60. Hence from analogy we infer that the number of permutations of n letters when r are taken together is expressed by

$$n(n-1)(n-2) \ldots . \text{ to } r \text{ factors.}$$

We propose to show that both of these inferences are correct.

107. Let $a, b, c \ldots . n$ be n different letters, and let us adopt the symbol $n\text{P}m$ to stand for "the number of permutations of n letters with m letters in a group."

(1.) If we place only one letter in a group we can evidently have n groups and no more; $\therefore$ $n\text{P}1 = n$.

(2.) Put a aside and we have $n-1$ letters left; and these taken in groups of one give $n-1$ groups. Now place a *before* each one of these letters, and we have $n-1$ groups of two letters in which a comes first. Similarly by operating on b we will have $n-1$ groups of two in which b comes first; then $n-1$ in which c comes first; and so on. But there are n different letters to come first, and each of these gives us $n-1$ groups; $\therefore$ the whole number of groups of two letters will be $n(n-1)$. $\therefore$ $n\text{P}2 = n(n-1)$.

(3). Setting a aside again we have $n-1$ letters left; out of these taking two at a time we may form $(n-1)(n-2)$ groups. For if $n2r=n(n-1)$, then $(n-1)P2=(n-1)(n-2)$.

Now put a *before* each groug, and we have $(n-1)(n-2)$ groups of three letters, with a first; and a like number with b first, and with c first, &c., and as there are n letters to stand first the whole number of groups is $n(n-1)(n-2)$.

$\therefore\ nP3=n(n-1)(n-2)$; and the law is manifest.

Suppose this law holds for r things in a group, then $nPr =n(n-1)(n-2)\ \ldots\ldots$ to r factors,

Putting a aside we have $n-1$ letters, and these taken r together give $(n-1)Pr=(n-1)(n-2)\ \ldots\ldots$ to r factors. Now putting a before each group we introduce an additional letter and thus have $r+1$ letters in a group. Hence there are $\{(n-1)(n-2)\ \ldots\ldots$ to r fact.$\}$ groups of $r+1$ letters with a standing first. Similarly there is the same number with b standing first; with c standing first; and so on. Hence there are $n\{(n-1)(n-2)\ \ldots\ r$ fact.$\}$ groups of $r+1$ letters altogether. Or

$$nP(r+1)=n(n-1)(n-2)\ \ldots\ldots r+1 \text{ factors,}$$

since we introduce the additional factor n.

If then the law holds for n letters taken r together, it holds when taken $r+1$ together. But it holds when $r=3$, and therefore for $r+1$ or 4, also for $4+1$ or 5 and so on for any number. $\therefore$ generally,

$$\underline{nPr=n(n-1)(n-2)\ \ldots\ldots (n-r+1)\ \ldots\ldots A}$$

Making r equal to n we have for the number of permutations of n things when taken all in each group,

$$nPn, \text{ or simply } P=n(n-1)(n-2)\ \ldots\ldots 3.2.1$$
$$=1.2.3\ \ldots\ldots n.$$

108. The continuous product of n consecutive natural numbers beginning with 1 is called *factorial* n, and is indicated by the symbol $n!$ or $\underline{|n}$. Thus $4!$ or $\underline{|4}$ means $1 \cdot 2 \cdot 3 \cdot 4$.

Taking the formula $nPr=n(n-1)(n-2)\ \ldots\ldots (n-r+1)$, and multiplying and dividing by $1 \cdot 2 \cdot 3\ \ldots\ldots (n-r-1)(n-r)$, we have,

Best Formula $n(n-1)(n-2)$ ...... $n-r+1$

r = no taken together

$$nPr = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)(n-r)\ldots\ldots3\,.\,2\,.\,1}{1\,.\,2\,.\,3\,\ldots\ldots(n-r)}\,;$$

$$\text{or } nPr = \frac{\underline{|n}}{\underline{|n-r}}\ \ldots\ldots B.$$

Making $r=n$ in Art. 107, A, we have for the number of permutations when all the articles are included in each group,

$$P = \underline{|n}.$$

But making $r=n$ in B, we have $P = \dfrac{\underline{|n}}{\underline{|0}}$. Hence we must interpret $\underline{|0}$ as meaning unity.

Ex. 139. If $nP4 : (n+2)P5 :: 3 : 56$, find n.

$$\frac{n(n-1)(n-2)(n-3)}{(n+2)(n+1)n(n-1)(n-2)} = \frac{3}{56}\,;$$

$$\therefore\ 56(n-3) = 3(n+2)(n+1)\,;$$

$$\text{whence } n = 6 \text{ or } 9\tfrac{2}{3}.$$

Of which, although both *numbers* satisfy the condition, the integer only will apply to *articles*.

109. *If* u *of the articles be alike.* If the u articles were all different they would give rise to $\underline{|u}$ permutations, each of which could be combined with each permutation from the remaining articles, and this would give the complete number of permutations of n different objects taken all together.

If we denote the number of permutations of n articles taken all together, of which u are alike, by $P(u)$ we have

$$P(u)\,.\,\underline{|u} = P = \underline{|n}\,;\ \text{and}\ \therefore\ P(u) = \frac{\underline{|n}}{\underline{|u}}.$$

Similarly if v other articles be alike,

$$P(u)\ (v) = \frac{\underline{|u}}{\underline{|u}\ \underline{|v}}.$$

Ex. 140. How many permutations can be made from the letters in the word Ontario ?

Here $n=7$ and $u=2$, since there are two O's ;

$$\therefore\ P(2) = \frac{7.6.5.4.3.2.1}{1.2} = 2520.$$

COMBINATIONS.

110. Let nCr denote the number of combinations of n things taken r together. Then from the definition of a *combination* each one would give rise to $\underline{|r}$ permutations. For *abcd* forms only one combination however you arrange the letters, while it can give 1.2.3.4 different permutations. Hence, (the number of combinations) $\times$ (the number of permutations which can be made from each combination) = the total number of permutations:

$$\text{that is, } nCr \times \underline{|r} = \frac{\underline{|n}}{\underline{|n-r}}\text{; Art 108, } B.$$

$$\therefore nCr = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} \ldots\ldots C$$

This may be put in another form ;

$$\frac{\underline{|n}}{\underline{|n-r}} = \frac{n(n-1)\ldots\ldots(n-r+1)(n-r)\ldots\ldots 2.1}{(n-r)\ldots\ldots 2.1}$$

$$= n(n-1)\ldots\ldots(n-r+1)$$

$$= n(n-1)\ldots\ldots \text{to } r \text{ factors};$$

$$\text{And } \underline{|r} = 1.2.3 \ldots\ldots \text{to } r \text{ factors};$$

$$\therefore nCr = \frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}\ldots\ldots \text{to } r \text{ factors} \ldots\ldots D.$$

From this it appears that the product of any n consecutive integers is divisible by factorial n, since nCr must necessarily be an integer.

Ex. 141. How many different guards of 4 men can be chosen from a company of 10 men ?

Here $n = 10$, $r = 4$; $\therefore\ 10C4 = \frac{10}{1}.\frac{9}{2}.\frac{8}{3}.\frac{7}{4} = 210.$

111. If in Art. 110, C, we make $n-r=p$, we have $r=n-p$,

Best Formula $\frac{n(n-1)(n-2)\ldots n-r+1}{1.2.3.\ldots\ldots r}$

$$\text{and} \quad nCn - p = \frac{\underline{|n}}{\underline{|n-p}\;\underline{|p}}.$$

and substituting r for p,

$$nCn - r = \frac{\underline{|n}}{\underline{|r}\;\underline{|n-r}} = nCr.$$

Hence the number of combinations of n things taken r together is the same as that of n things taken $n-r$ together.

