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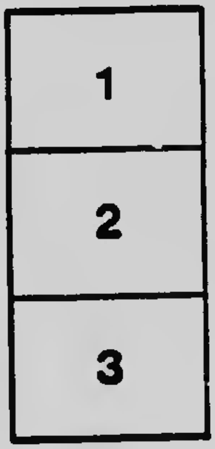
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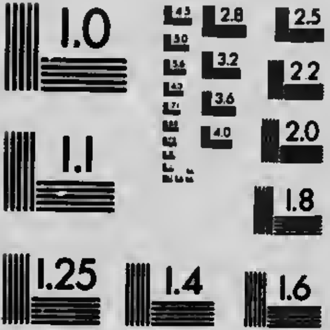

A Geometrical Problem

The Trisection
of any
Rectilineal Angle

By

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complments of
Geo. Goodwin

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INDEX.

Introduction

Prior to the publication of any Literary or Scientific Work, it is customary for the author to give an outline of his subject in the form of a preface.

In the present instance, the Trisection of any rectilineal angle was first observed by the author several years ago, when reading the following (see page 105 of the Elements of Geometry, in reference to the subdivision of the circumference of a circle, and which work is hereinafter more amply identified in the opening page hereof), viz : "It is obvious that any regular polygon whatever might be inscribed "in a circle, provided that its circumference could be divided into "any proposed number of equal parts ; but such division of the cir- "cumference like the trisection of an angle, which indeed depends on "it, is a problem which has not yet been effected."

A possible solution of the problem was suggested by the reading of the following : "If the side of any triangle be produced, the ex- "terior angle is equal to the two interior and opposite angles. (See Euclid 32-1).

It did not require much reflection to discover that, if a line could be drawn so that the interior and opposite angles would be to each other in the ratio of two to one, the problem could be solved, but a method by which a line could be so placed did not seem possible. Further consideration soon revealed the condition that the same result could be attained if a construction could be made by which, as in the following Analysis, the straight line A P would intersect the circumference of the semi-circle in the point V so that V P = A B the problem could be solved. However, a construction to comply with that condition, did not seem feasible. During the interval, much time has been periodically devoted by the author to the study of this problem, and with the final result as disclosed by the demonstration expounded in this treatise.

Subsequently to or about the year 1882, Mr. Andrew Doyle published several solutions of the Trisection problem. Some time in or about the year 1895, he published and proved that, if a straight line be drawn (which as before mentioned, is described in the following treatise), so that the portion of it situate between a perpendicular such as B F′ and the circumference of the semi-circle would be equal to the radius, and that to place a line in such position would be equivalent to the solution. Mr. Doyle did not make such a construction to give effect to his theory, and consequently failed to solve the problem.

In justice to Mr. Doyle, it is considered proper for this note of recognition to appear in this publication.

Ottawa, August, 1910 GEO. GOODWIN.

Trisection of Any Rectilineal Angle.

SECTION NO. 1.

Clause No. 1.—As an explanatory note, tbe qualifying words not greater tban a rigbt angle may be used, but do not mean a limitation of the problem, because, if the given angle should be greater than a right angle the angle, could be bisected, and the half of the angle, so divided, could be trisected, and the two-thirds of the half would be equal to the one-tbird of the wbole.

In order to facilitate tbe solution of the several parts of this problem, it will be found convenient to divide it into SECTIONS and CLAUSES, wbich can be referred to according to the numbers under whicb tbey appear.

Tbe geometrical authority, herein quoted, is "The Six Books of "Euclid, by John Playfair, F.R.S., Lond. and Edin. Professor of "Natural Pbilosophy, formerly of Mathematics, in tbe University of "Edinburgb. From the last London Edition, Enlarged. Philadelphia, "1860."

Clause No. 2.—Before proceeding to the consideration of this problem it may be observed, that Geometrical writers directly after tbe Enunciation of a Problem, proceed to show how the construction can be accomplished, the demonstration, as to its accuracy, following in general order.

In this case that which is designated as the first diagram is intended to show the mode of construction with the least number of lines to be drawn and the least number of Points necessary to establish tbe position and magnitude of these lines. Then afterwards will follow the main diagram, on which will appear a greater number of lines required for the purposes of demonstration, and which will herein be subsequently identified as Construction No. 1.

First Diagram.

Let the angle A B C be the given angle. Produce the line C B indefinitely to say Z (2 post). From the point B, in tbe straight line C Z (or C B Z) draw the perpendicular B F′ (11—1), let A B =B C. Then place the point E, in the line C Z, so that the line B E = 2 A B, then draw the line E A produced indefinitely to say A‴. Then from the point H at the intersection of the line E A with the perpendicular B F′ and with a radius equal to B E (or 2 A B) describe the arc intersecting the line C Z in tbe point R. Then draw the line R A intersecting the perpendicular B F′ in the point T. Then draw the line

T L′ parallel to C Z intersecting E A (or E H′ H A). Then from the point H with a radius equal to H′ A describe the arc A′ (or make H H′ = A A′). Then the line drawn from A′ to R will form an angle with the line C Z that will be equal to one-third ($\frac{1}{3}$) of the given angle (or A′ R C = $\frac{\text{A B C}}{3}$).

Analysis.

Clause No. 3.—In order to ascertain the conditions that prevail so that the interior and opposite angles may be divided in the proportion of one to two, which if accomplished would be equivalent to the solution of the trisection problem (32—1).

By Hypothesis.

It may be assumed that an angle, as, for example, the angle A B C, is already divided into three equal parts by the two lines, viz., B X and B Y. Let A B = B C. Produce the straight line C B indefinitely (2 Post.) to say Z. From the point B draw B F′ perpendicular to C Z (or C B) produced (11—1) make B E = 2 C B (or 2 A B). From the point A draw A P parallel to B X intersecting the line C Z in the point P (31—1). Then A P B = X B C (28—1) and A B C = A P C + B A P (32—1).

From the point B as a centre and with A B as a radius, describe the semicircle C A D (on C B produced) intersecting A P in the point V. Then B V = A B (being radius of the same semicircle), and the angle B A V = B V A (5—1) and A B C = B A P + A P B (32—1) and B V A = V B P + A P B and by the Hypothesis X B C = A P B and X B C = $\frac{\text{A B C}}{3}$. The angle A P B = $\frac{\text{B A P}}{2}$ and B A P = B V A (5—1) and B V A = A P B + V B P (32—1), and as A P B = $\frac{\text{B A P}}{2}$ $\frac{\text{B V A}}{2}$ = A P B. Then from the point V where the straight line A P intersects the semicircle draw the line V W parallel to B C (or C Z) (31—1). Then B V = B A and then B A V = B V A (5—1) and V B P = B V W (27—1) and B V A = 2 A P B. The angle B V A is bisected by the line V W and as C B F′ is a right angle, the angle V W B is also a right angle because V W is parallel to C Z (or C E) (13—1) or the angle B A P is equal to two of the parts of which the angle A B C contains three.

The alternate V B P = B V W and B V A = 2 A P B and A V W = A P B and B V A = A P B + V B P and B V A—A P B = B V A —V B P. Therefore A P B = V B P and V B = V P. It will be seen that the point U is at the intersection of the straight line A U V P (or A P) with the perpendicular B F′. The two triangles B V W and

U V W have two angles in the one equal to two angles in the other each to each, and the line V W common to both, V W B and U W V are both right angles (13—1), and B V W = U V W (or A V W).

