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NEWCOMB'S MATHEMATICAL COURSE.

ELEMENTS

Prof Dupuis
Queen's College

OF

GEOMETRY

BY

SIMON NEWCOMB

Professor of Mathematics, United States Navy

THIRD EDITION, REVISED.

NEW YORK

HENRY HOLT AND COMPANY

1884.

PREFACE.

In the present work is developed what is commonly known as the ancient or Euclidian Geometry, the ground covered being nearly the same as in the standard treatises of Euclid, Legendre, and Chauvenet. The question of the best form of development is one of such interest at the present time, among both teachers and thinkers, as to justify a statement of the plan which has been adopted.

It being still held in influential quarters that no real improvement upon Euclid has been made by the moderns, a comparison with the ancient model will naturally be the first subject of consideration. The author has followed this model in its one most distinctive feature, that of founding the whole subject upon clearly enunciated definitions and axioms, and stating the steps of each course of reasoning in their completeness. By the common consent of a large majority of educators the discipline of Euclid is the best for developing the powers of deductive reasoning. If the work had no other object than that of teaching geometry, a more rapid and cursory system might have been followed; but where the general training of the powers of thought and expression is, as it should be, the main object, it becomes important to guard the pupil against those habits of loose thought and incomplete expression to which he is prone. This can be best done by teaching geometry on the time-honored plan.

Notwithstanding this excellence of method, there are several points in which the system of Euclid fails to meet modern requirements, and should therefore be remodeled. The most decided failure is in the treatment of angular magnitude. We find neither in Euclid nor among his modern followers any recognition of angles equal to or exceeding 180°, or any explicit definition of what is meant by the sum of two or more angles. The additions to the old system of angular measurement are the following two:

Firstly. An explicit definition of the angle which is equal to the sum of two angles.

Secondly. The recognition of the sum of two right angles as itself an angle. The term "straight angle" has been adopted from the *Syllabus* of the English *Association for the Improvement of Geometrical Teaching*. Although not unobjectionable, it seems to be as good a term as our language affords. The term *gestreckte Winkel*, used by the Germans, is more expressive.

One of the most perplexing questions which the author has met in the preparation of the work is that of distinguishing the definitions of plane figures as lines and surfaces. In our recent text-books it is becoming more and more common to define triangles, circles, etc., as portions of a plane surface. But, as soon as analytic geometry is reached, the circle is considered as a curve line and the triangle as three straight lines, while, even in elementary geometry, these terms, in a large majority of cases, refer only to the bounding lines. It has seemed to the author that the confusion thus arising can best be avoided by defining plane figures neither as mere lines nor mere surfaces, but as things *formed* by lines; to use the specific term *area* when extent of surface alone is referred to; and to use the words *circumference, perimeter*, etc., only in the sense in which they are used in higher geometry.

Other leading features of the work, which may be briefly pointed out, are the following:

I. The addition of an introductory book designed not only to present the usual fundamental axioms and definitions, but to practice the student in the analysis of geometric relations by means of the eye before instructing him in formal demonstrations. The exercises in sections 24 to 34 are first attempts in this direction, to which the teacher may add at pleasure until he finds that the pupil has thoroughly mastered the conceptions necessary for subsequent use.

II. The application of the symmetric properties of figures in demonstrating the fundamental theorem of parallels. This system has been adopted from the Germans.

III. After the second book, the analysis of the problems of construction, whereby the pupil is led to discover the construction by reasoning.

IV. The division of each demonstration into separate numbered steps, and the statement of each conclusion, where practicable, as a relation between magnitudes. It is believed that this system will make it much easier to carry the steps of the demonstration in mind.

Each step is, when deemed necessary, accompanied by a reference to the previous proposition on which the conclusion is founded, not, however, to encourage the too frequent habit of requiring the pupil to memorize the numbers, but simply to enable him to refer to the proposition. He should always be ready, if required, to cite the proposition, but its number in the book is not of such importance that his memory need be burdened with it. A reference has not been considered necessary after a few repetitions.

V. The theorems for exercise have been selected from native and foreign works with a view to present those best adapted, either by their elegance or their applications in the higher geometry, to interest the student. An attempt has been made to arrange those of each book in the order of their difficulty.

VI. Some of the first principles of conic sections have been developed for the purpose of enabling pupils who do not intend to study analytic geometry to have some knowledge of these curves. It is believed that a previous study of these principles will be a valuable preparation for the advanced treatment of conic sections.

VII. The most difficult subject to treat has been that of Proportion. The ancient treatment as found in Euclid is perfectly rigorous, but has the great disadvantages of intolerable prolixity, unfamiliar conceptions, and the non-use of numbers. The system common in our American works, of treating the subject merely as the algebra of fractions, has the advantage of ease and simplicity. But, assuming, as it does, that geometric magnitudes can be used as multipliers and divisors on a system which is not demonstrated, even for algebraic quantities, it is not only devoid of geometric rigor, but is not properly geometry at all. The author has essayed a middle course between these extremes which he submits to the judgment of teachers with some reserve.

On the ancient system, magnitudes are compared with respect to their ratios by means of their multiples. For instance, the magnitude A is considered to have to the magnitude B the ratio of 2 to 3 when $3A = 2B$. This system has the undeniable advantage of admitting commensurable and incommensurable quantities to be treated on a uniform plan. But it has the disadvantage of not according with the natural and customary way of thinking of the subject. When we say that the magnitude A is to B as 2 to 3, we mean that if A is represented by the number 2, or is divided into 2 parts, B will be represented by 3 of those parts. The author has considered it more important to base the subject on natural and customary modes of thought than to adopt a system simple and rigorous, but not so based. The mode in which he has endeavored to avoid the difficulty, and to render the natural system as rigorous and nearly as simple as the other, will be seen by an examination of the chapter on Proportion.

VIII. Another difficult subject is the fundamental relations of lines and planes in space. In presenting it the author has been led to follow more closely the line of thought in Euclid than that in modern works. At the same time he is not fully satisfied with his treatment, and conceives that improvements are yet to be made.

A collection of notes on the fundamental principles of geometry upon which the work has been based will be found in the Appendix.

The author believes, from some trials, that the study of geometry as here presented can be advantageously commenced at the age of twelve or thirteen years. No especial knowledge of algebra is required for the first three books, but a previous familiarity with symbolic notation will facilitate the study of the second and following books, and may be found necessary to their advantageous use. From the fourth book onward a knowledge of simple equations is sometimes presupposed.

TABLE OF CONTENTS.

BOOK VI. REGULAR POLYGONS AND THE CIRCLE.

BOOK VII. OF LOCI AND CONIC SECTIONS.

GEOMETRY OF THREE DIMENSIONS.

BOOK VIII. OF LINES AND PLANES.

BOOK IX. OF POLYHEDRONS.

BOOK X. OF CURVED SURFACES.

BOOK XI. THE MEASUREMENT OF SOLIDS.

E.

SYMBOLS AND ABBREVIATIONS USED IN DEMONSTRATIONS.

When a step of a demonstration leads to a relation of two lines or other magnitudes, the relation is expressed by symbols.

$=$ *equals:* states that two magnitudes are equal.

$\parallel$ *parallel to:* states that two lines are parallel.

$\perp$ *perpendicular:* states that two lines are perpendicular to each other.

$\cong$ *coincides with,* or *falls upon:* states that two points, lines, surfaces, or figures coincide with each other.

In recitation, the teacher may find it advantageous to have the student recite the reasoning orally, but write the conclusion of each step on the blackboard. In this case symbols or abbreviations of the more common words will shorten the work. The following are recommended, though others are frequently used:

$\angle$, *angle.*	/, *line.*	$\cdot$, *point.*
Ar, *area.*		R, or 90°, *right angle.*
C, or 360°, *circumference.*		S, or 180°, *straight angle.*

These abbreviations are not generally used in the printed book, the author believing that the full word, in its usual form, will make a stronger impression on the mind of the beginner than any symbolic representation of it.

BOOK I.

GENERAL NOTIONS.

CHAPTER I.

THE PRIMARY CONCEPTS OF GEOMETRY.

1. *Definition.* **Geometry** is the science which treats of magnitude, position, and form.

Def. A **geometric magnitude** is that which has extension in space, or what is familiarly called *size.*

Geometric magnitudes are of three orders: **solids, surfaces,** and **lines.** Besides these, there is a fourth concept—that of a **point.**

Solids.

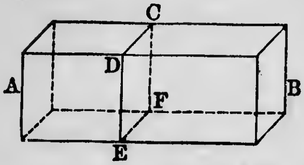
2. *Def.* A **solid** is that which has length, breadth, and thickness. **Length, breadth,** and **thickness** are called the **three dimensions** of the solid.

All material bodies are solids because they have these three dimensions and no more. The solids of geometry are bodies supposed to have the size, form, and mobility of material solids, but no other properties.

Surfaces.

3. *Def.* A **surface** is that which has length and breadth, but is not supposed to have thickness.

EXAMPLE 1. Let us conceive the solid AB to be divided into two parts through *CDEF* without removing any part of it, so that the two parts touch each other. The division *CDEF* is then a surface. If the surface had any thickness it would be a part either of the one solid or of the other. But it is only the boundary between them, and therefore no part of either, and therefore has no thickness.



Ex. 2. A sheet of paper is really a solid, because it must have some thickness. But the surface on which we write does not extend into the paper at all, and so has no thickness.

Lines.

4. *Def.* A **line** is that which has length, but is not supposed to have either breadth or thickness.

If we suppose a surface cut into two parts, touching each other, that which divides them is a line. It forms no part of either surface, and therefore can have no breadth.

A straight line.

Def. A **straight line** is one which has the same direction throughout its whole length.


A curve line.

Def. A **curve line** is one no part of which is straight.

The Point.

5. *Def.* A **point** is that which is supposed to have position, but neither length, breadth, nor thickness.

If we suppose a line cut into two parts touching each other, that which divides the parts is a point. It forms no part of either line, and therefore has no length.

We cannot make a real point, line, or surface out of matter, because all matter has the three dimensions, length, breadth, and thickness. But we may make a dot and think of it as a point. The real point would be the centre of the dot. We may also regard the sharp end of a pencil as representing a point.

Generation of Magnitudes by Motion.

6. A line may be generated by the motion of a point, as when we press the point of a pencil on paper and move it.

A surface may be generated by the motion of a line as a line is generated by the motion of a point.

A solid may be considered as generated by the motion of a surface. The surface, as it moves, must be supposed to leave a mark in every point through which it passes.

The Plane.

7. *Def.* A **plane** is a surface such that a straight line between any two of its points lies wholly on the surface.

A plane is perfectly flat and even, like the surface of still water, or of a smooth floor.

Geometric Figures.

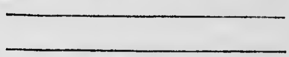
8. *Def.* A **figure** is any definite combination of points, lines, surfaces, or solids.

Def. A **plane figure** is one which lies wholly in a plane. It is formed by points and lines.

Plane geometry treats of plane figures.

Parallel Lines.

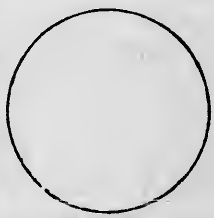
9. *Def.* **Parallel straight lines** are such as lie in the same plane, and never meet, how far soever they may be extended in both directions.



Parallel lines.

The Circle.

10. *Def.* A **circle** is a figure formed by a plane curve line, every point of which is equally distant from a point within it called the **centre**.



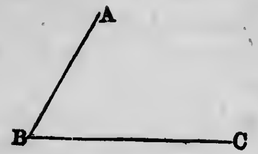
A circle.

The **circumference** of a circle is the line which forms it.

Def. An **arc** of a circle is a part of the circumference.

The Angle.

11. *Def.* An **angle** is a figure formed by two straight lines extending out from one point in different directions.



An angle.

Def. The **sides** of an angle are the two lines which form it.

Def. The **vertex** of an angle is the point where the sides meet.

Geometrical Symbols.

12. Any geometric concept, whether a *point*, *line*, *surface*, *solid*, or *angle*, may be represented by one or more letters of the alphabet.

A point is represented by a single letter near it.

A line is represented by one letter, or by two or more letters showing its course.

Other magnitudes or figures are represented by letters showing their outlines.

Designation of angles. A particular angle in a figure is designated by three letters, as ABC, of which the middle one B is at the vertex, and one of the other two on each side. The angle is then read ABC.

When there is only one angle formed at a vertex, it may be designated by a single letter at the vertex.

CHAPTER II.

COMPARISON OF GEOMETRIC MAGNITUDES.

Mode of Comparison.

13. *Def.* Two magnitudes which can be so applied to each other that each shall coincide with the other throughout its whole extent are said to be **identically equal.**

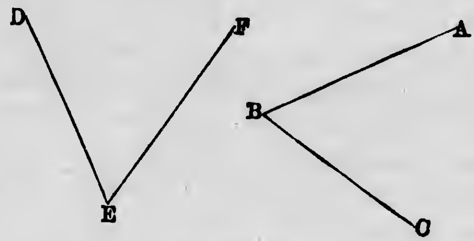
Def. If two magnitudes can be so divided into parts that each part of the one is identically equal to a separate part of the other, they are said to be **equal.**

Def. That which divides a magnitude into two equal parts is said to **bisect** the magnitude, and is called a **bisector.**

The point B bisects the line A C.

To **trisect** a magnitude means to divide it into three equal parts.

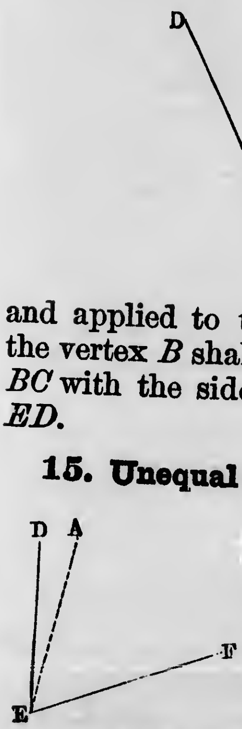
14. Equal Angles. Two angles *ABC* and *DEF* are said to be **equal** if the angle *ABC* can be taken up



Equal angles.

and applied to the angle *DEF* in such manner that the vertex *B* shall coincide with the vertex *E*, the side *BC* with the side *EF*, and the side *BA* with the side *ED*.

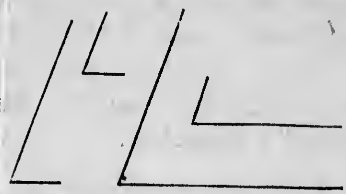
15. Unequal Angles. If, on thus applying the angles to each other, the side *BA* falls between the sides *ED* and *EF*, as in the dotted line, then the angle *CBA* (which is the same as *FEA*) is said to be **less** than the angle *FED*, and the angle *FED* is said to be **greater** than the other which falls inside of it.



Unequal angles.

16. Remark. The magnitude of an angle does not depend upon the length of its sides, but only upon their direction.

When we make the sides EF and BC coincide, it is only necessary that they shall coincide through the length of the shorter side, in order to test the equality or inequality of the angles.



These four angles are all equal, notwithstanding the difference in the lengths of their sides.

Symbols of Comparison.

17. The statement that any two magnitudes are equal is expressed by writing the sign = between the letters or words which indicate them.

The statement expressed by the sign =, that two magnitudes are equal, is called an **equation**.

The statement that one magnitude is greater or less than another is expressed by writing the sign > or < between them, the opening of the angle being toward the greater magnitude.

Examples. The expression

$$A = B$$

means that the magnitude A is equal to the magnitude B.

The expression

$$A > B$$

means that the magnitude A is greater than the magnitude B.

The expression

$$A < B$$

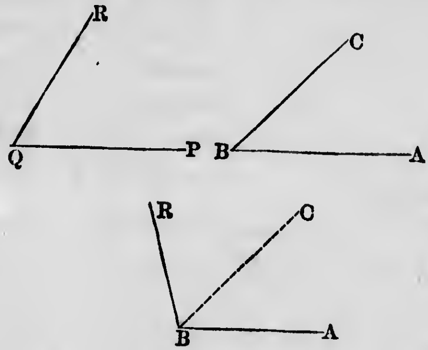
means that the magnitude A is less than the magnitude B.

Sum and Difference of Magnitudes.

18. *Def.* The magnitude formed by joining two or more magnitudes together is called their **sum**.

The sum of two or more straight lines is the line obtained by putting them end to end in the same straight line.

The sum of two angles ABC and PQR is the angle ABR



Sum of angles.

formed by applying the side QP to the side BC, so that the vertex Q shall fall on the vertex B, and the side QR on the opposite side of BC from BA.

Def. If the angles ABC and CBR are equal, each of them is said to be the **half** of the angle ABR, and the line BC is said to **bisect** the angle ABR.

When from one magnitude a part equal to another is taken, that which is left is called their **difference.**

Notation of Sum and Difference.

19. The sum of two magnitudes is expressed by writing the sign $+$, *plus*, between them.

EXAMPLES.

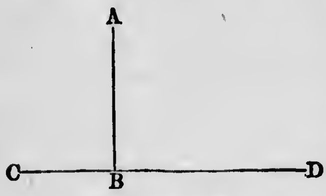
$$\text{Angle } ABC + \text{angle } CBR = \text{angle } ABR.$$
$$\text{Line } AB + \text{line } BC = \text{line } AC.$$

The difference between two magnitudes is expressed by writing their symbols with the sign $-$, *minus*, between them, the magnitude taken away being on the right.

EXAMPLES. Angle $ABR -$ angle $ABC =$ angle CBR.
Line $AC -$ line $AB =$ line BC.

Kinds of Angles.

20. *Def.* When a straight line AB standing on another straight line CD makes the angles ABC and ABD equal, each of these angles is called a **right angle**, and the line AB is said to be **perpendicular** to the line CD.



Right angles.

21. *Def.* When the two sides OA and OB of an angle go out in opposite directions, so as to be in the same straight line, the angle is called a **straight angle.**

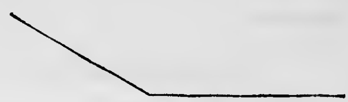
Straight angle.

We may conceive a straight angle to have its vertex at any point of a straight line.

A straight angle is by definition the sum of two right angles, because the sum of the two right angles ABC and ABD is (**18**) the angle CBD, which is a straight angle.

22. *Def.* An **acute** angle is one which is less than a right angle.


Acute angle.


Obtuse angle.

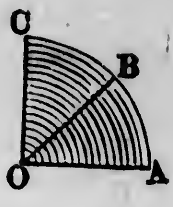
23. *Def.* An **obtuse** angle is one which is greater than a right angle.

Example of forming Angles by Addition.

24. Let us take a surface with a circular boundary $ABCDEFGH$, and cut it into eight equal parts by four lines all passing through its centre O.

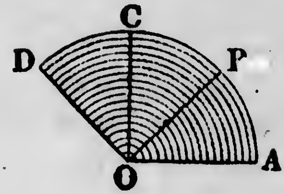
A circular disk of paper or pasteboard may be used to represent this surface.

Then let us put the pieces together again, one by one, beginning at A and going round in alphabetical order, and let us study the angles thus formed.

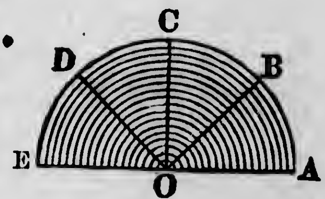


On adding the angle once we shall have a right angle AOC.

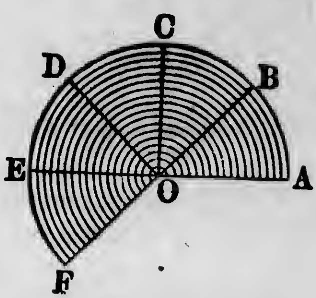
On adding it again we shall have the obtuse angle AOD, greater than a right angle, but less than a straight angle.



On adding it again we shall have a straight angle AOE, because we cut the figure so that AOE should be in a straight line.

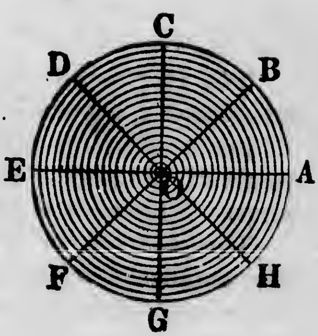


On adding it again we shall have the convex angle AOF. This angle will be greater than a straight angle if we count it round in the direction we have added its parts, but it will be less if we count it in the shortest direction from A through H and G to F. The relation of these two ways of considering an angle will be shown presently.



By one more addition the angle formed will be the sum of a right angle and a straight angle, or of three right angles, when measured the one way, but equal to a right angle when measured the other way.

If we add the angle twice more the whole space around O will be filled up, and the sum of the eight angles will be the angle AOA, counted all the way round, which is called a *perigon*.



Def. A **perigon** is equal to the sum of four right angles or of two straight angles.

Angular Measure.

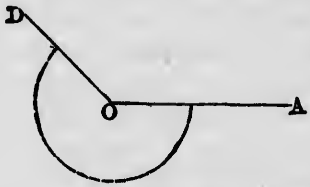
The following summary includes a recapitulation of results from the preceding sections:

25. An angle is measured by how much one of the sides must be turned to make it coincide with the other side.

Since one side can be brought into coincidence with the other by turning it in either direction, there are two measures to every angle.

EXAMPLE. In the figures the side *OA* can be brought into coincidence with *OD* by turning it either in the opposite direction to that in which the hands of a watch move, or in the same direction.

The lesser measure of the angle *AOD*.



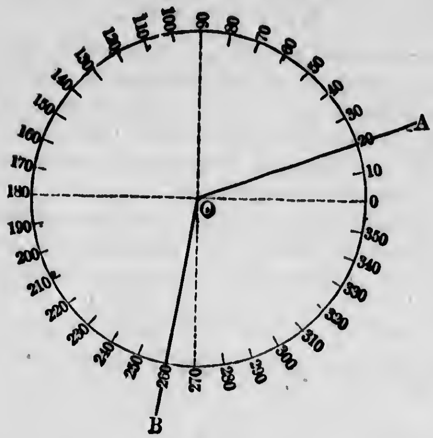
The greater measure of the angle *AOD*.

These two directions can be distinguished and the amount of motion be measured by describing an arc of a circle from one side to the other around the vertex of the angle as a centre. This arc must pass through the space over which the one arm must turn in order to coincide with the other.

26. In practice, angles are measured by degrees and subdivisions of a degree, in the following way:

Let a complete circle be drawn with its centre on the vertex of the angle. Let this circle be divided into 360 equal parts. Then each of these parts is called a degree.

The sides of the angle will cut the circle in two points. The number of degrees between these points is the measure of the angle, and the angle is said to be of that number of degrees.



The two sides of the angle divide the circle into two arcs, corresponding to the two measures of the angle just described.

Example. In the figure the side OA of the angle AOB cuts the circle at 20°, and the side OB at 260°. Counting the degrees in both directions we see that the angle measures 240° in one direction, and 120° in the other.

Remark. The sum of the two measures will always be 360°, which is therefore a perigon.

27. *Def.* The two measures of an angle are said to be **conjugate** to each other, or to represent **conjugate angles.**

One of the conjugate measures will always be less than 180°, and the other greater, except when each is equal to 180°.

The greater measure is called a reflex angle. Hence,

Def. A reflex angle is one which is greater than a straight angle, or greater than 180°.

28. The following relations of angles flow from what has been said:

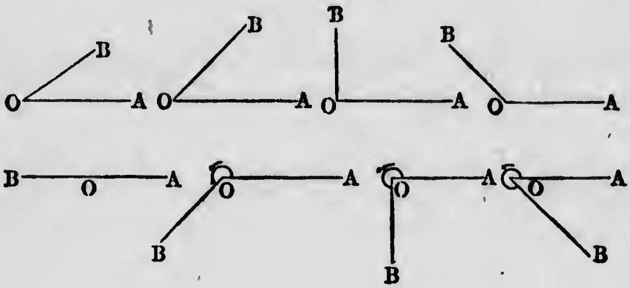
$$
\begin{aligned}
&& && 1\text{ right angle} &= 90^\circ\\
&& 1\text{ straight angle} &= 2\text{ right angles} &&= 180^\circ\\
1\text{ perigon} &= 2\text{ straight angles} &&= 4\text{ right angles} &&= 360^\circ
\end{aligned}
$$

EXERCISES.

NOTE. The following exercises are intended to familiarize the pupil with the idea of the magnitudes and measures of angles by causing him to make an eye-estimate of the magnitude of each angle, and, where applicable, a computation of their relations. It will be well for him to make a small paper protractor in order that he may check his estimates by some kind of measures, though rude.

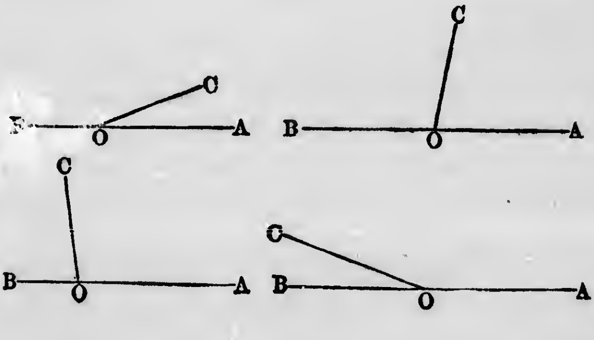
Where he is asked to draw angles, it is intended that he shall practice the drawing without instruments, repeating his first attempts until he obtains a drawing as accurate as he can make it by the unaided eye.

1. What kind of an angle is each of the following, and how many degrees do you judge it measures, the magnitude

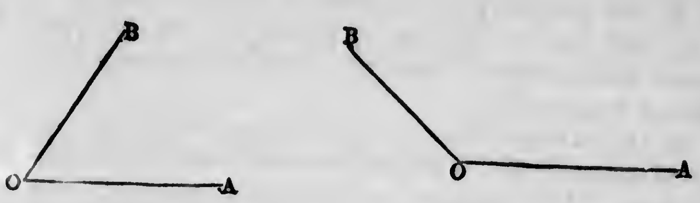


of each angle being measured from *OA* to *OB* in a direction the opposite of that of the motion of the hands of a watch?

2. What is the magnitude of each of the following angles, *AOC* and *COB*?



3. Draw an acute angle *AOB*. Bisect it. Draw another, and trisect it.

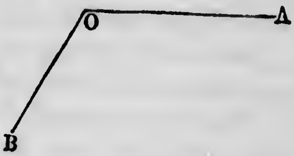


4. Draw an obtuse angle. Bisect it. Draw another and trisect it.

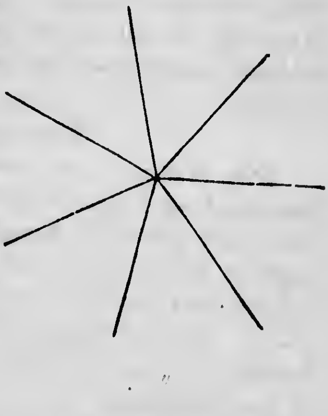
5. Draw an angle of 175° and bisect it.

6. Draw a straight angle and bisect it. Draw another and trisect it.

7. Draw a reflex angle and bisect it on the convex side. Then bisect the conjugate angle on the other side. Estimate the number of degrees in each of the angles thus formed.



8. Here are seven straight lines going out from the same point and making equal angles with each other. Now draw five other figures formed respectively of 6, 5, 4, 3, and 2 straight lines going out from the same point and making equal angles with each other. How many degrees in each angle thus formed?



9. Draw, by the eye, angles of 60°, 90°, 120°, 150°, 210°, 240°, 270°, 300°, 330°.

Comparison of Geometric Figures.

29. The only way in which we can decide whether two magnitudes are equal or unequal is by applying one to the other, or applying some third magnitude to both.

We are to think of the geometric figures as made of perfectly stiff lines which can be picked up from the paper and moved about without bending or undergoing any change of form or magnitude.

If two straight lines are to be compared we lay one upon the other, and find whether the two ends can be made to coincide. If so they are equal; if not, unequal. We may also take some measure (a scale of equal parts, for example) and apply it first to one line and then to the other.

If two planes are to be compared, they may be applied without change to each other. If they are of different shapes, one may be cut to pieces and the parts laid upon the other. If the latter can thus be exactly covered, the two surfaces are equal; if more than covered, the first is the larger; if not covered, the second is the larger.

Solids are compared by finding whether they will fill the same space, one or both of them being cut to pieces if necessary.

But the geometer does not actually apply his figures to each other, but only imagines them so applied. He is thus able to learn things which are true of all figures of a certain kind, whereas by actual measurement one can only learn what is true of the particular figure which he measures. This will be better understood when it is seen how theorems are demonstrated.

Trace of a Figure.

30. When we imagine a figure moved away, we may also imagine that it leaves its outline fixed upon the paper. Such an outline is called a **trace.** The trace will occupy exactly the position which the figure itself occupied before being moved, will be equal to the figure in every respect, and will be represented by the drawing of the figure. If another figure is found to coincide with the trace, it will be identically equal to the first figure.

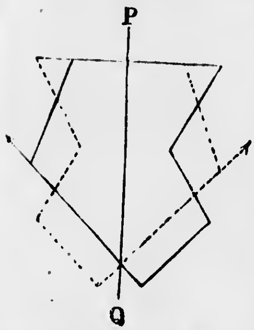
Since figures are supposed to be movable, the beginner may grasp the relation between a figure and its trace by imagining that he marks around the figure with a pencil. Then when the figure is taken away the marks will remain.

CHAPTER III.

OF SYMMETRY.

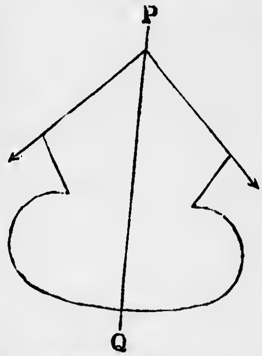
Symmetry with Respect to an Axis.

31. Let us take the figure in the margin, turn it over, and put it back so that the line PQ shall fall in its original position, but the figure shall be turned right side left. It will then fall into the position represented by the dotted lines. To make this change of position we may suppose the ends of the line PQ to be pivots, so that the figure can turn on this line as on an axis. In turning, one side of the figure must be supposed to sink below the paper and the other to rise above it, while the axis PQ remains unmoved.



An unsymmetrical figure.

If we take this figure and turn it over on the axis PQ, the right side will fall on the trace of the left side, and *vice versa*, so that the figure will occupy the same lines on the paper as before it was moved. Such a figure is said to be symmetrical with respect to the axis PQ. Hence the following definition:



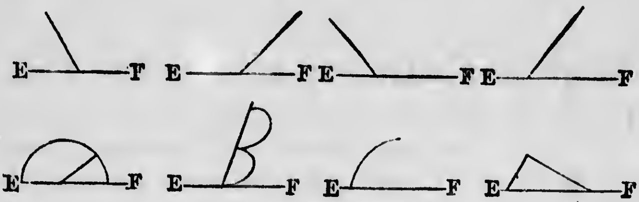
A figure symmetrical with respect to the axis PQ.

32. *Def.* A figure is said to be **symmetrical with respect to an axis** when, on being turned over on that axis, every part of the figure is in the position which the opposite part occupied before being moved.

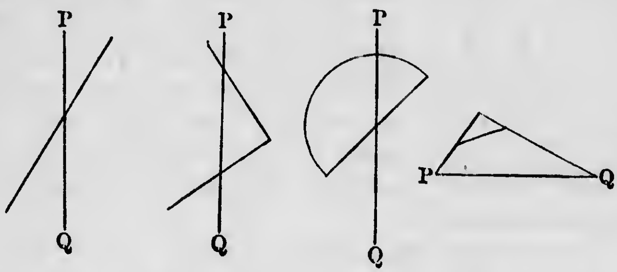
Def. The axis of revolution is called an **axis of symmetry.**

EXERCISES.

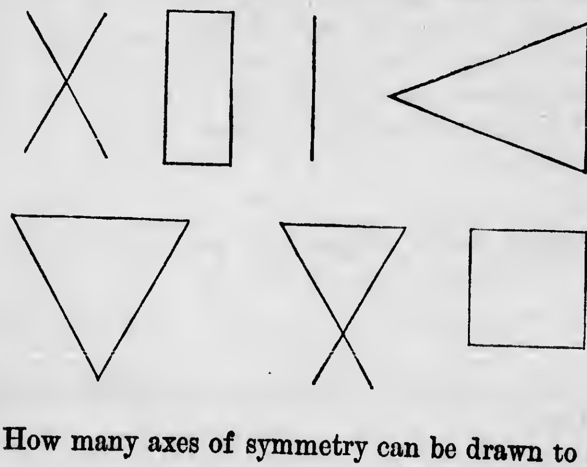
1. Copy the following figures. Then imagine each one turned over so that the line *EF* shall be changed end for end, and draw dotted lines showing where the rest of the figure would fall.



2. Let each of the following figures be turned over on the line *PQ* as an axis. Then draw dotted lines showing where the figure will fall.



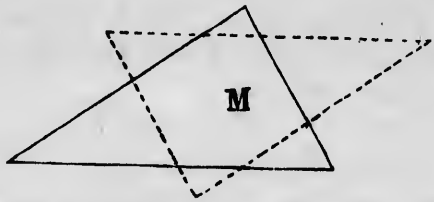
3. Draw the axis of symmetry of each of the following figures. If there is more than one such axis, draw them all.



4. How many axes of symmetry can be drawn to a circle?

Symmetry with Respect to a Point.

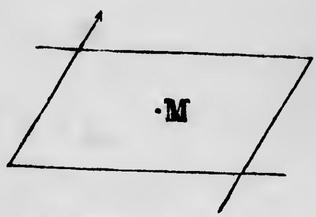
33. We may next suppose that the figure, instead of being turned over, is turned half way round on a fixed point without leaving the paper. For instance, suppose the an-



A figure unsymmetrical with respect to the point *M*.

nexed figure to have a pin stuck through it at the point *M*, and to be turned half round on that pin. It will then take up the position shown by the dotted outline.

If we turn this figure half way round in the same way, every part of it will occupy the position which the opposite part occupied before the motion, and the position of the figure will be represented by the same drawing. Such a figure is said to be symmetrical with respect to the point *M*.



A figure symmetrical with respect to the point *M*.

Hence the following definition:

34. *Def.* A figure is said to be **symmetrical with respect to a point** when, being turned half way round on this point, every part of the figure is in the position which the opposite part occupied before being moved.

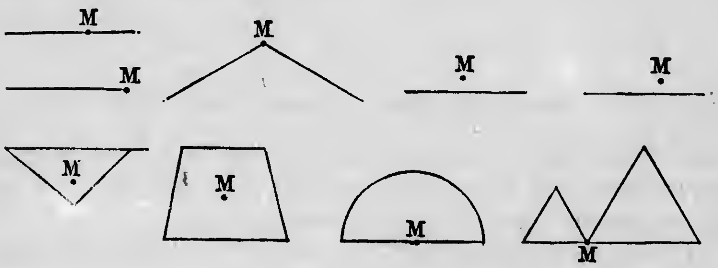
Def. The point on which the figure turns is called a **centre of symmetry.**

The different motions in §§ 31 and 33 must be studied. In the former the figure is turned over so that the side at first on the paper is turned up after the motion, and each part changes places with the corresponding part on the other side of the axis.

In the latter the figure does not leave the paper, but simply turns on it without turning over. Every part of the figure changes places with the part which is at an equal distance on the other side of the pivot point.

EXERCISES.

Copy the following figures. Then suppose them turned half way round on the point *M*, and draw dotted lines showing where the figure will fall.



CHAPTER IV.

LOGICAL ELEMENTS OF GEOMETRY.

Definitions.

35. *Def.* A **proposition** is either a statement that something is true, or a requirement that something shall be done.

A proposition affirming something to be true may be either an *axiom* or a *theorem.*

36. *Def.* An **axiom** is a statement which we assume to be true without proof.

For the axioms of geometry we try to take propositions which are self-evident and so need no proof.

37. *Def.* A **theorem** is a statement which requires to be proved.

A proposition requiring something to be done may be a *postulate* or a *problem.*

38. *Def.* A **postulate** is something which we suppose capable of being done without showing how.

39. *Def.* A **problem** is something which we must show how to do.

40. *Def.* A **demonstration** is the course of reasoning by which we prove a theorem to be true.

41. *Def.* A **corollary** is a theorem which follows from some other theorem.

42. *Def.* A **lemma** is an auxiliary theorem, to be used in demonstrating some other theorem.

43. *Def.* A **scholium** consists of remarks upon the application of theorems.

Axioms of Geometry.

44. *Axioms of magnitude in general.*

Axiom 1. Magnitudes which are each equal to the same magnitude are equal to each other.

Symbolic expression of this axiom. From

Magnitude $X =$ magnitude A,
Magnitude $Y =$ that same magnitude A,

we conclude

Magnitude $X =$ magnitude Y.

Ax. 2. If equals be added to equals, the sum will be equal.

Symbolic expression. From

$$X = Y,$$
$$A = B,$$

we conclude

$$A + X = B + Y,$$

and

$$A + Y = B + X.$$

Ax. 3. If equals be subtracted from equals, the remainders will be equal.

Symbolic expression. From

$$X = Y,$$
$$A = B,$$

we conclude

$$X - A = Y - B.$$

Ax. 4. Similar multiples of equals are equal to each other.

Symbolic expression. If n be any number, then from

$$X = A$$

we conclude

$$n \text{ times } X = n \text{ times } A.$$

This may be regarded as a corollary from Axiom 2.

Ax. 5. Similar fractions of equal magnitudes are equal.

Ax. 6. If equals be added to unequals, that sum will be the greater which has been obtained from the greater magnitude.

Symbolic expression. From

$$A > B,$$
$$X = Y,$$

we conclude

$$A + X > B + Y.$$

Ax. 7. If equals be subtracted from unequals, that remainder will be the greater which is obtained from the greater minuend.

Symbolic expression. From

$$A > B,$$
$$X = Y,$$

we conclude

$$A - X > B - Y.$$

Ax. 8. The whole is greater than its part.

45. *Axioms of geometric relation.*

Ax. 9. A straight line is the shortest distance between any two of its points.

Ax. 10. If two straight lines coincide in two or more points, they will coincide throughout their whole length.

Corollary. Two straight lines can intersect in only a single point.

Ax. 11. Through a given point one straight line can be drawn, and only one, which shall be parallel to a given straight line.

The Demonstration of Theorems.

46. A theorem of geometry first supposes something to be true of a figure, and then concludes, from this supposition, that something else must be true.

That which is supposed to be true of a figure is called the **hypothesis.**

That which is proved to follow from the hypothesis is called the **conclusion.**

The hypothesis is explained for demonstration by reference to a figure.

In general, the figures as drawn need not correspond to the hypothesis. On the other hand, the hypothesis applies not simply to the figure drawn, but to every possible figure fulfilling the conditions.

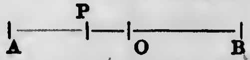
The drawn figure is used to assist the beginner. In the higher investigations of geometry no figures are drawn, but letters only are used to designate them, as they are supposed to be conceived in the mind of the reader.

47. *Def.* When two propositions are so related that the hypothesis of each is the conclusion of the other, they are said to be the **converse** of each other.

THEOREM I.*

48. *A straight line can be bisected in only a single point.*

Here the hypothesis supposes that we take any straight line whatever and bisect it.



To enunciate the hypothesis we call one end of the line A and the other end B, and the point of bisection O.

Then the hypothesis means that the point O is equally distant from A and B.

The conclusion asserts that there is no other point than O on the line which is equally distant from A and B.

The proof is effected by showing that to suppose any other point having this property is impossible.

If there is such a point, call it P, and suppose it between A and O (because we may call either end of the line A).

Let us then suppose that PA is equal to PB.

Because P is between A and O, AP will be less than AO.

Because OB is by hypothesis equal to OA, PB, which is greater than OB, will be greater than OA.

Therefore, if we suppose PA and PB equal, PA will be greater than OA and less than OA at the same time, which is absurd. Therefore there is no point on the line except O which is equally distant from the ends of the line.

THEOREM II.

49. *A straight line is symmetrical with respect to the perpendicular passing through its middle point.*

Hypothesis. AB, a straight line; O, its middle point; PQ, a perpendicular passing through O.

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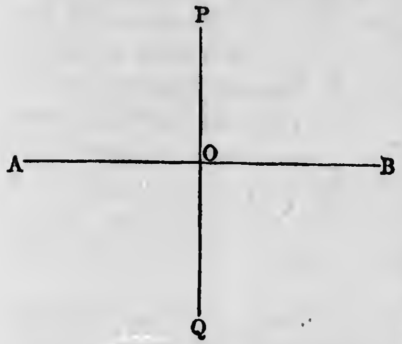
* These simple theorems are presented partly as exercises and explanations for the beginner, and partly as the basis of subsequent theorems. The demonstrations are not necessarily to be recited in full as given, but the student should be encouraged and assisted in stating the substance of the reasoning in his own language.

Conclusion. The line AB is symmetrical with respect to the axis PQ.

By reference to the definition of symmetry, § 32, the conclusion is found to mean that if the line AB be turned over on the line PQ as an axis, it will fall on its own trace; that is, into its original position; being merely changed end for end.



Demonstration. Suppose the line turned over on the axis PQ. By hypothesis and definition the angles POB and POA are equal. Therefore, after the line is turned over, the side OA will fall into the position OB, and *vice versa.* (§ 14).

Because the lengths OA and OB are equal (by hypothesis), the point A will fall on B, and *vice versa.*

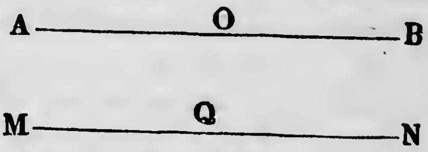
So the conclusion is proved.

Exercise for the pupil. Prove in the same way that the line AB is symmetrical with respect to the point O as a centre of symmetry (§ 34).

Theorem III.

50. *All straight angles and all right angles are equal to each other.*

To prove the first part of this proposition it is sufficient to show that any two straight angles we choose to take are equal.



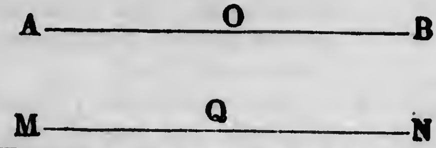
The hypothesis will be that we have any two straight angles which we may call AOB and MQN.

By the definition of a straight angle the hypothesis will mean that OA and OB go out from O in opposite directions so that AOB is a straight line, and that the same is true of QM and QN.

Note. The meaning of the hypothesis apart from its enunciation must always be clearly apprehended by the student.

The conclusion is that the angles AOB and MQN are equal.

By the definition of equality in angles, (§ 14), this will mean that if we apply AOB to MQN so that the side AO shall coincide with MQ, then the side OB will coincide with QN.

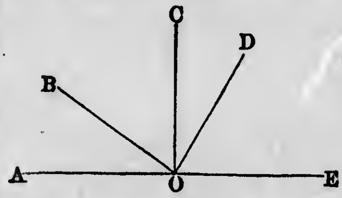


This must be the case, because two straight lines coincide throughout when any two of their points coincide. (§ 45, Ax. 10). Therefore the conclusion is proved, because, from the fact that any two straight angles are equal, it follows that all are equal.

Because a right angle is, by definition, the half of a straight angle, and because all straight angles are equal, it follows from § 44, Axiom 5, that all right angles are equal.

Theorem IV.

51. *The sum of all the angles formed on one side of a straight line by lines emanating from a point on it is a straight angle.*



Proof. If O be the point from which the lines emanate and OB, OC, etc., the lines emanating from it, then, by definition (§ 18), the sum of all the angles, AOB, BOC, etc., to DOE, will be the angle AOE; that is, a straight angle; because AOE is a straight line.

52. *Corollary.* The sum of all the angles around a point is equal to two straight angles, or a circumference.

BOOK II.

FUNDAMENTAL PROPERTIES OF RECTILINEAL FIGURES.

CHAPTER I.

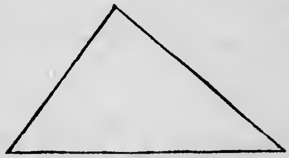
RELATIONS OF ANGLES.

Definitions.

53. *Def.* A **rectilineal figure** is one which is formed by straight lines.

54. *Def.* A **triangle** is a figure formed by three straight lines joined end to end.

55. *Def.* The three lines which form a triangle are called its **sides.**

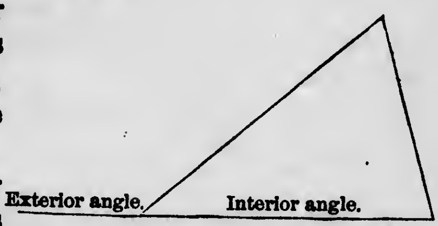


A triangle.

56. The sides of a triangle may be produced indefinitely. It is then called a **general triangle.**

REMARK. Any three indefinite straight lines which intersect each other in three different points form a general triangle. See figure on following page.

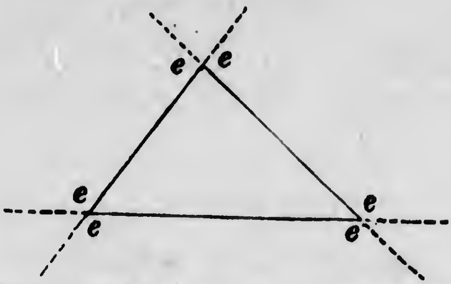
57. An **interior angle** of a triangle is one between two sides, measured inside the triangle.



58. An **exterior angle** of a triangle is one which is formed between any side and the continuation of another side.

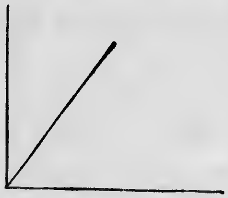
NOTE. The general triangle has six exterior angles in all.

REMARK. When no adjective is applied to the angle of a triangle an interior angle is meant.

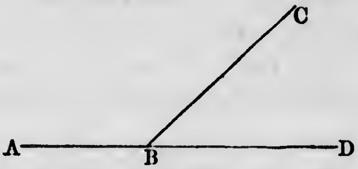


The six exterior angles, *e e e e e e*, of the general triangle.

59. *Def.* When the sum of two angles is a right angle, each is said to be the **complement** of the other, and the two are called **complementary angles.**



Complementary angles.

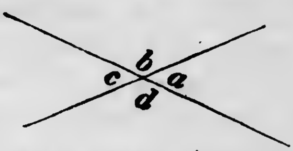


ABC and *CBD* are supplementary angles.

60. *Def.* When the sum of two angles is a straight angle, each is said to be the **supplement** of the other, and the two are called **supplementary angles.**

By definition, the sum of two angles *ABC* and *CBD* will be a straight angle when the two sides *AB* and *BD* lie in the same straight line. Therefore:

Corollary. If two supplementary angles be added, their extreme sides will form a straight line.



a and *b*, *b* and *c*, *c* and *d*, *d* and *a* are adjacent angles; *a* and *c* are opposite angles, and so are *b* and *d*.

61. When two straight lines cross each other four angles are formed, which we may call *a*, *b*, *c*, and *d*.

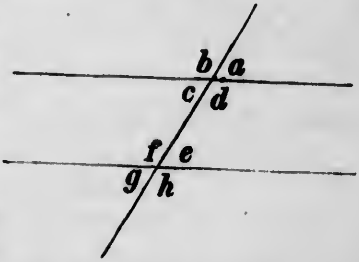
Any two of these angles which adjoin each other, as *a* and *b*, are called **adjacent angles.**

By § 60 two such adjacent angles are supplementary.

62. *Def.* A pair of angles contained between the same two lines on opposite sides of the vertex are called **opposite angles.**

63. *Def.* A **transversal** is a straight line crossing several other lines.

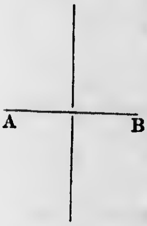
64. *Angles formed by a transversal.* When a transversal crosses two parallel lines it forms with them eight angles, four on each line. Pairs of these angles are designated thus:



The angles *a, b, g,* and *h* are called **exterior angles.** The angles *c, d, e,* and *f* are called **interior angles.** The pair *c* and *e* and the pair *d* and *f*, on opposite sides of the transversal and between the parallels, are called **alternate angles.** The pairs *a* and *e*, *b* and *f*, *c*, and *g*, *d* and *h*, on corresponding sides of the parallels and transversal, are called **corresponding angles.**

Remarks on Straight Lines.

65. Every straight line may extend without end in both directions. It is then called an **indefinite straight line.**



Sometimes we have to consider a line extending out only in one direction, and terminating at a point in the other direction.

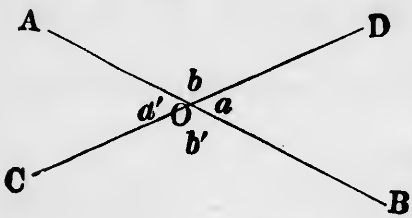
Example. In considering an angle, the two sides are suppose to terminate at the vertex.

Sometimes we have to consider a straight line contained between two definite points which are its ends. Such a piece is called a **finite straight line.**

66. *Def.* The **perpendicular bisector** of a finite straight line is an indefinite straight line passing at right angles through its middle point.

THEOREM I.

67. *If two straight lines intersect each other, the opposite angles will be equal.*



Hypothesis. AB and CD, two straight lines; O, their point of intersection; a, the angle DOB; a', the angle AOC; b, the angle DOA; b', the angle BOC.

Conclusions. Angle a = opp. angle a'.
Angle b = opp. angle b'.

Proof. 1. Because the sum of the angles BOD and DOA is the angle BOA, and OA and OB are in a straight line,

Angle a + angle b = straight angle. (§§ 18, 51)

2. In the same way it is shown that

Angle a' + angle b = straight angle.

3. Comparing (1) and (2),

Angle a + angle b = angle a' + angle b. (§ 44, Ax. 1)

4. Take away from these equal sums the common angle b, and we have

Angle a = angle a' (§ 44, Ax. 3). Q.E.D.

In the same way we may prove that

Angle b = angle b'. Q.E.D.

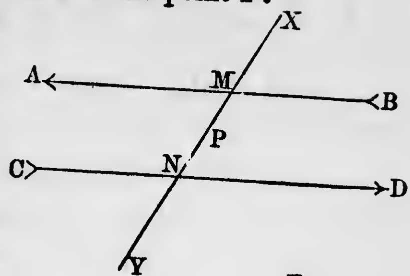
THEOREM II.

68. *If a transversal crossing two straight lines makes the alternate angles equal, the two straight lines are parallel.*

Hypothesis. XY, a transversal crossing the lines AB and CD at the points M and N, and making the angle AMN equal to its alternate angle MND.

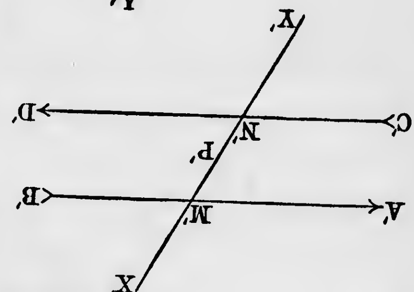
Conclusion. $AB \parallel CD$.

Proof. Bisect the length MN at the point P.



Let the figure be turned half way round, forming a new figure with the accented letters.

Apply the new figure to the old one so that the transversal shall be turned end for end, and the point N' shall fall upon M. Then—



1. Because $N'M' = MN$,
Point $M' \equiv$ position N.

2. Because, by hypoth.,
Ang. $A'M'N' =$ ang. MND,
and
Ang. $M'N'D' =$ ang. AMN,
therefore
$$\left.\begin{array}{l}\text{Line } M'A' \equiv \text{trace } ND; \\ \text{Line } N'D' \equiv \text{trace } MA.\end{array}\right\} \qquad (\S\,14)$$

3. Therefore the whole line $A'B'$ will fall upon CD, and $C'D'$ upon AB (§ 45, Ax. 10).

4. Suppose, if possible, that the lines AB and CD, when produced, meet in the direction B and D. Then, when the figure is inverted, the lines $A'B'$ and $C'D'$ will meet in the direction B' and D'. Because the new and old figures coincide when applied, BA and DC must also, when produced, meet in the direction A and C as well as in the direction B and D.

5. But the two straight lines AB and CD cannot meet each other in two points (§ 45, Ax. 10, Cor.).

Therefore they do not meet on either side.

Therefore they are parallel, by definition (§ 9). Q.E.D.

Corollary 1. If any two corresponding angles, as CNY and AMN, are equal, then, because MND is opposite to CNY, it is equal to it (Th. I.), and the alternate angles MND and AMN are also equal. Hence—

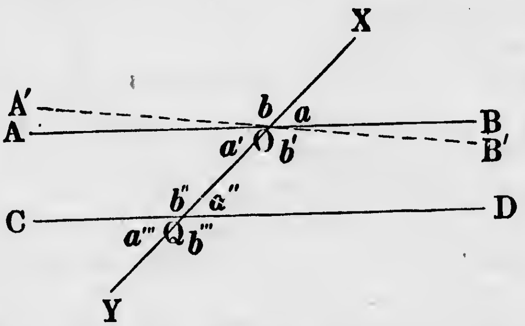
69. *If a transversal crossing two straight lines makes any two corresponding angles equal, those lines will be parallel.*

70. *Corollary 2. Any two perpendiculars to the same straight line are parallel.*

For such straight line is a transversal crossing the two perpendiculars, and making the angles all right angles.

THEOREM III.

71. *If a transversal cross two parallel straight lines, the four alternate and corresponding angles are equal to each other, and the other four are each equal to the common supplement of the first four.*



Hypothesis. XY, a transversal crossing the parallel lines AB and CD in the points O and Q, and forming with them the four alternate and corresponding angles

$$a,\quad a',\quad a'',\quad a''',$$

and the other four alternate and corresponding angles

$$b,\quad b',\quad b'',\quad b'''.$$

Conclusions.

I. Angle $a =$ angle $a' =$ angle $a'' =$ angle a'''.

II. Angle $b =$ angle $b' =$ angle $b'' =$ angle b'''.

III. Any angle $a +$ any angle $b =$ straight angle.

Proof. If the alternate angles a' and a'' are not equal, draw through O the line $A'B'$, making the angle $A'OQ$ equal to its alternate angle OQD. Then—

I. Because the alternate angles are equal,

Line $A'B' \parallel$ line CD. (§ 68)

But $AB \parallel CD$, by hypothesis.

Therefore we should have passing through O two straight lines AB and $A'B'$ each parallel to CD, which is impossible (§ 45, Ax. 11). Therefore

$$\text{Angle } a' = \text{angle } a''.$$

Also,

$$\left.\begin{aligned}\text{Angle } a &= \text{angle } a'.\\ \text{Angle } a''' &= \text{angle } a''.\end{aligned}\right\} \qquad (\S\ 67)$$

Therefore

$$\text{Angle } a = \text{angle } a' = \text{angle } a'' = \text{angle } a'''. \quad \text{Q.E.D.}$$

II. In the same way we may prove that

$$\text{Angle } b = \text{angle } b' = \text{angle } b'' = \text{angle } b'''. \quad \text{Q.E.D.}$$

III. Because AOB is a straight line,

$$\text{Angle } a + \text{angle } b = \text{straight angle}. \qquad (\S\ 51)$$

But all of the four angles a and b are equal. Therefore

$$\text{Any angle } a + \text{any angle } b = \text{straight angle}. \quad \text{Q.E.D.}$$

72. *Corollary. If a line be perpendicular to one of two parallels, it will be perpendicular to the other also.*

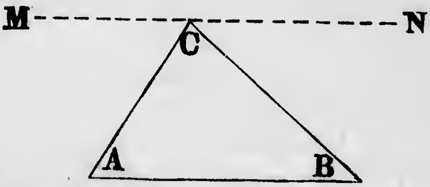
Theorem IV.

73. *The sum of the three interior angles of a triangle is equal to a straight angle.*

Hypothesis. ABC, any triangle.

Conclusion. Angle A + angle B + angle C = straight angle.

Proof. Through C draw a straight line MN parallel to the opposite side AB. Then—



1. Because CA is a transversal between the parallels AB and MN,

Angle A = alt. angle MCA (§ 72).

2. Because CB is a transversal between the same parallels,

$$\text{Angle } B = \text{alternate angle } BCN.$$

3. $$\text{Angle } C = \text{angle } ACB \text{ (identically).}$$

4. Adding these three equations,

$$\begin{aligned}\text{Angle } A + \text{angle } B + \text{angle } C &= MCA + ACB + BCN,\\ &= \text{angle } MCN,\\ &= \text{straight angle}.\end{aligned}$$

Therefore

Angle $A+$ angle $B+$ angle $C=$ straight angle. Q.E.D.

74. *Corollary* 1. *If two angles of a triangle are given, the third angle may be found by subtracting their sum from* 180°.

75. *Corollary* 2. *If two triangles have two angles of the one equal respectively to two angles of the other, the third angles will also be equal.*

EXERCISES.

1. If a triangle has two angles each equal to 60°, what will be the third angle?

2. If one angle of a triangle is a right angle, and one of the remaining angles is double the other, what will be the value of these two angles?

3. Prove that a triangle cannot have more than one right angle.

THEOREM V.

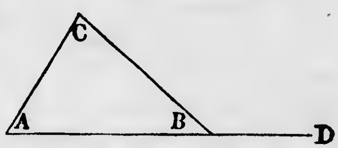
76. *Each exterior angle of a triangle is equal to the sum of the two interior and opposite angles.*

Hypothesis. ABC, any triangle. D, any point on AB produced.

Conclusion. Exterior angle $CBD=$ angle $A+$ angle C.

Proof. 1. Because ABD are in one straight line,

Angle $B+$ exterior angle $CBD=$ straight angle.



2. Angle $B+$ angle $A+$ angle $C=$ straight angle. (§ 73)

3. Comparing (1) and (2),

Angle $A+$ angle $B+$ angle $C=$ angle $B+$ ext. angle CBD.

4. Taking away the common angle B,

Angle $A+$ angle $C=$ exterior angle CBD. Q.E.D.

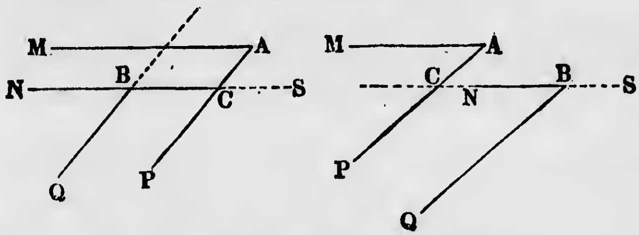
77. *Corollary. Any exterior angle of a triangle is greater than either of the interior and opposite angles.*

78. *Def.* Two parallel lines, each going out from a point, are said to be **similarly directed** or **oppositely**

directed according as they go out in the same direction or in opposite directions from their starting-points.

THEOREM VI.

79. *If the two sides of one angle are respectively parallel to the two sides of another, and similarly directed, these angles are equal.*



Hypothesis. Two angles MAP and NBQ having the sides AM and BN parallel and similarly directed, and the sides AP and BQ also parallel and similarly directed.

Conclusion. Angle NBQ = angle MAP.

Proof. Produce the side NB, if necessary, in either direction until it shall intersect the side AP of the other angle, also produced if necessary.

Produce NB past B to any point S.

Let C be the point of intersection of NB and AP. Then—

1. Because MA is parallel to NS, and AC is a transversal crossing them,

Angle MAP = alternate angle ACS. (§ 71)

2. Because AP and BQ are parallel, and NS is a transversal crossing them,

Angle NBQ = angle ACS.

3. Comparing (1) and (2),

Angle NBQ = angle MAP. Q.E.D.

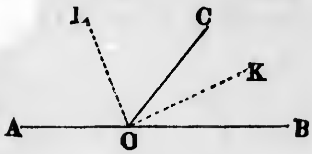
NOTE. In expressing this hypothesis in a diagram, two diagrams are necessary, according as the one angle is or is not contained within the other. But the same reasoning can be applied to both.

80. *Corollary* 1. *If the sides of two angles are parallel and oppositely directed, the angles will be equal.*

81. *Corollary 2. If the sides are parallel, and the one pair are similarly directed but the other pair oppositely directed, the angles will be supplementary.*

THEOREM VII.

82. *The bisectors of two adjacent angles on the same straight line are perpendicular to each other.*



Hypothesis. BOC and COA, adjacent angles on the straight line AB; OK, OL, the bisectors of these angles.

Conclusion. $OL \perp OK$; that is, $LOK =$ right angle.

Proof. 1. By hypothesis,

$$\text{Angle } COK = \tfrac{1}{2} \text{ angle } BOC.$$
$$\text{Angle } COL = \tfrac{1}{2} \text{ angle } COA.$$

2. Therefore

$$\begin{aligned} \text{Angle } KOL &= COK + COL, \\ &= \tfrac{1}{2}(BOC + COA), \\ &= \tfrac{1}{2} \text{ straight angle } BOA, \\ &= \text{right angle.} \end{aligned}$$

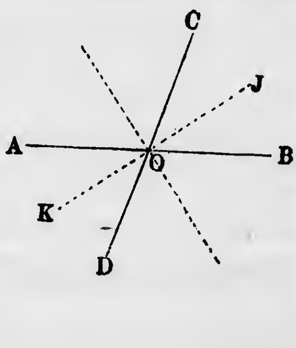
Q.E.D.

Corollary. Since an angle can have but one bisector, we conclude:

83. *A line through the vertex of an angle, perpendicular to the bisector, bisects the adjacent angle.*

THEOREM VIII.

84. *The bisectors of two opposite angles are in the same straight line.*



Hypothesis. AOD, BOC, two opposite angles formed by the straight lines AB and CD; OJ, the bisector of BOC; OK, the bisector of AOD.

Conclusion. OJ and OK are in the same straight line.

Proof. 1. Because the angles

BOJ and AOK are halves of the equal opposite angles BOC, AOD, they are equal (§ 67).

2. Angle $AOJ =$ angle $AOB -$ angle BOJ.

Angle $KOJ =$ angle $AOJ +$ angle AOK,

$=$ straight angle $AOB - BOJ + AOK$.

Or, BOJ and AOK being equal,

Angle $KOJ =$ straight angle,

and the lines OK and OJ are in one straight line. Q.E.D.

85. *Corollary. The four bisectors of the four angles formed by two intersecting straight lines form a pair of straight lines perpendicular to each other.*

EXERCISES.

1. If, in the diagram of Th. II., angle $BMX = 85°$, what will be the values of the other seven angles of the figure?

2. If, in the diagram of Th. VIII., angle $BOJ = 20°$, compute the values of the angles COA, AOD, and DOB.

3. Draw a triangle ABC, and suppose angle $A = 60°$, angle $B = 40°$. What angle will the bisectors of these angles A and B form with each other?

NOTE. Apply Th. IV. to the triangle formed by the bisectors and the side AB.

4. What will be the values of the three exterior angles of the preceding triangle ABC?

5. If, in the preceding triangle ABC, the bisector of the angle A be continued until it cuts the side BC, what angles will it form with that side?

6. Compute the same angles algebraically in terms of the angles A, B, and C, and show that the difference between the two adjacent angles formed by the bisector with the side BC is equal to the difference between the angles B and C.

7. If, in a triangle ABC, exterior angle $A = 85°$ and exterior angle $B = 150°$, what will be the three interior angles?

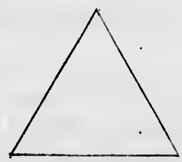
CHAPTER II

RELATIONS OF TRIANGLES.

Definitions.

86. *Definition.* An **equilateral** triangle is one in which the three sides are equal.

Def. An **isosceles** triangle is one which has two equal sides.



Equilateral triangle.

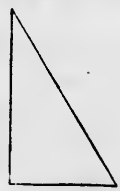
Isosceles triangle.

Def. An **acute-angled** triangle is one which has three acute angles.

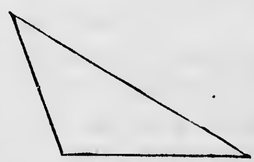
Def. A **right-angled** triangle is one which has a right angle.

Def. An **obtuse-angled** triangle is one which has an obtuse angle.

87. *Def.* In a right-angled triangle the side opposite the right angle is called the **hypothenuse.**



Right-angled triangle.



Obtuse-angled triangle.

88. *Def.* When one side of a triangle has to be distinguished from the other two it is called a **base,** and the angle opposite the base is called the **vertex.**

Either side of a triangle may be taken as the base, but we commonly take as the base a side which has some distinctive property.

In an isosceles triangle the base is generally the side which is not equal to another.

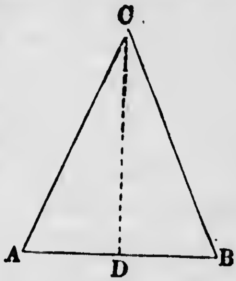
In other triangles the base is the side on which it is supposed to rest.

89. *Def.* An **oblique** line is one which is neither perpendicular nor parallel to some other line.

90. *Def.* **Segments** of a straight line are the parts into which it is divided.

Theorem IX.

91. *In an isosceles triangle the angles opposite the equal sides are equal to each other.*



Hypothesis. ABC a triangle in which

$$CA = CB.$$

Conclusion. Angle $A =$ angle B.

Proof. Bisect the angle C by the line CD, meeting AB in D. Turn the triangle over on CD as an axis. Then—

1. Because, by construction,
   Angle $BCD = ACD$,
   Side $CA \equiv$ trace CB, and *vice versa.*
2. Because, by hypothesis, $CA = CB$,
   Point $B \equiv$ position A,
   and Point $A \equiv$ position B.
3. Therefore line $AB \equiv$ trace BA, being turned end for end.
4. Therefore angle $CAB \equiv$ trace CBA.
5. Therefore angle $CAB =$ angle CBA (§ 14). Q.E.D.

Corollary 1. Since, after being turned over, the triangle falls upon its own trace, the triangle is symmetrical with respect to the bisecting line. Hence—

92. *The bisector of the vertical angle of an isosceles triangle is an axis of symmetry, and bisects the base at right angles.*

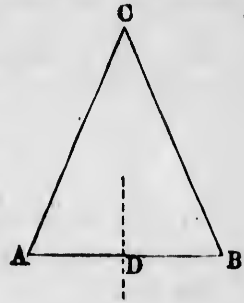
93. *Cor.* 2. *Every equilateral triangle is also equiangular.*

Theorem X.

94. *Conversely, if two angles of a triangle are equal, the sides opposite these angles are equal, and the triangle is isosceles.*

Hypothesis. ABC, a triangle in which angle A = angle B.

Conclusion. Side CA = side CB.



Proof. Through the middle point D of the side AB pass a perpendicular, and turn the figure over upon this perpendicular as an axis. Then—

1. Because the axis bisects AB perpendicularly,

$$\left.\begin{array}{l}\text{Point } A \equiv \text{position } B. \\ \text{Point } B \equiv \text{position } A.\end{array}\right\} \quad (\S\ 49)$$

Therefore $AB \equiv$ trace BA.

2. Because angle A = angle B,

$$\text{Side } AC \equiv \text{trace } BC.$$
$$\text{Side } BC \equiv \text{trace } AC.$$

3. Therefore the point of intersection C will fall into its original position.

4. Therefore $AC = BC$. Q.E.D.

95. *Corollary* 1. *A line bisecting the base of an isosceles triangle perpendicularly*

Passes through its vertex,
Is an axis of symmetry, and
Bisects the angle opposite the base.

96. *Cor.* 2. *Every equiangular triangle is also equilateral.*

Theorem XI.

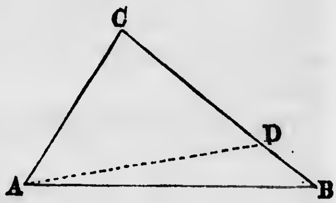
97. *If in any triangle one side be greater than another, the angle opposite that side will be greater than the angle opposite the other.*

Hypothesis. ABC, a triangle in which

$$AB > BC > CA.$$

Conclusion.

Angle C > angle A > angle B.



Proof. On a greater side, CB, take CD equal to a lesser side, CA, and join AD. Then—

1. Because the line AD falls within the triangle,

$$\text{Angle } CAB > \text{angle } CAD.$$

2. Because $CA = CD$,

$$\text{Angle } CAD = \text{angle } CDA. \qquad (\S\ 93)$$

3. Because CDA is an exterior angle of the triangle ABD,

$$\text{Angle } CDA > \text{angle } ABD. \qquad (\S\ 78)$$

4. From (1) and (2) we have angle $CAB > CDA$, whence, from (3),

$$\text{Angle } CAB > \text{angle } ABD.$$

In the same way may be shown,

$$\text{Angle } C > \text{angle } B. \quad \text{Q.E.D.}$$

THEOREM XII.

98. *Conversely, if one angle of a triangle be greater than another, the side opposite the greater angle will exceed that opposite the lesser angle.*

Hypothesis. ABC, a triangle in which

$$\text{Angle } C > \text{angle } A > \text{angle } B.$$

Conclusion. Side $AB >$ side $BC >$ side AC.

Proof. From C draw CD, making angle $ACD =$ angle CAD. Then—

1. Because angle $ACD =$ angle CAD,

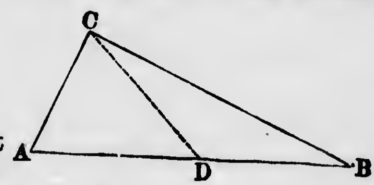
$$AD = CD. \qquad (\S\ 95)$$

2. Because angle $ACD < ACB$, the point D falls between A and B. Therefore

$$AB = AD + DB,$$

or, from (1),

$$AB = CD + DB.$$



3. Because CB is a straight line,

$$CD + DB > CB. \qquad (\text{Ax. } 9)$$

4. Therefore, from (2),

$$AB > CB. \quad \text{Q.E.D.}$$

5. In the same way may be shown,

$$BC > AC. \quad \text{Q.E.D.}$$

99. *Corollary.* Theorems X. and XI. may be combined into the single proposition :

The order of magnitude of the three sides of a triangle is the same as the order of magnitude of their opposite angles.

That is, if

Angle $A >$ angle $B >$ angle C, then
Side $BC >$ side $AC >$ side AB,

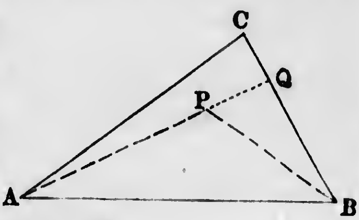
and *vice versa.*

THEOREM XIII.

100. *If from any point within a triangle lines be drawn to the ends of the base, the sum of these lines will be less than the sum of the other two sides of the triangle, but they will contain a greater angle.*

Hypothesis. ABC, any triangle; P, any point within it.
Conclusions. I. $AP + PB < AC + CB$.
II. Angle $APB >$ angle ACB.

Proof. Continue the line AP until it meets the side CB in Q. Then—



(I) 1. $AQ < AC + CQ$. (Ax. 9)

2. Adding QB to both sides of this inequality, we have

$AQ + QB < AC + CQ + QB$; (Ax. 6)

that is,

$$AQ + QB < AC + CB.$$

3. Also, in the same way,

$$PB < PQ + QB. \qquad \text{(Ax. 9)}$$

4. Adding AP to both sides,

$$AP + PB < AP + PQ + QB,$$

or $$AP + PB < AQ + QB.$$

Comparing (4) and (2),

$$AP + PB < AC + CB. \quad \text{Q.E.D.}$$

(II) 5. Because PQB is an exterior angle of the triangle ACQ,

Angle $PQB >$ angle ACB.

6. Because APB is an exterior angle of the triangle PQB,

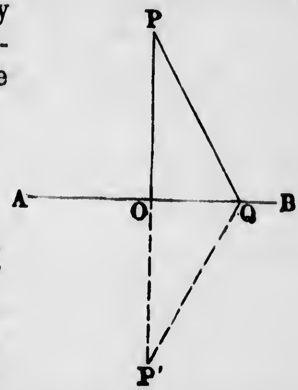
Angle $APB >$ angle PQB.

7. Comparing (5) and (6),

Angle $APB >$ angle ACB. Q.E.D.

THEOREM XIV.

101. *From a point outside a straight line only one perpendicular can be drawn to such straight line, and this perpendicular is the shortest distance from the point to the line.*



Hypothesis. AB, any line; P, any point without it; PO, a perpendicular from P on AB; PQ, any other line from P to AB.

Conclusions.

I. PQ is not perpendicular to AB.
II. $PO < PQ$.

Proof. Turn the figure over on AB as an axis, so that the point P shall fall into the position P'. Then—

1. Because AOP is a right angle, AOP' is also a right angle, and POP' lie in a straight line (§ 60, Cor.).

2. If PQA were also a right angle, it could be shown in the same way that PQP' is a straight line, and there would be two straight lines POP' and PQP' having the points P and P' common, which is impossible (§ 45, Ax. 10, Cor.). Therefore PQA is not a right angle. Q.E.D.

3. Because $PO = OP'$, and $PQ = QP'$, we have

$$PO = \tfrac{1}{2}PP'; \; PQ = \tfrac{1}{2}(PQ + QP').$$

But $$PP' < PQ + QP'. \qquad (§ 45, Ax. 9)$$

Therefore, taking the half of these unequal quantities, we have $$PO < PQ. \quad \text{Q.E.D.}$$

THEOREM XV.

102. *Two oblique lines from a point, cutting a straight line at equal distances from the foot of the perpendicular, are equal in length, and make equal angles with the line.*

Hypothesis. AB, any straight line; P, any point outside it; PO, the perpendicular from P to AB, meeting AB in O. M, N, two points on the line AB, such that $OM = ON$.

Conclusions. I. $PN = PM$.

II. Angle $PMO =$ angle PNO.

Proof. Turn the figure over on PO as an axis. Then—

1. Because angle $POM =$ angle PON (hypothesis),

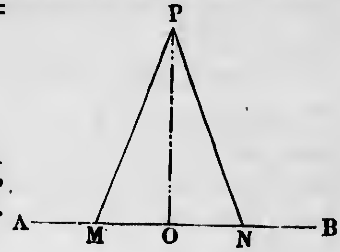
$OA \equiv$ trace OB.

2. Because $OM = ON$,

Point $M \equiv N$.

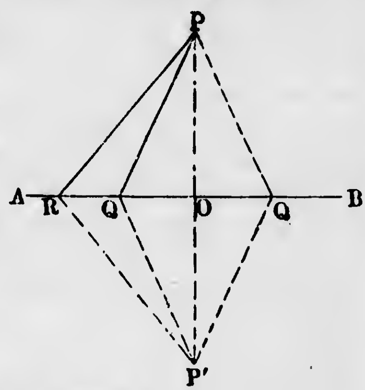
3. Therefore $PM = PN$, and angle $PMO =$ angle PNO.

Q.E.D.



Theorem XVI.

103. *Of two oblique lines drawn from a point to a straight line, that which meets the line at the greater distance from the foot of the perpendicular is the longer, and makes the lesser angle with the line.*



Hypothesis. AB, any straight line; P, any point without it; PO perpendicular to AB; Q, R, any two points on AB, such that $OR > OQ$.

Conclusions. I. $PR > PQ$.
II. Angle $PRO <$ angle PQO.

Proof. Turn the figure over on AB as an axis. Let P' be the point on which P shall fall. Then—

1. Because $P'Q = PQ$, and $P'R = PR$,

$$PQ = \tfrac{1}{2}(PQ + QP').$$
$$PR = \tfrac{1}{2}(PR + RP').$$

2. If Q is on the same side of O with R, the point Q will be within the triangle PRP'. Therefore

$$PR + RP' > PQ + QP', \qquad (\S\ 100)$$

and, taking the halves of these unequal quantities,

$$PR > PQ. \quad \text{Q.E.D.}$$

3. If Q is on the opposite side of O from R, take another point Q on the same side.

The two lines PQ will then be equal to each other (§ 102). The second point Q will fall between O and R (hypoth.).

4. Therefore we shall still have

$$PR > PQ. \quad \text{Q.E.D.}$$

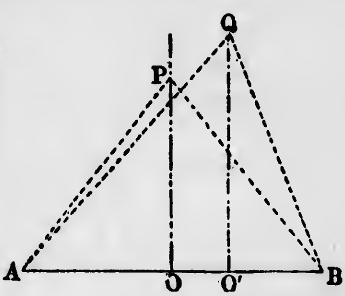
5. Because PQO is an exterior angle of the triangle PQR,

$$\text{Angle } PRQ < \text{angle } PQO \text{ (§ 77). Q.E.D.}$$

Theorem XVII.

104. I. *Every point on the perpendicular bisector of a straight line is equally distant from the extremities of the line.*

II. *Every point not on the perpendicular bisector is nearer that extremity toward which it lies.*



Hypothesis. AB, a straight line; O, its middle point; OP, a perpendicular from O; P, any point on this perpendicular; Q, a point on the same side of the perpendicular with B.

Conclusions. I. $PA = PB$.

II. $QB < QA$.

Proof. 1. Because $PO \perp AB$, and $AO = OB$, we have

$$PA = PB. \quad \text{(§ 102)}$$

2. From Q drop a perpendicular QO' upon AB. Then

$$QO' \parallel PO. \quad \text{(§ 70)}$$

Therefore O' falls on the side of O toward B, and

$$O'B < OA'.$$

Therefore $QB < QA$ (§ 103). Q.E.D.

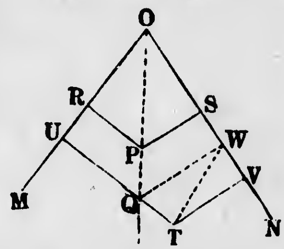
105. *Corollary. Every point equally distant from the extremities of a line lies upon the perpendicular bisector of the line.*

For, if it did not lie on this bisector, it could not be equally distant from the extremities without violating conclusion II. of the theorem.

THEOREM XVIII.

106. *Every point in the bisector of an angle is equally distant from the sides of the angle; and every point within the angle, but not on the bisector, is nearer that side toward which it lies.*

Hypothesis. MON, any angle; and OQ, its bisector, so that angle $MOQ = NOQ$; P, any point on OQ; T, a point within the angle, but not on OQ; PR, PS, perpendiculars to OM and ON; TU, TV, perpendiculars from T to OM and ON.



Conclusions. I. $PR = PS$.

II. $TU > TV$.

Proof I. Turn the figure over on the axis OP. Then—

1. Because angle $PON = POM$,

 Line $OM \equiv$ trace ON, and *vice versa.*

2. Because PR is a perpendicular dropped from P on R,

 $PR \equiv$ trace PS. (Th. 101)

3. Therefore Point $R \equiv S$, and

 $PR = PS$. Q.E.D.

Proof II. Let Q be the point in which TU crosses the bisector OQ. From Q drop the perpendicular QW upon ON. Join TW. Then—

4. Because TV is a perpendicular and TW an oblique line to ON,

 $$TW > TV. \qquad (\S\ 101)$$

5. Also, $$TQ + QW > TW. \qquad (\text{Ax. } 9)$$

Or, because $QU = QW$ (I.),

$$TQ + QU = TU > TW.$$

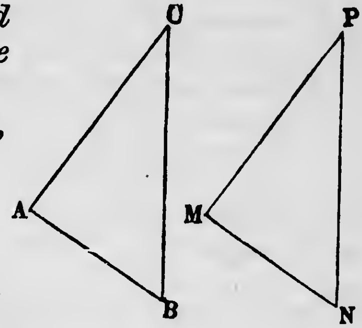
6. Comparing with (4),

 $$TU > TV. \quad \text{Q.E.D.}$$

107. *Corollary. Every point equally distant from two non-parallel lines in the same plane lies on the bisector of the angle formed by those lines.*

THEOREM XIX.