This must necessarily be true for the following reasons :— When from n things we take out r to form a combination, we leave another combination of $n-r$ things, and therefore the number of each must be the same. These are called *supplementary* combinations.

$$\text{Thus } 6C2 = \frac{6}{1}.\frac{5}{2} = 15\ ;\quad 6C4 = \frac{6}{1}.\frac{5}{2}.\frac{4}{3}.\frac{3}{4} = 15.$$

112. Forming the combinations of 6 articles 1 at a time, 2 at a time, &c., we have,

$$6C1 = 6,\ 6C2 = 15,\ 6C3 = 20,\ 6C4 = 15,\ 6C5 = 6.$$

Hence *if* n *is an even number the largest number of combinations can be made by taking* $\frac{n}{2}$ *articles at a time.*

Again, forming the combinations of 7 articles 1 at a time, 2 at a time, &c., we have,

$$7C1 = 7,\ 7C2 = 21,\ 7C3 = 35,\ 7C4 = 35,\ 7C5 = 21, \text{ \&c.}$$

Hence, *if* n *is an odd number the maximum number of combinations occurs when the articles are taken* $\frac{n+1}{2}$ or $\frac{n-1}{2}$ *at a time.* In this case there are two greatest terms.

113. To find how often any one thing occurs in the combinations of n things taken r together.

If from all the combinations containing a we take out a we will have left the combinations of $n-1$ things taken $r-1$ together. Hence in the combinations,

$$nCr, \text{ any one thing occurs } n-1Cr-1 \text{ times.}$$

Similarly any two articles will occur together $n-2Cr-2$ times, &c.

Ex. 142. The number of combinations of n letters 5 together in all of which a, b, c occur is 21. Find the number when taken 6 together and in all of which a, b, c, d occur.

$n-4 C 6-4 = 6C2$

Here $\quad n-3C5-3=21$, and $n-4C6-4=?$

$$\frac{n-3}{1}\cdot\frac{n-4}{2}=21, \text{ whence } n=10,$$

and $\quad 10-4C2=\frac{6}{1}\cdot\frac{5}{2}=15.$

Ex. 143. If the combinations of $n+1$ things taken $n-1$ together be 36; find the permutations of n things altogether.

From Art. 110, C,

$$n+1Cn-1=\frac{|\underline{n+1}}{|\underline{n-1}\;|\underline{(n+1)-(n-1)}}=\frac{|\underline{n-1}}{2\,|\underline{n-1}}$$

$$=\frac{(n+1)n}{2}=36;$$

$\therefore \quad n=8$, and $\mathrm{P}=|\underline{8}=40320.$

Combinations find their application in the Binomial theorem, in Probabilities, &c.

BINOMIAL THEOREM.

114. The Binomial theorem is a formula by which we are enabled to write down the expression of a binomial to any power without the actual labor of multiplication.

We have, $(1+ax)(1+bx)(1+cx)(1+dx)=1+(a+b+c+d)x$
$+(ab+ac+ad+bc+bd+ed)x^2+(abc+abd+acd+bcd)x^3$
$+abcd.x^4.$

Now a, b, c, d are the combinations of 4 things taken 1 at a time, $\therefore$ the number of terms in the coefficient of $x=4C1$.

ab, ac, ad, bc, bd, cd are the combinations of 4 things taken two at a time, $\therefore$ the number of terms in the coefficient of $x^2=4C2$.

Similarly, the number of terms in the coefficient of $x^3 = 4C3$; and the number of terms in the coefficient of $x^4 = 4C4 = 1$.

$\therefore$ making $a=b=c=d=1$, we have

$$(1+x)^4 = 1 + 4C1.x + 4C2.x^2 + 4C3.x^3 + 4C4.x^4.$$

In like manner by starting with 5 factors we may show that,

$$(1+x)^5 = 1 + 5C1.x + 5C2.x^2 + 5C3.x^3 + 5C4.x^4 + \&c.$$

The regularity of these expressions suggests at once that

$$(1+x)^n = 1 + nC1.x + nC2.x^2 + nC3.x^3 + \ldots + nCn.x^n, \ldots A.$$

which is one form of the Binomial theorem.

115. Putting for $nC1$, $nC2$, &c., their values in factorials, we have,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots B.$$

which is a second and commoner form of the Binomial theorem.

Ex. 144. Find the 5th term in the expression of $(1+n)^n$

5th term is $\dfrac{n(n-1)(n-2)(n-3)}{1.2.3.4}.n^4$.

Ex. 145. Show that when x is very small, $(1+x)^{10} = 1 + 10x$ approximately.

$$\left((1+x)^{10} = 1 + 10x + \frac{10.9}{1.2}x^2 + \ldots\right)$$

But x being very small, x^2 and all higher powers of x may be rejected in comparison with 1 and $10x$.

116. In B, Art. 115, write $\dfrac{x}{a}$ for x, and we have,

$$\left(1+\frac{x}{a}\right)^n = 1 + n.\frac{x}{a} + \frac{n(n-1)}{1.2}.\frac{x^2}{a^2} + \frac{n(n-1)(n-2)}{1.2.3}.\frac{x^3}{a^3} + \ldots$$

But $\left(1+\dfrac{x}{a}\right)^n = \dfrac{1}{a^n}(a+x)^n$

$\therefore$ multiplying both sides by a^n,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3 \ldots C.$$

And this is a third form in which the Binomial theorem is written.

117. Dividing both sides in C, Art. 116, by $\lfloor n$, we get,

$$\frac{(a+x)^n}{\lfloor n} = \frac{a^n}{\lfloor n} + \frac{a^{n-1}}{\lfloor n-1} \cdot \frac{x}{\lfloor 1} + \frac{a^{n-2}}{\lfloor n-2} \cdot \frac{x^2}{\lfloor 2} + \frac{a^{n-3}}{\lfloor n-3} \cdot \frac{x^3}{\lfloor 3} + \dots . D.$$

A fourth and very symmetrical form of the theorem.

118. We have drawn these expressions for the Binomial theorem from the expansions of $(1+x)^4$ and $(1+x)^5$. We shall now prove that if the theorem is true for $(1+x)^n$, it is also true for $(1+x)^{n+1}$.

Put $n+1=m$, then $n=m-1$; and writing this for n in B, Art. 115,

$$(1+x)^{m-1} = 1 + (m-1)x + \frac{(m-1)(m-2)}{1 \cdot 2}x^2 + \frac{(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3}x^3 + \dots .$$

Multiplying both sides by $1+x$, using detached co-efficients;

$$(1+x)^m = \left\{1 + (m-1) + \frac{(m-1)(m-2)}{1 \cdot 2} + \frac{(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3} + \right\} \times (1+1)$$

$$= 1 + m + \frac{m(m-1)}{1 \cdot 2} + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} + \dots .$$

Hence the formula is true for m; and $m=n+1$, $\therefore$ &c.

But the formula is true for $n=4$ as we have seen, $\therefore$ it is true for $n=5$, and if for $n=5$ then for $n=6$ and so on; i.e., it is generally true when n is any positive integer.