Then there are two triangles having two angles in the one equal to two angles in the other and one side of the one equal to one side of the other (or common to both), then the remaining angle of the one (V U W = V B W) is equal to the remaining angle of the other and the remaining sides of the one are equal to the remaining sides of the other (26—1). Therefore V B = V U and B W = U W.

It has been shown that V B = V P and U V = V P. The three lines, viz., B V, V P and U V are equal, therefore P V = V U and P V + V U = 2 A B. Note, it will be observed that, if a line be drawn from any other point than from the point P (in the line B E) to the point A, the angle would be less or greater than $\frac{A\,B\,C}{3}$, according as that point should be situate any place between B and P, or, any place between P and E.

SECTION NO. 2.

From the foregoing Analysis the following Theorem is derived, viz.:

If one of the sides of any given rectilineal angle (not greater than a right angle) be produced, indefinitely, and from the point where the two sides forming the given angle intersect each other a line be drawn perpendicular to the side produced, and from that point as a centre a semicircle be described, on the side produced, and if the straight line drawn from the point where the other side of the given angle intersects the circumference (of the semicircle) to a point in the side, so produced, so that the straight line so drawn will be divided by the perpendicular, so that the portion of it that lies, or is situate, between its intersection with the perpendicular and the point where it intersects the side produced will be equal to the diameter (or twice the radius), the angle formed by that line so drawn and the side so produced will be equal to one-third of the given angle and the line so drawn is constant with regard to any angle. The demonstration, in relation thereto, given in the following sections will show that the straight line A P and lines parallel to A P can be so placed independently and without the assumption as set forth by the Hypothesis in Section No. 1.

SECTION NO. 3.

Construction No. 1.

Clause No. 1.—The same diagram may be used as in Section No. 1. From the point E with a radius equal to the diameter (or 2 A B) describe the arc B G (indefinitely). From the point H at the

intersection of the line A E with the perpendicular B F′ draw the line H L parallel to B E or C Z (31—1) intersecting the arc B G in the point L. From the point H and with a radius equal to the diameter of the semicircle (or 2 A B) describe. the (small) arc at R (in the line C Z) then draw the line A R intersecting the perpendicular B F′ in the point T. Through the point A draw A M parallel to C Z (or B E) and through T draw T L′ parallel to C Z (or A M). Draw the line L J parallel to A E intersecting the line A M in J (and produced both ways towards A and E) and intersecting the line T L′ in the point L″. Produce L J to K, so that J K will be equal to L L″. and the line drawn from K to E will form an angle with the line B E (or C Z) that will be equal to one-third ($\frac{1}{3}$) of the given angle or $\frac{A\,B\,C}{3}$

SECTION NO. 4.

Demonstration.

Clause No. 1.—By Hypothesis the straight line A U P forms with C Z (or C B E) an angle $= \frac{A\,B\,C}{3}$, and, for the sake of convenience, in reference to the reasoning, the line A P or any line parallel to it or B X may be referred to or called a trisecting line.

Clause No. 2.—The portion of the line A U P that is situate between the perpendicular B F′ and the point P by Hypothesis is U P = B E (or 2 A B). The angle H B E = U B P (both being right angles) the side U P is greater than the side B P (19—1) and U P = B E. Therefore B E is greater than B P and H E is greater than B E (18—1). H E is included in and is a portion of the line A H E and U P is included in and is a portion of the straight line A U P. The angle A R B is greater than the angle A P B and the angle B A P is greater than the angle B A R and the angle B A E is greater than the angle B A P and A E intersects B F′ in the point H at a greater distance from B than A P intersects B F′ in the point U. Therefore B H is greater than B U and B U is greater than B T and the angle A P B is greater than the angle A E B.

The following is submitted as explanatory of the undermentioned terms used herein, viz:—divergent value, convergent value, horizontal convergent value, and vertical value. When certain lines of unequal length are drawn from a point, not situate in the line called the trisecting line, so as to intersect the trisecting line and from that same point if another line be drawn at a different angle so as to intersect the trisecting line in some other point, such lines, while different in length, are said to be of équal value, as, for example, viz:—The line E A converges towards the trisecting line and intersects P A in the point A

and the line E P converges from E to P, so that those two lines while different in length are said to be of equal value. That is to say, the convergent value of E A is said to be equal to the convergent horizontal value of E P. Then the divergent value of A R is said to be equal to convergent horizontal value of R P.

The meaning of the terms may be further explained, viz:—The term divergent means a line drawn from any point situate in the trisecting line to any point that is not situate in that line.

The term convergent means a line drawn from any point not situate in the trisecting line to any point that is situate in that line.

The term horizontal means the line C Z or the distance between any two points situate in C Z or any line parallel to it.

The term vertical means the line B F′ or the distance between any two points situate in B F′ or any line parallel to it, or any line drawn at right angles to C Z, or drawn at right angles to any line parallel to C Z.

The following may be used as illustrative of the foregoing explanations. The line E A converges towards the trisecting line A P and the same line A E diverges from A P and E P converges from E to P, so that the use of the term convergent or divergent is determined by whether the line is supposed to be drawn from E to A or from A to E. The same may be said as to whether the line is supposed to be drawn from R to A or from A to R. The divergent value of A R is equal to the convergent horizontal value of R P or the divergent horizontal value of P R is equal to convergent value R A. Also the convergent value of H A is equal to the horizontal convergent value of H U′, and the horizontal value of H U′ is equal to the vertical convergent value of H U, and the divergent value of A H is equal to the divergent vertical value of U H.

Clause No. 3.—The convergent value between the line A H and the trisecting line A U P measured on the perpendicular B F′ is H U and when measured on the horizontal L H produced to U′ is equivalent to the convergent horizontal value accrued between the intersection of A P with the horizontal straight line L H U′ and having a horizontal convergent value of H U′, so that the horizontal value of H U′ is equal to the convergent value of H A and the triangle H U U′ may be considered as a right angle triangle composed of a portion of the different lines as per sketch No. 1. The lines forming this triangle are viz., H U′, included in and forming a part of the horizontal straight line L H U′, the line H U, included in and forming a part of the perpendicular B F′, and the line U′ U, included in and forming a part of the trisecting line A P (or A U′ U P).

Observe H T is greater than H U, therefore the portion of A P situate between H L and T L′ is greater than U′ U. It may be mentioned the line A H converges towards or diverges from the trisecting line A P according as that line is drawn from A to H or from H to A.