108. *If two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, they are identically equal.*



Hypothesis. ABC and MNP, two triangles in which

$$PM = CA.$$
$$PN = CB.$$
$$\text{Angle } P = \text{angle } C.$$

Conclusion. The two triangles are identically equal.

Proof. Apply the triangle MNP to ABC in such manner that the vertex P shall fall on C, and PM on CA. Then—

1. Because $PM = CA$,
   Point $M \equiv$ point A.
2. Because angle $P =$ angle C,
   Side $PN \equiv CB$. (§ 14)
3. Because $PN = CB$,
   Point $N \equiv$ point B.
4. Because $M \equiv A$ and $N \equiv B$,
   Line $MN \equiv$ line AB. (§ 45, Ax. 10, Cor.)

Therefore every part of the one triangle will coincide with the corresponding part of the other, and the two triangles are identically equal by definition (§ 13). Q.E.D.

THEOREM XX.

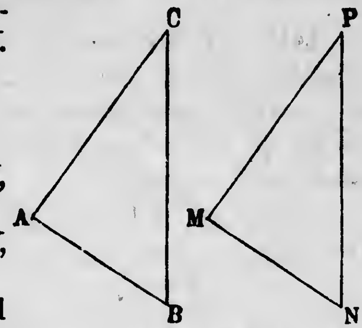
109. *If two triangles have a side and the two adjacent angles of the one equal to a side and the two adjacent angles of the other, they are identically equal.*

Hypothesis. ABC and MNP, two triangles in which

$$AB = MN.$$
$$\text{Angle } A = \text{angle } M.$$
$$\text{Angle } B = \text{angle } N.$$

Conclusion. The two triangles are identically equal.

Proof. Apply the triangle ABC to the triangle MNP in such manner that A shall coincide with M, and AB with MN. Then—



1. Because $AB = MN$,
   Point $B \equiv$ point N.
2. Because angle $A =$ angle M,
   Side $AC \equiv$ side MP.
3. Because angle $B =$ angle N,
   Side $BC \equiv$ side NP.
4. Because the sides AC and BC fall upon MP and NP respectively, the vertex C will fall upon the vertex P.

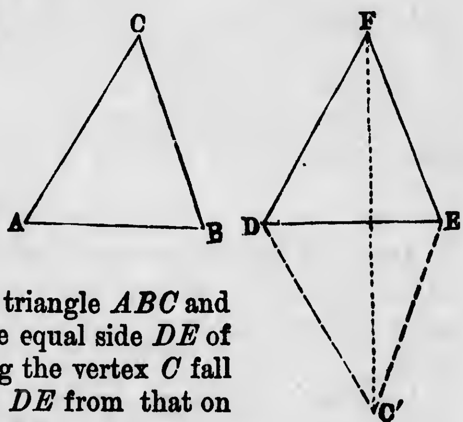
Therefore the two triangles coincide in all their parts and are identically equal. Q.E.D.

THEOREM XXI.

110. *If two triangles have the three sides of the one respectively equal to the three sides of the other, they are identically equal, and have the angles opposite the equal sides equal.*

Hypothesis. Two triangles, ABC and DEF, in which

$$AB = DE.$$
$$BC = EF.$$
$$CA = FD.$$



Conclusion. The two triangles are identically equal.

Proof. Take up the triangle ABC and apply the side AB to the equal side DE of the other triangle, letting the vertex C fall on the opposite side of DE from that on which the triangle DEF lies.

Let C' be the point in which the vertex C falls.

Join FC'. Then—

1. Because $FD = AC$, and $DC' = AC$, it follows that

$FD = DC'$, and the triangle FDC' is isosceles. Therefore

$$\text{Angle } DFC' = \text{angle } DC'F. \qquad (\S\ 91)$$

2. For the same reason the triangle FEC' is isosceles, and

$$\text{Angle } EFC' = \text{angle } EC'F.$$

3. Adding the equations (2) and (1), we find

$$\text{Angle } DFC' + \text{angle } EFC' = \text{angle } DC'F + \text{angle } EC'F.$$

But

$$\text{Angle } DFC' + \text{angle } EFC' = \text{angle } DFE.$$

$$\text{Angle } DC'F + \text{angle } EC'F = \text{angle } DC'E.$$

Therefore

$$\text{Angle } DC'E = \text{angle } DFE.$$

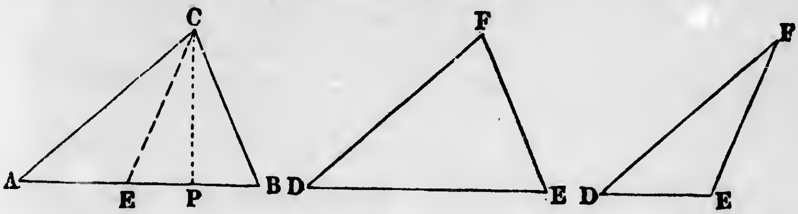
4. But angle ACB = angle $DC'E$, by construction. Therefore

$$\text{Angle } ACB = \text{angle } DFE.$$

5. The two given triangles, having the angle C = angle F and the sides which contain these angles equal, are identically equal (§ 108). Q.E.D.

Theorem XXII.

111. *If two triangles agree in the lengths of two sides and also in the angle opposite one of these sides, the angles opposite the other of the equal sides will be either equal or supplementary, and if they are equal the triangles are identically equal.*



Hypothesis. Two triangles ABC and DEF, in which

$$CA = FD.$$

$$CB = FE.$$

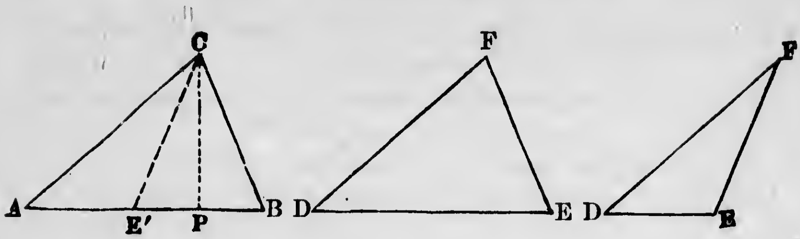
$$\text{Angle } A = \text{angle } D.$$

Conclusion. Either angle E = angle B or

$$\text{Angle } E = \text{straight angle} - \text{angle } B,$$

and in the former case the two triangles are identically equal.

Proof. Apply the triangle DEF to the triangle ABC in such manner that DF shall fall on the equal side AC. Then—



1. Because $DF = AC$,

Point $D \equiv A$; point $F \equiv C$.

2. Because angle D = angle A,

Base $DE \equiv$ base AB.

3. Because $FE = CB$, the point E will fall on a point of the base AB which is at a distance from C equal to CB.

4. There will be two such points equally distant from the foot P of the perpendicular from C on AB (§ 102).

Because $FE = CB$, one of these points will be B. Let E' be the other.

5. If E falls on B,

Triangle ABC = triangle DEF, *identically.*

6. If E falls on E', then, because $CE' = CB$, the triangle $CE'B$ will be isosceles, and

Angle $CE'P$ = angle CBP. (§91)

7. Because $AE'P$ is a straight line,

Angle $CE'A$ = supplement of angle $CE'B$,
= supplement of angle CBP. (6)

8. But in this case angle $CE'A$ = angle E. Therefore

Angle E = supplement of angle $CE'P$,
= supplement of angle B (7). Q.E.D.

112. *Corollary.* If the triangle ABC should be right-angled at B, we shall have

Angle B = straight angle − angle B,

and the two possible angles E would have the same value. Hence the two triangles would then be identically equal.

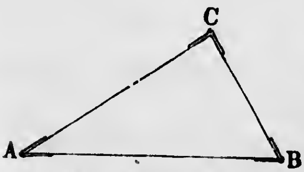
Scholium.

113. The preceding theorems of the identity of triangles are also expressed by saying that when certain parts of a triangle are given, the other parts are determined. The parts of a plane triangle are the three sides and the three angles.

The three angles are not all independent, because whenever two of them are given the third may be found by subtracting their sum from a straight angle (§ 74).

Whenever three independent parts of a triangle are given, the remaining parts may be found. In other words, when three independent parts are given there is only one triangle (or, in the case of § 111, two triangles) having those parts.

When the three sides of a triangle are given, we may imagine ourselves to have three stiff thin rods which we can fasten end to end in the form of a triangle. When the angles are not given, we may suppose the rods to be fastened together by hinges at the angles.

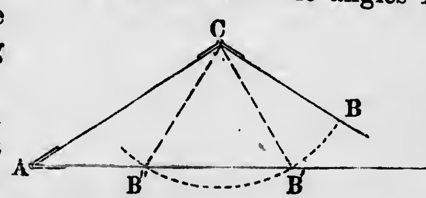


A hinged triangle.

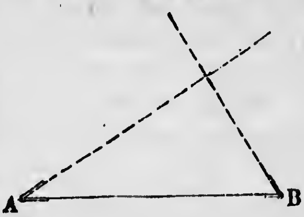
Theorem XXI. shows that although the hinges may be quite free, the rods cannot turn upon them when linked together. If they could turn, we could make the rods into several triangles by turning the rods on the hinges, and these triangles would not be identically equal.

When two sides and the included angle, as AC, BC, and the angle C are given, which is the case corresponding to Theorem XIX., we must suppose the side AB removed and the hinge at C tightened, so that the two rods cannot turn, and we are required to find a third rod of such length as to fit into the space AB. Theorem XIX. shows that this rod must have a definite length.

Suppose next, as in Theorem XXII., that AC, BC, and the angle A are given, while the base AB and the angles B and C are not given. We may then suppose a long rod extending out from A. The side AC of given length must be fastened at A, and the hinge tightened so that AC cannot turn, because the angle A is given.



The rod CB of given length is hinged at C, and this hinge is left loose because the angle C is not given. We are then to swing the side CB around on C until the end B touches the base, when it is to be fastened and the angle well fixed. There will be two points, B' and B'', where the junction may be made, and only two. We may choose which point we will, and the triangle will then be fixed. The two angles at B will, by the last theorem, be supplementary.



If CB should be shorter than the perpendicular from C upon AB, there would be no triangle which could be formed from the given parts.

Suppose, lastly, that one side AB and the two adjacent angles A and B are given, the other two sides being of indefinite length. We must then turn the two sides on the hinges and tighten the latter at the required angles, when the sides will cross each other at a definite point, and will make a definite angle with each other. This corresponds to the case of Theorem XX.

Theorem XXIII.

114. *If two triangles agree in the length of two sides, that triangle in which these two sides include the greater angle will have the greater base.*

Hypothesis. ABC and DEF, two triangles in which

$$AB = DE,$$
$$BC = EF,$$
$$\text{Angle } B < \text{angle } E.$$

Conclusion. Base $DF >$ base AC.

Proof. Apply the side AB of the one triangle to the equal side DE of the other in such manner that B shall fall upon E, and A upon D. Let C' be the position in which C falls.

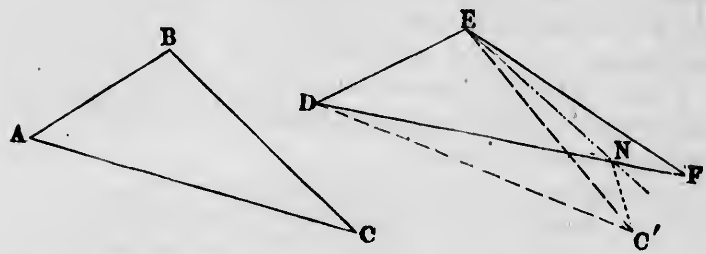
Bisect the angle $C'EF$, and let N be the point in which the bisector meets the base DF. Join $C'N$. Then—

1. In the two triangles $C'EN$ and FEN,

Angle $NEC' =$ angle NEF, by construction.

Side $EC' =$ side EF, by hypothesis.

Side $EN =$ side EN, identically.



Therefore triangle $ENC' =$ triangle ENF, *identically*, (§ 110)
and $$NC' = NF.$$

2. Therefore $DF = DN + NF = DN + NC'$.

3. Because AC is a straight line,

$$DN + NC' > DC'. \qquad \text{(Ax. 9)}$$

Comparing (2) and (3),

$$DF > DC',$$

or $$DF > AC. \quad \text{Q.E.D.}$$

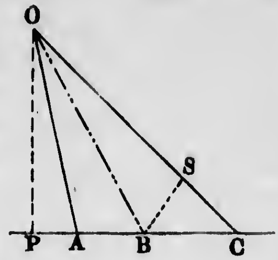
115. *Corollary.* *Conversely, if two triangles have two sides of the one equal respectively to two sides of the other, but the third sides unequal, the angle opposite the greater of the unequal sides will be the greater.*

For these angles could not be equal without violating Theorem XIX., nor could the angle opposite the lesser side be greater without violating Theorem XXIII.

Theorem XXIV.

116. *If three or more lines, making equal angles with each other, be drawn from a point to a straight line, that pair of lines will intercept the greater length which is farther from the perpendicular.*

Hypothesis. PC, a straight line; O, any point outside of it; OA, OB, OC, three lines from O to PC, making angle AOB = angle BOC; OP, the perpendicular from O upon P.



Conclusion. The intercept BC, on the side of OB away from the perpendicular, will be greater than the intercept AB, on the side toward the perpendicular.

Proof. From B draw the line BS, making angle OBS = angle OBA, and meeting OC in S. Then—

1. Because, in the triangles OBA and OBS,

Angle BOA = angle BOS (hypothesis),
Angle OBA = angle OBS (construction),
Side OB = side OB identically,

these triangles are identically equal (§ 109), and

Angle OSB (opp. OB) = angle OAB (opp. OB).
Side BA = side BS.

2. Therefore angle CSB (supplement of OSB) = angle OAP (supplement of OAB).

3. Because C is farther from P than A is,

Angle SCB < angle OAP. (§ 103)

Therefore Angle SCB < angle CSB,

and side BC (opposite greater angle CSB) > side BS (opposite lesser angle SCB). (§ 98)

4. Comparing with (1),

$BC > AB$. Q.E.D.

CHAPTER III.

PARALLELS AND PARALLELOGRAMS.

Definitions.

117. *Def.* A **quadrilateral** is a figure formed by four straight lines joined end to end.

The **sides** of a quadrilateral are the lines which form it.

118. *Def.* A **parallelogram** is a quadrilateral in which the opposite sides are parallel.

Whenever two parallels cross two other parallels, the intercepted portions of the parallels form a parallelogram.

A parallelogram.

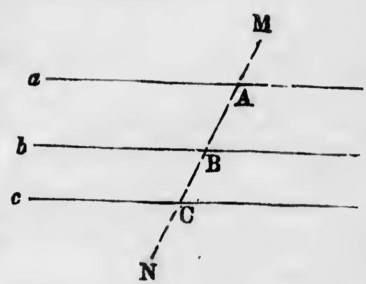
119. *Def.* The **diagonals** of a quadrilateral are two lines joining its opposite angles.

THEOREM XXV.

120. *Straight lines which are parallel to the same straight line are parallel to each other.*

Hypothesis. The line b parallel to the line a. The line c also parallel to the line a.

Conclusion. The lines b and c are parallel to each other.



Proof. Draw any transversal as MN across the three lines, intersecting them in the points A, B, C.

1. Because b is parallel to a,

Angle B = corresponding angle A. (§ 71)

2. Because c is parallel to a,

Angle C = corresponding angle A.

3. Comparing (1) and (2),

Angle B = corresponding angle C.

4. Therefore line $b \parallel$ line c (§ 69). Q.E.D.

THEOREM XXVI.

121. *The opposite angles of a parallelogram are equal to each other.*

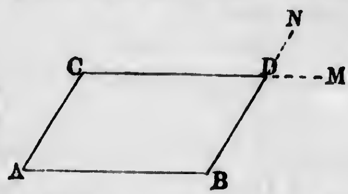
Hypothesis. $ABCD$, any parallelogram.

Conclusion.

Angle A = opposite angle D.

Angle B = opposite angle C.

Proof. Continue CD to any point M, and BD to any point N.



Then—

1. Because DN is parallel to AC and similarly directed, and DM parallel to AB and similarly directed,

Angle BAC = angle MDN. (§ 79)

2. Angle MDN = opp. angle BDC. (§ 67)

3. Comparing (1) and (2),

Angle BDC = angle BAC.

In the same way it may be proved that

Angle ACD = angle ABD. Q.E.D.

THEOREM XXVII.

122. *Any two adjoining angles of a parallelogram are supplementary.*

Proof. Any such pair of angles as A and B are interior angles between the parallels AC and BD, and are therefore supplementary (§ 71).

123. *Corollary* 1. *All the angles of a parallelogram may be determined when one is given, the angle opposite to the given one being equal to it, and the other two angles each equal to its supplement.*

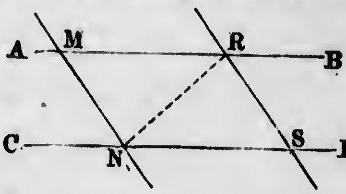
124. *Corollary* 2. *If two parallelograms have one angle of the one equal to one angle of the other, all the remaining angles of the one will be equal to the corresponding angles of the other.*

125. *Corollary* 3. *If one angle of a parallelogram is a right angle, all the other angles are right angles.*

THEOREM XXVIII.

126. *A pair of parallel straight lines intercept equal lengths of parallel transversals.*

Hypothesis. AB and CD, any pair of parallel straight lines; MN, RS, parallel transversals crossing them at the points M, N, R, and S.



Conclusion. $MN = RS$.
$MR = NS$.

Proof. Join the two opposite points R and N by a third transversal RN, and compare the two triangles MNR and NRS.

1. $MRNS$ is a parallelogram, by definition. Therefore

Angle $RMN =$ angle RSN. (§ 118)

2. Because RN is a transversal crossing the parallels AB and CD,

Angle $MRN =$ alternate angle RNS. (§ 71)
Angle $MNR =$ alternate angle NRS.

3. Because the two triangles have the side RN common and the adjacent angles equal, they are identically equal (§109). Therefore

Side RS (opp. angle N) $=$ side MN (opp. equal angle R).
Side $MR =$ corresponding side NS. Q.E.D.

127. *Corollary* 1. *The opposite sides of a parallelogram are equal.*

128. *Corollary* 2. *The diagonal of a parallelogram divides it into two identically equal triangles.*

If the two transversals are perpendicular to the parallels, the intercepted lengths will measure the distance of the parallels. Hence

129. *Corollary* 3. *Two parallels are everywhere equally distant.*

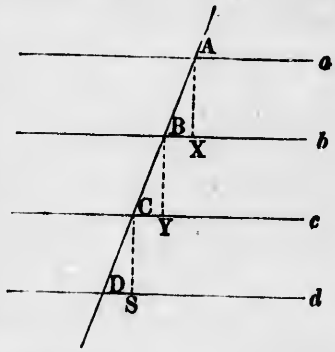
THEOREM XXIX.

130. *If three or more parallels intercept equal lengths upon a transversal crossing them, they are equidistant.*

Hypothesis. A transversal crossing the parallel lines a, b, c, d, etc., at the respective points A, B, C, D, etc., in such wise that

$$AB = BC = CD, \text{ etc.}$$

Conclusion. The distance between any two neighboring parallels, as a, b, is equal to the distance between any other two, as c, d.



Proof. From two or more of the points of intersection A, B, C, etc., drop perpendiculars upon the neighboring parallels and consider the triangles thus formed.

1. Because the angles A, B, C, etc., are corresponding angles between parallels,

$$\text{Angle } A = \text{angle } B = \text{angle } C, \text{ etc.} \qquad (\S\ 71)$$

2. The angles X, Y, S, etc., are equal by construction, because they are all right angles.

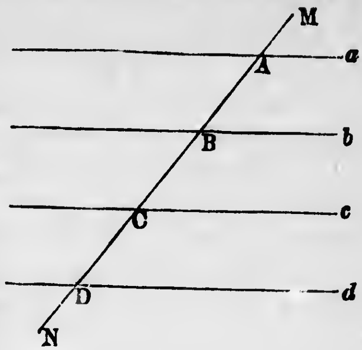
3. $AB = BC = CD$, etc., by hypothesis.

4. Therefore any two triangles, as ABX, CDS, have the angles and one side of the one equal to the angles and one corresponding side of another, and are identically equal.

5. Therefore $AX = BY = CS$, and the parallels are equidistant. Q.E.D.

THEOREM XXX.

131. Conversely, *Equidistant parallels intercept equal lengths upon any transversal.*



Hypothesis. The lines a, b, c, d, etc., parallel and equidistant; MN, a transversal intersecting them at the points A, B, C, D, etc.

Conclusion. Any one intercepted length on the transversal, as AB, equal to any other intercepted length, as CD.

Proof. Take up the figure and lay it down again in such manner that the point A shall fall into the position C, and the line a coincide with the trace c. Then—

1. Because angle A = angle C,

Transversal $MN \equiv$ its own trace.

2. Because the parallels a, b are equidistant with the parallels c, d,

Point $B \equiv$ position D.

4. Therefore $AB = CD$. Q.E.D.

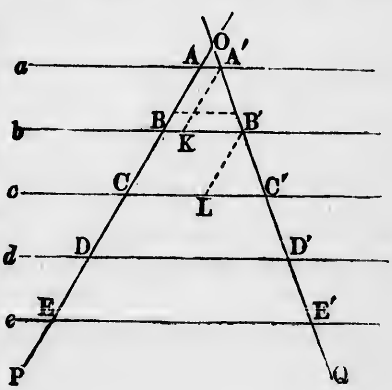
Scholium. This proposition may also be proved by dropping perpendiculars, as in Theorem XXIX.

Theorem XXXI.

132. *If three or more parallels are crossed by two transversals and intercept equal lengths on one, then—*

I. *The parallels will intercept equal lengths on the other transversal.*

II. *The intercepted part of each parallel will be longer or shorter than the neighboring intercept by the same amount.*



Hypothesis. OP, OQ, any two transversals; A, B, C, equidistant points on OP;

a, b, c, d, e, etc., parallels through these points, meeting OQ in the points A', B', C', etc.

Conclusions. I. $A'B' = B'C' = C'D' = D'E'$, etc.

II. $BB' - AA' = CC' - BB' = DD' - CC'$, etc.

Proof I. 1. Because the parallels a, b, c, etc., intercept equal intervals on OP, they are equidistant (§ 130).

2. Because the parallels are equidistant they intercept equal intervals on the transversal OQ. Therefore the intercepted intervals $A'B'$, $B'C'$, etc., are equal (§ 131).

II. Through A' draw a line parallel to AB, meeting BB' in K, and through B' draw another line parallel to BC, meeting C,C' in L; then—

3. Because, by construction, $AA'BK$ and $BB'CL$ are parallelograms,

$$\left.\begin{aligned} BK &= AA', \\ CL &= BB'. \end{aligned}\right\} \qquad (\S\ 124)$$

4. Because $B'A'K$ and $C'B'L$ are corresponding angles between the parallels $A'K$ and $B'L$, they are equal.

5. Because $A'B'K$ and $B'C'L$ are corresponding angles between the parallels BB' and CC', they are equal.

6. Comparing with (2) the triangles $A'KB'$ and $B'LC'$, have one side and the adjacent angles of the one equal to a corresponding side and two adjacent angles of the other. Therefore they are identically equal, and

$$B'K = C'L.$$

7. By construction,

$$BB' - BK = B'K.$$
$$CC' - CL = C'L.$$

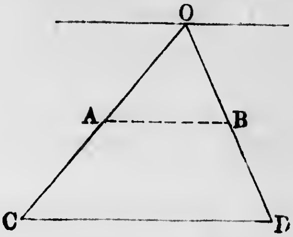
Comparing with 3 and 6,

$$BB' - AA' = B'K = CC' - BB', \text{ etc. } \quad \text{Q.E.D.}$$

133. *Corollary* 1. The amount by which each length exceeds the preceding one may be found by laying off from the point of intersection O, on the line OP, a length equal to AB, and drawing a parallel to the lines a, b, etc. The length of this parallel between the sides OP and OQ will be equal to the difference between the lengths of any two consecutive parallels.

134. *Corollary* 2. If the points A, B, C, etc., are found by measuring off equal distances from the point O, so that

$$OA = AB = BC, \text{ etc.},$$



we shall have

$$BB' = 2AA',$$
$$CC' = 3AA',$$
$$DD' = 4AA',$$
etc. etc.

135. *Corollary* 3. *If through the middle point of one side of a triangle a parallel be drawn to a second side, it will bisect the third side and will be half as long as the second side.*

For, let OCD be the triangle; A, the middle point of OC; and AB, a line parallel to CD, meeting OD in B. Let a third parallel pass through O. Then O, AB, and CD are three parallels intercepting equal lengths upon the transversal OC. Therefore they also intercept equal lengths on OD.

Moreover, CD exceeds AB as much as AB exceeds the intercepted length of O. But this length is nothing. Therefore

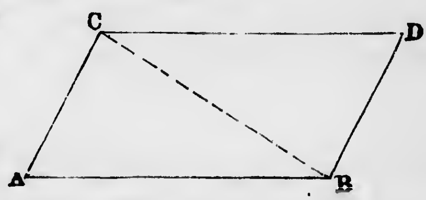
$$CD = 2AB.$$

Corollary 4. Because through A only one parallel to CD can be drawn, it follows that if B be the middle point of OD, AB will be that parallel. Therefore:

136. *The line joining the middle points of any two sides of a triangle is parallel to the third side.*

THEOREM XXXII.

137. *If the opposite sides of a quadrilateral are equal to each other, it is a parallelogram.*



Hypothesis. $ABCD$, a quadrilateral, in which

$$AB = CD.$$
$$AC = BD.$$

Conclusion. The figure $ABCD$ is a parallelogram.

Proof. Draw the diagonal BC. Then—

1. The three sides of the triangle ABC are by construc-

tion and hypothesis respectively equal to those of the triangle DBC.

2. Because the angles BCD and CBA are opposite the equal sides CA and BD,

$$\text{Angle } BCD = \text{angle } CBA. \qquad (\S\,110)$$

3. But these angles are alternate angles between the lines CD and AB on each side of the transversal CB. Therefore

$$CD \parallel AB.$$

4. In the same way may be shown $AC \parallel BD$.

Therefore $ABCD$ is a parallelogram, by definition. Q.E.D.

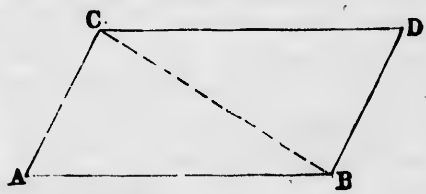
Theorem XXXIII.

138. *If any two opposite sides of a quadrilateral are equal and parallel, it is a parallelogram.*

Hypothesis. $ABCD$, a quadrilateral in which

$CD =$ and $\parallel AB$.

Conclusion. $AC =$ and $\parallel BD$, and therefore $ABCD$ is a parallelogram.



Proof. Draw the diagonal BC. Then—

1. Because AB and CD are parallels, and BC is a transversal,

$$\text{Angle } ABC = \text{alternate angle } BCD.$$

2. In the triangles ABC and BCD,

$$AB = CD, \text{ by hypothesis;}$$
$$BC \text{ is common;}$$

and (1) the angles between these equal sides are equal. Therefore these triangles are identically equal, and

$$AC = BD. \quad \text{Q.E.D.}$$
$$\text{Angle } ACB = \text{angle } CBD.$$

3. These angles ACB and CBD being alternate angles between AC and BD,

$$AC \parallel BD. \quad \text{Q.E.D.}$$

Theorem XXXIV.

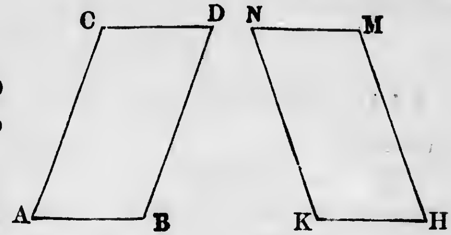
139. *If two parallelograms agree in the lengths of their sides and in one angle, they are identically equal.*

Hypothesis. $ABCD$ and $HKMN$, parallelograms in which

$AB = HK$.

$AC = HM$.

Angle A = angle H.



Conclusion. $ABCD$ is identically equal to $HKMN$.

Proof. Apply $ABCD$ to $HKMN$, turning it over, if necessary, in such manner that the angle A shall coincide with the equal angle H, and the side AB with the equal side HK. Then—

1. Because these sides are equal,

$$\text{Point } B \equiv \text{point } K.$$

2. Because angle A = angle H,

$$AC \equiv HM.$$

3. Because $AC = HM$,

$$\text{Point } C \equiv \text{point } M.$$

4. Because the parallelograms have the angle C = angle M (§ 121),

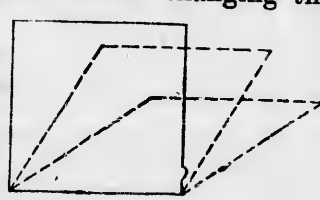
$$\text{Side } CD \equiv MN.$$

5. In the same way it may be shown that every side and angle of the one will coincide with a corresponding side and angle of the other. Therefore

The two parallelograms are identically equal. Q.E.D.

Scholium 1. A more simple but less general demonstration may be given by drawing diagonals between any equal angles.

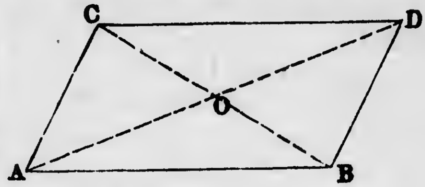
140. *Scholium* 2. We have shown that a triangle is completely determined when its three sides are given. From the preceding propositions it follows that a quadrilateral is not completely determined when its four sides are given, but that the angles may change to any extent without changing the lengths of the sides.



Thus the parallelogram in the margin may be made to assume the successive forms shown by the dotted lines. This is the geometrical expression of the well-known fact that a frame of four pieces is not rigid unless fastened by a diagonal brace.

THEOREM XXXV.

141. *The diagonals of a parallelogram bisect each other.*



Hypothesis. $ABCD$, a parallelogram of which the diagonals intersect at O.

Conclusions. $CO = OB$.
$AO = OD$.

Proof. In the triangles AOB and COD,

Angle AOB = opposite angle COD. (§ 67)
Angle BAO = alternate angle ODC,
Angle ABO = alternate angle DCO. (§ 71)
Side AB = side CD. (§ 124)

Therefore the two triangles are identically equal, and the sides opposite the equal angles equal; namely,

$BO = OC$.
$AO = OD$. Q.E.D.

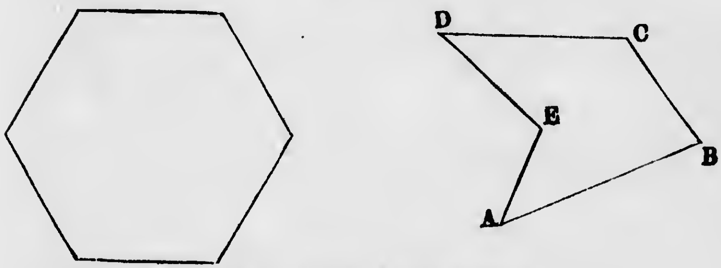
CHAPTER IV.

MISCELLANEOUS PROPERTIES OF POLYGONS.

Definitions.

142. *Def.* **A polygon** is a figure formed by a chain of straight lines returning into itself and inclosing a part of the plane on which it lies.

The lines are called **sides** of the polygon.

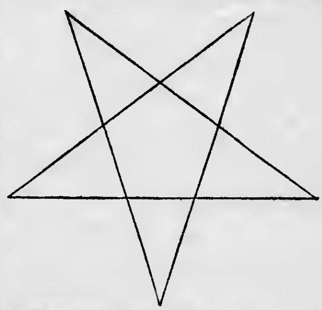


Polygons of elementary geometry.

In the higher geometry the sides of a polygon may cross each other, but elementary geometry treats only of polygons the sides of which dc not cross.

143. The angles of the polygon measured on the side containing the inclosed space are called **interior angles.**

EXAMPLE. One interior angle of the polygon $ABCDE$ is the angle EAB, measured by turning the side AB through the interior of the polygon until it coincides with AE. In the first polygon the angles are all less than straight angles. But in the polygon $ABCDE$ the interior angle AED is greater than a straight angle.



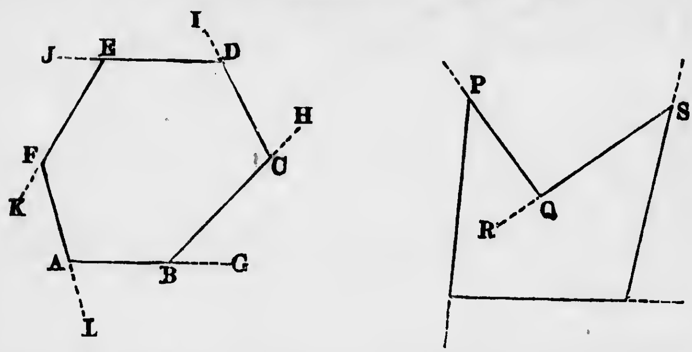
Polygon the sides of which cross.

144. An **exterior angle** of a polygon is an angle between one side and the continuation of the adjacent side.

REMARK 1. To form all the different exterior angles of a polygon it is sufficient to produce each side, taken in regular order, in *one direction*. The number of exterior angles will then be the same as that of the interior angles.

The angles BAL, CBG, etc., are exterior angles. Also,

Angle ABC + angle CBG = straight angle.



The angle PQR is an exterior angle which falls inside the polygon, because the interior angle PQS is greater than a straight angle.

REMARK 2. It is evident that a polygon has as many angles as sides.

REMARK 3. If the interior angle of a polygon is greater than a straight angle, the continuation of one of the sides will fall within the polygon.

The corresponding exterior angle is then regarded algebraically as having the negative sign.

REMARK 4. It is evident that the sum of any interior angle and of the corresponding exterior angle is a straight angle; that is,

$$\text{Int. angle} + \text{ext. angle} = 180°.$$

By transposing the second term of the first members, we have

$$\text{Ext. angle} = 180° - \text{int. angle}.$$

If the interior angle is greater than 180°, the members of this equation will be negative.

Classification of Polygons.

145. Polygons are classified according to the number of their sides.

The least number of sides which a polygon can have is three, and it is then a triangle.

146. *Def.* A **quadrilateral** is a polygon of four sides.

147. *Def.* A **pentagon** is a polygon of five sides.

148. *Def.* A **hexagon** is a polygon of six sides.

149. *Def.* An **octagon** is a polygon of eight sides.

150. Polygons of any number of sides may be designated by the Greek numerals expressing the number of sides.

151. *Def.* A **diagonal** of a polygon is a line joining any two non-adjacent angles.

Question for the student. How many diagonals can be drawn from one angle of a polygon having n sides?

152. A **regular polygon** is one of which the sides and angles are all equal.

Classification of Quadrilaterals.

153. *Def.* A **trapezoid** is a quadrilateral of which two opposite sides are parallel.

154. If both pairs of opposite sides are parallel, the quadrilateral is a **parallelogram.**

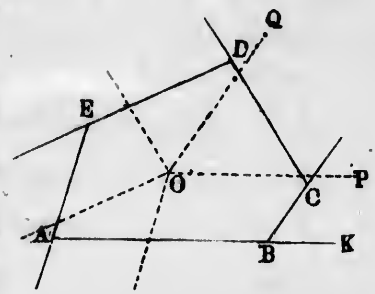
155. *Def.* A **rectangle** is a parallelogram in which the four angles are equal.

156. *Def.* A **rhombus** is a parallelogram of which the four sides are equal.

157. *Def.* A **square** is a rectangle of which the sides are equal.

THEOREM XXXVI.

158. *If each side of a polygon be produced in one direction, the sum of all the exterior angles is equal to a circumference.*



Hypothesis. $ABCDE$, a polygon.

Conclusion. Sum of n exterior angles ABC, etc. = two straight angles.

Proof. From any point O draw n straight lines, each parallel to one of the sides of the polygon. There will then be n angles around the point O.

1. Because OP is parallel to BK and similarly directed, and OQ to BC and similarly directed,

Angle POQ = exterior angle KBC. (§ 79)

2. In the same way it may be proved that each of the exterior angles of the polygon is equal to one of the angles around O.

3. Because the number of exterior angles and of angles around O is equal, the sum of all the exterior angles will be equal to the sum of all the angles around O; that is, to a circumference. Q.E.D.

159. *Scholium.* If any of the interior angles of the polygon should be reflex, so that exterior angles fall within the polygon, such exterior angles must be regarded as algebraically negative in forming the sum (§ 144).

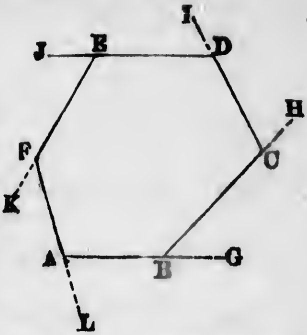
THEOREM XXXVII.

160. *The sum of the interior angles of a polygon is equal to a number of straight angles two less than the number of sides of the polygon.*

Hypothesis. $ABCDEF$, a polygon having n sides and n angles.

Conclusion. Sum of the n interior angles equal to $n-2$ straight angles, or

$$ABC + BCD + CDE + \text{etc.} = (n-2)\,180°.$$



Proof. 1. The sum of each interior angle and its adjacent exterior angles, as $ABC + CBG$, is a straight angle (§ 51).

2. Because the polygon has n such pairs of angles, the sum of all the interior and exterior angles is n straight angles.

3. But the sum of the exterior angles alone is a circumference; that is, two straight angles (§ 155).

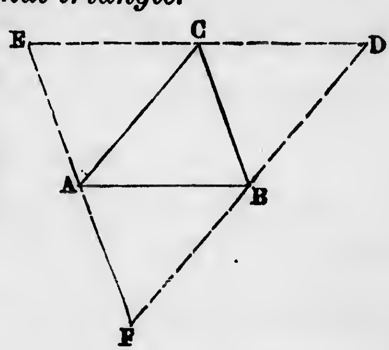
4. Taking these two straight angles from the sum (2), we have left the sum of the interior angles alone, equal to $n-2$ straight angles. Q.E.D.

161. *Corollary* 1. The sum of all the interior angles of a quadrilateral is equal to two straight angles; that is, to four right angles.

162. *Corollary* 2. Since in a rectangle all the four angles are equal, and their sum is four right angles, each of the angles is a right angle.

Theorem XXXVIII.

163. *If through each angle of a triangle a line be drawn parallel to the opposite side, the three lines will form a triangle the sides of which will be bisected by the vertices of the original triangle.*



Hypothesis. ABC, any triangle; DEF, another triangle formed by drawing,
through C, ED parallel to AB;
through A, EF parallel to CB;
through B, FD parallel to AC.

Conclusion. $EC = CD$.
$EA = AF$.
$FB = BD$.

Proof. The quadrilaterals $ABCD$ and $ABEC$ are parallelograms, by construction.

Therefore

$$\left.\begin{aligned} CD &= AB; \\ EC &= AB. \end{aligned}\right\} \qquad (\S\ 127)$$

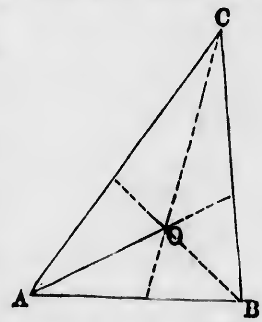
Whence

$$EC = CD. \quad \text{Q.E.D.}$$

In the same way the other conclusions may be proved.

Theorem XXXIX.

164. *The bisectors of the three interior angles of a triangle meet in a point equally distant from the sides of the triangle.*



Hypothesis. ABC, any triangle; AO, BO, CO, the bisectors of its interior angles, A, B, and C.

Conclusions. I. These three bisectors meet in a single point O.

II. This point O is equally distant from the three sides of the triangle.

Proof. Let O be the point in which the bisectors of A and B meet. Then—

1. Because O is on the bisector of the angle A, O is equally distant from the sides AB and AC of the angle A (§ 106).

2. Because O is on the bisector of the angle B, O is equally distant from the sides BA and BC of the angle B.

3. Therefore O is equally distant from AC and BC.

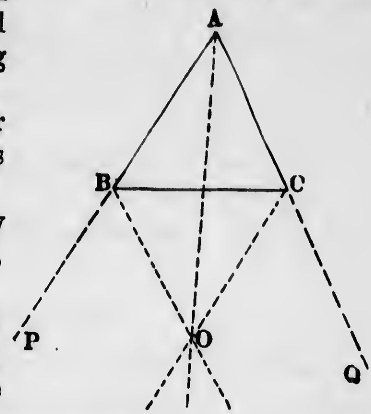
4. Therefore it is upon the bisector of the angle formed by AC and BC; namely, of the angle C (§ 107).

5. Therefore the point O is equally distant from the three sides, and the bisectors all pass through it. Q.E.D.

Theorem XL.

165. *The bisectors of any two exterior angles of a general triangle meet the bisector of the opposite interior angle in a point which is equally distant from the three sides of the triangle.*

Hypothesis. ABC, any triangle, of which the sides AB and AC are produced indefinitely in the directions P and Q; BO, CO, the bisectors of the exterior angles CBP and BCQ; O, the point of meeting of these bisectors.



Conclusions. I. The bisector of the angle BAC passes through O.

II. The point O is equally distant from the lines BC, BP, and CQ.

Proof. 1. Because O is on the bisector of the angle CBP, it is equally distant from the sides BC and BP.

2. Because O is on the bisector of the angle BCQ, it is equally distant from the sides BC and CQ.

3. Therefore O is equally distant from the three lines AP, BC, and AQ. Q.E.D.

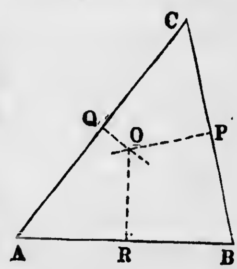
4. Because O is equally distant from AP and AQ, it is on the bisector of the angle made by those lines (§ 107).

Therefore this bisector passes through O. Q.E.D.

Theorem XLI.

166. *The perpendicular bisectors of the three sides of a triangle meet in a point, which point is equally distant from the three vertices of the triangle.*

Hypothesis. ABC, any triangle; R, P, Q, the middle points of the respective sides; PO, RO, lines passing through P and R perpendicular to BC and AB respectively.



Conclusions. I. The point O is equally distant from A, B, and C.

II. The perpendicular bisector of AC passes through O.

Proof. 1. Because O is on the perpendicular bisector of the line BC, it is equally distant from the ends, B and C, of this line (§ 104).

2. Because O is on the perpendicular bisector of the line AB, it is equally distant from the points A and B.

3. Therefore it is equally distant from the three points A, B, and C. Q.E.D.

4. Because it is equally distant from A and C, it lies on the perpendicular bisector of the line AC (§ 105).

Therefore this bisector also passes through O. Q.E.D.

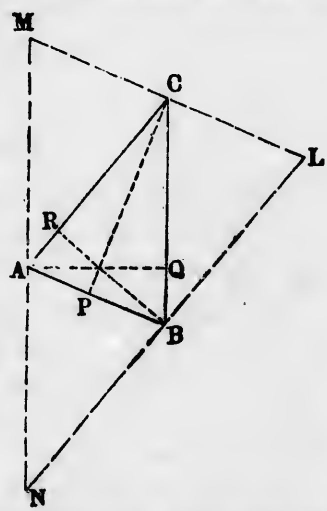
Theorem XLII.

167. *The perpendiculars dropped from the three angles of a triangle upon the opposite sides pass through a point.*

Hypothesis. ABC, any triangle; AQ, BR, CP, perpendiculars from A, B, C, upon BC, CA, AB, respectively.

Conclusion. These perpendiculars pass through a point.

Proof. Through A, B, and C, respectively, draw parallels to the opposite sides of the triangle, forming the triangle LMN. Then—



1. A, B, and C will be the middle point of the sides of the triangle LMN (§ 163).

2. Because LM is parallel to AB, and CP perpendicular to AB, CP is also perpendicular to LM. (§ 72)

3. Therefore CP is the perpendicular bisector of LM. In the same way BR and AQ are perpendicular bisectors of LN and MN respectively.

4. Because the three lines AQ, BR, and CP are the perpendicular bisectors of the sides of the triangle LMN, they pass through a point (§ 166). Q.E.D.

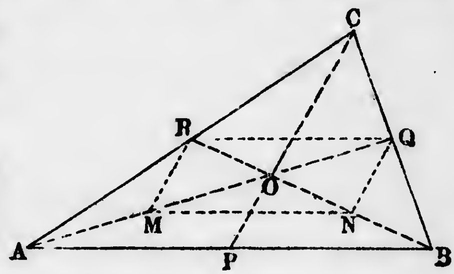
168. *Def.* The line drawn from any angle of a triangle to the middle point of the opposite side is called a **medial line** of the triangle.

Corollary. Since a triangle has three angles it may have three medial lines.

THEOREM XLIII.

169. *The three medial lines of a triangle meet in a point which is two thirds of the way from each angle to the middle of the opposite side.*

Hypothesis. ABC, a triangle; P, Q, R, the middle points of its respective sides; BR, AQ, two medial lines of the triangle; O, their point of intersection.



Conclusions. I. The third medial line CP also passes through the point O.

II. $$\begin{cases} PO = \frac{1}{2}OC. \\ QO = \frac{1}{2}OA. \\ RO = \frac{1}{2}OB. \end{cases}$$

Proof. Bisect AO in M, and OB in N. Join RQ, RM, QN, MN. Then—

1. Because MN is a line joining the middle points of the sides OA and OB of the triangle OAB,

$$MN = \tfrac{1}{2}AB. \qquad (\S\ 135)$$
$$MN \parallel AB. \qquad (\S\ 136)$$

2. Because RQ is a line joining the middle points of the sides CA and CB of the triangle ABC,

$$RQ = \tfrac{1}{2}AB.$$
$$RQ \parallel AB.$$

3. Therefore RQ is parallel and equal to MN, whence the quadrilateral $RQMN$ is a parallelogram (§ 138).

4. Because RN and QM are diagonals of the parallelogram $RQMN$,

$$\left.\begin{array}{l} OQ = OM. \\ OR = ON. \end{array}\right\} \qquad (\S\ 141)$$

But, by construction, M and N are the bisectors of OA and OB. Therefore

$$OM = \tfrac{1}{2}OA.$$
$$ON = \tfrac{1}{2}OB.$$

Whence

$$QO = \tfrac{1}{2}OA.$$
$$RO = \tfrac{1}{2}OB. \quad \text{Q.E.D.}$$

5. In the same way it may be shown that the point in which the medial line CP cuts the medial line AQ is two thirds of the way from A to Q; that is, it cuts AQ in the point O. Therefore

The three medial lines pass through the point O. Q.E.D.

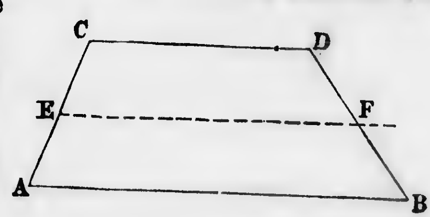
6. It may then be shown, in the same way as with AQ and BR, that

$$PO = \tfrac{1}{2}OC. \quad \text{Q.E.D.}$$

Theorem XLIV.

170. *The line drawn from the middle of one of the non-parallel sides of a trapezoid parallel to the parallel sides is equidistant from them, equal in length to half their sum, and bisects the opposite side.*

Hypothesis. $ABCD$, a trapezoid of which AB and CD are the parallel sides; E, the middle point of AC; EF, a parallel to AB and CD, meeting BD in F.



Conclusions.

I. EF is equidistant from AB and CD.

II. $BF = FD$.

III. $EF = \frac{1}{2}(AB + CD)$.

Proof. 1. Because AB, EF, and CD are parallels intercepting equal lengths AE and EC upon the transversal AC, they are equidistant (§ 130). Q.E.D.

2. Because they are equidistant, they intercept equal lengths upon the transversal BD (§ 131). Hence

$$BF = FD. \quad \text{Q.E.D.}$$

3. Therefore $EF - CD = AB - EF$ (§ 132, II.), and by transposition

$$2EF = AB + CD,$$

or

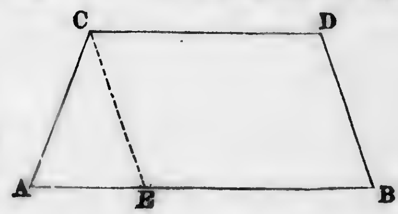
$$EF = \tfrac{1}{2}(AB + CD). \quad \text{Q.E.D.}$$

Def. The line EF is called the **middle parallel** of the trapezoid.

THEOREM XLV.

171. *If the two non-parallel sides of a trapezoid are equal, the angles they make with the parallel sides are equal.*

Hypothesis. $ABCD$, a trapezoid in which AB and CD are parallel, and $CA = DB$.



Conclusion.

Angle CAB = angle DBA.
Angle ACD = angle BDC.

Proof. From C draw CE parallel to DB, meeting AB in E. Then—

1. Because CE is parallel to DB, and CD to EB,
$DBCE$ is a parallelogram.
2. Because $DBCE$ is a parallelogram,
$CE = DB$.
3. Because by hypothesis $DB = CA$,
Therefore $CE = CA$.
4. Therefore ACE is an isosceles triangle, and
Angle CAE = angle CEA.
5. Because CE and DB are parallel,
Angle DBE = corresponding angle CEA.
6. Comparing with (4),
Angle CAE = angle DBE. Q.E.D.

In a similar way, by drawing through A a parallel to BD, is shown

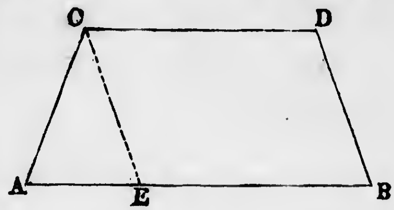
Angle ACD = angle BDC. Q.E.D.

THEOREM XLVI.

172. *Conversely, if the angles at the base of a trapezoid are equal, the non-parallel sides and the other angles are equal.*

Hypothesis. $ABCD$, a trapezoid in which the sides AB and CD are parallel and

Angle $CAB =$ angle DBA.



Conclusion.

I. $CA = DB$.

II. Angle $ACD =$ angle BDC.

Proof. Make the same construction as in the last theorem. Then—

1. Because BA is a transversal crossing the parallels CE and DB, Angle $CEA =$ angle DBE.

2. By hypothesis,

Angle $DBE =$ angle CAE.

Whence Angle $CAE =$ angle CEA.

3. Therefore ACE, having the angles at the base equal, is an isosceles triangle, and

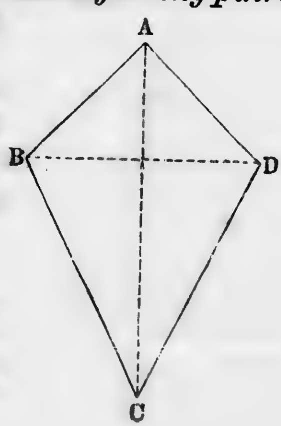
$CA = CE = DB$. Q.E.D.

Therefore angle $ACD =$ angle BDC (§ 171). Q.E.D.

REMARK. This form of trapezoid is sometimes called an **antiparallelogram.**

THEOREM XLVII.

173. *A quadrilateral of which two adjoining pairs of sides are equal is symmetrical with respect to the diagonal joining the angles formed by the equal sides, and the diagonals cut each other at right angles.*



Hypothesis. $ABCD$, a quadrilateral in which $AB = AD$.

$CB = CD$.

Conclusion. AC is an axis of symmetry. AC cuts BD at right angles.

Proof. Because, in the triangles ABC and ADC,

$$\left.\begin{aligned} AB &= AD, \\ BC &= DC, \end{aligned}\right\} \text{(hypothesis,)}$$

AC is common,

these triangles are identically equal, and

Angle BAC (opp. BC) = angle DAC (opp. AC).

Angle BCA (opp. AB) = angle DCA (opp. AD).

Therefore, if the figure be turned over on the line AC—

The lines AB and CB will fall upon AD and CD respectively.
The point B " " the point D.
And BD " " its own trace.

Therefore the figure is symmetrical and the line BD at right angles to AC. Q.E.D.

Lemma respecting Identical Figures.

174. In identically equal figures, corresponding lines are equal.

Note. Corresponding lines are those which coincide when the figures are applied to each other. From this definition the conclusion follows without demonstration.

Corollary. In identical figures, any lines so defined that there can be but one line in each figure answering to the definition are corresponding lines.

For if such lines did not coincide when the figures were brought into coincidence, the two lines would equally correspond to the definition.

175. *Special applications.* In identically equal triangles—

The perpendiculars from equal angles upon the opposite sides are equal.

The bisectors are of equal length.

In identically equal quadrilaterals the diagonals are equal.

CHAPTER V.

PROBLEMS.

POSTULATES.

176. It is assumed—

I. That a straight line may be drawn between any two points.

II. That a finite straight line may be produced indefinitely in either direction.

III. That a circle may be drawn around any point as a centre, with a radius equal to any given finite straight line.

The requirements of these postulates may be fulfilled with a ruler and a pair of compasses, which are the only instruments recognized in pure geometry.

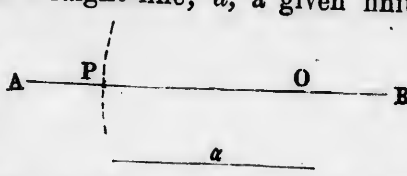
But it is not necessary to confine one's self to these instruments in solving all problems. When it is once well understood how a given problem is to be solved by them, other instruments may be used for the actual drawing, such as the protractor, the square, and the parallel ruler.

PROBLEM I.

177. *On a given straight line to mark off a length equal to a given finite straight line.*

Given. AB, an indefinite straight line; a, a given finite straight line.

Required. To mark off on AB a length equal to a.

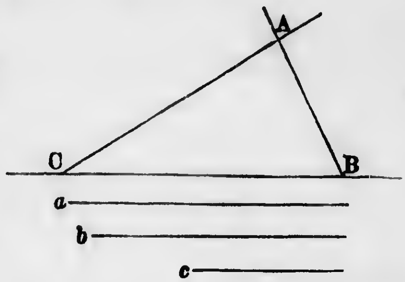


Construction. From any point O as a centre, on AB describe the arc of a circle with a radius equal to a. If P be the point in which the circle intersects AB, OP will be the required length.

Proof. All the radii of the circle around O are by construction equal to a. OP is one of these radii; therefore it is equal to a.

PROBLEM II.

178. *To construct a triangle of which the sides shall be equal to three given straight lines.*



Given. The three lines a, b, c.

Required. To draw a triangle with sides equal to these lines.

Construction. On an indefinite line take a length CB equal to any one of the three given lines (Problem I.).

From B as a centre, with a radius equal to one of the remaining lines, c, describe the arc of a circle.

From the point C as a centre, with a radius equal to the third given line describe another arc of a circle intersecting the former one in a point A. Join CA and BA.

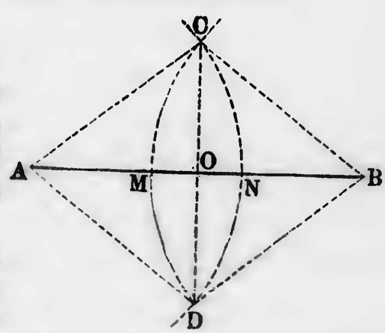
ABC will then be the triangle required.

Proof. From the mode of construction, the three lines AB, AC, and BC will be equal to the three given lines.

REMARK. The two circles may intersect on either side of the line AB. Therefore two triangles may be drawn which shall fulfill the given conditions. But these triangles will, by § 110, be identically equal.

PROBLEM III.

179. *To bisect a given finite straight line.*



Given. The line AB.

Required. To bisect it.

Construction. From the end A as a centre, with a radius greater than the half of AB, draw the arc of a circle CND.

From B as a centre, with the same radius draw the arc CMD, intersecting the first circle in the points C and D. Join CD.

The point O in which the line CD intersects AB will then bisect AB.

Proof. Join AC, BC, AD, BD.

Because AC, AD, BC, BD, are radii of the same or equal circles, they are equal, and the figure $ADBC$ is a parallelogram (§ 134).

Therefore the diagonals AB and CD bisect each other. (§ 138)

Therefore AB is bisected at O. Q.E.F.

180. *Corollary.* Because the parallelogrom $ABCD$ has all its sides equal, it is a rhombus, and its diagonals intersect at right angles. Therefore the above construction also solves the problem:

To draw the perpendicular bisector of a given line.

Problem IV.

181. *To bisect a given angle.*

Let ACB be the given angle.

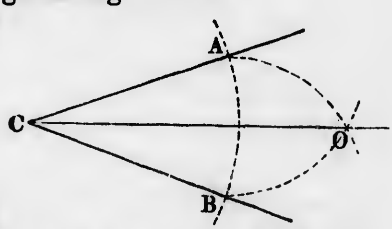
Construction. From C as a centre, with any radius CA describe the arc of a circle, cutting the sides of the angles in the points A and B.

From A as a centre, with the radius AB draw an arc of a circle.

From B as a centre, with an equal radius draw another arc intersecting the other in O.

Join CO.

The line CO will bisect the given angle ACB.



Proof. In the triangles CAO and CBO we have

$$\left.\begin{array}{l} CA = CB, \\ OA = OB, \end{array}\right\} \text{by construction.}$$

$CO = CO$, identically.

Therefore these two triangles are identically equal, and the angle OCA, opposite the side AO, is equal to OCB, opposite the equal side in the other triangle. Therefore the line CO bisects the angle ACB. Q.E.F.

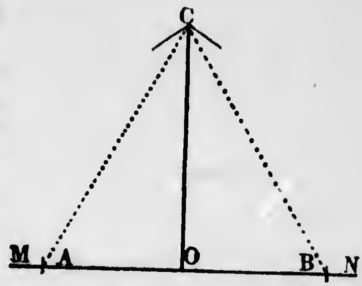
Problem V.

182. *Through a given point on a straight line to draw a perpendicular to this line.*

Let MN be the given line; O, the given point upon it.

Construction. From O as a centre, with any radius OA describe arcs of a circle cutting the given line at A and B.

From A as a centre, with the radius AB draw the arc of a circle.



From B as a centre, with the equal radius BA describe another arc intersecting the former one at C. Join OC.

OC will be the required perpendicular.

Proof. In the triangles CAO and CBO the three sides of the one are, by construction, equal to the three sides of the other.

Therefore the angle CAO is equal to the angle COB, and both these angles are right angles, by definition.

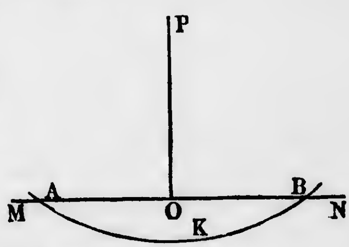
Therefore the line OC is perpendicular to MN. Q.E.F.

Problem VI.

183. *From a given point without a given line to drop a perpendicular upon the line.*

Let P be the given point; MN, the given line.

Construction. Take any point K on the opposite side of the line. From P as a centre, with the radius PK describe an arc cutting the given line at A and B.



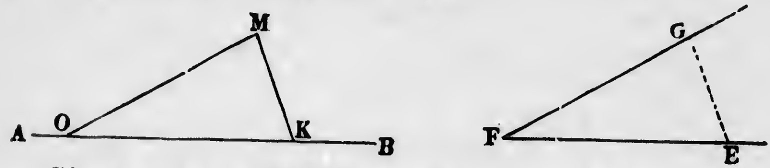
Bisect AB in the point O. Join PO.

The line PO will be the perpendicular required.

Proof. As in the last problem, and to be supplied by the student.

Problem VII.

184. *At a given point in a straight line to make an angle equal to a given angle.*



Given. An angle, EFG; a straight line, AB; a point O on that line.

Required. At O to make an angle equal to EFG.

Construction. 1. Join any two points in the sides FE and FG, thus forming a triangle.

2. From O take on OB a distance $OK = FE$.

3. On OK describe a triangle OKM whose sides KM, OM shall be equal to the sides EG, FG.

4. The angle MOK will be the required angle.

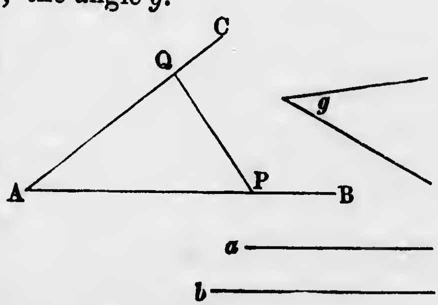
Proof. Because the triangles OKM and FEG have all the sides of the one equal to corresponding sides of the other, they are identically equal.

Therefore the angle MOK, opposite MK, is equal to the angle EFG, opposite the equal side EG of the other triangle. Q.E.F.

Problem VIII.

185. *To construct a triangle, having given two sides and the included angle.*

Given. Two sides, a, b; the angle g.



Required. To construct a triangle having the sides equal to a, b, and their included angle equal to g.

Construction. Draw an indefinite line AB.

At A make the angle BAC equal to g.

On AB take a length AP equal to b, and on AC take a length AQ equal to a.

Join PQ.

APQ will then be the required triangle.

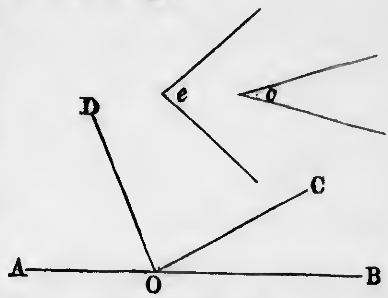
The result is evident, but should be shown by the pupil.

PROBLEM IX.

186. *Two angles of a triangle being given, to construct the third angle.*

Given. Two angles, c and e, of a triangle.

Required. To find the third angle of the triangle.



Construction. 1. At any point O in an indefinite line AB make the angle BOC equal to the given angle c.

2. At the same point make the angle $COD =$ given angle e.

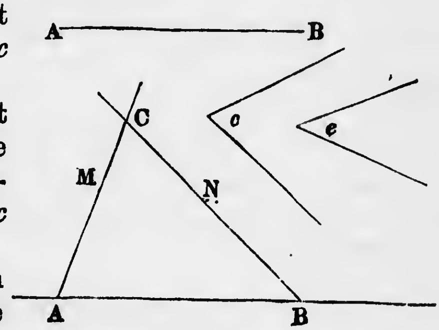
Then the angle DOA will be the angle required.

Proof. From Theorem IV., to be supplied by the student.

PROBLEM X.

187. *To construct a triangle, having given one side and the two adjacent angles.*

Given. A finite straight line, AB; two angles, c and e.



Required. To construct a triangle having its base equal to AB, and the angles at its base equal to c and e respectively.

Construction. 1. On an indefinite straight line mark off the length AB.

2. At A make the angle $BAM =$ angle c.

3. At B make the angle $ABN =$ angle e.

4. Continue the lines AM and BN until they meet, and let C be their point of meeting.

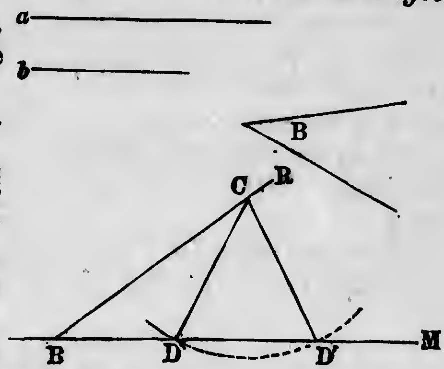
ABC will then be the required triangle.

The proof should be supplied by the student.

Corollary. Since the third angle of a triangle can always be found when two angles are given, this problem, combined with the preceding one, will suffice for the construction of the triangle when one side and any two angles are given.

PROBLEM XI.

188. *Two sides of a triangle and the angle opposite one of them being given, to construct the triangle.*



Given. The two sides, *a*, *b*; the angle *B* opposite the side *b*.

Required. To construct the triangle.

Construction. 1. At the point *B* on the indefinite line *BM* make an angle *MBR* equal to the given angle *B*.

2. From *B* on the line *BR* cut off a length *BC* equal to the given line *a*, which is not opposite the given angle *B*.

3. From *C* as a centre, with a radius equal to the line *b* describe a circle cutting *BM* at the points *D* and *D′*.

Either of the triangles *BCD* or *BCD′* will then be the required triangle.

The proof follows at once from the construction.

189. *Scholium.* The fact that there may be two triangles formed from the given data has been explained in the scholium § 113.

PROBLEM XII.

190. *Through a given point to draw a straight line which shall be parallel to a given straight line.*

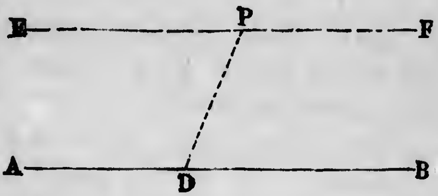
Given. A straight line, *AB*; a point, *P*.

Required. To draw a straight line through *P* parallel to *AB*.

Construction. 1. Take any point D in AB, and join PD.

2. At P in the line PD make angle DPE equal to the angle PDB.

3. Produce EP in the direction F.



EPF will then be the required straight line passing through P parallel to AB.

Proof. By Theorem III., because the angles PED and PDB are alternate angles.

PROBLEM XIII.

191. *To divide a finite straight line into any given number of equal parts.*

Given. A finite straight line, AB; a number, n.

Required. To divide AB into n equal parts.

Construction. 1. From one end of AB draw an indefinite straight line, making an angle with AB different from a straight angle.

2. Upon the indefinite line lay off any equal lengths, A 1, 1 2, 2 3, etc., until n lengths are laid off.

3. Join B to the end n of the last length.

4. Through each of the points 1, 2, 3, ... n, draw a parallel to nB intersecting AB.

The line AB will then be divided into n equal parts by the points of intersection.

Proof. The parallels intercept equal lengths on the transversal An, by construction.

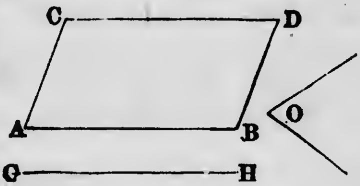
Therefore they also intercept equal lengths on AB, and the number of intercepted lengths is n (§ 132).

Therefore AB is divided into n equal parts. Q.E.F.

Problem XIV.

192. *Two adjacent sides of a parallelogram and the angle which they contain being given, to describe the parallelogram.*

Given. Two lines, AC and GH; an angle, O.



Required. To form a parallelogram having GH and AC for two adjacent sides, and O for the angle between these sides.

Construction. 1. At one end A of the line AC make an angle equal to O.

2. On the side of this angle take $AB =$ the given line GH.

3. Through B draw a line BD parallel to AC.

4. Through C draw CD parallel to AB, intersecting BD in D.

$ABCD$ will be the required parallelogram.

Proof. May be supplied by the student.

193. *Corollary. To construct a square upon a given straight line.*

This problem is a special case of the preceding one. in which the given sides are equal and the given angle is a right angle. To solve it:

Draw AC and $BD \perp AB$.
Join CD.

And the square will be complete.

CHAPTER VI.

EXERCISES IN DEMONSTRATING THEOREMS.

The following theorems should be demonstrated by the student in his own way, so far as he is able.

Analysis of a given Theorem.

The first step in the process of finding a demonstration is to state the hypothesis, referring to a diagram, which it is generally best the pupil should draw for himself. The statement should include not simply what the theorem says, but what it implies. Reference must be made to definitions until the hypothesis is resolved into its first elements.

Next, the conclusion must be analyzed in the same way, in order to see not only what it says, but also what it implies.

By the analyses of these two statements they must as it were be brought together, in order to see in what way they are related. The process of discovering this relation is one which the student must find for himself in each case, and for which no rule can be given. Frequently, however, it will be necessary to draw additional lines in the figure, and to call to mind the various theorems which apply to the figures thus formed. To facilitate this, references to previous theorems which come into play are added.

The relation being found, the demonstration must next be constructed in the simplest manner, but without the omission of any logical step. This, also, is a matter of practice in which no general rule can be given.

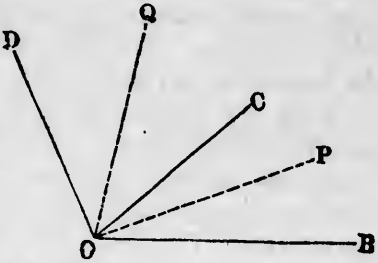
It is recommended that the teacher require the pupil to express each step of the demonstration with entire completeness and fullness. Some of the first theorems are so simple that the only serious exercise is that of constructing an artistic demonstration. The work thus becomes a valuable exercise in language and expression as well as in geometry.

The most common fault is that of passing over steps in the demonstration because the conclusion seems to be obvious. One of the great objects of practice in geometry is to cultivate the habit of examining the logical foundations of those conclusions which are accepted without critical examination. The feeling of security that a conclusion is right before its foundation has been examined is a most fruitful source of erroneous opinions; and the person who neglects the habit of inquiring into what appears obvious is liable to pass over things which, had they been carefully examined, would have changed the conclusion.

Remark. The theorems are arranged nearly in the order of their supposed difficulty. The references give the theorems on which the demonstration may be founded, or of which the method of proof has some analogy. It is not to be expected that the beginner will prove more than the first fifteen or twenty.

Theorem 1. If a line be divided into any two parts and each of these parts be bisected, the distance of the points of bisection will be one half the length of the original line.

Theorem 2. If any angle be divided into two angles and each of these angles be bisected, the angle between the bisectors will be half the original angle.

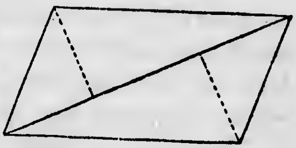


Hypothesis. BOD, the original angle divided into the two angles BOC and COD by the line OC.

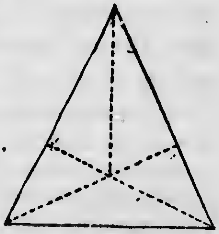
OP, OQ, the bisectors of BOC and COD.

Conclusion. Angle $POQ = \frac{1}{2}$ angle BOD.

Theorem 3. The perpendiculars dropped from two opposite angles of a parallelogram upon the diagonal joining the other angles are equal (§§ 128, 175).

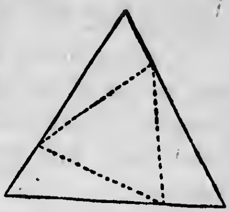


Theorem 4. If perpendiculars be drawn from the angles at the base of an isosceles triangle to the opposite sides, the line from the vertex to their point of crossing bisects both the angle at the vertex and the angle between the perpendiculars.



Theorem 5. If from any point of the base of an isosceles triangle perpendiculars be dropped upon the two equal sides, they will make equal angles with the base.

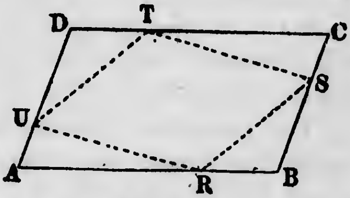
Theorem 6. If, on the three sides of an equilateral triangle, points equally distant from the three angles be taken in regular order and joined by straight lines, these lines will form another equilateral triangle (§ 108).



Theorem 7. If the perpendicular from any angle of a triangle upon the opposite side bisects this side, the triangle is isosceles.

Theorem 8. If the diagonals of any quadrilateral bisect each other, it is a parallelogram (§§ 67, 68, 108).

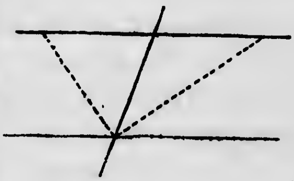
Theorem 9. If, on each pair of opposite sides of a parallelogram, we take two points equally distant from the opposite angles and join them by straight lines, these lines will form another parallelogram.



Hyp. $AU = SC$.
$DT = RB$.

Theorem 10. If the alternate angles formed by a transversal crossing two parallels be bisected, the bisectors will be parallel to each other (§§ 68, 71).

Theorem 11. If either bisector of an interior angle between two parallels be continued until it meets the opposite parallel, it forms the base of an isosceles triangle of which the equal sides are the transversal and the intercepted part of that parallel.



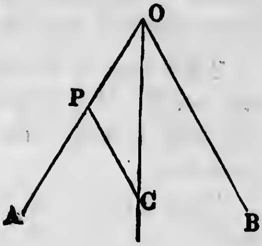
Corollary. The two bisectors of the angles which a transversal makes with one parallel cut off equal segments of the other parallel on the two sides of the transversal.

Theorem 12. If the four interior angles formed by a transversal crossing two parallels be bisected, and the bisectors produced until they meet, what figure will be formed? (§ 82)

This theorem is to be enunciated by the student.

THEOREM 13. If the bisectors of the four interior angles of a parallelogram be continued until each one meets two others, they will form a rectangle.

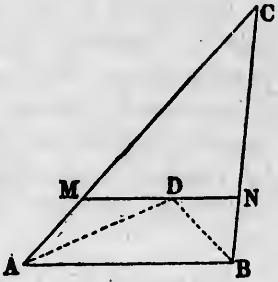
THEOREM 14. A line drawn from any point of the bisector of an angle parallel to one side, and meeting the other side, will form an isosceles triangle.



Hypothesis. Angle AOC = angle COB.
$CP \parallel BO$.

Conclusion. $PO = PC$.

THEOREM 15. If any two interior angles of a triangle be bisected and a line parallel to the included side be drawn through the point of meeting of the bisectors, the length of this parallel between the sides will be equal to the sum of the segments which it cuts off from the sides.



Hypothesis. Angle DAC = angle DAB.
Angle DBC = angle DBA.
$MDN \parallel AB$.

Conclusion. $MN = MA + NB$.

THEOREM 16. In an antiparallelogram—

I. The angles at the ends of the upper side are equal.

II. The sum of each pair of opposite angles is equal to a straight angle.

III. The diagonals are equal to each other (§ 172, Rem.).

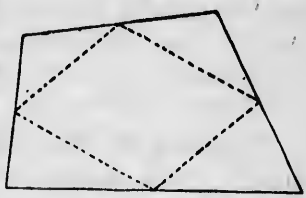
THEOREM 17. That portion of the middle parallel of a trapezoid which is intercepted between the diagonals is equal to half the difference of the parallel sides (§ 170, Def.).

THEOREM 18. If the diagonals of a trapezoid are equal, it is an antiparallelogram.

THEOREM 19. The sum of the diagonals of any quadrilateral is less than the sum of the four sides, but greater than the half sum.

THEOREM 20. If from any point in the base of an isosceles triangle a parallel to each side be drawn until it meets the other side, the sum of these parallels will be equal to either of the equal sides (§ 121).

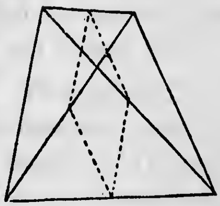
THEOREM 21. If the middle points of the sides of any quadrilateral be joined by straight lines, these lines will form a parallelogram (§ 136).



If the given quadrilateral has its pairs of adjacent sides equal (cf. § 173), the parallelogram formed from it will be a rectangle; and if it is a rectangle, the parallelogram formed from it will be a rhomboid.

THEOREM 22. If one of the equal sides of an isosceles triangle be produced beyond the vertex, and the exterior angle thus formed be bisected, the bisector will be parallel to the base of the triangle.

THEOREM 23. If the middle points of any two opposite sides of a quadrilateral be joined to each of the middle points of the diagonals, the four joining lines will form a parallelogram (§ 136).



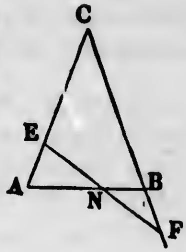
THEOREM 24. If one diagonal of a quadrilateral bisects both of the angles between which it is drawn, the other diagonal will cross it at right angles.

THEOREM 25. If from the right angle of a right-angled triangle a perpendicular be dropped upon the hypothenuse, the two triangles thus formed will be equiangular to the original one.

THEOREM 26. If one of the acute angles of a right-angled triangle be double the other, the hypothenuse will be double the shortest side.

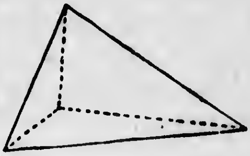
THEOREM 27. Each side of any triangle is less than half the sum of the three sides.

THEOREM 28. If one side of an isosceles triangle be produced below the base to a certain length, and an equal length be cut off above the base from the other equal side and the two ends be joined together by a straight line, this line will be bisected by the base.

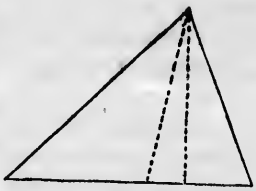


Hypothesis. $AC = BC$; $AE = BF$.
Conclusion. $EN = NF$.

THEOREM 29. The sum of the three straight lines drawn from any point within a triangle to the three vertices is less than the sum of the sides, but greater than half their sum.



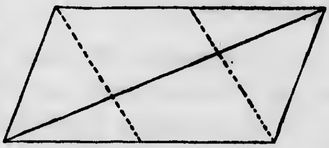
THEOREM 30. If from the vertex of any triangle two lines be drawn, one of which bisects the angle at the vertex, and the other is perpendicular to the base, the angle between these lines will be half the difference of the angles at the base of the triangle.



THEOREM 31. If from any point inside of an equilateral triangle perpendiculars be dropped upon the three sides, their sum will be equal to the perpendicular from the vertex upon the base.

What corollary may be deduced from this theorem?

THEOREM 32. If from two opposite angles of a parallelogram lines be drawn to the middle points of two opposite sides, these lines will divide the diagonal joining the other angles into three equal parts (§137).



THEOREM 33. If from either angle at the base of an isosceles triangle a perpendicular be dropped upon the opposite side, the angle between this perpendicular and the base will be one half the angle at the vertex of the triangle.

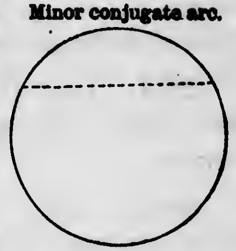
BOOK III.

THE CIRCLE.

CHAPTER I.

GENERAL PROPERTIES OF THE CIRCLE.

Definitions.



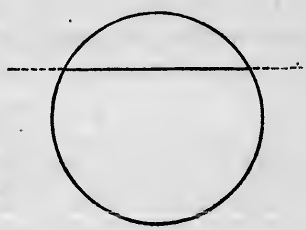
194. *Def.* The **circumference** of a circle is the total length of the curve-line which forms it.

195. *Def.* An **arc** of a circle is a part of the curve which forms it.

196. *Def.* When two arcs together make an entire circle, they are called **conjugate arcs**, and the one is said to be the **conjugate** of the other.

197. *Def.* When two conjugate arcs are equal, each of them is called a **semicircle.**

198. *Def.* When two conjugate arcs are unequal, the lesser is called the **minor arc,** and the greater the **major arc.**



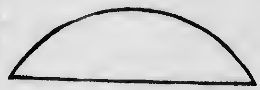
199. *Def.* A **chord** is a straight line between two points of a circle.