We have thus proved that the Binomial theorem holds when n is any positive integer. It may also be proved that it holds when n is any quantity whatever, but the general proof is beyond the scope of this work.

119. The following generalizations are readily drawn from the form of the theorem.

i. If n be a positive integer the series is finite and consists of $n+1$ terms, $\because$ n terms contain x and one term is without x.

ii. If n be not a positive integer the series can never terminate, as reducing n by units can never give a factor equal to zero.

iii. If n be a positive fraction and x negative, all the terms after the first are negative.

iv. If n be negative and x negative, all the terms are positive.

Ex. 146. $$\sqrt{1+x}=(1+x)^{\frac{1}{2}}=1+\tfrac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}x^2+\dots.$$

$$=1+\tfrac{1}{2}x-\frac{1}{4}\cdot\frac{1}{1.2}\cdot x^2+\dots.$$

$$=1+\frac{x}{2}-\frac{1}{1.2}\cdot\left(\frac{x}{2}\right)^2+\frac{1.3}{1.2.3}\cdot\left(\frac{x}{2}\right)^3-+\dots.$$

Ex. 147.

$$\sqrt{1-x}=(1-x)^{\frac{1}{2}}=1-\frac{x}{2}-\frac{1}{1.2}\left(\frac{x}{2}\right)^2-\frac{1.3}{1.2.3}\left(\frac{x}{2}\right)^3-,-\dots.$$

Ex. 148.

$$\frac{1}{1-x}=(1-x)^{-1}=1+x+\frac{-1(-1-1)}{1.2}x^2+\frac{-1(-2)(-3)}{1.2.3}x^3\dots.$$

$$=1+x+x^2+x^3+\dots.\ \text{(Art. 9.)}$$

Ex. 149.

$$(a^3-x^3)^{\frac{1}{3}}=a\left(1-\frac{x^3}{a^3}\right)^{\frac{1}{3}}=a\left\{1-\tfrac{1}{3}\cdot\left(\frac{x}{a}\right)^3+\frac{\frac{1}{3}(-\frac{2}{3})}{1.2}\cdot\left(\frac{x}{a}\right)^6-\dots.\right\}$$

$$=a-\frac{x^3}{3a^2}-\frac{2x^6}{1.2.3^2a^5}-\dots.$$

120. The Binomial theorem may be used for the expansion of the power of a trinomial or polynomial.

Ex. 150.

$$(1+ax+bx^2)^n=1+n(ax+bx^2)+\frac{n(n-1)}{1.2}(ax+bx^2)^2+\dots.$$

$$=1+nax\quad \left.\begin{array}{l}+nb\\[4pt] +\dfrac{n(n-1)}{1.2}a^2\end{array}\right|\,x^2+\left.\begin{array}{l}\dfrac{n(n-1)}{1.2}2ab\\[4pt] +\dfrac{n(n-1)(n-2)}{1.2.3.}a^3\end{array}\right|\,x^3+\dots.$$

Ex. 151. $\sqrt{1+x+x^2}=(1+x+x^2)^{\frac{1}{2}}$

$$=1+\tfrac{1}{2}(x+x^2)+\frac{\frac{1}{2}(-\frac{1}{2})}{1.2}(x-x^2)^2+\ldots.$$

$$=1+\frac{x}{2}+\frac{3x^2}{8}-\frac{3x^3}{16}\ldots.$$

121. The Binomial theorem may sometimes be employed to approximate to the roots of numbers.

Ex. 152. Required the fifth root of 12.

$$12=32-20=2^5(1-\tfrac{5}{8})$$

$$\therefore \sqrt[5]{12}=2(1-\tfrac{5}{8})^{\frac{1}{5}}=2\left\{1-\frac{1}{5}.\tfrac{5}{8}-\frac{4}{2.5^2}(\tfrac{5}{8})^2-\frac{4\cdot 9}{2.3.5^3}.(\tfrac{5}{8})^3-\ldots.\right\}$$

$$=2\left\{1-\tfrac{1}{8}-\tfrac{1}{32}-\tfrac{3}{256}-\ldots.\right\}=1.65 \text{ nearly.}$$

LOGARITHMS.

We propose to deal here with the nature and use of Logarithms, and not with their development.

122. Take the equalities, $2^0=1$, $2^1=2$, $2^2=4$, $2^3=8$, $2^4=16$, $2^5=32$, $2^6=64$, $2^7=128$, &c.; the quantities 1, 2, 4, 8, 16, 32, 64, 128, &c., are *numbers*; the indices of 2, *i.e.*, 0, 1, 2, 3, 4, 5, 6, 7 are the corresponding *logarithms*, and 2, the number raised to the several powers, is the *base*.

By tabulating these, as in the margin, we have a table of logarithms to the base 2. In like manner we may form a table of logarithms to the base 3, or to any other base which one may choose.

For common purposes the base employed is 10, for being at the same time the base of our numeral system, it possesses certain practical advantages over every other number.

To *illustrate* the practical applications of logarithms we may employ a table to any base

TABLE.

No.	Log.
1	0
2	1
4	2
8	3
16	4
32	5
64	6
128	7
&c.	&c.

whatever, for *the general properties of logarithms are the same for all bases.* Taking the table above, then, let it be required (1) to multiply 16 by 8.

$$\left.\begin{array}{l}\log.\ 16=4\\ \log.\ \ \,8=3\end{array}\right\}\text{add}$$
$$\overline{7}$$

Number of which 7 is the log. = 128, $\therefore 8\times16=128$.

(2) To divide 64 by 4.

$$\left.\begin{array}{l}\log.\ 64=6\\ \log.\ \ \,4=2\end{array}\right\}\text{subtract}$$
$$\overline{4}$$

Number of which 4 is the log. = 16, $\therefore 64\div4=16$.

We thus see that multiplication of numbers corresponds to addition of logarithms, and division of numbers to subtraction of logarithms. This will be shown more generally hereafter.

123. The above table is not complete, even as far as it goes, since the numbers do not follow each other in order. Thus it lacks the numbers 3, 5, 7, 9, &c. To find the logarithm of one of these numbers we notice that the numbers in our table are in geometric progression while the logarithms are in arithmetic progression. Hence the geometric mean between two numbers must correspond to the arithmetic men between their respective logarithms. Thus $3\frac{1}{2}$ is the logarithm of $\sqrt{8\times16}$ or 11 3136 . . . .

This may be readily shown as follows :

$$2^7=128=8\times16\ ;\ \therefore\ 2^{\frac{7}{2}}=\sqrt{8\times16},\ \text{or}\ 2^{3\frac{1}{2}}=11.31\ \ldots$$
$$\therefore\ 3\tfrac{1}{2}=\log.\ 11.31\ \ldots$$

By this means we may calculate the logarithm of 3.

1. $\sqrt{2\times4}=2.8284\ ;\ \frac{1}{2}(1+2)=1.5\ \ \therefore\ 1.5=\log.\ 2.8284,$

2. $\sqrt{4\times2.8284}=5.6568\ ;\ \frac{1}{2}(2+1.5)=1.75=\log.\ 5.6568,$

3. $\sqrt{2\times5.6568}=3.363\ ;\ \frac{1}{2}(1+1.75)=1.375=\log.\ 3.3630,$

4. $\sqrt{2.8284\times3.3630}=3.0842\ ;\ \frac{1}{2}(1.5+1.375)=1.4375$ $=\log.\ 3.0842\ ;$

And by continually approximating towards 3 we at last find log. $3 = 1.585$ . . . approximately. And in this way, although exceedingly operose, the logarithms of the prime numbers were once calculated.