Clause No. 4.—There is, so far considered, no method (exclusive of the Hypothesis) by which the length of the lines H U′ + R P can be measured or shown horizontally or how either of them can be measured separately. However, it has been shown that a line drawn from L to J is equal and parallel to H A, and that, as A H = L J, the convergent value of H A is equal to the convergent horizontal value of H U′. See Clause No. 3. Therefore the convergent value of L J is equal to the horizontal value of H U′ and by the drawing of the line L J the distance or line equivalent to H U′ has been eliminated and the line L J has converged from L to J. so that the distance from J to the point where the trisecting line drawn from E would intersect L J produced would be equivalent to the horizontal value of line R P. The (small) arc at R has been drawn from the point H with a radius equal to the diameter (or 2 A B). The arc B G has been drawn from the point E with a radius equal to the diameter, and R H is parallel to E L and R H = E L and A J = R E and R E = L H. It will be seen that if the line J L be produced it will intersect B E produced in the point Z because J L being parallel to A E the three lines, viz., A J and H L and E Z, are equal to each other, also because H R is parallel to E L (and are between the parallel lines H L and C Z), the line R E = H L (33—1). Therefore the four lines, viz., R E = H L = E Z = A J are equal to each other (see Axiom No. 1, Book 1) "things which are equal to the same thing are "equal to one another," A R is parallel to E J and A R = E J (33—1).

It has been shown that L J has converged towards the line drawn from E parallel to A P so as to eliminate H U′ or the equivalent of H A, so that the convergent value of H A is equal to the horizontal value of H U′. The horizontal difference measured along L H from L towards H, or, in other words, the difference between the line E L and the line drawn from E parallel to P A and intersecting the line L H, would be equal to U′ H + R P but R P = J J′. Therefore the line L J (or Z L J) requires to be produced from J to such a distance so that it will intersect E J′ produced so as to be the horizontal equivalent of J J′ (or R P), and the line drawn from E parallel to P A will intersect the line L H at a horizontal distance from the point L = U′ H + J J′.

SECTION NO. 5.

Clause No. 1.—It is now to be determined to what distance the straight line Z L J shall be produced so that it will intersect the straight line drawn from the point E parallel to the line P A. It may be mentioned the line L J is included in and is a portion of the line Z L J and coincides therewith.

The lines H L and T L′ also the lines A M and A′ M′ are by construction parallel to B E (or C Z) and are therefore parallel to each other (30—1). The lines A M and A′ M′ are (by construction) the same distance from each other as are the lines H L and T L′.

Clause No. 2.—The, so called, intervening line marked U U″ is situate between the lines H L and T L′, and the other, so called, intervening line marked O N is situate between the lines A M and A′ M′. Both of these, so called, intervening lines are parallel to the lines between which they are situate, and by Hypothesis the said intervening line U U″ is situate at a greater distance from H L than it is from T L′ and intervening line O N is situate at a greater distance from A M than it is from A′ M′. The intervening line U U″ so nearly coincides with T L′ and the intervening line O N so nearly coincides with the line A′ M′ that it would be difficult (particularly if a smaller angle had been given) to observe any difference in position between these, so called, intervening lines and the lines with which they so nearly coincide. The two sketches marked Sketch No. 1 and Sketch No. 2 are marked on the main diagram and are necessarily drawn in an exaggerated form and are not according to any scale, but are for purposes of illustration to enable the reader to follow without difficulty.

Clause No. 3.—Produce the straight line Z L J indefinitely intersecting the line A′ M′ in the point K, draw the straight line K J″ B′ parallel to A B (31—1) intersecting B Z (or C B Z) in the point B′. Then the angle K B′ Z = A B E and the angle B′ Z K = B E A and Z K B′ = E A B, K B′ is parallel to A B and A E is parallel to K Z and the line B Z (or the straight line C B Z) being common to both triangles, therefore the triangle K B′Z is similar to the triangle A B E (Dif. 1—6) (see Section No. 1, Clause No. 2), by construction 2 A B = B E then Z B′ = 2 K B′, the line A R is parallel to J E and E R = A J and R E = E P + P R, and if a line parallel to A P be drawn from the point E it will intersect A J (or A M) in the point J′, so that R P = J J′ and the line R E = E P + P R and the line A J = A J′ + J J′ or R E — R P = P E and A J — A J′ = J′ J. It will be observed that at the point where the line K B′ intersects the line J A (or A M) in the point J″ the angle K J″ A is equal to the angle A B C and the angle K J″ J = A B E and J″ J K = B E A (or B′ Z K). Therefore the triangle K J″ J is similar to the triangle A B E (or the triangle K B′ Z), the triangle K B′ Z is similar to the triangle K J″ J, the latter being included in the former with the point K being common to both triangles.

The triangle A P B is situate within the triangle A B E with the point A common to both triangles, the line A B being common to both triangles and the line B P coinciding with and forming a part of B E. Furthermore in the triangle K J″ J the line J″ J = 2 K J″ (according to Section No. 2), J″ J is equal to that portion of the line drawn from K (parallel to A P) that is situate between the line A M and the perpendicular J″ K′, but it will be seen that 2 K J″ or J″ J is less than the portion of A P that is situate between the lines H L and T L′.

Again, the portion of the line E J′ produced (and parallel to A P) that is situate between the lines A M and A′M′ which (see Section No. 2) is equal to J″ J *together* with the remaining portion of E J′ produced situate between the perpendicular J″ K′ raised from J″ and the line A′ M′, so that J″ J is equal to that portion of E J′ produced from J′ to intersect the line O N, and J″ J is also equal to that portion of A P situate between the lines H L and U U″. Then the distance between the lines A M and A′ M′ is equal to that between H L and T L′, because if equals be taken from equals the remainders are equal (see Axiom 3, Book 1).

According to construction (see Section No. 1), in the triangle A B E the line B E = 2 A B and in the triangle K B′ Z (similarly constructed) the line Z B′ = 2 K B′. The line Z K is parallel to A E. According to Clause No. 1 hereof, the distance between H L and T L′ is equal to that between the lines A M and A′ M′. The portion of the line Z K situate between A M and A′ M′ is equal to that situate between H L and T L′, and that portion of E A situate between H L and T L′ is also equal to that portion of Z K situate between H L and T L′. If a line be drawn from the point E (as by the Hypothesis) parallel to P A it will intersect the line A M in the point J′ and that portion of E J′ produced will intersect A′ M′ so that the portion of E J′ produced to A′ M′ will be equal to that portion of P A situate between the lines H L and T L′ and will also be equal to that portion of E J′ situate between H L and T L′ (see first Axiom). Furthermore, it would be impossible for the portion of E J′ produced to A′ M′ to be equal to that portion of A P situate between equal parallel lines equally distant from each other, if it was not parallel to A P.

Clause No. 4.—According to Section No. 2, in the triangle K B′ Z, if a line be drawn from the point B′at right angles to the line B Z (or C B Z), and if a line be drawn from the point K parallel to A P and produced to intersect the line B Z (or C B Z), the portion of that line situate between the perpendicular raised from B′ to its intersection with the line B Z (or C B Z) will be equal to 2 K B′ B′ Z = 2 K B′ (see Section No. 2). Then, if the line E J′ parallel to

P A be produced to intersect the line A′ M′, and from that point of intersection a line be drawn parallel to A B, it will also be parallel and equal to K B′, and presumably will coincide with K B′.

Clause No. 5.—Assume that the line E J′ produced to intersect A′ M′ and drawn from that point of intersection parallel to K B′ (or A B) intersecting the line C Z and at a greater distance from Z than B′ is from Z or at a less distance from Z than B′ is from Z. Then by construction Z B′ = 2 K B′ and the line drawn from said point of E J′ produced to A′ M′ and drawn from such point parallel to K B′ intersecting C Z would be equal to K B′.