200. *Def.* A **secant** is a straight line which intersects a circle.

REMARK. A secant may be considered as a chord with one or both of its ends produced, and a chord as that part of a secant contained within the circle.

201. *Def.* The **diameter** of a circle is a chord which passes through its centre.

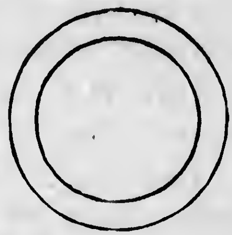
202. *Def.* A **segment** of a circle is composed of a chord and either of the arcs between its extremities.



203. *Def.* A **sector** is formed of two radii and the arc included between them.

To a pair of radii may belong either of the two conjugate arcs into which their ends divide the circle.

204. *Def.* **Concentric circles** are those which have the same centre.



205. *Def.* A **tangent** to a circle is any straight line which touches the circle without intersecting it.

206. Special Axioms relating to the Circle.

I. A circle has only one centre.

II. Every point at a distance from the centre less than the radius is within the circle.

III. Every point at a distance from the centre equal to the radius is on the circle.

IV. Every point at a distance from the centre greater than the radius is without the circle.

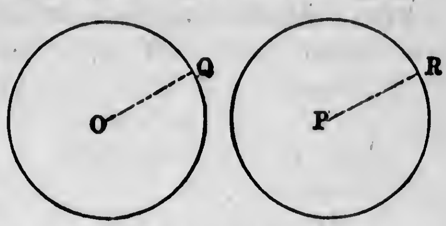
THEOREM I.

207. *Circles of equal radii are identically equal.*

Hypothesis. Two circles of which O and P are centres, and Radius OQ = radius PR.

Conclusion. The circles are identically equal.

Proof. Apply the one circle to the other in such manner that the centre O shall coincide with P, and OQ with PR. Then—



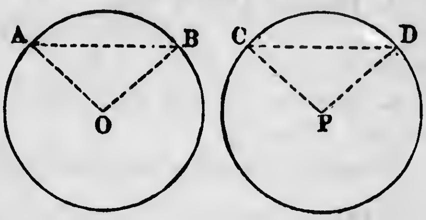
1. Because $OQ = PR$,
Point $Q \equiv$ point R.

2. Because each point of the one circle is at the distance OQ from the centre, it will fall on the other circle. (§ 206, Ax. III.)

Therefore the circles are identically equal. Q.E.D.

THEOREM II.

208. *Equal arcs of equal circles are identically equal, subtend equal angles at centre, and contain equal chords.*



Hypothesis. AB, CD, equal arcs around the centres O and P.
$OA = OB = PC = PD$.

Conclusion. The angles AOB and CPD and the chords AB and CD are equal.

Proof. Apply the sector OAB to the sector PCD so that the centre O shall fall on P, and the radius OA on the radius PC. Then—

1. Because $OA = PC$,

Point $A \equiv$ point C.

2. Because the radii are all equal, every part of the arc AB will fall on some part of the circle to which the arc CD belongs (§ 206, III.).

3. Because arc $AB =$ arc CD, the point B will fall on D, and chord $AB \equiv$ chord CD. Therefore

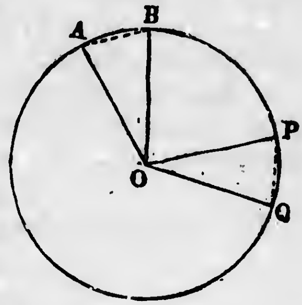
Chord $AB =$ chord CD.
Angle $AOB =$ angle CPD. Q.E.D.

THEOREM III.

209. *Equal angles between radii include equal arcs on the circle and equal chords.*

Hypothesis. OA, OB, OP, OQ, four radii of a circle such that

Angle AOB = angle POQ.



Conclusions.

Arc AB = arc PQ.

Chord AB = chord PQ.

Proof. Apply the sector AOB to the sector POQ in such manner that OA shall coincide with OP. Then—

1. Because $OA = OP$,

Point $A \equiv$ point P.

2. Because angle AOB = angle POQ,

$OB \equiv OQ$.

3. Because $OB = OQ$,

Point $B \equiv$ point Q.

Therefore $AB = PQ$. Q.E.D.

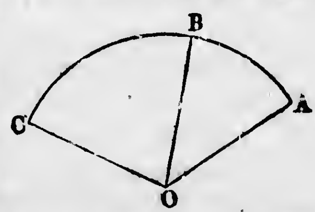
4. Because all the radii are equal, the arcs will coincide between P and Q.

Therefore the arcs are identically equal. Q.E.D.

210. *Corollary. Sectors of equal angles in equal circles are identically equal, and every line in the one sector is identically equal to the corresponding line in the other* (§ 174).

Lemma.

211. *A sum of two arcs at the centre subtends an angle equal to the sum of the angles which each arc subtends separately.*



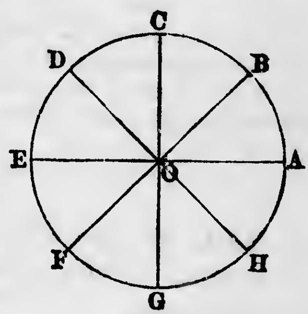
Proof. The arc AB subtends the angle AOB, the arc BC subtends the angle BOC, and the arc ABC subtends the angle AOC. But AOC is by definition the sum of the angles, and ABC is the sum of the arcs; which proves the lemma.

The Measurement of Angles by means of Arcs.

212. From the preceding theorems it follows that to every arc of given length in a given circle corresponds a definite angle, and to every angle corresponds a definite length of arc. To express corresponding arcs and angles in the shortest way, we call the arc corresponding to the angle AOB the **arc angle** AOB, and we call the angle corresponding to an arc the **angle arc.**

Thus, in the following figure,

$$\text{Angle } AOB = \text{angle arc } AB.$$
$$\text{Angle } AOC = \text{angle arc } ABC.$$
$$\text{Arc } ABC = \text{arc angle } AOC.$$
$$\text{Arc } AB = \text{arc angle } AOB.$$



Combining Theorem III. with the above lemma, it follows that arcs can be taken as the measure of the corresponding angles, and *vice versa.*

In the figure the circle is divided into eight sectors, and since $\frac{360}{8}° = 45°$, each of these sectors subtends an angle of 45°. Therefore

Angle AOB =	45°;	arc AB	= 45°.
Angle AOC =	90°;	arc ABC	= 90°.
Angle AOD =	135°;	arc $ABCD$	= 135°.
Angle AOE =	180°;	arc $ABCDE$	= 180°.
Angle AOF =	225°;	arc $ABCDEF$	= 225°.
Angle AOG =	270°;	arc $ABCDEFG$	= 270°.
Angle AOH =	315°;	arc $ABCDEFGH$	= 315°.
Angle AOA =	360°;	arc $ABCDEFGHA$	= 360°.

213. The following are the principles to which we are thus led:

I. *In the same circle or in equal circles, the greater arc measures the greater angle.*

II. *A minor arc, or an arc less than a semicircle, measures an angle less than a straight angle.*

III. *A major arc measures a reflex angle or one greater than a straight angle.*

IV. *When an arc measures an angle, the conjugate arc measures the conjugate angle.*

V. *The sum of an arc and its conjugate measures the sum of an angle and its conjugate, and each sum is a circumference, or* 360^c.

The use of arcs to express angles has a great advantage of making plain to the eye the difference between an angle and its conjugate, because we can always draw either of two conjugate arcs between the sides of the angle.

When we say "the angle *AOF*," we should not, without some means of distinction, know which of the two conjugate angles is meant.

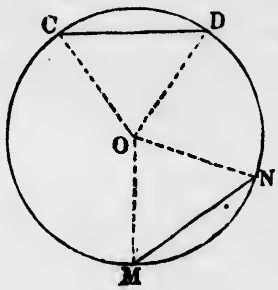
But when we say the angle arc *ABCDEF*, we do know which of the two conjugate angles is meant, because the arc measures only one of them, not both.

When we do not use arcs, the angle expressed without any adjective will mean the lesser of the conjugate angles.

When we mean the greater conjugate angle and do not use an arc, we shall call it a reflex angle.

Theorem IV.

214. *In a circle equal chords subtend equal arcs and equal angles at the centre.*



Hypothesis. *CD*, *MN*, two equal chords of a circle having its centre at *O*.

Conclusion.

Angle COD = angle MON.

Arc CD = arc MN.

Proof. In the triangles *COD* and *MON* we have

$CD = MN$, by hypothesis.

$\left.\begin{array}{l} OC = ON, \\ OD = OM, \end{array}\right\}$ because they are radii of the circle.

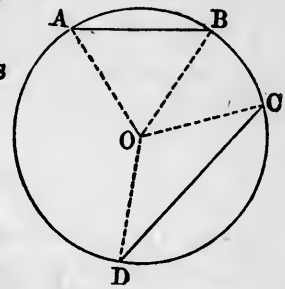
2. Therefore these two triangles are identically equal.
3. Therefore

Angle COD, opp. CD = angle MON, opp. equal side MN.

4. Therefore arc CD = arc MN (§ 209). Q.E.D.

THEOREM V.

215. *In a circle the greater chord includes the greater minor arc and the lesser major arc.*



Hypothesis. AB, CD, two chords of a circle such that

$$CD > AB.$$

Conclusion.

Minor arc $CD >$ minor arc AB.
Major arc $CBAD <$ major arc $BCDA$.

Proof. From the centre of the circle draw the radii OA, OB, OC, OD. Then—

1. In the triangles AOB and COD we have

$$OA = OC;\ OB = OD;\ AB < CD.$$

Therefore angle $AOB <$ angle COD. (§ 115)
Therefore minor arc $AB <$ minor arc CD (§ 213, I.).
Q.E.D.

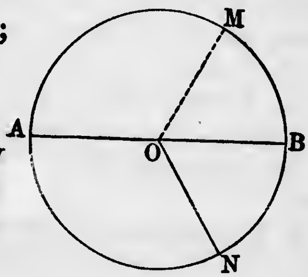
2. Because the major and minor arc together make up the entire circle, to a greater minor arc will correspond a lesser major arc. Q.E.D.

216. *Corollary.* Conversely, *to the greater minor arc of a circle will correspond the greater chord.*

For, when the arc is greater, its chord cannot be equal without violating Theorem IV., nor less without violating Theorem V. Therefore it must be greater.

THEOREM VI.

217. *Every diameter divides the circle into two identically equal semicircles.*



Hypothesis. $AMBN$, a circle; AB, a diameter; O, the centre

Conclusion.

Arc $AMB =$ arc ANB.

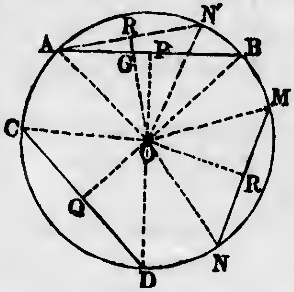
Proof. Draw any two radii OM and ON making equal angles with AB. Turn the semicircle AMB over on AB as an axis. Then—

1. Because angle $BOM =$ angle BON,
Radius $OM \equiv$ radius ON.
2. Because $OM = ON$,
Point $M \equiv$ point N.
3. Since M may be any point whatever on the circle, every point of the arc AMB will fall on a point of ANB.

Therefore the two arcs coincide and are identically equal.
Q.E.D.

THEOREM VII.

218. *Equal chords are equally distant from the centre, and of unequal chords the greater is nearer the centre.*



Hypothesis. AB, CD, and MN, chords of a circle such that $AB = CD > MN$; O, the centre of the circle; OP, OQ, OR, perpendiculars from O on AB, CD, and MN, respectively.

Conclusions. I. $OP = OQ$.
II. $OP < OR$.

Proof. I. Draw the radii OA, OB, OC, OD. Then—

1. In the triangles AOB and COD we have
Radius $OC = OB$; $OD = OA$; $CD = AB$ (hyp.).
2. Therefore these triangles are identically equal, and the perpendicular OP is equal to the corresponding perpendicular OQ (§ 175). Q.E.D.

II. Turn the figure $ORMN$, composed of the chord and its perpendicular, around the centre O in such manner that OM shall coincide with OA, and let G be the point in which OR intersects AB. Then—

3. Because the radii are equal, $M \equiv A$.
4. Because $MN < AB$, it subtends a less minor arc (§215), and the point N will fall within the minor arc AB.
5. Therefore MN will fall inside the minor segment AB, and
$$OG < OR.$$
6. But because OP is a perpendicular on AB,
$$OP < OG.$$

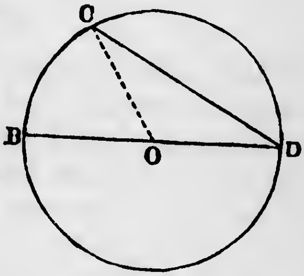
Comparing 5 and 6,

$$OP < OR. \quad \text{Q.E.D.}$$

219. *Corollary. A chord nearer the centre than another is the longer.*

THEOREM VIII.

220. *The diameter of a circle is greater than any other chord.*



Hypothesis. CD, a chord meeting a circle in the points C and D, and not passing through the centre.

Conclusion. CD is less than a diameter.

Proof. Let O be the centre of the circle. Join OC and OD, and continue DO across the centre to B. Then—

1. Because CD is a straight line,

$$CD < OC + OD.$$

2. Because OC and OB are radii,

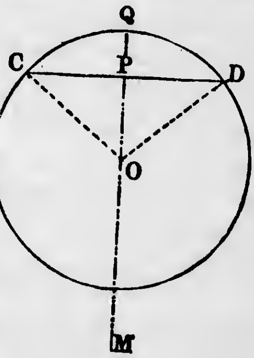
$$OC + OD = BO + OD = \text{a diameter.}$$

3. Comparing (1) and (2),

$$\text{Diameter} > CD. \quad \text{Q.E.D.}$$

THEOREM IX.

221. *The perpendicular from the centre of a circle upon a chord bisects the chord and the arc which contains it.*



Hypothesis. CD, a chord of a circle; O, its centre; OPQ, a perpendicular from O on CD, cutting the circle in Q.

Conclusion. $PC = PD$.

Arc CQ = arc QD.

Proof. Join OC and OD. Then—

1. Because OC and OD are radii, they are equal.

2. Because the triangles COP and DOP have $OC = OD$, OP common,

and OPC and OPD right angles, they are identically equal. (§ 112)

Therefore $PC = PD$. Q.E.D.

And Angle COP = angle DOP.

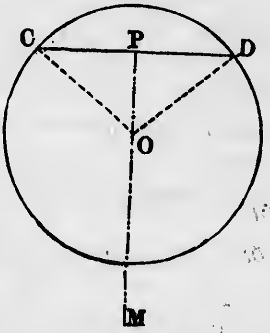
3. Because the arcs CQ and QD are subtended by the equal angles COQ and DOQ,

Arc CQ = arc QD. Q.E.D.

Theorem X.

222. *Conversely, a line bisecting a chord at right angles passes through the centre.*

Hypothesis. CD, a chord of a circle; PM, its perpendicular bisector.



Conclusion. The centre of the circle lies on the line PM.

Proof. 1. Because C and D are each upon the circle, the centre is, by definition, equally distant from C and D.

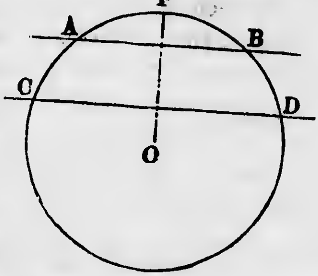
2. Because PM is the perpendicular bisector of CD, every point equally distant from C and D lies upon this line (§105).

3. Therefore the centre of the circle lies upon the line PM. Q.E.D.

Theorem XI.

223. *Parallel chords or secants intercept equal arcs between them.*

Hypothesis. AB, CD, parallel straight lines, of which the first meets the circle in the points A and B, and the second in the points C and D.



Conclusion. Arc AC = arc DB.

Proof. Let O be the centre of the circle. From O drop the radius OF perpendicular to one of the parallels. Then—

1. Because OF is perpendicular to one of the parallels, it will be perpendicular to the other also (§ 72).

2. Because OF is perpendicular to AB,

Arc AF = arc BF. (§ 221)

3. Because OF is perpendicular to CD,

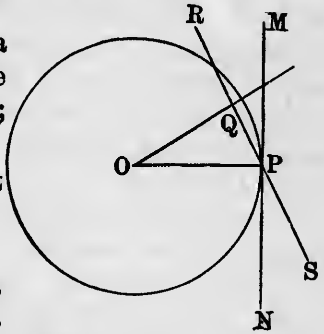
Arc CF = arc DF.

4. Subtracting (2) from (3),

Arc AC = arc DB. Q.E.D.

THEOREM XII.

224. *Of lines passing through the end of any radius the perpendicular is a tangent to the circle, and every other line is a secant*



Hypothesis. O, the centre of a circle; OP, a radius; MN, a line through P perpendicular to OP; RS, any other line through P.

Conclusion. I. MN is a tangent to the circle at P.

II. RS is a secant.

Proof I. 1. Because OP is a perpendicular from O upon MN, it is less than any other line from O to MN (§ 101).

2. Therefore every other point of MN is farther from the centre O than P is.

3. Therefore every point of MN except P is outside the circle, while P is on the circle (§ 206, Ax. III.).

4. Therefore MN is a tangent (§ 205, def.). Q.E.D.

Proof II. From O drop a perpendicular upon RS, and let Q be the point of meeting. Then—

5. Because the line PR is, by hypothesis, different from PM, and OPM is a right angle, the angle OPR cannot be a right angle.

6. Therefore OP is not a perpendicular upon RP.

7. Therefore OQ, the perpendicular, will be a different line from OP.

8. Because OQ is a perpendicular, it will be less than OP, an oblique line (§ 101).

9. Therefore the point Q is inside the circle (§206, Ax. II.).

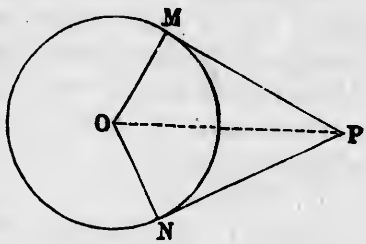
10. Therefore RS is a secant of the circle. Q.E.D.

225. *Corollary* 1. *The radius to the point of contact of any tangent is perpendicular to the tangent.*

226. *Corollary* 2. *The perpendicular from the point of tangency passes through the centre of the circle.*

Theorem XIII.

227. *Two tangents drawn to a circle from the same external point are equal, and make equal angles with the line joining that point to the centre.*



Hypothesis. P, a point outside a circle; PM, PN, two tangents from P touching the circle at M and N; O, the centre of the circle.

Conclusion. $PM = PN$.

Angle $OPM =$ angle OPN.

Proof. 1. In the triangles OMP and ONP we have

Side $OM =$ side ON.

Side $OP =$ side OP.

Angle $OMP =$ angle $ONP =$ right angle.

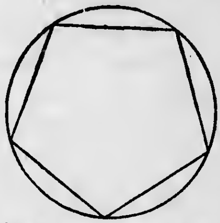
2. Therefore these two triangles are identically equal; namely, the side PM is equal to its corresponding side PN, and the angle OPM to OPN. Q.E.D.

CHAPTER II.

INSCRIBED AND CIRCUMSCRIBED FIGURES

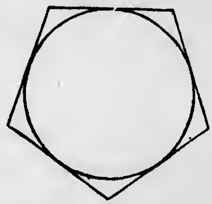
Definitions.

228. *Def.* A rectilineal figure is said to be **inscribed in a circle** when all its vertices lie on the circle. The circle is then said to be **circumscribed about** the figure.



An inscribed polygon and a circumscribed circle.

229. *Def.* A figure is said to be **circumscribed about** a circle when all its sides are tangents to the circle. The circle is then said to be **inscribed in** the figure.

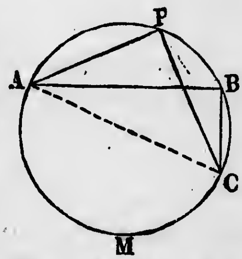


A circumscribed polygon and an inscribed circle.

230. *Def.* An **inscribed angle** is one of which the sides are two chords going out from the same point on a circle.

231. *Def.* An inscribed angle is said to stand upon the arc included between the ends of its sides.

If the sides of the inscribed angle are PA and PC, and the circle is divided into two segments by the third chord AC, the angle APC is said to be inscribed in the segment $ACBPA$ and to stand upon the arc AMC.



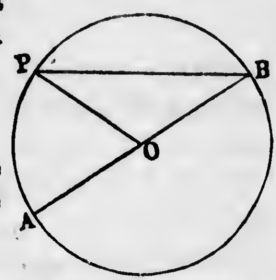
The inscribed angles ABC and APC are inscribed in the segment ACB and stand upon the arc AMC.

232. *Def.* A line is said to **subtend** a certain angle from a certain point when the lines drawn from the point to the ends of the line form that angle.

The line AC subtends the angle APC from the point P.

THEOREM XIV.

233. *If from any point on a circle lines be drawn to the end of a diameter and to the centre, the angle at the end of the diameter will be half that at the centre.*



Hypothesis. AB, a diameter of a circle; O, the centre; P, any point on the circle.

Conclusion.

Angle $PBO = \frac{1}{2}$ angle POA.

Proof. 1. Because OP and OB are radii, they are equal. Therefore the triangle POB is isosceles; whence

Angle $OPB =$ angle PBO.

2. POA is an exterior angle of the triangle POB. Therefore

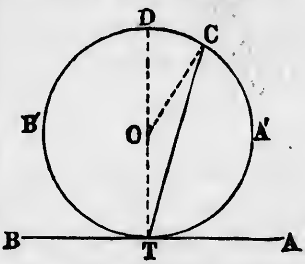
Angle $OPB +$ angle $PBO =$ angle POA. (§ 76)

3. Comparing (1) and (2),

Angle $PBO = \frac{1}{2}$ angle POA. Q.E.D.

THEOREM XV.

234. *Each angle between a chord and the tangent at its end is measured by half the arc cut off by the chord on the corresponding side.*



Hypothesis. AB, a tangent touching the circle at T; TC, a chord from T to C.

Conclusions. 1. Angle $ATC = \frac{1}{2}$ angle of arc $TA'C$ on the side A.

2. Angle $BTC = \frac{1}{2}$ angle of arc $TB'DC$ on the side B.

Proof. Let O be the centre of the circle. From T draw the diameter TOD, and join OC. Suppose the chord from T to fall between TO and TA. Then—

1. Because TA is a tangent and TO a radius, ATD is a right angle. Therefore

Angle $BTC =$ right angle $+$ angle OTC.
Angle $ATC =$ right angle $-$ angle OTC.

2. Angle TOC = straight angle TOD − angle COD,
= 2 right angles − angle COD.

Reflex angle TOC = straight angle TOD + angle COD,
= 2 right angles + angle COD.

3. But

Angle COD = 2 angle OTC. (§ 233)

4. Comparing (2) and (3),

Angle TOC = 2 right angles − 2 angle OTC.

Reflex angle TOC = 2 right angles + 2 angle OTC.

5. Comparing with (1),

Angle TOC = 2 angle ATC.

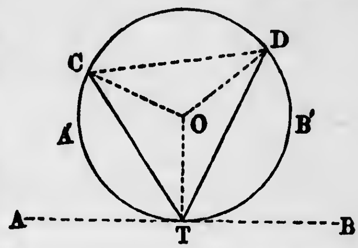
Reflex angle TOC = 2 angle BTC.

Or Angle $ATC = \frac{1}{2}$ angle TOC,
$= \frac{1}{2}$ angle arc $TA'C$.

Angle $BTC = \frac{1}{2}$ reflex angle TOC,
$= \frac{1}{2}$ angle arc $TB'C$. Q.E.D.

Theorem XVI.

235. *An inscribed angle is one half the angle of the arc on which it stands.*



Hypothesis. TC, TD, two chords meeting at a point T on a circle.

Conclusion. Angle CTD = $\frac{1}{2}$ angle of arc CD (that arc being taken on which T does not lie).

Proof. Let O be the centre of the circle. From O draw the radii OC, OD, OT. Through T draw the tangent ATB. Then—

1. Angle $CTB = \frac{1}{2}$ angle of arc $CDB'T$;
2. Angle $DTB = \frac{1}{2}$ angle of arc $DB'T$. (§ 234)

3. Subtracting the second equation from the first, and remarking that

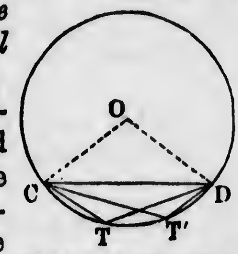
Angle CTB − angle DTB = angle CTD,

Arc $CDB'T$ − arc $DB'T$ = arc CD,

we have angle $CTD = \frac{1}{2}$ angle of arc CD which does not include T. Q.E.D.

Corollary 1. The angle of the arc CD is, by definition, the angle COD between the radii OC and OD. Therefore the preceding theorem may be expressed thus:

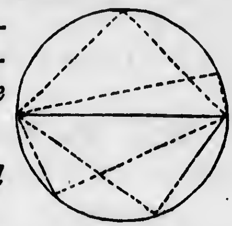
236. *If, from two points on a circle, lines be drawn to the centre and to any third point on the circumference, the angle at the centre will be double the angle at the circumference.*



The angles CTD, $CT'D$, inscribed in the same segment, are equal.

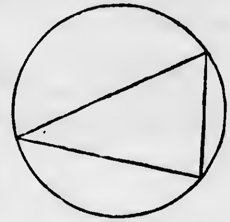
But, in applying the theorem, the angle at the centre, COD, must be counted round in such a direction as not to include the radius OT to the angle at the circumference. This angle will therefore be greater than 180° whenever the arc CTD is a minor arc.

237. *Corollary* 2. *All angles inscribed in the same segment are equal, because they are all halves of the same angle at the centre.*



238. *Corollary* 3. *All angles inscribed in a semicircle are right angles.*

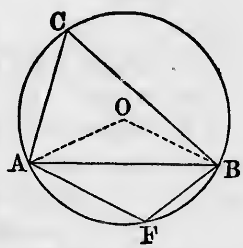
For they are all measured by half a semicircle.



239. *Corollary* 4. If a triangle be inscribed in a circle, its angles will divide the circle into three arcs.

The angle of each of these arcs will be double the opposite angle of the triangle.

240. *Corollary* 5. *Every pair of angles inscribed in conjugate segments are supplementary.*

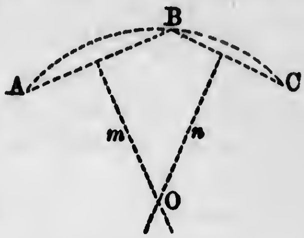


For if ACB and AFB are two such angles, each of them is measured by one half the opposite arc, and therefore their sum is half a circumference, or a straight angle; whence they are supplementary, by definition (§ 60).

Theorem XVII.

241. *Through three given points not in the same straight line, one circle, and only one, may be drawn.*

Hypothesis. A, B, C, three given points.



Conclusion. There is only one point, O, so situated that it may be the centre of a circle passing through these points.

Proof. The centre of the circle must be equally distant from A, B, and C.

Join AB and BC, and let the lines m and n be the perpendicular bisectors of AB and BC. Then—

1. Every point which is equally distant from A and B lies on the line m (§ 105).

2. Every point which is equally distant from B and C lies on the line n.

3. Therefore every point which is equally distant from all three points, A, B, and C, lies on both the lines m and n; that is, on their point of intersection O.

4. But there is only one point of intersection. Therefore there is one point, and only one, equally distant from A, B, and C; namely, the point O.

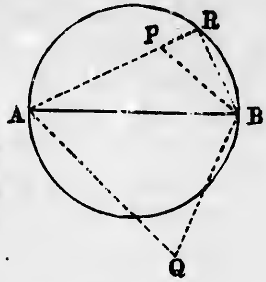
5. Because $OA = OB = OC$, if with the centre O and the radius OA we describe a circle, it will pass through A, B, and C. Q.E.D.

Scholium. If the three points A, B, and C are in a straight line, the perpendiculars to the lines AB and BC are parallel (§ 70). Therefore in this case no point can be found which shall be equally distant from A, B, and C.

Theorem XVIII.

242. *From any point within a circle every diameter subtends an angle greater than a right angle, and from any point without the circle it subtends an angle less than a right angle.*

Hypothesis. AB, any diameter of a circle; P, any point within the circle; Q, any point without the circle.



Conclusions.

I. Angle $APB >$ right angle.

II. Angle $AQB <$ right angle.

Proof. I. Continue either side of the angle APB, say AP, until it meets the circle. Let R be the point of meeting. Join BR.

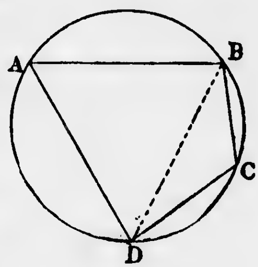
Then—

1. Because the angle APB is an exterior angle of the triangle BRP, it is greater than the interior angle PRB (§ 77).
2. Because the angle $PRB = ARB$ is inscribed in a semicircle, it is a right angle (§ 238).
3. Therefore APB is greater than a right angle. Q.E.D.

II. The proof of this case is so near like that of case I. that it is left as an exercise for the student.

Theorem XIX.

243. *If a quadrilateral be inscribed in a circle, the sum of each pair of opposite angles is two right angles.*



Hypothesis. $ABCD$, a quadrilateral of which the four angles lie on a circle.

Conclusions.

Angle A + opposite angle C = 2 right angles.

Angle B + opposite angle D = 2 right angles.

Proof. Draw the diagonal BD. This diagonal will be a chord dividing the circle into two segments. Then—

1. Angle $BCD = \frac{1}{2}$ angle arc BAD. (§ 235)
   Angle $BAD = \frac{1}{2}$ angle arc BCD.
2. Adding these two equations,

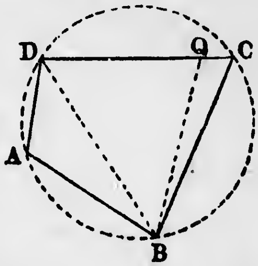
$$\text{Angle } BCD + \text{angle } BAD = \tfrac{1}{2} \text{ circumference,}$$
$$= 2 \text{ right angles.}$$

In the same way, by drawing the diagonal AC, may be shown

Angle B + angle $C = 2$ right angles. Q.E.D.

THEOREM XX.

244. *Conversely, if the sum of two opposite angles of a quadrilateral is equal to two right angles, the four angles lie on a circle.*



Hypothesis. $ABCD$, a quadrilateral in which

Angle A + angle C = two right angles.

Conclusion. The points A, B, C and D lie on the same circle.

Proof. Describe a circle through the three points B, A, D.

If this circle does not pass through C, it must intersect DC, or DC produced, at some other point than C.

Let Q be this point. Join BQ. Then—

1. Because the quadrilateral $ABQD$ is inscribed in a circle,

Angle BAD + angle BQD = two right angles.

2. Angle BAD + angle BCD = two right angles (hyp.).

Therefore Angle BQD = angle BCD,

which is impossible, because BQD is an exterior angle of the triangle BQC (§ 77).

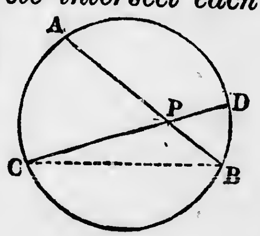
3. In the same way it may be shown that the circle cannot intersect DC produced at any point beyond C. Therefore the circle must pass through C, and $ABCD$ lie on one circle.

Q.E.D.

245. *Corollary. Each exterior angle of an inscribed quadrilateral is equal to the opposite interior angle.*

THEOREM XXI.

246. *When two chords of a circle intersect each other, each angle is measured by half the sum of the arcs intercepted by its sides and the sides of its vertically opposite angle.*



Hypothesis. AB, CD, two chords intersecting at the point P.

Conclusions.

Angle $DPB = APC = \frac{1}{2}$ angle arc $BD + \frac{1}{2}$ angle arc AC.
Angle $APD = CPB = \frac{1}{2}$ angle arc $DA + \frac{1}{2}$ angle arc CB.

Proof. Join BC. Then—

1. Because APC is an exterior angle of the triangle BCP,
$$\text{Angle } APC = \text{angle } PBC + \text{angle } PCB. \quad (\S\ 76)$$
2. Because PBC is an inscribed angle standing on the arc AC,
$$\text{Angle } PBC = \tfrac{1}{2} \text{ angle arc } AC. \quad (\S\ 235)$$
3. Because PCB is an inscribed angle standing on the arc BD,
$$\text{Angle } PCB = \tfrac{1}{2} \text{ angle arc } BD.$$
4. Comparing (2) and (3) with (1),
$$\text{Angle } APC = \tfrac{1}{2} \text{ angle arc } AC + \tfrac{1}{2} \text{ angle arc } BD. \quad \text{Q.E.D.}$$
5. In the same way may be shown
$$\text{Angle } APD = \tfrac{1}{2} \text{ angle arc } DA + \tfrac{1}{2} \text{ angle arc } CB. \quad \text{Q.E.D.}$$

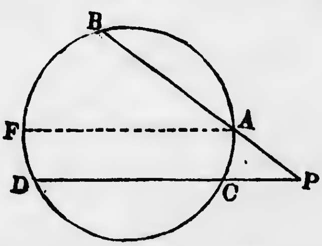
Corollary. Since vertically opposite angles are equal, we conclude—

247. *The sum of each pair of vertically opposite angles is measured by the sum of the corresponding intercepted arcs on any circle which includes the vertex of the angle.*

THEOREM XXII.

248. *If two secants be drawn from a point outside a circle, the angle between them is measured by half the difference of the intercepted arcs.*

Hypothesis. PAB, PCD, two secants emanating from the point P without a circle, and intersecting the latter at the respective points A, B and C, D.



Conclusion.

Angle $APC = \frac{1}{2}$ angle arc BD $- \frac{1}{2}$ angle arc CA.

Proof. Through A draw a parallel to PD, intersecting the circle in the points A and F.

Then—

1. Because BP is a transversal of the parallels FA and DP,
$$\text{Angle } APC = \text{corresponding angle } BAF.$$

2. Because BAF is an inscribed angle standing on the arc BF,

$$\text{Angle } BAF = \tfrac{1}{2} \text{ angle arc } BF,$$
$$= \tfrac{1}{2} \text{ angle arc } BD - \tfrac{1}{2} \text{ angle arc } FD.$$

3. Because CA and FD are intercepted between the parallels AF and CD,

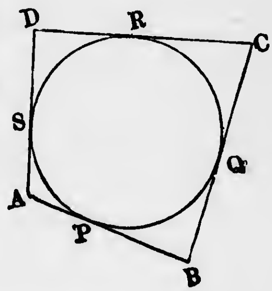
$$\text{Angle arc } FD = \text{angle arc } CA. \quad (§\ 223)$$

4. Comparing (1), (2), and (3),

Angle $APC = \frac{1}{2}$ angle arc $BD - \frac{1}{2}$ angle arc CA. Q.E.D.

Theorem XXIII.

249. *If a quadrilateral be circumscribed about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.*



Hypothesis. $ABCD$, a quadrilateral touching a circle in the points P, Q, R, S.

Conclusion.

$$AB + CD = BC + DA.$$

Proof. 1. Because AP and AS, BP and BQ, etc., are tangents drawn to the circle from the points A, B, etc., we have

$$AP = AS.$$
$$BP = BQ.$$
$$CR = CQ.$$
$$DR = DS. \quad (§\ 227)$$

2. Taking the sum of these equations, we find

$$AP + BP + CR + DR = AS + BQ + CQ + DS.$$

3. But we have, identically,

$$AP + BP + CR + DR = AB + CD.$$
$$BQ + CQ + DS + AS = BC + AD.$$

Therefore
$$AB + CD = BC + AD. \quad \text{Q.E.D.}$$

CHAPTER III.

PROPERTIES OF TWO CIRCLES.

250. *Def.* Two circles are said to touch each other, or to be tangent to each other, when they meet in a single point but do not intersect.

THEOREM XXIV.

251. *Two circles cannot intersect in more than two points.*

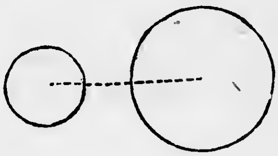
Hypothesis. O, P, the centres of two circles; M, N, Q, three of their points of intersection.

Conclusion. The hypothesis is impossible.

Proof. If O and P were the centres of two circles passing through M, N, and Q, then the two points O and P would be equally distant from all three points M, N, and Q, which is impossible (§ 241).

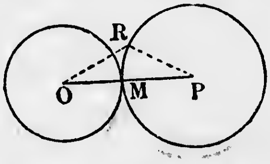
AXIOM V.

252. *If the distance between the centres of two circles is greater than the sum of their radii, they will not meet each other.*



THEOREM XXV.

253. *If the distance of the centres of two circles is equal to the sum of their radii, they will be tangent to each other.*



Hypothesis. O and P, the centres of two circles, the sum of whose radii is equal to the line OP.

Conclusion. The two circles have one point in common, and no more.

Proof. 1. On OP take a point M, such that OM shall be equal to the radius of the circle whose centre is at O. The point M will be on that circle (§ 206, Ax. III.).

2. Because OP is equal to the sum of the radii, the distance MP will be equal to the radius of the other circle.

3. Therefore the point M will be at the same time on both circles.

4. Take any other point R on the circle P, and join PR and OR. Because O, M, P are in one straight line, we shall have
$$OR + PR > OM + PM.$$

5. But because PM and PR are radii of the same circle,
$$PR = PM.$$

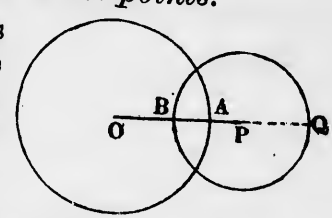
6. Taking (5) from (4),
$$OR > OM.$$

7. Therefore the point R lies without the circle around O. (§ 206, Ax. IV.)

8. Therefore the two circles meet in no point but M. Because they meet in this point without intersecting, they are tangent to each other (§ 250). Q.E.D.

Theorem XXVI.

254. *If the distance of centres of two circles is less than the sum but greater than the difference of their radii, they will intersect in two points.*



Hypothesis. O, P, the centres of two circles; OA, a radius of the greater one, on the line OP; PB, a radius of the lesser, on the same line;
$$OA - BP < OP < OA + BP.$$

Conclusion. The circles intersect in two points.

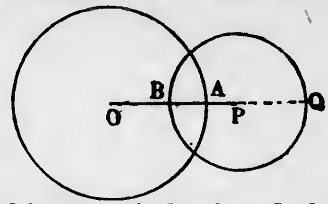
Proof. 1. The line OP is made up of the two parts $OB + BP$. The condition $OP < OA + BP$ is therefore the same as
$$OB + BP < OA + BP.$$

2. Taking away the common part BP,
$$OB < OA.$$

Therefore OB is less than the radius of the circle O, and the point B on the circle P is within the circle O (§ 206, Ax. II.).

Continue OP until it intersects the circle P in Q. Then—

3. Because BP and PQ are radii of the same circle, the condition $OA - BP < OP$ gives



$$OA - BP + BP < OP + PQ,$$

or, which is the same thing,

$$OQ > OA.$$

Therefore the point Q is without the circle O.

4. Because the point B is within the circle O and the point Q without it, if we move a point along the circle P from B to Q, this point must cross the circle O.

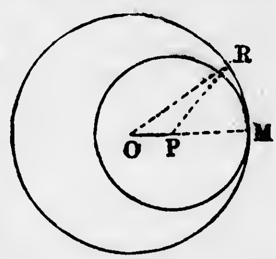
5. But there are two ways in which we can go from B to Q; namely, around either semicircle. Therefore there must be at least two points of intersection of the circles.

6. There cannot be more than two such points, because two circles would then pass through the same three points, which is impossible (§ 241).

Therefore there are two. Q.E.D.

Theorem XXVII.

255. *If the distance of centres of two circles is equal to the difference of their radii, they will touch each other in a single point.*



Hypothesis. O, P, the centres of two circles such that the line OP is equal to the difference of their radii.

Conclusion. These circles touch in a single point, and no more.

Proof. 1. Produce the line OP, and on it take a point M, such that PM shall be equal to the radius of the circle P. M will then be on that circle.

2. Because OP is equal to the difference of the radii, the point M will also be on the circle O.

3. Therefore the point M will be common to both circles.

Now, take any other point R on the greater circle, and join OR and PR. Then—

4. Because OR is a straight line,

$$OP + PR > OR.$$

5. Because OR and OM are radii of the same circle, and $OM = OP + PM$,

$$OR = OP + PM.$$

6. Comparing with (4) and taking away the common part OP,

$$PR > PM.$$

7. Because PM is a radius of the circle P, and PR is greater than this radius, the point R falls without the circle P. (§ 206, Ax. IV.)

8. Therefore every point of the circle P except M falls without the circle O, and M is the only point in common. Q.E.D.

Axiom VI.

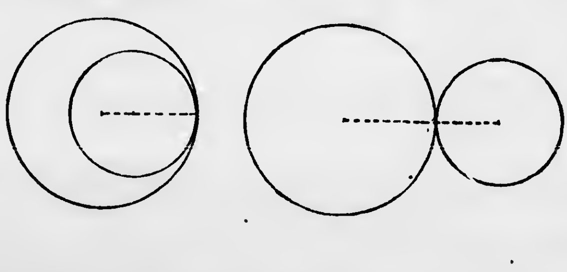
256. *If the distance of centres of two circles is less than the difference of their radii, the lesser will be wholly within the greater, and they will not meet at any point.*

Scholium. The five cases enumerated in the two axioms and three theorems preceding include all possible cases of the relative positions of two circles. For in a plane, whatever be the distance of the centres, it must be either

greater than the sum of the radii,
or equal to the sum of the radii,
or less than the sum and greater than the difference of the radii,
or equal to the difference,
or less than the difference.

We therefore may deduce the following corollaries from the five cases.

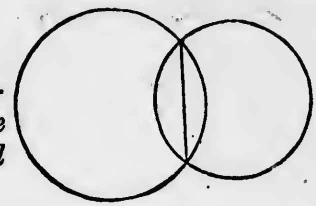
257. *Corollary* 1. Two circles may touch each other externally or internally. In the first case (§ 253) the one circle lies wholly outside the other; in the second case (§ 255) it is wholly inside the other.



258. *Corollary* 2. *When two circles touch each other, the two centres and the point of contact are in the same straight line.*

259. *Corollary* 3. *Two circles cannot touch each other in more than one point, unless they coincide so as to form but one circle.*

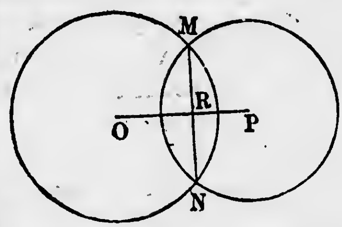
AXIOM VII.



260. *When two circles intersect, the straight line which joins the two points of intersection is a chord of each circle.*

THEOREM XXVIII.

261. *When two circles intersect each other, the straight line joining their centres bisects their common chord at right angles.*



Hypothesis. O, P, the centres of two circles which intersect each other; M, N, their points of intersection.

Conclusion. The line OP bisects the line MN at right angles.

Proof. Let R be the middle point of the chord MN. Through R draw a perpendicular to the chord. Then—

1. Because MN is a chord of the circle P, its perpendicular bisector will pass through the centre P (§ 222).

2. Because MN is a chord of the circle O, its perpendicular bisector will pass through the centre O.

3. Therefore the perpendicular bisector passes through the centres of both circles.

4. But there can be only one straight line between these centres. Therefore the straight line OP bisects MN perpendicularly in the point R. Q.E.D.

262. *Corollary. Conversely, the perpendicular bisector of a common chord passes through the centres of both circles.*

CHAPTER IV.

PROBLEMS RELATING TO THE CIRCLE.

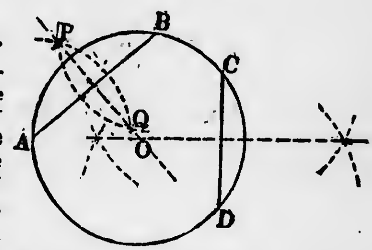
Hereafter we shall generally present with each problem an analysis; that is, a course of reasoning by which the solution may be arrived at. In an analysis we generally begin by supposing the problem solved, and reasoning out the conditions which must thus be fulfilled.

It is expected that the analysis will generally enable the student to supply the proof himself, since the latter will comprise the same reasoning as the analysis, but generally in the reverse order.

PROBLEM I.

263. *To find the centre of a given circle.*

Given. A circle, $ABCD$.



Required. To find its centre.

Analysis. The perpendicular bisector of every chord of the circle passes through the centre (§ 222). Therefore if we draw two such chords and bisect each of them at right angles, the centre will lie on each bisector; that is, it will be their point of intersection. Hence the following

Construction. Draw any two chords of the circle, as AB and CD.

Bisect each of these chords at right angles (§ 179, Cor.).

The point O in which they intersect will be the centre of the circle.

264. *Scholium.* It is not necessary actually to draw the chords. The construction may be found as follows: From any point A on the circle as a centre, with any radius describe the arc of a circle. From any other point B, with the same radius describe another arc intersecting the first at the points P and Q.

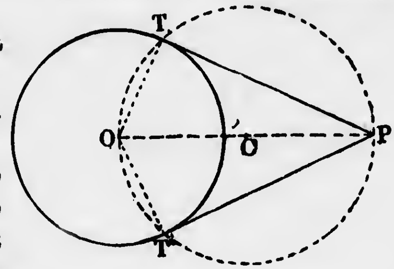
The straight line PQ, produced if necessary, will pass through the centre of the circle.

In the same way another line passing through the centre can be found, and the centre itself will then be their point of intersection.

Corollary. Since we need not use any definite portion of the circle in this construction, we may in the same way find the centre when only an arc of the circle is given. Then from this centre we may describe the whole circle. Hence this construction also enables us to complete a circle of which an arc is given.

Problem II.

265. *From a given point without a circle to draw a tangent to the circle.*



Given. A circle, TCT'; a point P outside of it.

Required. To draw through P a tangent to the circle.

Analysis. Any tangent to the circle is at right angles to the radius drawn to the point of tangency (§225). Therefore

the centre of the given circle,
the point P, and
the point of tangency

will be at the vertices of a right-angled triangle.

But a right-angled triangle is that inscribed in a semicircle (§ 238). Therefore the point of tangency will be on the circle of which OP is a diameter.

Construction. From the centre O of the given circle draw the line OP, and bisect it at the point C.

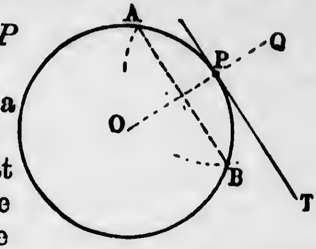
From C as a centre, with the radius $CO = CP$ describe a circle $OTPT'$, intersecting the given circle in T and T'.

Join PT and PT'.

The lines PT and PT' will each be a tangent to the given circle and will pass through P as required.

PROBLEM III.

266. *Through a given point on a circle to draw a tangent to the circle.*



Given. A circle and a point P upon it.

Required. Through P to draw a tangent to the circle.

Analysis. The tangent will be at right angles to the radius from the centre to P (§ 224). Hence we have only to find this radius and draw a perpendicular to it through P.

Construction. From P as a centre, with any radius less than the diameter of the circle, cut off the equal arcs PA, PB.

Join AB.

Bisect AB at right angles by a line OQ.

Through P draw a line perpendicular to OQ and therefore parallel to AB.

This line will be the tangent required.

Proof. 1. Because the line OP bisects the chord AB at right angles, it passes through the centre of the circle and bisects the arc AB (§§ 221, 222).

2. Because the arcs PA and PB are equal, P is the bisecting point of the arc AB. Therefore OQ passes through P.

3. Because PT is perpendicular to this line, it is the tangent required.

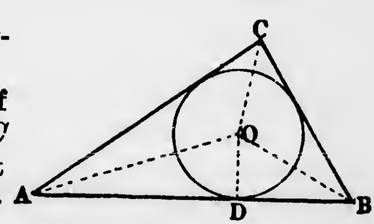
Scholium. The radius OQ is not necessary to the mere construction, since it suffices to draw $PT \parallel AB$.

PROBLEM IV.

267. *To inscribe a circle in a given triangle.*

Given. A triangle, ABC.

Required. To inscribe a circle within it.



Analysis. The bisectors of the three angles A, B, and C meet in a point equally distant from the three sides of the triangle (§ 164). Therefore this point is the centre of the required circle.

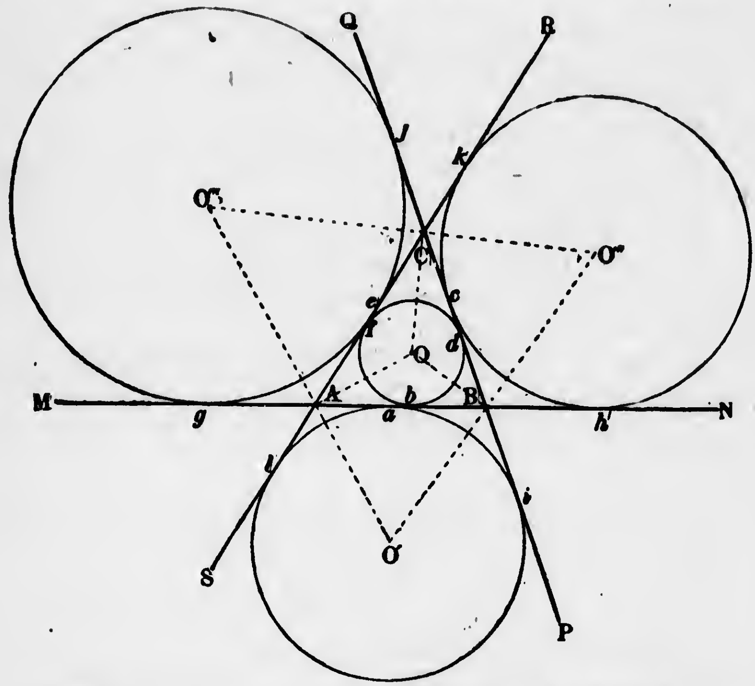
Construction. 1. Bisect any two angles of the triangle as A and B by the lines AO and BO, and let O be their point of meeting.

2. From O drop a perpendicular OD upon any side, as AB.

3. From O as a centre, with the radius OD describe a circle. This circle will be the required inscribed circle.

Proof. The perpendiculars from O upon each of the three sides of the circle are equal (§ 164).

Therefore each of these sides is a tangent to the circle.



268. *Scholium.* In the general triangle (§ 58) there are four circles, each fulfilling the condition of touching the three sides of the triangle. One of these is within the triangle as just described, and the other three are without it, each of them touching one of the sides from without. The latter are called **escribed circles.**

The centre of each escribed circle is on the bisectors of two exterior angles of the triangle. It has been shown that these bisectors meet the bisector of the third interior angle in a point. (Compare § 165.)

Therefore if O', O'', and O''' be the centres of the escribed circles, we shall have

AO'	bisector	of	exterior	angle	SAB.
BO'	"	"	"	"	ABP.
BO''	"	"	"	"	NBC.
CO''	"	"	"	"	BCR.
CO'''	"	"	"	"	QCA.
AO'''	"	"	"	"	CAM.

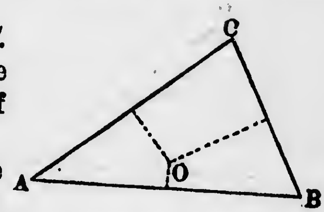
From this the construction of the escribed circles may be deduced by the student.

Problem V.

269. *To describe a circle which shall pass through three given points.*

Given. Three points, A, B, C.

Required. To describe a circle which shall pass through each of them.



Analysis. The centre of the circle must be equally distant from each of the three points. There is only one such centre, which centre lies on the perpendicular bisectors of the lines joining the given points (§241). Therefore the point of intersection of the bisectors is the centre of the required circle, passing through A, B, and C.

Construction. 1. Join AB, BC, CA, forming a triangle.

2. Bisect any two sides of this triangle at right angles by lines meeting in O.

3. From O as a centre, with either of the equal radii OA, OB, or OC describe a circle.

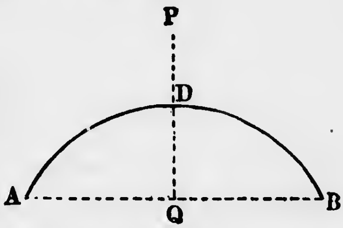
This circle will be that required, passing through the points A, B, and C.

Proof. As in §§ 166, 241.

270. *Corollary.* Since, in the construction of this problem, we describe a triangle having its angles at the given points, this problem is the same as that of circumscribing a circle about a given triangle.

PROBLEM VI.

271. *To bisect a given arc of a circle.*



Given. An arc AB.

Required. To bisect it.

Analysis. 1. The radius which is perpendicular to the chord of the arc bisects both the chord and the arc.

2. The line which bisects the chord at right angles passes through the centre of the circle, so that that part of it which is contained between this centre and the circle itself is a radius.

3. Therefore this line will bisect the arc of the chord.

Construction. 1. Draw the chord AB between the two ends of the given arc.

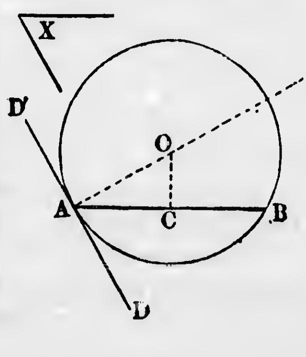
2. Bisect this chord at right angles by the line PQ.

3. Let D be the point in which this bisector intersects the given arc AB.

The point D will bisect the arc.

PROBLEM VII.

272. *Upon a given line as chord, to describe an arc of a circle of which the inscribed angle shall be equal to a given angle.*



Given. A line, AB; an angle, X.

Required. On AB as a chord, to draw an arc of a circle such that any angle inscribed in it shall be equal to X.

Analysis. 1. Suppose the whole circle of which an arc is required to be drawn, and let a tangent DAD' be drawn at one end of the chord AB, suppose A.

2. The angles BAD and BAD' will be equal to the angles in the alternate segments of the circle (§§ 234, 235).

3. Therefore the required circle will be one of which the tangent at A makes an angle equal to X with the chord AB.

4. The perpendicular to the tangent at A and the perpendicular bisector of the chord AB will both pass through the centre of the circle (§§ 222, 226). Hence

Construction. 1. At one extremity, as A, of the line AB make an angle BAD equal to the given angle X (§ 184).

2. At A erect a perpendicular AO to the line AD.

3. Bisect the line AB at right angles by the line CO.

The point of intersection O will be the centre of the required arc, which may be drawn with the radius AO.

If the given angle is acute, the required arc will be that on the opposite side of AB from O.

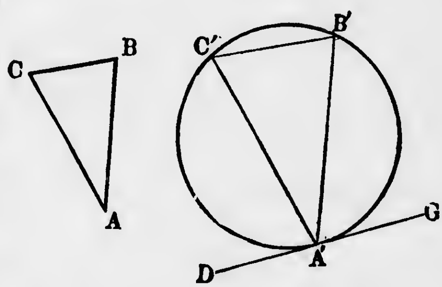
If the given angle is a right angle, the line AO will coincide with AB, and the line AB will be a diameter of the circle, and C its centre.

Problem VIII.

273. *In a given circle to inscribe a triangle which shall be equiangular to a given triangle.*

Given. A circle; a triangle, ABC.

Required. To inscribe in the circle a triangle which shall have its angles equal to the respective angles A, B, C, of the triangle ABC.



Analysis. 1. Suppose the triangle inscribed, and let it be $A'B'C'$. At A' draw a tangent DG to the circle.

2. We shall then have

$$\left.\begin{array}{l}\text{Angle } DA'C' = \tfrac{1}{2} \text{ angle arc } A'C' = \text{angle } B'; \\ \text{Angle } GA'B' = \tfrac{1}{2} \text{ angle arc } A'B' = \text{angle } C'.\end{array}\right\} \quad (\S\S\ 234,\ 239)$$

3. Therefore the three angles of the triangle will be the same as the three angles $DA'C'$, $C'A'B'$, $B'A'C$. Hence

Construction. 1. Draw a tangent to the circle at any point A'.

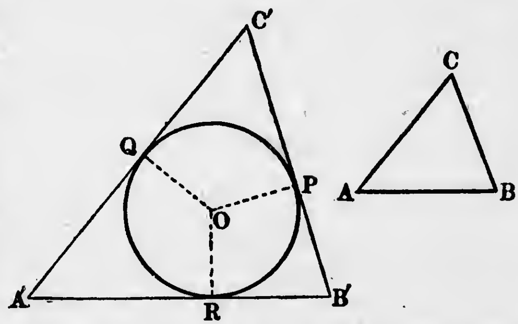
2. At the point of tangency A make the angle $DA'C'$ equal to the angle B of the given triangle; the angle $C'A'B'$ equal to A; when the angle $B'A'G$ will be equal to C (§ 73).

3. Produce the sides until they intersect the circle in B' and C', and join $B'C'$

$A'B'C'$ will be the required inscribed triangle.

Problem IX.

274. *About a given circle to circumscribe a triangle which shall be equiangular to a given triangle.*



Given. A triangle, ABC; a circle, PQR.

Required. To circumscribe about PQR a triangle which shall have angles equal to A, B, C, respectively.

Analysis.

1. Suppose the problem solved. Let P, Q, R, be the points in which the sides of the required triangle touch the circle, and let O be the centre of the circle.

2. Join OP, OQ, OR.

3. In the quadrilateral $OPC'Q$ we have

$$\left.\begin{array}{l}\text{Angle } OQC' = \text{right angle;}\\ \text{Angle } OPC' = \text{right angle.}\end{array}\right\} \qquad (\S\ 225)$$

And because the sum of all four angles is equal to four right angles, if we take away these two angles from all four angles we have left

$$\text{Angle } QOP + \text{angle } QC'P = \text{two right angles.}$$

4. Therefore QOP is the supplement of C', or of its equal, C; and in the same way POR is the supplement of B, and ROQ of A.

5. But the exterior angles of the given triangle are the supplements of their respective interior angles, and therefore equal to the respective angles around the centre O. Hence

Construction. 1. Produce each side of the given triangle so as to form its three exterior angles.

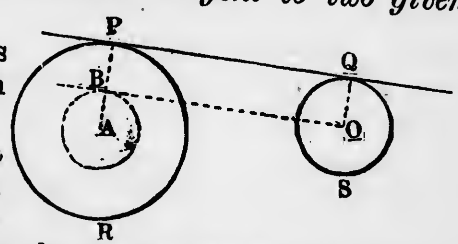
2. From the centre of the given circle draw three radii making with each other angles respectively equal to these three exterior angles.

3. At the end of each radius draw a tangent to the circle, and produce these tangents until they form a triangle.

This triangle will be that required.

PROBLEM X.

275. *To draw a common tangent to two given circles.*



Given. Two circles PR and QS, of which QS is the lesser.

Required. To draw a straight line which shall be tangent to both circles.

Analysis. 1. Suppose the required tangent to be drawn, and to touch the circles at the respective points P and Q.

2. Let A and O be the centres of the circles.

3. Draw the radii AP and OQ. These radii will be perpendicular to the required tangent (§ 225), and therefore parallel to each other (§ 70).

4. Through the centre O suppose a line to be drawn parallel to PQ, and let B be the point in which it intersects the radius AP.

5. Because BO is perpendicular to this radius, it will be a tangent to this circle.

6. Because BP and OQ are parallel and contained between the parallels BO and PQ, they are equal (§ 126).

Therefore AB is equal to the difference between the radii AP and OQ. Hence

Construction. 1. Around the centre A of the greater circle, with a radius AB equal to the difference of the radii of the two circles, draw a circle.

2. Through the centre O of the smaller circle draw a tangent to the inner circle, and let B be the point of tangency.

3. Join AB, and produce the joining line to P.

4. From the centre O draw the line OQ parallel to AP and meeting the circle in Q.

5. Join PQ. The line PQ will then be one of the tangents required.

Proof. To be supplied by the student from the analysis.

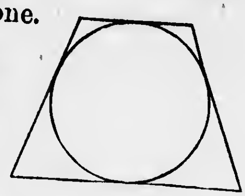
EXERCISES.

THEOREM 1. If two chords be drawn in a circle intersecting each other at right angles, their ends will divide the circle into four arcs. Show that the sum of each pair of opposite arcs is a semicircle.

How will the theorem be modified if the chords do not intersect within the circle?

THEOREM 2. If, from the two ends of a chord, chords perpendicular to it be drawn, they will be equal in length.

THEOREM 3. The shortest line between two concentric circles is part of the radius of the outer one.



THEOREM 4. If the angles at the base of a circumscribed trapezoid are equal, each non-parallel side is equal to half the sum of the parallel sides (§ 227).

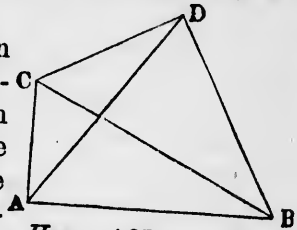
THEOREM 5. If from the centre of a circle a perpendicular be dropped upon either side of an inscribed triangle, and a radius be drawn to one end of this side, the angle between the radius and perpendicular will be equal to the opposite angle of the triangle.

THEOREM 6. If two equal chords intersect, the segments of the one are respectively equal to the segments of the other.

THEOREM 7. The only parallelogram which can be inscribed in a circle is a rectangle.

THEOREM 8. From any point outside a circle a chord subtends an angle less than half its arc; from any point inside the circle an angle greater than half its arc.

Explain the relation of the two conjugate arcs into which the chord divides the circle to the side on which the vertex of the subtended angle lies.

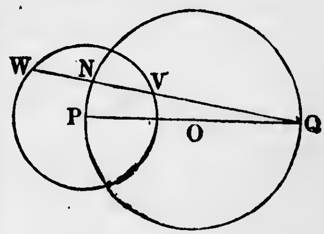


Hyp. $ACB = ADB$.
Conc. $CAD = CBD$.
$CDA = CBA$.
$DAB = DCB$.
$ABCD$ on a circle.

THEOREM 9. If an angle between a diagonal and one side of a quadrilateral is equal to the angle between the other diagonal and the opposite side, the same will be true of the three other pairs of angles corresponding to the same description, and the four vertices of the quadrilateral lie on a circle (§ 237).

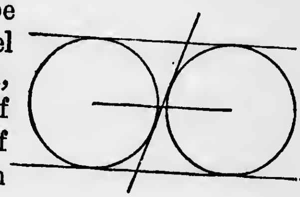
THEOREM 10. A circle described around one of the equal sides of an isosceles triangle as a diameter bisects the base.

THEOREM 11. Two chords of a circle cannot mutually bisect each other unless they are both diameters.

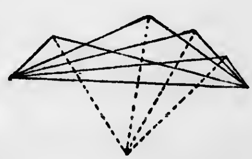


Conclusion. $WN = NV$.

THEOREM 12. If a circle O pass through the centre of another circle P, and from the centre of P a diameter to the circle O be drawn, every chord of P passing (when produced) through the other end Q of this diameter is bisected by the circle O (§§ 221, 238).



THEOREM 13. If two circles be drawn each touching a pair of parallel lines and a transversal crossing them, the distance between the centres of the circles is equal to the length of the transversal intercepted between the parallels (§ 227.)



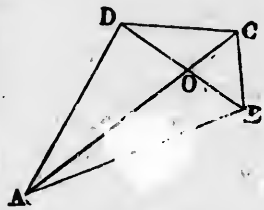
THEOREM 14. If any number of triangles have the same base and equal angles at the vertices, the bisectors of these angles pass through a point (§ 237).

How is this point defined?

THEOREM 15. Of all chords passing through a given point within a circle, the least is that which is bisected by the point.

THEOREM 16. The centres of the four circles circumscribed about the four triangles formed by the sides and diagonals of a quadrilateral lie on the vertices of a parallelogram (§ 166).

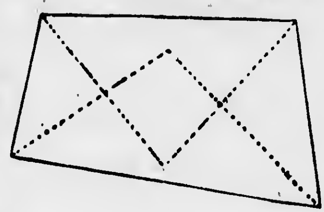
Define the lengths of the sides of this parallelogram and its angles.



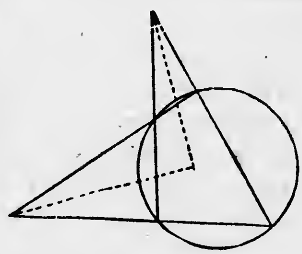
The four circles circumscribed about the triangles *AOB, BOC, COD, DOA*, have their centres on the vertices of a parallelogram.

THEOREM 17. The tangents at the four vertices of an inscribed rectangle form a rhombus.

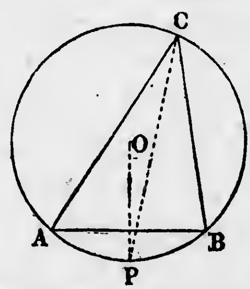
THEOREM 18. The quadrilateral formed by the bisectors of the four angles of another quadrilateral has its four vertices on a circle.



THEOREM 19. If each pair of opposite sides of an inscribed quadrilateral be produced until they meet, the bisectors of the angles formed at the point of meeting will be perpendicular to each other.

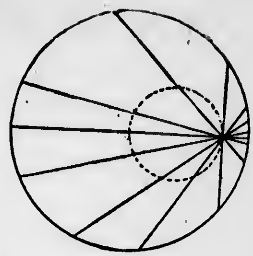


THEOREM 20. If the arc cut off by the base of an inscribed triangle be bisected, and from the point of bisection be drawn a radius and a line to the opposite vertex, the angle between these lines will be half the difference of the angles at the base of the triangle.

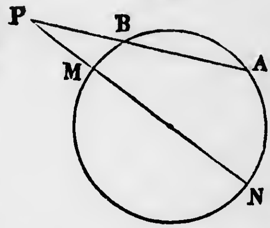


THEOREM 21. If perpendiculars be dropped from the ends of a diameter upon any secant, their feet will be equally distant from the points in which the secant intersects the circle.

THEOREM 22. The middle points of all chords passing through a fixed point lie on a circle (§§ 136, 238).



THEOREM 23. If a chord be extended by a length equal to the radius, and from the end a secant be drawn through the centre of the circle, the greater included arc will be three times the lesser.



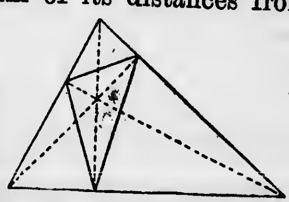
Hyp. $PB =$ radius.
Conc. Arc $AN = 3$ arc BM.

THEOREM 24. If a chord be produced equally each way, and from its ends tangents be drawn to the circle on opposite sides, the line joining the point of tangency will bisect the chord.

THEOREM 25. In a right-angled triangle the sum of the hypothenuse and the diameter of the inscribed circle is equal to the sum of the two sides.

THEOREM 26. If lines be drawn from the centre of a circle to the vertices of any circumscribed quadrilateral, each pair of opposite angles at the centre will be supplementary.

THEOREM 27. If an equilateral triangle be inscribed in a circle, the distance of any point on the circle from the farther side of the triangle is equal to the sum of its distances from the two nearer sides.



THEOREM 28. If in any triangle the feet of the perpendiculars from the angles upon the opposite sides be joined, the three angles of the new triangle thus formed will be bisected by the perpendiculars.

BOOK IV.

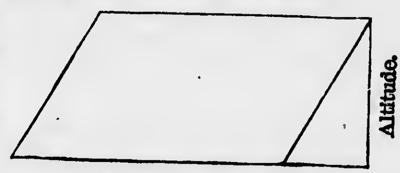
OF AREAS.

Definitions.

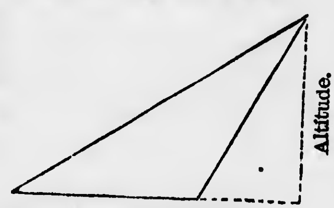
276. *Base and Altitude.* *Def.* The **base** of a figure is that one of its sides on which we conceive it to rest.

Any side of a figure may be taken as its base.

277. *Def.* The **altitude** of a figure is the perpendicular distance of its highest point above its base.



The altitude of a parallelogram is the length of the perpendicular dropped from any point of one side to the opposite side, produced if necessary. The latter side is then considered as the base.

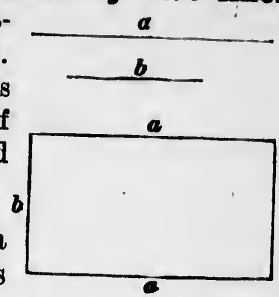


The altitude of a triangle is the length of the perpendicular dropped from either angle to the opposite side, produced if necessary. The latter side is then considered as the base.

The terms *base* and *altitude* are therefore correlative, altitude meaning a perpendicular distance from the base.

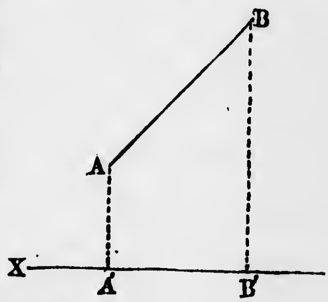
278. *Def.* The **perimeter** of a polygon is the combined length of all its sides.

279. *Def.* A rectangle **contained by two lines** is a rectangle of which two adjacent sides are equal to these lines.



The rectangle contained by the lines a and b is that in which one pair of opposite sides are each equal to a, and the other pair to b.

280. *Def.* The **projection** of a finite line upon an indefinite line is the distance between the perpendiculars dropped from the ends of the finite line upon the indefinite line.



EXAMPLE. $A'B'$ is the projection of the line AB upon the indefinite line X.

281. *Def.* The **area** of a plane figure is the extent of surface of which it forms the boundary.

A plane figure cannot have an area unless it completely incloses a portion of the plane in which it lies.

EXAMPLE 1. An angle has no area.

Ex. 2. Two parallel lines do not inclose an area.

Ex. 3. But polygons and circles have areas.

282. In elementary geometry we regard the area as the measure of the figure, the latter being thought of as a disk, or portion of the plane bounded by the lines which form the figure. Hence

The term **equal figures** means figures of equal area.

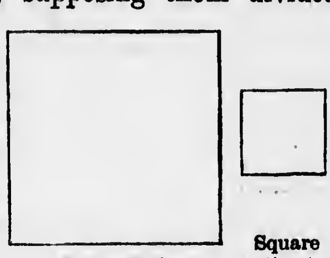
To **divide** a figure into two or more parts means to divide its area.

CHAPTER I.

AREAS OF RECTANGLES.

283. Areas are measured by supposing them divided up into units.

To form a unit of area, we take any unit of length and erect a square upon it. The area of this square is the corresponding unit of area.



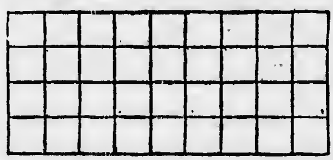
Square inch. Square centimetre.

We may thus form a square millimetre, a square centimetre, a square inch, etc.

LEMMA.

284. *The number of units of area in a rectangle is equal to the product of the numbers of units of length in its containing sides.*

Proof for whole numbers. In the figure let the base be m units in length, and the vertical sides each n units. Then if we divide the sides into units of length and join all the corresponding division points by straight lines, the whole rectangle will be divided up into units of area. It is evident that there will be in the whole rectangle n rows, each containing m units. Therefore the whole number of units will be mn.



285. *Corollary. The area of a square of which each side contains m units is m^2 units.*

NOTE. The above demonstration presupposes that each side of the rectangle contains a whole number of units. The general case in which the sides contain fractions of a unit must be deferred until after the subject of proportion is taught.

286. *Scholium.* The preceding result is sometimes expressed by saying that the area of a rectangle is equal to the product of two of its containing sides. But when we speak of the product of two lines, what we really mean is the product of the number of units which the lines contain. This product is always equal to the number of square units in the rectangular area. Hence we may consider such an area as a kind of product of lines, and for shortness use a form of algebraic product to represent it, as follows. Instead of saying

The rectangle contained by the lines AB and AD,

we may say

Rectangle $AB \cdot AD$,

or *Area* $AB \cdot AD$,

or simply $AB \cdot AD$.

If the lines are represented by single letters, as a, b, we may write simply ab

to express the area of the rectangle.

To these expressions of algebraic products we may assign either a geometric or an algebraic signification.

Geometrically, the expression

$$AB \cdot CD$$

means *the area of the rectangle contained by the lines AB and CD.* This meaning may be considered independently of any idea of a product.

Algebraically, the same expression means *the product of the number of units in CD by the number of units in AB.*

Since this product is equal to the number of units of area in the rectangle, the two meanings are entirely consistent.

Again, geometrically, the expression

$$AB^2$$

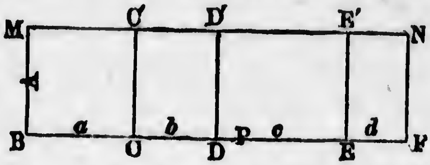
means *the area of the square erected upon the line AB.*

This meaning may be assigned without any idea of the product $AB \times AB$, which is the algebraic meaning of the expression.

NOTE. Since the following theorems are designed to be primarily of geometrical application, we should write or understand the word "area" before the symbols of products, to show that the student is expected to think of them geometrically.

THEOREM I.

287. *The area of the rectangle formed by two lines, one of which is divided into several parts, is equal to the sum of the areas of all the rectangles formed by the undivided line and the several parts of the divided line.*



Hypothesis. A line, $BM = A$; another line, $BF = P$, divided into the parts a, b, c, d, etc., at the points C, D, E, etc.

Conclusion. Area $A.P$ = area $(A.a + A.b + A.c + A.d)$.

Proof. On BF erect the rectangle $BFNM$, of which the sides BM, FN, shall each be equal to the line A, and MN and BF to P. At C, D, E, etc., erect the perpendiculars CC', DD', EE', etc., meeting MN in C', D', E', etc. Then—

1. Because the angles at C, D, E, etc., are all right angles, each of the quadrilaterals $MBCC'$, $C'CDD'$, etc., is a rectangle.

2. The sum of all these rectangles is equal to the rectangle $BM \,.\, BF$, by construction.

3. Therefore area $BM \,.\, BF$ = sum of areas $A \,.\, a$, $A \,.\, b$, etc., or, because $BM = A$, and $BF = P$,

Area $A.P$ = area $(A.a + A.b + A.c + A.d)$. Q.E.D.

Scholium. When we give the symbols A, a, b, c, d, which we have supposed to represent lines, their algebraic signification, we have

$$P = a + b + c + d,$$

which gives the well-known formula

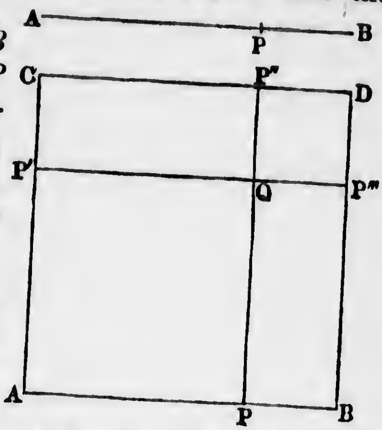
$$A\,(a + b + c + d) = Aa + Ab + Ac + Ad,$$

and expresses the distributive law in multiplication.

THEOREM II.

288. *If a straight line be made up of two parts, the square of the whole line is equal to the sum of the squares of the two parts plus twice the rectangle contained by the parts.*

Hypothesis. AB, a straight line divided at P into the parts AP and PB.



Conclusion. Square on AB equals sum of squares on AP and PB plus twice the rectangle $AP \,.\, PB$. Or, in symbols, $AB^2 = AP^2 + PB^2 + 2AP.PB$.

Proof. On AB erect the square $ABCD$.

On AC take $AP' = AP$.

Through P draw PP'' parallel to AC, and through P' draw $P'P'''$ parallel to AB. Then—

1. Taking away from the equal lines AB, AC, the equal parts AP, AP', we have

$$PB = P'C.$$

2. Because the lines AB, $P'P'''$, and CD are all parallel, and also the lines AC, PP'', and BD, the quadrilaterals AQ, CQ, DQ, and BQ are parallelograms.

3. Because the angles at A, B, C, and D are all right angles (by construction), each of these parallelograms has a right angle. Therefore they are all rectangles (§ 125).

4. Because of the parallelism of the lines AC, PP'', and BD, and also of the lines AB, $P'P'''$, and CD, we have

$$CP'' = P'Q = AP.$$
$$BP''' = PQ = AP' = AP. \qquad (§\ 126)$$
$$P''D = QP''' = PB.$$
$$P'''D = QP'' = P'C. \qquad (1)$$

5. Therefore Area $APQP' = AP^2$.

Area $QP'''DP'' = PB^2$.

Area $P'QP''C = AP \,.\, PB$.

Area $PBP'''Q = AP \,.\, PB$.

6. But these four areas make up the area of the square on AB. Therefore

$$AB^2 = AP^2 + PB^2 + 2AP \,.\, PB. \quad \text{Q.E.D.}$$

Corollary. By transposition we have, from this last equation,

$$AB^2 - AP^2 - PB^2 = 2AP \,.\, PB.$$

Hence:

289. *If from the square of the sum of two lines we take away the sum of their squares, we shall have left twice their rectangle.*

290. *Scholium.* By hypothesis we have

$$AB = AP + PB.$$

Substituting this in the conclusion, we have

$$(AP + PB)^2 = AP^2 + 2AP \,.\, PB + PB^2,$$

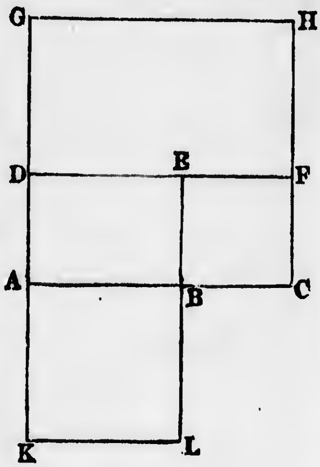
a well-known algebraic expression.

The geometric construction serves to exhibit to the eye the different parts of which this algebraic expression is made up.

Theorem III.

291. *The square upon the difference of two lines is equal to the sum of the squares upon the lines, diminished by twice the rectangle contained by them.*

Hypothesis. AB, AC, two lines of which AC is the longer; BC, their difference.



Conclusion.

$BC^2 = AB^2 + AC^2 - 2AB \,.\, AC.$

Proof. On AC erect the square $ACGH$.

On BC erect the square $BCEF$.

On AB erect the square $ABKL$.

Produce FE till it meets AG in D. Then—

1. The whole area $AKLBCHG = AB^2 + AC^2$.

2. Because $EB = BC$, and $BL = AB$, we have

$$EL = AB + BC = AC.$$

Therefore

$$\text{Area } KLDE = \text{area } AB \,.\, AC.$$

3. Because $CH = AC$, and $CF = BC$, we have

$$FH = AC - BC = AB.$$

Hence

$$\text{Area } DFGH = \text{area } AB \,.\, AC.$$

4. If from the whole area (1) we take away the areas (2) and (3), we have left the square $BCEF$; that is, BC^2. Therefore

$$BC^2 = AB^2 + AC^2 - 2AB \, . \, AC. \quad \text{Q.E.D.}$$

292. *Scholium.* Since $BC = AC - AB$, we have

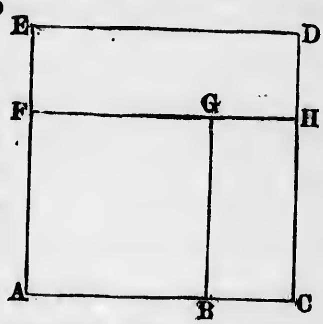
$$(AC - AB)^2 = AC^2 - 2AB \, . \, AC + AB^2,$$

the algebraic formula for the square of the difference of two numbers.