We infer then that $2^{1.585} = 3$, *i.e.* $2^{\frac{1585}{1000}} = 3$, or $2^{1585} = 3^{1000}$. Of course we have no means of proving this except through logarithms themselves.

124. The *Base.* In the computation of logarithms by means of series. we come naturally upon a system having the strange number 2.7182818 . . ., generally designated by e or ϵ, as a base. These are called *natural* logarithms, *Napierian* logarithms, and sometimes *hyperbolic* logarithms.

This system is usually employed in mathematical analysis. The only other system in use is the one having 10 as a base. These are *common* or *decimal* logarithms.

Let a denote any base ; then,

$\because\ a^0 = 1$, the logarithm of 1 is always zero.

If $a > 1$, then $a^n > 1$ and $a^{-n} < 1$.

And, since a is greater than 1 in both systems of logarithms, *the logarithm of a quantity greater than* 1 *is positive, and of a quantity less than* 1, *negative.*

Thus log 3 is a positive quantity ;
but log .3 is a negative "

Since $a^{-\infty} = \frac{1}{a^\infty} = 0 \quad \therefore$ log. $0 = -\infty$. Hence the logarithms of all proper fractions lie between 0 and $-\infty$. And since $a^\infty = \infty$, the logarithms of numbers above unity lie between 0 and $+\infty$.

Since if a is positive no power of a can be negative it follows that *negative quantities have no special logarithms.*

125. The number which we found for log. 3 to the base 2 is composed of two parts, an integer 1 called the *characteristic,* and a fractional part .585 . . . called the *mantissa.*

In decimal logarithms the distinction between these parts is important.

126. The *characteristic.* Since, $10^{-3} = .001$, $10^{-2} = .01$, $10^{-1} = .1$, $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, &c., we have,

number,	.001	.01	.1	1	10	100	1000	&c.
logarithm,	-3	-2	-1	0	1	2	3	&c.

We see from this that the characteristics are the logarithms of numbers made up of unity and ciphers only.

Also, for a number between	100 and	1000,	log = 2 +	a decimal
" "	10 "	100,	log = 1 +	"
" "	1 "	10,	log = 0 +	"
" "	.1 "	1,	log = − 1 +	"
" "	.01 "	.1,	log = − 2 +	"
&c.	&c.,		&c.	

Hence we may write down the characteristic of the logarithm of any given number at sight by the following rule :

If the number is a decimal the characteristic is negative and greater by unity than the number of ciphers to the right of the decimal point.

If the number is integral or contains an integral part the characteristic is positive and less by unity than the number of figures in the integral part.

Or by the following rule :

Call the units place zero and count from it to the significant figure farthest upon the left. The number of that figure is the characteristic, positive if counted leftward, negative if rightward.

Ex. 153. To find the characteristics of, .00000734, 386.5, 943007.0162.

0 123456	210	543210
0.00000754	386.5	943007.0162
|	|	|
units place.	units pl.	units pl.
∴ −6	∴ 2	∴ 5

For reasons now readily seen the characteristic is not usually written in tables of common logarithms.

127. The *Mantissa*. Let log 425 be $2+m$, where m is the mantissa or decimal part.

Dividing 425 by 10, we must subtract the log of 10 from that of 425, (Art. 122). ∴ log of $42.5 = 1 + m$.

Dividing by 10 again, log of $4.25 = 0 + m$.

Dividing by 10 again, log of $.425 = -1 + m$.

&c., &c,

We notice that the mantissa remains constant, the only change being in the characteristic. Hence we may sum up the significance of the parts of a logarithm as follows :

The mantissa *is connected with the group of figures and their arrangement; the* characteristic, *with the position of the decimal point.*

128. A table of decimal logarithms registers only mantissæ; and since these start from zero at every power of 10, the table extends only between two consecutive powers of 10. For 7-place logarithms, *i.e.*, for those with 7 decimals in the mantissæ, the usual extent is from 10^4 to 10^5.

We give below a portion of a table of 7-place logarithms taken from Hutton's tables as published by Chambers.

No.	0	1	2	3	4	5	6	7	8	9	D.
2397	3796680	6686	7043	7224	7405	7586	7767	7918	8130	8311	181
98	8492	8673	8854	9035	9216	9397	9578	9759	9940	$\bar{0}121$	
99	3800302	0484	0665	0846	1027	1208	1389	1570	1750	1931	
2400	2112	2293	2474	2655	2836	3017	3198	3379	3560	3741	
01	3922	4102	4283	4464	4645	4826	5007	5188	5368	5549	
02	5730	5911	6092	6272	6453	6634	6815	6995	7176	7357	
03	7538	7718	7899	8080	8261	8441	8622	8803	8983	9164	
04	9345	9525	9706	9887	$\bar{0}067$	0248	0428	0609	0790	0970	
05	3811151	1331	1512	1693	1873	2054	2234	2415	2595	2776	
D { 181	P.	18	36	54	72	91	109	127	145	163	

129. *The working of a table of logarithms* consists in two operations the converse of one another, viz: (*a*) given an arrangement of figures to find the corresponding mantissa, and (*b*) given a mantissa to find the corresponding arrangement of figures; for the characteristic not being registered has no immediate connection with the table.

(*a*) Given an arrangement of figures to find the corresponding mantissa.

The table above mentioned gives the mantissæ of all arrangements of 5 figures at sight; four of these are found in the column of numbers marked No., and the fifth in the horizontal row at the top.

i. *When the arrangement contains 5 figures.*

To find the mantissa of 23987, start at 2398 in the first column and proceed horizontally until in the column marked 7 at the top. To the figures 9759 there found prefix the 379 which the first column shows to be common to several rows. We thus have, mantissa of 23987 = 3799759.

ii. *When the arrangement contains less than 5 figures.*

Add ciphers or suppose them to be added to raise the number of figures to 5, and then proceed as in i.

Thus, mantissa of 24 = mantissa of 24000 = 3802112.

And, log 24 = 1.3802112.

iii. *When the arrangement contains more than 5 figures.*

To find the mantissa of 2403872.

We find the mantissa of the first 5 figures to be 3808983.

In order to show what is to be done with the remaining figures 72 we shall explain the column and row of the table marked D and P respectively.

$$\left.\begin{array}{l}\text{Mant. of } 24038 = 3808983 \\ \text{“} \quad \text{“} \quad 24039 = 3809164\end{array}\right\} \begin{array}{l}\text{D = difference of} \\ \text{mantissæ} = 181.\end{array}$$

Now the 72 occurring here is $\frac{72}{100}$ of the difference between 24038 and 24039. ∴ we should add to 3808983, $\frac{72}{100} \times 181$.

$$\text{But } \tfrac{72}{100} \times 181 = 7 \times \tfrac{181}{10} + \tfrac{1}{10}(2 \times \tfrac{181}{10}).$$

The row marked P (proportional parts) gives the multiples of $\frac{181}{10}$ from 1 to 9. Thus under 7 at the top we find 127 which is $7 \times \frac{181}{10}$ to the nearest unit. Under 2 we have 36, one-tenth of which is 3.6 or 4 to the nearest unit. Hence the mantissa of 2403872 is $3808983 + 127 + 4 = 3809114$.

iv. As a special case let it be required to find the mantissa to the arrangement 24044.