Then if the line drawn from said point of intersection of E J′ produced to A′ M′ and drawn parallel to K B′ from that said point should be found to intersect C Z at a greater distance from Z than B′ is from Z, in which case a triangle so formed with the side Z B′ being produced so as to be greater than Z B′, a condition inconsistent with the construction, viz., that the line B′ Z = 2 K B′, and therefore could not be similar to the triangle K B′ Z, in which case its corresponding sides 2 K B′ = Z B′. Neither could it be similar to the triangle A B E (because 2 A B = B E) which is similar to K B′ Z. Next, should the line parallel to and equal to K B′ intersect the line C Z in a point at a less distance from Z than B′ is from Z, then the triangle so formed could not be similar to the triangle K B′ Z, neither could it be similar to the triangle A B E.

Clause No. 6.—If a line be drawn through the point J′ (in the line A M) parallel to K Z it will intersect the line B′ Z in the point Z′, so that E Z′ = E P and Z′ Z = J′ J (or R P). Then the line drawn from Z′ parallel to Z K intersecting A M in the point J′ will if produced intersect the line A′ M′ in the point indicated by K‴. Then the line K K‴ will be equal to Z′ Z or J′ J.

Observe sketch No. 2 when reading clause No. 6, section No. 5. Therefore J′ J = K K‴, and the line drawn from the point K‴ (in the line A′ M′) parallel to K B′ will intersect C Z in the point indicated by B″, but B″ is a greater distance from Z than B′ is from Z, but, as will be seen, B″ Z′ = B′ Z, and the line drawn from K‴ parallel to A P will intersect C Z in the point E′, so that E′ Z′ = E Z and E′ E = Z′ Z. Again, if the line be drawn from the point J (in the line A M) parallel to A P, it will intersect the line A′ M′ in the point indicated by K⁗. If a line be drawn from the point K⁗ (in the line A′ M′) parallel to P A it will intersect C Z at the same distance from E that P is from R. As will be seen, it does not matter whether the line be drawn from J or from K⁗ when drawn parallel to P A it will intersect C Z in the same point, viz., E″, and therefore K K‴ = K K⁗ and K K‴ = J J′ and J J′ = K K⁗ and K‴ K⁗ = 2 J J′, and if a line be drawn from K‴ parallel to A P it will intersect C Z in the point indicated by E′ and E′ E″ = 2 E E′.

If a line be drawn from the point K″″ parallel to K Z it will intersect C Z produced in the point indicated by Z″ then K K″″ = Z Z″ and E′ Z = E Z + J J′ and E′ E = K″ K, and the line drawn from K parallel to A P intersects the line C Z in the point E, therefore the line E J′ produced intersects A′M′ in the point K. Consequently there is only one point where Z J produced can intersect the line E J′ produced and that is the point K in the line A′ M′. The same theory holds good as to any point on either side of the point K in the line A′ M′.

Clause No. 7.—In the case of the triangles A B P and A B E, if the line P A be produced it will intersect A′ M′ in the point A″. the same theory holds good, viz., A′ A″ = R F, and if a line be drawn from R to A′ it will be parallel to P A, so that the angle formed by A′ R with the line C Z will be equal to the angle $\frac{A\,B\,C.}{3}$

Clause No. 8.—Continuing, in reference to similar triangles, the first two, viz., the triangle A B E and the triangle A B P, the latter triangle being included in the former and having a common angle in A with A B common to both of these two triangles and the line B P included in and forming a part of the line B E.

Then there are the other triangles, viz., K B′ Z and K B′ E, the latter being included in the former and having a common angle in K with K B′ common to both of these two triangles and the line B′ E included in and forming a part of the line B′ Z. The triangle A B E is similar to the triangle K B′ Z and the triangle K B′ E is similar to the triangle A B P. By the Hypothesis the angle A P B = $\frac{A\,B\,C}{3}$. The line drawn from the point K parallel to A P to intersect C Z will also form an angle with the line C Z equal to $\frac{A\,B\,C}{3}$ or $\frac{K\,B'C}{3}$. Then again the triangle K B′ Z is similar to the triangle K J″J, the latter being included in the former, with the point K common to both of these triangles, the line K J″ being included in and forming a part of the straight line K J″ B′, the line K J being included in and forming a part of the straight line K J Z, and the line J J″ being parallel to B′ Z.

The line Z J = E A and the line Z K = Z J + J K and J″ B′ = A B and the whole K B′ [illegible] J″ B′ + J″ K. The line drawn from K parallel to A P will int[illegible] the line [illegible] M in the point J′. It will be seen, in the triangles K [illegible] and K J″ J′, the latter being included in the former with the point K common to both triangles and the line K J″ common to both of these triangles as, also is the line

J″ J which includes both lines, viz., J″ J = J″ J′ + J′ J, then the line J J″ = 2 K J″

The portion of the line E J′ produced to A′ M′ is equal to J J″ together with the remaining portion of E J′ produced to A′ M′ that is situate between the perpendicular J″ K′ and the line A′ M′ (see Section No. 2).

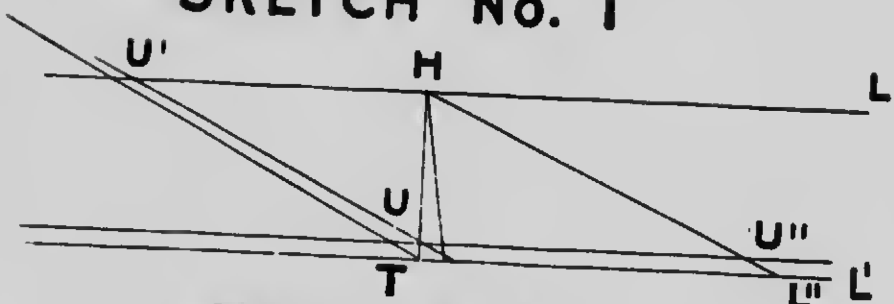
Clause No. 9.—In order to comply with conditions (of Clause No. 1, Section No. 5), the line Z J requires to be produced so as to intersect the line E J′ produced, and therefore that intersection must necessarily be at a common altitude or at an equal vertical distance from the (base) C Z as is the case in the triangles E A B and P A B, which have the point A common to both triangles and at the common altitude or at an equal vertical distance as the point A is from the line C Z. It has been shown the line R P is equal to the several lines, viz., J J′—E E′—E E″—Z′Z—Z Z″—K K‴—K K⁗—B″ B′—A′A″. It has also been shown (see Clause No. 4, Section No. 4) the following lines are equal to each other, viz., R E = E Z = H L = J A. It is well to observe that the position of the different points, in connection with the magnitude of the different lines, just mentioned as being equal to each other, can be found independently of and without having recourse to the Hypothesis (see Section No. 6 hereinafter). The line A E is parallel to K Z, the line E R = E Z and E R = E P + P R, the line Z J = E A, and the whole line Z K = E A + J K as shown (in Clause No. 8 hereof continued), the line K J diverges from the line drawn from K parallel to A P, so that at the intersection of these lines with the line A M (or J″ J) shows a distance equal to the horizontal value of J J′ and J″ J = J″ J′ + J′ J. It has been shown E A converges from E to A equal to the horizontal value of E P as measured on the line C Z and E P + P R = E R and E R = E Z. Then the whole line Z K = Z J + J K, therefore the whole line Z K will converge towards the line drawn from the point E parallel to the line P A which intersects the line Z K in the point K. That is a condition of the proportions which the sides of similar triangles bear to each other viz. Then as the line E A : E P :: Z K : Z E (4—6).