Theorem IV.

293. *The difference of the squares described on two lines is equal to the rectangle contained by the sum and difference of the lines.*

Hypothesis. AB, AC, two straight lines of which AC is the greater, and each of which is to have a square described upon it.



Conclusion.

$$AC^2 - AB^2 = (AC + AB)(AC - AB).$$

Proof. On AC describe the square $ACDE$. On AE take $AF = AB$. From F draw FH parallel to AC, meeting CD in H, and from B draw BC parallel to AE and meeting FH in G. Then—

1. In the same way as in the last theorem it may be shown that EH and GC are rectangles, and AG a square.

2. Because $AE = AC$ (by construction), and $AF = AB$, FE is equal to $AC - AB$, and ED is, by construction, equal to AC. Therefore

$$\text{Rectangle } EH = AC\,(AC - AB).$$

3. Because FH and AC are parallel, $CH = AF = AB$, while BC is, by construction, equal to $AC - AB$. Therefore

$$\text{Rectangle } GC = AB\,(AC - AB).$$

4. The sum of the rectangles $AC\,(AC - AB)$ and $AB\,(AC - AB)$ = rectangle $(AC + AB)\,(AC - AB)$. (§ 287)

5. The difference between the squares on the lines AB and AC is made up of the sum of these rectangles.

Therefore

$$AC^2 - AB^2 = (AC + AB)(AC - AB). \quad \text{Q.E.D.}$$

294. *Scholium.* Expressing the areas of the squares and rectangles in algebraic language, this theorem gives

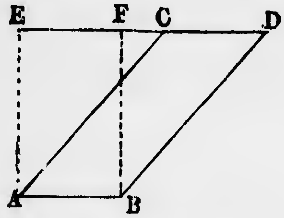
$$a^2 - b^2 = (a + b)(a - b).$$

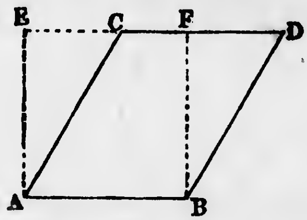
CHAPTER II.

AREAS OF PLANE FIGURES.

THEOREM V.

295. *The area of a parallelogram is equal to that of the rectangle contained by its base and its altitude.*





Hypothesis. $ABCD$, any parallelogram of which the side AB is taken as the base; AE, the altitude of the parallelogram.

Conclusion. Area $ABCD$ = rectangle $AB \cdot AE$.

Proof. From A and B draw perpendiculars to the base AB, meeting CD produced in E and F. Then—

1. Because $ABCD$ and $ABEF$ are both parallelograms,

$$EF = AB, \text{ and } CD = AB. \qquad (\S\ 127)$$

Therefore $$EF = CD.$$

$$BF = AE.$$

$$BD = AC.$$

2. If from the line ED we take away EF, FD remains; and if we take away CD, EC remains. Because the parts taken away are equal (1),

$$FD = EC.$$

3. Comparing with the last two equations of (1), it is seen that the triangles BFD and AEC have the three sides of the one equal to the three sides of the other.

Therefore

$$\text{Triangle } AEC = \text{triangle } BFD.$$

4. From the trapezoid $ABED$ take away the triangle AEC, and there is left the parallelogram $ABCD$. From the same trapezoid take away the equal triangle BFD, and there is left the rectangle $ABEF$. Because the triangles are equal,

$$\text{Rectangle } ABEF = \text{parallelogram } ABCD.$$

Therefore

$$\text{Area } ABCD = \text{rectangle } AB \,.\, AE. \quad \text{Q.E.D.}$$

296. *Corollary* 1. *All parallelograms upon the same base and between the same parallels are equal in area, because they are all equal to the same rectangle.*

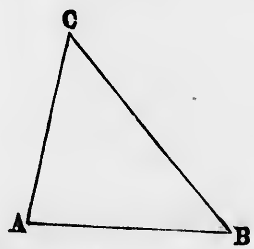
297. *Cor.* 2. *Parallelograms having equal bases and equal altitudes are equal in area.*

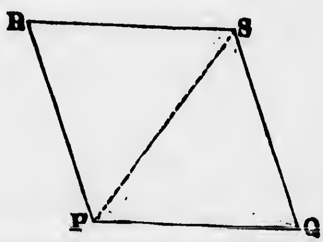
298. *Cor.* 3. *Of two parallelograms having equal bases, that has the greater area which has the greater altitude.*

Of parallelograms having equal altitudes, that has the greater area which has the greater base.

THEOREM VI.

299. ***The area of a triangle is equal to half the area of the parallelogram formed from any two of its sides, having an angle equal to that between those sides.***





Hypothesis. ABC, any triangle; $PQRS$, a parallelogram in which

$$PQ = RS = AB.$$
$$PR = QS = AC.$$
$$\text{Angle } PQS = \text{angle } CAB.$$

Conclusion. Area $ABC = \frac{1}{2}$ area $PQRS$.

Proof. Draw the diagonal PS. Then—

1. Because of the equations supposed in the hypothesis, the triangles PQS and ABC have two sides and the included angle of the one equal to two sides and the included angle of he other. Therefore the triangles are identically equal, and

Area ABC = area PQS. (§ 108)

2. In the same way is shown

Area ABC = area PRS.

3. The sum of the areas PQS and PRS makes up the whole area of the parallelogram $PQRS$. Therefore, comparing (1) and (2),

Area $ABC = \frac{1}{2}$ area $PQRS$. Q.E.D.

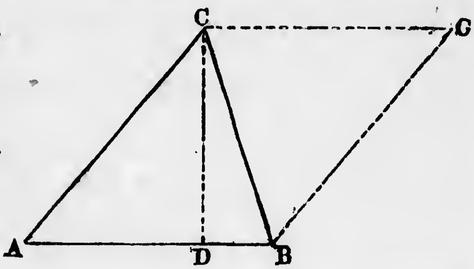
300. *Corollary. A diagonal of a parallelogram divides it into two triangles of equal area.*

Theorem VII.

301. *The area of a triangle is one half the area of the rectangle contained by its base and its altitude.*

Hypothesis. ABC, a triangle having the base AB and the altitude CD.

Conclusion. Area $ABC = \frac{1}{2}$ rect. $AB.CD$.



Proof. Through B draw BG parallel to AC, and through C draw CG parallel to AB, meeting BG in G. Then—

1. $ABCG$ is a parallelogram having the base AB and the altitude CD. Therefore

Area $ABCG$ = rect. AB . CD. (§ 295)

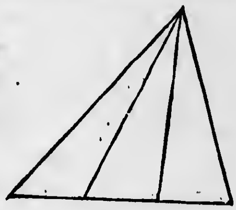
2. Because AB and CD are each sides of this parallelogram,

Area $ABC = \frac{1}{2}$ area $ABGC$. (§ 299)

3. Comparing (1) and (2),

$$\text{Area } ABC = \tfrac{1}{2} \text{ rect. } AB \cdot CD. \quad \text{Q.E.D.}$$

302. *Corollary* 1. *All triangles on the same base, having their vertices in the same straight line parallel to the base, are equal to each other in area.*

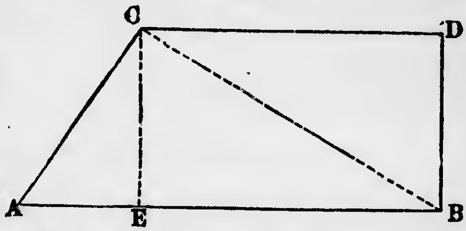


303. *Cor.* 2. *If several triangles have their vertices in the same point, and their bases equal segments of the same straight line, they are equal in area.*

304. *Cor.* 3. *If a triangle and a parallelogram stand upon the same base and between the same parallels, the area of the parallelogram will be double that of the triangle.*

THEOREM VIII.

305. *The area of a trapezoid is equal to that of the rectangle contained by its altitude and half the sum of its parallel sides.*



Hypothesis. $ABCD$, a trapezoid of which the sides AB and CD are parallel; CE, the altitude of the trapezoid.

Conclusion. Area $ABCD = \frac{1}{2}(AB + CD) \cdot CE$.

Proof. Draw either diagonal of the trapezoid, say BC. Then—

1. Because ABC is a triangle having AB as its base and CE as its altitude,

$$\text{Area } ABC = \tfrac{1}{2}AB \cdot CE. \qquad (\S\ 301)$$

2. Because BCD is a triangle having CD as a base and an altitude equal to the distance of the vertex B from CD—that is (because AB and CD are parallel), to CE—

$$\text{Area } BCD = \tfrac{1}{2}\, CD \cdot CE.$$

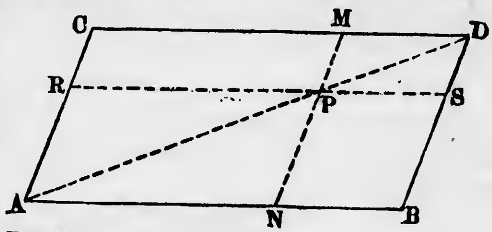
3. The sum of these areas makes up the whole area of the trapezoid. Therefore

$$\text{Area of trapezoid} = \tfrac{1}{2}AB \cdot CE + \tfrac{1}{2}CD \cdot CE$$
$$= \tfrac{1}{2}(AB + CD)\, CE \ (\S\ 187). \quad \text{Q.E.D.}$$

THEOREM IX.

306. *If through any point on the diagonal of a parallelogram two lines be drawn parallel to the sides, the two parallelograms on each side of the diagonal will be equal.*

Hypothesis. $ABCD$, a parallelogram; P, any point on the diagonal AD; RS, MN, lines passing through P, parallel to AB and AC respectively, and meeting the four sides in the points R, S, M, N.



Conclusion. Area $RPMC$ = area $NBSP$.

Proof. 1. Because the lines AD, AP, and PD are the diagonals of the respective parallelograms $ABCD$, $ANRP$, and $PSMD$, we have

Area ACD = area ABD.
Area ARP = area ANP; } (§ 300)
Area PMD = area PSD. }

2. From the area ACD take away the areas ARP and PMD, and we have left the area $RPMC$.

3. From the equal area ABD take away the equal areas ANP and PSD, and we have left the area $NBSP$. Therefore

Area $RPMC$ = area $NBSP$. Q.E.D.

307. *Definition.* In the foregoing constructions the parallelograms $ANRP$ and $PSMD$ are called **parallelograms about the diagonal** AD.

$RPCM$ and $NBPS$ are called the **complements of parallelograms about the diagonal** AD.

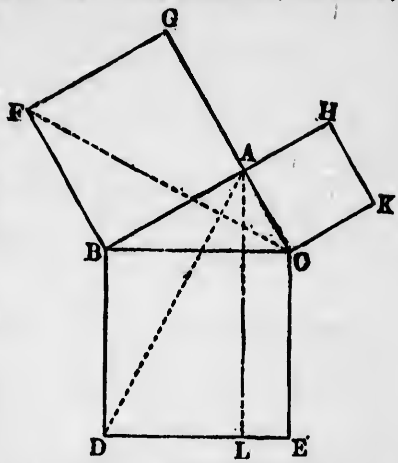
THEOREM X.

308. *In a right-angled triangle the square of the hypothenuse is equal to the sum of the squares of the other two sides.*

Hypothesis. ABC, a triangle, right-angled at A; $BAGF$, $ACKH$, $BCED$, squares on its respective sides.

Conclusion.

Area $BAGF$ + area $ACKH$ = area $BCED$.



Proof. Through A draw AL parallel to BD and CE, meeting DE in L. Join FC and AD.

The proof will now be arranged as follows:

We shall show (1) that the triangles FBC and ABD are identically equal; (2) that the area $BAGF$ is double that of the triangle FBC; (3) that the area BL is double that of the equal triangle ABD. From this will follow area $ABGF$ = area BL. It may be shown in the same way that the area of the square on AC is equal to that of the rectangle CL, from which the theorem will follow.

1. In the triangles ABD and FBC we have

$$\left.\begin{aligned} BA &= BF, \\ BD &= BC, \end{aligned}\right\} \text{by hypothesis.}$$

Angle DBA = right angle DBC + angle ABC.
Angle FBC = right angle FBA + angle ABC.

Therefore the two triangles having two sides and the included angle of the one equal to two sides and the included angle of the other are identically equal, so that

$$\text{Area } ABD = \text{area } FBC.$$

2. Because BAG and BAC are both right angles (hypothesis), GA and AC are in the same straight line.

Therefore the triangle FBC is on the same base FB, and between the same parallels FB and GC, as the square $BAGF$. Therefore

$$\text{Area } BAFG = 2 \text{ area } FBC. \qquad (\S\ 304)$$

3. Because AL is (by construction) parallel to BD, the triangle ABD is upon the same base BD, and between the

same parallels BD and AL, as the rectangle BL. Therefore

$$\text{Area } BL = 2 \text{ area } ABD.$$

4. Comparing (2) and (3) with (1),

$$\text{Area } BAGF = \text{area } BL.$$

5. In the same way, by joining BK and AE, it may be shown that

$$\text{Area } ACKH = \text{area } CL.$$

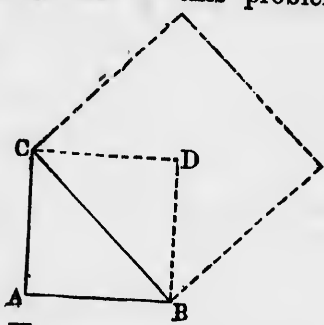
6. Adding (4) and (5),

$$\text{Area } (BAGF + ACKH) = \text{area } (BL + CL)$$
$$= \text{square } BCED. \quad \text{Q.E.D.}$$

309. *Scholium.* This proposition is called the *Pythagorean proposition,* because it is said to have been discovered by Pythagoras, who sacrificed a hecatomb of oxen in gratitude for so great a discovery. It is one of the most important propositions in geometry, as upon it is founded a great part of the science of measurement. It also furnishes the basis of trigonometry.

Corollary. An important special case of this problem occurs when the two sides of the triangle are equal, or when $AB = AC$. Since the squares on AB and AC are then equal, we have

$$BC^2 = AB^2 + AC^2 = 2AB^2.$$



If we complete the square by drawing BD parallel to AC and CD parallel to AB, $ABCD$ will be a square, and BC its diagonal. Hence:

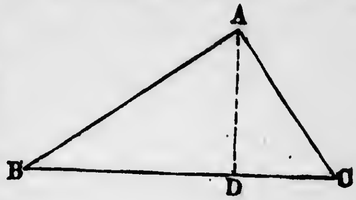
310. *The square on the diagonal of a square is double the square itself.*

THEOREM XI.

311. *If from the right angle of a right-angled triangle a perpendicular be dropped upon the hypothenuse, the square of this perpendicular will be equal to the rectangle of the two parts of the hypothenuse.*

Hypothesis. ABC, a triangle, right-angled at A; AD, a perpendicular from A on BC.

Conclusion. $AD^2 = BD \cdot DC$.



Proof. 1. Because BAD and CAD are both right-angled at D,

$$\left.\begin{array}{l} AB^2 - BD^2 = AD^2; \\ AC^2 - DC^2 = AD^2. \end{array}\right\} (\S 308)$$

2. Adding these two equations,

$$AB^2 + AC^2 - BD^2 - DC^2 = 2AD^2.$$

3. Because BAC is right-angled at A,

$$AB^2 + AC^2 = BC^2. \qquad (\S\ 308)$$

4. Comparing (2) and (3),

$$BC^2 - BD^2 - DC^2 = 2AD^2.$$

5. Because the line BC is the sum of the lines BD and DC,

$$BC^2 - BD^2 - DC^2 = 2BD \cdot DC. \qquad (\S\ 289)$$

6. Comparing (4) and (5),

$$2AD^2 = 2BD \cdot DC,$$

or

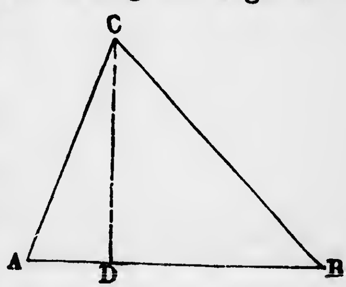
$$AD^2 = BD \cdot DC. \quad \text{Q.E.D.}$$

Theorem XII.

312. *The square on a side opposite any acute angle of a triangle is less than the sum of the squares on the other two sides by twice the rectangle contained by either of those sides and the projection of the other side upon it.*

Hypothesis. ABC, any triangle having the angle at A acute; CD, the perpendicular dropped from C on D, and therefore

AD the projection of AC on AB.



Conclusion.

$$BC^2 = AC^2 + AB^2 - 2AB.AD.$$

Proof. 1. Because CDB is right-angled at D,

$$BC^2 = BD^2 + CD^2. \quad (\S\ 308)$$

2. Because ACD is right-angled at D,

$$CD^2 = AC^2 - AD^2.$$

3. Putting this value of CD^2 in (1),

$$BC^2 = BD^2 - AD^2 + AC^2.$$

4. $BD^2 - AD^2 = (BD + AD)(BD - AD)$, (§ 293)
$= AB(BD - AD)$, (§ 287)
$= AB(BD + AD - 2AD)$,
$= AB(AB - 2AD)$,
$= AB^2 - 2AB \cdot AD$. (§ 287)

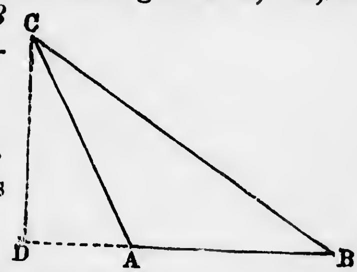
5. Substituting this last value in (3),

$$BC^2 = AC^2 + AB^2 - 2AB \cdot AD. \quad \text{Q.E.D.}$$

THEOREM XIII.

313. *In an obtuse-angled triangle the square on the side opposite the obtuse angle is greater than the sum of the squares on the other two sides by twice the rectangle contained by either of those sides and the projection of the other side upon it.*

Hypothesis. ABC, a triangle, obtuse-angled at A; CD, the perpendicular from C upon AB produced, so that DA is the projection of CA on AB.



Conclusion.

$$BC^2 = AC^2 + AB^2 + 2AB.AD.$$

Proof. 1. Because CDB is right-angled at D,

$$BC^2 = BD^2 + CD^2.$$

2. Because CDA is right-angled at D,

$$CD^2 = AC^2 - AD^2.$$

3. Putting this value of CD^2 in (1),

$$BC^2 = AC^2 + BD^2 - AD^2.$$

4. $BD^2 - AD^2 = (BD - AD)(BD + AD)$, (§ 293)
$= AB(BD + AD)$,
$= AB(AB + 2AD)$,
$= AB^2 + 2AB \cdot AD$. (§ 287)

5. Substituting this last value in (3),

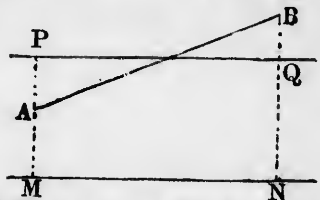
$$BC^2 = AC^2 + AB^2 + 2AB \cdot AD. \quad \text{Q.E.D.}$$

314. *Scholium.* The method of demonstration is the same in the last two problems, except that in Th. XII. the line AB is the *sum* of the lines AD and BD, and in Th. XIII. it is equal to their *difference*. But if we regard the projection AD as algebraically negative when it falls outside the triangle, as in Th. XIII., then Th. XII. will express both theorems, because the subtraction of the negative rectangle $AB \,.\, AD$ would mean that it was to be added arithmetically.

THEOREM XIV.

315. *The projections of a straight line upon parallel straight lines are equal.*

Proof. Let AB be the line projected, and MN, PQ, its projections upon two parallel lines.



1. Because these lines are parallel, the perpendiculars AM and AP, BQ and QN, form two straight lines.

2. Because the lines PM and QN are perpendicular to the same straight line MN,

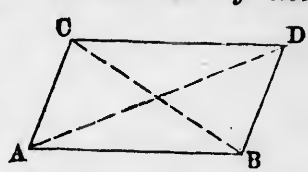
$$MP \parallel NQ.$$

3. Therefore $MNPQ$ is a parallelogram, and

$$MN = PQ \quad (\S\ 127). \qquad \text{Q.E.D.}$$

THEOREM XV.

316. *The sum of the squares upon the two diagonals of a parallelogram is equal to the sum of the squares upon the four sides.*



Proof. Let $ABCD$ be the parallelogram, having an acute angle at A. Then—

1. In the triangle ABC,

$$BC^2 = AB^2 + AC^2 - 2AB \times \text{proj. of } AC \text{ on } AB. \qquad (\S\ 312)$$

$$AD^2 = AC^2 + CD^2 + 2CD \times \text{proj. of } AC \text{ on } CD. \qquad (\S\ 313)$$

2. Because AB and CD are parallel,

$$\text{Proj. of } AC \text{ on } CD = \text{proj. of } AC \text{ on } AB. \qquad (\S\ 315)$$

Also, because $ABCD$ is a parallelogram,

$$AB = CD.$$

Therefore the last two terms of the equations (1) are equal.

3. Adding the equations (1), the last terms of the equations (1) cancel each other, and we have

$$BC^2 + AD^2 = AB^2 + AC^2 + AC^2 + CD^2,$$
$$= AB^2 + BD^2 + CD^2 + AC^2 \text{(because } AC = BD\text{)}.$$

That is, the sum of the squares on the diagonals AD and BC is equal to the sum of the squares upon the four sides. Q.E.D.

CHAPTER III.

PROBLEMS IN AREAS

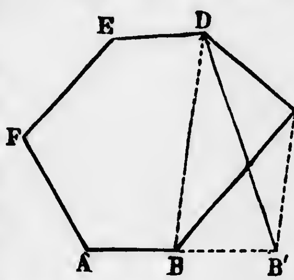
PROBLEM I.

317. *To construct a triangle which shall be equal in area to a given polygon.*

Given. A polygon, $ABCDEF$.

Required. To construct a triangle equal to it in area.

Construction. 1. Join the ends of any pair of adjacent sides, say BC and CD, by the line DB.



2. Through the intermediate angle C draw a line parallel to BD, meeting AB produced in B'.

3. The polygon $AB'DEF$ will have the number of its sides one less than $ABCDEF$, because the two sides BC, CD are replaced by the one side $B'D$, and it will be equal in area.

4. By performing the same operation upon $AB'DEF$, the number of sides will be still further diminished by one, and the operation may be repeated until the number of sides is reduced to three.

Proof. 1. Because the triangles DCB and $DB'B$ are on the same base, DB, and have their vertices on the line CB' parallel to that base, they are equal in area (§ 302).

2. The original polygon is made up of the parts

Area $ABDEF +$ area DCB,

and the new one, $AB'DEF$, is made up of

Area $ABDEF +$ area $BB'D$.

3. Because the area $DCB = BB'D$,

Area $AB'DEF =$ area $ABCDEF$.

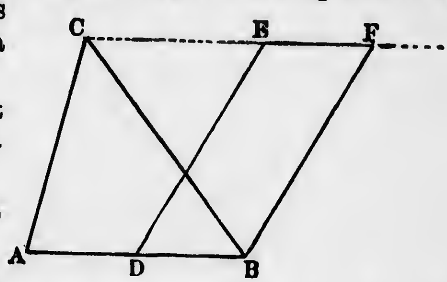
4. In the same way it may be shown that each transformed polygon is equal to the one from which it is formed, so that the last one of all, which is a triangle, is equal in area to the original polygon.

Problem II.

318. *To describe a parallelogram which shall be equal in area to a given triangle, and have one of its angles equal to a given angle.*

Given. A triangle, ABC; an angle, X.

Required. To construct a parallelogram having one of its angles equal to X, and its area equal to the area ABC.



Construction. 1. Bisect the base AB of the triangle at the point D.

2. Through C draw a line CF parallel to AB.

3. At D make the angle $BDE = X$, and continue the side until it meets CF in E.

4. Through B draw BF parallel to DE. $DBFE$ will then be the parallelogram required.

The proof is left as an exercise for the student.

319. *Corollary.* If it be required to construct a rectangle which shall be equal to the triangle, we have only to make the line DE perpendicular to AB.

Problem III.

320. *To describe a square which shall be equal in area to a given rectangle.*

Given. A rectangle, $ABCD$.

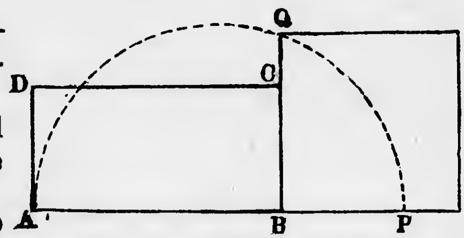
Required. To construct a square of the same area.

Analysis. Theorem X. teaches that in the right-angled triangle ABC, the square upon AD is equal to the rectangle $BD \cdot DC$.

Therefore, if we can construct a right-angled triangle in which the perpendicular from the right angle upon the hypothenuse shall divide the latter into two parts equal respectively to two adjacent sides of the given rectangle, this perpendicular will be the side of the square required.

This result will be reached by describing a semicircle upon a diameter equal to the sum of two adjacent sides of the rectangle, because all the angles in the semicircle are right angles.

Construction. 1. Produce AB to the point P, making $BP = BC$.



2. On AP as a diameter describe a semicircle.

3. Produce BC until it cuts the semicircle in Q.

The square on BQ will be the square required.

Proof. 1. If we join AQ, PQ, the triangle AQP will be right-angled, because it is inscribed in a semicircle.

2. Because BQ is a perpendicular from the right angle Q upon the base AP, we have

$$BQ^2 = AB \cdot BP = AB \cdot BC. \quad \text{Q.E.D.}$$

321. *Corollary.* By the three preceding constructions a square may be constructed equal in area to any given polygon. The polygon is first transformed into a triangle by Problem I.; this triangle into a rectangle by Problem II., Cor.; this rectangle into a square by Problem III.

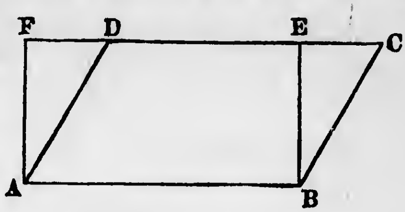
Problem IV.

322. *To describe a rectangle which shall be equal in area to a given parallelogram.*

Given. A parallelogram, $ABCD$.

Required. To describe a rectangle having the same area.

Construction. Produce the side CD to F, and at A and B erect perpendiculars to AB, meeting EF in E and F.



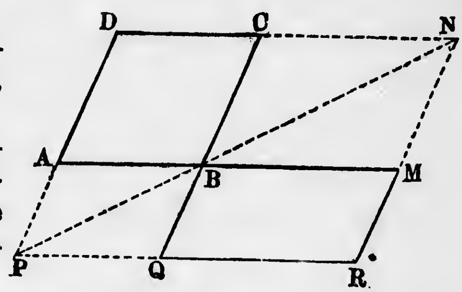
$ABEF$ will be the rectangle required.

Proof. The proof is given by Theorem V.

323. *Corollary.* If, instead of a rectangle, we wish to describe a parallelogram having a given angle, we have only to make the angle BAF equal to the given angle.

Problem V.

324. *On a given line as a base to describe a parallelogram equal to a given parallelogram in area and in angles.*



Given. A parallelogram, $ABCD$; a line, BM.

Required. To describe on BM a parallelogram having the same area as $ABCD$ and equiangular to it.

Construction. Let BM be drawn in the same straight line with AB. Then—

1. Through M draw MN parallel to BC, meeting DC produced in N.

2. Draw the diagonal NB, and produce it until it meets DA produced in P.

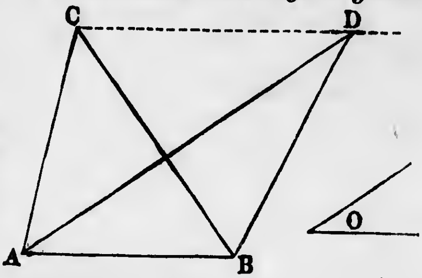
3. Through P draw PQR parallel to AM, and meeting CB produced in Q and NM produced in R.

$BMRQ$ will be the required parallelogram, having $QR = BM$ as its base, and equal to $ABCD$ in area and in its angles.

Proof. From Theorem IX.

PROBLEM VI.

325. *On the base of a given triangle to describe another triangle equal in area and having a given angle.*



Given. A triangle, ABC; an angle, O.

Required. On the base AB to describe a triangle equal to ABC in area, and having an angle equal to O.

Construction.

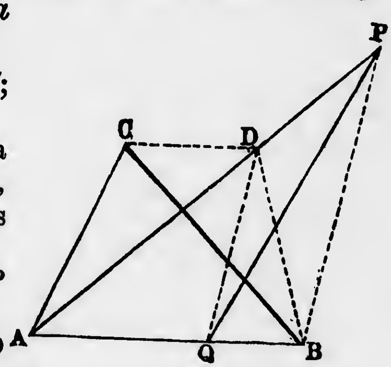
1. Through C draw CD parallel to AB.
2. At A make the angle $BAD = O$, and produce the side until it meets CD in D.
3. Join BD.

ABD will be the triangle required.

Proof. From Theorem VII., Cor. 1.

PROBLEM VII.

326. *To form a triangle equal in area to a given triangle, and having its base on the same straight line and its vertex in a given point.*



Given. A triangle, ABC; a point, P.

Required. To describe a triangle equal to ABC in area, having its base on AB and its vertex in P.

Construction. 1. Join AP and PB.

2. Through C draw CD parallel to AB, meeting AP in D.
3. Draw DQ parallel to PB, meeting AB in Q.
4. Join PQ.

AQP will be the required triangle on the base AB; equal in area to ABC, and having its vertex in P.

Proof. Join BD. Then—

1. Because AB and CD are parallel,

   Area $ABC =$ area ABD.

2. Because DQ and PB are parallel,

   Area $BPD =$ area BPQ.

3. Area $ABD + BPD - BPQ =$ area AQP.

4. Comparing with (2),

   Area $ABD =$ area AQP.

5. Comparing this with (1),

   Area $ABC =$ area AQP. Q.E.D.

The construction and demonstration of the following problems are left as exercises for the student.

Problem VIII.

327. *To construct a square which shall be equal to the sum of two given squares.*

Remark. If we form a right-angled triangle of which the sides about the right angle are equal to the sides of the given squares, the square upon the hypothenuse will be that required (§ 308).

Problem IX.

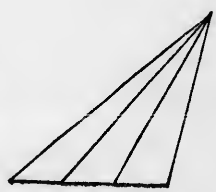
328. *To construct a square which shall be equal to the difference of two given squares.*

Problem X.

329. *To construct a square which shall be equal to one half a given square.*

Problem XI.

330. *To divide a triangle into any given number of equal triangles by lines drawn from the vertex to the base.*



CHAPTER IV.

THE COMPUTATION OF AREAS.

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331. Geometrical problems may be divided into two general classes, depending on the kind of solution which is to be obtained.

I. *Problems of pure geometry.* In problems of construction the solution consists in drawing a figure which is to conform to the conditions of the problem. The answer to such a problem is given, not in numbers or algebraic expressions, but as a geometrical figure simply. The problems we have hitherto considered belong to this class, and they are the only kind recognized as belonging to pure geometry.

II. *Problems of numerical geometry.* In problems of the second class the solution appears not merely as a line or figure drawn upon a plane, but as a calculated length or a calculated extent of area. For example, the result may be expressed by saying that a line the length of which is required is seven centimeters or other units in length, or that a surface contains a certain number of square units. The number of units in either case may be expressed either by algebraic symbols or by the numbers of arithmetic. Such problems may be considered as belonging to numerical or algebraic geometry.

Relations of the two methods. In pure geometry the division of magnitudes into definite units is not recognized. If an angle is given, it is supposed to be given by drawing it, not by stating the number of degrees. The angle itself may not be drawn at all except in imagination. So, also, a given length is a length of a given line, and not a number of units of any kind.

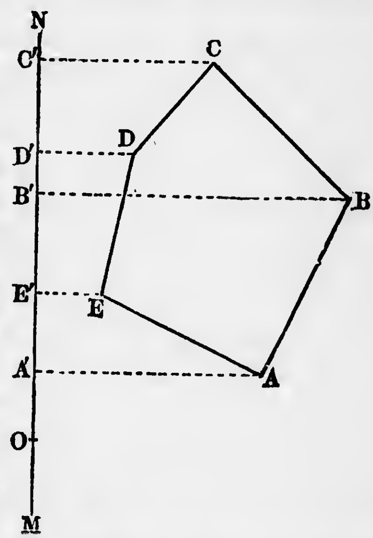
Pure geometry was almost the only kind cultivated by the ancients, because the methods of algebra were not known to them. Hence it is sometimes called the ancient geometry. But it does not suffice for modern wants, where numbers of miles, feet, acres, etc., are required.

One great advantage of the modern method arises from the application of algebraic signs to lines. In the ancient geometry, whenever the position of a point is changed to the opposite side of a line, we have to suppose a different theorem or a different case of the same proposition. But in the modern geometry the difference is expressed by changing the algebraic sign of the distance of the point from the line, and the general statement of the proposition remains the same.

The general investigation of lines and areas by algebraic methods requires an application of trigonometry, and, in the cases of curve lines, of the integral calculus; but there is a general method applicable to the computation of areas both in the surveying of land and in the integral calculus, the principles of which can now be explained.

Problem XII.

332. *To find by measurement and calculation the area of a given polygon.*



Let $ABCDE$ be the polygon.

Draw any straight line MN. From each angle of the polygon drop a perpendicular upon the line MN. Let A', B', C', etc., be the points at which these perpendiculars meet the line.

The area $A'ABCC'$ includes the whole polygon plus an area $AEDCC'A'$ between the polygon and the line. Therefore, if from the first of these areas we take away the second, the remainder will be the area of the polygon.

Each of these two areas is made up of several trapezoids, namely:

$$\begin{aligned}
\text{First area} &= \text{trapezoid } A'ABB' \\
&+ \text{trapezoid } B'BCC''. \\
\text{Second area} &= \text{trapezoid } A'AEE' \\
&+ \text{trapezoid } E'EDD' \\
&+ \text{trapezoid } D'DCC'.
\end{aligned}$$

The first area includes the quantities to be added, the second those to be subtracted.

The area of each trapezoid is the rectangle of its altitude into half the sum of its parallel sides (Th. VIII.). In particular,

$$\begin{aligned}
\text{Area } A'ABB' &= \tfrac{1}{2}(A'A + B'B)\, A'B'. \\
\text{Area } B'BCC' &= \tfrac{1}{2}(B'B + C'C)\, B'C'. \\
\text{Area } C'CDD' &= \tfrac{1}{2}(C'C + D'D)\, C''D'. \\
\text{etc.} \qquad & \text{etc.} \qquad \text{etc.}
\end{aligned}$$

To express these areas in algebraic form let us put p_1, p_2, p_3, etc., for the lengths of the several perpendiculars; that is,

$$\begin{aligned}
p_1 &= A'A. \\
p_2 &= B'B. \\
p_3 &= C''C. \\
p_4 &= D'D. \\
p_5 &= E'E.
\end{aligned}$$

Let us also take an arbitrary point O on the line, to measure distances from, and put

$$\begin{aligned}
y_1 &= OA'. \\
y_2 &= OB'. \\
y_3 &= OC''. \\
y_4 &= OD'. \\
y_5 &= OE'.
\end{aligned}$$

Then

$$\begin{aligned}
A'B' &= y_2 - y_1. \\
B'C' &= y_3 - y_2. \\
C'D' &= y_3 - y_4. \\
D'E' &= y_4 - y_5. \\
E'A' &= y_5 - y_1.
\end{aligned}$$

The expressions for the areas will then be:

$$\begin{aligned}
\text{Area } A'ABB' &= \tfrac{1}{2}(p_2 + p_1)(y_2 - y_1). \\
B'BCC' &= \tfrac{1}{2}(p_3 + p_2)(y_3 - y_2). \\
CC'DD' &= \tfrac{1}{2}(p_4 + p_3)(y_3 - y_4). \\
D'DEE'' &= \tfrac{1}{2}(p_5 + p_4)(y_4 - y_5). \\
E'EAA' &= \tfrac{1}{2}(p_6 + p_5)(y_5 - y_1).
\end{aligned}$$

The required area of the polygon we have found to be given by subtracting the last three areas from the first two. Now this subtraction may be indicated by simply changing the algebraic signs of the quantities to be subtracted—a change which will be effected by changing the factor

$$y_3 - y_4 \quad \text{into} \quad y_4 - y_3.$$
$$y_4 - y_5 \quad \text{into} \quad y_5 - y_4.$$
$$y_5 - y_1 \quad \text{into} \quad y_1 - y_5.$$

The expression for the area will then be:

$$\begin{aligned} \text{Area } ABCDE = & \tfrac{1}{2}(p_2 + p_1)(y_2 - y_1) \\ & + \tfrac{1}{2}(p_3 + p_2)(y_3 - y_2) \\ & + \tfrac{1}{2}(p_4 + p_3)(y_4 - y_3) \\ & + \tfrac{1}{2}(p_5 + p_4)(y_5 - y_4) \\ & + \tfrac{1}{2}(p_1 + p_5)(y_1 - y_5). \end{aligned}$$

It will be seen that the formula is uniform with respect to the different values of p and y taken in order, each value of p being added to that next following in order, and each value of y subtracted from that next following in order.

If we execute the multiplications indicated, one half the partial products will cancel each other, and the area will reduce to

$$\begin{aligned} \tfrac{1}{2}\{ & p_1y_2 - p_2y_1 \\ & + p_2y_3 - p_3y_2 \\ & + p_3y_4 - p_4y_3 \\ & + p_4y_5 - p_5y_4 \\ & + p_5y_1 - p_1y_5\} \end{aligned}$$

In principle this method is that used by surveyors in computing the area of irregular pieces of land. It also involves the best system of measuring areas in more advanced mathematical investigations.

The student may be supposed to find the values of p_1, p_2, etc., y_1, y_2, etc., by measurement on the actual figure. Their calculation, when all the sides and angles are given, is a problem of trigonometry.

The criterion whether a trapezoid is to be put into the additive or the subtractive column is this:

If, in crossing over any side of the polygon from the outside to the inside, we pass to the inside of the trapezoid bounded by that side, the area of that trapezoid is additive, or algebraically positive.

If, in passing inside of the polygon, we pass out of the trapezoid, the area of that trapezoid is subtractive, or algebraically negative.

Examples. If we pass into the polygon over the side AB, we pass into the trapezoid $A'ABB'$. Therefore the area of this trapezoid is additive. (See diagram on p. 155.)

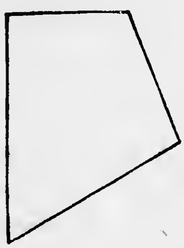
The same applies to $B'BCC'$.

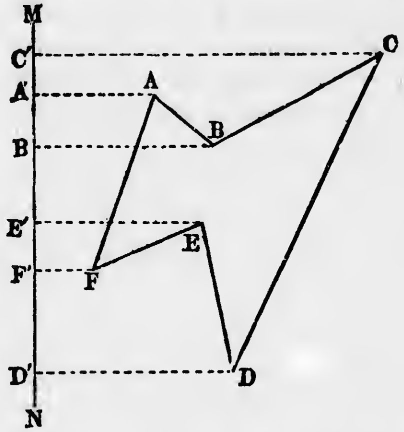
If we pass into the polygon over the side CD, we pass out of the trapezoid $C'CDD'$. Therefore the area of this trapezoid is negative.

The same remark applies to the trapezoid bounded by DE and by EA.

EXERCISES.

Measure off and compute the area of each of the following polygons in square centimetres, inches, or other scale measure.



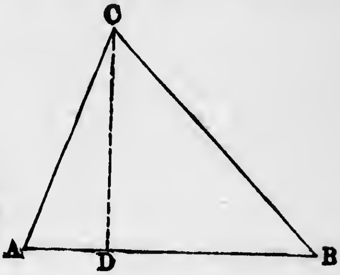


It will be noticed that owing to the re-entrant angles of the second figure there is a double overlapping of some of the trapezoids. But this makes no change in the application of the formula, which always gives correct results when the algebraic signs of the quantities are properly interpreted.

333. *Algebraic expression for the Area of a Triangle.* Because a triangle is completely determined when three of its sides are given (§ 110), its area must admit of being expressed algebraically in terms of its sides. The required expression is found as follows:

Let ABC be the triangle, and CD the perpendicular from C upon D. Put

a, the side BC;
b, " " AC;
c, " " AB;
p, the perpendicular CD.



Then

Area $ABC \times 2 = cp$. (§ 301)

To find p we have, from the right-angled triangles CDA and CDB,

$$p^2 = AC^2 - AD^2 = BC^2 - BD^2 = BC^2 - (AB - AD)^2,$$

or

$$p^2 = b^2 - AD^2 = a^2 - (c - AD)^2 = a^2 - c^2 + 2c.AD - AD^2.$$

We must use these equations to eliminate the quantity AD from the expression for p^2. Equating the second and fourth members of the last line of equations, we find

$$b^2 = a^2 - c^2 + 2c.AD;$$

whence

$$AD = \frac{b^2 + c^2 - a^2}{2c}.$$

Substituting the square of this in the expression for p^2, we have

$$p^2 = b^2 - \frac{(b^2 + c^2 - a^2)^2}{4c^2} = \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4c^2}.$$

Squaring the above expression for the area,

$$4(\text{Area } ABC)^2 = c^2p^2.$$

$$16(\text{Area } ABC)^2 = 4c^2p^2 = 4b^2c^2 - (b^2 + c^2 - a^2)^2.$$

This expression, being the difference of two squares, may be transformed into the product

$$(2bc + b^2 + c^2 - a^2)\ (2bc - b^2 - c^2 + a^2).$$

The first three terms in the first factor are a perfect square; namely, the square of $b + c$; and the first three of the second factor are the negative of the same square. Therefore each factor can again be factored, making the product

$$(b + c + a)\ (b + c - a)\ (a + b - c)\ (a - b + c).$$

Now, if we put s for half the perimeter of the triangle, namely (§ 282),

$$s = \tfrac{1}{2}(a + b + c) \text{ or } 2s = a + b + c,$$

we have

$$b + c + a = 2s.$$
$$b + c - a = 2(s - a).$$
$$a + b - c = 2(s - c).$$
$$a - b + c = 2(s - b).$$

Substituting these values and dividing by 16, we have

$$(\text{Area } ABC)^2 = s(s-a)(s-b)(s-c)$$

and

$$\text{Area } ABC = \sqrt{s(s-a)(s-b)(s-c)},$$

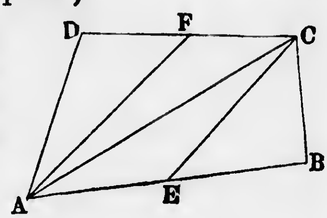
the required expression.

Theorems for Exercise.

Theorem 1. The difference of the squares upon any two sides of a triangle is equal to the difference of the squares of the projections of these sides upon the third side.

Theorem 2. The sum of the squares upon the diagonals of a quadrilateral is equal to twice the sum of the squares upon the four lines joining the middle points of its sides, taken consecutively. (See Th. 21, p. 89.)

Theorem 3. If we join the middle points of two opposite sides of a quadrilateral to two opposite angles, the two triangles thus formed will have half the area of the quadrilateral.

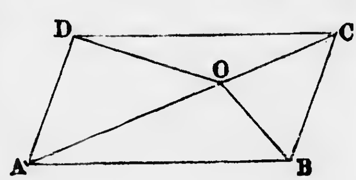


Hypothesis. $FC = FD$; $AE = EB$.

Conclusion. Area $AFD + EBC$ = area $AECF$
$= \frac{1}{2}$ area $ABCD$.

Theorem 4. The four triangles into which a parallelogram is divided by its diagonals are of equal area.

Theorem 5. If from any point on the diagonal of a parallelogram lines be drawn to the opposite angles, the parallelogram will be divided into two pairs of equal triangles.



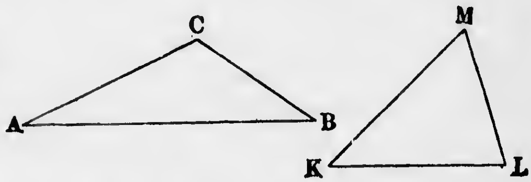
Area OAD = area OAB.
Area OCD = area OCB.

THEOREM 6. The parallelogram formed by joining the middle points of the consecutive sides of a quadrilateral has half the area of the quadrilateral (§ 136).

THEOREM 7. If through the middle point of one of the non-parallel sides of a trapezoid we draw a line parallel to the opposite side, and complete the parallelogram, the area of the parallelogram will be equal to that of the trapezoid.

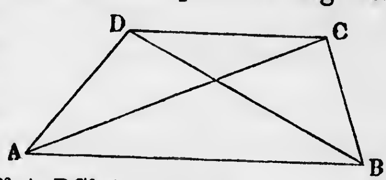
THEOREM 8. If we join the middle of one of the non-parallel sides of a trapezoid to the ends of the opposite side, the middle triangle will have half the area of the trapezoid.

THEOREM 9. If two triangles have two sides of the one equal to two sides of the other respectively, and the included angles supplementary, they are equal in area.



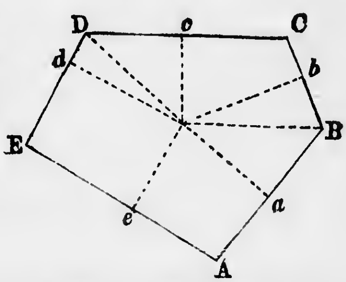
Hypothesis. $CA = MK$. $CB = ML$.
Angle ACB + angle $KML = 180°$.
Conclusion. Area ABC = area KLM.

THEOREM 10. The sum of the squares upon the diagonals of a trapezoid are equal to the sum of the squares upon the non-parallel sides plus twice the rectangle of the parallel sides.



Conclusion. $AC^2 + BD^2 = AD^2 + BC^2 + 2AB.CD.$

THEOREM 11. If from any point within a polygon perpendiculars be dropped upon the sides, the sum of the squares of one set of alternate segments is equal to the sum of the squares of the other set.



$$Aa^2 + Bb^2 + Cc^2 + Dd^2 + Ee^2 = aB^2 + bC^2 + cD^2 + dE^2 + eA^2.$$

NUMERICAL EXERCISES.

1. In a right-angled triangle the lengths of the sides containing the right angle are 9 and 12 feet. What is the length of the hypothenuse? What is the area of the triangle?

2. If the length of the hypothenuse is 10 feet, and that of one side 8 feet, what is the length of the remaining side? What is the area of the triangle?

3. In a right-angled triangle the perpendicular from the right angle upon the hypothenuse divides the latter into segments which are respectively 9 and 16 feet. Find the lengths of the perpendicular and of the two sides, and the area of the triangle.

4. What three different expressions for the area of a triangle may we obtain from § 301 by taking different sides as the base? What theorem hence follows?

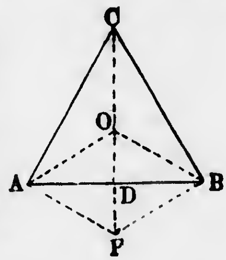
5. What is the area of the triangle of which the respective sides are 15, 41, and 52 metres?

6. If the diagonal of a rectangle is 13 feet, and one of the sides 12 feet, what is the area?

7. Show how the altitude and area of a trapezoid may be computed when its four sides are known.

Refer to the computation of the altitude p of a triangle in § 333.

8. If each side of an equilateral triangle is unity, find its altitude.



9. Draw an equilateral triangle, ABC. Show that the bisectors of each interior angle will bisect the opposite side perpendicularly. Show that if the bisector of C be produced beyond the point O in which it meets the other bisectors and intersect the opposite side in D, and if we take $DF = DO$ and join AF, BF, then—

I. OAF and OBF will be equilateral triangles.
II. The points A, C, B, F lie on a circle.
III. The lines OA, OB, and OC will all be equal.

Also, supposing the length of each side of the triangle ABC to be unity, compute the lengths of OC and OD.

BOOK V.

THE PROPORTION OF MAGNITUDES.

CHAPTER I.

RATIO AND PROPORTION OF MAGNITUDES IN GENERAL.

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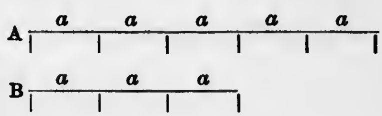
334. *Definition.* When a greater magnitude contains a lesser one an exact number of times, the greater one is said to be a **multiple** of the lesser, and the lesser is said to **measure** the greater, and to be an **aliquot part** of the greater.

335. *Def.* When a lesser magnitude can be found which is a measure of each of two greater ones, the latter are said to be **commensurable**, and the former is said to be a **common measure** of them.

336. *Def.* When two magnitudes have no common measure they are said to be **incommensurable.**

337. *Def.* When one of two commensurable magnitudes contains the common measure m times, and the other contains it n times, they are said to be to each other as m to n.

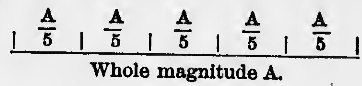


EXAMPLE. If the magnitude A contains the measure a 5 times, and B the same measure 3 times, then A is to B as 5 to 3, and a is a common measure of A and B.

EXERCISES. Draw, by the eye, pairs of lines which shall be to each other as 3 to 4; as 2 to 5; as 4 to 7; as 5 to 6; as 7 to 2.

338. *Corollary.* If A is to B as m to n, then, by definition (§ 337), the mth part of A will be equal to the nth part of B; or, in symbolic language,

$$\frac{A}{m} = \frac{B}{n}.$$



NOTE. The mth part of a magnitude is indicated by a fraction of which the symbol of the magnitude is the numerator and m the denominator.

NOTATION. The statement that two magnitudes A and B are to each other as the numbers m and n is written symbolically

$$A : B :: m . n,$$

or

$$A : B = m : n.$$

NOTE. The second form, or that of an equation, is preferable, and is most used by mathematicians; but the first form is more common in elementary books.

339. *Def.* If a pair of magnitudes A and B are to each other as two numbers m and n, and another pair P and Q are also to each other as $m : n$, then we say that A is to B as P to Q, and the four magnitudes A, B, P, and Q are said to be **proportional** or to **form a proportion.**

NOTATION. The statement that the four magnitudes A, B, P, and Q are proportional is expressed in the symbolic form

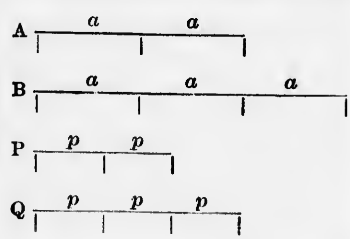
$$A : B :: P : Q,$$

or

$$A : B = P : Q;$$

which is read: A is to B as P is to Q.

340. *Def.* The symbolic statement that four magnitudes are proportional is called a **proportion.**



EXAMPLE. If A contains a twice, and B contains it three times; if also P contains p twice, and Q contains it three times, then

$$A : B :: P : Q.$$

341. *Def.* The four quantities which form a proportion are called **terms** of the proportion.

342. *Def.* The first and fourth terms of a proportion are called the **extremes;** the second and third, the **means.**

EXAMPLE. In the last proportion A and Q are the extremes, B and P the means.

343. *Def.* The first and third terms, which precede the symbol :, are called **antecedents;** the second and fourth, which follow the symbol :, are called **consequents.**

EXAMPLE. In the last proportion A and P are the antecedents, B and Q the consequents.

344. *Def.* If the means are equal, each of them is said to be a **mean proportional** between the extremes, and the three quantities are said to be **in proportion.**

Axioms.

345. *Ax.* 1. If there be a greater and a lesser magnitude of the same kind, the greater may be divided into so many equal parts that each part shall be less than the lesser magnitude.

NOTE. By magnitudes of the same kind are meant those which are both numbers, both lines, both surfaces, or both solids.

Ax. 2. If a greater magnitude be a certain number of times a lesser one, then any multiple of that greater one will be the same number of times the corresponding multiple of the lesser.

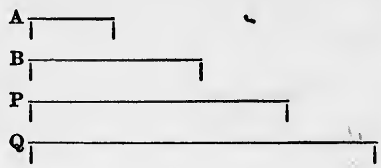
Symbolic expression of this axiom. If mag. $G = i \times$ mag. L, then
$$nG = i \times nL.$$

Ax. 3. If a lesser magnitude be a certain aliquot part of a greater one, then any multiple of the lesser one will be the same aliquot part of the corresponding multiple of the greater.

Symbolic expression of this axiom. If mag. $L = \frac{\text{mag.}\,G}{i}$, then
$$n \times L = \frac{nG}{i}.$$

346. THEOREM. *Equimultiples of commensurable magnitudes are proportional to the magnitudes themselves.*

Hypothesis. A and B, two commensurable magnitudes; P, a magnitude i times as great as A; Q, a magnitude i times as great as B, i being any number whatever.



Conclusion. $P : Q :: A : B$.

Proof. 1. Let the magnitude A be to B as m to n.

This will mean that if we divide A into m parts and B into n parts, these parts will be equal, or

$$\frac{A}{m} = \frac{B}{n}.$$

2. Because $P = iA$, if we divide P into m parts, we shall have for each part

$$\frac{P}{m} = \frac{iA}{m} = i \times \frac{A}{m}. \qquad (\S\ 345, \text{Ax. } 3)$$

3. In the same way, if we divide Q into n parts,

$$\frac{Q}{n} = i \times \frac{B}{n}.$$

4. Comparing these results with (1),

$$\frac{P}{m} = \frac{Q}{n}.$$

Therefore $P : Q :: m : n$. (§ 339)

Comparing with (1), $P : Q :: A : B$.

347. *Corollary* 1. In a similar way it may be shown that *similar aliquot parts of magnitudes are to each other as the magnitudes themselves.*

That is, whatever be the whole number k,

$$\frac{P}{k} : \frac{Q}{k} :: P : Q.$$

348. *Cor.* 2. *Similar fractions of magnitudes are proportional to the magnitudes themselves.*

For, by *Corollary* 1, we have

$$\frac{i}{k} P : \frac{i}{k} Q :: iP : iQ,$$

and, by the original theorem,

$$iP : iQ :: P : Q,$$

from which follows

$$\frac{i}{k}P : \frac{i}{k}Q :: P : Q.$$

Ratio of Two Magnitudes.

349. Consider any two numbers, which we may call m and n. If we divide each of them into n parts, each part of m will be $\frac{m}{n}$, and each part of n will be $\frac{n}{n} = 1$. By *Corollary* 1 of the last theorem these parts will be to each other as the original numbers; that is,

$$m : n :: \frac{m}{n} : 1.$$

Therefore, *if two magnitudes A and B are to each other as m to n, they will also be to each other as $\frac{m}{n}$ to 1, or*

$$A : B :: \frac{m}{n} : 1.$$

350. *Def.* When two magnitudes are to each other as m to n, the fraction $\frac{m}{n}$ is called the **ratio** of the magnitude A to the magnitude B.

Corollary. When we say

$$A : B :: m : n,$$

we mean that A contains m parts, and B contains n equal parts (§ 338). Hence:

351. *The ratio of a magnitude A to another magnitude B is the quotient formed by dividing the number of parts in A by the number of equal parts in B.*

352. *Scholium.* There are three ways of conceiving of the ratio of two magnitudes, which all lead to the same result.

I. If we have two magnitudes A and B, the ratio of A to B is the numerical factor by which we must multiply the consequent, B, in order to produce the antecedent, A. There may then be three cases:

1. If A is a multiple of B, the ratio is a whole number. The multiplication is then effected by adding B to itself the proper number of times.

2. If A and B are commensurable, the ratio is a vulgar fraction. If A contains the common measure m times, and B contains it n times, the ratio is $\frac{m}{n}$. We may conceive the multiplication to be effected by dividing B into n parts, and taking m of these parts to make A.

3. If A and B are incommensurable, the ratio will neither be a whole number nor a vulgar fraction. If we attempt to express it as a decimal, the figures will go on without end.

II. The ratio of A to B may also be conceived of as a number expressing the magnitude of A when we take B as the unit of measure. This amounts to the same thing as I., because when we multiply unity by any factor we produce the factor itself.

III. If A and B are numbers, instead of geometric magnitudes, the ratio of A to B is the quotient $\frac{A}{B}$.

The consistency of these ways of conceiving a ratio is established by the following definition:

353. *Def.* To **multiply** a magnitude B by a numerical factor r means to find a magnitude which shall have the same ratio to B that r has to unity.

Hence the expressions

$$A : B :: r : 1$$

and

$$A = rB$$

are equivalent.

The preceding definition of a ratio gives us another definition of a proportion, namely:

354. Four magnitudes are proportional when the ratio of the first to the second is equal to the ratio of the third to the fourth.

355. *Def.* If the terms of a ratio are interchanged, the new ratio is called the **inverse** of the original one.

Thus the ratio $B : A$ is the inverse of $A : B$.

If $A : B :: m : n$, then $A : B = \frac{m}{n}$ and $B : A = \frac{n}{m}$; the product of these ratios is $\frac{mn}{nm} = 1$. Therefore:

356. THEOREM. *The product of two inverse ratios is unity.*

Ratios of Incommensurable Magnitudes.

357. If two magnitudes are incommensurable (§ 336), they may still be considered as having a ratio, but this ratio cannot be exactly expressed by a fraction. Let us suppose that in dividing the magnitude B into n parts, A is found to contain m of these parts and a fraction of another part. Then the ratio of A to B will be greater than $\frac{m}{n}$, and less than $\frac{m+1}{n}$; that is, less than $\frac{m}{n} + \frac{1}{n}$. The number n may here be as great as we please.

358. THEOREM. *If four incommensurable magnitudes A, B, P, and Q are so related that, on dividing the antecedents A and P each into n equal parts, Q shall contain the same whole number of parts of P that B contains of A, fractions being neglected, and this however great the number n,—then the ratio of Q to P is equal to the ratio of B to A, and A, B, P, and Q form a proportion.*

Hypothesis.
$$\frac{m}{n} B < A < \left(\frac{m}{n} + \frac{1}{n}\right) B,$$
$$\frac{m}{n} Q < P < \left(\frac{m}{n} + \frac{1}{n}\right) Q,$$

how great soever the numbers m and n.

Conclusion. $A : B :: P : Q$.

Proof. If the ratios $A : B$ and $P : Q$ be unequal, let α be their difference. Since we can make the number n as great as we please, let us make it so great that $\frac{1}{n}$ shall be less than α (§ 345, Ax. 1).

If m be the whole number of times which A contains the nth part of B,

$$A : B > \frac{m}{n} \text{ and } A : B < \frac{m}{n} + \frac{1}{n}.$$

By hypothesis P contains the nth part of Q this same number m of times, plus a fraction. Therefore

$$P : Q > \frac{m}{n} \text{ and } P : Q < \frac{m}{n} + \frac{1}{n}.$$

Since both ratios are greater than $\frac{m}{n}$ and less than $\frac{m}{n} + \frac{1}{n}$, their difference must be less than $\frac{1}{n}$ and therefore less than α, because $\frac{1}{n} < \alpha$. Therefore the difference of the ratios would be at the same time equal to α and less than α, which is absurd. Therefore the ratios do not differ at all.

359. *Corollary. If any theorem respecting the equality of ratios be proved for the ratios of all commensurable magnitudes, however small the common measure, it will hold true for the ratios of all incommensurable magnitudes.*

360. *Def.* The ratio of two incommensurable magnitudes is called an **irrational** number.

Although an irrational number cannot be expressed as the quotient of two entire numbers, yet by taking such numbers sufficiently great we can find quotients which shall come as near as we please to the irrational number. Thus:

To come within $\frac{1}{1000}$ we take a divisor > 1000.

To come within $\frac{1}{100000}$ we take a divisor > 100000,

etc. etc. etc.

Transformation of Proportions.

361. *Def.* **Inversion** is when the terms of each ratio in a proportion are interchanged to form a new proportion.

Theorem of Inversion. From the proportion

$$A : B :: P : Q$$

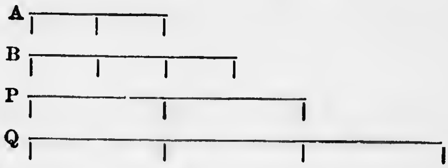
we may conclude by inversion

$$B : A :: Q : P.$$

Proof. From § 355.

362. *Def.* **Alternation** is when the means of a proportion are interchanged to form a new proportion. The new proportion is then said to be the **alternate** of the original proportion.

Theorem of Alternation. *In any proportion the antecedents have the same ratio to each other as the consequents.*



Hypothesis. If

$A : B :: P : Q$—

Conclusion. Then

$A : P :: B : Q.$

Proof. 1. Let the ratios $A : B$ and $P : Q$ each be $\frac{m}{n}$, so that

mth part of $A = n$th part of B, which call a.
mth part of $P = n$th part of Q, which call p.

2. Then $A = ma. \quad B = na.$
$P = mp. \quad Q = np.$

3. Hence $\left.\begin{array}{l} A : P :: ma : mp :: a : p; \\ B : Q :: na : np :: a : p. \end{array}\right\}$ (§ 346)

4. Therefore $A : P :: B : Q.$ Q.E.D.

363. *Corollary. If the extremes be interchanged, the proportion will still be true.*

For by alternation we have

$$A : P :: B : Q,$$

and then by inversion, and putting the second ratio first,

$$Q : B :: P : A.$$

364. *Def.* **Composition** is when the sums of antecedents and consequents are compared with either antecedents or consequents to form a new proportion.

THEOREM OF COMPOSITION. If we have the proportion

$$A : B :: P : Q, \qquad (1)$$

we may conclude

$$\left.\begin{array}{l} A : A + B :: P : P + Q; \\ A + B : B :: P + Q : Q. \end{array}\right\} \qquad (2)$$

Proof. Let the equal ratios in (1) be $m : n$. Using the same notation as in the preceding theorem, we find

$$A = ma; \; B = na. \quad A + B = (m + n)a.$$
$$P = mp; \; Q = np. \quad P + Q = (m + n)p.$$
$$A : A + B = m : m + n.$$
$$P : P + Q = m : m + n$$
$$A + B : B = m + n : n.$$
$$P + Q : Q = m + n : n.$$

Whence the conclusions (2) follow by comparing the equal ratios.

365. *Def.* **Division** is when the difference of antecedents and consequents is compared with either antecedents or consequents to form a new proportion.

THEOREM OF DIVISION. If we have the proportion

$$A : B :: P : Q, \qquad (1)$$

we may conclude

$$\left.\begin{array}{l} A : A - B :: P : P - Q; \\ A - B : B :: P - Q : Q. \end{array}\right\} \qquad (2)$$

Proof. By the same process as in the last theorem.

366. THEOREM. If we have the several proportions

$$A : B :: P : Q,$$
$$A' : B :: P' : Q,$$
etc. etc.,

we may conclude

$$A + A' + \text{etc.} : B :: P + P' + \text{etc.} : Q.$$

Proof. The proportions show that if

$$A = \frac{m}{n}B,$$

$$A' = \frac{m'}{n'}B, \text{ etc.,}$$

then
$$P = \frac{m}{n}Q,$$
$$P' = \frac{m'}{n'}Q, \text{ etc.},$$
whence
$$A + A' + \text{etc.} = \left(\frac{m}{n} + \frac{m'}{n'} + \text{etc.}\right)B,$$
$$P + P' + \text{etc.} = \left(\frac{m}{n} + \frac{m'}{n'} + \text{etc.}\right)Q.$$
The conclusion now follows from the equality of the coefficients.

Multiple Proportions.

367. When three or more ratios are equal, a proportion may be formed between any two of them. Thus, if
$$A : B = M : N = P : Q = X : Y, \text{ etc.}, \quad (1)$$
we may form the proportions
$$A : B :: M : N,$$
$$A : B :: X : Y,$$
$$M : N :: X : Y,$$
$$\text{etc.} \quad \text{etc.}$$
The equality of such ratios is generally expressed by writing all the antecedents with the sign : between them, followed by the consequents in the same order.

Thus (1) would be expressed in the form
$$\left.\begin{aligned} &A : M : P : X = B : N : Q : Y, \\ \text{or} \quad &A : M : P : X :: B : N : Q : Y. \end{aligned}\right\} \quad (2)$$
Here the first consequent (B) corresponds to the first antecedent (A); the second (N) to the second (M), etc.

368. *Def.* A proportion of six or more terms expressed in the form (2) is called a **multiple proportion.**

Simple proportions may be formed as follows from a multiple proportion.

369. THEOREM. *In a multiple proportion any antecedent is to its consequent as any other antecedent to its consequent.*

Example. In the proportion (2), as the first antecedent, A, is to the first consequent, B, so is the third antecedent, P, to the third consequent, Q; or, in symbolic language,

$$A : B :: P : Q.$$

Proof. This theorem follows at once from the form of expression in (1) and (2).

370. THEOREM. *In a multiple proportion any two antecedents are to each other as the corresponding consequents.*

Example. In (2), as A is to P, so is B (the consequent of A) to Q (the consequent of P); or

$$A : P :: B : Q.$$

Proof. Any such proportion is the alternate of one of the original proportions expressed by the continued proportion. Thus one of the original proportions expressed in (1) is $A : B :: P : Q$, and the above proportion is its alternate.

371. THEOREM. *In any proportion the sum of any number of antecedents is to the sum of the corresponding consequents as any one antecedent is to its consequent.*

Proof. Let the proportion be that in (2), and let the ratio of each antecedent to its consequent be $\frac{m}{n}$, so that

$$\begin{aligned} A : B &:: m : n, \\ M : N &:: m : n, \\ P : Q &:: m : n, \\ \text{etc.} &\quad \text{etc.} \end{aligned}$$

Then

$$\begin{aligned} \frac{A}{m} &= \frac{B}{n}, \\ \frac{M}{m} &= \frac{N}{n}, \\ \frac{P}{m} &= \frac{Q}{n}, \\ \text{etc.} &\quad \text{etc.} \end{aligned}$$

By adding these equations we have

$$\frac{A + M + P + \text{etc.}}{m} = \frac{B + N + Q + \text{etc.}}{n};$$

that is, the mth part of $A + M + P +$ etc. is equal to the nth part of $B + N + Q +$ etc., so that

$$A + M + P + \text{etc.} : B + N + Q + \text{etc.} :: m : n,$$

which is by hypothesis the same as the ratio of each antecedent to its consequent.

Because this reasoning is correct how great soever we suppose the numbers m and n, the theorem is true whether the magnitudes are commensurable or incommensurable (§ 359).

372. THEOREM. *In any proportion the difference of any two antecedents is to the difference of the corresponding consequents as any antecedent is to its consequent.*

Proof. By taking the difference of the first two equations of § 371, we have

$$\frac{A - M}{m} = \frac{B - N}{n},$$

which shows that

$$A - M : B - N :: m : n,$$

the same ratio which each antecedent has, by hypothesis, to its consequent.

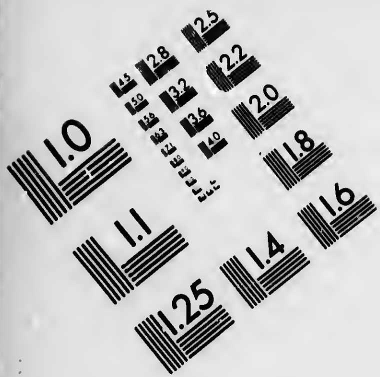
In the same way it may be shown that any other difference of the corresponding magnitude has this ratio.

373. THEOREM. *If in a series of ratios the consequent of each is the antecedent of the next, the ratio of the first antecedent to the last consequent is equal to the product of the separate ratios.*

Hypothesis. We have the separate ratios

$$A : B.$$
$$B : C.$$
$$C : D.$$

Conclusion. The ratio of A to D is the product of the ratios $A : B$, $B : C$, $C : D$.



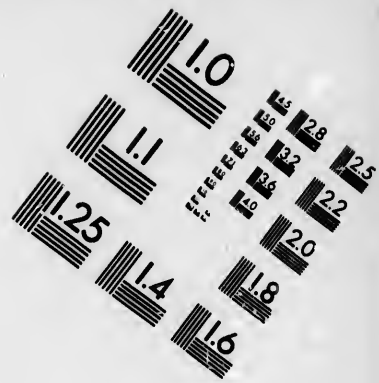
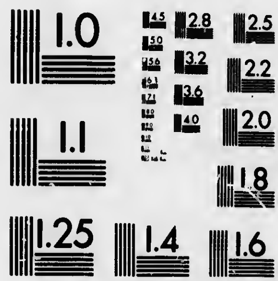


IMAGE EVALUATION TEST TARGET (MT-3)



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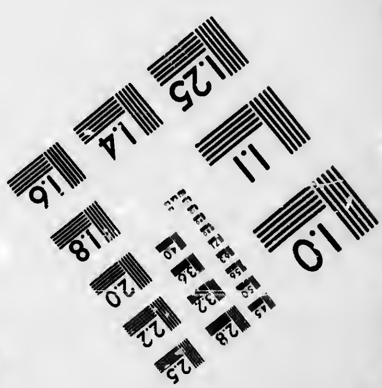


Proof. Let the values of the respective ratios $A : B$, $B : C$, $C : D$ be $\frac{m}{n}$, $\frac{i}{j}$, $\frac{p}{q}$, so that

$$A : B :: m : n, \text{ or ratio } A : B = \frac{m}{n}.$$

$$B : C :: p : q, \text{ or ratio } B : C = \frac{p}{q}.$$

$$C : D :: i : j, \text{ or ratio } C : D = \frac{i}{j}.$$

Then

$$\left.\begin{aligned} \frac{A}{m} &= \frac{B}{n}; \\ \frac{B}{p} &= \frac{C}{q}; \\ \frac{C}{i} &= \frac{D}{j}. \end{aligned}\right\} \qquad (1)$$

Divide the first equation by p and the second by n,

$$\frac{A}{mp} = \frac{B}{np}; \qquad \frac{B}{np} = \frac{C}{nq}.$$

Therefore

$$\frac{A}{mp} = \frac{C}{nq}.$$

Divide this equation by i and the last of (1) by nq,

$$\frac{A}{mpi} = \frac{C}{nqi}; \qquad \frac{C}{nqi} = \frac{D}{nqj}.$$

Therefore

$$\frac{A}{mpi} = \frac{D}{nqj};$$

that is,

$$A : D = \frac{mpi}{nqj} = \frac{m}{n} \times \frac{p}{q} \times \frac{i}{j}.$$

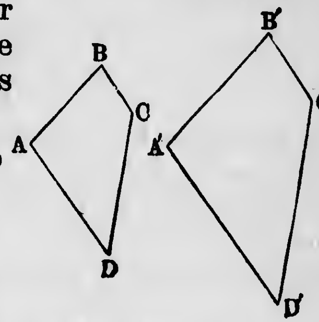
374. *Def.* When the ratio of the first antecedent to the last consequent is formed by multiplying a series of intermediate ratios, the ratio thus obtained is said to be **compounded** of these intermediate ratios.

CHAPTER II.

LINEAR PROPORTIONS.

Definitions.

375. *Def.* **Similar** figures are those of which the angles taken in the same order are equal, and of which the sides between the equal angles are proportional.



EXAMPLE. The figure $ABCD$ is similar to $A'B'C'D'$ when

Angle $A =$ angle A',
Angle $B =$ angle B', etc.,

and

$$AB : BC : CD : DA :: A'B' : B'C' : C'D' : D'A'.$$

376. *Def.* In two or more similar figures any side of the one is said to be **homologous** to the corresponding side of the other.

EXAMPLE. In the above figure,
the sides AB and $A'B'$ are homologous,
the sides BC and $B'C'$ are homologous,
etc. etc. etc.

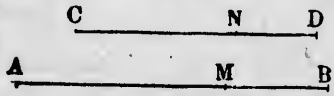
377. *Def.* When a finite straight line, as AB, is cut at a point P between A and B it is said to be **divided internally** at P, and the two parts AP and BP are called **segments.**

378. *Def.* If the straight line AB is produced, and cut at a point Q outside of A and B, it is said to be **divided externally** at Q, and the lines AQ and BQ are called **segments.**

Corollary 1. A line cut internally is equal to the sum of its segments.

Cor. 2. A line cut externally is equal to the difference of its segments.

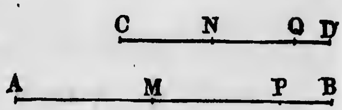
379. *Def.* Two straight lines are said to be **similarly divided** when the different segments of the one have the same ratios as the corresponding segments of the other.



EXAMPLE 1. If the line AB is divided at M and the line cd at N in such manner that

$$AM : MB :: CN : ND,$$

the lines AB and CD are similarly divided at M and N.



EXAMPLE 2. If the lines AB and CD are divided at M, P, N, and Q in such wise that

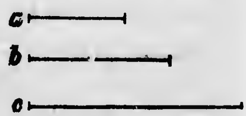
$$AM : MP : PB :: CN : NQ : QD,$$

they are similarly divided.

380. *Def.* If three straight lines, a, b, c, are so related that

$$a : b :: b : c,$$

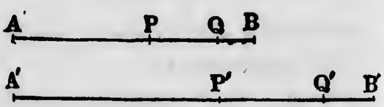
the line b is said to be a **mean proportional** between a and c.



THEOREM I.

381. *If two straight lines are similarly divided, each part of the first has the same ratio to the corresponding part of the second that the whole of the first has to the whole of the second.*

Hypothesis. Two straight lines, AB and $A'B'$, divided at P, Q, P', and Q', so that



$$AP : PQ : QB :: A'P' : P'Q' : Q'B'.$$

Conclusion. $\left.\begin{array}{l} AP : A'P' :: AB : A'B', \\ PQ : P'Q' :: AB : A'B'; \end{array}\right\}$

or, expressed as a multiple proportion,

$$AP : PQ : QB : AB :: A'P' : P'Q' : Q'B' : A'B'.$$

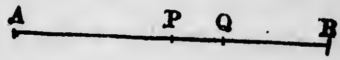
Proof. In the proportion of the hypothesis the sum of the antecedents is AB, and the sum of the consequents $A'B'$. Therefore (§ 371)

$$AB : A'B' :: AP : A'P' :: PQ : P'Q', \text{ etc.} \quad \text{Q.E.D.}$$

THEOREM II.

382. *A line cannot be divided at two different points, both internal or both external, into segments having the same ratio to each other.*

Hypothesis I. A line, AB, divided internally at the points P and Q.



Conclusion. The ratio $AP : PB$ will be different from the ratio $AQ : QB$.

Proof. Let the ratio $AP : PB$ be $\frac{m}{n}$. Then AP will contain m parts, and PB n equal parts.

Because AQ is greater than AP, it will contain more than m parts; and because QB is less than PB, it will contain less than n parts.

Therefore the numerator of the ratio $AQ : QB$ will be greater than m, and its denominator less than n, whence it must be greater than $\frac{m}{n}$ and cannot be equal to it.

Therefore there is no other point of division than P for which the ratio of the segments will be the same as $AP : PB$.

Hypothesis II. A line, AB, divided externally at the points P and Q.

Conclusion. The ratio $AP : BP$ will be different from the ratio $AQ : BQ$.

Proof. Let m be the number of equal parts in AP; n, the number in BP; and s, the number in PQ. Then

$$AP : BP = \frac{m}{n}$$

$$AQ : BQ = \frac{m+s}{n+s}.$$

If we reduce these fractions to a common denominator and take their difference, we find it to be

$$\frac{(m-n)\,s}{n\,(n+s)}.$$

Because m and n are necessarily different, this fraction cannot be zero, and the ratio $AP : BP$ is different from $AQ : BQ$. Q.E.D.

383. *Corollary* 1. When the point of division P is nearer to B than to A, AP is greater than BP, and the ratio $AP : BP$ is greater than unity.

When it is nearer to A than to B, AP is less than BP, the ratio is less than unity.

384. *Cor.* 2. If we suppose the point P to move from A toward B, the ratio $AP : PB$ will be equal to zero as P starts from A, will be unity when P is half way between A and B, and will increase without limit as P approaches B.

385. *Cor.* 3. A line cannot be divided externally into segments having the ratio unity.

386. *Cor.* 4. Two different points may be found, the one internal and the other external, which shall divide a line into segments having the same given ratio.

EXERCISES.

1. Draw a line, AB, and cut it internally in several points so that the ratios of the segments shall be

$$1 : 6,\ 2 : 5,\ 3 : 4,\ 4 : 3,\ 5 : 2,\ 6 : 1.$$

2. Cut the same line externally in the ratios

$$2 : 9,\ 1 : 8,\ 8 : 1,\ 9 : 2,\ 11 : 4.$$

3. A line 7 inches long is to be divided into segments having the ratio 4 : 5. How long are the segments?

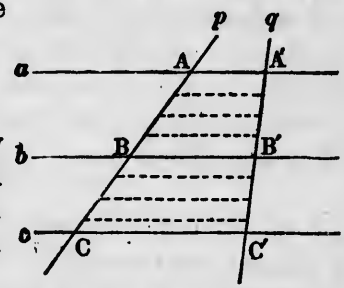
4. A line 6 inches long is to be divided externally into segments having the ratio 5 : 8. How far is the point of division from each end of the line.

5. If the line AB (§ 377) is 3 centimetres in length, and the points P and Q divide it both internally and externally into segments having the same ratio 1 : 2 (§ 386), find the lengths AP, AB, and AQ.

THEOREM III.

387. *If two straight lines are cut by three or more parallel straight lines, any two intercepts on the one are as the corresponding intercepts on the other.*

Hypothesis. a, b, c, three parallels intersecting the line p in the points A, B, C, and the line q in A', B', C'.



Conclusion.

$$AB : BC :: A'B' : B'C'.$$

Proof. Divide AB into any number m of equal parts, and through the points of separation draw lines parallel to AA' and BB'.

Cut off from BC parts equal to those of AB. Let the number of parts in BC be n plus a *fraction*. Through the points of separation draw lines parallel to BB'. Then—

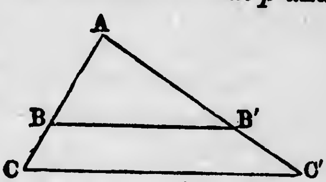
Because the lines p and q are cut by parallels intercepting equal lengths on p, the intercepts on q are also equal (§132, I.).

Therefore $A'B'$ is divided into m equal parts, and $B'C'$ is divided into n equal parts plus a fraction.