Referring to our table we find the first cipher overlined $\bar{0}$. This indicates that the three figures to be prefixed to the four there given change at this point from 380 to 381. The mantissa is accordingly 3810067.

(*b*) Given a mantissa to find the corresponding arrangement of figures.

Take the mantissa 3806745, for example.

The highest mantissa in the table capable of being subtracted from this is 3806634; and we proceed as follows:

Mant. given . . .	3806745	
tab. mant.	3806634	arrrang't = 24025
Diff. of mant.	111,	
highest subtractive number from P	109	. . . number . . . 6
Diff. . . .	2,	
Subtract number from P after dividing by 10 . . .	1.8	. . . number . . . 1.

∴ arrangement = 2402561.

130. *To find the logarithm of a number.* Find the mantissa of the arrangement without any reference to the decimal point, and then prefix the characteristic according to rule Art. 126.

To find the number answering to a given logarithm. Find the arrangement corresponding to its mantissa and then fix the decimal point by means of the characteristic and rule Art. 126.

131. *To perform multiplication by logarithms.*

$$\text{Let } a^x = m, \text{ then } x = \log m;$$

$$a^y = n, \text{ then } y = \log n.$$

$$\text{Then, } mn = a^x \cdot a^y = a^{x+y};$$

$$\text{and } \log mn = x + y = \log m + \log n.$$

∴ *to multiply numbers we add their logarithms and take the number answering to their sum.*

Ex. 154. To multiply 23.974 by .024056.

$$\log\ 23.974 = 1.3797405$$

$$\log\ .024056 = \bar{2}.3812234$$

$$\log\ .57671 = \bar{1}.7609639, \text{ sum.}$$

∴ .57671 is the product.

As in this example, the negative sign of the characteristic is placed above it to save room, and it must be borne in mind that although the characteristic may be negative the mantissa is always positive.

132. To perform division by logarithms.

with the notation of Art. 131,

$$\frac{m}{n}=\frac{a^x}{a^y}=a^{x-y}\,;$$

$$\text{and } \log\frac{m}{n}=x-y=\log m-\log n.$$

$\therefore$ *To divide one number by another subtract the logarithm of the divisor from that of the dividend and take the number answering to the difference.*

Ex. 155. To divide 1.4936 by .007453.

$$\begin{aligned}
\log\ 1.4936 &= 0.1742343\\
\log\ .007453 &= \bar{3}.8723311\\
\log\ 200.4025 &= 2.3019032, \text{ difference.}
\end{aligned}$$

$\therefore$ 200.4025 is the quotient.

133. To raise a number to any power.

$$a^x=n\ ;\ \therefore\ n^y=(a^x)^y=a^{xy}\,;$$

$$\text{and } \log n^y=xy=y\log n.$$

$\therefore$ *To raise a number to any power multiply the logarithm of the number by the index of the power required, and take the number answering to the product.*

Ex. 156. To find the 21st power of 1.06.

$$\begin{aligned}
\log\ 1.06 &= 0.0253059\\
&\quad\ 21=\text{index.}\\
\log\ 3.39957 &= 0.5314239
\end{aligned}$$

$\therefore\ (1.06)^{21}=3.39957.$

Ex. 157. Find the value of $(.4726)^8$.

$$\begin{aligned}
\log\ .4726 &= \bar{1}.6744937\\
&\quad\ 8\\
\log\ .00248857 &= \bar{3}.3959496
\end{aligned}$$

$\therefore\ (.4726)^8=.00248857\ldots$

In this example the mantissa being positive we have, upon multiplying, $-8+5.3959496=-3+.395\ldots$

134. To extract any root of a number.

Since $a^x=n$,

$$(n)^{\frac{1}{y}}=(a^x)^{\frac{1}{y}}=a^{\frac{x}{y}}.$$

$$\therefore \log n^{\frac{1}{y}} = \frac{x}{y} = \frac{\log m}{y}.$$

$\therefore$ *To extract any root of a number, divide the logarithm of the number by the number denoting the root to be extracted, and take the number corresponding to the quotient.*

Ex. 155. Find the value of $(.017325)^{\frac{1}{7}}$

$$\log .017325 = \bar{2}.2386732,$$

$$\text{Divide by 7 gives, } \bar{1}.\ 7483819,$$

$$\text{corresponding number } = .56025.$$

In this case having a negative characteristic we make it evenly divisible thus:

$\bar{2}.2386732 = \bar{7} + 5.2386732$, which divided by 7 gives the quotient found. This is the equivalent to

$$-\tfrac{2}{7} + .034 : \ldots = -1 + \tfrac{5}{7} + .034 \ldots = -1 + .748 \ldots$$

Ex. 159. To find the value of $\dfrac{\sqrt{3} \times (1.05)^6}{2^{\frac{2}{5}} \times (216)^{\frac{1}{5}}}$, being given the logs of 2, 3, 5 and 7.

$$\text{Numerator} = 3^{\frac{1}{2}} \times 3^6 \times 7^6 \times .05^6 = 3^{\frac{13}{2}} \times 7^6 \times .05^6 \ ;$$

$$\therefore \log \text{num.} = \tfrac{13}{2} \log 3 + 6 \log 7 + 6 \log .05 = 0.36569.$$

$$\text{Denominator} = 2^{\frac{2}{5}} \times (2^3 \times 3^3)^{\frac{1}{5}} = 2^{\frac{2}{5}} . \times 2^{\frac{3}{5}} \times 3^{\frac{3}{5}} = 2 \times 3^{\frac{3}{5}} \ ;$$

$$\therefore \log \text{denomr.} = \log 2 + \tfrac{3}{5} \log 3 = 0.58730.$$

$$\therefore \log \text{of the value} = .36569 - .58730 = \bar{1}.77839 \ ;$$

$$\text{and value} = .60033.$$

Those who make very great use of logarithms, as astronomers and navigators, do not usually employ negative indices for the logarithms of fractions, but make use of a system much more convenient in practice, although probably more difficult to master at first.

An explanation of that system, as well as of other conventions in logarithmic practice, can scarcely find a place in this work.

EXPONENTIAL EQUATIONS.

135. An equation in which the unknown quantity is involved as an index or exponent is called an exponential equation.

These usually require the application of logarithms in their solution.

Ex. 160. In how many years will a sum of money double itself at 3 per cent. compound interest.

From Art. 39, $A = P(1+r)^t$.

But $A = 2P \therefore (1+r)^t = 2$.

And going to logarithms,

$$t \log (1.03) = \log 2, \; \therefore t = \frac{\log 2}{\log 1.03} = 23 + \text{years}.$$

Ex. 161. Given $a^x + a^{-x} = b$ to find x.

Multiplying by a^x, $a^{2x} - ba^x = -1$;

$$\therefore a^x = \frac{b \pm \sqrt{b^2-4}}{2},$$

$$\text{And } x = \frac{\log(b \pm \sqrt{b^2-4a}) - \log 2}{\log a}.$$

Ex. 162. Given $\left(a^{\frac{n}{m}}\right)^x = b^{nx-a}$ to find x.

$$\frac{nx}{m} \log a = (nx-a) \log b;$$

$$\therefore x\left(\frac{n}{m}\log a - n \log b\right) = -a \log b;$$

$$\therefore x = \frac{ma \log b}{n(m \log b - \log a)}.$$

CONTINUED FRACTIONS.