Clause No. 10.—These proportions, in conjunction with the foregoing demonstration, show that a line drawn from the point E to the point K forms an angle with the line C Z equal to one-third of the given angle. However, before reaching a final conclusion of the solution of this important problem, it is well to amplify what has gone before by expounding the principles involved by the consideration of the two sketches already mentioned (see Section No. 5, Clause No. 2). The consideration of these sketches will form the subject of another section identified as Section No. 6.

SECTION NO. 6.

Clause No. 1.

SKETCH No. 1



SKETCH No. 2.

The two sketches marked on this page, as above, are also marked on the main diagram, viz., "Sketch No. 1 and Sketch No. 2" as already mentioned (in Section No. 5, Clause No. 2). These sketches are necessarily drawn in an exaggerated form and therefore cannot be compared with any measurements on the main diagram, but are merely intended as a guide to distinguish the difference or space between certain lines on the diagram which could not easily be seen on account of their being very close to or nearly coinciding with each other. The sketches just mentioned are for the purposes of illustration to enable the reader to follow without difficulty.

The lines in these sketches are lettered in the same manner as the lines in the main diagram which they represent.

The line situate between the lines A′ M′ and A M is called the intervening line O N, and the other line situate between the lines H L and T L′ is called the intervening line U U″. Both of these intervening lines are parallel to the lines between which they are situate and parallel to each other (see Section No. 5, Clause No. 2).

Clause No. 2.—The line K Z (or K J Z) is parallel to A E, and the portion of K Z, viz., K J, that is situate between the lines A′ M′ and A M is equal to that part of that same line situate between the lines H L and T L′, and each of them is also equal to the portion of A E that is situate between the lines H L and T L′. The portion of E J′ produced to A′ M′ (parallel to P A) is equal to that part of that same line situate between the lines H L and T L′, and each of them is also equal to that portion of P A situate between the lines H L and T L′. The portion of R A situate between the lines H L and

T L′ is also equal to the portion E J situate between the lines H L and T L′, which is also equal to the portion of E J produced to A′ M′.

The different pairs of lines that are parallel to each other, such as R H and E L, R A and E J, A P and E J′, Z K and E A, K B′ and A B. The whole of the five pairs of lines just mentioned are intersected by the other two pairs of lines viz:—A′ M′ and A M, H L and T L′ that are also parallel to each other and at equal distance apart (see Section No. 5, last part of Clause No. 3). The following axiom may be used, viz., "Things which are equal to the same thing are equal to one another" (Axiom No. 1, 1—1).

Clause No. 3.—According to the Hypothesis, the line A P forms with the line C Z an angle equal to one-third of the given angle or A P C = $\frac{A\,B\,C}{3}$. In the straight line A U P the line U P = 2 A B. The line U U″ is parallel to H L and T L′. It is clear that the arc described from the point E with a radius equal to 2 A B will intersect the line U U″ (called an intervening line) so that the line drawn from the point E to the intersection of the arc B G with the line U U″ will be parallel, and equal, to A P and will form an angle with C Z, equal to $\frac{A\,B\,C}{3}$.

Clause No. 4.—It will be observed that the vertical distance from the intersection of the line A P with B F′ in the point U to the point B in the line C Z is equal to the perpendicular or vertical distance from the intersection of the arc B G with the line U U″ to the line C Z, because U U″ is parallel to C Z. Therefore the line P U = 2 A B (or U P), and the intersection of the arc B G drawn from the point E with a radius equal to 2 A B (or P U) shows that the intersection of the line drawn from E to the intersection of the arc B G with the line U U″ is parallel to P U and which if produced will intersect the line A M in the point J′, that is to say, at the same distance from J that R is from P or J J′ = R P.

The whole of the distance between A′ M′ and A M is equal to the whole of the distance between the lines H L and T L′. Furthermore, the whole of that portion of the line A P that is situate between the lines H L and T L′ is also equal to the whole of that portion of E J′ produced that is situate between the lines A′ M′ and A M (see Clause No. 3, Section No. 5). (Observe the sketches in order to assist in following the diagram.)

It will be observed E J′ is parallel to A P and the portion of E J′ produced to intersect the line A′ M′ is equal to that portion of that same line situate between the lines H L and T L′ and each of

them are equal to that portion of A P that is situate between the lines H L and T L′, and the portions of A P and E J′ situate between the lines H L and U U″ are equal to each other (see Section No. 5, Clause No. 3). The vertical distance between the intersection of A E and A P with the perpendicular B F′ is the vertical value H U, and the whole of the divergence between lines A E and A R at their intersection with the perpendicular B F′ is the vertical divergent value of H U + U T and H U + U T = H T.

The portion of A P situate between the lines H L and U U″ is less than that situate between the lines (U′ H L or) H L and T L′. As will be seen, the difference between H T and H U is U T and comes off the lower part of H T and is indicated by the difference between the intervening line U U″ and the line T L′. In the case of E J′ produced from A M to A′ M′, the difference is taken from the upper part of E J′ when produced to A′ M′, and which difference is that portion of E J′ produced to the point K that is situate between the intervening line O N and A′ M′ and which is also equal to the difference or space between the intervening line U U″ and T L′. The difference in each of these cases is equal to the divergence of the line A R from A P that accrues between the point A to the intersection of A R with the perpendicular B F′.

The right angle triangle (marked on Sketch No. 1) described (in Section No. 4, Clause No. 3), as will be seen, is situate between the lines H L and U U″, and another right angle triangle, but in an inverted form, may be observed (by reference to Sketch No. 2) is situate between the lines A M and O N.

The line E J′ produced to intersect O N is equal to 2 K J″ (or J″ J) and intersects the perpendicular J″ K′, and the portion of E J′ produced that is situate between its intersection with the perpendicular J″ K′ is also equal to that portion of A P that is situate between its intersection with the perpendicular B F′ and the line L H U′ (or L H produced to U′); that is to say 2 J″ K (or J″ J) = U U′ (see Sketch No. 1).

Clause No. 5.—It has been shown (see Section No. 5, Clause No. 3) that the portion of A P situate between H L and U U″ is equal to J″ J and J″ J = 2 K J″ and 2 A B + 2 K J″ is equal to E L + J″ J. Therefore, if from the point E, with a radius equal to E L + J″ J, an arc be described intersecting the line H L, and a line drawn from the point E to that point of intersection (in the line H L), it will be parallel to A P, and will also form an angle with the line C Z (or C B E Z) that will be equal to one-third of the given angle, viz., A B C or $\frac{A\ B\ C.}{3}$

Furthermore that line so drawn, as described, coincides with and is included in and is a portion of the straight line E J′ produced to K. Other reasoning could be submitted to amplify and support the foregoing, but would involve a prolonged and unnecessary dissertation.

SECTION NO. 7.