Because this is true however great the numbers m and n, we conclude

$$AB : BC :: A'B' : B'C' \text{ (§ 358).} \quad \text{Q.E.D.}$$

Corollary. If the points A and A' coincide so that p and q cross each other at A, the figure ACC' will form a triangle; and the conclusion will, by the same demonstration, be



$$AB : BC :: AB' : B'C'.$$

But BB' is parallel to the side CC' of the triangle. Therefore:

388. *A straight line parallel to one side of a triangle divides the other two sides similarly.*

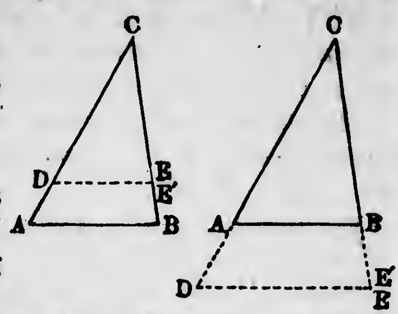
THEOREM IV.

389. *If two sides of a triangle be similarly divided, the line joining the points of division will be parallel to the remaining side of the triangle.*

Hypothesis. ABC, a triangle of which the sides CA and CB are cut (internally or externally) in the points D and E in such manner that

$CD : AD :: CE : BE.$

Conclusion. $DE \parallel AB.$



Proof. If DE is not parallel to AB, draw DE' parallel to AB and meeting CB in E'. Then

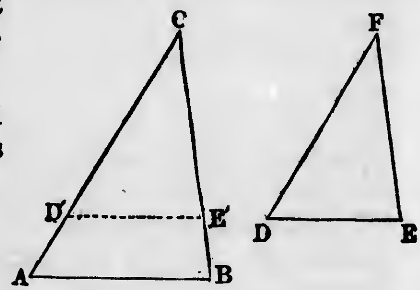
$$CD : AD :: CE' : BE', \qquad (\S\ 388)$$

and the line CB is divided at two points, E and E'; into parts having the same ratio $CD : AD$, which is impossible (§ 382).

Therefore the points E and E' coincide, and the line DE is the same as the parallel DE'. Q.E.D.

Theorem V.

390. *Equiangular triangles are similar, and the sides between the equal angles are homologous to each other.*



Hypothesis. ABC and DEF, equiangular triangles such that

Angle $C =$ angle F.
Angle $A =$ angle D.
Angle $B =$ angle E.

Conclusion. $AB : BC :: DE : EF,$

$AB : AC :: DE : DF,$

or $AB : BC : CA :: DE : EF : FD.$

Proof. Apply the triangle EDF to ABC so that F shall coincide with C and FD shall fall on CA. Let D' be the point of CA on which D falls. Then—

1. Because angle $F =$ angle C,

$$\text{Line } FE \equiv CB.$$

Let E' be the point on which E falls.

2. Because the angle $CD'E'$ is equal to the corresponding angle CAB (hyp.),

$$D'E' \parallel AB. \qquad (\S\ 69)$$

4. Therefore

$$CD' : AD' :: CE' : BE';$$

and, by composition (§ 364),

$$CD' : CA :: CE' : CB;$$

or, because $CD' = FD$ and $CE' = FE$,

$$DF : AC :: EF : BC. \quad \text{Q.E.D.}$$

In the same way it may be shown that the other proportions of the conclusion are true.

391. *Corollary. If from one triangle another be cut off by a line parallel to one of its sides, the triangle thus cut off will be similar to the original triangle.*

Theorem VI.

392. *If the sides of one triangle have to each other the same ratios as the sides of another, these triangles are equiangular and similar.*

Hypothesis. Two triangles, ABC and DEF, such that

$$AB : BC : CA :: DE : EF : FD.$$

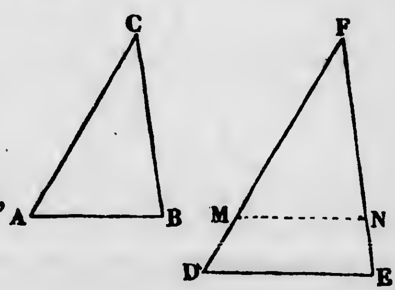
Conclusions.

Angle C (opposite AB) = angle F (opposite DE).
Angle A (" BC) = angle D (" FF).
Angle B (" CA) = angle E (" FD).

Proof. On the sides FD and FE of the triangle DEF take the points M and N such that

$$FM = CA,$$
$$FN = CB,$$

and join MN. Then—



1. Because $FM = CA$ and $NF = BC$, and because $BC : CA :: EF : FD$ (hyp.), we have

$$NF : FM :: EF : FD.$$

2. Therefore the sides FD and FE are divided similarly at M and N, whence

$$MN \parallel DE. \qquad (\S\,389)$$

3. Because these lines are parallel, the triangles FDE and FMN are similar (§ 391). Therefore

$$MN : NF : FM :: DE : EF : FD. \qquad (\S\,390)$$

4. Because $NF = BC$ and $FM = CA$,

$$MN : BC : CA :: DE : EF : FD.$$

5. Comparing with the hypothesis,

$$MN = AB.$$

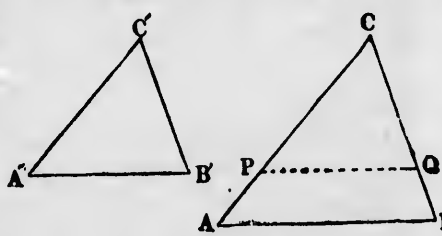
Therefore the triangle FMN is identically equal to CAB, and, comparing the angles opposite the equal sides,

$$\text{Angle } C = \text{angle } F.$$
$$\left.\begin{array}{l}\text{Angle } A = \text{angle } FMN = \text{angle } D;\\ \text{Angle } B = \text{angle } FMN = \text{angle } E.\end{array}\right\} \qquad (3)$$

Q.E.D.

Theorem VII.

393. *Two triangles having one angle of the one equal to one angle of the other, and the sides containing these angles proportional, are similar.*



Hypothesis. ABC and $A'B'C'$, two triangles in which

$$\text{Angle } C = \text{angle } C'.$$
$$C'A' : C'B' :: CA : CB.$$

Conclusion. The triangles are similar; that is,

$$\text{Angle } A = \text{angle } A'.$$
$$\text{Angle } B = \text{angle } B'.$$
$$AB : BC : CA :: A'B' : B'C' : C'A'.$$

Proof. In CA take $CP = C'A'$, and draw PQ parallel to AB. Then—

1. Because PQ is parallel to AB—

I. The triangle CAB is similar to CPQ. (§ 391)
II. $CP : CQ :: CA : CB$. (§ 388)

2. By hypothesis, $C'A' : C'B' :: CA : CB$, and $CP = C'A'$, by construction. Therefore

$$CP : C'B' :: CA : CB.$$

3. Comparing with (1), II.,

$$C'B' = CQ.$$

Therefore the triangles $C'A'B'$ and CPQ have two sides and the included angle equal, and are identically equal.

4. Comparing with (1), I.,

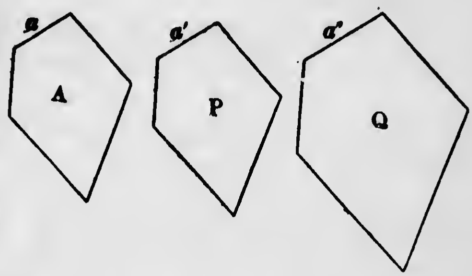
Triangle $C'A'B'$ similar to triangle CAB. Q.E.D.

Theorem VIII.

394. *Rectilineal figures similar to the same figure are similar to each other.*

Hypothesis. Two figures, P and Q, each similar to the figure A.

Conclusion. P and Q are similar to each other.



Proof. 1. Let any side of P, a' for instance, be to the homologous side a of A as $m : n$, and the side a be to the homologous side a'' of Q as $p : q$, so that

$$a' : a = \frac{m}{n},$$

$$a : a'' = \frac{p}{q}.$$

Then, because the antecedent of one of these ratios is the consequent of the next,

$$a' : a'' = \frac{mp}{nq}. \qquad (§ 373)$$

In the same way it may be shown that every side of P has to the homologous side of q the same ratio $mp : nq$.

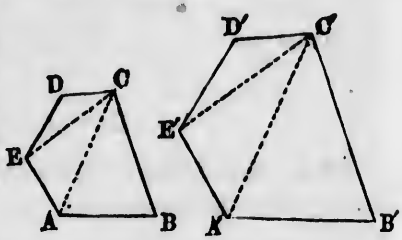
2. Because the angles of P and Q are severally equal to the corresponding angles of A, they are equal to each other.

3. The figures P and Q having their homologous sides in the same ratio $mp : nq$, and their angles equal, are similar by definition (§ 375). Q.E.D.

THEOREM IX.

395. *Similar polygons may be divided into the same number of similar triangles.*

Hypothesis. Two similar polygons, $ABCDE$ and $A'B'C'D'E'$, divided into triangles by lines drawn from the corresponding angles C and C' to all the non-adjacent angles.



Conclusion.

Triangle CDE similar to $C'D'E'$,
Triangle CEA similar to $C'E'A'$,
etc. etc.

Proof. 1. In the triangles CDE and $C'D'E'$, angle $D = D'$ (hyp. and def.), and

$$CD : C'D' :: DE : D'E'.$$

Therefore these triangles are similar and equiangular, and

$$CE : C'E' :: CD : C'D' :: DE : D'E'. \quad (\S\ 393)$$

2. From the equal angles DEA and $D'E'A'$ take the equal angles DEC and $D'E'C'$, and we have left

$$\text{Angle } CEA = \text{angle } C'E'A'.$$

Also, from (1) and the hypothesis of similarity of the two figures,

$$CE : C'E' :: EA : E'A'.$$

Therefore the triangles CAE and $C'A'E'$ are also similar.

3. In the same way it may be shown that all the other triangles into which the polygons are divided are similar.

Q.E.D.

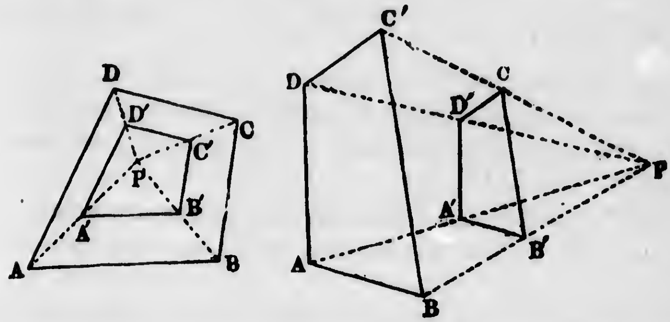
396. *Def.* Similar figures are said to be **similarly placed** when so placed that each side of the one shall be parallel to the homologous side of the other.

THEOREM X.

397. *If two similar figures are similarly placed, then—*

I. *All straight lines joining a vertex of one to the corresponding vertex of the other meet in a point when produced.*

II. *The point of meeting divides the lines externally into segments having the same ratio as the homologous sides of the figures.*



Hypothesis. $ABCD$ and $A'B'C'D'$, two similar figures in which

$AB : BC : CD : DA :: A'B' : B'C' : C'D' : D'A'$,

and of which each side is parallel to its homologous side in the other.

Conclusion. I. The lines AA', BB', etc. (which we shall call *junction lines*), being produced, meet in a point P.

II. $AP : A'P :: BP : B'P$, etc. $:: AB : A'B'$.

Proof. 1. If the junction lines AA' and BB' are not parallel, they must meet in some one point. Let P be this point.

2. Because AB and $A'B'$ are parallel, the triangles PAB and $PA'B'$ are similar, so that

$AP : A'P :: BP : B'P :: AB : A'B'$. (§ 391)

3. It may be shown in the same way that if we call Q the

point in which the junction lines BB' and CC' meet, we shall have

$$BQ : B'Q :: BC : B'C';$$

or, because $BC : B'C' :: AB : A'B'$ (hyp.),

$$BQ : B'Q :: AB : A'B'.$$

4. Comparing with (2),

$$BQ : B'Q :: BP : B'P.$$

5. Therefore the line BB' is cut externally at P and at Q into segments having the same ratio; namely, the ratio of the homologous sides of the figures.

But a line can be cut only at a single point into segments having a given ratio (§ 382). Therefore the points P and Q are the same; that is, the lines AA' and CC' cross BB' at the same point.

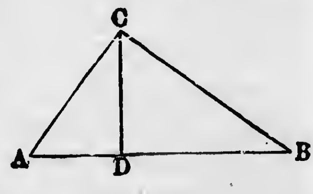
6. In the same way it is shown that all the other junction lines intersect at the point P, and that the segments terminating at P have the same ratio as the homologous sides of the figures. Q.E.D.

398. *Corollary.* If the similar figures are equal, the junction lines will all be parallel and the point of meeting will not exist.

399. *Def.* The point in which the lines joining the equal angles of two similar and similarly placed figures meet each other is called the **centre of similitude** of the two figures.

THEOREM XI.

400. *A perpendicular from the right angle to the hypothenuse of a right-angled triangle divides it into two triangles, each similar to the whole triangle.*



Hypothesis. ABC, a triangle, right-angled at C; CD, a perpendicular from C on AB.

Conclusion. The triangles ABC, ACD, and CBD are all similar to each other, so that

$$CB : CA : AB :: DC : DA : AC :: DB : DC : BC.$$

Proof 1. In the triangles ABC and ACD the angle A is common, and the angle $C =$ angle D because each of them is a right angle.

Therefore the third angles are also equal, and the triangles are equiangular. Comparing the sides opposite equal angles,

$$CB : CA : AB :: DC : DA : AC. \quad \text{Q.E.D.}$$

2. In the same way is shown

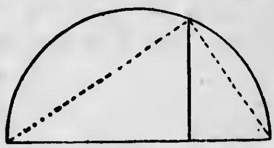
$$CB : CA : AB :: DB : DC : BC. \quad \text{Q.E.D.}$$

Corollary 1. Comparing the equiangular triangles ADC and CDB, we have

$$AD : DC :: DC : DB.$$

Therefore DC is a mean proportional between AD and DB, or:

401. *The perpendicular from the right angle upon the hypothenuse is a mean proportional between the segments into which it divides the hypothenuse.*

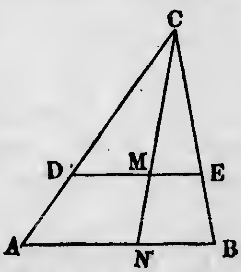


Corollary 2. It has been shown that lines from any point of a circle to the ends of a diameter form a right angle with each other. Therefore:

402. *If from any point of a circle a perpendicular be dropped upon a diameter, it will be a mean proportional between the segments of the diameter.*

THEOREM XII.

403. *If between two sides of a triangle a parallel to the base be drawn, any line from the vertex will divide the base and its parallel similarly.*



Hypothesis. ABC, a triangle; DE, a parallel to AB, intersecting AC in D and BC in E; CN, a line from C, intersecting DE in M and AB in N.

Conclusion. $DM : ME :: AN : NB$.

Proof. 1. Because in the triangles CDM and CAN the sides DM and AN are parallel, these two triangles are equiangular and similar (§ 391).

2. Comparing the homologous sides opposite the equal angles,

$$DM : AN :: CM : CN. \qquad (§ 390)$$

3. In the same way it may be shown that the triangles CEM and CBN are similar, so that

$$ME : NB :: CM : CN.$$

4. Comparing with (2),

$$DM : AN :: ME : NB,$$

or, by alternation,

$$DM : ME :: AN : NB \text{ (§ 362).} \qquad \text{Q.E.D.}$$

Corollary. If $AN = NB$, the ratio will be one of equality and we shall have $DM = ME$. Therefore:

404. *The line drawn from any vertex of a triangle so as to bisect the opposite side will also bisect any line in the triangle parallel to that opposite side.*

Theorem XIII.

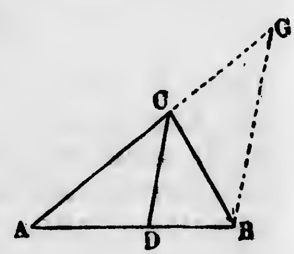
405. *The bisector of an interior angle of a triangle divides the opposite side into segments having the same ratio as the two adjacent sides.*

Hypothesis. ABC, any triangle; CD, the bisector of the angle C, meeting the side AB in D, so that

Angle ACD = angle DCB.

Conclusion.

$$AD : DB :: AC : CB.$$



Proof. Through B draw BG parallel to DC, meeting AC produced in G. Then—

1. Because DC and BG are parallel,

Angle CGB = corresponding angle ACD.
Angle CBG = alternate angle DCB.

2. Comparing with the hypothesis,

$$\text{Angle } CGB = \text{angle } CBG.$$

Therefore the triangle BGC is isosceles, and

$$CB = CG.$$

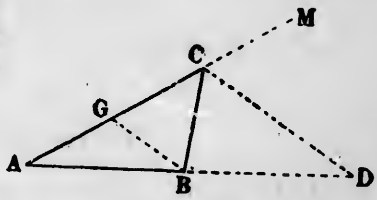
3. Because DC and BG are parallel,

$$AD : DB :: AC : CG,$$
$$:: AC : CB \text{ (by 2).} \quad \text{Q.E.D.}$$

Theorem XIV.

406. *The bisector of an exterior angle of a triangle divides the opposite side externally into segments having the same ratio as the two adjacent sides.*

Hypothesis. ABC, a triangle; CM, the continuation of AC; CD, the bisector of the exterior angle at C, meeting AB produced in D, so that angle BCD = angle MCD.

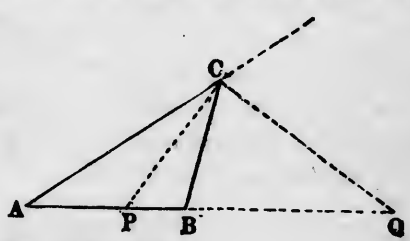


Conclusion.

$$AD : BD :: AC : BC.$$

Prooof. Through B draw BG parallel to DC, meeting AC in G. The proof will then be so much like that of Theorem V. that it is left as an exercise for the student.

Corollary. Since the bisectors of the interior and exterior angles of a triangle each divide the opposite side into segments having the ratio of the other two sides, the ratios of the two divisions are the same. That is,



$$AP : BP :: AQ : BQ.$$

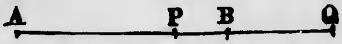
407. *Def.* When a line is divided internally and externally into segments having the same ratio, it is said to be divided **harmonically**.

The preceding corollary may therefore be expressed thus:

408. *The bisectors of an interior and exterior angle at the vertex of a triangle divide the base harmonically.*

Theorem XV.

409. *If a line AB be divided harmonically at the points P and Q, the line PQ will be divided harmonically at the points A and B.*



Hypothesis. A line AB divided internally at P and externally at Q, so that

$$AP : BP :: AQ : BQ.$$

Conclusion. $PB : QB :: PA : QA.$

Proof. From the ratio of the hypothesis we have, by inversion,

$$BP : AP :: BQ : AQ.$$

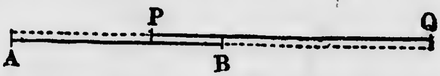
Then, by alternation,

$$BP : BQ :: AP : AQ. \quad \text{Q.E.D.}$$

Harmonic Points.

410. *Def.* The four points A, B and P, Q, of which each pair divide harmonically the line terminated by the other pair, are called **four harmonic points.**

Scholium. The relation of four harmonic points may be made clear to the beginner by supposing the line terminating in one pair to partly overlap the line terminating in the other;

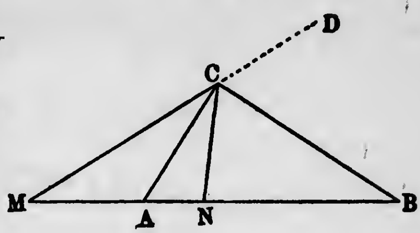


thus, when the points are harmonically aranged, the line AB is divided harmonically at the points P and Q, and the line PQ at the points A and B.

Theorem XVI.

411. *The hypothenuse of a right-angled triangle is divided harmonically by any pair of lines through the right angle, making equal angles with one of the sides.*

Hypothesis. ABC, a triangle, right-angled at C; CM, CN, two lines from C, making angle $MCA = ACN$, and meeting the base in M and N.



Conclusion. The base AB is divided harmonically at M and N.

Proof. Produce MC to any point D. Then—

1. Because angle $MCA = ACN$, CA is the bisector of MCN. Therefore CB, perpendicular to CA by construction, is the bisector of the exterior angle NCD (§ 83).

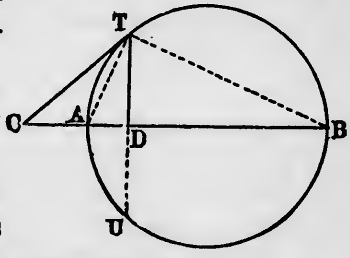
2. Because CA and CB are the bisectors of an interior and exterior angle of the triangle MCN, the base MN of this triangle is divided harmonically at the points A and B (§408).

3. Therefore the line AB is divided harmonically at the points M and N (§ 409). Q.E.D.

THEOREM XVII.

412. *The diameter of a circle is divided harmonically by any tangent and a perpendicular passing through the point of tangency.*

Hypothesis. AB, a diameter of a circle; CT, a tangent at T, cutting the diameter (produced) in C; TD, a perpendicular from T upon AB.



Conclusion. The diameter AB is cut harmonically at C and D.

Proof. Join TA and TB, and produce TD until it cuts the circle in U. Then—

1. Because AB is a diameter perpendicular to the chord TU, it bisects the arc TU (§ 221). Therefore

$$\text{Arc } TA = \tfrac{1}{2} \text{ arc } TU = \text{arc } AU.$$

2. Also, $\quad$ Angle $CTA = \frac{1}{2}$ arc TA. $\quad$ (§ 234)

$\quad\quad\quad\quad$ Angle $ATD = \frac{1}{2}$ arc AU. $\quad$ (§ 235)

Therefore $\quad$ Angle $CTA =$ angle ATD.

3. Because the angle ATB is inscribed in a semicircle, it is a right angle. And because the angles CTA and ATD on each side of TA are equal, the line AB is divided harmonically at C and D (§ 411). Q.E.D.

CHAPTER III.

PROPORTION OF AREAS.

Theorem XVIII.

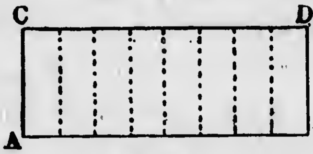
413. ***If two rectangles have equal altitudes, their areas are to each other as their bases.***

Hypothesis. $ABCD$ and $PQRS$, two rectangles in which the altitude PR is equal to the altitude AC.

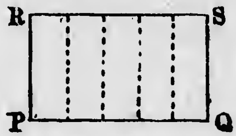
Conclusion.

Area PS : area AD :: PQ : AB.

Proof. 1. Let PQ and AB be to each other as $m : n$. Then, if PQ be divided into m parts and AB into n parts, the mth part of PQ will be equal to the nth part of AB.



n parts.



m parts.

Through the points of division draw lines parallel to the sides of each rectangle.

Then the area PS will be divided into m equal parts and AD into n parts, each equal to the parts of PS. Therefore

Area PS : area AD :: PQ : AB.

2. Because this proportion is true how great soever may be the numbers m and n, it remains true when the sides AB and PQ are incommensurable. Therefore the proportion

Area PS : area AD :: PQ : AB

is true in all cases (§ 359). Q.E.D.

Corollary 1. Because the area of a triangle is one half the area of a rectangle having the same base and altitude (§ 301), and because aliquot parts of magnitudes have the same ratio as the magnitudes themselves (§ 347), therefore:

414. *The areas of triangles having equal altitudes are to each other as their bases.*

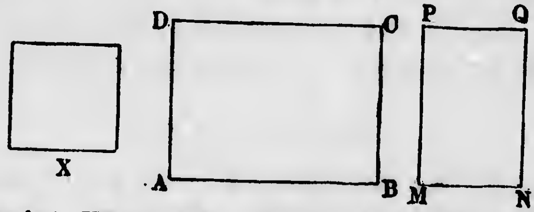
415. *Cor. 2. The areas of all triangles having their vertices in the same point and their bases in the same straight line are to each other as their bases.*

For all such triangles have the same altitude.

Theorem XIX.

416. *The area of a rectangle is expressed by the product of its base and altitude.*

Hypothesis. $ABCD$, a rectangle; X, the unit of area.



Conclusion. When AB and CD are expressed in numbers of which the side of X is the unit, then the area $ABCD$ is expressed in units by the product

$$AB \times AD.$$

Proof. Let us put

a, the number of units in AB;

b, the number of units in AD;

that is (§ 352, II.), $a =$ ratio AB : side X,

$b =$ ratio AD : side X.

The numbers a and b may be either entire, fractional, or incommensurable.

Construct a rectangle $MNPQ$, having $MN =$ side X and $MP = b$. Consider MP as a base of this rectangle. Then—

1. Because $MN =$ side X,

$$\text{Area } MNPQ : X :: MP : \text{side } X;$$

that is, $$\text{Area } MNPQ : X = b.$$

2. Because $MP = AD$,

$$\text{Area } ABCD : \text{area } MNPQ :: AB : MN :: AB : \text{side } X;$$

that is, $$\text{Area } ABCD : \text{area } MNPQ = a.$$

Compounding the ratios (2) and (1),

$$\text{Area } ABCD : X = ab. \qquad (\S\ 373)$$

That is, the area $ABCD$ is expressed by ab when X is taken as the unit (§ 352). Q.E.D.

Corollary 1. Because a triangle has one half the area of a rectangle, with the same base and altitude, we conclude:

417. *The area of a triangle is represented by one half the product of its base and altitude.*

Cor. 2. If $AB = AD$, then $a = b$ and $ab = a^2$. Hence:

418. *The area of the square on a line is expressed algebraically by the square of the number of units in the line.*

We thus prove, for all cases, the theorems proved for whole numbers only in Book IV., §§ 284, 285.

419. *Cor.* 3. *The areas of rectangles or triangles having equal bases are to each other as their altitudes.*

THEOREM XX.

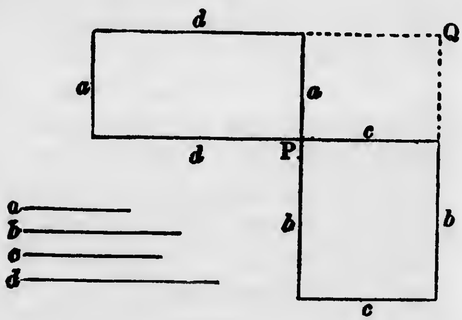
420. *If four straight lines are proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means.*

Hypothesis. Four straight lines a, b, c, d, such that

$$a : b :: c : d.$$

Conclusion. Rect. $a.d$ = rect. $b.c$.

Proof. Form the rectangles ad and bc, and place them so that the sides d and c shall be in one straight line and the sides a and b in another straight line, crossing the first at P. Complete rectangle PQ. Then—



1. Because the rectangles PQ and $a.d$ have the same altitude a, and the bases c and d,

$$\text{Rect. } PQ : \text{rect. } a.d :: c : d. \qquad (\S\ 413)$$

2. Because, taking a and b as bases, the rectangles PQ and $b.c$ have the same altitude c, and the bases a and b,

$$\text{Rect. } PQ : \text{rect. } b.c :: a : b;$$

or, because $a : b :: c : d$,

$$\text{Rect. } PQ : \text{rect. } b.c :: c : d.$$

3. Comparing with the proportion (1), three terms are found equal. Therefore the fourth are also equal, and

$$\text{Rect. } a.d = \text{rect. } b.c. \quad \text{Q.E.D.}$$

Scholium. This theorem corresponds to the theorem of algebra that, if four numbers are proportional, the product of the extremes is equal to the product of the means.

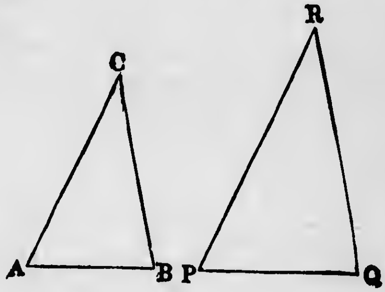
Corollary. If ABC and PQR be two similar polygons in which the sides AB, BC, and CA are respectively homologous to PQ, QR, and RP, we shall have

$$AB : BC :: PQ : QR,$$

and therefore

$$\text{Rect. } AB.QR = \text{rect. } BC.PQ.$$

Hence:



421. *In two similar polygons the rectangle formed by any side of the one and any side of the other is equal in area to that formed by the homologous sides.*

Theorem XXI.

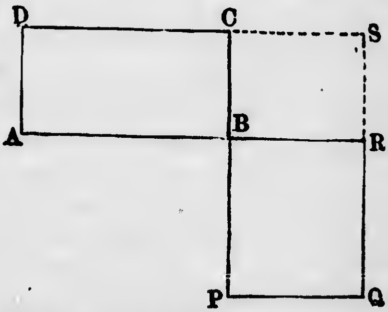
422. *Conversely, if two rectangles are equal in area, the sides of the one will form the extremes, and the sides of the other the means, of a proportion.*

Hypothesis.

$$\text{Rect. } ABCD = \text{rect. } PQRB.$$

Conclusion.

$$AB : BR :: PB : BC.$$



Proof. Place the rect-angles so that the sides AB and BR shall be in one straight line and the sides PB and BC in another straight line, and complete the rectangle $BRSC$. Then—

1. Because the rectangles AC and PR are equal,

$$\text{Rect. } AC : \text{rect. } BS :: \text{rect. } PR : \text{rect. } BS.$$

2. Because the rectangles AC and BS have the same altitudes BC, and stand on the bases AB and BR,

$$\text{Rect. } AC : \text{rect. } BS :: AB : BR.$$

3. In the same way,

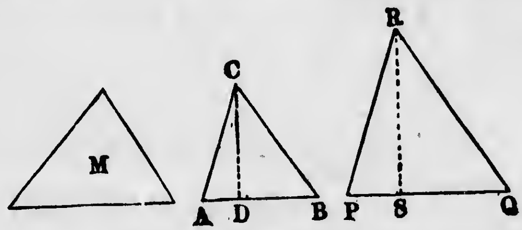
$$\text{Rect. } PR : \text{rect. } BS :: PB : BC.$$

4. Comparing with (1) and (2),

$$AB : BR :: PB : BC. \quad \text{Q.E.D.}$$

Theorem XXII.

423. *The areas of similar triangles are to each other as the squares of their homologous sides.*



Hypothesis. ABC and PQR, two triangles in which

$$AB : BC : CA :: PQ : QR : RP.$$

Conclusion. Area $ABC : PQR :: AB^2 : PQ^2$.

Proof. Construct a third triangle M, of which the base shall be equal to PQ, and the altitude to CD, the altitude of the triangle ABC. Then—

1. Because the triangles M and ABC have the same altitude CD, they are to each other as their bases (§ 414). Therefore

$$\text{Area } ABC : \text{area } M :: AB : PQ.$$

2. Because the triangles M and PQR have equal bases, they are to each other as their altitudes. Therefore

$$\text{Area } M : \text{area } PQR :: CD : RS. \qquad (\S\, 419)$$

3. In the triangles CAD and RPS we have

Angle A = angle P (by hypothesis).
Angle D = angle S (right angles).

Therefore angle $ACD =$ angle PRS, and the two triangles are similar, so that

$$CD : RS :: CA : RP :: AB : PQ.$$

Therefore, from (2),

$$\text{Area } M : \text{area } PQR :: AB : PQ.$$

4. Compounding the ratios (1) and (3), we have

$$\text{Area } ABC : \text{area } PQR :: AB^2 : PQ^2. \quad \text{Q.E.D.}$$

Corollary. The preceding theorem may be expressed in the following form:

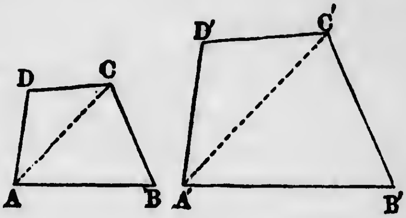
424. *The ratio of the areas of two similar triangles is the square of the common ratio of each side of the one to the homologous side of the other.*

Theorem XXIII.

425. *The areas of similar polygons are to each other as the squares upon their homologous sides.*

Hypothesis. $ABCD$ and $A'B'C'D'$, two similar polygons, in which AB and $A'B'$ are homologous sides.

Conclusion. Area $ABCD$: area $A'B'C'D'$ $:: AB^2 : A'B'^2$.



Proof. Let the polygons be divided into similar triangles by lines drawn from A (Th. IX.). Then—

1. Because the triangles ABC and $A'B'C'$ are similar, and the sides AB and $A'B'$ are homologous,

$$\text{Area } ABC : \text{area } A'B'C' :: AB^2 : A'B'^2.$$

2. In the same way we have

$$\text{Area } ADC : \text{area } A'D'C' :: AD^2 : A'D'^2;$$

or, because AD and AD', as well as AB and $A'B'$, are homologous,

$$\text{Area } ADC : \text{area } A'D'C' :: AB^2 : A'B'^2.$$

3. Continuing these proportions through the whole polygon, and then adding all the antecedents and consequents, we have (§ 371)

$$\text{Area } ABCD : \text{area } A'B'C'D' :: AB^2 : A'B'^2. \quad \text{Q.E.D.}$$

Corollary 1. The result of this theorem may be expressed in the following form:

426. *The ratio of the areas of similar polygons is the square of the ratio of each side of the one to the homologous side of the other.*

427. *Cor.* 2. *If three lines form a proportion, the area of a polygon upon the first is to the area of the similar polygon upon the second as the first is to the third.*

Also, *the areas of the similar polygons upon the second and third lines are in the same ratio.*

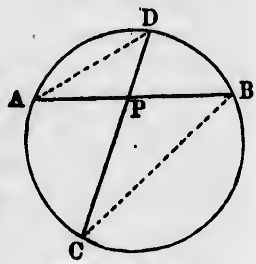
THEOREM XXIV.

428. *If two chords in a circle cut each other, the rectangle of the segments of the one is equal in area to the rectangle of the segments of the other.*

Hypothesis. AB and CD, two chords intersecting at P.

Conclusion.

Rect. $AP.PB$ = rect. $CP.PD$.



Proof. Join AD and BC. Then—

1. Because the angles ADC and ABC stand on the same arc AC,

Angle ADC = angle ABC. (§ 237)

Therefore, in the triangles APD and BPC, we have

Angle APD = opp. angle BPC.

Angle ADP = angle PBC (on same arc AC).

Angle PAD = angle PCB (on same arc BD).

Therefore these triangles are similar (§ 390), and the sides opposite the equal angles are proportional, so that

$$AP : PD :: CP : PB.$$

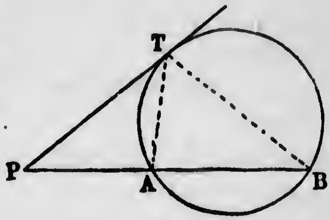
2. Because of this proportion,

Rect. $AP.PB$ = rect. $CP.PD$ (§ 420). Q.E.D.

Scholium. This theorem and the corollary of the following are identical when we consider the chords as cutting each other externally when they do not meet within the circle.

THEOREM XXV.

429. *If from a point without a circle a secant and tangent be drawn, the rectangle of the whole secant and the part outside the circle is equal to the square of the tangent.*



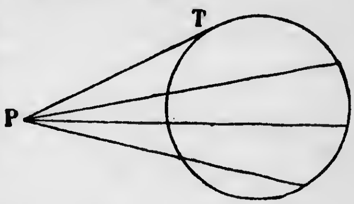
Hypothesis. P, a point without a circle; PT, a tangent touching the circle at T; PB, a secant cutting the circle at A and B.

Conclusion. $PA.PB = PT^2$.

Proof. Join TA and TB. Then—

1. Because the angle ABT stands upon the arc TA,
 Angle $ABT = \frac{1}{2}$ angle arc AT. (§ 235)
2. Because PT is a tangent, and AT a chord,
 Angle $PTA = \frac{1}{2}$ angle arc AT. (§ 234)
3. Therefore angle $ABT =$ angle PTA, and the triangles PAT and PTB have the angles at P identical and two other angles equal. Therefore the third angles PTB and PAT are also equal, and the triangles are similar (§ 390).
4. Comparing homologous sides, we have
 $$PA : PT :: PT : PB.$$
5. Therefore
 Rect. $PA.PB = PT^2$ (§ 420). Q.E.D.

430. *Corollary.* Because the rectangles formed by all secants from P are equal to the square of the same tangent PT, they are all equal to each other.



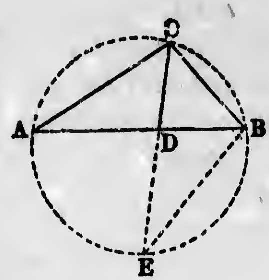
THEOREM XXVI.

431. *When the bisector of an angle of a triangle meets the base, the rectangle of the two sides is equal to the rectangle of the segments of the base plus the square of the bisector.*

Hypothesis. ABC, any triangle; CD, the bisector of the angle at C, cutting the base at D.

Conclusion.

Rect. $CA.CB =$ rect. $AD.DB + CD^2$.



Proof. Circumscribe a circle $ACBE$ around the given triangle, and continue the bisector till it meets the circle in E. Join BE. Then—

1. In the triangles CAD and CEB we have

Angle $ACD =$ angle BCE (hyp.).

Angle $CAD =$ angle BEC (on same arc BC).

Therefore these triangles are equiangular and similar.

2. Comparing the sides opposite equal angles,

$$CA : CD :: CE : CB.$$

Whence

$$\begin{aligned} \text{Rect. } CA.CB &= \text{rect. } CE.CD, \\ &= \text{rect. } (CD + DE)CD, \\ &= CD^2 + \text{rect. } CD.DE, \qquad (\S\ 287) \\ &= CD^2 + \text{rect. } AD.DB\ (\S\ 428). \qquad \text{Q.E.D.} \end{aligned}$$

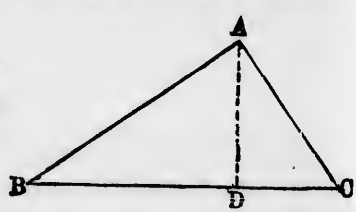
Theorem XXVII.

432. *In a right-angled triangle the area of any polygon upon the hypothenuse is equal to the sum of the areas of the similar and similarly described polygons upon the two other sides.*

Hypothesis. ABC, a triangle, right-angled at A. We also put

$$a,\ a',\ a'',$$

the areas of any three similar and similarly described polygons, upon the sides BC, CA, and AB, respectively.



Conclusion. $a = a' + a''$.

Proof. From A drop $AD \perp BC$. Then—

1. Because ABC is right-angled at A, and AD is a perpendicular upon the hypothenuse,

$$DC : CA :: CA : BC.$$
$$DB : BA :: BA : BC.$$

2. Because a', a'', and a are the areas of similar polygons described upon CA, BA, and BC,

$$a' : a :: DC : BC.$$
$$a'' : a :: BD : BC. \qquad (\S\, 427)$$

3. Therefore, taking the sum of the ratios (§ 366),

$$a' + a'' : a :: BD + DC : BC.$$

4. Because $BD + DC = BC$, the second ratio is unity; therefore the first also is unity, and

$$a' + a'' = a. \quad \text{Q.E.D.}$$

Scholium. This result includes the Pythagorean proposition (§ 308), as a special case in which the polygons are squares.

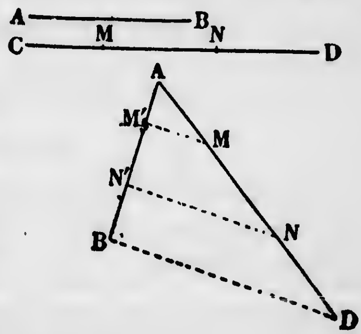
CHAPTER IV.

PROBLEMS IN PROPORTION.

Problem I.

433. *To divide a straight line similarly to a given divided straight line.*

Given. A line, AB; another line, CD, divided at the points M and N.



Required. To divide AB similarly to CD.

Analysis. By § 388 two sides of a triangle are similarly divided by any lines parallel to the base. Therefore, if we put together the lines AB and CD in such a way as to form two sides of a triangle, all lines parallel to the third side will divide these two sides similarly. Hence

Construction. 1. Form a triangle ABD, such that

$AB =$ given line AB.
$AD =$ given line CD.
$BD =$ any convenient length.

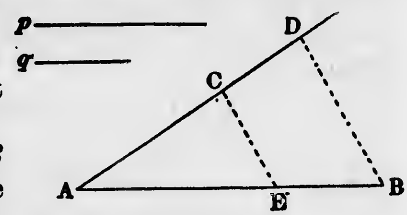
2. Through the points M and N draw MM' and NN' parallel to BD, meeting AB in M' and N'. Then

$$AM' : M'N' : N'B :: AM : MN : ND. \qquad (\S\ 388)$$

Therefore the line AB is divided at the points M' and N' similarly to CD. Q.E.F.

PROBLEM II.

434. *To divide a straight line internally into segments which shall be to each other as two given straight lines.*



Given. Two straight lines, p and q; a third straight line, AB.

Required. To divide AB into segments having the same ratio as p to q.

Construction. 1. From one end of the line AB draw an indefinite straight line AD.

2. From this line cut off $AC = p$ and $CD = q$.

3. Join DB.

4. From C draw a line parallel to DB, and let E be the point in which it cuts AB.

The line AB will be cut internally at E into the segments AE and EB, having to each other the same ratio as the lines p and q.

Proof. As in Problem I.

PROBLEM III.

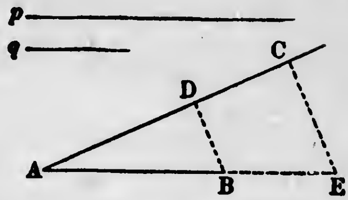
435. *To divide a straight line externally so that the segments shall be to each other as two given straight lines.*

Given. A straight line, AB; two other lines, p, q.

Required. To cut AB externally so that the segments shall have to each other the ratio $p : q$.

Construction. 1. From either end of the line AB, as A, draw an indefinite straight line AC.

2. On this line cut off $AC =$ the greater line p, and from C, toward A, cut off $CD =$ the lesser line q.



3. Join DB.

4. From C draw a line parallel to DB, and let E be the point in which it cuts AB produced.

The line AB will be divided externally at E, so that

$$AE : BE :: p : q.$$

Proof. As in Problem I.

436. *Corollary* 1. If $p = q$, the point D would fall upon A, and the line DB would coincide with AB. The line drawn through C parallel to DB would then be parallel to AB, so there would be no point of intersection E. Therefore there would be no external point for which the ratio of the segments would be unity.

437. *Cor.* 2. If we combine Problems II. and III. on the same straight line, using the same ratio, we shall divide the line harmonically, and the ends of the line, together with the points of division, will form four harmonic points.

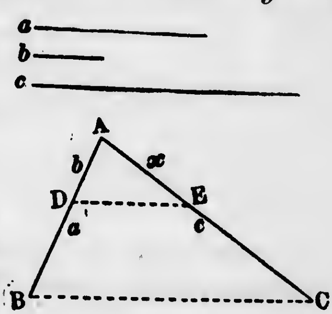
Problem IV.

438. *To find a fourth proportional to three given straight lines.*

Given. Three straight lines, a, b, c.

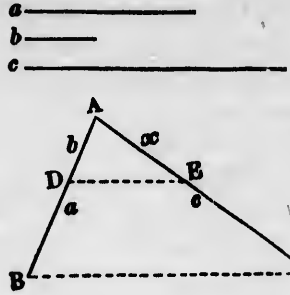
Required. To find a fourth line, x, such that

$$a : b :: c : x.$$



Analysis. To solve the problem it is only necessary to form two similar triangles, one of which shall have a and c for two of its sides, while the other shall have b as the side homologous to a. The side homologous to c will then be x.

Construction. 1. Draw two straight lines from the same point A.



2. From one of them cut off $AB = a$ and $AD = b$, and from the other cut off $AC = c$.

3. Join BC.

4. Through D draw $DE \parallel BC$.

AE will be the required line x.

Proof. The similar triangles ABC and ADE give

$$AB : AD :: AC : AE,$$

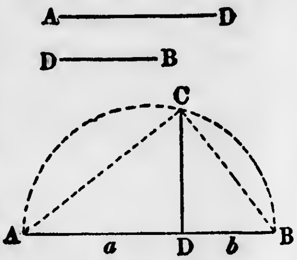
which is the required proportion.

PROBLEM V.

439. *To find a mean proportional between two given straight lines.*

Given. Two straight lines, AD and DB.

Required. To find a third line which shall be a mean proportional between them.



Analysis. The perpendicular from any point of a circle upon the diameter is a mean proportional between the segments into which it divides the diameter (§ 402). Hence

Construction. 1. On an indefinite line take the segments AD and DB equal to the given lines.

2. On AB as a diameter describe a semicircle.

3. At D erect the perpendicular DC, meeting this circle in C. DC will then be the required mean proportional between AD and DB.

Proof. This may be supplied by the student from §§ 208 and 401.

440. *Def.* A straight line is said to be divided in **extreme and mean ratio** when it is divided into two segments, such that the greater segment is a mean proportional between the lesser one and the whole line.

PROBLEM VI

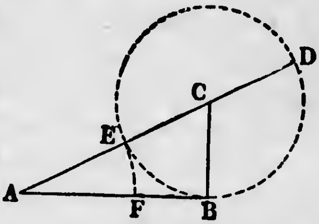
441. *To divide a straight line in extreme and mean ratio.*

Construction. Let AB be the given straight line. Then—

1. At one end B of the given straight line erect a perpendicular BC, and take $BC = \frac{1}{2}AB$.

2. Around C as a centre, with a radius CB describe a circle.

3. Join AC, produce the line AC, and let E and D be the points in which it intersects the circle.



4. From A as a centre draw an arc cutting off from AB the length $AF = AE$.

The line AB will then be cut at F in such manner that

$$FB : AF :: AF : AB;$$

that is, it will be cut at F in extreme and mean ratio.

Proof. 1. Because AB is perpendicular to the radius CB at its extremity, B, it is a tangent to the circle. Therefore

$$AE : AB :: AB : AD. \qquad (\S\ 429,\ 4)$$

2. By division,

$$AE : AB - AE :: AB : AD - AB. \qquad (\S\ 365)$$

3. But, by construction, $AE = AF$; and because $AB = 2CB$, AB is equal to the diameter ED, so that $AD - AB = AD - ED = AE = AF$. Making these substitutions in (2),

$$AF : FB :: AB : AF;$$

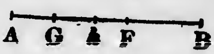
or, by inversion,

$$FB : AF :: AF : AB.$$

4. Therefore the segment AF is a mean proportional between the segment FB and the whole line AB.

442. *Scholium.* This division of the straight line has such interesting properties that the ancient geometers called it the *golden section.*

One of the properties is this: The line AB being divided in extreme and mean ratio at F, if we take $FG = FB$, the line AF will be cut in extreme and mean ratio at G. For, by hypothesis,



$$FB : AF :: AF : AB;$$

whence, by inversion and division,

$$AF - FB : FB :: AB - AF : AF.$$

Because $GF = FB$, this proportion is the same as

$$AG : GF :: GF : AF.$$

In the same way, by taking on GF a line Gh equal to AG we shall divide GF in extreme and mean ratio at h, and so on indefinitely.

443. *Corollary. The two segments formed by the "golden section" are incommensurable with each other.*

For suppose AF were composed of m parts, and FB of n equal parts. Then when we cut off FG, we should have

$$AG = m - n \text{ parts}.$$

Cutting this off from GF, we should have in hF a still smaller number of parts. But no part would ever be divided by cutting, because when we subtract one whole number from another, the remainder is a whole number.

Therefore, because every time we cut off we reduce the number of parts by one or more, our last section would take away all the line that was left, and so would not cut it in extreme and mean ratio.

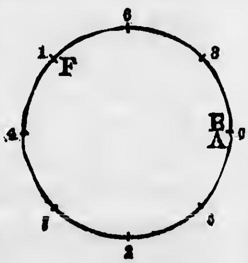
Illustration. If AF had eight parts, and FB five parts, then by successively cutting off we should have left

$$AG = 3 \text{ parts} = Gh.$$
$$hF = 2 \text{ parts}.$$

Cutting off these two parts from Gh, we should have one part left, and after two more cuttings nothing would be left.

The Phyllotaxis.

444. Let us bend the line AB around into a circle, the two ends, A and B, coming together. Let us then take the distance BF in our dividers, and, starting from the point AB, keep measuring off equal steps round and round the circle. The end of each step is numbered from 0 through 1, 2, 3, etc. Then—



1. At every step we shall find the circle divided into arcs of two or three different lengths.

2. At every step the dividers will fall upon one of the longer arcs, and will divide it in extreme and mean ratio.

3. The division-marks will be scattered around the circle more evenly than by any other system of division.

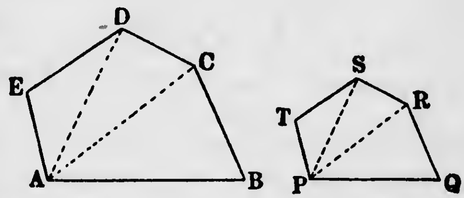
It has been considered by botanists that the leaves of plants are arranged around the stem on such a plan as this. This arrangement is then called the *phyllotaxis*.

PROBLEM VII.

445. *Upon a given straight line to construct a polygon similar to a given polygon.*

Given. A polygon, $ABCDE$; a straight line, PQ.

Required. To construct upon PQ a polygon similar to $ABCDE$.



Construction. Let AB be the side of the given polygon to which PQ is to be homologous. Divide the given polygon into triangles by diagonals from the point A.

Upon PQ describe the triangle PQR equiangular to ABC.

Upon PR describe the triangle PRS equiangular to ACD, and continue the process through all the triangles into which $ABCDE$ is divided.

The polygon $PQRST$ will be the one required.

Proof. By § 395.

PROBLEM VIII.

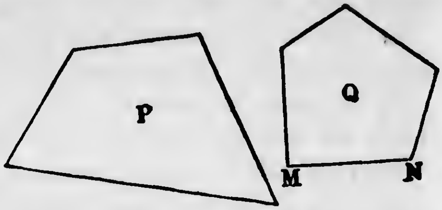
446. *To describe a polygon which shall be equal to one and similar to the other of two given polygons.*

Given. Two polygons, P and Q.

Required. To construct a third polygon, equal in area to P and similar to Q.

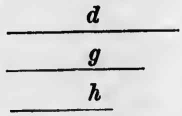
Analysis. Because the required figure is similar to Q, the square of any one of its sides will have the same ratio to the square of the homologous side of Q that its area has to the area of Q.

Because the area is that of P, this ratio of the squares of homologous sides is the ratio of the area of P to the area of Q.



Therefore if we construct two squares, the one equal in area to P and the other to Q, the sides of these squares will have the same ratio which each side of the required figure has to the homologous side of Q.

Construction. Construct the side d of a square equal in area to P, and the side g of another square equal in area to Q (§ 321).



Take any side MN of Q and find a fourth proportional h to g, d, and MN (§ 438).

Upon h describe a polygon X similar to Q, and having h as the side homologous to MN.

This polygon X will be equal in area to P, as well as similar to Q.

Proof. Because h and MN are homologous sides of the similar polygons X and Q,

$$\text{Area } X : \text{area } Q :: h^2 : MN^2;$$

or, because by construction $h : MN :: d : g$,

$$\text{Area } X : \text{area } Q :: d^2 : g^2.$$

But, by construction, $d^2 = \text{area } P$ and $g^2 = \text{area } Q$.

Therefore $\qquad \text{Area } X : \text{area } Q :: \text{area } P : \text{area } Q.$

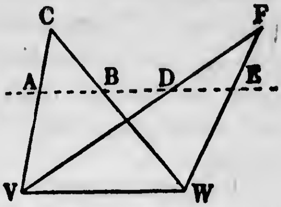
Whence $\qquad \text{Area } X = \text{area } P.$ Q.E.F.

Theorems for Exercise.

Theorem 1. If the ends of two intersecting chords be joined by straight lines, the two triangles thus formed will be similar to each other.

Theorem 2. If two chords of circles subtend arcs which together make up a semicircle, the sum of their squares is equal to the square of the diameter.

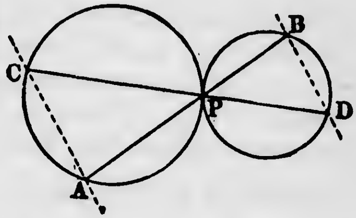
Theorem 3. If from any point within a parallelogram lines be drawn to the four vertices, the sum of the areas of each pair of opposite triangles is half the area of the parallelogram.



Theorem 4. If two equal triangles on the same base be cut by a line parallel to the base, equal areas will be cut off from them.

Area $ABC = DEF$.

Theorem 5. Lines drawn through the point of contact of two circles, and terminated by the circles, form four chords which are proportional, and the lines through the ends of which are parallel.

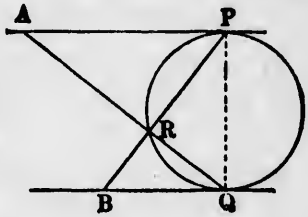


Conclusions.

I. $AP : PC :: BP : PD$.

II. $AC \parallel DB$.

Theorem 6. If a circle be described touching two parallels, and from the points of tangency secants be drawn intersecting the circle in the same point, and terminated by the opposite parallel, the diameter of the circle will be a mean proportional between the segments of the parallels.



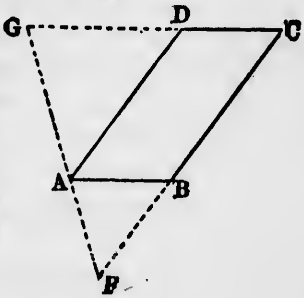
Hypothesis. The lines PB and QA intersect at R, a point of the circle.

Conclusion. $BQ : PQ :: PQ : AP$.

Corollary. The same thing being supposed, prove

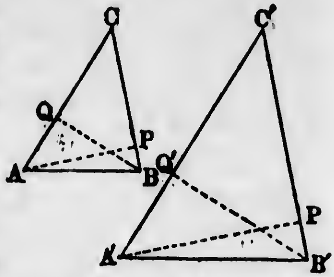
Rect. $RA.RB =$ rect. $RP.RQ$.

Theorem 7. If through any vertex of a parallelogram a line be drawn meeting the two opposite sides produced without the parallelogram, the rectangle of the produced portion of such sides is equal to the rectangle of the sides of the parallelogram.



Hypothesis. $GD \,.\, BF = AB \,.\, BC$.

THEOREM 8. If two triangles have one angle of the one equal to one angle of the other, and the perpendiculars from the other two angles upon the opposite sides proportional, they are similar.

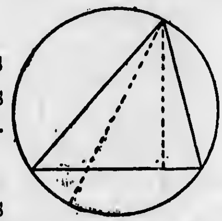


Hypothesis. Angle $C =$ angle C'.
$AP : BQ :: A'P' : B'Q'$.

Conclusion. The triangles are similar.

THEOREM 9. If the four sides of an inscribed quadrilateral taken consecutively form a proportion, the diagonal having the means on one side and the extremes on the other side divides it into two triangles of equal area. (Book IV., Ex. Th. 9.)

THEOREM 10. The rectangle of two sides of a triangle is equal to the rectangle of its altitude above the third side and the diameter of the circumscribed circle.



THEOREM 11. The area of a triangle is equal to the product of the three sides divided by twice the diameter of the circumscribed circle.

THEOREM 12. If two parallel tangents to a circle are intercepted by a third tangent, the rectangle of the segments of the latter is equal to the square of the radius of the circle.

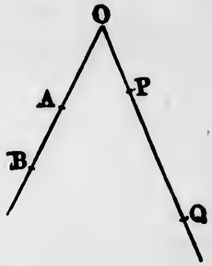
THEOREM 13. If a chord be drawn parallel to the tangent at the vertex of an inscribed triangle, the portion of the triangle cut off by the chord is similar to the original triangle.

THEOREM 14. If from the middle point A of the arc subtended by a fixed chord a second chord be drawn intersecting the fixed one, the rectangle contained by the whole of that second chord and the part of it intercepted between the fixed chord and the point A is a constant whatever be the direction of the second chord.

Show to what square or area the constant area is equal.

THEOREM 15. If in two triangles any angle of the one is equal to some angle of the other, their areas are to each other as the rectangles of the sides which contain the equal angles.

THEOREM 16. If two chords of a circle intersect each other at right angles, the sum of the squares of the four segments is equal to the square of the diameter.



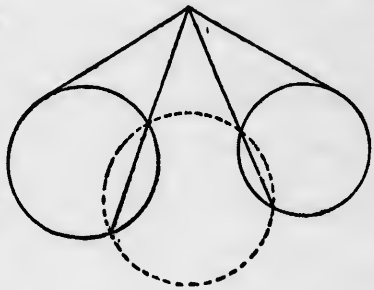
THEOREM 17. If on each of the sides of an angle having its vertex at O two points A and B on the one side and P and Q on the other be taken such that

$$OP : OA :: OB : OQ,$$

the four points A, B, P, and Q will lie on a circle (§§ 244, 392).

THEOREM 18. If at any point outside of two circles a point be chosen from which the tangents to the two circles shall be equal in length, and from this point secants to each circle be drawn, the four points of intersection will lie on a third circle.

Apply § 429, 4, to the case of each circle.



THEOREM 19. Conversely, if two circles be intersected by a third, and secants be drawn through each pair of points of intersection, the tangents to the circles from the point of intersection of the secants will be equal in length.

THEOREM 20. If the common secant of two intersecting circles be drawn, the tangents to the two circles from each point of this secant will be equal in length.

NUMERICAL EXERCISES.

1. If one of two similar triangles has its sides 50 per cent longer than the homologous sides of the other, what is the ratio of their areas?

2. The owner of a rectangular farm containing 10,000 square yards finds that it measures 5 inches × 20 inches on a map. What are its length and breadth?

BOOK VI.

REGULAR POLYGONS AND THE CIRCLE.

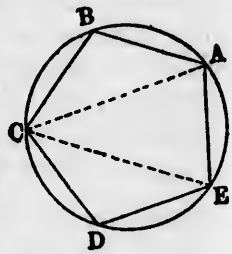
CHAPTER I.

PROPERTIES OF INSCRIBED AND CIRCUMSCRIBED REGULAR POLYGONS.

Theorem I.

447. ***If a circle be divided into any number of equal arcs, and a chord be drawn in each arc, these chords will form a regular polygon.***

Hypothesis. $A, B, C, D, E,$ equidistant points around a circle, separating it into equal arcs; AB, BC, etc., the chords of those arcs.



Conclusion. The polygon $ABCDE$ is regular (§ 152).

Proof. I. Because the sides are by hypothesis all chords of equal arcs, they are all equal to each other.

II. Take any two angles of the polygon, say ABC and CDE. Join AC and CE. Then—

2. Because the arcs AC and CE are equal, being sums of equal arcs,

$$\text{Chord } AC = \text{chord } CE. \qquad (\S\ 208)$$

3. Hence in the triangles ABC and CDE we have

$$AB = CD,$$
$$BC = DE,$$
$$AC = CE.$$

Therefore these triangles are identically equal, and

$$\text{Angle } ABC = \text{angle } CDE.$$

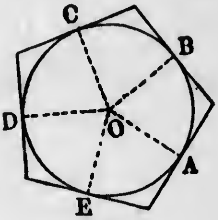
4. In the same way it may be shown that any other two angles of the inscribed polygon we choose to take are equal.

5. Comparing (1) and (4), the polygon is shown to be regular (§ 152). Q.E.D.

Scholium. The equality of the angles of the polygon may be proved with yet greater elegance by showing that they are all inscribed in equal segments.

THEOREM II.

448. *If a circumscribed polygon touch a circle at equidistant points around it, it is regular.*



Hypothesis. A circumscribed polygon whose sides touch the circle at the equidistant points $ABCDE$.

Conclusion. This polygon has all its sides and angles equal.

Proof. Let O be the centre of the circle. Join OA, OB, etc. Then—

1. Because the intercepted arcs AB, BC, etc., are equal, we shall have

$$\text{Angle } AOB = BOC = COD, \text{ etc.}$$

Turn the figure around on the point O until the radius OA coincides with the trace OB. Then—

2. Because of the equality of the angles AOB, BOC, etc., OB will fall upon the trace OC; $OC \equiv$ trace OD, etc.

3. Because the radii are equal, the point A will fall on B, B on C, etc.

4. Because each radius is perpendicular to the tangent at its extremity, each side will fall upon the trace of the side next following.

5. Therefore each point of intersection will fall on the trace of the point next following.

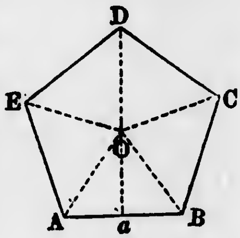
6. Therefore each side and angle is equal to the side and angle next following, and the polygon has all its sides and angles equal. Q.E.D.

Remark. Another demonstration may be found by drawing lines from O to the angles of the polygon, and proving the equality of all the triangles thus formed.

THEOREM III.

449. *A circle may be inscribed in any regular polygon, or circumscribed about it.*

Proof. I. Let $ABCDE$ be the regular polygon. Bisect any two adjacent angles of the polygon, as A and B, and let O be the point of meeting of the bisecting lines. Then—



1. In the triangle AOB the angles OAB and OBA are by construction the halves of the equal angles EAB and ABC (hyp.).

Therefore the angles are equal and the triangle is isosceles.

2. Join OC. In the triangles AOB and BOC we have

$$BC = AB \text{ (hyp.)}.$$
$$OB = OB \text{ (common side)}.$$
$$\text{Angle } OAB = \text{angle } OBC \text{ (halves of equal angles)}.$$

Therefore these two triangles are identically equal, and

$$OC = OB.$$
$$\text{Angle } OCB = \text{angle } OBA = \tfrac{1}{2} \text{ angle } B = \tfrac{1}{2} \text{ angle } C.$$

3. In the same way it may be shown that if we join O to the other angles of the polygon, the triangles thus formed will all be identically equal. Therefore

$$OA = OB = OC = OD = OE.$$

Hence if a circle be drawn around O as a centre with a radius equal to either of these lines, it will pass through all the points A, B, C, D, E, and will be circumscribed around the polygon. Q.E.D.

II. Let Oa be the perpendicular from O upon AB.

4. Because the triangles OAB, OBC, etc., are identically equal (2), the perpendiculars from O upon AB, BC, etc., are equal (§ 175).

Therefore if a circle be drawn around O as a centre, with a radius Oa, this circle will also pass through the feet of the perpendiculars dropped from O upon BC, upon CD, etc.

5. Because each of the sides AB, BC, etc., is perpen-

dicular to a radius at its extremity, it is a tangent to the circle.

Therefore the circle is inscribed in the polygon. Q.E.D.

450. *Corollary* 1. *The inscribed and circumscribed circles of a regular polygon are concentric.*

451. *Cor.* 2. *The bisectors of the angles of a regular polygon all meet in a point, which point is the centre of both the inscribed and circumscribed circles, and is equally distant from all the angles and all the sides of the polygon.*

452. *Def.* The common centre of the inscribed and circumscribed circles is called the **centre** of the regular polygon.

453. *Cor.* 3. *The perpendicular bisectors of the sides of a regular polygon all pass through its centre.*

454. *Cor.* 4. *If lines be drawn from the centre of a regular polygon to each of its vertices, the polygon will be divided into as many identically equal isosceles triangles as it has sides.*

Theorem IV.

455. *All regular polygons having the same number of sides are similar to each other.*

Proof. 1. If the polygons have each n sides, the sum of all the n angles of each is equal to $n-2$ straight angles (§ 160).

2. Because the angles of each polygon are equal, each angle of each polygon is the nth part of $n-2$ straight angles; that is, the angles of one polygon are each equal to the angles of the other.

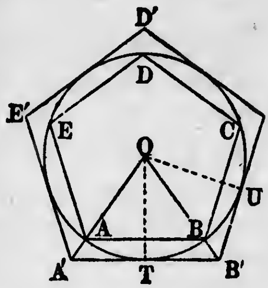
3. Because the sides of each polygon are equal to each other (hypothesis), the ratio of any side of the one to any side of the other is the same whatever side be chosen.

Therefore the polygons are similar (§ 375). Q.E.D.

Theorem V.

456. *Regular inscribed and circumscribed polygons of the same number of sides may be so placed that their sides shall be parallel, and each vertex of the one on the same radius with a vertex of the other.*

Hypothesis. $ABCDE$, an inscribed regular polygon; $A'B'C'D'E'$, a circumscribed polygon of the same number of sides, having the side $A'B'$ tangent to the circle at T, the middle point of the arc AB.



Conclusions. I. $A'B' \parallel AB$;
$B'C' \parallel BC$, etc.

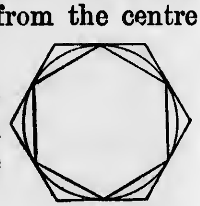
II. The vertex A' is on the radius OA produced, and all the other vertices, B', C', etc., are on the radii OB, OC, etc., produced.

Proof I. 1. Because $A'B'$ is tangent at the middle point of the arc AB, it is parallel to the chord of that arc.

2. Because the two polygons are equiangular (Th. IV.) and have a pair of homologous sides parallel, all the other homologous sides are parallel. Q.E.D.

Proof II. Because the line $B'O$ is drawn from the intersection of the two tangents, $B'U$ and $B'T$, to the centre of the circle, it bisects the arc TU.

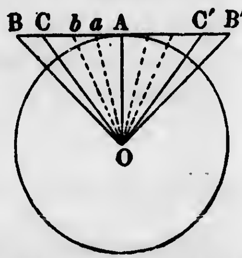
Therefore it passes through the middle point B of this arc, and OBB' are in the same straight line from the centre of the circle.



457. *Corollary.* The polygons may also be so placed that the circumscribed polygon shall touch the circle at the angles of the inscribed polygon.

THEOREM VI.

458. *The greater the number of sides of a regular circumscribed polygon the less will be its perimeter.*



Hypothesis. BB', one side of a regular circumscribed polygon having n sides; CC', one side of another such polygon having $n+1$ sides; OA, a perpendicular from the centre upon BB'.

Conclusion. The perimeter of the polygon having n sides is greater than that having $n+1$ sides.

Proof. 1. Because the one polygon has n equal sides, and the other $n+1$ equal sides, we have

$$\text{Angle } C'OC = \frac{360°}{n+1}.$$

$$\text{Angle } B'OB = \frac{360°}{n}.$$

Therefore

$$\text{Angle } C'OC : \text{angle } B'OB :: n : n+1. \qquad (\S 337)$$

And because AOC and AOB are respectively the halves of $C'OC$ and $B'OB$,

$$\text{Angle } AOC : \text{angle } AOB :: n : n+1.$$

That is, if we divide the angle AOC into n equal parts, AOB will be $n+1$ of these parts. Hence COB will be one of the parts.

2. Divide the angle AOC into n parts by straight lines meeting AB in the points a, b, etc.

Because OA is a perpendicular upon BB', each of the segments Aa, ab, bC, etc., will be longer than the segment next preceding (§ 116). Therefore n times segment CB will be greater than the sum of the n segments which make up AC.

3. The perimeter of the polygon of n sides is

$$nAB = nAC + nCB,$$

and that of the polygon of $n+1$ sides is

$$(n+1)\,AC = nAC + AC.$$

4. Because nCB is greater than AC,

$$nAB > (n+1)\,AC;$$

that is, the perimeter of the polygon of n sides is the greater.

Q.E.D.

CHAPTER II.

CONSTRUCTION OF REGULAR POLYGONS.

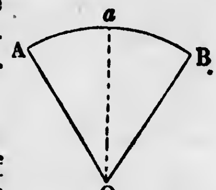
459. Theorem I. of this book shows that a regular polygon of any number of sides may be inscribed in a circle by dividing the latter into as many equal parts as the polygon has sides, and joining the points of division by chords. Hence the problem of constructing such polygons is reduced to that of dividing a circle into any required number of equal parts.

The following theorem may be seen almost without demonstration.

If we can divide a circle into any number of equal parts, we can also divide it into twice that number.

For if we divide it into n parts, we have only to bisect each of these parts, which we do by § 271, and it will be divided into $2n$ parts.

460. It is also easily seen that the problem of dividing an arc into any number of parts may be reduced to that of dividing the angle corresponding to the arc into the same number of parts.

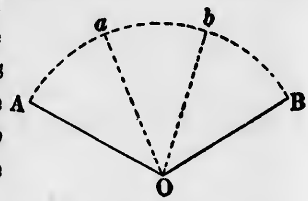


For let AB be the arc, and O its centre.

If we divide the arc into any number of equal parts, and join O to the points of division, the angle AOB will be divided into that same number of equal parts.

Hence, *by dividing the arc we divide the angle.*

461. Conversely, if, having an angle AOB, we draw the arc AB of a circle around the vertex O as a centre, and divide the angle AOB into any number of equal parts by the lines Oa and Ob, meeting the arc in a and b, the points a and b will divide the arc into that same number of equal parts.



Hence, *by dividing the angle we divide the arc.*

The problem of bisecting the angle (or arc) is so simple that it offered no difficulty to the ancient geometers. By bisecting the halves of each arc it might be divided into fourths, and so on; therefore there was no difficulty in dividing any angle or arc into 2, 4, 8, 16, etc., equal parts.

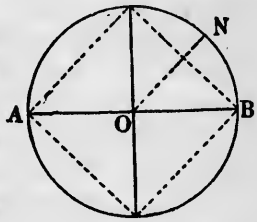
This being so easy, it was naturally sought to *trisect* the angle, or divide it into three equal parts.

This problem of the trisection of the angle was long celebrated. But geometers never succeeded in solving it, and it is now considered impossible by the constructions of elementary geometry.

PROBLEM I.

462. *To divide a given circle into* 2, 4, 8, 16, *etc., equal parts.*

Any diameter, as AB, will divide the circle into two equal parts at the points A and B.



Drawing through the centre O another diameter perpendicular to this, the circle will be divided into four equal parts by four radii, at right angles to each other.

By joining the ends of these radii a square will be described in the circle.

Bisecting each of the right angles at the centre by radii like ON, the circle will be divided into eight equal parts.

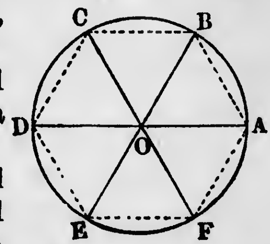
By joining the points of division, a regular octagon will be inscribed in the circle.

The process of bisection may be continued indefinitely, so as to divide the circle into 2^m equal parts, where m may be any positive integer.

PROBLEM II.

463. *To divide the circle into* 3, 6, 12, 24, *etc., equal parts.*

Analysis. 1. Suppose the division into six parts effected, and the points of division to be A, B, C, D, E, F.



2. Draw the radii OA, OB, etc., and join AB, BC, CD, etc., forming a regular hexagon.

3. Because each of the three equal angles AOB, BOC, COD, is one third the straight angle AOD, they are each angles of 60°. And because the sides OA, OB, are equal, the angles OAB and OBA are also equal.

4. But the sum of the three angles is 180°. Therefore the three angles are each equal to 60°; that is, the triangle OAB

is equiangular and therefore equilateral, and the other five triangles, being identically equal to it, are also equilateral.

5. Therefore each of the sides, AB, BC, CD, DE, EF, FA, is equal to the radius OA of the circle. Hence

Construction. 1. Starting from any point A on the circle, cut off the distances AB, BC, CD, etc., each equal to the radius.

2. Six equal measures will [illegible] to the point A, and the circle will be divided into six equal parts at the points A, B, C, etc.

3. The alternate points A, C, E, or B, D, F, divide the circle into three equal parts.

4. By bisecting the arcs AB, BC, etc., the circle will be divided into 12 parts; by bisecting these arcs, the number of parts will be 24, etc.

Corollary. The perimeter $AB + BC + CD + DE + EF + FA$ of the hexagon $ABCDEF$ is six times the radius of the circle, and therefore three times its diameter.

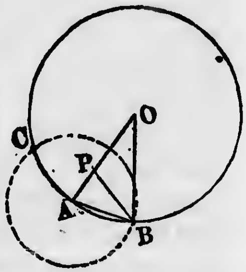
Because each of the six sides is a straight line, it is less than the corresponding arc; that is, side $AB <$ arc AB, etc. Therefore

The sum of the six arcs, or the whole circumference of the circle, is more than three times the diameter.

Problem III.

464. *To divide a circle into* 5, 10, 20, *etc., equal parts.*

Analysis. Let O be the centre of the circle, and the arc AB one tenth part of the circumference, or 36°.



Join OA, OB, and AB. Then—

1. Because the sum of the three angles of the isosceles triangle AOB is 180°, the sum of the angles A and B is $180° - 36° = 144°$, and each of these angles is 72°. Therefore each of the angles OAB and OBA is double the angle at O.

2. If we bisect the angle OBA by the line BP, meeting OA in P, the angles OBP and PBA will each be 36°, and we shall have

$$\text{Angle } APB = 180° - \text{angle } PAB - \text{angle } PBA \qquad (\S\ 74)$$
$$= 180° - 72° - 36° = 72°.$$

3. Therefore the triangle BAP is isosceles and equiangular, and therefore similar, to the triangle OAB. Also, because angle POB = angle PBO, the triangle OPB is isosceles. Hence

$$PO = PB = AB.$$

4. Because of the similarity of the triangles BAP and OAB,

$$AP : AB :: AB : AO,$$

or

$$AP : PO :: PO : AO.$$

5. Therefore the radius OA is divided in extreme and mean ratio at the point P (§ 440). Hence we conclude:

465. *If the radius of a circle be divided in extreme and mean ratio, the greater segment will be the chord of one tenth of the circle.*

Construction. Divide the radius OA of the circle in extreme and mean ratio at the point P (§ 441).

Around A as a centre, with a radius equal to the greater segment, OP, describe a circle, and let B and C be the points in which it intersects the first circle.

The arc AB will be one tenth of the circle and BC will be one fifth of the circle. By measuring BC off five times around the circle, the latter will be divided into five equal parts.

By successive bisection the circle will be divided into 10, 20, 40, etc., equal parts. Q.E.F.

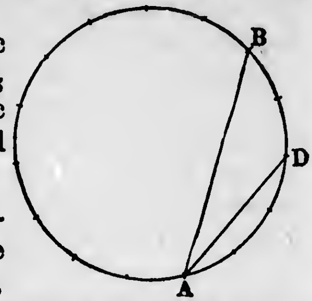
Problem IV.

466. *To divide a circle into fifteen equal parts.*

Construction. 1. From any point A cut off AB equal to one third the circle (§ 463).

2. From the same point A cut off AD, equal to one fifth the circle (§ 465).

3. Arc BD will then be $(\frac{1}{3} - \frac{1}{5})$ of the circle; that is, $\frac{2}{15}$ of it.



4. Therefore, if we bisect the arc BD, we shall have an arc equal to $\frac{1}{15}$ of the circle. By measuring this arc off 15 times, the circle will be divided into 15 equal parts. Q.E.F.

Scholium. The foregoing divisions of the circle are all that were known to the ancient geometers. But, about the beginning of the present century, Gauss, the great mathematician of Germany, showed that whenever any power of 2, increased by 1, made a prime number, the circle could be divided into that number of parts by the rule and compass. Thus:

$$2^1 + 1 = 3, \text{ a prime number.}$$
$$2^2 + 1 = 5 \quad \text{"} \quad \text{"}$$
$$2^4 + 1 = 17 \quad \text{"} \quad \text{"}$$
$$2^8 + 1 = 257 \quad \text{"} \quad \text{"}$$

Therefore, besides the old solutions, the circle can be divided into 17 or into 257 equal parts.

The division into 17 parts by construction is, however, too complicated for the present work, and that into 257 parts is so long that no one has ever attempted to really execute the construction.

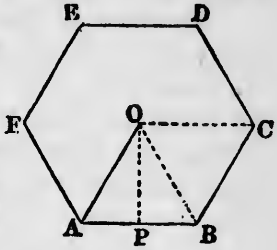
CHAPTER III.

AREAS AND PERIMETERS OF REGULAR POLYGONS AND THE CIRCLE.

467. *Def.* The **apothegm** of a regular polygon is the perpendicular from its centre upon any one of its sides.

THEOREM VII.

468. *The area of a regular polygon is equal to half the rectangle contained by its perimeter and its apothegm.*



Hypothesis. $ABCDEF$, a regular polygon; OP, its apothegm.

Conclusion.
Area $ABCDEF = \frac{1}{2} OP \times$ perimeter.

Proof. Join OA, OB, etc. Then—

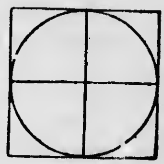
1. Because OP is the altitude of the triangle AOB,

$$\text{Area } AOB = \tfrac{1}{2} OP \,.\, AB.$$

2. In the same way it may be shown that the area of each of the other triangles into which the polygon is divided is equal to one half of the side into the apothegm. But the apothegms are all equal. Therefore

$$\text{Area } ABCDEF = \tfrac{1}{2} OP \,.\, AB + \tfrac{1}{2} OP \,.\, BC + \tfrac{1}{2} OP \,.\, CD + \text{etc.}$$
$$= \tfrac{1}{2} OP \times \text{perimeter.} \qquad \text{Q.E.D.}$$

469. *Corollary* 1. Because the perimeter of each circumscribed regular polygon is less the greater the number of its sides (§ 458), it follows that the area of the circumscribed regular polygon is less the greater the number of its sides.



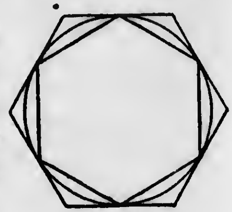
Cor. 2. It is easily shown that the area of the circumscribed square is equal to the square upon the diameter of the circle. Therefore:

470. *The area of any circumscribed regular polygon of more than four sides is less than the square upon the diameter of the circle.*

471. *Scholium.* If a regular polygon be inscribed in a circle, and another regular polygon of the same number of sides be circumscribed about it, the area of the outer polygon will be greater than that of the inner one by the surface contained between the perimeters of the two polygons.

This surface will be called the *included area.*

When the two polygons are so placed that their respective sides are parallel (§ 456), the included area will be formed of as many identically equal trapezoids, $AA'B'B$, $BB'C'C$, etc., as the polygons each have sides; and each apothegm, as OT, will again divide each of these trapezoids into two identically equal trapezoids.

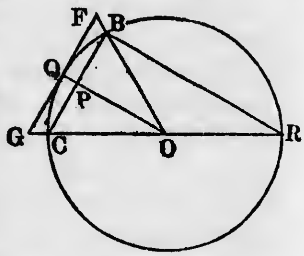


When the polygons are so placed that the sides of the circumscribed polygon shall touch the circle at the vertices of the inscribed polygon (§457), the included area is made up of as many identically equal triangles as the polygons have sides.

Theorem VIII.

472. *When the inscribed and circumscribed regular polygons have more than four sides, the included area is less than the square upon one side of the inscribed polygon.*

Hypothesis. O, the centre of the circle; BC, a side of the inscribed polygon; FG, a side of the circumscribed polygon, placed parallel to BC.



Conclusion. If the polygons be completed, the included area will be less than the square upon CB.