136. Let us take for illustration the fraction $\frac{45}{101}$.

$$\text{Then, } \frac{45}{101} = \frac{1}{\frac{101}{45}} = \frac{1}{2+\frac{11}{45}}.$$

$$\text{Again, } \frac{11}{45} = \frac{1}{\frac{45}{11}} = \frac{1}{4+\frac{1}{11}}.$$

$$\therefore \frac{45}{101} = \cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{11}}}$$

This latter expression is usually written $\frac{1}{2+\frac{1}{4+\frac{1}{11}}}$, or more compactly $\frac{1}{2+}\frac{1}{4+}\frac{1}{11}$ and is called a *continued fraction*, which is *rational* when the number of terms is limited, and *irrational* when not limited.

137. *To convert any fraction into a continued fraction.*

In the example of the preceding article we divide 101 by 45 with a quotient 2 and a remainder 11; we then divide 45 by 11 with a quotient 4 and a remainder 1. And this being identical with the operation for finding the G.C.M. of 101 and 45, we deduce the following rule: *Proceed as in finding the G.C.M. of the numerator and denominator of the given fraction; the quotients taken in order form the denominators of the terms of the continued fraction.*

Ex. 163. To convert $\frac{472}{681}$ into a continued fraction.

Proceeding to find the G.C.M. we obtain the quotients 1, 2, 3, 1, 6, 1, 2, 2 in order.

$\therefore$ continued fraction is,

$$\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{1+}\frac{1}{6+}\frac{1}{1+}\frac{1}{2+}\frac{1}{2}$$

	472	681	1
2	54	209	3
1	7	47	6
1	2	5	2
2	0	1	∞

If we proceed to divide by the remainder 0 we get $\frac{1}{0}=\infty$, and the corresponding term of the continued fraction is $\frac{1}{\infty}$, which is zero. But as the process of finding the G.C.M. of any fraction must finally give a remainder 0, the equivalent continued fraction must always be limited or rational. Hence *any fraction may be converted into a rational continned fraction.*

138. If we take the values of one, two, three, four, &c., terms in the continued fraction of Ex. 163, we have,

$$\frac{1}{1}=1,\ \frac{1}{1+}\frac{1}{2}=\frac{2}{3},\ \frac{1}{1+}\frac{1}{2+}\frac{1}{3}=\frac{7}{10},\ \frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{1}=\frac{9}{13},\ \frac{1}{1+}\frac{1}{2}$$

$$+\frac{1}{6+}\frac{1}{1+}\frac{1}{3}=\frac{61}{88},\ \&c.$$

The quantities 1, $\frac{2}{3}$, $\frac{7}{10}$, $\frac{9}{13}$, $\frac{61}{88}$, &c., are successively closer approximations to the value of the original fraction. They are consequently called convergents to the fraction $\frac{472}{681}$.

Thus the successive differences are:

(1) $1-\frac{472}{681}=\frac{209}{681}$; (2) $\frac{2}{3}-\frac{472}{681}=-\frac{18}{681}$;
(3) $\frac{7}{10}-\frac{472}{681}=\frac{4.7}{681}$; (4) $\frac{9}{13}-\frac{472}{681}=-\frac{1.4}{681}$ nearly;
(5) $\frac{61}{88}-\frac{472}{681}=\frac{.06}{681}$ nearly; &c., &c., &c.

We thus see that the 5th convergent differs from the original fi action by only $\frac{6}{68100}$ or $\frac{1}{11350}$ nearly.

We see moreover that the odd convergents are too great and the even ones too small, so that the successive convergents are alternately too great and too small, the true value of the fraction always lying between those of any two consecutive convergents.

139. To find the convergents.

Let $\frac{1}{a+}\frac{1}{b+}\frac{1}{c+}\frac{1}{d+}$ &c. be the conveying fraction.

Then 1st conv. $=\frac{1}{a}$. For the second convergent we must put $a+\frac{1}{b}$ for a in the first; for the third convergent we must put $b+\frac{1}{c}$ for b in the second; &c., &c.

We thus get,

1st convergent $= \frac{1}{a}$, ... which denote by $\frac{N_1}{D_1}$.

2nd " $= \frac{1}{a+\frac{1}{b}} = \frac{b}{ba+1}$ " " " $\frac{N_2}{D_2}$.

3rd " $= \dfrac{b+\frac{1}{c}}{a(b+\frac{1}{c})+1} = \dfrac{bc+1}{abc+c+a} = \dfrac{c.b+1}{c(ab+1)+a},$

$$= \frac{cN_2+N_1}{cD_2+D_1},$$

which denote by .... $\frac{N_3}{D_3}$.

4th " $= \dfrac{(c+\frac{1}{d})N_2+N_1}{(c+\frac{1}{d})D_2+D_1} = \dfrac{cdN_2+dN_1+N_2}{cdD_2+dD_1+D_2}$

$$= \frac{d(cN_2+N_1)+N_2}{d(cD_2+D_1)+D_2} = \frac{dN_3+N_2}{dD_3+D_2}, \text{ \&c.}$$

We thus see that every convergent after the second is formed from the two proceeding convergents according to a fixed law, which may be stated as follows: Calling a, b, c, d, &c., partial quotients, the numerator of the n^{th} convergent is formed by multiplying the number of the $(n-1)^{\text{th}}$ convergent by the n^{th} partial quotient and adding the numerator of the $(n-2)^{\text{nd}}$ convergent. The denominator of the n^{th} convergent is formed from the denominators of the $(n-1)^{\text{th}}$ and $(n-2)^{\text{nd}}$ convergents in a precisely similar manner.

The operation may be carried out as in the following example.

Ex. 164. Find the convergents to the fraction, $\frac{40}{111}$.

The partial quotients are 2, 1, 3, 2, 4.

Assume $\frac{0}{1}$ as the first convergent; then $\frac{1}{2}$ is the second convergent.

		1	4	9	40
$\frac{0}{1}$	$\frac{1}{2}$	1	3	2	4
		3	11	25	111

Write these two convergents in order and the remaining partial quotients in a row following them.

Then starting with the partial quotient 1 as a multiplier, 1×1 (the numerator of $\frac{1}{2}$) $+ 0$ (the numerator of $\frac{0}{1}$) $= 1$, which write above for the third numerator.

1×2 (denominator of $\frac{1}{2}$) $+ 1$ (denominator of $\frac{0}{1}$) $= 3$, which write below for the denominator of the third convergent.

Next starting with 3, $3 \times 1 + 1 = 4$ for numerator, and $3 \times 3 + 2 = 11$ for denominator, &c.

We thus find the convergents to be $\frac{1}{2}$, $\frac{1}{3}$, $\frac{4}{11}$, $\frac{9}{25}$ and finally the fraction itself $\frac{40}{111}$.

Or the working may be arranged as in the margin, where the various steps are readily made out without any additional explanation.

0		1
1	1	2
1	3	3
4	2	11
9	4	25
40		111

Ex. 165. To find approximate values for 3.14159.

Take the reciprocal $\frac{100000}{314159}$ for which we find the partial quotients 3, 7, 15, 1, 25, 1, &c.

∴ the convergents are,

3, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, $\frac{9230}{2931}$, &c.