Clause No. 1.—According to the foregoing, some of the lines and points have been obtained by having recourse to the Hypothesis, which, according to geometrical reasoning, is a valuable method of investigation. It is, however, absolutely essential that the drawing of any line or the locating or placing of any point, as in this Trisection problem, shall be accomplished entirely and independently of the Hypothesis, viz., by the method of construction adopted in the present instance. The points viz., R — E —Z — K — A′ — A and the lines, viz., A E — A B — E A — Z K — R A′ — R A have been found independently of the Hypothesis (see Section 5, Clause 9).

Clause No. 2.—It is therefore concluded that the line drawn from the point K to the point E will form an angle with the line C Z equal to one-third ($\frac{1}{3}$) of the given angle, that is to say K E C = $\frac{A\,B\,C}{3}$ or $\frac{K\,B'\,C}{3}$ and K B′ C = A B C.

It is therefore concluded that a line drawn through the point B parallel to the line E K will coincide with the line B X (in the triangle A B C), and it is likewise concluded that a line drawn through the point B′ parallel to the line E K will form an angle with the line C Z (or C B B′ Z) that will be equal to the one-third ($\frac{1}{3}$) of the angle K B′ C or $\frac{K\,B'\,C}{3}$ then, by bisecting the remaining portion of K B′ C (the angle A B C being equal K B′ C), the solution of the Trisection Problem has been accomplished.

SECTION NO. 8.

Clause No. 1.—Although the foregoing fully establishes the solution of the Trisection Problem, this Section No. 8 is submitted as an amplification, and a somewhat different construction from that hereinbefore adopted to determine the point where the trisecting line drawn from the point E intersects L H.

Produce the line E L to intersect A M in the point X′, and from the point J (in the line A M) draw the line from J parallel to E L intersecting the line E A (in the point E‴) and produce to intersect C Z in the point Y′. The straight line J E‴ Y′ will intersect the line L H (or H L), so that the line drawn from E to that point of intersection with the line L H in the point X‴ will be parallel to A P, and will form an angle with the line C Z that will be equal to one-third of the given angle A B C.

It has been shown (see last part of Section No. 4) that the horizontal divergent value of E L from the trisecting line drawn from E to its intersection with H L, measured from L along the line L H, is equivalent to H U′ + J′ J. Then E L being produced to intersect A M in the point X′, and the line J E‴ produced to intersect C Z in the

point Y′ and the straight line J E‴ Y′ being drawn parallel to the straight line E L X′, E X′ = J Y′.

From the point X′ (in the line X′ E) make X′ X″ = E L. Then E X″ = L X′, H A = E E‴, H A = L J (see Section No. 4, Clause No. 4). Then E E‴ = L J, J E‴ = E L, E L = X″ X′. The divergence of E L from the trisecting line drawn from the point E to its intersection with the line H L has been shown to be equal to the horizontal value of U′ H + J′ J. Then E L = X″ X′ and E L + X″ X′ = 2 E L and 2 E L is equal to the horizontal value of 2 U′ H + 2 J′ J.

It will be observed that E L + L X′ is equal to the whole of the line E X′, but 2 E L is greater than the line E X′, 2 E L = E X′ + L X″ and 2 E L — L X″ is equal to the horizontal value of U′ H + 2 J′ J, which show L X″ is equivalent to the horizontal value of U′ H, and L X″ is equal to that portion of E L situate between the points X″ and L, and the line drawn from E‴ to the intersection of the trisecting line with H L will be equal and parallel to the line X″ L or X″ L = E‴ X‴.

As already shown, E X′ = J Y′ and J Y′ = Y′ E‴ + E‴ X‴ + X‴ J. Then Y′ E‴ = E X″, X″ L = E‴ X‴ and X‴ J = L X′. Therefore the divergent value of E L is equal to the convergent value of Y′ X‴ and E‴ X‴ = X″ [illegible] Then the horizontal line L X‴ is equal to the horizontal line [illegible] (or L X‴ = E Y′) and L X‴ = U′ H + J′ J and the diverge[illegible] of X″ L is equal to the horizontal value of U′ H and X″ L = E X‴.

Furthermore, if a line parallel to E L be drawn from the point J′ (in the line A M) instead of from the point J, it will intersect the line H L at a horizontal distance measured from the point L that will be equal to U′ H + 2 J′ J instead of U′ H + J′ J. Therefore it is clear that the line E L produced to X′ (in the line A M) diverges from the trisecting line drawn from E to J′, and X′ J′ = U′ H + 2 J′ J. This reasoning establishes that the divergence of L X′ is equal to the horizontal J′ J or L X′ = the horizontal J′ J.

Clause No. 2.—The principle involved in this section, as already shown, is that the divergence of E L from the trisecting line drawn from the point E to its intersection with the line L H is the horizontal equivalent of U′ H + J′ J (see Clause No. 1), and that the line E L is divided in the point X″, so that L X″ is the horizontal equivalent of U′ H and E X″ is the horizontal equivalent of J′ J.

Therefore, if a line equal to L X″ be measured from the point E along the line E L and from that point of section (not marked) a line be drawn parallel to L J, it will intersect L H in the same point that the trisecting line, drawn from the point E, intersects L H, and the line drawn from the point J parallel to E L will also intersect L H in the same point that the trisecting line drawn from the point E intersects

the line L H. As already stated, the straight line J Y′ has been drawn from the point J parallel to the line E L, and it is concluded that a line drawn from the intersection of that line with L H to the point E will form an angle with the line C Z that will be equal to one-third of the given angle A B C.

ADDENDUM.

There are other constructions by which the Trisection Problem can be solved and explained by reference to the main diagram, which constructions will be consistent with the principles involved in the solution of the problem as already presented, viz.:

1. From the point H (at the intersection of the line A E with the perpendicular B F′) and with a radius equal to 2 A B + 2 K J″ (see Section 6, Clause 5) describe an arc intersecting the line C Z, and the line drawn from that point of intersection to the point H will form an angle with the line C Z that will be equal to one-third of the given angle A B C.

2. The line R H is equal and parallel to the line E L. From the point R in the line R H make R R′ = E X″ (see Section No. 8) and from the point A draw A R′ produced to intersect the line C Z and that line will form an angle with C Z that will be equal to one-third of the given angle A B C that is to say it will intersect C Z in the point P.