Proof. Join OBF. Drop the perpendicular OQ from O upon FG; draw the diameter COR and join BR. Let n be the number of sides of each polygon. Then—

1. Because the area of the inscribed polygon is made up of $2n$ triangles identically equal to OBP, and the circumscribed polygon of $2n$ triangles equal to OFQ, the inscribed polygon will be to the circumscribed one as the area OBP is to the area OFQ. That is, if we put

A, the area of the circumscribed polygon,

a, the area of the inscribed polygon,

we shall have

$$A : a :: \text{area } OFQ : \text{area } OBP.$$

2. Because the lines BC and FG are parallel, the triangles OBP and OFQ are similar. Because of this similarity,

$$\text{Area } OFQ : \text{area } OBP :: OQ^2 : OP^2. \quad (\S\, 423)$$

3. Comparing with (1), and because $OQ = OB$, both being radii of the same circle,

$$A : a :: OB^2 : OP^2.$$

4. The included area is equal to $A - a$. Hence, by division,

$$A - a : A :: OB^2 - OP^2 : OB^2. \quad (\S\, 365)$$

5. Because OPB is a right-angled triangle,

$$OB^2 - OP^2 = PB^2.$$

Making this substitution in (4), and putting

D, the diameter of the circle (whence $OB = \frac{1}{2}D$),

s, the length of the side CB of the inscribed polygon (whence $PB = \frac{1}{2}s$),

we shall have

$$A - a : A :: \tfrac{1}{4}s^2 : \tfrac{1}{4}D^2 :: s^2 : D^2. \quad (346)$$

Therefore

$$A - a = \frac{s^2 \times A}{D^2} = \frac{A}{D^2} \times s^2.$$

6. But $A < D^2$, or $A \div D^2 < 1$ (§ 470). Therefore

$$A - a < s^2. \quad \text{Q.E.D.}$$

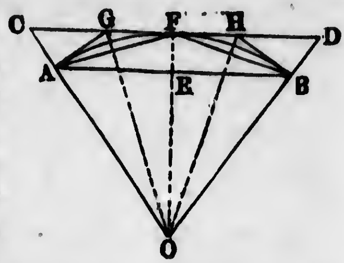
Corollary. By sufficiently increasing the number of sides of the polygons, we can make each side as short as we please, and therefore its square as small as we please. Hence:

473. *If the number of sides of the inscribed and circumscribed polygons be indefinitely increased, the included area will become less than any assignable quantity.*

Problem V.

474. *From the areas of the inscribed and circumscribed polygons of n sides to find the areas of those having $2n$ sides.*

Given. O, the centre of the circle; AB, one of the sides of the inscribed polygon of n sides; CD, one of the sides of the circumscribed polygon of n sides, placed parallel to AB, and tangent to the circle at F; OEF, the perpendicular from the centre upon the tangent; AF, FB, two sides of the inscribed polygon of $2n$ sides; AG, BH, tangents to the circle at A and B; wherefore GH is one side of the circumscribed polygon of $2n$ sides (though not parallel to any side of the inscribed polygon of $2n$ sides). Also, the areas of the triangles OAB and OCD are supposed given, and from them those of the inscribed and circumscribed polygons are found by multiplying by n.



Required. I. To find the area OAF (from which the area of the second inscribed polygon is obtained by multiplying by $2n$).

II. To find the area OGH (from which the area of the second circumscribed polygon is found by multiplying by $2n$).

Solution. Let us put

t, the area of the triangle OEA, which is one half that of the given area OAB;

T, the area of the triangle OFC, which is one half that of the given area OCD;

t', the required area OAF;

T', the required area OGH. Then—

I. 1. Because the triangles OAE and OAF have the same vertex A, and their bases on the same straight line OF, their areas are as OE to OF, or

$$t : t' :: OE : OF. \qquad (\S\ 415)$$

2. Because the triangles OAF and OCF have the common vertex F, and their bases on the same straight line OC,

$$t' : T :: OA : OC.$$

3. Because AE and CF are parallel,

$$OE : OF :: OA : OC.$$

4. Comparing with (1) and (2),

$$t : t' :: t' : T;$$

that is, the area t' is a mean proportion between t and T, or

$$t' = \sqrt{tT}.$$

II. 5. Because the triangles FGO and AGO have $GA = GF$ $OA = OF$, and OG common, they are identically equal. Also because OGA and OGC have the same vertex G, they are to each other as OA to OC. Hence

$$\text{Area } FOG : \text{area } GOC :: OA : OC :: t' : T. \qquad (2)$$

6. But

$$\text{Area } FOG = \tfrac{1}{2} \text{ area } GOH = \tfrac{1}{2}T'.$$
$$\text{Area } GOC = \text{area } FOC - FOG = T - \tfrac{1}{2}T'.$$

7. Comparing with (5),

$$\tfrac{1}{2}T' : T - \tfrac{1}{2}T' :: t' : T;$$

and by composition,

$$\tfrac{1}{2}T' : T :: t' : t' + T,$$

whence
$$T' = \frac{2t'T}{t' + T}.$$

8. If we put A, A', the areas of the circumscribed polygons; a, a', the areas of the inscribed polygons, we shall have

$$A = 2nT, \qquad a = 2nt,$$
$$A' = 2nT', \qquad a' = 2nt',$$

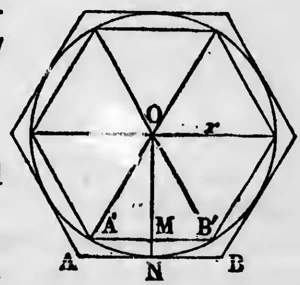
and the relations between A, A', a, and a' will be the same as those between T, T', t, and t' in (4) and (7).

We therefore conclude:

If we have given the areas A and a of the circumscribed and inscribed polygons of n sides, those of the corresponding polygons of $2n$ sides will be given by the equations

$$a' = \sqrt{aA},$$
$$A' = \frac{2a'A}{a' + A}.$$

475. *Application of the preceding solution to the computation of the area of inscribed and circumscribed polygons of a continually increasing number of sides.*



Let us put

r, the radius of the circle.

I. Let us begin with the inscribed and circumscribed regular hexagons.

Each of these hexagons can be divided into six equal equilateral triangles by lines drawn from the centre (§ 463). Each side of the triangles in the inscribed hexagon will be equal to r. The area of each equilateral triangle, found by the method of § 333, will be

$$\frac{\sqrt{3}}{4} r^2,$$

and the area of the whole inscribed hexagon will be

$$\frac{6\sqrt{3}}{4} r^2 = \frac{3\sqrt{3}}{2} r^2.$$

To find the area of the circumscribed hexagon we have

$$OM : OB' :: ON : OB,$$

or

$$OM : r :: r : OB,$$

whence for the side of the circumscribed hexagon

$$OB = \frac{r^2}{OM} = \frac{r^2}{\frac{1}{2}\sqrt{3}\,r} = \frac{2r}{\sqrt{3}}.$$

Because OAB is an equilateral triangle,

$$\text{Area } OAB = \frac{\sqrt{3}}{4} OB^2 = \frac{\sqrt{3}}{4}\frac{4}{3} r^2$$

$$= \frac{1}{\sqrt{3}} r^2,$$

whence for the area of the whole circumscribed hexagon

$$\frac{6}{\sqrt{3}} r^2 = 2\sqrt{3}\, r^2.$$

Then, using the previous notation, we have

$$a = \frac{3\sqrt{3}}{2} r^2.$$

$$A = 2\sqrt{3}\, r^2.$$

II. *Polygon of* 12 *sides.* Next we pass to the polygons of 12 sides by the formulæ

$$a' = \sqrt{aA}\,; \; A' = \frac{2a'A}{a'+A}.$$

Making these substitutions, and reducing by algebraic methods, we have

$$a' = 3r^2.$$

$$A' = \frac{12}{2+\sqrt{3}} r^2 = 3.2153910r^2.$$

This value of a' gives the remarkable result that the area of the inscribed regular polygon of 12 sides is exactly three times the square upon the radius; that is, three fourths the square upon the diameter.

III. *Polygon of 24 sides.* To pass from the polygon of 12 sides to that of 24 sides, we substitute the preceding values of a' and A' for a and A in the general formulæ.

If we put a'' and A'' for the areas required, we have

$$a'' = \sqrt{a'A'} = \sqrt{3 \times 3.215391}\,r^2,$$

$$A'' = \frac{2a''A'}{a'' + A'};$$

or, reducing to numbers,

$$a'' = 3.105829r^2,$$
$$A'' = 3.159661r^2.$$

IV. Repeating the same process, we may compute the areas of the inscribed and circumscribed polygons of 48 sides, 96 sides, and so on indefinitely.

Subtracting the area of each inscribed one from that of the corresponding circumscribed one, we have the included area in each case. The results to 192 sides are shown in the following table.

No. of sides.	a. Area of inscribed polygon.	A. Area of circumscribed polygon.	$A - a$. Included area.
12....	$3.000000r^2$	$3.215391r^2$	$0.215391r^2$
24....	$3.105829r^2$	$3.159661r^2$	$0.053832r^2$
48....	$3.132630r^2$	$3.146087r^2$	$0.013457r^2$
96....	$3.139350r^2$	$3.142715r^2$	$0.003365r^2$
192....	$3.141032r^2$	$3.141873r^2$	$0.000841r^2$

By continually increasing the number of sides of the polygons, we may diminish the included area to any extent (§ 473), though we can never reduce it to zero.

Area of the Circle.

476. *Def.* The **limit** of a varying magnitude is a fixed quantity to which the varying magnitude may approach so as to differ from the limit as little as we please, but to which it can never become equal.

477. *Theorems of Limits.* The theorems relating to the subject of limits are proved in algebra. The following propositions are applied in geometry:

AXIOM I. *Any quantity may be multiplied by a factor so great as to make the product exceed any quantity we may assign.*

AX. II. *Any quantity may be multiplied by a factor so small as to make the product less than any quantity we may assign.*

THEOREM. *If two varying quantities each approach a limit, the limit of their product will be the product of their limits.*

Hypothesis. A quantity X approaching the limit L.
A quantity X' approaching the limit L'.

Conclusion. The product XX' will approach the product LL' as its limit.

Proof. Let α and α' be the respective amounts by which X and X' differ from their limits L and L'. We then have

$$X = L - \alpha,$$
$$X' = L' - \alpha'.$$

Multiplying,

$$XX' = LL' - \alpha L' - \alpha' L + \alpha\alpha'.$$

Let β be a quantity as small as we please. How small soever it may be, we may take the quantities α and α' so small that the products $\alpha L'$, $\alpha' L$, and $\alpha\alpha'$ shall each be less than one third of β. (Ax. II.)

The quantity XX' will then differ from LL' by less than β.

Because the difference β may be as small as we please, the product LL' is the limit of XX'. Q.E.D.

Cor. Because the area of a rectangle is represented by the product of the lengths of its containing sides we conclude:

If the containing sides of a rectangle approach two lines L and L' as their limits, the area of the rectangle will approach the area LL' as its limit.

LEMMA.

478. *When the number of sides of the inscribed and circumscribed polygons is increased indefinitely, each of their areas approaches the area of the circle as its limit.*

Proof. In order to prove this lemma we have to show:

I. That the area of the inscribed polygon must always be less than that of the circle, how great soever the number of its sides.

II. That the area of the circumscribed polygon must always be greater than that of the circle.

III. That if we assume an area α, we can by increasing the number of sides of the polygons make each of their areas differ from that of the circle by less than α, how small soever α may be.

1. Because the apothegm OM (§ 475) of the inscribed regular polygon is a perpendicular from O upon $A'B'$, it is less than the line OA', which is the radius of the circle.

Therefore some part of the area of the circle will always be outside the polygon, and the area of the polygon must always be less than that of the circle.

2. In a similar way we may show that the area of the circumscribed polygon is always greater than that of the circle.

3. How small soever the area α, we may increase the number of sides of the polygons until the included area shall be less than α (§ 473).

Then, because the area of the circle is greater than that of the inscribed polygon, but less than that of the circumscribed polygon, it will differ from each of them by less than α.

Therefore the area of the circle is a quantity which the area of each polygon may approach, so as to differ from it by less than any assignable quantity, but to which it can never become equal.

Therefore the area of the circle is the limit of the area of each polygon when the number of its sides is indefinitely increased. Q.E.D.

479. By continuing the table of § 475 we may approximate as nearly as we please to the area of the circle. But there is a theorem of approximation which we give without proof, and which will enable us to make a more rapid approximation.

I. *We obtain an approximate area of the circle by adding to the inscribed polygon two thirds of the included area.*

II. *This approximation is nearer the truth the greater the number of sides.*

Applying this rule to the preceding table, we have—

No of sides.	$\frac{2}{3}(A-a)$.	$a+\frac{2}{3}(A-a)$.
12..........	$\cdot 143594r^2$	$3\cdot 143594r^2$
24..........	$\cdot 035888r^2$	$3\cdot 141717r^2$
48..........	$\cdot 008971r^2$	$3\cdot 141601r^2$
96..........	$\cdot 002243r^2$	$3\cdot 141593r^2$
192..........	$\cdot 000561r^2$	$3\cdot 141593r^2$

We see that we get the same result from 96 sides and 192 sides, so that they both may be regarded as correct to the sixth place of decimals.

480. The coefficient of r^2, which we have found to six places of decimals, is represented by the symbol π. That is, we put

$$\pi = 3.141593 \ldots .$$
$$\pi r^2 = \text{area of circle of radius } r.$$

481. *Corollary The areas of any two circles are proportional to the squares of their diameters.*

Circumference of the Circle.

482. Axiom. When we increase indefinitely the number of sides of the inscribed and circumscribed polygons, the perimeter of each of these polygons approaches the circumference of the circle as its limit.

Theorem IX.

483. *The area of a circle is equal to one half its radius into its circumference.*

Proof. 1. Let a regular polygon of n sides be circumscribed about the circle.

The area of this polygon is equal to half its apothegm into its perimeter, and its apothegm is equal to the radius of the circle.

Let the number of sides of the polygon be increased indefinitely. Then—

2. The area of the polygon will approach the area of the circle as its limit.

3. The perimeter of the polygon will approach the circumference of the circle as its limit (§ 482).

4. Therefore the limit of area (area of the circle) will be equal to half the radius into the limit of the perimeter (circumference of the circle) (§ 478). Q.E.D.

PROBLEM VII.

484. *To find the ratio of the circumference of the circle to its diameter.*

Put C, the circumference; D, the diameter, or $2r$; A, the area of the circle.

By § 480 we have

$$A = \pi r^2 = \tfrac{1}{4}\pi D^2.$$

But we have just proved that

$$A = \tfrac{1}{2}rC = \tfrac{1}{4}DC.$$

Therefore $\quad \tfrac{1}{4}DC = \tfrac{1}{4}\pi D^2;$

or $\quad C = \pi D,$

$$\frac{C}{D} = \pi;$$

that is, *the number* $\pi = 3.14159$ . . . *is itself the ratio of the circumference of the circle to its diameter.*

This number π is of such fundamental importance in geometry that mathematicians have devoted great attention to its calculation. The preceding method, by which we have found it to six decimals, is the easiest afforded by elementary geometry, but more rapid methods are afforded by the higher mathematics. DASE, a German computer, carried the calculation to 200 places of decimals. The following are the first 36 figures of his result:*

3.141 592 653 589 793 238 462 643 383 279 502 884.

The result is here carried far beyond all the wants of mathematics. Ten decimals are sufficient to give the circumference of the earth to the fraction of an inch, and thirty decimals would give the circumference of the whole visible universe to a quantity imperceptible with the most powerful microscope.

EXERCISES.

1. Assuming the radius of a circle to be 5 metres, compute, by the process of § 475, the area of the inscribed and circumscribed regular hexagon, dodecagon, and polygon of 24 sides.

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* *Crelle's Journal*, Vol. 27, p. 198.

NOTE. In computations like this, the student should not be satisfied by working blindly with the formulæ, but should reason the results out by the same process employed to reason out the formulæ. In the present case the computation of the area of the hexagon is easy; and that of the figures of 12 and 24 sides can then be executed as in § 475.

2. If an equilateral triangle be inscribed in a circle, show that the perpendicular from any vertex upon the opposite side is three fourths the diameter of the circle.

3. Using the preceding theorem, compute the length of sides and the area of the equilateral triangle inscribed in the circle whose radius is unity.

4. Show that the altitude of a circumscribed equilateral triangle is three times the radius of the circle.

5. Without using any of the preceding theorems, show that the radius of the circle circumscribed about an equilateral triangle is double the radius of the inscribed circle.

6. What conclusion thence follows respecting the relation of the areas of the two circles? (§ 481.)

7. If the radius of a circle is r, what is the length of each side of the circumscribed equilateral triangle?

8. In a circle of radius r, find the sides of the inscribed and circumscribed squares and their areas.

9. From the results of the preceding example find the areas of the inscribed and circumscribed regular polygons of 8, 16, 32, and 64 sides, and thence the area of the circle, as in §§ 475, 479.

10. A bought a piece of pasturage 30 yards $\times$ 40 yards in B's field, and then tied his cow in the centre with a rope just long enough to reach to the corners of his piece. Over how much of B's part of the field could A's cow feed?

11. Four equal circles of radius a have their centres on the corners of a square, and touch each other. What is the radius of the circle in the centre touching each of them?

12. What must be the diameter of a circle in order that its area may be 100 square feet? (Apply § 480.)

13. In a regular polygon of n sides, what angle (in degrees) does a line from any vertex to the centre make with the sides meeting at that vertex? (§ 160.)

14. If from one vertex of a regular polygon of n sides lines be drawn to all the other vertices, what angles will they form with each other? (Apply § 235.)

15. What is the area of a circle circumscribed about a square whose side is a?

16. If the apothegm of a regular hexagon is h, what is the area of the ring included between its inscribed and circumscribed circles?

CHAPTER IV.

MAXIMUM AND MINIMUM FIGURES.

485. *Def.* A **maximum** figure is the greatest figure of a given class.

486. *Def.* A **minimum** figure is the least of a given class.

Remark. If a figure is entirely unrestricted, there can be no such thing as a maximum or a minimum, because a figure, if not restricted, can be made as great or as small as we please.

Hence a maximum or minimum figure means one subject to certain conditions; for example, required to have a certain perimeter, or to be included between certain limits, or to have some relations among its parts which prevent it from becoming indefinitely great or indefinitely small.

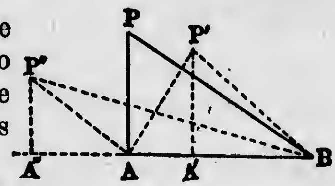
Having defined the conditions of the figure, we may imagine ourselves to construct all possible figures fulfilling these conditions. This collection of possible figures will constitute a *class*. The greatest among them will be the *maximum;* the least, the *minimum.*

487. *Def.* **Isoperimetrical** figures are those which have the same perimeter.

THEOREM X.

488. *If two sides of a triangle be given, its area will be a maximum when these sides are at right angles.*

Proof. Let AB and AP be the two given sides of the triangle.



At whatever angle we fix these sides the area will be equal to $AB \times$ altitude, and so will be the greatest when the altitude is greatest (§ 301).

If AP is perpendicular to AB, AP will itself be the altitude. In any other position, as AP' or AP'', the altitude $A'P'$ or $A''P''$ will be less than AP (§ 101).

Therefore the triangle of greatest area is BAP, in which $AP \perp AB$. Q.E.D.

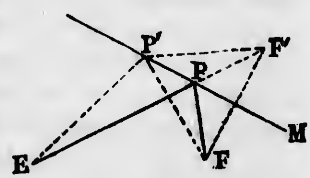
489. PROBLEM VIII. Having given

A straight line M,

Two points E and F on the same side of the line:

It is required *to find the point P on the line M for which the sum of the distances $PE + PF$ shall be a minimum.*

Solution. From one of the given points, as F, drop a perpendicular upon the line M, and produce it to the point F' at an equal distance on the other side.



Because the line M is the perpendicular bisector of FF', every point upon it will be equally distant from F and F' (§ 104).

Therefore, if P' be any point upon the line, we shall have

$$EP' + P'F = EP' + P'F'.$$

The distance $EP' + P'F'$ will be a minimum when P' is in the straight line from E to F'. Therefore the required point P is the point in which the straight line from E to F' intersects the given line.

Draw PF.

Because the line M is the perpendicular bisector of the line FF', we have

$$\text{Angle } MPF' = \text{angle } MPF;$$

also, $\quad \text{Angle } MPF' = \text{opp. angle } EPP;$

whence $\quad \text{Angle } EPP' = \text{angle } MPF.$

The solution of the problem is therefore expressed in the following lemma:

490. Lemma. *The sum of the distances from a movable point on a straight line to two fixed points on the same side of the line is a minimum when those distances make equal angles with the straight line.*

Scholium. If the line M is a section of a mirror, the lines $EP + PF$ are those which would be followed by a ray of light emanating from a candle at E and reflected to F, because it is a law of optics that the angles of incidence and reflection, namely EPP' and FPM, are equal. Hence:

The course taken by a ray of light emanating from one point and reflected by a plane surface to another point is the shortest path from the one point to the reflector, and thence to the other point.

Theorem XI.

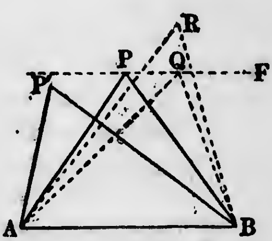
491. *If the base of a triangle and the sum of the other two sides be given, the area will be a maximum when these sides are equal.*

Hypothesis. APB, an isosceles triangle on the base AB; $AP'B$, another triangle on the same base AB, in which

$$AP' + P'B = AP + PB.$$

Conclusion.

$$\text{Area } AP'B < \text{area } APB.$$



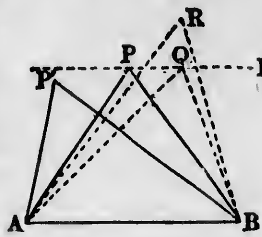
Proof. Through P draw $PF \parallel AB$. Because the areas of the triangles are proportional to their altitudes, it is sufficient to show that the vertex P' must fall below the parallel PF.

1. Because angle $PAB =$ angle PBA (§ 91), and $PF \parallel AB$, the sides AP and PB make equal angles with PF.

2. The vertex P' cannot lie on PF, because then we should have

$$AP' + P'B > AP + PB. \qquad (\S\ 490)$$

3. If possible, suppose the vertex P' to fall at any point R above PE. The sides RA and RB will then include a segment of the line PF between them. From any point Q of this segment draw QA and QB. Then



$$AR + RB > AQ + BQ. \qquad (\S\ 100)$$
$$AQ + QB > AP + PB. \qquad (\S\ 490)$$

Therefore

$$AR + RB > AP + PB.$$

Because R may be any point above PF, the vertex P' cannot fall above the line PF.

4. Since it can fall neither above nor upon this line, it must fall below it, and we must have

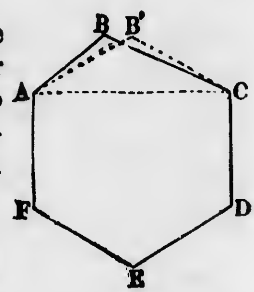
$$\text{Alt. of } P' < \text{alt. of } P;$$

whence $\quad$ Area $AP'B <$ area APB. Q.E.D.

Theorem XII.

492. *Among all isoperimetrical polygons of a given number of sides, that of maximum area has all its sides equal.*

Proof. If possible, let $ABCDEF$ be the maximum polygon of given perimeter and number of sides in which some two adjacent sides, as AB and BC, are unequal. Join AC, and describe on AC an isosceles triangle $AB'C$, such that $AB' + B'C = AB + BC$. Then—



1. Because ABC is isosceles and $AB' + B'C = AB + BC$,

$$\text{Area } AB'C > \text{area } ABC. \qquad (\S\ 491)$$

2. Because the area $ACDEF$ remains unchanged,

$$\text{Area } AB'CDEF > \text{area } ABCDEF.$$

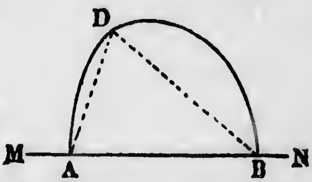
3. But the polygon $AB'CDEF$ has the same perimeter and number of sides as the polygon $ABCDEF$. Because the former has a greater area, the latter cannot be the polygon of maximum area.

4. Therefore no polygon having two adjacent sides unequal can be a polygon of maximum area; and because any polygon with unequal sides must have some two adjacent sides unequal, it follows that the polygon of maximum area must have all its sides equal. Q.E.D.

Theorem XIII.

493. *If a line of given length, which may be curved at pleasure, is required to have its extremities upon an indefinite straight line, it will inclose a maximum area when bent into a semicircle.*

Hypothesis. MN, an indefinite straight line; ADB, a curve line which may be bent at pleasure and have its extremities, A and B, rest upon MN.

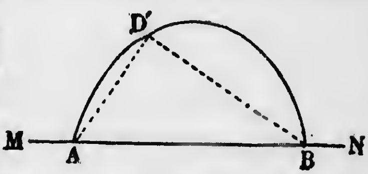


Conclusion. The inclosed area $ADBA$ cannot be a maximum unless ADB is a semicircle.

Proof. If the line is not a semicircle, there must be some point D upon it such that the angle ADB shall not be a right angle (§§ 238, 241). Join DA, DB.

The area $ADBA$ between the curve and the straight line is then divided into three parts, which we may call AD, DB, and the triangle ADB.

Bend the curve at the point D, leaving the two branches AD and BD unchanged, and sliding the ends A and B along the line MN, so that the curve shall take up the form $AD'B$, in which $AD'B$ is a right angle.



Because the triangles ADB and $AD'B$ have their sides AD and $DB = AD'$ and $D'B$, and $AD'B$ is a right angle,

$$\text{Area } AD'B > \text{area } ADB, \qquad (\S\ 488)$$

while the other two areas, AD and DB, remain unchanged.

Therefore the inclosed area $ADBA$ can be increased without changing the length of the curve, whence this area is not a maximum.

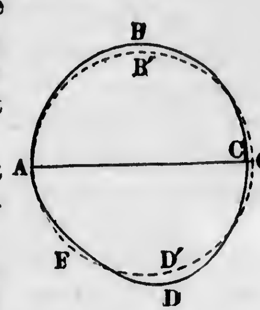
Hence the curve cannot inclose a maximum area unless AB subtends a right angle from every one of its points, and it is then a semicircle. Q.E.D.

THEOREM XIV.

494. *Of all areas inclosed by equal perimeters, the circle is a maximum.*

Hypothesis. $ABCD$, a closed line of given length.

Conclusion. $ABCD$ cannot inclose the maximum area unless it is a circle.



Proof. Take any two points, A and C, on the curve so as to divide it into two equal parts. Join AC.

Now if ABC and ADC are not both semicircles, suppose the curve-line ABC bent into a semicircle without changing its length, the foot A remaining unchanged in position, and let this semicircle be $AB'C'$. We shall then have

Area $AB'C'A >$ area $ABCA$. (§ 493)

If ADC is not a semicircle, we may bend it into the semicircle $AD'C'$ such that

Area $AD'C'A >$ area $ADCA$.

Because the two semicircles are equal in length, they are halves of equal circles and the diameters are equal, so that the two points C' coincide.

Adding the two areas, we shall have

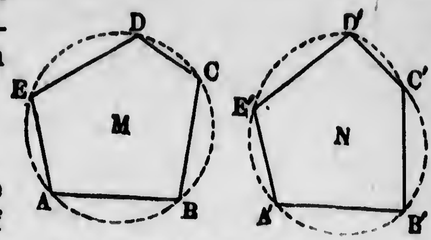
Area of circle $AB'C'D'A >$ area $ABCDA$.

Therefore the area $ABCD$ will not be a maximum when it is not a circle. Q.E.D.

THEOREM XV.

495. *A polygon of which all the sides are given incloses a maximum area when it can be inscribed in a circle.*

Hypothesis. M, a polygon inscribed in a circle; N, any other polygon, having its sides equal in length, number, and arrangement with those of the polygon M.



Conclusion.

Area $M >$ area N.

Proof. 1. Upon the sides of N describe arcs of circles equal to the arcs upon the sides of M. Then

Area $M =$ area of circle $-$ area of segments.

Area $N =$ area of distorted circle $-$ area of segments.

2. Because each segment around a side of N is identically equal to the segment around the corresponding side of M, the areas of the two sets of segments are equal.

3. Because the circumferences of the true circle around M and the distorted circle around N are equal,

Area of circle $>$ area of distorted circle. (§ 494)

4. Comparing with (1),

Area $M >$ area N. Q.E.D.

Corollary. It has been shown that the maximum polygon of given perimeter and number of sides has equal sides.

If a polygon with all its sides equal be inscribed in a circle, it must have its angles equal and be regular (§ 447). Hence:

496. *A polygon of which the perimeter and number of sides are given incloses the maximum area when it is regular.*

THEOREMS FOR EXERCISE.

1. The inclosed area between two concentric circles is equal to the area of a circle whose diameter is that chord of the outer circle which is tangent to the inner one.

2. The area of a circle is to that of any circumscribed polygon as its circumference to the perimeter of the polygon.

3. The area of the regular inscribed hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

4. The inscribed regular octagon is equal to the rectangle of the sides of the inscribed and circumscribed squares.

BOOK VII.

OF LOCI AND CONIC SECTIONS.

CHAPTER I.

LINES AND CIRCLES AS LOCI.

497. To fix the position of a point on a plane, two independent conditions are required.

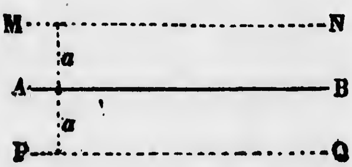
Example. If a point is subject to the condition that it must be two inches below one line and one inch to the right of another perpendicular line, its position is completely fixed.

But if the only condition is that it must be two inches below a given horizontal line, its position is not fixed, but it may move along a line two inches below the given one. This last line is then called the locus of the point.

498. *Def.* The **locus** of a point is a line or group of lines to which the point must be confined when subject to some one condition.

Problem I.*

499. *To find the locus of a point which must be at a given distance from a given straight line.*



Let AB be the given straight line, and call a the given distance.

Draw MN and PQ parallel to AB, at the distance a on either side of it.

* It is recommended to the student that, before beginning to draw the locus in these problems, he mark a number of points each fulfilling the required condition, and continue marking until he sees what the locus will be.

Every point on either of the lines MN and PQ is at the given distance a from the given line AB (§ 129).

It is evident that no other point in the plane can be at that distance a.

Therefore the two lines MN and PQ form the required locus of the point at the distance a from AB.

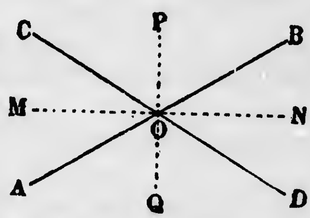
500. *Corollary.* If the condition were that the point must be at the distance a *above* the line AB, the locus would be the line MN alone.

If the point must be at the distance a *below* the line AB, the locus would be PQ alone.

PROBLEM II.

501. *To find the locus of the point which is equidistant from two given straight lines.*

Let AB and CD be the two lines, and O their point of intersection.



Let each of the four angles at O—namely, BOD, DOA, AOC, COB—be bisected by the respective lines ON, OQ, OM, OP.

Every point on the bisecting lines will be equally distant from the two given lines (§ 106), and every other point will be unequally distant.

Therefore these bisectors form the required locus. By § 85 they form a pair of straight lines at right angles to each other.

Therefore the locus of the point which is equidistant from two given straight lines is a pair of lines at right angles to each other, bisecting the angles formed by the given lines.

Scholium. There are two ways of thinking of the relation of a point to its locus which both amount to the same thing.

1. That a row of points, as numerous and close as we choose, lie on the locus. Every one of these points will then fulfill the given condition.

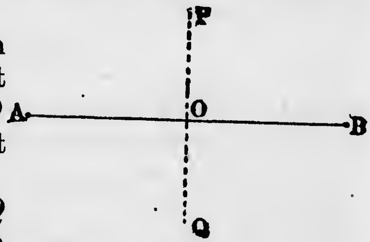
2. That the point slides along the locus. The point will then fulfill the condition so long as it does not leave the locus.

EXAMPLES. 1. If we make any number of points on the lines MN and PQ, every one of these points will be equally distant from the lines AB and CD.

2. If we slide a point along the lines MN and PQ, it will always be as far from the line AB as from the line CD.

PROBLEM III.

502. *To find the locus of the point subject to the condition that it shall be equally distant from two given points.*



Let A and B be the given points. Join them by a straight line, and bisect this line at O by another line POQ at right angles to it.

Then every point on PQ will be equally distant from the two points A and B, and every other point will be unequally distant (§ 104).

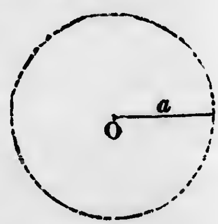
Therefore PQ is the locus of the point which is equally distant from A and B.

Therefore the locus of a point equally distant from two fixed points is the perpendicular bisector of the straight line joining the points.

PROBLEM IV

503. *To find the locus of the point which is at a given distance from a given point.*

Let O be the given point, and a the given distance.



Around O as a centre describe a circle with the radius a.

Every point on this circle will be at the distance a from O, and every point either inside or outside the circle will be at a less or greater distance from O (§ 206).

Therefore every point on the circle fulfills the required condition of being at the distance a from O, and no other point does.

Therefore the locus of the point at the distance a from the point O is a circle of which O is the centre and a the radius.

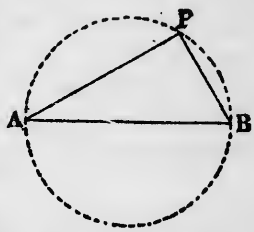
Problem V.

504. *To find the locus of the point from which a given line subtends a right angle.*

Let AB be the given line.

On AB as a diameter describe the circle APB.

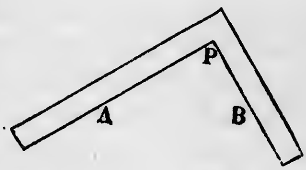
If P be any point on this circle, the angle APB will be a right angle (§ 238). From any point inside or outside of the circle the angle will be greater or less than a right angle (§ 242).



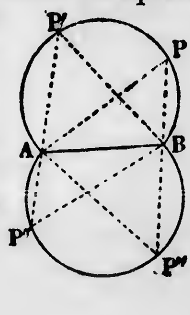
Therefore every point on the circle fulfills the required condition, and no other point does.

Therefore the locus of the point from which the line AB subtends a right angle is the circle described around AB as a diameter.

We may also say: If the point P slides around the circle APB, the angle APB will always be a right angle.



Corollary 1. This result teaches us a curious method by which a circle may be described. Drive two pins A and B into the surface representing the plane. Take a common square, and fasten a pencil-point into its interior angle P. Then slide the square around on the two pins, and the pencil-point will describe a circle. The pins will be at the extremities of a diameter of the circle.

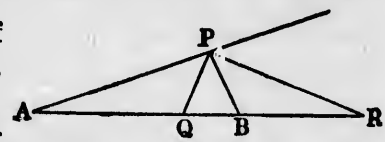


505. *Cor.* 2. It may be shown in the same way that the locus of all the points from which a given line subtends a given angle different from a right angle is formed of two arcs of circles. (Compare § 236.)

Problem VI.

506. *To find the locus of the point subject to the condition that its distances from two given points shall have a given ratio to each other.*

Let A and B be the two given points, and let the given ratio be that of $m : n$.



Let P be any position of the required point. Join AB, PA, and PB.

Bisect the angle APB internally by the line PQ, cutting AB internally at Q, and bisect the adjacent exterior angle by PR, cutting AB externally in R.

Then, by the given condition,

$$PA : PB :: m : n.$$

Therefore

$$AQ : BQ :: m : n. \qquad (\S\ 405)$$

$$AR : BR :: m : n. \qquad (\S\ 406)$$

Because of the equality of these ratios, the line AB is cut harmonically in the points Q and R (§ 407).

Because the condition requires that the lines PA and PB constantly have this same ratio $m : n$, it follows that the bisectors in question constantly pass through the same points Q and R, wherever the point P may move.

But these bisectors are at right angles to each other (§ 82).

Therefore the angle QPR is a right angle, and the locus of P is the same as the locus of the point from which the line QR subtends a right angle.

Therefore the required locus is the circle described around QR as a diameter, the points Q and R being fixed by the conditions

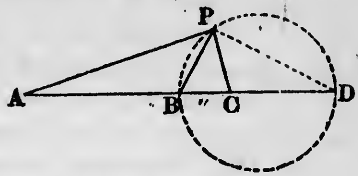
$$AQ : BQ :: m : n.$$

$$AR : BR :: m : n.$$

Problem VII.

507. *To find the locus of the point from which two adjacent segments of the same straight line subtend equal angles.*

Let AB and BC be the two adjacent segments, and P any position of the point.



By the condition we must have

Angle $APB =$ angle BPC.

Therefore PB is the bisector of the angle APC, and in consequence the lines PA and PC fulfill the condition

$$PA : PC :: AB : BC.$$

Because the points A, B, and C are fixed, the ratio $AB : BC$ is a constant. Therefore the ratio $PA : PC$ is also a constant, and the locus is that of the point whose distances from A and C have a given ratio, $AB : BC$, to each other. This locus is a circle (§ 506).

NOTE. The locus may be found independently of Prob. VI. by drawing PD at right angles to PB, and then reasoning as in Prob. VI.

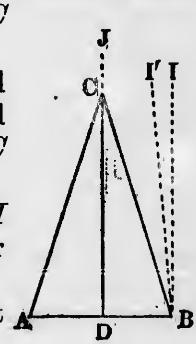
CHAPTER II.

LIMITS OF CERTAIN FIGURES.

THEOREM I.

508. *If the vertex of an isosceles triangle be carried away from the base indefinitely, each angle at the base will approach a right angle as its limit.*

Hypothesis. ABC, an isosceles triangle in which $CA = CB$; CD, the perpendicular from the vertex C upon the base, bisecting the latter.



Conclusion. If the vertex C be carried out indefinitely along the line DC, produced past J, each of the angles DBC and DAC will approach a right angle as their limit.

Proof. Through B draw a line BI parallel to DC, and therefore perpendicular to BD.

1. If BI is not the limit of BC, let some other line, BI', be that limit.

2. Because BI' is not parallel to DC, it will meet it if sufficiently produced (§ 45, Ax. 11). Call the point of meeting y.

3. By carrying the vertex C beyond y, the angle DBC will become greater than DBI'.

4. Because this is true however small the angle $I'BI$, the angle DBC has no limit less than the right angle DBI.

5. DBC can never become equal to a right angle, because then the triangle CDB would have the angles D and B both right angles.

6. Therefore the right angle DBI is the limit of the angle DBC, as the vertex C goes out indefinitely along the line DJ.

In the same way it may be shown that the limit DAC is a right angle. Q.E.D.

509. *Corollary* 1. As the vertex C goes out indefinitely, each of the sides BC and AC will approach the position of parallelism to the perpendicular CD as their limit, and will therefore approach indefinitely near to parallelism with each other.

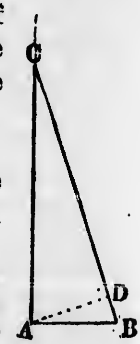
Cor. 2. The same thing being supposed, because the angle DCB and DCA are each supplements of CAD and CBD, and these angles approach indefinitely near to right angles, we conclude:

510. *The angle at the vertex approaches zero as its limit.*

THEOREM II.

511. *If the vertex of a right-angled triangle be lengthened out indefinitely, the adjacent side will approach the length of the hypothenuse as its limit.*

Hypothesis. ABC, a right-angled triangle of which the base AB is fixed, but of which the vertex C may be carried out indefinitely along the line AC produced.



Conclusion. However great the distance AB, the vertex C may be carried out to such a distance that the difference $CB - CA$ shall be less than any length we can assign.

REMARK. We may express the conclusion in this form: How many miles soever may be the base AB, we can carry the vertex C so far out that the excess of CB

over CA shall be less than a foot, less than an inch, less than the hundredth of an inch, and so on indefinitely.

Proof. From A draw $AD \perp BC$. Then—

1. Because CDA and CAB are right angles, CA is greater than CD but less than CB.

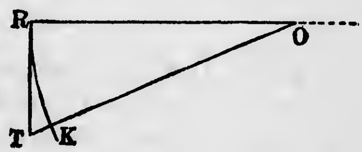
2. The triangles BDA and BAC are similar (§ 400).

3. As the vertex C moves out indefinitely, the ratio $BA : BC$ will approach zero as its limit. Therefore the ratio $BD : BA$ will also approach zero as its limit; that is, AB being constant, the point D will approach B as its limit.

4. Then CA, being between CD and CB, will approach CB as its limit. Q.E.D.

Theorem III.

512. *If the radius of a circle increase indefinitely, an arc of the circle of given length will approach a straight line as its limit.*



Proof. 1. Let RT be the length of the given arc.

2. At R erect the line RO perpendicular to RT, and join OT.

3. From O as a centre describe an arc equal to RT and passing through R.

4. Let the centre O move out indefinitely along the line RO produced, the circle still passing through R, so that OR is its radius.

5. If K be the point in which the circle intersects the line OT, we shall have $OK = OR$. Therefore, as O moves out indefinitely, the point K will approach T as its limit (§ 511), and the arc of the circle will approach the straight line RT as its limit. Q.E.D.

Theorem IV.

513. *If each of two lines, which differ by a constant quantity, are extended indefinitely, their ratio will approach unity as its limit.*

Proof. Call A the lesser line; D, the constant difference of the two lines; $A+D$, the greater line; r, the ratio of $A+D$ to A; α, the quantity by which r exceeds unity.

Then $$A+D=rA=(1+\alpha)A=A+\alpha A.$$
Therefore $$D=\alpha A,$$
or $$D:A::\alpha:1.$$

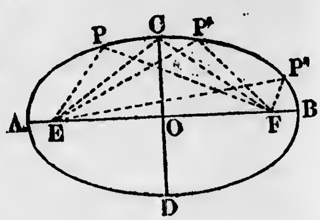
Now, we can increase A so that the ratio $D:A$ shall be less than any quantity we may assign. Therefore α may be made less than any assignable quantity, whence r may differ from unity by less than any such quantity; that is, the limit of r is 1. Q.E.D.

CHAPTER III.

THE ELLIPSE.

514. *Def.* An **ellipse** is the locus of the point, the sum of whose distances from two fixed points is a constant. Each of the two fixed points is called a **focus** of the ellipse.

515. *To describe an ellipse.* Let E and F be the foci.



Take a thread of which the length shall be equal to the sum of the distances of each point of the curve from the foci, which sum is supposed to be given, and fasten one end in each focus.

Stretch the thread tight by pressing a pencil-point against it, and move the latter round, keeping it pressed against the thread. The pencil-point will describe an ellipse.

Proof. Let P be any point of the curve described by the pencil-point. The sum of the distances of this point from the foci is $PE+PF$. But $PE+PF$ makes up the whole length of the thread which remains constant. Therefore the sum of the distances of P from the foci is equal to this constant, whence P is by definition a point of the ellipse.

516. *Axes of the ellipse.* In drawing the ellipse there will be two points, A and B, where the two parts of the thread will overlap each other. The line AB is called the **major axis** of the ellipse.

Let us put $l =$ the length of the thread. Then

$$\left.\begin{array}{l} AE + AF = l; \\ EB + FB = l. \end{array}\right\} \qquad (a)$$

Adding these equations, and noting that $AF = AE + EF$ and $EB = EF + FB$, we have

$$2AE + 2EF + 2FB = 2l.$$

Dividing this equation by 2,

$$AE + EF + FB = l,$$

or $$AB = l.$$ Hence:

517. *The major axis of the ellipse is equal to the sum of the distances of each point of the ellipse from the foci.*

The same equations (a) also give

$$AE + AF = EB + FB,$$

or $$2AE + EF = EF + 2FB;$$

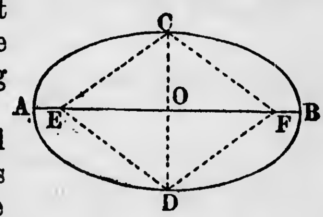
whence $$AE = FB.$$

Hence *the foci are equally distant from the ends of the major axis.*

If O be the middle point of the major axis, the distance OA is called the semi-major axis, and is represented by the letter a. Then $2a$ is the major axis; whence

$$2a = l, \text{ the length of the string.}$$

518. *Minor axis.* In drawing the ellipse there will be two points, C and D, equidistant from the two foci. Join these points by the line CD, intersecting the major axis in O.



Because, in the quadrilateral $ECFD$, $CE = ED = DF = FC$, this quadrilateral is a rhombus, and the diagonals EF and CD bisect each other at right angles (§173).

Hence CD is the perpendicular bisector of the major axis AB. CD is called the minor axis of the ellipse.

519. *Def.* The **minor axis** of the ellipse is that segment of the perpendicular bisector of the major axis which is terminated by the curve.

Also:

The point O is called the **centre** of the ellipse.

The major and minor axes are called the **principal axes** of the ellipse.

The distance of the centre O from each of the foci is called the **linear eccentricity** of the ellipse.

The ratio of the linear eccentricity to the semi-major axis is called the **eccentricity** of the ellipse. That is,

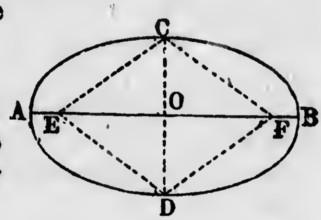
$$\text{Eccentricity} = OE : OA = \frac{OE}{a}.$$

520. It is common to use the notation:

b, the semi-minor axis of the ellipse $= CO$.

c, its linear eccentricity.

e, the eccentricity of the ellipse.



The relation between the two eccentricities is then expressed by the equations

$$e = \frac{c}{a} \text{ (because } c = OE).$$

$$c = ae.$$

521. *To find the length of the minor axis of an ellipse.*

Because EOC is a right-angled triangle, and $EC = a$,

$$b^2 = a^2 - c^2 = a^2(1 - e^2).$$

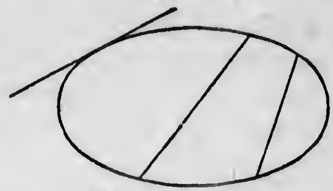
Whence, by extracting the square root,

$$b = a\sqrt{1 - e^2},$$

which enables us to determine the length of the minor axis when the major axis and the eccentricity are known.

NOTE. In the older geometry the length OE, which we have called the linear eccentricity, was called the "eccentricity" simply. But the moderns use the word eccentricity to designate the ratio of this length OE to the major axis, according to the above definition.

522. *Def.* A **chord** of an ellipse is any straight line terminated by two points of the ellipse.



523. *Def.* A **diameter** of an ellipse is any chord passing through the centre.

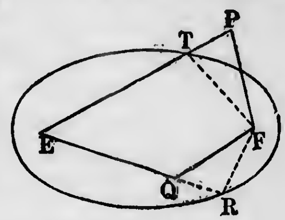
524. A **tangent** to an ellipse is any straight line which touches it but does not intersect it.

THEOREM V.

525. *Of every point without the ellipse, the sum of the distances from the foci is greater than the major axis.*

Of every point within the ellipse, that sum is less than the major axis.

Hypothesis. E, F, foci of an ellipse; P, a point without the ellipse; Q, a point within the ellipse.



Conclusions. I. $PE + PF > 2a$.
II. $QE + QF < 2a$.

Proof. I. Let T be the point in which the line EP intersects the ellipse. Join TF. Then—

1. $$EP + PF = ET + TP + PF.$$
$$TP + PF > TF.$$

Therefore
$$EP + PF > ET + TF.$$

2. $$ET + TF = 2a. \quad \text{(Def. of ellipse.)}$$

Therefore
$$EP + PF > 2a. \quad \text{Q.E.D.}$$

II. Produce EQ until it meets the ellipse in R. Join FR. Then we prove, as in (I.),
$$EQ + QF < ER + RF.$$
$$ER + RF = 2a.$$

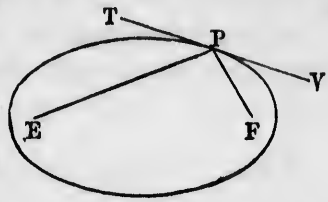
Whence
$$EQ + QF < 2a. \quad \text{Q.E.D.}$$

THEOREM VI.

526. *If through any point of an ellipse we draw a line making equal angles with the lines from that point to the foci, that line will be a tangent, and the only tangent, to the ellipse at that point.*

Hypothesis. E, F, foci of an ellipse; P, any point of the ellipse; TPV, a straight line through P such that

Angle $TPE =$ angle VPF.



Conclusion. TPV will be a tangent to the ellipse, and every other line passing through P will intersect the ellipse.

Proof. 1. Because P is a point of the ellipse,

$$EP + PF = 2a.$$

2. Because angle $TPE =$ angle VPF, the sum of the distances from P to the foci, that is, $EP + PF$, or $2a$, is less than the sum of the distances from any other point of TV to the foci (§ 490).

3. Because the sum of the distances of every other point of TV from the foci is greater than $2a$, every such other point is without the ellipse (§ 525).

4. Therefore the line TV touches the ellipse at P without intersecting it, and is therefore a tangent. Q.E.D.

5. If any other line than TPV passes through P, it cannot make equal angles at P with the lines PE and PF. Therefore there will be some point of the line for which the sum of the distances from E and F will be less than $EP + PF$ (§ 490); that is, the line will pass inside of the ellipse and cannot be a tangent. Q.E.D.

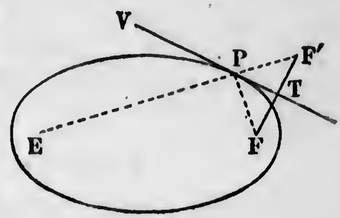
527. *Def.* Two points so situated that an indefinite line is the perpendicular bisector of the line joining them are said to be **opposite points** with respect to that indefinite line.

Corollary. The indefinite line is the perpendicular bisector of the line joining two opposite points with respect to it.

THEOREM VII.

528. *The line from any focus to the opposite point of the other focus, with respect to a tangent, passes through the point of tangency.*

Hypothesis. VT, a tangent to an ellipse having E and F as foci; F', the opposite point of F with respect to the tangent; P, the point in which the line EF' intersects VT.



Conclusion. P is the point of tangency.

Proof. 1. Because VT is the perpendicular bisector of FF',

$$PF' = PF. \qquad (\S\ 104)$$
$$\text{Angle } TPF = \text{angle } TPF' \qquad (\S\ 102)$$
$$= \text{angle } EPV. \qquad (\S\ 67)$$

2. Because the angles FPT and EPV are equal, P is the point at which the tangent touches the ellipse (§ 526). Q.E.D.

Corollary. Because P is a point of the ellipse, we have

$$EP + PF = 2a,$$

and because $PF' = PF$, we have also

$$EF' = 2a.$$

Therefore the opposite point of any one focus is at the distance $2a$ from the other focus, and we conclude:

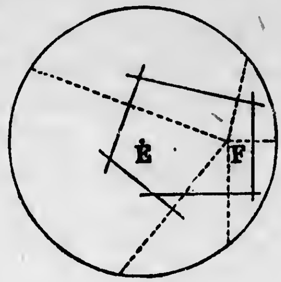
529. *The locus of the opposite point of one focus, with respect to a moving tangent, is a circle around the other focus with the radius $2a$.*

In other words, if a tangent roll round on an ellipse, the opposite point of either focus will describe a circle round the other focus as a centre with the radius $2a$.

530. This theorem and corollary afford an elegant method of drawing any number of tangents to an ellipse without drawing the ellipse itself. We need only to know the positions of the foci and the length of the major axis.

Construction. Let E and F be the given foci.

Around either focus, as E, with a radius equal to the major axis we describe a circle. This circle will then be the locus of all the opposite points of F with respect to the tangents.



We draw any line from F to the circle, and bisect this line at right angles by another line.

The bisecting line will be a tangent to the ellipse.

By drawing a number of such lines any number of tangents may be drawn.

Theorems for Exercise.

I. Each principal axis of an ellipse is an axis of symmetry.

II. The ellipse is symmetrical with respect to its centre as a centre of symmetry.

III. Every diameter is bisected at the centre.

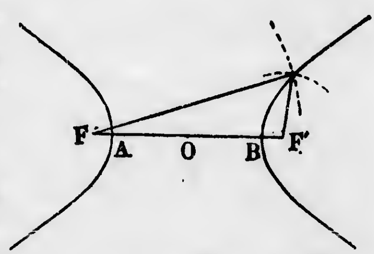
IV. The tangents at the two ends of a diameter are parallel.

CHAPTER IV.

THE HYPERBOLA.

531. *Def.* An **hyperbola** is the locus of the point the difference of whose distances from two fixed points is a constant.

Each of the two fixed points is called a **focus** of the hyperbola.



532. Any number of points of an hyperbola may be found by the intersection of two circular arcs, thus:

Let $2a$ be the constant difference between the distances of a point of the curve from the foci. From either focus as a centre, with an arbitrary radius r describe an arc of a circle.

From the other focus, with the radius $r + 2a$, describe an arc intersecting the other arc.

The point of intersection will be at the distance r from one focus and $r + 2a$ from the other; the difference of those distances is $2a$, whence the point lies on the hyperbola.

533. *Corollary.* Since there is no limit to the radius r, the hyperbola extends out to infinity.

534. *Major axis of the hyperbola.* If the constant $2a$ were greater than the distance between the foci F and F', there would be no point the difference of whose distances from F and F' could be as great as $2a$, and so there would be no hyperbola. Therefore $2a$ must be less than FF'.

Again, if we pass along the line FF' from F to F', the difference of the distances will be FF' when we start, it will diminish to zero at the middle point of the line, and will then increase to FF' at the end F'. Hence there must be two points on the line for which this difference is $2a$; that is, two points of the hyperbola. Let A and B be these points. We must then have, by the conditions of the locus,

$$BF - BF' = 2a; \text{ that is, } FA + AB - BF' = 2a.$$

$$AF' - AF = 2a; \text{ that is, } -FA + AB + BF' = 2a.$$

The sum of these equations divided by 2 gives

$$AB = 2a.$$

Their difference gives

$$FA = BF'.$$

From these two equations we readily see that the curve cuts the line FF' at the distance a on each side of the middle point O of that line.

535. *Def.* The distance between the points at which the hyperbola cuts the line joining its foci is called the **major axis** of the hyperbola.

From what has been said we see that the major axis is equal to the common difference of the distances from each point of the curve to the foci.

Since a point of the hyperbola may be either nearer to F than to F' by $2a$, or nearer to F' than to F, the hyperbola consists of two branches.

Also, if we draw the perpendicular bisector of FF' through

O, every point of this bisector being equally distant from F and F', no point of it can be a point of the hyperbola: Hence

536. *The two branches of the hyperbola are completely separated.*

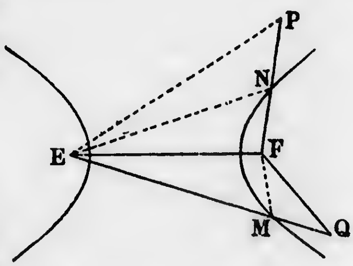
REMARK 1. Most of the properties of the ellipse and hyperbola correspond to each other in that where one has sums of lines, the other has differences; where an angle is formed in one, the adjacent angle will be formed in the other, etc. The student should compare the corresponding theorems.

REMARK 2. Since each branch of the hyperbola extends out to infinity, it may be considered as dividing the plane into three distinct parts, one within each branch and one between the branches. The two first portions may be considered as belonging to one class, and as lying within the hyperbola—*i.e.*, within one of its branches—and the last as lying without the hyperbola.

THEOREM VIII.

537. *From every point without the hyperbola the difference of the distances from the foci is less than the major axis.*

From every point within the hyperbola that difference is greater than the major axis.



Hypothesis. E, F, foci of an hyperbola; P, any point without the hyperbola; Q, any point within the hyperbola.

Conclusions.

I. $PE - PF < 2a$.

II. $QE - QF > 2a$.

Proof. I. Let N be the point in which the line PF intersects the curve, and call Δ the amount by which PE exceeds PF. Then $\quad \Delta = PE - PF = PE - PN - NF.$

Because N is on the curve,

$$2a = EN - NF.$$

Because PE is a straight line,

$$EN + PN > PE; \text{ whence } PE - PN < EN.$$

Therefore

$$PE - PN - NF < EN - NF, \text{ or } \Delta < 2a. \quad \text{Q.E.D.}$$

II. Let M be the point in which QE first intersects the curve, and call Δ the excess of QE over QF. Then

$$\Delta = QE - QF.$$

Because M is on the curve,

$$\begin{aligned} 2a &= ME - MF \\ &= QE - (MQ + MF). \end{aligned}$$

Because QF is a straight line,

$$MQ + MF > QF.$$

Therefore to form Δ we take from QE a less line than we do to form $2a$, whence

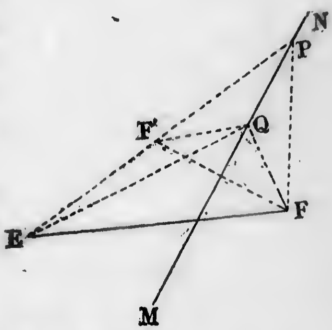
$$\Delta > 2a. \quad \text{Q.E.D.}$$

538. *Corollary.* Since every point on the plane must be either within the hyperbola, without it, or upon it, we conclude that, conversely:

Every point the difference of whose distances from the foci is less than 2a lies without the hyperbola.

Every point the difference of whose distances from the foci is greater than 2a lies within the hyperbola.

539. PROBLEM. *Having a straight line passing between the foci, it is required to find that point upon it at which the difference of the distances from the foci shall be the greatest.*



Solution. Let E and F be the foci; MN, the line; F, the focus nearest the line.

Let F' be the opposite point of F relatively to the line, and let EF' produced intersect the line MN on P.

Let Q be any point at pleasure on the line. Call Δ the difference of the distances. Then—

1. Because F' and F are opposite points, we have

$$QF = QF',$$

whence $\quad \Delta = QE - QF = QE - QF'.$

2. So long as Q is at any point of the line except P, QEF will form a triangle, and we shall have

$$\varDelta = QE - QF' < EF'.$$

But when Q coincides with P, we have

$$\varDelta = PE - PF' = EF'.$$

Because $\varDelta$ is equal to EF' at P, and less than EF' at every other point of the line, we conclude that P is the point of maximum difference of distance.

Because F and F' are opposite points, we have

$$\text{Angle } EPQ = \text{angle } FPQ.$$

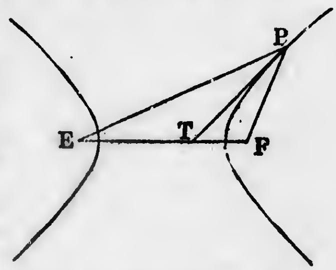
Therefore *the point of maximum difference of distances is that from which lines to the foci make equal angles with the line.*

At this point the line will be the bisector of the angle EPF.

Theorem IX.

540. *If from any point of an hyperbola lines be drawn to the foci, the bisector of the angle between those lines will be a tangent to the hyperbola.*

Hypothesis. E, F, foci of an hyperbola; P, any point of the curve; PT, the bisector of the angle EPF.



Conclusion. PT is a tangent to the hyperbola at the point P.

Proof. 1. Because PT bisects the angle EPT, the difference of distances from the foci is less at every other point of PT than it is at P (§ 539).

2. Because P is on the curve, the difference is there equal to $2a$. Hence it is less than $2a$ at every other point of the line.

3. Therefore every other point of PT except P is without the curve, so that PT touches the curve at P (§ 538). Q.E.D.

Scholium. Comparing this theorem with § 526, we see that while in the hyperbola the tangent bisects the interior angle of the triangle EPF, in the ellipse it bisects the exterior adjacent angle. Therefore if through the point P both an ellipse and an hyperbola be passed, having E and F as the foci, the tangents to the two curves at P will be perpendicular to each other (§ 82).

541. *Def.* The ellipse and hyperbola are called the **conic sections,** or simply **conics.**

542. *Def.* **Confocal** conics are those which have the same foci.

543. *Def.* A **family** of confocal conics means an indefinite number of conics having the same foci.

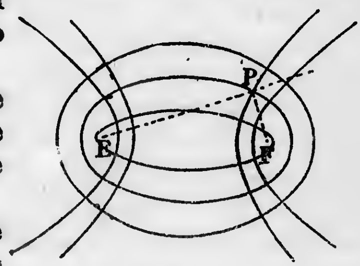
544. *Scholium.* Two curves which intersect are said to cut each other at an angle equal to the angle between their tangents at the point of intersection.

The reason of this appellation is that a curve at any point is considered to have the same direction as its tangent at that point.

Theorem X.

545. *In a family of confocal ellipses and hyperbolas, all the ellipses cut all the hyperbolas at right angles.*

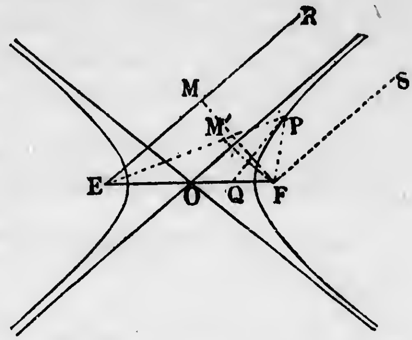
Proof. Let P be any point of intersection of an ellipse with a confocal hyperbola, and PE, PF the lines from P to the foci.



Because P is a point of the ellipse, the tangent to the ellipse at P bisects the exterior angle at P.

Because P is a point of the hyperbola, the tangent to the hyperbola at P bisects the interior angle EPF.

Therefore these tangents are perpendicular to each other (§ 82), and the curves intersect at right angles (§ 544).



546. *Asymptotes of the hyperbola.* Because the tangent to the hyperbola at the point P bisects the angle FPE, it divides the line EF between the foci at the point Q into two such segments that

$EQ : QF :: EP : FP$. (§405)

Now suppose the point P to move out upon the hyperbola to infinity. The ratio $EP : FP$ will then approach unity as its limit, because the difference between its terms is the finite quantity $2a$, while each term may increase to infinity (§ 513).

Therefore the point of intersection Q approaches the centre O as its limit; and, using the convenient language of infinity, we may say:

547. *A tangent to the hyperbola at infinity passes through the centre of the hyperbola.*

548. *Def.* The tangents at infinity are called **asymptotes** of the hyperbola.

549. *To construct the asymptotes.* From E and F let the lines ER and FS, parallel to the asymptotes, be drawn.

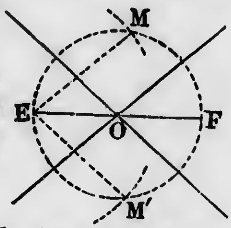
As the point P moves along the hyperbola to infinity, EP and FP will approach ER and FS as their limits (§ 508).

On the lines ER and EP take segments EM and EM' each equal to $2a$; then, since $PE - PF = 2a$, we have $PM' = PF$, so that the triangle $M'PF$ is isosceles and angle $M' =$ angle F. Therefore, as P recedes to infinity, the point M' approaches M as its limit, the angles M' and F both approach right angles as their limit (§ 508). The triangle EMF is therefore right-angled at M. Hence the direction of the asymptote is found thus:

On the line EF as a base erect a right-angled triangle EMF, of which the side EM shall be equal to $2a$.

The asymptote will be the line through the centre of the hyperbola parallel to the side EM.

To construct the required triangle we notice that, since EMF is a right angle, the vertex M lies on the circle of which EF is the diameter. Hence the construction:



On EF as a diameter describe a circle.

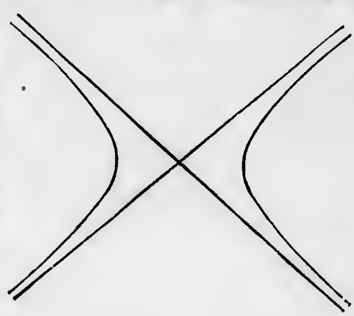
From either E or F as a centre, with a radius $2a$ describe arcs of a circle cutting the first circle in the points M and M'.

Join EM and EM'.

Through the middle point O of the circle draw lines parallel to EM and EM'.

These lines will be asymptotes of the hyperbola of which E and F are the foci, and $2a$ the major axis.

550. It is easy to prove that if we draw the arcs of circles around the centre F, the chords then found will be parallel to EM and EM', and will therefore lead to the same pair of asymptotes.

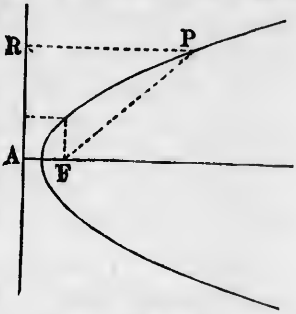


We therefore conclude that *the asymptotes of the hyperbola consist of a pair of straight lines, intersecting each other in the centre of the figure.*

CHAPTER V.

THE PARABOLA.

551. *Definition.* A **parabola** is the locus of a point equidistant from a fixed point and a straight line.



The distances PR and PF are equal on whatever point of the curve P may be placed. AF is the axis of the parabola.

552. *Def.* The fixed point is called the **focus** of the parabola.

553. *Def.* The straight line is called the **directrix** of the parabola.

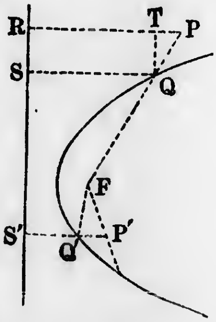
554. *Def.* A straight line through the focus, and perpendicular to the directrix, is called the **axis** of the parabola.

REMARK. Since there is no limit to the possible distance of a point from both the focus and directrix, every parabola extends out to infinity.

THEOREM XI.

555. I. *Every point without the parabola is nearer to the directrix than to the focus.*

II. *Every point within the parabola is nearer to the focus than to the directrix.*



Proof I. Let P be a point without the parabola. The line from this point to the focus must then intersect the curve. Let Q be the point of intersection. Drop the perpendiculars QS upon the directrix and QT upon PR. Then—

Because Q is on the parabola,

$$QF = QS.$$

Adding PQ to these equal lines,

$$PF = QS + PQ,$$
$$= TR + PQ;$$

also, $$PR = TR + TP.$$

Because QTP is right-angled at T,

$$PQ > TP.$$

Therefore $$PF > PR. \quad \text{Q.E.D.}$$

Proof II. Let P' be a point within the parabola. From P' drop a perpendicular $P'S'$ upon the directrix. Let Q' be the point in which it intersects the parabola. Join FQ'. Then we prove, as in the case of the ellipse and hyperbola,

$$S'Q' = Q'F,$$
$$S'P' = Q'F + Q'P',$$
$$Q'F + Q'P' > P'F;$$

whence $$S'P' > P'F. \quad \text{Q.E.D.}$$

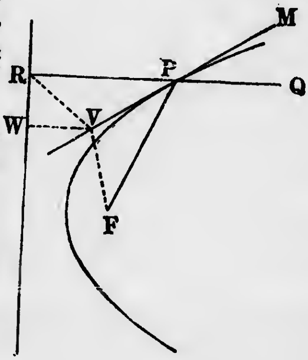
Corollary. Since every point in the plane must be either on the parabola, within it, or without it, we conclude that, conversely:

556. I. *Every point nearer to the focus than to the directrix lies within the parabola.*

II. *Every point nearer to the directrix than to the focus lies without the parabola.*

Theorem XII.

557. *At any point of a parabola a line making equal angles with the line to the focus and the perpendicular upon the directrix is a tangent to the parabola.*



Hypothesis. P, any point of a parabola of which F is the focus and RW the directrix; PR, the perpendicular upon the directrix; PV, a line through P making angle $RPV =$ angle FPV.

Conclusion. PV is a tangent to the parabola at P.

Proof. Let V be any point of the line MPV. Join VF, VR, and from V drop the perpendicular VW on the directrix. Then—

1. In the triangles RPV and FPV,

$$\text{Angle } RPV = \text{angle } FPV \text{ (hyp.)}.$$
$$PR = PF \text{ (} P \text{ being on the curve)}.$$
$$PV \text{ common.}$$

Therefore the triangles are identically equal, and

$$VR = VF.$$

2. Because VW is a perpendicular upon the directrix,

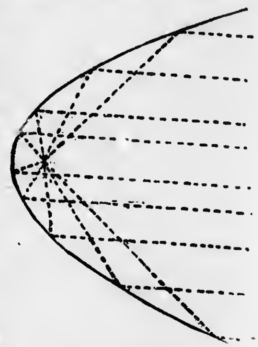
$$VW < VR;$$

whence $$VW < VF,$$

and the point V is therefore without the parabola (§ 556, II.).

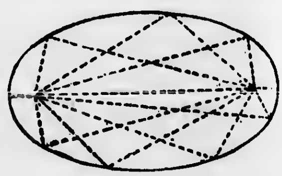
3. Because V may be any point of the line MPV except P, every point of this line except P is without the parabola, and the line touches the parabola at P without intersecting it. Q.E.D.

558. *Scholium.* The property of the ellipse and parabola expressed in this theorem and in Theorem VI., relating to the ellipse, leads to the use of these curves in reflectors designed to bring rays of light to a focus. Since the curve and the tangent have the same direction at the point of tangency, rays of light are reflected by the curve as they would be by the tangent at the point of reflection.



Because the angles of incidence and reflection are equal, it follows that if parallel rays of light, perpendicular to the directrix and therefore parallel to the axis, fall upon a parabolic reflector, they will all be reflected toward the focus.

Conversely, if a light be placed in the focus of a parabolic reflector, all the rays from the focus will be parallel after reflection.



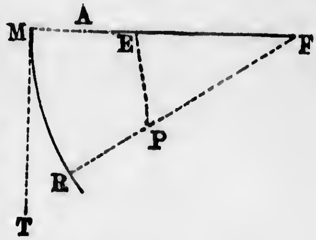
In the case of the ellipse, the corresponding property leads to all the rays which emanate from one focus being reflected to the other focus.

Relations of the Ellipse, Parabola, and Hyperbola.

Theorem XIII.

559. *The parabola may be regarded as an ellipse of which the major axis is infinite.*

Proof. Let E and F be the foci of an ellipse, and A one end of its major axis.



On the line FA produced take $AM = EA$. FM will then be equal to the major axis (§§ 516, 517).

From the farther focus F as a centre, with a radius FM describe an arc of a circle.

Let P be any point of the ellipse. Join EP, FP, and produce FP until it meets the circle in R. Then—

1. Because $FP + EP =$ major axis $= FR$, we have

$$PE = PR.$$

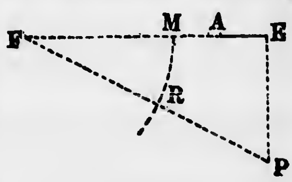
Therefore each point of the ellipse is equally distant from the focus E and from the arc MR.

2. Now let the focus F recede to infinity along the line AF produced.

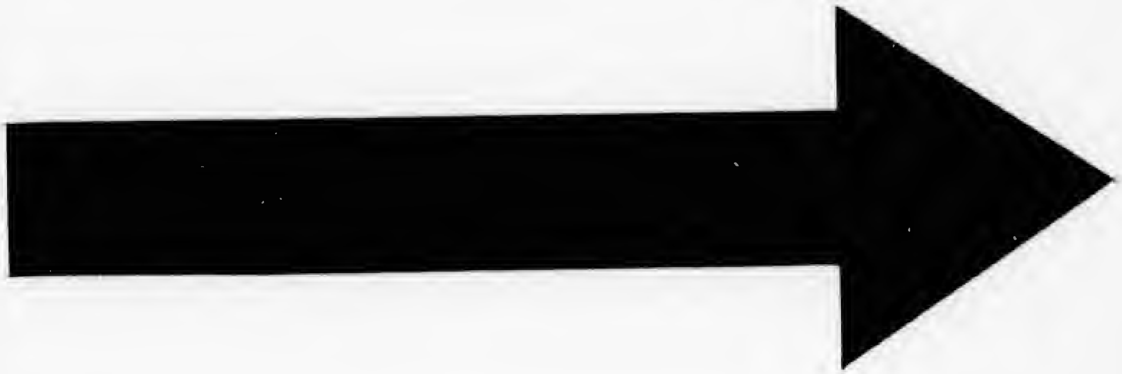
3. If MT be the perpendicular to MA at M, the arc MR will approach MT as its limit (§ 512); and PR will approach parallelism to MF, and therefore perpendicularity to MT as its limit (§ 509). Hence each point P of the ellipse will approach a position in which it is equally distant from the focus E and the line MT. That is, it will approach a parabola having E as its focus and MT as its directrix. Hence:

560. *If one focus of an ellipse recedes to infinity, the ellipse will become a parabola about the other focus.* Q.E.D.

561. *Passage from the ellipse to the hyperbola.* Starting as in the last section, let us place the second focus, F, on the opposite side of the point M from E; and let us, as before, draw an arc around the centre F with a radius FM.



Let FR be any radius of this arc;




2.8
2.5
3.2
2.2
3.6
2.0
1.8
1.1
1.0

and produce FR to a point P, such that $PR = PE$. We shall then have $$PF - PE = FR.$$

Therefore P will lie upon an hyperbola of which E and F are the foci, and $FM = FR$, the major axis (§ 535).

Then, supposing F to recede to infinity in the direction MF, we show, as before, that P will approach a parabola of which E is the focus, and a perpendicular to FM through M the directrix.

Scholium. The ellipse, parabola, and hyperbola therefore all belong to one class of curves. It is shown in solid geometry that they may all be formed by the intersection of a cone with a plane, from which property is derived the term *conic section.*

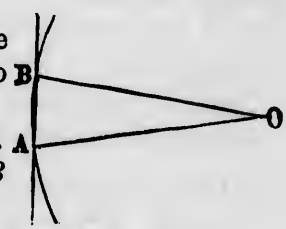
Tangents as Limits of Secants.

562. Because one straight line, and no more, may be drawn between two points, two points determine a straight line passing through them.

If the two points lie on a curve, the straight line passing through them is a secant of the curve.

563. *Def.* The tangent to a curve is the line which the secant approaches, as its limit, when the two points which determine the secant come indefinitely near together.

This is a more general definition of a tangent than that heretofore given for the circle, and applied to the ellipse. By means of it the fundamental properties of tangents to the circle and conic sections may be established, as follows:



564. *The Circle.* Let O be the centre of a circle, and OA, OB, two of its radii.

Through A and B draw a secant. Then in the isosceles triangle OAB we have

$$\text{Angle } A + \text{angle } B + \text{angle } O = 180°,$$

or $$2 \text{ angle } A + \text{angle } O = 180°;$$

whence $$\text{Angle } A = \text{angle } B = 90° - \tfrac{1}{2} \text{ angle } O.$$

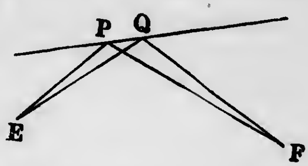
Now let the point B approach indefinitely near to A.

The angle O will then approach zero as its limit.

Therefore the angle A will approach 90° as its limit, and, the tangent being the limit toward which the line AB approaches, must be perpendicular to OA.

We thus arrive at the property of the tangent demonstrated in § 225.

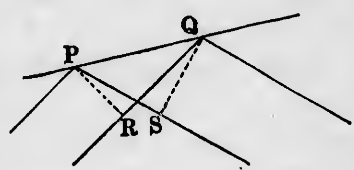
565. *The Ellipse.* Let E and F be the foci of an ellipse, and P and Q two points upon it.



Through P and Q pass a secant. Join PE, PF, QE, QF.

Now let the point Q approach indefinitely near to P, and, as the approach becomes nearer, let us look at P and Q through a microscope of which the magnifying power may increase indefinitely.

Then at the limit, because the angles at E and F become zero, EP and EQ, as also FP and FQ, will seem parallel in the microscope.



From P drop the perpendicular PR upon EQ, and from Q the perpendicular QS upon FP. Then—

1. $$EQ - EP = RQ. \qquad (§ 511)$$
$$FP - FQ = SP.$$

From the fundamental condition of the ellipse,

$$EQ + FQ = EP + FP,$$

we have
$$EQ - EP = FP - FQ,$$

whence
$$RQ = SP.$$

2. Because the right-angled triangles PRQ and PSQ have

Hypothenuse PQ common,

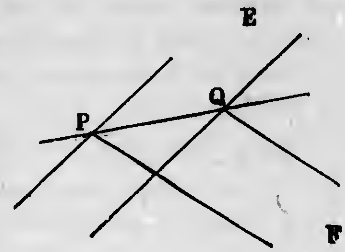
Side QR = side PS,

they are identically equal, and

Angle PQR = angle QPS.

But PQR is the angle which the tangent makes with the line EQ to the focus at E, and QPS is the angle with PF. Because at the limit $EP \parallel EQ$ and $PF \parallel QF$, the tangent PQ makes equal angles with the lines to the foci.

566. *The Hyperbola.* The same reasoning will apply to the hyperbola, except that the foci E and F will be on opposite sides of the tangent, and in consequence the tangent will bisect the angles to the foci.



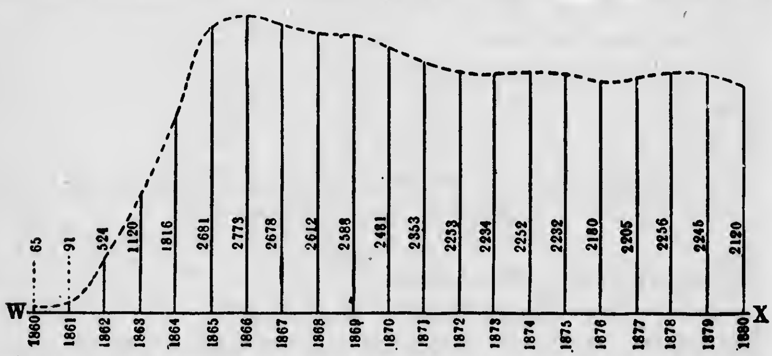
In the case of the parabola the same reasoning will always apply, the lines EP and EQ being replaced by perpendiculars to the directrix.

CHAPTER VI.

REPRESENTATION OF VARYING MAGNITUDES BY CURVES.

567. The changes of a varying magnitude may be represented to the eye by a curve on a system which we shall illustrate by showing the changes of the National Debt of the United States between 1860 and 1880.

In the following figure the horizontal line WX is divided up to represent the different years. On th middle of each



year we erect a perpendicular proportional to the magnitude of the debt at that time, as given in millions of dollars on each perpendicular. Then by noting the length of these perpendiculars we have the different magnitudes of the debt presented to the eye.

These perpendiculars are called *ordinates*.

The ordinates only show what the debt was on July 1st of each year, and we may wish to know what it was at other times, supposing it to have varied in a regular way. This we do by drawing a curve through the tops of all the ordinates. Then the height of the curve above the base WX will show the length of the ordinate, and therefore the amount of the debt.

Having drawn the curve, we may erase the ordinates entirely, and get the amount of the debt at any time by measuring the height of the curve above the base line at the point corresponding to that time.