0		1
1	7	3
7	15	22
106	1	333
113	25	355
2931	1	9230
&c.,		&c.

π being the ratio of the circumference of a circle to its diameter is approximately 3.1415926. The approximate value $\frac{22}{7}$ was discovered by Archimedes, and $\frac{355}{113}$ by Metius.

140. The difference between two consecutive convergents is equal to unity divided by the product of the denominators of the convergents.

Taking the convergents to Ex. 164, $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$; $\frac{4}{11} - \frac{1}{3} = \frac{1}{33}$; $\frac{4}{11} - \frac{9}{25} = \frac{1}{275}$; and this may be proved to be universally true.

Hence we may solve the following problem.

Ex. 166. To find multiples of 71 and 131 which shall differ by unity.

The convergents to $\frac{71}{131}$ are, $\frac{1}{1}$, $\frac{1}{2}$, $\frac{6}{11}$, $\frac{13}{24}$, $\frac{71}{131}$.

And $24 \times 71 - 13 \times 131 = 1$.

This principle may be employed in solutions like that of Ex. 112. Applying it to that example, we find 3 as a multiplier adapted to the question in hand.

Ex. 167. Two wheels, A and B, of a clock being geared together should move with the relative velocities of 1401 and 1945. No more than 120 teeth being allowed in any wheel, to find the numbers to be employed. After 100 revolutions of A, how much will B be either in advance of or behind its true place?

The convergents to $\frac{1401}{1945}$ *are* $\frac{1}{1}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{7}$, $\frac{13}{18}$, $\frac{18}{25}$, $\frac{85}{118}$, $\frac{188}{261}$, &c.

$\therefore$ 85 and 118 are the numbers to be employed.

Now when A makes one rev. B should make $\frac{1401}{1945}$ rev.

But when A makes 1 rev. B does make $\frac{85}{118}$.

$$\therefore\ 100\left(\frac{85}{118} - \frac{1401}{1945}\right) = \frac{700}{1945 \times 118} = \frac{1}{328} \text{ nearly.}$$

Or B would be before its true place by $\frac{1}{328}$th of a revolution.

If the greatest number of teeth allowed were 100, our convergents give us 18 and 25 as the numbers to be employed. 67 and 93, however, give a closer approximation. The true solution of a question of this kind can only be obtained in all cases by the use of what are called *intermediate converging fractions*, the theory of which is beyond the scope of this work.

141. To develope the square root of a number into a continued fraction.

Let the number be 15 for example,

$$\sqrt{15} = 3 + \sqrt{15} - 3; \text{ and } \sqrt{15} - 3 = \frac{\sqrt{15}-3}{1} = \frac{1}{\dfrac{1}{\sqrt{15}-3}} = \frac{1}{\dfrac{\sqrt{15}+3}{6}}$$

$$\frac{\sqrt{15}+3}{6} = 1 + \frac{\sqrt{15}-3}{6}; \text{ and } \frac{\sqrt{15}-3}{6} = \frac{1}{\dfrac{6}{\sqrt{15}-3}} = \frac{1}{\sqrt{15}+3}.$$

$$\therefore \sqrt{15} = 3 + \frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{6+} \ldots\ldots \text{ad infinitum.}$$

$\sqrt{15}+3 = 6 + \sqrt{15} - 3$; and $\sqrt{15} - 3 =$ &c., as in the upper line.

In the above, 3 being the largest integer in $\sqrt{15}$ we put $\sqrt{15} = 3 + \sqrt{15} - 3$, so that $15 - 3$ may be fractional. Then

$$\frac{\sqrt{15}+3}{6} = \frac{\sqrt{15}-3+6}{6} = 1 + \frac{\sqrt{15}-3}{6} \text{ \&c.}$$

In the third line $\sqrt{15}+3 = 6 + 15 - 3$. and this being the same as the first line, the quotients 1 and 6 will be continually repeated.

Ex. 168. $\sqrt{7} = 2 + \sqrt{7} - 2$; and $\sqrt{7} - 2 = \dfrac{1}{\frac{\sqrt{7}+2}{3}}$.

$$\frac{\sqrt{7}+2}{3} = \frac{\sqrt{7}-1+3}{3} = 1 + \frac{\sqrt{7}-1}{3};\ \text{and}\ \frac{\sqrt{7}-1}{3} = \frac{1}{\frac{\sqrt{7}+1}{2}}.$$

$$\frac{\sqrt{7}+1}{2} = 1 + \frac{\sqrt{7}-1}{2};\ \text{and}\ \frac{\sqrt{7}-1}{2} = \frac{1}{\frac{\sqrt{7}+1}{3}}.$$

$$\frac{\sqrt{7}+1}{3} = 1 + \frac{\sqrt{7}-2}{3};\ \text{and}\ \frac{\sqrt{7}-2}{3} = \frac{1}{\sqrt{7}+2}.$$

$\sqrt{7}+2 = 4 + \sqrt{7} - 2$, &c., as in the first line.

$$\therefore \sqrt{7} = 2 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+} \text{\&c., ad inf.}$$

SERIES OF SQUARE NUMBERS.

142. The squares of the natural numbers are called square numbers, and the series of square numbers is accordingly

$$1\ \ 4\ \ 9\ \ 16\ \ 25\ \ldots\ldots\ n^2.$$

To find the sum of n terms we do as follows:

$$(n+1)^3 = \qquad n^3 + 3n^2 + 3n + 1$$
$$n^3 = (\overline{n-1}+1)^3 = (n-1)^3 + 3(n-1)^2 + 3(n-1) + 1$$

$$(n-1)^3 = (\overline{n-2}+1)^3 = (n-2)^3 + 3(n-2)^2 + 3(n-2) + 1$$

$$\cdots\cdots \qquad\qquad \cdots\cdots$$

$$3^3 = (2+1)^3 = 2^3 + 3.2^2 + 3.2 + 1$$
$$2^3 = (1+1)^3 = 1^3 + 3.1^2 + 3.1 + 1$$
$$1^3 = (0+1)^3 = 0^3 + 3.0^2 + 3.0 + 1.$$

Adding, the quantities in the first column upon the right cancel all upon the left except the first term.

Putting Σn^2 to denote the sum of the square numbers to n terms, and Σn to denote that of the natural numbers to the same extent, we have,

$$(n+1)^3 = 3\Sigma n^2 + 3\Sigma n + n + 1\ ;$$

$$\therefore\ 3\Sigma n^2 = (n+1)^3 - 3\cdot\frac{n(n+1)}{2} - (n+1) \quad \text{Art. 93.}$$

$$= (n+1)(n^2 + \frac{n}{2});$$

$$\text{Or,} \qquad \Sigma n^2 = n\frac{(n+1)(2n+1)}{6}.$$

If objects be arranged in squares upon a plane surface, as in the margin, the whole number of objects in any square block will be the square of the number upon the side.

If a number of balls be piled in the form of a pyramid with a square base, each layer contains the square of the number of balls forming its side, and the sides of two consecutive layers differ by unity.

Hence the balls in the layers give the series of square numbers begining at the top where there is but one ball; and the whole number of balls is the sum of the square numbers from 1 to n^2, n being the number of balls on the side of the basal layer.

1	*	1
2	* * * *	4
3	* * * * * * * * *	9
4	* * * * * * * * * * * * * * * *	16

Ex. 169. How many balls are in an unfinished square pyramidal pile, the basal row having 22 and the top row 14?