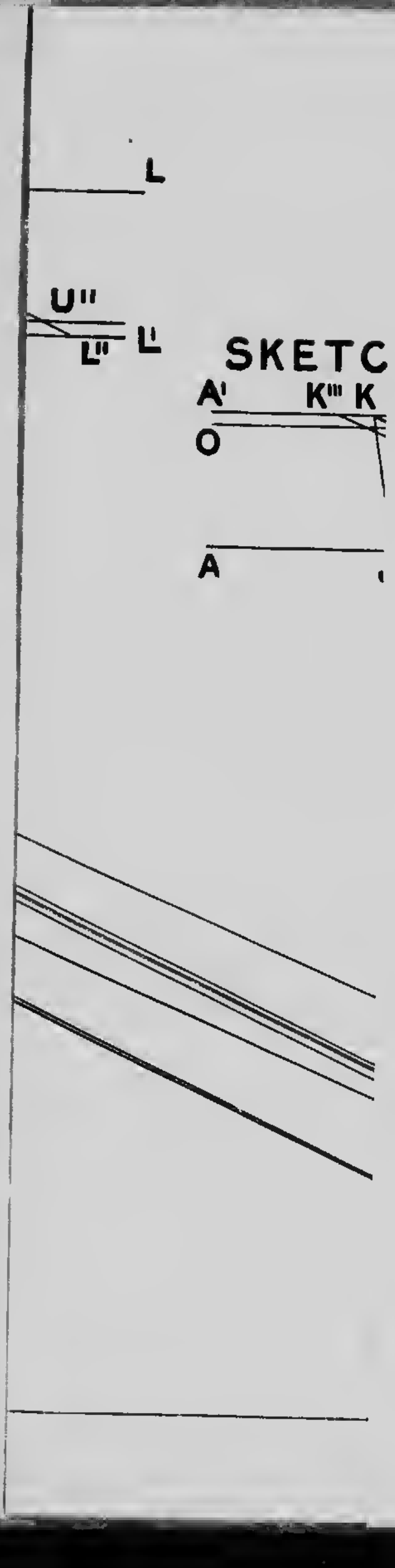
3. From the point where the arc B G intersects the line T L′ draw a line parallel to E J and produce it to intersect the line A′ M′, and from that point of intersection draw a line to the point E, and that line will form an angle with C Z that will also be equal to one-third of the given angle A B C.

4. From the point where the arc B G intersects the line T L′ erect a perpendicular (11—1) to intersect the line L H.

Then take the line situate between the point H and the point where the line A R intersects the line L H produced. The sum of these two lines, as described, the former, from L to the intersection of the perpendicular (erected from the intersection of the arc B G with T L′) being greater than R P and the latter being greater than H U′, their sum is greater than H U′ + R P, and R P + U′ H measured from L along the line L H, is equal to the horizontal distance from L that the trisecting line if drawn from the point E would intersect the line L H. Then the line drawn from the point J to that point where the trisecting line would intersect L H is equal or equivalent to the divergent or convergent horizontal value of J′ J (or R P).

Therefore, if a line equal to the sum of these two lines, described, be measured from the point L along the line L H, it will intersect L H at a greater distance from L than the trisecting line, drawn from E, intersects L H. Then if a line be drawn from J to this latter greater distance in L H, as described, is from L, that line will have a greater convergent or divergent horizontal value than J′ J (or R P). Then draw from the point E a line parallel and equal to the one just described as having a greater divergent or convergent horizontal value than R P, and from the extremity of that line draw a line parallel to E L intersecting the line H L produced, and from that point of intersection with H L (produced) take a distance equal to the sum of the two said lines (which is greater than R P + H U′), and that will give the point where the trisecting line drawn from the point E intersects the line L H (or H L).

The End.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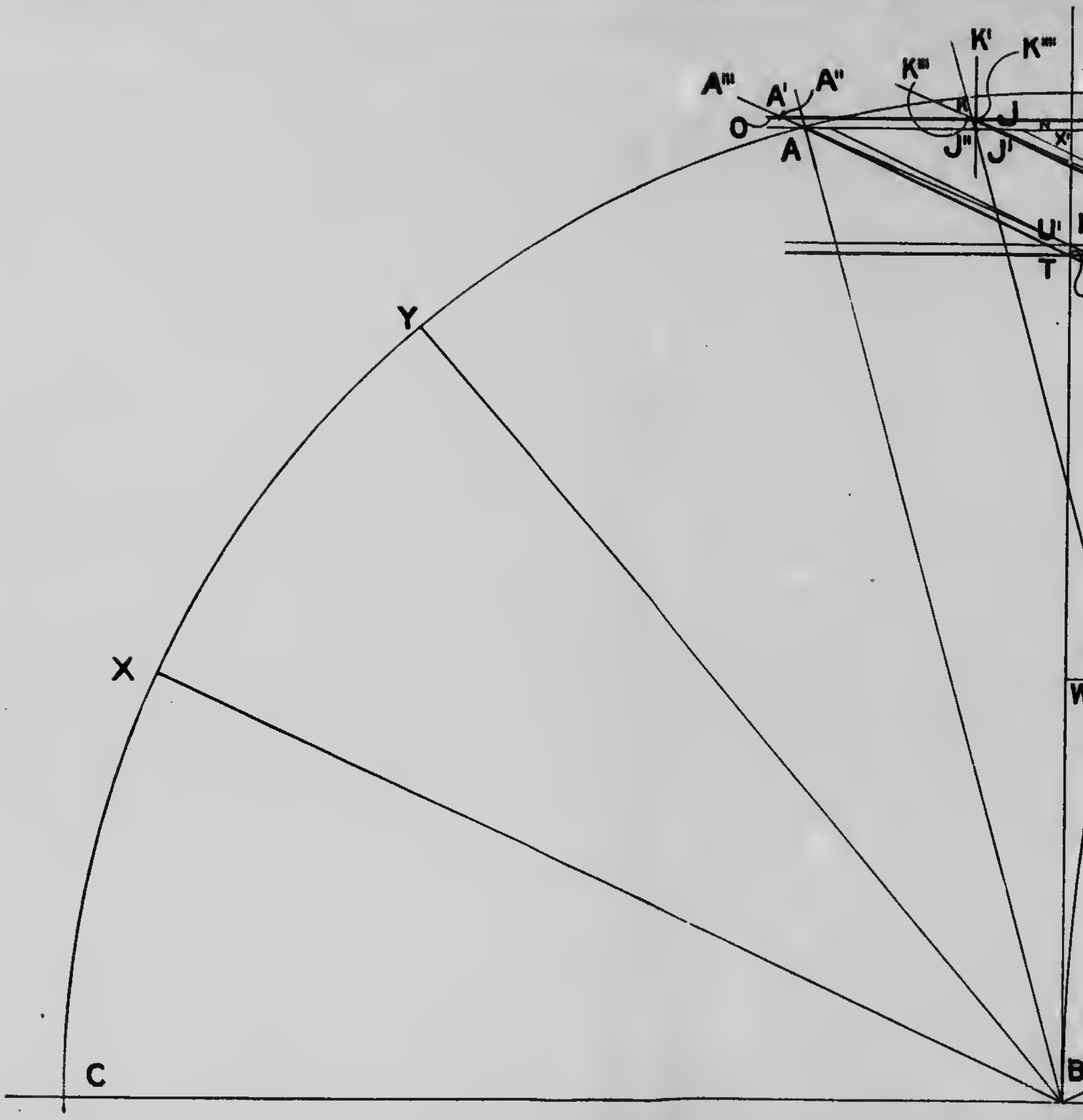