568. *Def.* **A curve showing the magnitude of a varying quantity at any time is called a graphic representation of that quantity.**

569. The preceding method may be used to show to the eye the relation between two varying quantities connected by an algebraic equation. The following is an example of the process. Let us have the equation

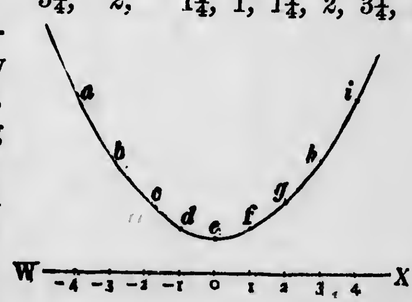
$$y = \frac{x^2}{4} + 1.$$

We suppose x to have in succession a number of different values, say -4, -3, -2, -1, 0, 1, 2, 3, and 4, and for each of these values we calculate the corresponding value of y.

We arrange these values together, thus:

Values of x..........	-4,	-3,	-2,	-1,	0,	1,	2,	3,	4.
Corresp. values of y..	5,	$3\frac{1}{4}$,	2,	$1\frac{1}{4}$,	1,	$1\frac{1}{4}$,	2,	$3\frac{1}{4}$,	5.

We next draw a horizontal base WX, and lay the values of x off on it, the positive values being laid off toward the right, the negative ones toward the left. Then from each point on the line we measure upwards a length equal to the corresponding value of y, and there make a point.



Let the points be a, b, c, etc. The heights of these points show the several values of y. Should any values of y be negative, they extend below the line. Then we draw a curve through all the points. The height of this curve above the base-line will then represent the value of y corresponding to any value of x between the limits, $+4$ and -4, so that the eye can see at a glance how y varies.

570. *Def.* **Abscissas** are the values of x, laid off upon the base-line.

571. *Def.* The base-line is called the **axis of abscissas**, or the axis of X.

572. *Def.* The equation which gives rise to a particular curve is called the **equation** of that curve.

EXERCISES. Plot in the above way the curves corresponding to the following equations between the limits $x = -4$ and $x = 4$.

$$y = x + 4.$$

$$y = \frac{x}{2} + 3.$$

$$y = \frac{x^2}{3}.$$

$$y = \frac{x^2}{4} + x + 1.$$

$$y = \frac{x^2}{5} - 2.$$

$$y = \frac{x^2}{4} - x - 2.$$

$$y = \frac{x^2}{3} - 2x + 1.$$

$$y = \frac{x^2}{2} - 3x.$$

$$y = \frac{x^2}{2} - 3x - 1.$$

$$y = \frac{x^2}{2} - 3x - 2.$$

GEOMETRY OF THREE DIMENSIONS.

BOOK VIII.

OF LINES AND PLANES.

573. *Def.* **Geometry of three dimensions** treats of figures whose parts are not confined to a single plane.

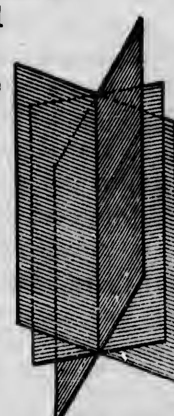
Geometry of three dimensions is also called the *geometry of space* and *solid geometry.*

574. *Def.* **Parallel planes** are those which never meet, how far soever they may be produced.

575. *Def.* A straight line is said to be **parallel to a plane** when it never meets the plane, how far soever it may be produced.

576. The different parts of a figure are said to lie in one plane when a plane can be passed through them all.

REMARK. Whenever a plane can be passed through a system of lines or points in space, the theorems of plane geometry apply to the figure formed by such lines and points. Otherwise they do not so apply.



Several planes passing through the same straight line.

577. AXIOM I. If two or more points of a straight line lie in a plane, the whole line lies in that same plane.

Ax. II. Any number of planes may be passed through the same straight line, and a plane may be turned round on any line lying in it.

Ax. III. Only one plane can pass through a line and a point without that line.

EXERCISES.

The following exercises are inserted here in order that the student may, by working upon them, acquire a clear conception of the difference between the relations of lines in space and in a plane before proceeding to the study of the theorems of the geometry of space. If four stiff rods be jointed together, or merely held together, so as to form a quadrilateral, it may assist in Exercises 3 and 4. A beginner in the subject will find it useful to construct the diagrams in space with wires, strings, and rods attached to a flat board.

A polygon in space may be formed by joining end to end any number of finite straight lines, as defined in Book II., § 142. The only change is that in Book II. the lines are all supposed to be confined to one plane, whereas there is no such restriction upon polygons in space. Now:

1. Explain that all the propositions of Book I. which relate to triangles are true of triangles in space, however situated.

This may be done by Theorem I., which follows, by showing that the three sides must be all in one plane.

2. Explain that these proportions are not true of polygons of four or more sides situated anyhow in space.

Show that a polygon of four or more sides (which may be formed of stiff straight rods) may be so joined that its sides shall not all be in one plane.

3. How many different planes may the sides of a quadrilateral in space contain when taken two and two?

4. Show that the two diagonals of a quadrilateral in space do not necessarily intersect.

Show that each pair of adjacent sides may be turned round on the vertices joining them to the opposite pair.

5. But if we draw three lines in a triangle, one from each angle to any point of the opposite side, each of these lines will intersect the two others.

6. Any pair of parallel lines must lie in the same plane. But there may be three lines each parallel to the other two, and yet they may not all three lie in any one plane. How many planes will they lie in when taken two and two?

7. If four lines are parallel, how many planes may they lie in when taken two at a time?

8. A transversal intersecting a pair of parallel lines lies in the same plane with them.

9. If two lines not in the same plane are intersected by the same transversal, how many planes may be determined by the three lines taken two and two?

CHAPTER I.

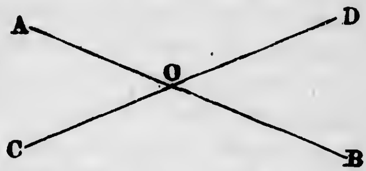
RELATION OF LINES TO A PLANE

THEOREM I.

578. *Through two intersecting straight lines one plane, and one only, may pass.*

Hypothesis. Two straight lines, AB and CD, intersecting in the point O.

Conclusion. One plane, and only one, can pass through them.



Proof. 1. Let any plane pass through the line AB.

2. Turn this plane around on AB as an axis until it meets any point C of the line CD (§ 577, II.).

3. The line CD then has two points in this plane; namely, C and O. Therefore the whole line CD is in the plane. (§ 577, I.)

4. Because the plane contains the line AB and the point C without this line, it is the only plane which satisfies the conditions (§ 577, III.). Q.E.D.

Scholium. This proposition is also expressed by saying that two lines which intersect each other lie in one plane.

Corollary 1. Because any third line which intersects both the lines AB and CD in any other point than O must have two points in the plane C (namely, the points of intersection), it must lie in the same plane. Hence:

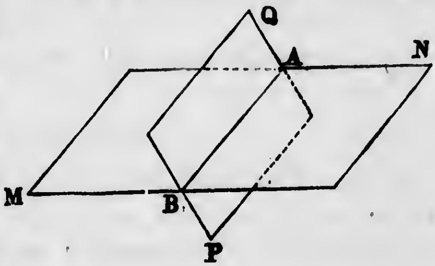
579. *Any number of straight lines, each of which intersects all the others at different points, lie in one plane.*

580. *Cor.* 2. *Through any three points not in the same straight line one plane, and only one, may pass.*

Theorem II.

581. *Two planes intersect each other in a straight line.*

Hypothesis. *MN, PQ*, two planes intersecting each other along the line *AB*.



Conclusion. The line *AB* is a straight line.

Proof. Let *A* and *B* be any two points common to both planes.

Draw a straight line from *A* to *B*. Then—

1. Because the points *A* and *B* are both in the plane *MN*, the straight line *AB* lies wholly in the plane *MN* (§ 577, I.).

2. Because the points *A* and *B* are both in the plane *PQ*, the straight line *AB* lies wholly in the plane *PQ*.

3. Hence this straight line lies in both planes, and therefore forms their line of intersection.

4. The planes cannot intersect in any point not in this line, because then two planes would each contain a line and a point without it, which is impossible (§ 577, III.).

Therefore *AB* is the only line of intersection. Q.E.D.

582. *Corollary. The line of intersection of two planes is a line lying wholly in both planes.*

Theorem III.

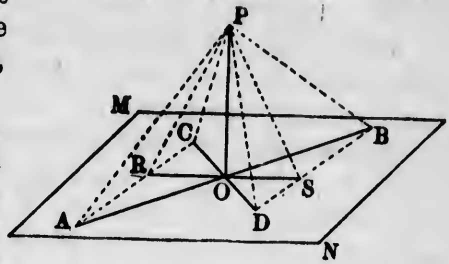
583. *If a straight line be perpendicular to two straight lines in a plane, it will be perpendicular to every other straight line lying in the plane and passing through its foot.*

Hypothesis. *MN*, any plane; *AB, CD*, two lines in this plane; *OP*, a common perpendicular to these lines at their point of intersection *O*; *RS*, any other line lying in the plane, and passing through *O*.

Conclusion. The line OP is also perpendicular to RS.

Proof. In the lines AB and CD take the points A, B, C, and D, so that

$$AO = BO, \quad CO = DO.$$



Join BD and AC, and let R and S be the points in which the joining lines intersect the third line RS. (These lines must intersect RS, because they are in the same plane with it and not parallel to it.)

Join PC, PR, PA, PB, PS, PD. Then—

1. Because, in the triangles OAC and OBD,

$$OC = OD, \quad OA = OB \text{ (by construction)},$$
$$\text{Angle } AOC = \text{opp. angle } BOD:$$

these triangles are identically equal, and

$$BD = AC.$$
$$\text{Angle } OAR = \text{angle } OBS.$$
$$\text{Angle } OCR = \text{angle } ODS.$$

2. Because of these equalities, and the equality of ROA to its opposite angle BOS, we have

$$\text{Triangle } AOR = \text{triangle } BOS \text{ (identically)},$$

whence $\quad OR = OS; \quad AR = BS.$

3. Because OP is perpendicular to AB, and $OA = OB$,

$$PA = PB,$$

and, in the same way, $\quad PC = PD.$

4. Because of these equalities, and of $BD = AC$,

$$\text{Triangle } PAC = \text{triangle} PBD \text{ (identically)},$$

whence $\quad$ Angle PAR = angle PBS.

5. Because, in the triangles PAR and PBS,

$$PA = PB, \qquad (3)$$
$$AR = BS, \qquad (2)$$
$$\text{Angle } PAR = \text{angle } PBS: \qquad (4)$$

these triangles are identically equal, and

$$PR = PS.$$

Therefore $\quad$ Angle POR = angle POS;

whence POR and POS are right angles, and $OP \perp RS$.

6. Because RS may be any line whatever lying in the

plane and passing through O, the line OP is perpendicular to every such line. Q.E.D.

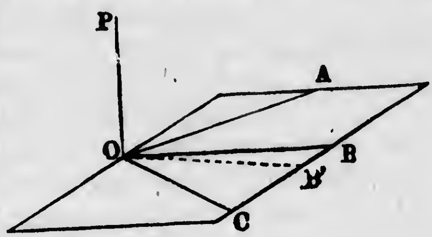
584. *Def.* A line meeting a plane so as to be perpendicular to every line lying in the plane and passing through the point of intersection is said to be **perpendicular to the plane.**

585. *Corollary. From a given point in a plane only one perpendicular to the plane can be erected, and from a point without a plane only one perpendicular can be dropped upon the plane.*

Theorem IV.

586. Conversely, *all lines perpendicular to another line at the same point lie in the same plane.*

Hypothesis. OP, any straight line; OA, OC, two lines perpendicular to OP at O; OB, any third line perpendicular to OP at O.



Conclusion. OB lies in the same plane with OA and OC.

Proof. 1. If OB is not in the plane AOC, pass a plane through PO and OB, and let OB' be the line in which it intersects the plane AOC.

2. Because OP is perpendicular both to OA and OC, it is also perpendicular to OB', which lies in this plane (§ 583).

3. Because OB' is in the plane POB, we have in this plane two straight lines, OB and OB', both perpendicular to OP, which is impossible.

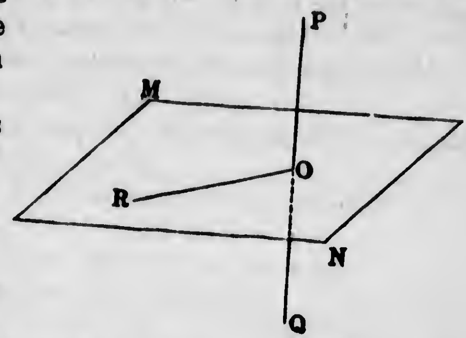
4. Therefore OB and OB' are the same straight line, and OB lies in the plane AOC. Q.E.D.

587. *Corollary. If a right angle be turned round one of its sides as an axis, the other side will describe a plane.*

Theorem V.

588. *If a plane bisect a line perpendicularly, every point of the plane is equally distant from the ends of the line.*

Hypothesis. PQ, a line perpendicular to the plane MN, intersecting it at the middle point O of the line; R, any point in the plane.



Conclusion. R is equally distant from P and Q.

Proof. Join OR. Then—

1. Because PQ is perpendicular to the plane, it is perpendicular to OR in that plane (§ 584).

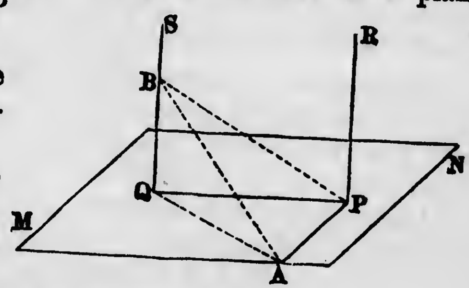
2. Because OR is perpendicular to PQ at its middle point O, it is equally distant from P and Q (§ 104). Q.E.D.

589. *Corollary.* Conversely, *every point which is equally distant from two fixed points is in the plane bisecting at right angles the line joining the points.*

Theorem VI.

590. *Straight lines perpendicular to the same plane are parallel to each other.*

Hypothesis. PR, QS, two perpendiculars to a plane MN at the points P and Q.



Conclusion. These perpendiculars are parallel.

Proof. In order to prove the parallelism of the lines, we must show—

I. That they can never meet;

II. That they are in the same plane.

I. If the two lines could meet at any point X, then from that point there would be two perpendiculars XP and XQ upon the same straight line PQ, which is impossible. Therefore the lines cannot meet.

II. From the point P draw in the plane MN the line PA, perpendicular both to PQ and to PR. In the line QS take $QB = PA$. Join BP, BA, QA. Then—

1. Because, in the right-angled triangles QPA and PQB,

$$AP = BQ,\ PQ \text{ common},$$

these triangles are identically equal, and

$$BP = AQ.$$

2. Because, in the triangles BQA and BPA,

$$AQ = BP,\ BQ = AP,\ AB \text{ common},$$

these triangles are identically equal. But BQA is a right angle (§ 584). Therefore the corresponding angle APB is also a right angle.

3. Therefore from the point P of the line AP there proceed three straight lines PQ, PB, and PR, all at right angles to AP. Hence these three lines are in one plane; that is, PR is in the plane fixed by the two lines PQ, PB.

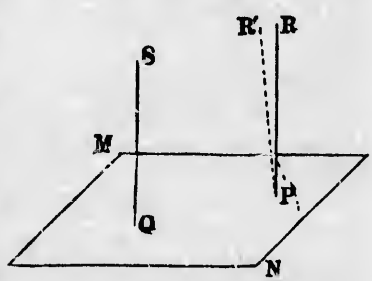
4. But QS is also in this plane, because it intersects these lines (§ 579). Therefore QS and PR are in the same plane.

5. Hence the lines PR and QS are in the same plane and never meet, and are therefore parallel. Q.E.D.

Theorem VII.

591. Conversely, *if one of several parallels is perpendicular to a plane, each of the others is also perpendicular to that plane.*

Hypothesis. A plane, MN; two parallel lines, PR and QS, intersecting the plane at P and Q in such manner that QS is perpendicular to the plane.



Conclusion. PR is also perpendicular to the plane.

Proof. 1. If PR is not perpendicular to the plane, let PR' be perpendicular to it.

2. Then PR' is parallel to QS (§ 590).

3. Therefore through the point P we have two straight lines, PR and PR', both parallel to QS, which is impossible.

4. Therefore the perpendicular to the plane at P is the line PR. Q.E.D

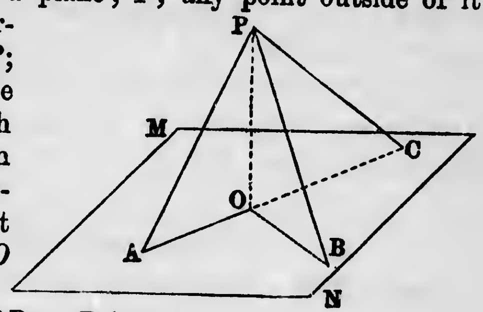
Corollary. If two or more lines are each parallel to the same straight line, and a plane be drawn perpendicular to this straight line, it will also be perpendicular to the other lines, and they will be parallel to each other. Hence:

592. *Lines parallel to the same straight line are parallel to each other.*

THEOREM VIII.

593. *From any point above a plane lines meeting the plane at equal distances from the foot of the perpendicular are equal, and the line meeting the plane at the greater distance from this foot is the greater.*

Hypothesis. MN, a plane; P, any point outside of it; O, the foot of the perpendicular from P; AOC, any straight line in the plane through O; A, B, two points in the plane equally distant from O; C, a point more distant from O than A is.



Conclusions. I. $PB = PA$.

II. $PC > PA$.

Proof. 1. In the triangles POA and POB,

PO is common.

$OA = OB$ (hyp.).

Angle POA = angle POB (both right angles).

Therefore hypothenuse PB = hypothenuse PA. Q.E.D.

2. Because O is the foot of the perpendicular from P on AC, and $OC > OA$,

$PC > PA$ (§ 103). Q.E.D.

594. *Corollary* 1. *If through the centre of a circle a line be passed perpendicular to its plane, each point of this line is equally distant from all points of the circle.*

595. *Cor.* 2. *Equal lines meet the plane at equal distances from the foot of the perpendicular, and greater lines at greater distances.*

This corollary may be expressed as follows:

596. *The locus of the point in a plane at a given distance from a fixed point without the plane is a circle drawn around the foot of the perpendicular as a centre.*

597. *Def.* The **projection** of a point upon a plane is the foot of the perpendicular dropped from the point upon the plane.

EXAMPLE. If MN be a plane, and P a point outside of it, and if the perpendicular from P upon the plane meet the latter in P', then P' is the projection of P upon the plane MN.

598. *Def.* The **projection** of a line upon a plane is the locus of the feet of the perpendiculars dropped from every point of the line upon the plane.

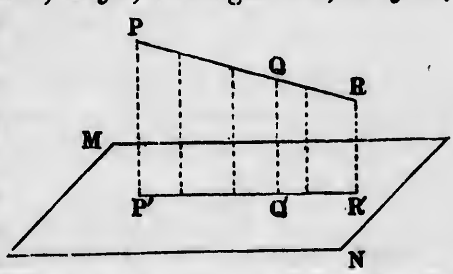
EXAMPLE. The line $P'R'$ is the projection of the line PR upon the plane MN.

THEOREM IX.

599. *The projection of a straight line upon a plane is itself a straight line, and the straight line and its projection are in one plane.*

Hypothesis. MN, a plane; PQR, a straight line; $P'Q'R'$, the projection of PQR upon MN.

Conclusion. $P'Q'R'$ is a straight line, and lies in one plane with PQ.



Proof. 1. Because the lines PP' and RR' are perpendiculars to the plane MN, they are parallel to each other, and therefore in one plane (§ 590).

2. Pass a plane through these lines. Because this plane contains the points P and R, it will contain the point Q, which lies on the line PR.

3. Because the line QQ' is perpendicular to MN, it is also parallel to PP' (§ 590); and because it contains the point Q, the plane $PP'QQ'$ is the same as the plane $PP'RR'$. Therefore the foot Q' lies in this same plane.

4. Because the intersection of two planes is a straight line (§ 581), the foot Q', which lies on the intersection of the two planes, is in a straight line with P' and R'.

5. Because Q may be any point on PR, the projection of every point of PR is in the straight line $P'R'$. Q.E.D.

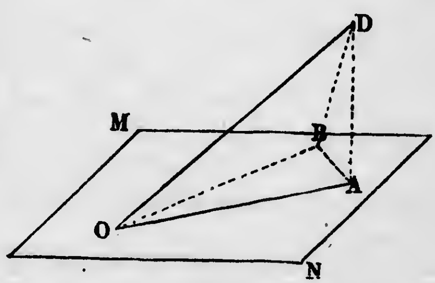
600. *Corollary* 1. *If a line intersect a plane, its projection passes through the point of intersection.*

601. *Cor.* 2. *If a line be perpendicular to a plane, its projection upon the plane is a point; namely, the point in which it intersects the plane.*

THEOREM X.

602. *If a line intersect a plane, it makes a less angle with its projection than with any other line in the plane passing through the point of intersection.*

Hypothesis. MN, a plane; OD, a line intersecting this plane in O; OA, the projection of OD upon the plane; OB any other line in the plane passing through O.



Conclusion. The angle DOA is less than DOB.

Proof. Take $OB = OA$, and join AB and DB. Then—

1. Because DAB is a right angle (A being the projection of D),

$$DB > DA. \qquad (\S\, 593)$$

2. Because the triangles DOA and DOB have the side DO common, and $OB = OA$, while the third side DB is greater than the third side DA,

Angle $DOB > DOA$ (§ 115). Q.E.D.

603. *Def.* The angle between a line and its projection on a plane is called the **inclination** of the line to the plane.

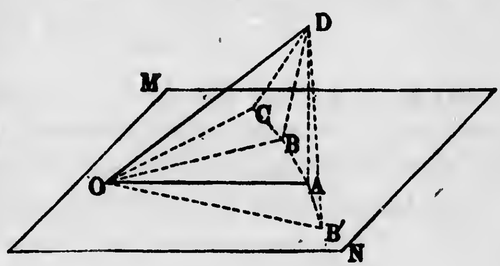
THEOREM XI.

604. *If a line intersect a plane—*

I. *The angle which it makes with a line in the plane passing through its point of intersection is greater, the greater the angle this last line makes with its projection.*

II. *The line makes equal angles with lines at equal angles on both sides of its projection.*

Hypothesis. MN, a plane; OD, a line intersecting it in O; OA, the projection of OD on the plane; OB, OB', two lines from O making equal angles with OA; OC, a line making a still greater angle with OA.



Conclusions. I. Angle $DOC >$ angle DOB.
II. Angle $DOB =$ angle DOB'.

Proof I. Take OB, OB', and OC all equal to OA. Join DB', DA, DB, DC, AB, AB', BC. Then—

1. Because the points B, A, B', C are all in the same plane and equally distant from O, they lie on a circle having O as its centre.

2. Because angle $AOC > AOB$, the distance AC is greater than the chord AB; therefore

$DC > DB$. (§ 593)

3. In the triangles ODB and ODC we have

$$OD \text{ common}; \; OC = OB; \; DC > DB.$$

Therefore

Angle DOC (opp. DC) > angle DOB (opp. DB). (§ 115)

Q.E.D.

Proof II. 4. In the triangles AOB and AOB' we have

OA common;

$OB = OB'$ (construction);

Angle AOB = angle AOB' (hyp.).

Therefore these triangles are identically equal, and

$$AB = AB'.$$

5. Because DAB and DAB' are both right-angled at A, and because DA is common and $AB = AB'$, these triangles DAB and DAB' are identically equal, and

$$DB = DB'.$$

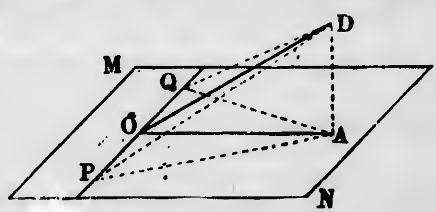
6. Therefore the triangles DOB and DOB', having their sides equal, are also identically equal, and

Angle DOB = angle DOB'. Q.E.D.

THEOREM XII.

605. *At the point of intersection, a line in the plane perpendicular to the projection of a line is perpendicular to the line itself.*

Hypothesis. OD, a line intersecting the plane in O; OA, the projection of OD upon the plane; POQ, a line in the plane perpendicular to OA.



Conclusion.

Line $POQ \perp OD$.

Proof. Take the points P and Q at equal distances from O, and join AP, AQ, DP, DQ. Then—

1. Because the points P and Q are at equal distances from the foot O of the perpendicular AO on PQ, we have

$$AP = AQ.$$

2. In the triangles DAP and DAQ,

DA is common;

$AP = AQ$; (1)

Angle DAP = angle DAQ (hyp.).

Therefore $DP = PQ$.

3. In the triangles DOP and DOQ,

DO is common;

$DP = DQ$; (2)

$OP = OQ$ (construction).

Therefore Angle DOP = angle DOQ;

and because POQ is a straight line, both these angles are right angles. Q.E.D.

Corollary. If the line OD is not perpendicular to the plane MN, there can be only one line in this plane perpendicular to OD. For if there were two such lines, OD would be perpendicular to the plane (§ 583). Hence, because POQ is the only perpendicular to OD in the plane:

606. Conversely, *a line in a plane perpendicular to an intersecting line is perpendicular to the projection of the intersecting line.*

Scholium. An astronomical illustration of these theorems is afforded by conceiving one's self to be looking at the sun in the south. The plane is that of the horizon, in which the observer must suppose himself situated at the point O. Let the line OD be that toward the sun. (It is not necessary to suppose it cut off at D or any other point, because our theorems do not refer to its length.)

Then the horizontal line OA from the observer to that point of the horizon under the sun will be the projection of the line to the sun.

By Theorem X. the angular distance of the sun from this point will be less than from any other point of the horizon. This angle is called the sun's *altitude.*

If we suppose a horizontal east and west line, Theorem XII. shows that this line will always be at right angles both to the direction of the sun and to the south line which passes directly below the sun.

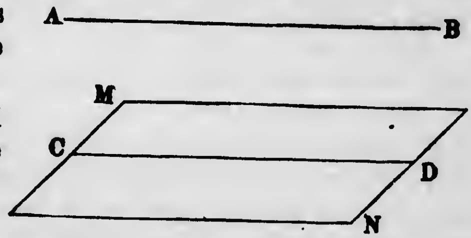
If we take a series of points along the horizon, starting from the point directly below the sun, Theorem XI. shows that the angular distance of these points from the sun will increase up to the opposite point of the horizon from that below the sun.

THEOREM XIII.

607. *When two straight lines are parallel, each of them is parallel to every plane passing through the other and not containing both lines.*

Hypothesis. AB, CD, two parallel straight lines; MN, any plane passing through CD.

Conclusion. AB is parallel to the plane MN.



Proof. Let us call P* the common plane of the parallels AB and CD. Then—

1. Because AB lies wholly in the plane P, if AB meets the plane MN at any point, that point will be common to the plane P and the plane MN.

2. But the only points common to these two planes are on their line of intersection; namely, the line CD (§ 582). Therefore if AB ever meets the plane MN, it must meet this line CD.

3. But, by hypothesis, it is parallel to CD, and so cannot meet it.

4. Therefore it cannot meet the plane MN, and therefore is parallel to it (§ 575). Q.E.D.

5. But should the plane MN coincide with the plane P, the line AB will then lie in MN as it does in P.

THEOREM XIV.

608. *Angles of which the sides are parallel and similarly directed are equal.*

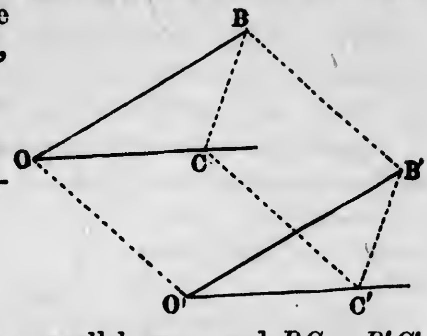
Hypothesis. BOC and $B'O'C'$, two angles in which $OB \parallel O'B'$ and $OC \parallel O'C'$.

Conclusion. Angle $BOC =$ angle $B'O'C'$.

---

* The letter P is here employed not as a mark on the diagram, but as a short and convenient appellation of the plane referred to. Such a use of letters is of constant occurrence in the higher geometry, and should be well understood.

Proof. On the sides B and B' take $OB = O'B'$, and on the sides C and C' take $OC = O'C'$. Join BB', CC', OO'. Then—



1. Because OB and $O'B'$ are equal and parallel, the figure $OO'BB'$ is a parallelogram (§ 138), and

$BB' =$ and $\parallel OO'$.

2. In the same way,

$CC' =$ and $\parallel OO'$.

3. Therefore $BB'CC'$ is a parallelogram, and $BC = B'C'$.

4. Therefore the triangles BOC and $B'O'C'$, having the three sides of the one equal to the sides of the other, are identically equal, and

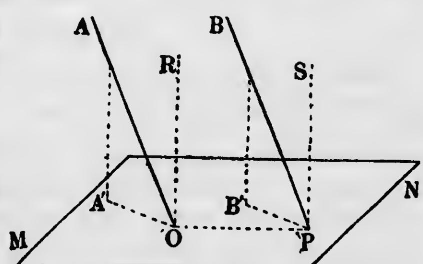
Angle $BOC =$ angle $B'O'C'$. Q.E.D.

609. *Corollary.* It may be shown, as in Book II., Th. VI., that if the sides of the angles are parallel and oppositely directed, the angles will still be equal, and that if one pair of sides is similarly directed and the other oppositely directed, the angles will be supplementary.

Theorem XV.

610. *Parallel lines intersecting the same plane make equal angles with it.*

Hypothesis. OA, PB, two parallel lines intersecting the plane MN in O and P; OA', PB', the projections of certain portions of these lines on the plane.



Conclusion.

Angle $AOA' =$ angle BPB'.

Proof. At the points O and P erect the perpendiculars OR and PS. Then—

1. Because OR and PS are perpendicular to the same plane, they are parallel (§ 590), while $OA \parallel PB$, by hypothesis. Hence Angle $AOR =$ angle BPS. (§ 608)

2. Because OA is a transversal crossing the parallels $A'A$ and OR, it is in the same plane with them. Also, because $OR \perp$ plane MN, $OR \perp OA'$ in that plane (§ 584). The same things are true of PB', PB, and PS.

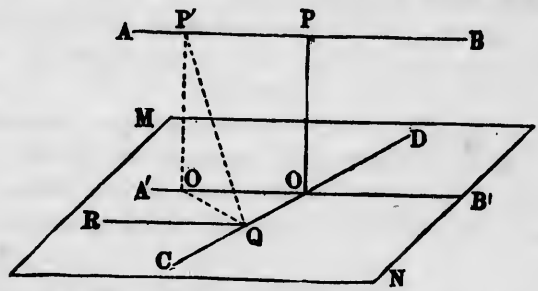
3. Because $A'OR$ and $B'PS$ are right angles, AOA' is the complement of the angle AOR, and BPB' is the complement of the equal angle BPS. Comparing with (1),

$$\text{Angle } AOA' = \text{angle } BPB'. \quad \text{Q.E.D.}$$

Theorem XVI.

611. *Between two lines not in the same plane one, and only one, common perpendicular can be drawn.*

Hypothesis. AB, CD, two lines not in the same plane, and therefore passing each other without intersecting.



Conclusion. There is one line, and no more, perpendicular to both AB and CD.

Proof. I. Through one line, say CD, pass a plane, and let it turn round on CD until it is parallel to AB. Let MN be this plane. Let $A'B'$ be the projection of AB on the plane MN, and let O be the point in which this projection intersects CD. Then—

1. Every point of $A'B'$ is fixed by dropping a perpendicular from some point of AB. Let P be the point, of which O is the projection. Then $PO \perp$ plane MN (§ 597).

2. Because PO is perpendicular to MN, it is perpendicular to both the lines $A'B'$ and CD, which lie in MN.

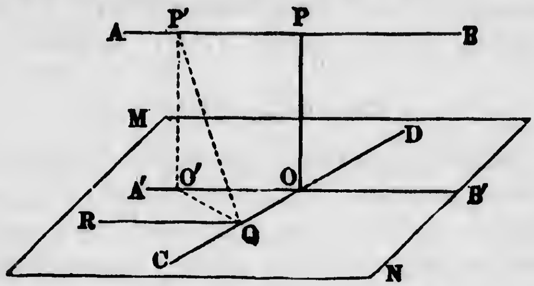
3. Because PO is perpendicular to $A'B'$, it is also perpendicular to AB, which is parallel to $A'B'$ (§ 72).

Therefore OP is perpendicular to both the lines AB and CD. Q.E.D.

II. If there is any other common perpendicular, let it be $P'Q$. Through Q draw, in the plane MN, $QR \parallel AB$. Then—

4. Because $P'Q$ is perpendicular to AB, it is also perpendicular to QR, which is parallel to AB.

5. Because $P'Q$ is perpendicular to both the lines QR and CD, it is perpendicular to their plane MN.



6. But, because $A'B'$ is the projection of AB, the foot of the perpendicular from P' on the plane must fall on some point of $A'B'$. Let O' be this point.

Therefore from the point P' are dropped two perpendiculars $P'O'$ and $P'Q$ upon plane MN, which is impossible (§ 585).

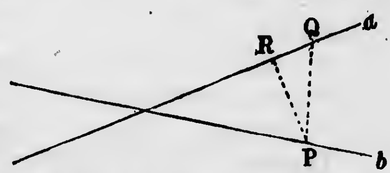
Therefore $P'Q$ is not a common perpendicular to the lines AB and CD, and PO is the only common perpendicular.

Q.E.D.

Theorem XVII.

612. *The least distance between two indefinite lines which do not intersect each other is their common perpendicular.*

Hypothesis. a, b, two lines in space, the one being supposed to lie behind the other, so that they do not intersect.



Conclusion. No line which is not perpendicular to both lines can be the shortest line between them.

Proof. If possible, suppose that some line PQ which does not make a right angle with a is the shortest line.

From P drop a perpendicular PR upon a. Then

$$PR < PQ.$$

Therefore PQ is not the shortest line.

But since the lines do not intersect, some line must be the shortest.

Therefore this line is one making a right angle with both lines. Q.E.D.

CHAPTER II.

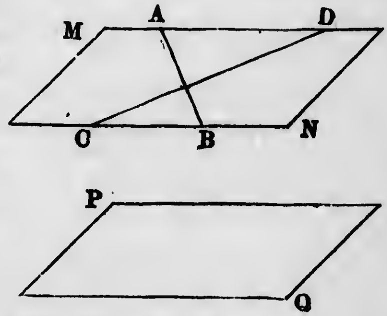
RELATIONS OF TWO OR MORE PLANES.

THEOREM XVIII.

613. *Two planes are parallel if any two intersecting lines on the one are both parallel to the other plane.*

Hypothesis. AB, CD, two intersecting lines lying in the plane MN, and each of them parallel to the plane PQ.

Conclusion. The planes MN and PQ are parallel.



Proof. 1. If the planes are not parallel, they must intersect in a straight line lying in both planes, and therefore in the plane MN. Let us call this line X.

2. Because the lines AB and CD are not parallel, the line X must intersect at least one of them.

3. Because the line X is also in the plane PQ, the line AB or CD, where it intersects X, would also intersect the plane PQ.

4. But, by hypothesis, these lines are parallel to the plane PQ, and therefore cannot meet it.

5. Therefore there can be no such line as X, and the plane MN can nowhere meet the plane PQ. Therefore these planes are parallel. Q.E.D.

THEOREM XIX.

614. *The lines in which two parallel planes intersect a third plane are parallel.*

Let us call A and B the parallel planes,* and X the third or intersecting plane. Then—

1. The lines of intersection are in one plane, because they both lie in the plane X.

2. Because one of the lines of intersection lies in the plane A, and the other in the plane B, parallel to it, and because these planes never meet, the lines can never meet.

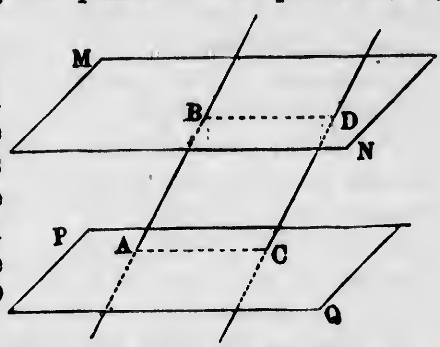
Therefore the lines are in one plane and can never meet, and so are parallel, by definition. Q.E.D.

THEOREM XX.

615. *Parallel planes intercept equal segments of parallel lines.*

Hypothesis. MN, PQ, two parallel planes; AB, CD, two parallel lines intersecting the planes in the points A, B, C, D.

Conclusion. $AB = CD$.



Proof. 1. Join AC and BD. Consider the plane containing the parallels AB and CD. Because the four points A, B, C, and D all lie in this plane, the joining lines AC and BD lie in it.

2. But because the lines AC and BD also lie in the respective planes MN and PQ, they are the lines of intersection of these planes with the plane $ABCD$.

3. Because the planes MN and PQ are parallel (hyp.), the lines of intersection AC and BD are parallel (§ 614).

4. Because $AB \parallel CD$ (hyp.) and $AC \parallel BD$, as just shown, $ABCD$ is a parallelogram. Therefore

$$AB = CD \text{ (§ 127).} \quad \text{Q.E.D.}$$

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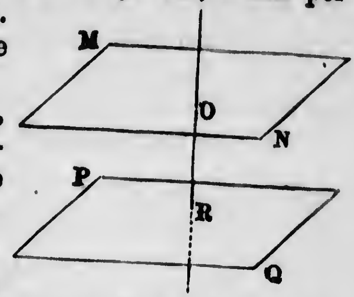
* This theorem is so simple that the student can imagine the figure which is to embody the hypothesis and conclusion better than it can be represented in a diagram. We therefore give the demonstration without a diagram.

THEOREM XXI.

616. *Planes perpendicular to the same straight line are parallel or coincident.*

Hypothesis. Two planes, MN and PQ; OR, a line perpendicular to each of these planes.



Conclusion. The planes are parallel.

Proof. If they are not parallel, they must intersect. If they intersect, call X any point on the line of intersection and join OX, RX. Then—

1. Because OX is in the plane MN, it is perpendicular to OR, a perpendicular line to the plane.

2. Because RX is in the plane PQ, it is also perpendicular to OR.

3. Therefore from the same point, X, we have two perpendiculars, XO and XR, to the same straight line, OR, which is impossible.

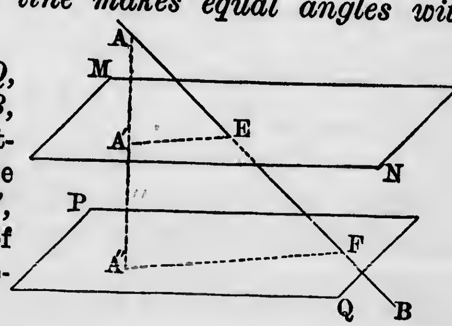
4. Therefore the planes never meet, and so are parallel.

Q.E.D.

617. *Corollary.* Conversely, *a straight line perpendicular to a plane is also perpendicular to every parallel plane.*

THEOREM XXII.

618. *A straight line makes equal angles with parallel planes.*



Hypothesis. MN, PQ, two parallel planes; AB, a straight line intersecting these planes at the points E and F; EA', FA'', the projections of this line upon the respective planes.

Conclusion. Angle AEA'—that is (§ 603), the inclination of AE to the plane MN—is equal to angle AFA'', the inclination of the line to the plane PQ.

Proof. 1. Because the points A' and A'' are the projections of the point A upon parallel planes, the point A' is in the straight line AA'' (§§ 597, 617).

2. Because the plane of the two lines AA'' and AE contains the four points A', E, A'', and F, the lines $A'E$ and $A''F$ are in this same plane, and are its line of intersection with the parallel planes MN and PQ. Therefore

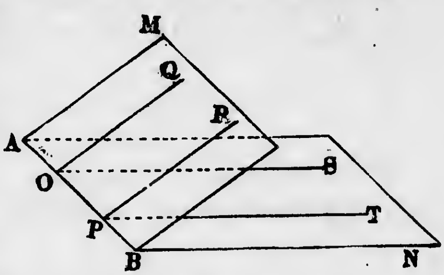
$$A'E \parallel A''F \qquad (\S\ 614)$$

and Angle AEA' = cor. angle AFA''. Q.E.D.

THEOREM XXIII.

619. *If from any points of the line of intersection of two planes two perpendiculars to that line be drawn, one in each plane, they will form equal angles.*

Hypothesis. M and N, two planes intersecting along the line AB; OQ, OS, two lines, one in each plane, perpendicular to AB; PR, PT, two other lines, one in each plane, perpendicular to AB.



Conclusion. Angle QOS = angle RPT.

Proof. 1. Because the sides OQ, PR lie in the same plane M, and are perpendicular to the same straight line AB,

$$OQ \parallel PR. \qquad (\S\ 70)$$

2. In the same way,

$$OS \parallel PT.$$

3. Therefore

Angle QOS = angle RPT (§ 608). Q.E.D.

620. *Def.* Two planes which intersect are said to form a **dihedral angle** along their line of meeting.

An angle formed by two lines is called a **plane angle** to distinguish it from a dihedral angle.

REMARK. A dihedral angle differs from a plane angle in that it is formed by planes instead of lines, and a line instead of a vertex.

621. *Def.* The **faces** of a dihedral angle are the two planes which form it.

The faces of a dihedral angle correspond to the sides of a plane angle.

622. *Def.* The **edge** of a dihedral angle is the line of meeting of the planes which form it.

623. A dihedral angle is measured by the angle between two perpendiculars, one in each face, from any part of its edge.

EXAMPLE. In the preceding diagram the dihedral angle formed along the line AB by the planes M and N is measured by either of the angles QOS or RPT between the perpendiculars.

If we pass a plane through O perpendicular to AB, it will contain both the sides OQ and OS of the angle QOS, which sides will be its intersection with the planes M and N (§ 584). Hence a dihedral angle is measured by the plane angle between the intersections of its faces with a plane perpendicular to its edge.

EXERCISE.

Show that two indefinite intersecting planes form four dihedral angles.

624. The following propositions respecting dihedral angles should be demonstrated or explained by the student from the corresponding propositions respecting ordinary plane angles.

I. *When two planes intersect, four dihedral angles are formed having the line of intersection as a common edge.*

II. *The planes being indefinitely extended, every point in space must lie in one of these angles or on one of the planes.*

III. *Opposite dihedral angles are equal.*

IV. *The sum of two adjacent dihedral angles is a straight angle.*

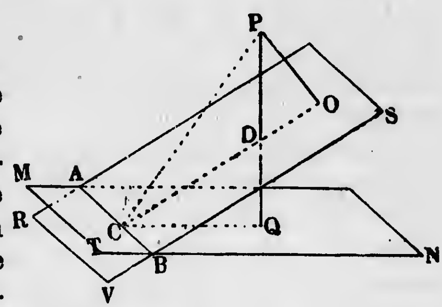
Corollary. When two planes intersect, we may take either of the two adjacent dihedral angles as measuring their inclination.

V. *If a plane intersect two parallel planes, the alternate and corresponding dihedral angles are equal.*

NOTE. When speaking of the angle between two planes, the adjective *dihedral* may be omitted when no ambiguity will arise from the omission.

THEOREM XXIV.

625. *If from any point perpendiculars be dropped upon two intersecting planes, the angle between these perpendiculars will be equal to the dihedral angle between the planes, adjacent to the angle in which the point is situated.*



Hypothesis. MN, RS, two planes (of which the parts in the diagram are supposed to be rectangular) intersecting along the line AB; P, any point in the obtuse dihedral angle TBS; PO, PQ, perpendiculars upon the planes from P.

Conclusion. Angle OPQ = dihedral angle SBN.

Proof. From P drop the perpendicular PC upon AB. Join CO, CQ. Then—

1. Because PO is perpendicular to the one plane and PQ to the other, CO is the projection of CP on the plane RS, and CQ is the projection of CP on the plane MN (§§ 598, 600).

Therefore, AB being perpendicular to CP, the angles ACO and ACQ are right angles (§ 606).

Therefore CP, CO, and CQ are in one plane, (§ 586) which plane must also contain PO and PQ.

2. Let D be the point in which PQ and OC intersect. Because, in the triangle POD, O is a right angle,

$$\begin{aligned} \text{Angle } OPQ = OPD &= \text{complement of } ODP, \\ &= \text{complement of } CDQ, \\ &= DCQ. \end{aligned}$$

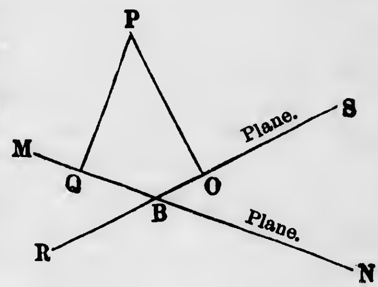
Therefore

$$\text{Angle } OPQ = \text{angle } OCQ.$$

4. Because CO and CQ are each perpendicular to AB, their angle measures the dihedral angle between the planes. Therefore Angle OPQ = dihedral angle SBN. Q.E.D.

626. *Scholium.* If we compare the angle ODP between the perpendiculars with the dihedral angle SBT, in which is situated the point from which they are dropped, we shall find them to be supplementary. This follows from the consideration that the angles SBN ($= OCQ$) and SBT are supplementary.

We may also, in the diagram, suppose the point P situated farther to the left, so that its perpendicular upon the plane MN shall fall to the left of AB, and therefore not intersect the plane RS. Then, if we imagine ourselves to look directly along the line AB so as to see both planes edgewise, the figure will present this appearance.



The plane figure $POBQ$ will then be a quadrilateral in which O and Q are right angles. Therefore the sum of the angles OPQ and OBQ will be two right angles, so that these two angles will be supplementary. OBQ is then the obtuse dihedral angle between the planes.

It is also readily shown that the angle SBN is still equal to OPQ.

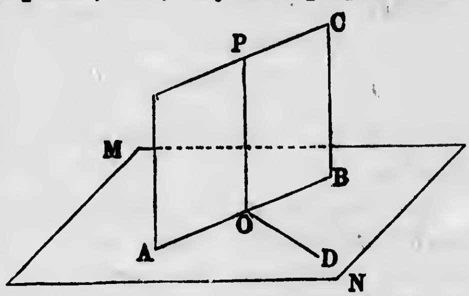
627. *Corollary* 1. *If from any point C of the line of intersection of two planes two perpendiculars, CO, CQ, be erected, one in each plane, and from O and Q perpendiculars to their respective planes be erected, these perpendiculars will intersect each other.*

628. *Cor.* 2. *If a plane bisect a dihedral angle, every point of the bisecting plane is equally distant from the planes which form the angle.*

Theorem XXV.

629. *If a line be perpendicular to a plane, every plane containing this line is also perpendicular to the plane.*

Hypothesis. *MN*, a plane; *PO*, any line perpendicular to it, intersecting it in *O*; *AC*, any plane containing the line *PO*.



Conclusion. The plane *AC* is perpendicular to the plane *MN*.

Proof. From *O* draw *OD* in the plane *MN* perpendicular to *AB*. Then—

1. Because $PO \perp$ plane *MN*, it is perpendicular to *AB* and *OD* in that plane, and

Angle $POD =$ right angle. (§ 584)

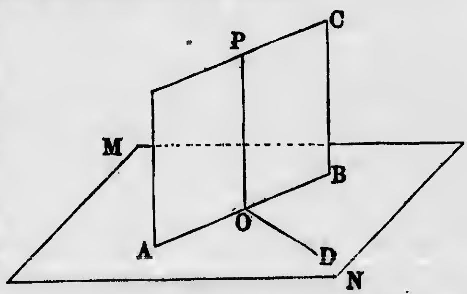
2. Because *OP* and *OD* are both perpendicular to *AB*, their angle *POD* measures the dihedral angle between the planes which intersect along *AB* (§ 623).

3. Therefore, from (1), this dihedral angle is a right angle, and the planes are therefore perpendicular. Q.E.D.

Theorem XXVI.

630. *If two planes be perpendicular to each other, every line in the one, perpendicular to their common intersection, is perpendicular to the other.*

Hypothesis. *AC*, *MN*, two planes intersecting at a right angle along the line *AB*; *OD*, a line in the plane *MN* perpendicular to *AB*.



Conclusion. *OD* is perpendicular to the plane *AC*.

Proof. In the plane *AC* draw $OP \perp AB$. Then—

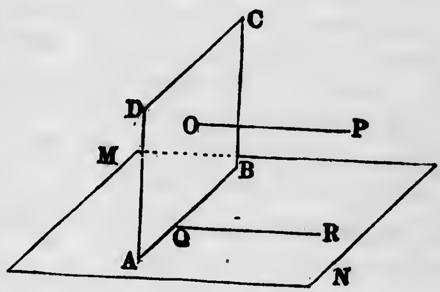
1. Because the plane AC is perpendicular to MN, the two lines OD, OP, one in each plane at right angles to the line of intersection, form a right angle (§ 623).

2. Because OD is perpendicular to AB (hyp.) and to OP (1), it is perpendicular to the plane of these lines; that is, to the plane AC. Q.E.D.

THEOREM XXVII.

631. *If two planes be perpendicular to each other, every line perpendicular to the one is either parallel to the other or lies in the other.*

Hypothesis. MN, any plane; $ABCD$, a plane perpendicular to it, intersecting it in the line AB; OP, a line perpendicular to the plane $ABCD$.



Conclusion.

$OP \parallel$ plane MN.

Proof. From any point R of the plane MN drop a perpendicular RQ upon AB. Then—

1. Because RQ is perpendicular to AB and lies in the plane MN, it is perpendicular to the plane $ABCD$ (§ 630).

2. Because the lines OP and QR are each perpendicular to the plane $ABCD$, they are parallel (§ 590).

3. Because the plane MN contains one of these parallels, QR, it is either parallel to the other, OP, or contains it (§ 607). Q.E.D.

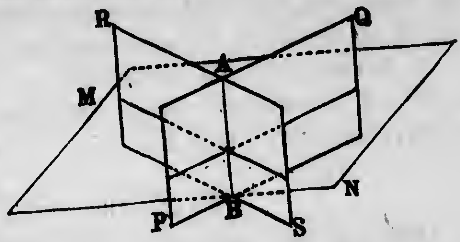
Corollary. The plane MN will contain the perpendicular OP when O lies on the line of intersection AB. Hence:

632. *If two planes be perpendicular, a line through their common intersection perpendicular to the one will lie in the other.*

THEOREM XXVIII.

633. *If each of two planes is perpendicular to a third plane, their line of intersection is also perpendicular to that third plane.*

Hypothesis. PQ, RS, two planes, each perpendicular to plane MN; AB, their line of intersection.



Conclusion.

$AB \perp$ plane MN.

Proof. 1. Because the plane PQ is perpendicular to MN, if we erect from the point B on their line of intersection a perpendicular to MN, it will lie in the plane PQ (§ 632).

2. Because $RS \perp MN$, this line will also lie in RS.

3. Therefore the line will be the line of intersection of the two planes, or AB; whence AB is the perpendicular to the plane MN from the point B. Q.E.D.

Corollary. Because a plane perpendicular to two planes is perpendicular to their line of intersection, and because all planes perpendicular to the same line are parallel or coincident (§ 616), we conclude:

634. *All planes perpendicular to the same two planes are either parallel or coincident.*

Theorem XXIX.

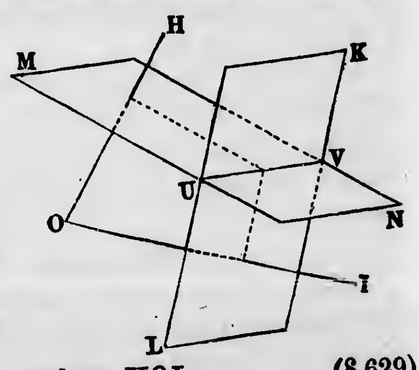
635. *If two planes are respectively perpendicular to two intersecting lines, their line of intersection is perpendicular to the plane of the lines.*

Hypothesis. OH, OI, two lines intersecting at O;

Plane $MN \perp$ line OH;

Plane $KL \perp$ line OI;

UV, the line of intersection of these planes.



Conclusion.

$UV \perp$ plane HOI.

Proof. 1. Because the plane MN is perpendicular to OH, it is perpendicular to every plane passing through OH. That is,

Plane $MN \perp$ plane HOI. (§ 629)

2. In the same way,

$$\text{Plane } KL \perp \text{ plane } HOI.$$

3. Because each of the planes is perpendicular to HOI, and UV is their line of intersection,

$$UV \perp \text{ plane } HOI \text{ (§ 633). Q.E.D.}$$

Relations of Three or more Planes.

636. Remark. When three planes, which we may call X, Y, and Z, mutually intersect, there will be three lines of intersection:

One line formed by the planes X and Y;
One line formed by Y and Z;
One line formed by Z and X.

Theorem XXX.

637. *The three lines of intersection of three planes are either parallel or meet in a point.*

Proof. Let us call the three planes X, Y, and Z. Let us also call

a, the line of intersection of X and Y;
b, the line of intersection of Y and Z;
c, the line of intersection of Z and X.

Then—

1. Because the lines a and b both lie in the plane Y, they are either parallel or intersect each other. The same may be shown for b and c, and for c and a.

2. Suppose a and b to intersect. Because a lies in both the planes X and Y, and b lies in both Y and Z, the point where they intersect must lie in all three planes X, Y, and Z. Therefore it must lie on both the planes X and Z, and therefore on their line of intersection c. The three lines a, b, and c will then all meet at this point.

3. If a and b are parallel, c cannot meet either of them, because, by (2), where it meets the one it must meet the other also. Therefore, in this case, none of the lines will ever meet any of the others, and because each pair lies in the same plane they must be all parallel.

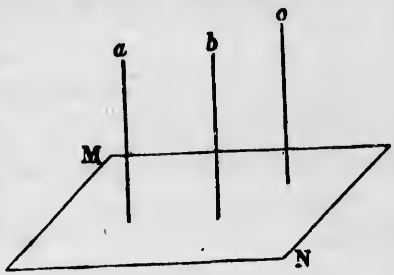
638. *Corollary* 1. *If the three lines of intersection of three planes meet in a point, the three planes all pass through that point.*

639. *Cor.* 2. *If two lines are parallel to each other, each of them is also parallel to the intersection of any two planes, one of which passes through each of the lines.*

THEOREM XXXI.

640. *If the lines of intersection of three planes are parallel, any fourth plane perpendicular to two of the three planes is also perpendicular to the third.*

Hypothesis. The parallel lines a, b, c are the lines of intersection of three planes, which we call the planes ab, bc, and ca; MN, a plane perpendicular to the two planes ab and bc.



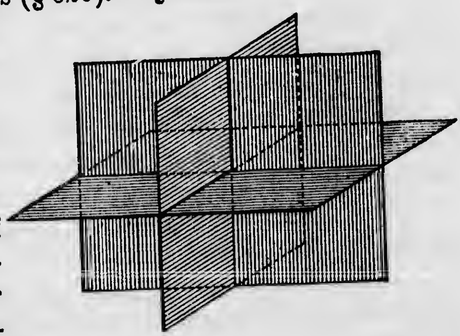
Conclusion. MN is also perpendicular to the plane ac.

Proof. 1. Because the plane MN is perpendicular to both the planes ab and bc, it is perpendicular to their line of intersection b (§ 633).

2. Because MN is perpendicular to b, it is perpendicular to the lines a and c, parallel to b (§ 591).

3. Therefore it is perpendicular to the plane ac, which passes through those lines (§ 629). Q.E.D.

Scholium. The most remarkable position of three planes is that in which each plane is perpendicular to the other two. By the preceding theorems each line of intersection will be perpendicular to the other two lines of intersection and to the third plane.



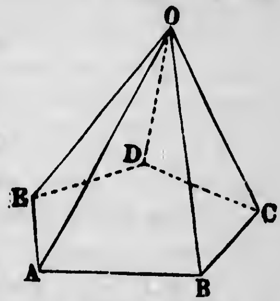
CHAPTER III.

OF POLYHEDRAL ANGLES.

641. *Def.* When three or more planes pass through the same point, they are said to form a **polyhedral angle** at that point.

A polyhedral angle is also called a *solid angle.*

Each plane which forms a polyhedral angle is supposed to be cut off along its lines of intersection with the planes adjoining it on each side.



A polyhedral angle.
O, the vertex; *OA*, *OB*, *OC*, etc., the edges; *AOB*, *BOC*, etc., the faces.

642. *Def.* **Edges** of a polyhedral angle are the straight lines along which the planes intersect.

643. *Def.* **Faces** of a polyhedral angle are the planes which form it.

644. *Def.* The **vertex** of a polyhedral angle is the point where the faces and edges all meet.

645. The edges of a polyhedral angle may be produced indefinitely. But to make the study of the angle easy, the faces and edges may be supposed cut off by a plane. The intersection of the faces with this plane will then form a polygon, as *ABCDE*.

This polygon is the *base* of the polyhedral angle.

646. Each pair of faces which meet an edge form a dihedral angle along that edge. There are as many edges as faces, and therefore as many dihedral angles as faces.

Hence two classes of angles enter into any polyhedral angle, namely:

I. The *plane angles AOB, BOC, COD,* etc., called also *face angles,* which the edges form with each other. The planes on which these angles are measured are the faces.

Example. The plane of the angle AOB is the face bounded by OA and OB.

II. The *dihedral angles* between the faces, called also *edge angles*. By § 623 each of these angles is measured by the plane angle between two lines, one in each face perpendicular to the edge of the dihedral angle.

If the cutting plane $ABCDE$ were perpendicular to one of the edges, say OB, then the dihedral angle along OB would be measured by the plane angle ABC.

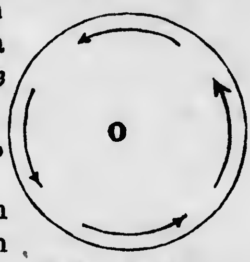
But this plane cannot be perpendicular to more than one edge, so that to measure the dihedral angles in this way we must have as many cutting planes as edges.

647. *Def.* Two polyhedral angles are **identically equal** when they can be so applied to each other that al the faces and edges of the one shall coincide with the corresponding faces and edges of the other.

In order that such coincidence may be possible, the face and dihedral angles of the one must all be equal to the face and dihedral angles of the other, taken in the same order, each to each.

Positive and Negative Rotations.

648. When a person looking down upon a point O sees a motion around that point in a direction the opposite of that of the hands of a watch, the motion is said to be *positive* relative to his standpoint.



Positive rotation.

If the motion is in the other direction, it is said to be *negative*.

A motion which is positive when seen from one side will appear negative when the observer views it from the other side of the plane, or when the figure is turned over so as to be seen from the other side.

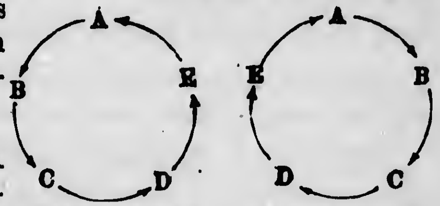
For illustration, imagine one's self seeing the hands of a watch by looking through it from behind.

To avoid ambiguity one side of the plane of motion may be taken as positive and the other side as negative. Then a positive rotation is that which appears positive when seen from the positive side, or negative as seen from the negative side.

Astronomical illustration. If one could look down upon the earth from above the north pole, the earth would appear rotating in the positive direction. If he should look down upon it from above the south pole, it would appear rotating in the negative direction.

REMARK. The habit of regarding the motion opposite that of the hands of a watch as positive arose from the direction of the north pole being taken as positive, because astronomy was developed among the people of the northern hemisphere; these people regarded as positive the direction in which the earth would appear to rotate when seen from the north.

649. As there are positive and negative rotations, so letters, angles, and lines may succeed each other in a positive or negative direction.

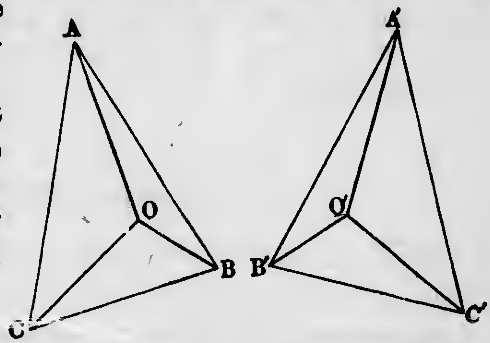


Letters succeed each other in the positive order.

Letters succeed each other in the negative order.

650. NOTATION. A polyhedral angle is designated by a letter at its vertex followed by a hyphen and the letters at the vertices of its base.

651. *Def.* **Symmetrical** polyhedral angles are those which have their plane and dihedral angles equal, each to each but arranged in reverse order, the one positive and the other negative when seen from the vertex.

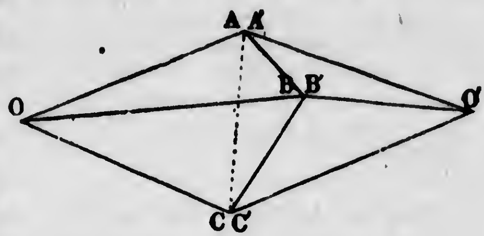


EXAMPLE. The trihedral angles $O\text{-}ABC$ and $O'\text{-}A'B'C'$ are symmetrical when

$$\begin{aligned} \text{Angle } AOB &= \text{angle } A'O'B', \\ \text{"} \quad BOC &= \quad \text{"} \quad B'O'C', \\ \text{"} \quad COA &= \quad \text{"} \quad C'O'A', \end{aligned}$$

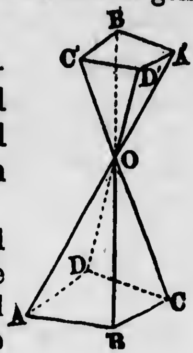
Dihedral angle along edge OA = dihedral angle along edge $O'A'$,
" " " " OB = " " " " $O'B'$,
" " " " OC = " " " " $O'C'$.

The two symmetrical polyhedral angles may be so cut that the base ABC shall be identically equal to the base $A'B'C'$. But, in order to bring these bases into coincidence, one of the figures must be turned over and the bases brought together with the vertices in opposite directions, as in the figure.



Hence two symmetrical polyhedral angles cannot in general be brought into coincidence.

652. *Def.* **Opposite** polyhedral angles are those each of which is formed by the continuation of the edges and faces of the other beyond the common vertex.



EXAMPLE. If the lines AO, BO, CO, and DO are produced through O to the respective points A', B', C', D', then the polyhedral angle $O\text{-}A'B'C'D'$ is the opposite of the angle $O\text{-}ABCD$.

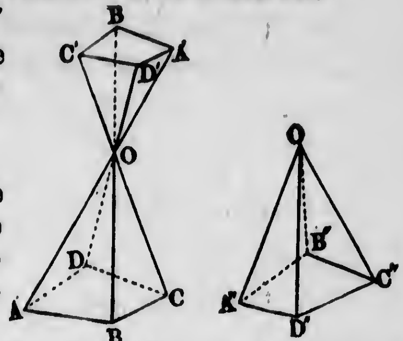
THEOREM XXXII.

653. *Opposite polyhedral angles are symmetrical.*

Hypothesis. $O\text{-}ABCD$, any polyhedral angle; $O\text{-}A'B'C'D'$, its opposite angle.

Conclusion.

Angle $O\text{-}A'B'C'D'$ is symmetrical to $O\text{-}ABCD$.



Proof. 1. Because AOA' and BOB' are in the same straight line,

Face angle $A'OB'$ = opp. angle AOB.

In the same way it may be shown that every other face angle of the one is equal to the opposite face angle of the other.

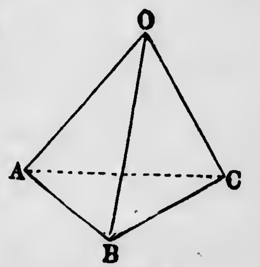
2. Because the lines AOA' and BOB' pass through the same point, they are in the same plane. Therefore the face $A'OB'$ is in the same plane with the face AOB. In the same way, every other pair of corresponding faces are in the same plane.

3. Because the dihedral angle between two planes is everywhere the same (§ 619), each of the edge angles OA', OB', etc., is equal to the corresponding one of the edge angles OA, OB, etc., of the other angle.

4. If one should look down upon the figure from above, the letters $ABCD$ and $A'B'C'D'$ would each succeed each other in the positive order. Hence if the opposite angle is turned over into the position $O\text{-}A''B''C''D''$, the order of the letters will appear negative, and therefore the opposite of those in the original polyhedral angle.

5. Therefore the two polyhedral angles, being equal in all their face and edge angles, but having them arranged in reverse order, are symmetrical.

Q.E.D.



A trihedral angle.

Face OAC is opp. edge OB.
Face OAB is opp. edge OC.
Face OBC is opp. edge OA.

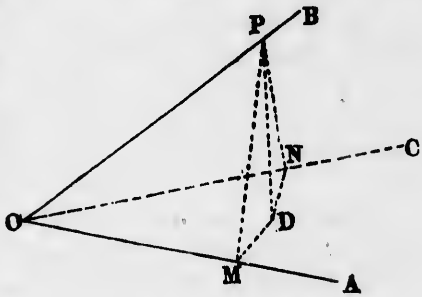
654. *Def.* A **trihedral** angle is a polyhedral angle which has three edges, and therefore three faces.

REMARK. In a trihedral angle each face has an opposite edge, and each edge an opposite face.

THEOREM XXXIII.

655. *If two face angles of a trihedral angle are equal, the edge angles opposite them are also equal.*

Hypothesis. $O\text{-}ABC$, a trihedral angle in which face angle BOA = face angle BOC.



Conclusion. Edge angle OA = edge angle OC.

Proof. From any point P of OB drop the perpendiculars $PM \perp OA$;

$PN \perp OC$;

$PD \perp$ plane AOC.

Join DM, DN. Then—

1. Because PM intersects the plane AOC in M, and PD is perpendicular to this plane, MD is the projection of PM upon this plane (§ 600). Therefore, because $OA \perp PM$,

$$OA \perp MD. \qquad (\S\,606)$$

2. Because MD and MP are pependicular to OA in the planes forming the dihedral angle OA,

Dihedral angle OA = plane angle PMD. (§ 623)

3. In the same way, we show

Dihedral angle OC = plane angle PND.

4. In the right-angled triangles POM and PON,

The side OP is common,

Angle PON = angle POM (hyp.);

therefore these triangles are identically equal, and

$$PN = PM.$$

5. In the right-angled triangles PDN and PDM,

Side PD is common,

$PN = PM$; (4)

therefore these triangles are identically equal, and

Angle PND = angle PMD.

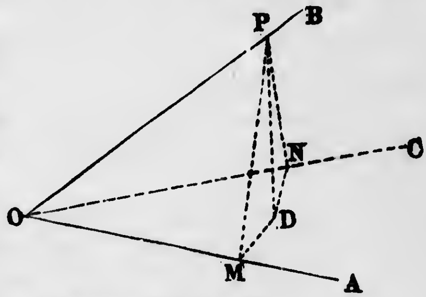
6. Comparing this result with (2) and (3),

Dihedral angle OA = dihedral angle OC. Q.E.D.

THEOREM XXXIV.

656. *In a trihedral angle the greater face angle and the greater edge angle are opposite each other.*

Hypothesis. $O\text{-}ABC$, a trihedral angle in which face angle $BOA >$ face angle BOC.



Conclusion. Edge angle $OC >$ edge angle OA.

Proof. Make the same constructions as in the last theorem. Then—

1. In the same way as in the last theorem follows:

$$\text{Edge angle } OA = \text{plane angle } PMD.$$
$$\text{Edge angle } OC = \text{plane angle } PND.$$

2. In the right-angled triangles POM and PON the line OP is the common hypothenuse; and because angle POM is greater than angle PON,

$$\text{Line } PM > PN.$$

3. In the right-angled triangles PDN and PDM, because the side PD is common and the hypothenuse PM greater than the hypothenuse PN,

$$\text{Angle } PND > \text{angle } PMD.$$

4. Comparing this result with (1),

$$\text{Edge angle } OC > \text{edge angle } OA. \quad \text{Q.E.D.}$$

Corollary 1. From these two theorems it follows, as in Bk. II., § 115:

657. *If one edge angle is greater than another, the face angle opposite it is greater than that opposite the other.*

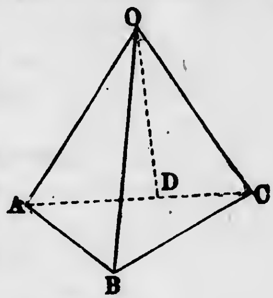
For the face angles could not be equal without violating Theorem XXXIII.; nor could that opposite the lesser edge angle be the greater without violating Theorem XXXIV.

658. *Cor.* 2. If the edge angles be arranged in the order of magnitude, the face angles opposite them will be in the same order of magnitude, so that the smallest, mean, and greatest angle of the one class will be opposite the smallest, mean, and greatest angle of the other, respectively.

THEOREM XXXV.

659. *In a trihedral angle in which each of the face angles is less than a straight angle, the sum of any two face angles is greater than the third.*

Hypothesis. $O\text{-}ABC$, a trihedral angle in which AOC is the greatest face angle.



Conclusion. The sum of the face angles AOB and BOC is greater than AOC.

Proof. Through O draw in the plane AOC a line OD, making angle $AOD =$ angle AOB. Let the base ABC be so cut off that we shall have $OB = OD$. Then—

1. Because the triangles OAB and OAD have

$$\left.\begin{array}{r}\text{Side } OA \text{ common,}\\ \text{Side } OD = OB,\\ \text{Angle } AOB = AOD,\end{array}\right\}\text{construction,}$$

they are identically equal, and

$$AB = AD.$$

2. Because ABC is a plane triangle,

$$\text{Sides } AB + BC > \text{third side } AC.$$

3. Taking away from this inequality the equal lengths AB and AD,

$$BC > DC.$$

4. Because, in the two triangles OCB and OCD,

$$\text{Side } OC \text{ is common,}$$
$$OD = OB \text{ (const.),}$$

and

$$CB > CD,$$

we have

$$\text{Angle } BOC > \text{angle } COD. \qquad (\S 115)$$

5. Adding the equal angles AOB and AOD,

$$\text{Angle } AOB + \text{angle } BOC > \text{angle } AOC. \quad \text{Q.E.D.}$$

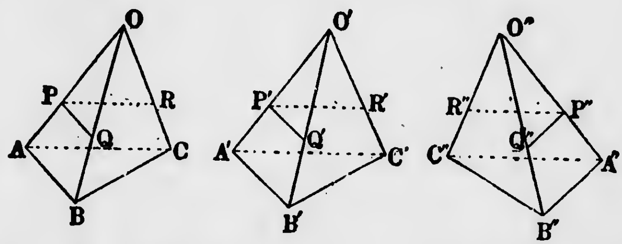
THEOREM XXXVI.

660. *Two trihedral angles are either equal or symmetrical when the three face angles of the one are respectively equal to the face angles of the other.*

Hypothesis. $O\text{-}ABC, O'\text{-}A'B'C', O''\text{-}A''B''C''$, three trihedral angles in which

$$\begin{array}{llll} \text{Face angle } AOB & = \text{angle } A'O'B' & = \text{angle } A''O''B''; \\ \text{`` \quad ``} \quad BOC & = \text{``} \quad B'O'C' & = \text{``} \quad B''O''C''; \\ \text{`` \quad ``} \quad COA & = \text{``} \quad C'O'A' & = \text{``} \quad C''O''A''; \end{array}$$

orders of angles ABC and $A'B'C'$ positive when viewed from the vertex, and of $A''B''C''$ negative.



Conclusion. The edge angles of the trihedral angles are also equal, and the trihedral angles $O\text{-}ABC$ and $O'\text{-}A'B'C'$ are equal, and $O''\text{-}A''B''C''$ is symmetrical with them.

Proof. In the sides OA, $O'A'$, and $O''A''$, respectively, take the equal distances OP, $O'P'$, and $O''P''$, and on OA, $O'A'$, and $O''A''$ erect the perpendiculars PQ, PR; $P'Q'$, $P'R'$; $P''Q''$, $P''R''$, in the respective faces which meet along the edges OA, $O'A'$, $O''A''$. Then—

1. In the triangles OPQ, $O'P'Q'$, $O''P''Q''$,

$$OP = O'P' = O''P'' \text{ (const.).}$$

$$\text{Angle } POQ = P'O'Q' = P''O''Q'' \text{ (hyp.).}$$

$$OPQ = O'P'Q' = O''P''Q'' \text{ (all being right angles).}$$

Therefore these triangles are identically equal, so that

$$OQ = O'Q' = O''Q'',$$
$$PQ = P'Q' = P''Q''.$$

2. In the same way, OPR, $O'P'R'$, and $O''P''R''$ being right angles by construction,

$$OR = O'R' = O''R'',$$
$$PR = P'R' = P''R''.$$

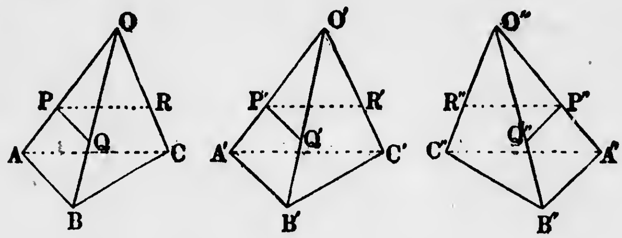
Imagine QR, $Q'R'$, and $Q''R''$ to be joined, then—

3. In the triangles OQR, $O'Q'R'$, $O''Q''R''$, etc.,

Face angle $QOR = Q'O'R' = Q''O''R''$ (hyp.),

and the sides which include these angles are equal by (1) and (2). Therefore these triangles are identically equal, and

$$QR = Q'R' = Q''R''.$$



4. Comparing with (1) and (2), the three triangles PQR, $P'Q'R'$, $P''Q''R''$ have their corresponding sides equal, and are therefore identically equal to each other. Hence

Angle QPR = angle $Q'P'R'$ = angle $Q''P''R''$.

5. Because QP, QR, etc., are each perpendicular to their edges, and lie in their respective faces, their angles measure the dihedral angle between those faces. Therefore

Edge angle OA = edge angle $O'A'$ = edge angle $O''A''$.

6. In the same way may be shown

Edge angle OB = edge angle $O'B'$ = edge angle $O''B''$.
Edge angle OC = edge angle $O'C'$ = edge angle $O''C''$.

Therefore the three trihedral angles have their edge angles all respectively equal.

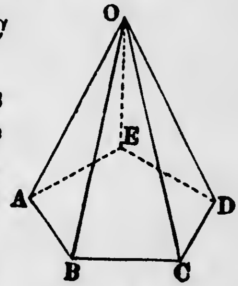
7. Because in the first two trihedral angles the arrangement of the equal angles is the same, while in the third this arrangement is reversed, the first two angles are equal, and the third symmetrical with them. Q.E.D.

THEOREM XXXVII.

661. *In a convex polyhedral angle the sum of the face angles is less than a perigon* (360°).

Hypothesis. *O-ABCDE*, a polyhedral angle of which all the angles of the base are convex.

Conclusion. Angles $AOB + BOC + \text{etc.} + EOA < 360°$.



Proof. Let n be the number of faces of the given polyhedral angle. The base *ABCDE* will then be a polygon of n sides. Let us also put Σ, the sum of the face angles AOB, BOC, etc. Then—

1. Because *ABCDE* is a convex polygon of n sides,

Angle ABC + angle BCD + etc. $= (n-2)$ straight angles. (§ 160)

2. Because the faces form n triangles, the sum of all the angles of these triangles is n straight angles. That is,

$$\Sigma + \text{angles } (OAB + OBA + OBC + OCB + \text{etc.})$$
$$= n \text{ straight angles.}$$

3. Because the planes BAO, BCO, ABC form a trihedral angle at B, of which the face angles are OBA, OBC, ABC,

$$\text{Angle } OBA + OBC > ABC.$$
$$\text{Angle } OCB + OCD > BCD. \qquad (\S\ 659)$$
$$\text{etc.} \qquad \text{etc.} \qquad \text{etc.}$$

4. Taking the sum of these inequalities, we find

Sum of the $2n$ base angles of triangles OAB, OBC, etc., greater than the sum of the angles of the polygon *ABCDE*; that is, greater than $n-2$ straight angles.

5. If we put B for the sum of these base angles, the results, (2) and (4), are

$$\Sigma + B = n \text{ straight angles.}$$
$$B > (n-2) \text{ straight angles.}$$

The difference of these shows that

$$\Sigma < 2 \text{ straight angles,}$$

or

$$\Sigma < 360° \quad \text{Q.E.D.}$$

BOOK IX.

OF POLYHEDRONS.

CHAPTER I.

OF PRISMS AND PYRAMIDS.

662. *Definition.* A **solid** is that which has length, breadth, and thickness.

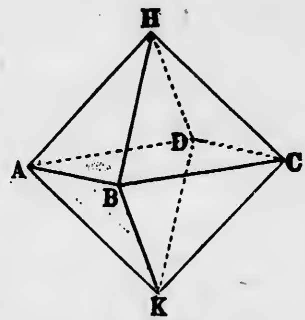
A solid is bounded by a surface.

REMARK. The form, magnitude, and position of a solid are completely determined by the form, magnitude, and position of its bounding surface. Hence we may consider the surface as defining the solid.

663. *Def.* A **polyhedron** is a solid bounded by planes.

664. *Def.* The **faces** of a polyhedron are its bounding planes.

665. *Def.* The **edges** of a polyhedron are the lines in which its faces meet.



A polyhedron.

The planes HAB, HBC, AKB, etc., are the faces. The lines HA, HB, etc., bounding the faces, are the edges. The points A, B, C, D, H, K are the vertices.

666. *Def.* The **vertices** of a polyhedron are the points in which its edges meet.

667. *Def.* **Diagonals** of a polyhedron are straight lines joining any two vertices not in the same face.

668. *Def.* A **plane section** of a polyhedron is the polygon in which its faces cut a plane passing through it.

669. *Def.* Two polygons are said to be **parallel** to each other when each side of the one is parallel to a corresponding side of the other.

Prisms.

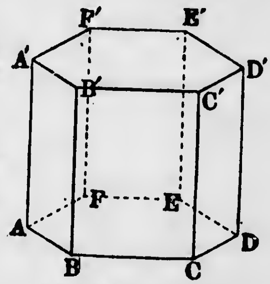
670. *Def.* A **prism** is a polyhedron of which the end faces are equal and parallel polygons, and the side faces parallelograms.

671. *Def.* The **bases** of a prism are its end faces.

672. *Def.* The **lateral faces** are all except its bases.

673. *Def.* The **lateral edges** of a prism are the intersections of its lateral faces.

674. *Def.* A **right section** of a prism is a section by a plane perpendicular to its lateral edges.



An hexagonal prism. *ABCDEF* and *A'B'C'D'E'F'* are the two bases.

675. *Def.* A prism is said to be **triangular, quadrangular, hexagonal,** etc., according as its bases are triangles, quadrilaterals, hexagons, etc.

676. *Def.* The **altitude** of a prism is the perpendicular distance between its faces.

677. *Def.* A **right** prism is one in which the lateral faces are perpendicular to its bases.