If the pile were complete there would be

$$\frac{22(22+1)(2\times 22+1)}{6} = \frac{22\times 23\times 45}{6}.$$

But the number required to finish the pyramid is

$$\frac{13(13+1)(2\times 13+1)}{6}=\frac{13\times 14\times 27}{6}.$$

$\therefore$ whole number in the pile $=\dfrac{22\times 23\times 45-13\times 14\times 27}{6}=2976.$

SERIES OF TRIANGULAR NUMBERS.

143. If objects be arranged in equilateral triangles upon a plane surface, the number required to form a complete triangle, as in the margin is called a triangular number. With 1 object upon a side we have 1 as the first triangular number. With 2 objects upon a side it requires 3 to complete the triangle; there being one row with one in it and a second row with two. With 3 upon a side we have 3 rows, of 1, 2 and 3 objects respectively; i.e., 6 in all. With 4 upon a side we have four rows of 1, 2, 3 and 4 objects, or 10 in all, &c.

Hence the series of triangular numbers is 1, 3, 6, 10, 15, 21, &c.

```
1      *       1

2      *       3
      * *

       *
3     * *      6
     * * *

       *
4     * *     10
     * * *
    * * * *
```

The numbers are evidently the successive sums of the series of natural numbers beginning at unity.

Thus, $1=1$, $3=1+2$, $6=1+2+3$, $10=1+2+3+4$
$15=1+2+3+4+5$, &c.

144. To find the sum of n terms of the triangular numbers.

Let Σn denote the sum of n terms of the series of natural numbers, Σn^2 that of the series of square numbers, and Σt the sum of n terms of the series of triangular numbers.

Then, $\Sigma n=1+2+3+4\;.\;.\;.\;.n,$

$\Sigma n^2=1+4+9+16\;.\;.\;.\;.\;n^2,$

$\Sigma n+\Sigma n^2=2+6+12+20+\;.\;.\;.\;.\;(n^2+n).$

$=2(1+3+6+10+\;.\;.\;.\;.\frac{n^2+n}{2})$

$$= 2\Sigma t.$$

$$\therefore\ \Sigma t = \tfrac{1}{2}(\Sigma n + \Sigma^2) = \tfrac{1}{2}\frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6} \quad \text{Arts. 93 \& 142.}$$

$$= \frac{n(n+1)(n+2)}{6}.$$

If a number of balls be piled in a triangular pyramid, the numbers in the successive layers will be the series of triangular numbers, and the whole number of balls in the pile, commencing at one upon the top, will be the sum of the first n triangular numbers, n being the number of balls in a side of the basal layer.

Ex. 170. How many balls in a complete triangular pyramid, the basal layer containing 10 upon a side.

Here $n = 10$, and,

$$\Sigma t = \frac{10 \,.\, 11 \,.\, 12}{6} = 220.$$

INDETERMINATE COEFFICIENTS.

145. The truth of the statement that

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots \text{ ad inf.}$$

is not limited to any particular value of x, but holds for all values, arithmetically if x is less than one, and algebraically if x is any quantity whatever.

In other words, the expression is an identity, and must, therefore, be true, quite independently of any particular values given to the symbols employed.

146. Proposition. If we have the identity

$$A + Bx + Cx^2 \ldots = a + bx + cx^2 + \ldots$$

Where A, B, C .... a, b, c .... are constant coefficients, and x is variable, then, $A = a$, $B = b$, $C = c$, &c., *i.e.*, the coefficient of any particular power of x upon one side of the identity is equal to the coefficient of the same power of x upon the other side.

For, $\quad A - a = (b - B)x + (c - C)x^2 + \dots .$

But the second member changes value as x changes its value, while the first member is constant.

Hence there cannot be equality unless each member is equal to zero. $\therefore A - a = 0$ or $A = a$, and by rejecting A and a as being equal and dividing by x we obtain in like manner $B - b = 0$, or $B = b$, &c. . . . .

The coefficients A, a, B, b, &c., are called *indeterminate* or undetermined coefficients, and the proposition now proved states the principle of indeterminate coefficients.

The principle of indeterminate coefficients is one of the most prolific in algebraic analysis. Some of its simpler application will be illustrated by a few examples.

Ex. 171. To expand $\dfrac{1+x}{(1-x)^2}$ into a series according to ascending powers of x.

Put $\dfrac{1+x}{(1-x)^2} = a + bx + cx^2 + dx^3 + \dots .$

then $1 + x = (1 - 2x + x^2)(a + b + cx^2 + dx^3 + \dots .)$

$$= a \quad \begin{array}{r|r|r|l} +b & x+c & x^2+d & x^3 \dots . \\ -2a & -2b & -2c & \\ & +a & +b & \end{array}$$

and equating coefficients of like powers of x,

$$\begin{array}{ll} a = 1 \, ; \; b - 2a = 1 & \therefore \; b = 3, \\ c - 2b + a = 0 & \therefore \; c = 2b - a = 5, \\ d - 2c + b = 0 & \therefore \; d = 2c - b = 7, \\ \text{\&c.,} & \qquad \text{\&c.} \end{array}$$

$$\therefore \frac{1+x}{(1-x)^2} = 1 + 3x + 5x^2 + 7x^3 + \dots .$$

Compare this result with Ex. 22.

Ex. 172. To expand the square root of $1 + x + x^2$.

Put $\sqrt{1 + x + x^2} = a + bx + cx^2 + dx^3 + \dots .$

Squaring, $1 + x + x^2 = a^2 + 2abx + \begin{array}{r|r|l} 2ac & x^2 + 2ad & x^3 \dots . \\ b^2 & 2bc & \end{array}$

Equating coefficients,

$a^2=1 \therefore a=1$

$2ab=1 \therefore b=\frac{1}{2}$

$2ac+b^2=1 \therefore c=\dfrac{1-b^2}{2a}=\dfrac{3}{8}$

$2ad+2bc=0 \therefore d=-\dfrac{bc}{a}=-\dfrac{3}{16}$

$\therefore \sqrt{1+x+x^2}=1+\frac{1}{2}x+\frac{3}{8}x^2-\dfrac{3}{16}x^4 \ldots.$

Ex. 173. What relation must exist among the quantities p, q, r, s in order that x^2+px+q and x^2+rx+s may have a common factor.

Let the common factor be $x+a$, then the expressions may be written

$$(x+a)(x+\frac{q}{a}), \text{ and } (x+a)(x+\frac{s}{a}),$$

since the last terms in the products will evidently be q and s as they should be.

Then we must have,

$$a+\frac{q}{a}=p, \qquad a+\frac{s}{a}=r.$$

$$\therefore \quad a^2-ap=-q, \qquad a^2-ar=-s.$$

And eliminating a^2 and a by determinants.

$$a=\frac{\begin{vmatrix} q & 1 \\ s & 1 \end{vmatrix}}{\begin{vmatrix} p & 1 \\ r & 1 \end{vmatrix}}=\frac{q-s}{p-r}$$

$$\text{and } a^2=\frac{\begin{vmatrix} p & q \\ r & s \end{vmatrix}}{\begin{vmatrix} 1 & p \\ 1 & r \end{vmatrix}}=\frac{qr-ps}{p-r}.$$

$$\therefore (p-r)(qr-ps)=(q-s)^2,$$

is the necessary relation.

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