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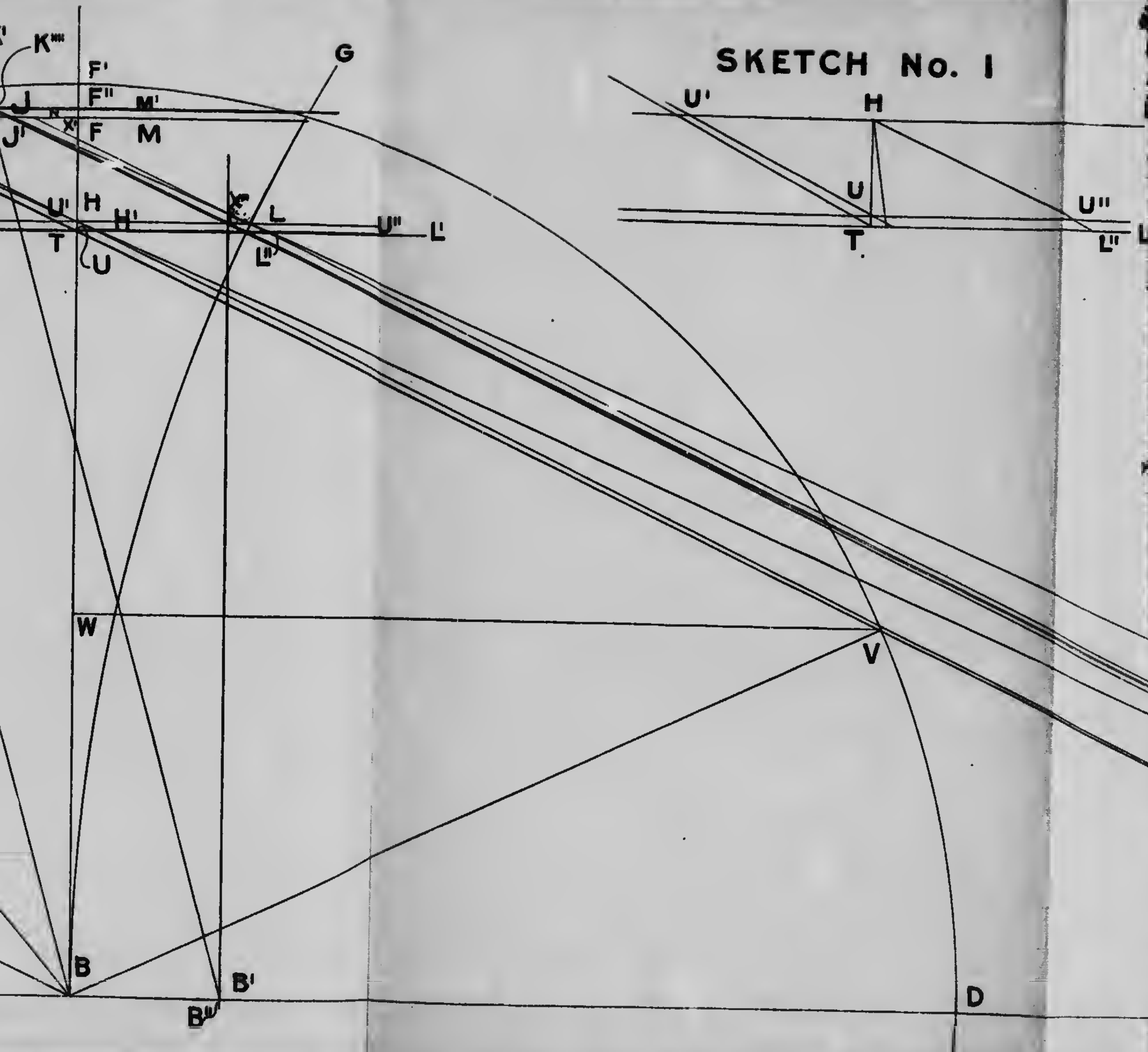
SKETCH No. 1



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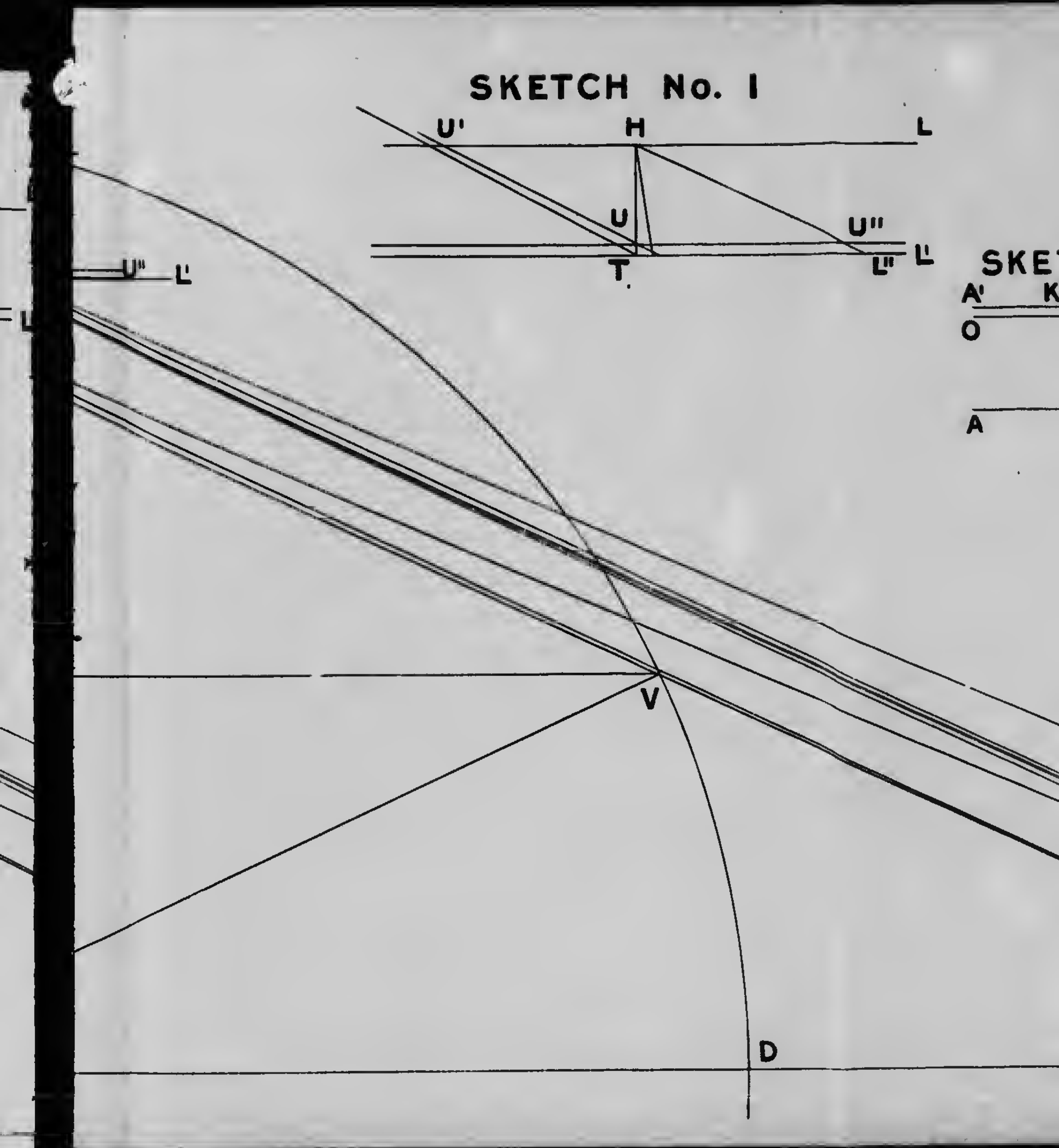

SKETCH No. 2.