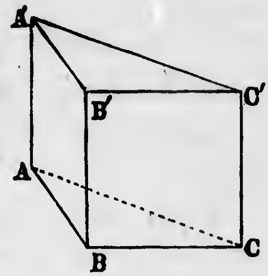
678. *Def.* An **oblique** prism is one in which the lateral faces are not perpendicular to the bases.

679. *Def.* A **regular** prism is a right prism whose bases are regular polygons.

THEOREM I.

680. *The lateral edges of a prism are equal and parallel, and make equal angles with the bases.*

Hypothesis. ABC, $A'B'C'$, two edges of the bases of a prism; $ABA'B'$, $BCB'C'$, the lateral faces joining those edges.



Conclusion. AA', BB', CC' are equal and parallel and make equal angles with the bases.

Proof. 1. Because $ABA'B'$ is a parallelogram (§ 670),

Line $BB' =$ and $\parallel AA'$.

2. Because $BB'CC'$ is a parallelogram,

Line $BB' =$ and $\parallel CC'$.

3. Because AA' and CC' are equal and parallel to the same line, they are equal and parallel to each other (§ 592).

4. In the same way it may be shown that all the other lateral edges are equal and parallel. Q.E.D.

5. Because the lateral edges are parallel lines, they make equal angles with either base (§ 610). Q.E.D.

6. Because the bases are parallel planes, the edges make equal angles with the two bases (§ 618). Q.E.D.

THEOREM II.

681. *The sections of the lateral faces of a prism by parallel planes are equal and parallel polygons.*

Hypothesis. $ABCD\text{-}A'B'C'D'$, a prism; $EFGH$ and $E'F'G'H'$, sections of the lateral faces by parallel planes.

Conclusion. $EFGH =$ and $\parallel E'F'G'H'$.

Proof. 1. Because AA' and BB' are parallel lines intersecting parallel planes,

$EE' =$ and $\parallel FF'$. (§ 615)

Therefore the four lines EF, FF', $F'E'$, and $E'E$ form a parallelogram and

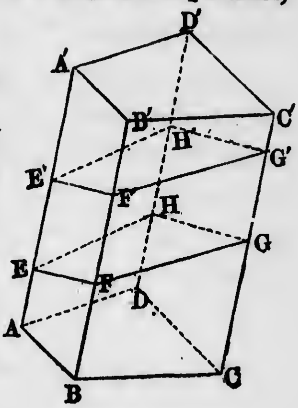
Line $EF =$ and $\parallel E'F'$.

2. In the same way we find

Line $FG =$ and $\parallel$ $F'G'$.
Line $GH =$ and $\parallel$ $G'H'$.
etc. etc.

3. Because the sides of the respective angles are parallel,

Angle $EFG =$ angle $E'F'G'$,
Angle $FGH =$ angle $F'G'H'$, (§ 608)
etc. etc.



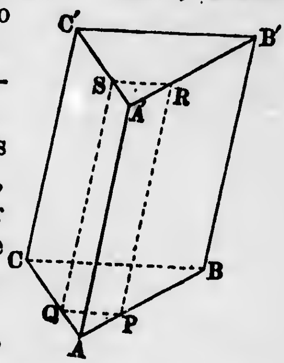
Therefore the polygons $EFGH$ and $E'F'G'H'$, having their sides and angles, taken in order, all equal, are identically equal. Q.E.D.

682. *Corollary. Any section of a prism by a plane parallel to the bases is identically equal to the bases.*

Theorem III.

683. *Every section of a prism by a plane parallel to the lateral edges is a parallelogram.*

Hypothesis. ABC-$A'B'C'$, any prism; $PQRS$, a section of this prism by a plane parallel to AA', BB'.



Conclusion. $PQRS$ is a parallelogram.

Proof. 1. Because the line AA' is parallel to the intersecting plane $PQRS$, it cannot meet either of the lines PR or QS which lie in that plane. Therefore the lines AA', PR, and QS are parallel. (§ 637)

2. Because the opposite sides AP and $A'R$, AA' and PR of the quadrilateral $APA'R$ are parallel, the quadrilateral is a parallelogram, and

$PR =$ and $\parallel$ AA'.

3. In the same way we may prove

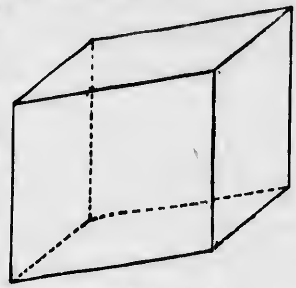
Line $QS =$ and $\parallel$ AA'.

4. Therefore $QS =$ and $\parallel$ PR, and the quadrilateral $PQRS$ is a parallelogram. Q.E.D.

Parallelopipeds.

684. *Def.* A **parallelopiped** is a solid contained by three pairs of parallel planes.

A parallelopiped is therefore a prism of which the lateral faces are two pairs of parallel planes.



A parallelopiped.

685. *Def.* A **rectangular parallelopiped** is one whose faces intersect at right angles.

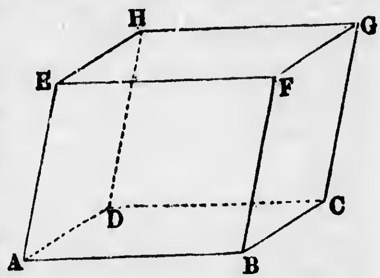
686. *Def.* A **cube** is a parallelopiped whose faces are all squares.

Theorem IV.

687. *The opposite faces of a parallelopiped are identically equal parallelograms.*

Hypothesis. $ABCD$-$EFGH$, any parallelopiped.

Conclusion. The opposite faces $ABCD$ and $EFGH$ are identically equal parallelograms.



Proof. 1. Because BC and AD are the lines in which the parallel planes $BCFG$ and $ADEH$ intersect the third plane $ADBC$,

$BC \parallel AD$. (§ 614)

2. In the same way it may be shown that the lines AB and DC are parallel. Therefore

$ABCD$ is a parallelogram. Q.E.D.

3. It may be shown in the same way that all the other faces are parallelograms. Therefore, by comparing opposite sides of successive parallelograms,

$AB =$ and $\parallel EF$,

and $BC =$ and $\parallel FG$.

4. Because the sides BA and BC of the angle ABC are parallel to the sides FE and FG of the angle EFG,

Angle $ABC =$ angle EFG. (§ 608)

5. Therefore the parallelograms $ABCD$ and $EFGH$ having their respective sides and one angle equal, are identically equal. In the same way it may be shown that every other pair of opposite faces are equal. Q.E.D.

Corollary 1. *The edges of a parallelopiped are twelve in number, and may be divided into three sets, each set comprising four equal and parallel lines.*

Cor. 2. *The vertices of a parallelopiped are eight in number.*

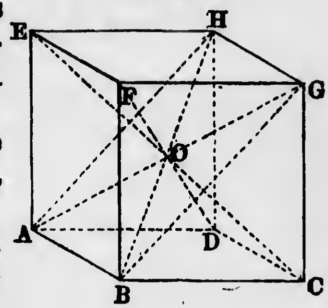
Cor. 3. *The diagonals of a parallelopiped are four in number, and may be drawn from any angle of each face to the opposite angle of the opposite face.*

Theorem V.

688. *The four diagonals of a parallelopiped all intersect in a point which bisects them all.*

Hypothesis. $ABCD\text{-}EFGH$, any parallelopiped.

Conclusion. The four diagonals AG, BH, CE, and DF all intersect in a point O, and are bisected by this point.



Proof. Through the opposite parallel edges AB and HG pass a plane. Join AH, BG. Then—

1. Because the sides AB and HG of the quadrilateral $ABHG$ are equal and parallel, $ABHG$ is a parallelogram. Therefore AG and BH, the diagonals of this parallelogram, intersect and bisect each other.

Let O be the point of intersection.

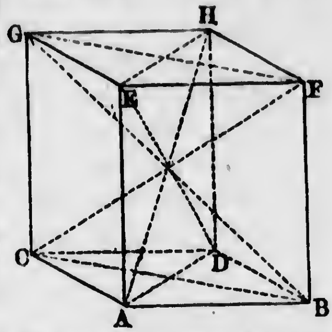
2. In the same way it may be shown that BH and CE bisect each other. Therefore CE passes through the point of bisection O, and O bisects CE.

3. In the same way it may be shown that DF passes through O and is bisected by O. Therefore all the diagonals pass through O and are bisected by that point. Q.E.D.

689. *Def.* The point O through which all the diagonals pass is called the **centre** of the parallelopiped.

THEOREM VI.

690. *The sum of the squares upon the four diagonals of a parallelopiped is equal to the sum of the squares upon its twelve edges.*



Hypothesis. $ABCD$ - $EFGH$, any parallelopiped.

Conclusion. $AH^2 + BG^2 + CF^2 + DE^2 = 4(AC^2 + AB^2 + AE^2)$.

Proof. Draw the diagonals of any pair of opposite faces, as AD and BC, EH and FG. Then—

1. AH and DE are diagonals of the parallelogram $ADHE$ (§ 688, 1). Therefore

$$AH^2 + DE^2 = 2AD^2 + 2AE^2. \qquad (§ 316)$$

2. In the same way,

$$BG^2 + CF^2 = 2BC^2 + 2BF^2$$
$$= 2BC^2 + 2AE^2.$$

3. Adding (1) and (2),

$$AH^2 + BG^2 + CF^2 + DE^2 = 2(AD^2 + BC^2) + 4AE^2.$$

4. Because AD and BC are diagonals of the parallelogram $ABCD$,

$$AD^2 + BC^2 = 2AB^2 + 2AC^2. \qquad (§ 316)$$

5. Substituting this result in (3), we have

Sum of squares of diagonals $= 4AB^2 + 4AC^2 + 4AE^2$.

6. Since there are four edges equal to AB, four equal to AC, and four equal to AE, this sum is equal to the sum of the squares of all the edges, and

Sum of squares of diagonals = sum of squares of edges.

Q.E.D.

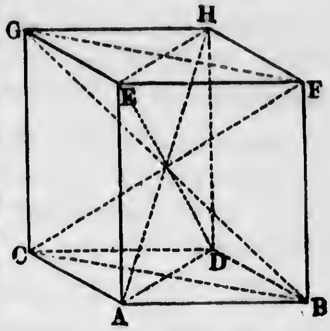
THEOREM VII.

691. *The four diagonals of a rectangular parallelopiped are equal to each other.*

Hypothesis. $ABCD$-$EFGH$, a rectangular parallelopiped.

Conclusion. The diagonals AH, BG, CF, and DE are all equal.

Proof. 1. Because the faces AF and AG are each at right angles to the face AD (§ 685), their line of intersection AE is also perpendicular to that face (§ 633), and to the line AD in that face (§ 584).



2. Therefore $ADEH$ is a rectangle, and its diagonals AH and DE are equal.

3. It may be shown in the same way that any other two diagonals are equal. Therefore these diagonals are all equal to each other.

Q.E.D.

THEOREM VIII.

692. *The square of each diagonal of a rectangular parallelopiped is equal to the sum of the squares of the three edges which meet at any vertex.*

Hypothesis. Same as in Theorem VII.

Conclusion. $AH^2 = AB^2 + AC^2 + AE^2$.

Proof. 1. Because AEH is a right angle (Th. VII., 1),

$$AH^2 = AE^2 + EH^2$$
$$= AE^2 + AD^2.$$

2. Because ABD is a right angle,

$$AD^2 = AB^2 + BD^2$$
$$= AB^2 + AC^2.$$

3. Comparing with (1),

$$AH^2 = AE^2 + AB^2 + AC^2. \quad \text{Q.E.D.}$$

693. *Scholium.* This theorem might have been regarded as a corollary from the two preceding ones. But we have preferred an independent demonstration, owing to its importance. It may be considered as an extension of the Pythagorean proposition from a plane to space.

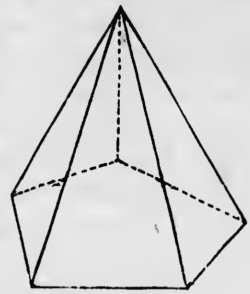
Pyramids.

694. *Def.* A **pyramid** is a polyhedron of which all the faces except one meet in a point.

The point of meeting is called the **vertex.**

REMARK. The face which does not pass through the vertex is taken as the base.

695. *Def.* The faces and edges which meet at the vertex are called **lateral** faces and edges.


A pyramid.

696. *Def.* The **altitude** of a pyramid is the perpendicular distance from its vertex to the plane of its base.

697. *Def.* A pyramid is said to be **triangular, quadrangular, pentagonal,** etc., according as its base is a triangle, a quadrilateral, a pentagon, etc.

698. *Def.* If the vertex of a pyramid is cut off by a plane parallel to the base, that part which remains is called **a frustum** of a pyramid.

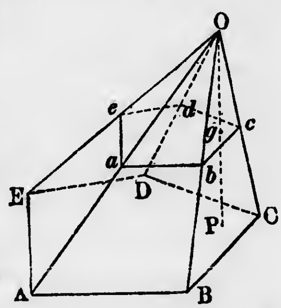
THEOREM IX.

699. *If a pyramid be cut by a plane parallel to the base, then—*

I. *The edges and the altitude are similarly divided.*

II. *The section is similar to the base.*

Hypothesis. *O-ABCDE*, a pyramid; *OP*, its altitude; *abcde*, a section of the pyramid by a plane parallel to the base *ABCDE*, cutting the altitude line at *g*.



Conclusions.

I. $OP : Og :: OB : Ob :: OC : Oc$, etc.

II. The polygon *abcde* is similar to the polygon *ABCDE*.

Proof. 1. Because *AB* and *ab* are the intersections of parallel planes with the plane *OAB*,

$$ab \parallel AB. \qquad (\S\,614)$$

2. Therefore the triangles OAB and Oab are similar, and

$$OA : Oa :: OB : Ob;$$

whence $$Oa : aA :: Ob : bB.$$

3. In the same way it may be shown that OC, OP, etc., are all divided similarly at c, g, etc. Q.E.D.

4. It may also be shown, as in (1), that each side of the polygon $abcde$ is parallel to the corresponding side of $ABCDE$. Therefore the angles of the two polygons are respectively equal (§ 608), and the polygons are equiangular to each other.

5. Because the bases ab and AB of the triangles Oab and OAB are parallel,

$$AB : ab = OA : Oa.$$

In the same way,

$$BC : bc :: OB : Ob :: OA : Oa,$$
$$CD : cd :: OC : Oc :: OA : Oa,$$
etc. etc.

6. Therefore the polygons $ABCDE$ and $abcde$, having their angles equal and the sides containing the equal angles proportional, are similar to each other. Q.E.D.

Corollary 1. Because $abcde$ and $ABCDE$ are similar polygons, their areas are as the squares of ab and AB; that is, as $Oa^2 : OA^2$, or $Og^2 : OP^2$ (§ 425). Hence:

700. *The areas of parallel sections of a pyramid are proportional to the squares of the distances of the cutting planes from the vertex.*

Cor. 2. The base of a pyramid may be regarded as one of the plane sections, so that if two pyramids have equal bases and altitudes, the plane sections made by the bases are equal, and the distances of these sections from the vertices are also equal. Hence:

701. *In pyramids of equal bases and altitudes, parallel plane sections at equal distances from the vertices are equal in area.*

CHAPTER II.

THE FIVE REGULAR SOLIDS.

---

702. *Def.* A **regular polyhedron** is one of which all the faces are identically equal regular polygons and all the polyhedral angles are identically equal.

REMARK. A regular polyhedron is familiarly called a *regular solid.*

703. PROBLEM. *To find how many regular solids are possible.*

1. Let us consider any vertex of a regular polyhedron. Since at least three faces must meet to form the polyhedral angle at each vertex, and since the sum of all the plane angles which make up the polyhedral angle must be less than 360° (§ 661), we conclude:

Each angle of the faces of a regular solid must be less than 120°.

2. Since the angles of a regular hexagon are each 120°, and the angles of every polygon of more than six sides yet greater, we conclude:

Each face of a regular solid must have less than six sides. Such faces must therefore be either triangles, squares, or pentagons.

3. If we choose equilateral triangles, each polyhedral angle may have either 3, 4, or 5 faces, because $3 \times 60°$, $4 \times 60°$, and $5 \times 60°$ are all less than 360°. It cannot have 6 faces, because each angle of the triangle being 60°, six angles would make 360°, reaching the limit. Therefore no more than three regular solids may be constructed with triangles.

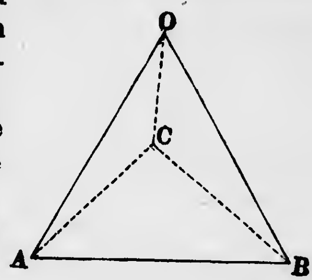
4. Because each angle of a square is 90°, three is the only number of squares which can form a polyhedral angle. Therefore only one regular solid can have square faces.

5. Because each angle of a regular pentagon is 108°, a

polyhedral angle cannot be formed of more than three pentagons. Therefore only one regular solid can be formed of pentagons.

6. We therefore conclude that not more than five regular solids are possible. That five can really be constructed we show in the following way.

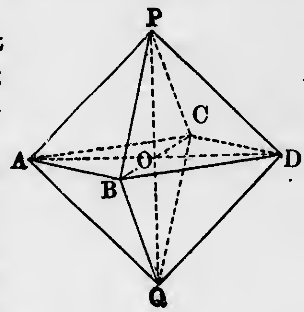
704. *The Tetrahedron.* If we take three equal equilateral triangles, AOB, BOC, and COA, and join them at the common vertex, O, ABC will be an identically equal equilateral triangle. Therefore a polyhedron will be formed having as faces four identical equilateral triangles.



This solid is a regular **tetrahedron.**

705. *The cube, or regular hexahedron,* is a regular parallelopiped of which the faces are all squares. Its construction is clear from §§ 685–693.

706. *The Octahedron.* Let $ABCD$ be a square; O, its centre; PQ, a line passing through O perpendicular to the plane of the square.



On this line take the points P and Q, such that PA, PB, PC, and PD, also QA, QB, QC, and QD, are each equal to a side of the square. The figure P-$ABCD$-Q will be a regular solid, having for its sides eight equilateral triangles.

Proof. 1. Because all the lines from P or Q to A, B, C, and D are equal, all the eight triangles which form the faces are equal and equilateral.

2. Let a diagonal be drawn from A to D.

Because O is the centre of the square $ABCD$, this diagonal will pass through O, and the lines PQ and AD, which intersect in O, are in the same plane. Therefore

P, A, Q, and D are in one plane.

3. In the triangles APD, ABD, AQD we have

Side AD common,

All the other sides equal.

Therefore the triangles are identically equal; and because ABD is a right angle, APD and AQD are also right angles, and the quadrilateral $APDQ$ is a square equal to the square $ABCD$.

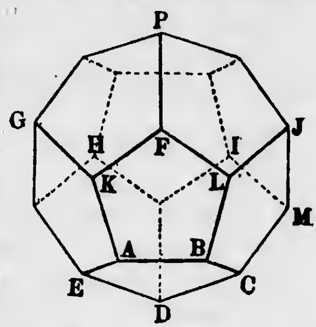
4. Because the polyhedral angle at B has its four faces and its base $APDQ$ equal to the four faces and the base $ABCD$ of the polyhedral angle at P,

Polyhedral angle B = polyhedral angle P.

5. In the same way it may be shown that any other two polyhedral angles are equal. Therefore the figure P-$ABCD$-Q is a regular solid having eight equilateral triangles for its faces.

This solid is called the regular **octahedron**.

707. *The Dodecahedron.* Taking the regular pentagon $ABCDE$ as a base, join to its sides those of five other equal pentagons, so as to form five trihedral angles at A, B, C, D, and E, respectively.



Because the face angles of these trihedral angles are equal, the angles themselves are identically equal. (§ 660)

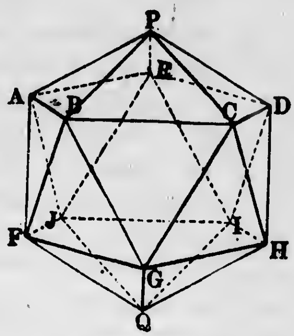
Therefore the dihedral angles formed along the edges AK, BL, CM, etc., are equal to the dihedral angles AB, BC, etc.

Therefore the face angles EAK, LBC, etc., are identically equal to the angles of the original regular pentagons.

Pass planes through JLF and GKF, etc., and let FP be their line of intersection. Then, continuing the same course of reasoning, it may be shown that the face angles GKF, FLJ, etc., are all angles of 108°, or those of a regular pentagon. Completing this second series of five pentagons, we shall have left a pentagonal opening, which being closed, the surface of the polyhedron will be completed as shown in the diagram.

The solid thus formed has 12 pentagons for its sides, and is called a regular **dodecahedron**.

708. *The Icosahedron.* Let five equilateral triangles form a polyhedral angle at P, such that the dihedral angles along PA, PB, etc., shall all be equal.



The base $ABCDE$ will then form a regular pentagon.

Complete the polyhedral angles at A, B, C, D, and E by adding to each three other equilateral triangles, and making the dihedral angles around A, B, C, etc., all equal.

It can be then shown, as in the case of the dodecahedron, that the lines F, G, H, I, J will form a second regular pentagon.

If upon this pentagon we construct five equilateral triangles, FQG, GQH, etc., having their vertices at Q, the solid will be completed, and its faces will be formed of 20 equilateral triangles.

This solid is called the **icosahedron.**

709. The five regular solids are therefore:

The tetrahedron, formed of	3	triangles.
The cube, or hexahedron, formed of	6	squares.
The octahedron, formed of	8	triangles.
The dodecahedron, formed of	12	pentagons.
The icosahedron, formed of	20	triangles.

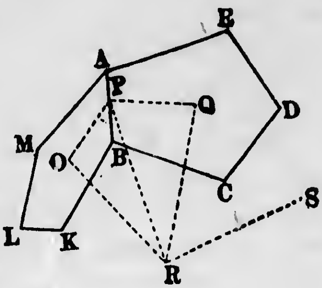
Theorem X.

710. *The perpendiculars through the centres of the faces of a regular solid meet in a point which is equally distant from all the faces, from all the edges, and from all the vertices.*

Hypothesis. $ABCDE$ and $ABKLM$, two faces of a regular polyhedron, intersecting along the edge AB; O, Q, the centres of these faces; OR, QR, perpendiculars to the faces.

Conclusion. These perpendiculars meet all the perpendiculars through the centres of the other faces in a point R equally distant from all the faces, edges, and vertices.

Proof. From O and Q drop perpendiculars upon the edge AB. Then—



1. Because these perpendiculars are dropped from the centres of regular polygons, they will fall upon the middle point P of the common side AB.

2. Because PO and PQ are perpendicular to AB at the same point P, and OR and QR are perpendiculars to the intersecting planes, they will meet in a point (§ 627).

Let R be their point of meeting. Join PR. Then—

3. In the right-angled triangles POR and PQR we have

Side PR common,

$PO = PQ$ (being apothegms of equal polygons).

Therefore these triangles are identically equal, and

$$OR = QR.$$

$$\text{Angle } PRO = \text{angle } PRQ.$$

4. If S be the centre of any other face adjacent to $ABCDE$, it can be shown in the same way that the perpendicular from S will meet QR in a point.

5. Because the angles between the faces Q and S are the same as between O and Q, it may be shown that the perpendicular from S will meet QR in the point R.

6. Continuing the reasoning, it will appear that all the perpendiculars from the centres of the faces meet in the same point R.

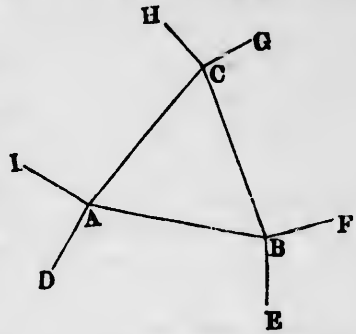
7. If from R perpendiculars be dropped upon all the edges and all the vertices, these perpendiculars, together with those upon the corresponding faces and the lines like QP and QB from the centres of the faces to the edges and vertices, will form identically equal triangles. Hence will follow the conclusion to be demonstrated.

NOTE. We have given but a brief outline of the demonstration, which the student may complete as an exercise. The conclusions may also be considered as following immediately from the symmetry of the polyhedron.

THEOREM XI.

711. *A regular solid is symmetrical with respect to all its faces, edges, and vertices.*

Proof. Let ABC be a face of any regular polyhedron; A, B, and C will then be vertices.



Let AD, BE, BF, etc., be the edges going out from these vertices.

Now move the polyhedron so as to bring any other face into the position ABC. This can be done, because the faces are all identically equal.

Because the polyhedral angles are all identically equal, whatever polyhedral angles take the positions A, B, C, their faces and edges will coincide with the positions of the faces and edges already marked in the figure.

Because the edges are all of equal length, the vertices at the ends of D, E, F, G will fall into the same positions where the former vertices were.

Continuing the reasoning, the whole polyhedron will be found to occupy the same space as before, every face, edge, and vertex falling where some other face, edge, or vertex was at first.

Because this is true in whatever way the positions of the faces may be interchanged, the polyhedron is symmetrical.

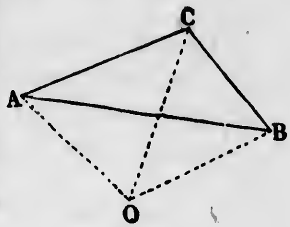
Q.E.D.

712. *Corollary.* Conversely, *if a polyhedron be such that, when any one face is brought into coincidence with the position of any other, every other face shall coincide with the former position of some face, the polyhedron is regular.*

THEOREM XII.

713. *If a plane be passed through each vertex of a regular solid, at right angles to the radius, these planes will be the faces of another regular solid.*

Proof. 1. Let A, B, and C be any vertices of a regular solid, and O its centre. Imagine planes passed through A, B, and C, perpendicular to OA, OB, and OC respectively, and cut off along their lines of intersection, so as to form the faces of another solid. We call this the *new solid*, and the original one the *inner solid*.



2. Because the inner solid is regular, if we bring any other of its faces into the position ABC, the whole solid wlll occupy the same position as before (§ 711).

3. Because each face of the new solid is at right angles to the end of a radius to some vertex of the inner solid, and these radii all coincide with the former positions, the plane of each face of the new solid will, when the change of position is made, take the position of the plane of some other face.

4. Therefore the edges in which these planes intersect will take the positions of other edges.

5. Therefore the vertices where these edges meet will take the positions of other vertices.

6. Therefore the new solid occupies the same space as before the change, and is consequently symmetrical with respect to all its faces, edges, and vertices.

Therefore it is a regular solid (§ 712). Q.E.D.

714. *Def.* A pair of polyhedrons such that each face of the one corresponds to a vertex of the other are called **sympolar polyhedrons.**

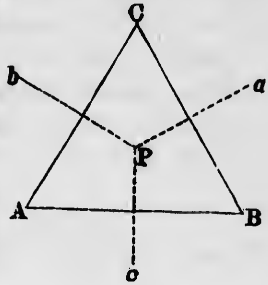
Theorem XIII.

715. *Every regular solid has as many faces as its sympolar has vertices, and as many edges as its sympolar has edges.*

Proof. 1. Continuing the reasoning of the last theorem, it is evident that the centres of the faces of the new solid coincide with the vertices of its sympolar.

2. Because each edge of a regular solid is equally distant from the centres of two adjoining faces, each edge of the new solid will be equally distant from two adjoining vertices of the sympolar.

3. Since every two such vertices are connected by an edge of the sympolar, the new solid will have as many edges as the sympolar, each edge of the one being at right angles and above the edge of the sympolar. Q.E.D.



Let *A*, *B*, and *C* be three vertices of the sympolar, and therefore the centres of three faces of the new solid.

4. By what has just been shown, *Pa*, *Pb*, and *Pc* will be three edges of the new solid, meeting in a vertex at *P*. Therefore—

5. The new solid will have a vertex over the centre of each side of the sympolar, and so will have as many vertices as the sympolar has faces. Q.E.D.

Theorem XIV.

716. *The sympolar of a polyhedron whose faces have S sides will have S-hedral angles.*

Proof. The vertex of one polyhedron being at *P* over the centre of the face of its sympolar, the edges meeting at this vertex are each perpendicular to an edge of the face of the sympolar.

Therefore if the face has *S* sides, the polyhedral angle above it will have *S* edges, and therefore *S* faces. Q.E.D.

717. *Corollary.* Conversely, *the sympolar of a polyhedron whose vertices are S-hedral will have S-sided faces.*

718. *Corollary.* What pairs of regular solids are sympolar to each other can be readily determined from the preceding theorems.

The tetrahedron has four vertices. Therefore its sympolar has four faces, and is therefore another tetrahedron.

The cube has 8 trihedral vertices and 6 four-sided faces. Therefore its polar has 8 triangular faces and 6 four-hedral vertices. This is the octahedron.

Conversely, the sympolar of the octahedron is the cube. They each have 8 edges.

The dodecahedron has 12 pentagonal faces and 20 trihedral vertices. Therefore its sympolar has 12 five-hedral vertices and 20 triangular faces. This is the icosahedron.

Each of these polyhedrons has 30 edges.

These results are shown in the following table, where the headings at the top of each column refer to the solid on the left, and those at the bottom to its sympolar on the right.

Solid.	Number of sides to each face.	Number of faces.	Number of edges.	Number of vertices.	Number of edges at each vertex.	
Tetrahedron.	3	4	6	4	3	Tetrahedron.
Cube.	4	6	12	8	3	Octahedron.
Octahedron.	3	8	12	6	4	Cube.
Dodecahedron.	5	12	30	20	3	Icosahedron.
Icosahedron.	3	20	30	12	5	Dodecahedron.
	Number of edges at each vertex.	Number of vertices.	Number of edges.	Number of faces.	Number of sides to each face.	Sympolar solid.

Note that each column applies to two solids. For instance, the left-hand column shows the number of sides to each face of the solid named at the left, and the number of edges at each vertex of the solid named at the right.

BOOK X.

OF CURVED SURFACES.

CHAPTER I.

THE SPHERE.

Definitions.

719. *Def.* A **curved surface** is a surface no part of which is plane.

720. *Def.* A **spherical surface** is a surface which is everywhere equally distant from a point within it called the centre.

721. *Def.* A **sphere** is a solid bounded by a spherical surface.

NOTE 1. A spherical surface may also be described as the locus of the point at a given distance from a fixed point called the centre.

NOTE 2. In the higher geometry a spherical surface is called a *sphere*. We shall use this appellation when no confusion will thus arise.

722. *Def.* The **radius** of a sphere is the distance of each point of the surface from the centre.

723. *Def.* A **diameter** of a sphere is a straight line passing through its centre, and terminated at both ends by the surface.

Corollary. Every diameter is twice the radius; therefore *all diameters of the same sphere are equal.*

724. *Def.* A **tangent plane** to a sphere is a plane which has one point, and one only, in common with the sphere.

725. *Def.* A line is **tangent** to a sphere when it touches the spherical surface at a single point.

726. *Def.* Two spheres are **tangent** to each other when they have a single point in common.

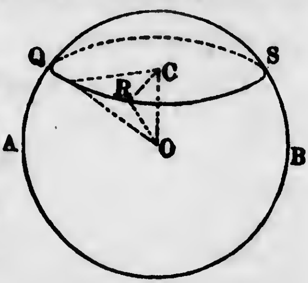
727. *Def.* A **section** of a sphere is the curve line in which any other surface intersects the spherical surface.

728. *Def.* **Opposite points** of a sphere are points at the ends of a diameter.

THEOREM I.

729. *Every section of a sphere by a plane is a circle of which the centre is the foot of the perpendicular from the centre of the sphere upon the plane.*

Hypothesis. AB, any sphere; O, its centre; QRS, the curve in which a plane intersects the spherical surface; OC, the perpendicular from O upon the cutting plane.



Conclusion. QRS is a circle having C as its centre.

Proof. 1. Because the lines OQ, OR, OS are radii of the sphere, they are equal.

2. Because they are equal, they meet the plane QRS at equal distances from the foot C of the perpendicular (§ 595).

Therefore the curve QRS is a circle around C as a centre.

Q.E.D.

730. *Corollary. The line through the centre of a circle of the sphere, perpendicular to its plane, passes through the centre of the sphere.*

731. *Def.* The circular section of a sphere by a plane is called a **circle of the sphere.**

732. *Def.* If the cutting plane passes through the centre of the sphere, the circle of intersection is called

a great circle of the sphere, and the two parts into which it divides the sphere are called **hemispheres.**

733. *Corollary. All great circles of the sphere are equal to each other.*

734. *Def.* The two points in which a perpendicular through the centre of a circle of the sphere intersects the surface of the sphere are called **poles** of the circle.

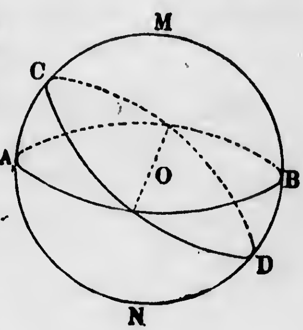
Cor. 2. Because the perpendicular passes through the centre of the sphere:

735. *The two poles of every circle of the sphere are at the extremities of a diameter, and so are opposite points* (§ 728).

Theorem II.

736. *Every great circle divides the sphere into two identically equal hemispheres.*

Hypothesis. AB, a circle of the sphere, having the centre *O* of the sphere in its plane and dividing the sphere into the parts *M* and *N*.



Conclusion. The parts *M* and *N* are identically equal.

Proof. Turn the part *M*, on *O* as a pivot so that the plane *AB* shall return to its own position but be inverted. Then—

1. Because the centre *O* remains fixed, the great circle *AB* will fall upon its own trace.

2. Because the surfaces *M* and *N* are everywhere equally distant from the centre, they will coincide throughout.

Therefore the parts are equal. Q.E.D.

Theorem III.

737. *Any two great circles bisect each other.*

Proof. Let *AB* and *CD* be the two great circles. Because the planes of these circles both pass through the centre of the sphere, their line of intersection is a diameter of the sphere, and therefore a diameter of each circle.

Hence it divides each circle into two equal parts. Q.E.D.

738. *Corollary. If any number of great circles pass through a point, they will also pass through the opposite point.*

THEOREM IV.

739. *Through three points on a sphere one circle, and only one, can be passed.*

Proof. 1. Three points determine the position of a plane passing through them (§ 580).

2. This plane cuts the sphere in a circle of the sphere. (§ 729)

3. Because the three points lie both upon the sphere and in this plane, they lie in this circle.

4. Only one circle can pass through these points (§ 241).

5. Therefore this circle, and no other, passes through the three points. Q.E.D.

THEOREM V.

740. *Through two points upon a sphere, not at the extremities of a diameter, one great circle, and only one, can pass.*

Proof. 1. Because the plane of the great circle must pass through the centre of the sphere, the centre and the two points on the surface determine its position (§ 580).

2. But if the points are at the extremities of a diameter, the centre is in the same straight line with them, and an infinite number of planes may pass through them (§ 577, II.).

741. *Def.* The arc between two points on a sphere means the arc of the great circle passing through these points.

THEOREM VI.

742. *Equal arcs upon the same sphere subtend equal angles at the centre.*

Proof. Because all great circles are equal, their arcs are arcs of equal circles (§ 733).

Because their centres are in the centre of the spheres, and because equal arcs subtend equal angles at the centre (§ 208), their equal arcs subtend equal angles at the centre of the sphere. Q.E.D.

743. *Corollary. Equal chords in the sphere subtend equal angles at the centre.*

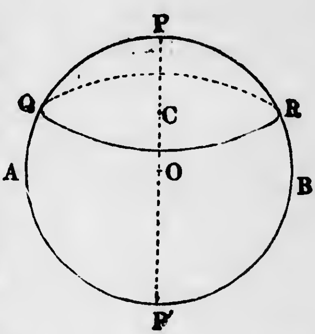
744. *Corollary. The angular distance between two points on the sphere may be measured either by the great circle joining them, or by the angle between the radii drawn to them.*

NOTE. By the distance of two such points is commonly meant their *angular distance*

THEOREM VII.

745. ***All points of a circle of the sphere are equally distant from a pole.***

Hypothesis. QR, a circle of the sphere AB; P, P', the poles of the circle.



Conclusion. Every point of the circle QR is equally distant from P and equally distant from P'.

Proof. 1. Because PP' is a line through the centre of the circle perpendicular to its plane, every point of this line is equally distant from all points of the circle (§ 594).

2. Therefore P and P', being on the line, are each equally distant from all points of the circle. Q.E.D.

746. *Corollary. Every point of a circle of the sphere is at an equal angular distance from the pole.*

747. *Def.* The **polar distance** of a circle of the sphere is the common angular distance of all its points from either pole.

Cor. 2. The angular distance of two poles is a semicircle, because they are at the extremities of a diameter. Hence:

748. *The sum of the distances of a circle from its two poles is a semicircle.*

749. *Cor.* 3. *Every point of a great circle of the sphere is half a semicircle or a right angle distant from each pole.*

750. *Def.* A **quadrant** is an arc of one fourth a great circle, or half a semicircle.

Corollary. A quadrant subtends a right angle from the centre of the sphere.

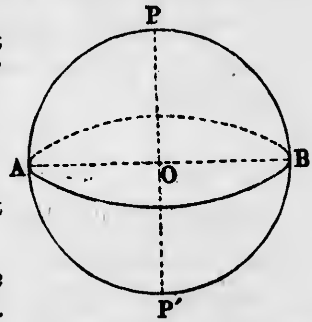
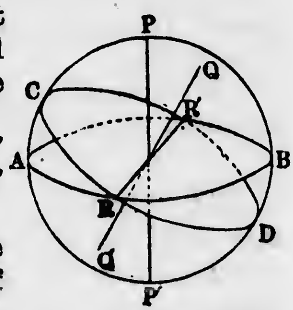


Illustration. If AB is a great circle of the sphere, and P and P' its poles, then
Arc PAP' = arc PBP' = semicircle.
Arc PB = arc BP' = quadrant.
Angle POB = angle BOP' = right angle.

751. *Corollary. On a sphere the locus of a moving point one quadrant distant from a fixed point is a great circle having the fixed point as its pole.*

THEOREM VIII.

752. *The poles of any two great circles lie on a third great circle, having their points of intersection as its poles.*



Hypothesis. AB, CD, two great circles intersecting in the points R and R'; P, P', the poles of AB; Q, Q', the poles of CD.

Conclusion. The poles P, Q, P', Q' lie on the great circle having R and R' as its poles.

Proof. 1. Because R and R' are points on the great circle AB, of which P and P' are the poles,
Arc PR = arc PR' = arc $P'R$ = arc $P'R'$ = quadrant (§ 749).

2. Because R and R' are points on the circle CD, of which Q and Q' are the poles,

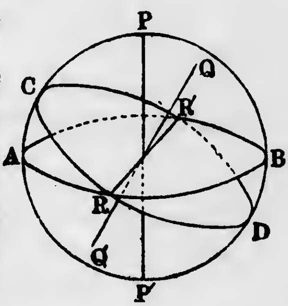
Arc QR = arc QR' = arc $Q'R$ = $Q'R'$ = quadrant.

3. Because P, Q, P', and Q' are points each one quadrant distant from R and R', they lie on the great circle having R and R' for its poles (§ 751). Q.E.D.

Theorem IX.

753. Conversely, *if the poles of two great circles lie on a third great circle, the two great circles will intersect in the poles of that third circle.*

Proof. 1. Because all points one quadrant distant from P lie on the great circle AB, the poles of every great circle through P must lie somewhere on the circle AB.



2. In the same way, the poles of every great circle through Q lie on the great circle CD.

3. Therefore the poles of the great circle through P and Q lie in both of the great circles AB and CD; that is, in the points R and R', in which AB and CD intersect. Q.E.D.

754. *Def.* A great circle having a point R as its pole is called the **polar circle** of the point R.

The great circle containing the poles of other great circles is called the polar circle of these other circles.

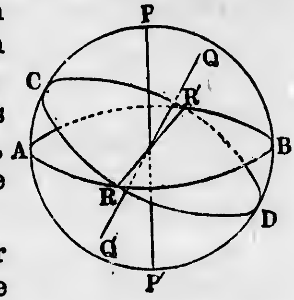
755. *Corollary* 1. *If any number of great circles have their poles upon another great circle, they will all intersect in the pole of that other circle.*

756. *Cor.* 2. *If any number of great circles pass through a common point, their poles will all lie on the polar circle of that point.*

THEOREM X.

757. *The angular distance between two poles of circles is equal to the dihedral angle between the planes of the circles.*

Proof. 1. If P and Q are two poles of circles, then the perpendiculars from P and Q upon the planes of the circles pass through the centre of the sphere (§§ 730, 734).



2. Because these perpendiculars pass through the centre of the sphere, the arc PQ is equal to the plane angle between them (§ 744).

3. Because they are perpendicular to the planes of the circles, the angle they form is equal to the dihedral angle between those planes (§ 625).

4. Comparing (2) and (3), the arc PQ is shown to be equal to the dihedral angle between the planes. Q.E.D.

THEOREM XI.

758. *If one great circle passes through the pole of another, their planes are perpendicular to each other.*

Proof. 1. Because one pole lies on the polar great circle of the other pole, the angular distance of the poles is a quadrant.

2. Therefore the dihedral angle between the planes of the circles is a right angle (§ 757). Q.E.D.

759. *Corollary. If any number of great circles pass through a common point, their planes will all be perpendicular to the plane of their polar circle.*

For, by definition (§ 754), their polar circle passes through all their poles, and its plane is therefore perpendicular to all of their planes by the theorem.

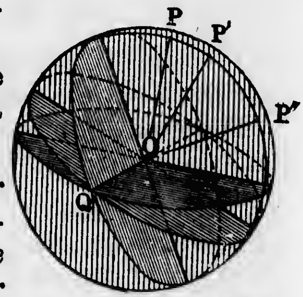
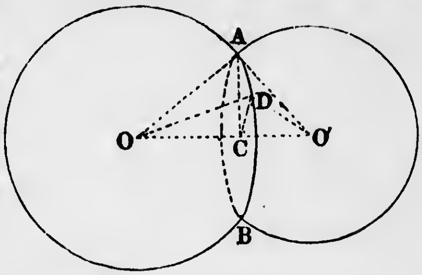


Illustration of §§ 758, 759.
If three great circles intersect at Q, their planes intersect along OQ, and their poles, P, P', P'', lie on another great circle of which Q is the pole, and of which the plane OPP'' is perpendicular to each of the given planes.

THEOREM XII.

760. *Two spherical surfaces intersect each other in a circle whose plane is perpendicular to the line joining the centres of the spheres, and whose centre is in that line.*

Hypothesis. O, O', the centres of two spheres; ADB, the curve line in which their surfaces intersect.



Conclusion. ADB is a circle having its centre on the line OO' and its plane perpendicular to that line.

Proof. Let A and D be any two points on the curve of intersection.

Join OA, $O'A$, OD, and $O'D$.

From A and D drop perpendiculars upon OO'. Then—

1. In the triangles OAO' and ODO' the side OO' is common; $OA = OD$ and $O'A = O'D$, because these lines are radii of the same sphere. Therefore

Triangle OAO' = triangle ODO' *identically.*

2. Because these triangles are identically equal, perpendiculars from A and D upon the base OO' are equal (§ 175), and the feet of these perpendiculars meet OO' at equal distances from O; that is, at the same point. Let C be this point.

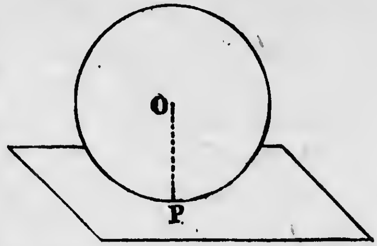
3. Because the lines CA and CD are both perpendicular to OO', they are in one plane perpendicular to this line (§ 586); and because they are equal, their ends are in a circle having its centre at C. Q.E.D.

THEOREM XIII.

761. *A plane perpendicular to a radius of the sphere at its extremity is a tangent to the sphere.*

Proof. Let OP be the radius to which the plane is perpendicular. Then—

1. Because OP is a radius, the point P is common to the sphere and the plane.



2. Because OP is perpendicular to the plane, it is the shortest line from O to the plane. Therefore every other point of the plane is without the sphere.

3. Therefore the plane is a tangent to the sphere at the point P. Q.E.D.

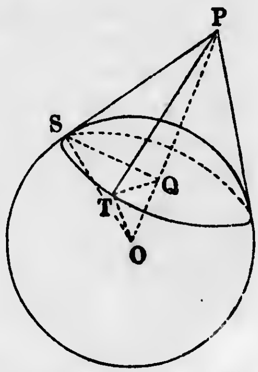
762. *Corollary* 1. *Every line perpendicular to a radius at its extremity is tangent to the sphere* (§ 725).

763. *Cor.* 2. *Conversely, every plane or line tangent to the sphere is perpendicular to the radius drawn to the point of contact.*

Theorem XIV.

764. *All lines tangent to a sphere from the same external point are equal, and touch the sphere in a circle of the sphere.*

Proof. Let O be the centre of the sphere; P, the point from which the tangents are drawn; PS, PT, any two tangents touching the sphere at S and T. Then—



1. In the triangles PSO and PTO the side PO is common; $OS = OT$ (because they are radii), and angle $OTP =$ angle OSP (both being right angles) (§ 763).

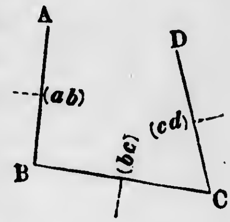
Therefore these triangles are identically equal, whence

$$PS = PT. \quad \text{Q.E.D.}$$

2. Because the triangles PSO and PTO are identically equal, the perpendiculars from S and T upon PO are equal, and meet PO at the same distance from P; that is, at the same point Q. Since S and T may be any two points in which tangents through P touch the sphere, all these points are in one plane, and in a circle having its centre at Q. Q.E.D

THEOREM XV.

765. *Through four points not in the same plane one spherical surface, and no more, may pass.*



Proof. Let $ABCD$ be the four points. Join AB, BC, CD, and bisect each of these lines by a plane perpendicular to them. Let us call these respective planes the planes (ab), (bc), (cd). Then—

1. If the sphere passing through A, B, C, and D exist, then, because its centre is equally distant from A and B, it lies in the plane (ab) bisecting AB perpendicularly (§ 589).

2. In the same way, it lies in the other two bisecting planes, and therefore in their point of intersection if they have one.

3. If they have no point of intersection, their three lines of intersection are parallel (§ 637).

4. Suppose these lines to be parallel. Because the plane ABC contains the line $AB \perp$ plane (ab),

Plane $ABC \perp$ plane (ab). (§ 629)

For the same reason,

Plane $ABC \perp$ plane (bc).

Because plane $ABC \perp$ to both the planes (ab) and (bc), and (cd) is a third plane having parallel lines of intersection with (ab) and (bc).

Plane $ABC \perp$ plane (cd). (§ 640)

Because plane $ABC \perp$ plane (cd), and line $CD \perp$ plane (cd), by construction, therefore CD lies in the plane ABC (§ 631), and A, B, C, D lie in one plane, which is contrary to the hypothesis; whence the lines of intersection are not parallel.

5. Therefore the three planes intersect in a point (§ 637), which point is equally distant from A, B, C, and D, and which may therefore be the centre of a sphere passing through A, B, C, and D. Q.E.D.

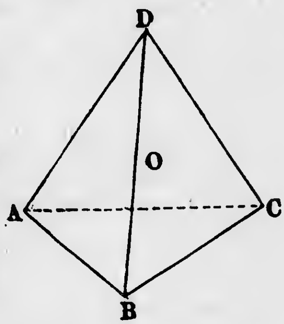
Corollary. The four points A, B, C, and D may be taken as the vertices of a tetrahedron. The edges will then be the six lines formed by joining every pair of vertices, and we may take any three of those edges to determine the positions of the planes whose point of intersection is the centre of the sphere. Hence:

766. *The six planes which bisect at right angles the six edges of a tetrahedron all pass through a point.*

THEOREM XVI.

767. *A sphere may be tangent to any four planes which do not intersect in a point, and of which the lines of intersection are not all parallel.*

Proof. Let AB, AC, AD, BC, BD, and CD be the six lines of intersection of the four planes, taken two and two.



Bisect any three of the dihedral angles which lie in one plane, as AB, BC, and AC, by other planes.

Let O be the point in which these planes meet. Then—

1. Because O is on the bisector of the dihedral angle BA, it is equally distant from the faces ABC and ABD (§ 628).

2. In the same way O is equally distant from the faces BCA and BCD, and from CAB and CAD.

3. Therefore the point O is equally distant from the four planes; and if a sphere be described having its centre at O and its radius equal to the common distance, it will be tangent to all four planes. Q.E.D.

Corollary. Since we may take any three dihedral angles not meeting in a point to determine the centre of the sphere, we conclude:

768. *The six planes which bisect the six edge angles of a tetrahedron intersect in a point.*

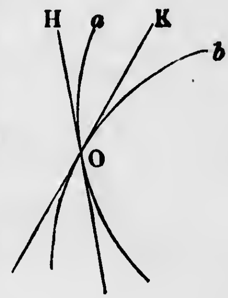
Scholium. It may be shown, as in the case of the triangle, that besides the sphere inscribed in the tetrahedron there are four escribed spheres, each touching one face externally and the other three faces produced.

CHAPTER II.

OF SPHERICAL TRIANGLES AND POLYGONS.

769. *Def.* If two great circles of the sphere intersect, they are said to make an angle with each other equal to the angle between their tangents at the point of intersection.

Illustration. If the great circles a and b intersect at O, and if H and K are their respective tangents at O, then their angle is measured by the angle HOK between the tangents.

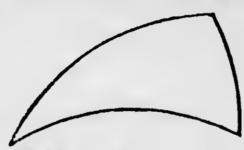


770. *Def.* A **spherical triangle** is the figure formed by joining any three points on the sphere by arcs of great circles.

771. *Def.* The points which are joined are called **vertices** of the triangle.

772. *Def.* The arcs of which a spherical triangle is formed are called **sides** of the triangle.

773. *Def.* The angles which the sides make with each other are called **angles** of the triangle.



A spherical triangle.

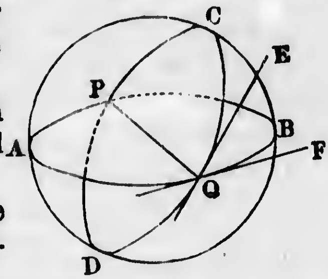
REMARK 1. Between any two points two arcs of a great circle may be drawn, the one greater and the other less than a semicircle. In forming a spherical triangle the arcs less than a semicircle are supposed to be taken, unless otherwise expressed.

REMARK 2. In a spherical triangle, as in a plane triangle, each side has an opposite angle and two adjacent angles, and each angle has an opposite side and two adjacent sides.

THEOREM XVII.

774. *The angle between two great circles is equal to the dihedral angle between their planes.*

Hypothesis. AB, CD, two great circles of the sphere; PQ, the line of intersection of their planes; QE, QF, their respective tangents at Q.



Conclusion. The angle between the circles is equal to the dihedral angles along PQ.

Proof. 1. By definition the angle between the circles is measured by the angle EQF (§ 769).

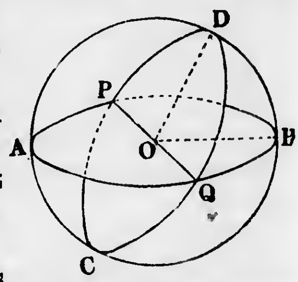
2. Because QF and QE are tangents to the circles AB and CD, they are perpendicular to the diameter PQ; and because they lie in the planes AB and DC, their angle measures the dihedral angle between those planes (§ 623).

3. Therefore the angle between the great circles AB and CD is equal to the dihedral angle along PQ. Q.E.D.

THEOREM XVIII.

775. *If on two great circles points be taken a quadrant distant from their points of intersection, the arc between the points measures the angle between the circles.*

Hypothesis. AB, CD, two great circles of the sphere intersecting along the line PQ; B, D, two points each a quadrant distant from P and Q.



Conclusion. The arc BD is equal to the angle BQD between the circles.

Proof. From the centre O erect in each plane a perpendicular to PQ. Then—

1. Because these perpendiculars are erected from the centre, they will meet the great circle in the points B and D.

2. Because OB and OD are radii of the sphere,

Arc BD = angle BOD.

3. Because OB and OD are perpendicular to PQ,

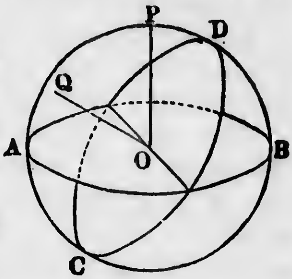
Angle BOD = dihedral angle of planes.

4. Because angle BQD = dihedral angle of planes (§ 623),

Angle BOD = angle BQD = angular distance BD. Q.E.D.

THEOREM XIX.

776. *The angular distance between the poles of two great circles is equal to the angle between the circles.*



Proof. Let P and Q be the poles of the great circles AB and CD, and O the centre. Join OP, OQ. Then—

1. Because P and Q are poles, by definition,

$$\left.\begin{array}{l} OP \perp \text{plane } AB; \\ OQ \perp \text{plane } CD. \end{array}\right\} \text{(§ 734)}$$

Therefore

Angle POQ = dihedral angle between planes,

= angle between circles. Q.E.D.

Corollary 1. From this and the preceding theorem, with Theorem XI., follows:

777. *If through the poles P and Q of two great circles a third circle be passed intersecting the other circles at A, C, B, and D, we shall have—*

I. Angle PQ = angle AC = angle BD,

= angle between circles AB and CD,

= dihedral angle BOD.

II. *The third circle will intersect both the other circles at right angles.*

778. *Cor.* 2. *If two sides of a spherical triangle are quadrants, the angles opposite these sides will be right angles.*

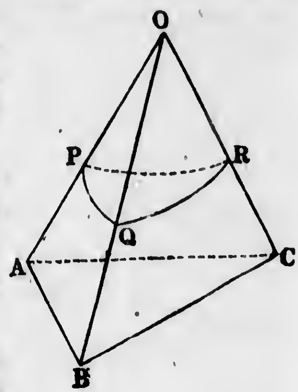
Relation of a Spherical Triangle to a Trihedral Angle at the Centre of the Sphere.

Because the three sides of a spherical triangle are arcs of great circles, their planes all intersect at the centre of the sphere, where they form a solid trihedral angle.

By § 744 the three sides of the triangle are measured by the three plane angles of the solid angle, and by § 774 the three angles of the triangle are measured by the three dihedral angles of the solid angle. Hence:

779. *To every spherical triangle corresponds a trihedral angle at the centre of the sphere, having its six parts equal to the parts of the spherical triangle.*

Conversely, if $O\text{-}ABC$ be any trihedral angle, we may imagine a sphere with its centre at O and an arbitrary radius OP. Then—



The surface of the sphere will in-intersect the edges OA, OB, OC at the points P, Q, and R.

The same surface will intersect the planes OAB, OBC, OCA in the arcs of great circles PQ, QR, RP, which arcs will form a spherical triangle. Hence:

780. *Every trihedral angle may be represented by a triangle on a sphere having its centre at the vertex.*

From this it follows that the relations proved in §§ 655–660 between the faces and angles of a trihedral angle are true of spherical triangles, when for the face angles of the trihedral angle we substitute the sides of the spherical triangle, and for the dihedral angles the angles of the triangle. We thus conclude:

781. *If two sides of a spherical triangle are equal, the opposite angles are equal* (§ 655).

782. *In a spherical triangle the greater side and the greater angle are opposite each other* (§ 656).

783. *The sum of any two sides of a spherical triangle is greater than the third side* (§ 659).

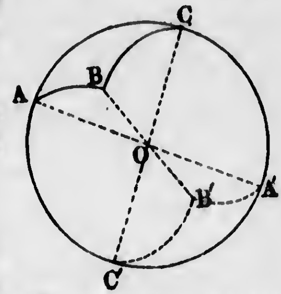
784. *Two spherical triangles are equal when their sides are equal and similarly arranged* (§ 660).

785. *The sum of the three sides of a spherical triangle is less than a circumference* (§ 661).

Symmetrical Spherical Triangles.

786. *Def.* Two spherical triangles are said to be **opposite** when the vertices of the one are at the ends of the diameters from the vertices of the other.

Corollary 1. Since the radii AO, BO, CO from the vertices of a spherical triangle are the edges of the trihedral angle corresponding to it, we conclude:



When AA', BB', CC' are diameters, ABC and $A'B'C'$ are symmetrical triangles.

787. *To two opposite spherical triangles correspond two opposite and symmetrical trihedral angles* (§ 653).

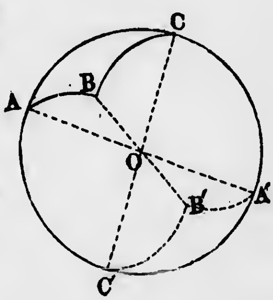
Corollary 2. Since the lines AA', BB', and CC' all intersect in the same point O, each pair of them is in one plane, passing the centre of the sphere and containing the corresponding arcs AB and $A'B'$, BC and $B'C'$, CA and $C'A'$. Hence:

788. *The corresponding sides of two opposite triangles are formed of arcs of the same great circle.*

789. Symmetrical spherical triangles are those in which the sides and angles of the one are respectively equal to those of the other, but arranged in the reverse order.

Theorem XX.

790. *Opposite spherical triangles are symmetrical.*



Proof. Let ABC and $A'B'C'$ be the triangles, and O the centre of the sphere. Then—

1. Because the angles AOB and $A'OB'$ are in one plane, we have

Angle $AOB =$ opp. angle $A'OB'$;
and for the same reason,
Angle $BOC =$ opp. angle $B'OC'$;
Angle $COA =$ opp. angle $C'OA'$.

2. For the same reason,

Dihedral angle OA = dihedral angle OA'.
Dihedral angle OB = dihedral angle OB'.
Dihedral angle OC = dihedral angle OC'.

Whence the corresponding angles of the two triangles are respectively equal (§ 774).

3. To an eye looking from O the vertices A, B, C and the sides AB, BC, CA succeed each other in the negative order (§ 648); while A', B', C' and the sides $A'B', B'C', C'A'$ succeed each other in the positive order. Therefore the triangles are symmetrical. Q.E.D.

NOTE. This theorem should be compared with § 653, which is the equivalent theorem in the case of a trihedral angle.

791. *Corollary. Two symmetrical triangles cannot in general be made to coincide identically.*

For if we slide the triangle $A'B'C'$ over to the other side of the sphere so that $A' \equiv A$ and $C' \equiv C$, the vertices B' and B will fall on opposite sides of AC.

If we turn one triangle round so that B and B' shall fall on the same side of AC, the vertex $A' \equiv C$ and $C' \equiv A$.

THEOREM XXI.

792. *If two symmetrical spherical triangles are isosceles, they are identically equal.*

Proof. In the preceding case suppose

$$\text{Side } BA = \text{side } BC;$$

we shall then have

$$\text{Angle } A = \text{angle } C. \qquad (\S\, 781)$$

Therefore, in the symmetrical triangle,

$$\text{Side } B'A' = \text{side } B'C' = \text{side } BC = \text{side } BA.$$
$$\text{Angle } A' = \text{angle } C' = \text{angle } C = \text{angle } A.$$

Then if we slide $A'B'C'$ over so that $A' \equiv C$ and $C' \equiv A$, we shall have

$$\text{Side } B'A' \equiv \text{side } BC.$$
$$\text{Side } B'C' \equiv \text{side } BA.$$
$$\text{Vertex } B' \equiv \text{vertex } B.$$

Therefore the two triangles are identically equal. Q.E.D.

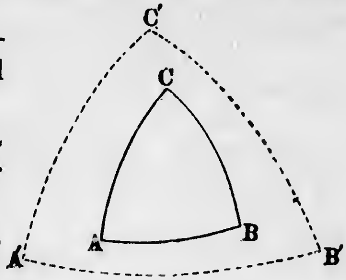
Polar Triangles.

793. *Def.* If the sides of one spherical triangle have their poles at the vertices of a second triangle, the first triangle is called the **polar triangle** of the second.

THEOREM XXII.

794. *The polar triangle of a polar triangle is the original triangle.*

Hypothesis. ABC, a spherical triangle; $A'B'C'$, its polar triangle.



Conclusion. The polar triangle of $A'B'C'$ is the original triangle ABC.

Proof. 1. Because the great circles $A'B'$ and $A'C'$ have their poles at C and B (hyp. and def.), their point of intersection A' is the pole of the circle BC (§ 753).

2. In the same way it is shown that C' is the pole of the circle AB, and B' of AC.

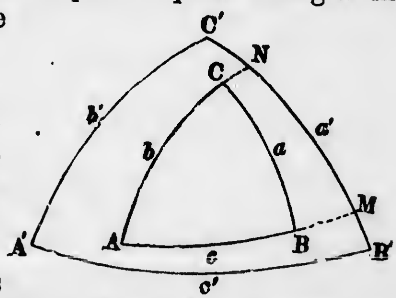
3. Because the three great circles AB, BC, and CA have their poles at C', A', and B', ABC is, by definition, the polar triangle of $A'B'C'$. Q.E.D.

795. *Corollary. The relation between two polar triangles may be expressed by saying that the vertices of each triangle are the poles of the sides of the other.*

THEOREM XXIII.

796. *In two polar triangles each side of the one is the supplement of the opposite angle of the other.*

Hypothesis. ABC, $A'B'C'$, a pair of polar triangles in which $A'B'$ is the pole of the vertex C, etc.



Conclusions.

Arc AB + angle $C' = 180°$.
Arc BC + angle $A' = 180°$.
Arc $A'B'$ + angle $C = 180°$.
Arc $B'C'$ + angle $A = 180°$.
etc. etc. etc.

Proof. Produce the sides AB and AC until they meet $B'C'$ in M and N. Then—

1. Because A is the pole of MN, AM and AN are quadrants and

$$\text{Arc } MN = \text{angle } A. \qquad (\S\ 775)$$

2. Because B' is the pole of ACN and C' is the pole of ABM,

$$\text{Arc } B'N = \text{arc } C'M = \text{quadrant, or } 90^\circ.$$

3. $$\text{Arc } B'C' = \text{arcs } B'N + C'M - MN \text{ (identically);}$$

or comparing with (2),

$$\text{Arc } B'C' = 180^\circ - \text{arc } MN;$$

and comparing with (1),

$$\text{Arc } B'C' = 180^\circ - \text{angle } A.$$

4. Therefore arc $B'C' + \text{angle } A = 180^\circ$. Q.E.D.

In the same way all the other relations may be proved.

Theorem XXIV.

797. *The sum of the three angles of a spherical triangle is greater than a straight angle and less than three straight angles.*

Proof. Let A, B, and C be the three angles of the spherical triangle, and a', b', and c' the opposite sides of the polar triangle. Then—

$$\begin{array}{llcll}
\text{I.} & A + a' & = & 130 & \\
 & B + b' & = & 180 & (\S\ 796) \\
 & \underline{C + c'} & = & \underline{180} & \\
\text{Sum} & A + B + C + a' + b' + c' & = & 3.180^\circ & \\
\text{Whence} & A + B + C & = & 3.180^\circ - (a' + b' + c'). &
\end{array}$$

But because a', b', and c' are sides of a spherical triangle,

$$a' + b' + c' < 2.180^\circ. \qquad (\S\ 785)$$

Therefore $\quad A + B + C > 180^\circ$. Q.E.D.

II. Because each angle of the triangle is less than a straight angle, the sum of the three angles is less than three straight angles. Q.E.D.

798. *Def.* The **spherical excess** of a spherical triangle is the excess of the sum of its three angles over a straight angle.

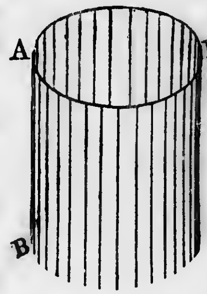
Corollary. The spherical excess may be equal to any positive angle less than a circumference.

CHAPTER III.

THE CYLINDER.

799. *Def.* A **cylindrical surface** is the surface which is generated by the motion of a straight line constantly touching a given curve, and remaining parallel to its original position.

Illustration. If the straight line AB move around the curve AM, remaining parallel to the position AB during the motion, it will generate a cylindrical surface.



800. *Def.* The **generatrix** is the line which generates the surface.

801. *Def.* The **directrix** is the curve which the generatrix touches.

802. *Def.* A **cylinder** is a solid bounded by a cylindrical surface and two parallel planes.

803. *Def.* **Elements** of the cylinder are the different positions of the generatrix.

REMARK 1. In elementary geometry the directrix is a circle whose plane is perpendicular to the generatrix.

REMARK 2. Since the generatrix may extend out indefinitely in two directions, a cylindrical surface may extend out indefinitely in two directions.

804. *Def.* The **axis** of a cylinder is a line through the centre of the directrix parallel to the elements.

805. *Def.* A **tangent plane** to a cylinder is one which touches the cylindrical surface without intersecting it.

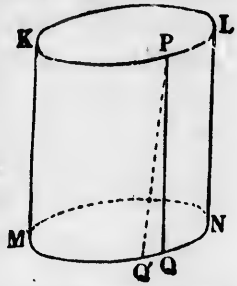
806. *Def.* A **right section** of a cylinder is the section by a plane perpendicular to the elements.

807. *Def.* A sphere is said to be **inscribed in a cylinder** when all the elements of the cylinder are tangents to the sphere.

THEOREM XXV.

808. *A plane tangent to a cylinder is parallel to all the positions of the generatrix, and touches the cylindrical surface along an element.*

Hypothesis. *KLMN*, a cylindrical surface; *P*, *Q′*, two points in which a plane may touch the surface without intersecting it.



Conclusion. The line *PQ′* is an element and lies in the plane, and all other elements are parallel to the plane.

Proof. 1. Let *KM* be any element. If *KM* were not parallel to the tangent plane, it could be produced so far as to intersect the plane, and then the cylindrical surface would also intersect the plane, which is contrary to the definition of a tangent plane. Therefore

$$KM \parallel \text{tangent plane.} \quad \text{Q.E.D.}$$

2. Let *PQ* be the element which passes through *P*. Because *P* is in the tangent plane, and *PQ* cannot intersect the plane (1), *Q* is in the plane. But *Q′* is also in the plane, by hypothesis. Then the plane would intersect the curve *MN* at the points *Q* and *Q′*, and therefore would intersect the surface also, which is contrary to the definition of a tangent plane.

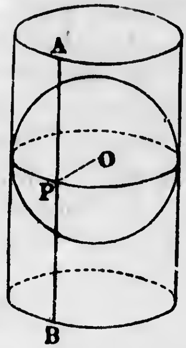
3. Therefore the points *Q* and *Q′* are the same, and *PQ′* is an element.

4. Because *P* and *Q* are both in the plane, the line *PQ* is in this plane, and is the line of contact between the cylinder and plane. Q.E.D.

THEOREM XXVI.

809. *If a sphere be inscribed in a cylinder, the surfaces will touch on a great circle the plane of which forms a right section of the cylinder.*

Proof. Let *O* be the centre of the sphere, and *AB* any element of the cylinder. Through *O* pass a plane perpendicular to the elements of the cylinder, which call *the plane* O. Then—



1. Because the sphere is inscribed in the cylinder, the line AB is tangent to the sphere (§ 807).

2. Let P be the point of tangency. Join OP. Then

$$OP \perp AB. \qquad (\S\ 763)$$

Therefore AB being $\perp$ plane O, OP lies in the plane O; whence P also lies in this plane, and is the only point in which AB meets the surface of the sphere.

3. Because AB may be any element, all the elements touch the sphere in the plane O perpendicular to them.

4. Because the plane O passes through the centre of the sphere, and the points of tangency are all on its intersection with the surface, these points are on a great circle of the sphere. Q.E.D.

Corollary. If two spheres are inscribed in the same cylinder, then the planes of contact, being perpendicular to the elements, are parallel (§ 616). Because the centres of the spheres are in the axis perpendicular to the planes, their distance is also equal to the distance between those centres. Hence:

810. *Two spheres inscribed in the same cylinder intercept lengths of the elements equal to the distance between the centres of the spheres.*

Theorem XXVII.

811. *Every plane section of a cylindrical surface is an ellipse.*

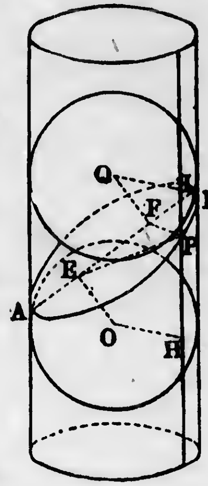
Hypothesis. APB, any plane section of a cylindrical surface.

Conclusion. APB is an ellipse.

Proof. In the cylinder inscribe two spheres of which O and Q are the centres, in such a position that each of them shall touch the plane AB.

Let E and F be the points of contact with the plane.

Let P be any point on the section APB, HI the element passing through P, and H and I the points at which this element touches the spheres O and Q.



Join PF and PE. Then—

1. Because the plane AB is tangent to the sphere O at E, and P is a point in this plane, PE is a line tangent to the sphere at E.

2. Because PH is another tangent from P to the same sphere,

$$PE = PH. \qquad (\S\,764)$$

3. We find in the same way, for the sphere Q,

$$PF = PI.$$

Whence

$$PE + PF = PH + PI = HI,$$

or

$$PE + PF = OQ. \qquad (\S\,810)$$

4. Since F may be any point of the section, the sum of the distances of every point of the section from E and F is equal to the same constant length OQ. Therefore the section is, by definition, an ellipse around E and F as foci (§ 514).

Q.E.D.

812. *Corollary* 1. *The distance of the centres of the inscribed tangent spheres is equal to the major axis of the ellipse.*

813. *Cor.* 2. *The tangent spheres touch the plane of the ellipse at its respective foci.*

814. *Cor.* 3. *Parallel plane sections of a cylindrical surface are identically equal.*

CHAPTER IV.

THE CONE.

815. *Def.* A **conical surface** is the surface generated by the motion of a straight line which constantly passes through a fixed point and touches a curve.

816. A **cone** is the solid formed by cutting off a portion of a conical surface by a plane.

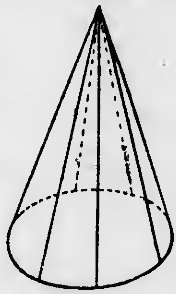
A cone is completely bounded by the conical surface and the plane.

817. The **base** of a cone is its plane surface.

REMARK. In the higher geometry a conical surface is called a *cone*, and we shall use this abbreviation when convenient.

818. *Def.* The **generatrix** is the generating line of a cone.

819. *Def.* The **directrix** of a cone is the curve along which the generatrix moves.



A cone.

820. The **vertex** of a cone is the fixed point through which the generatrix passes.

821. *Def.* **Elements** of the cone are the straight lines occupying the different positions of the generatrix.

REMARK 1. In elementary geometry the directrix is supposed to be a circle.

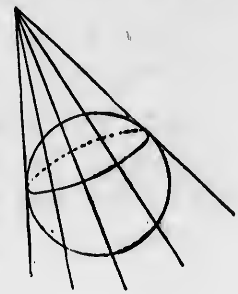
REMARK 2. Since the generating line may extend on both sides of the vertex, a *complete conical surface* consists of two surfaces meeting in a point at the vertex and extending out indefinitely in both directions.

822. *Def.* The two parts of a complete conical surface are called **nappes** of the cone.

823. *Def.* A **tangent plane** to a cone is a plane touching the conical surface without intersecting it.

824. *Def.* A sphere is said to be **inscribed** in a cone when all the elements of the cone are tangents to the sphere.

825. *Def.* The **axis** of a cone is the straight line from the vertex through the centre of the directrix.



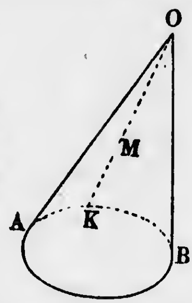
826. *Def.* A **right cone** is one in which the vertex is in the line passing through the centre of the directrix perpendicular to its plane.

NOTE. In the following propositions the word *cone* means a right cone, though some of the theorems are true of other cones.

THEOREM XXVIII.

827. *A plane tangent to a cone touches it along an element, and passes through the vertex.*

Hypothesis. O-AB, a conical surface; M, a point at which a plane touches the surface.



Conclusions.

I. The plane passes through O.

II. OM, and no other element, lies in it.

Proof. I. If the plane did not pass through O, then, because it does pass through M, OM would intersect the plane at M, and the plane could not be a tangent.

Therefore the plane passes through O.

Q.E.D.

II. Because the points O and M both lie in the plane, the element OM lies wholly in it.

If any other element than OM may lie in the plane, let OA be that element. The plane would meet the directrix AB at two points A and K, and therefore would intersect it and would not be a tangent plane.

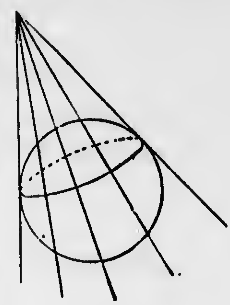
Therefore the plane touches the cone only along the element OK. Q.E.D.

Scholium. In II. of this demonstration it is assumed that no part of the directrix is a straight line. If such were the case, a portion of the conical surface would be a plane, and the tangent plane might coincide with this plane surface.

THEOREM XXIX.

828. *If a sphere be inscribed in a cone, the surfaces touch on a circle all points of which are equally distant from the vertex.*

Proof. When a sphere is inscribed in a cone, each element is, by definition, a tangent to the sphere. Hence all the elements are tangents to the sphere from the vertex, and are equal by Theorem XV. (§ 764).



From the same theorem it follows that the points of contact lie upon a circle. Q.E.D.

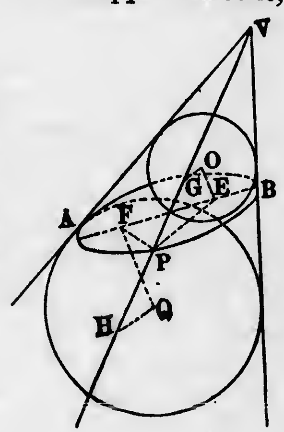
829. *Corollary. If two spheres be inscribed in the same cone, the segments of the elements intercepted between the points of tangency are all equal.*

THEOREM XXX.

830. *Every complete plane section of one nappe of a cone is an ellipse.*

Hypothesis. APB, a plane section of one nappe of a cone, passing through all the elements.

Conclusion. APB is an ellipse.



Proof. Let O and Q be the centres of two spheres inscribed in the cone, and tangents to the cutting plane at the respective points E and F.

Let P be any point of the section, and G and H the points in which the element VP touches the respective spheres.

Join PE, PF. Then—

1. Because PE and PG are tangents to the same sphere,

$$PE = PG.$$

2. In the same way,

$$PF = PH;$$

whence

$$PE + PF = GH.$$

3. Because G and H are the points in which an element touches two inscribed spheres, the line GH has the same length for all the elements (§ 829).

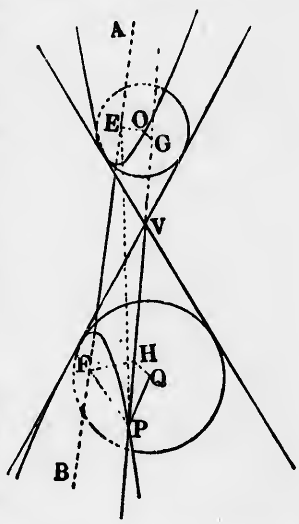
Therefore the sum of the distances $PE + PF$ is the same for every point of the section, and this section is an ellipse, by definition. Q.E.D.

831. *Corollary. The points in which the inscribed spheres touch the cutting plane are the foci of the ellipse.*

Theorem XXXI.

832. *Every section of both nappes of a cone by a plane is an hyperbola.*

Hypothesis. AB, a section of two nappes of a cone by the same plane; V, the vertex of the cone.



Conclusion. AB is an hyperbola.

Proof. Let O and Q be the centres of two spheres inscribed in the cone, and tangents to the cutting plane at the points E and F.

Let P be any point of the section; PVG, the element through P; and G and H, the points in which this element touches the spheres.

Join PE, PF. Then—

1. Because the plane AB is tangent to both spheres at the points E and F, and P is in this plane,

PE and PF are tangents to the respective spheres.

2. Because PE and PG are tangents to the sphere O from the same point P, $\quad PE = PG.$

3. Because PF and PH are tangents to the sphere Q,

$$PF = PH.$$

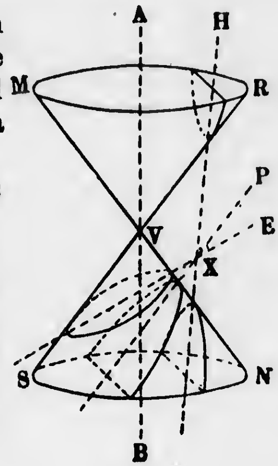
4. Subtracting this equation from (2),

$$PE - PF = PG - PH = GH = GV + VH.$$

5. But the lengths VG and VH are each constant for all the elements of the cone. Therefore since every point of the section must be on some element of the cone, the difference of the distances of all such points from E and F is the same.

6. Therefore the section is an hyperbola having its foci at E and F (§ 531). Q.E.D.

833. *Corollary. The major axis of the hyperbola is equal to the common length of the segments of the elements contained between the points in which they touch the inscribed spheres.*



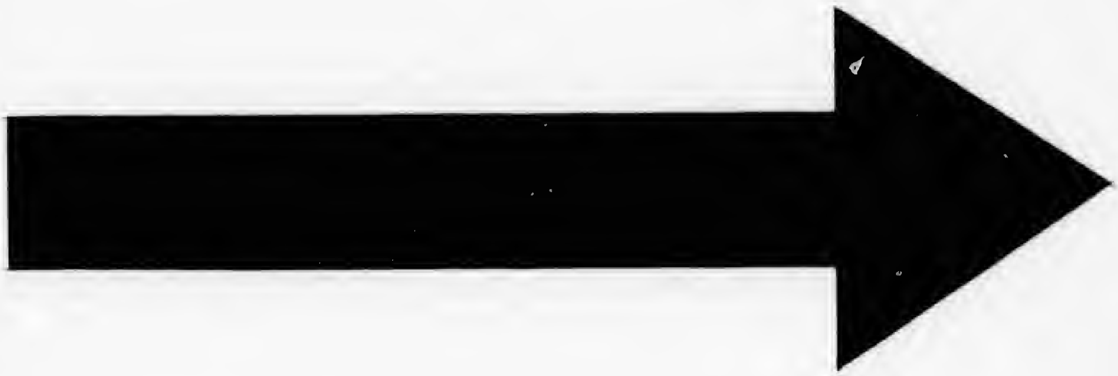
Scholium. Let $MNRS$ be a portion of a conical surface, of which AB is the axis, extending out indefinitely; and let an indefinite plane pass through a point X. Then—

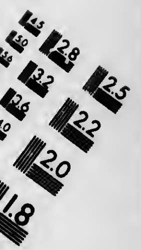
1. If the plane makes with the axis an angle greater than AVR, it will intersect one nappe of the cone completely, and will not touch the other nappe. The section is then an ellipse.

2. If the plane makes with the axis an angle less than that of the cone, it will partly intersect both nappes but will cut through neither of them. The section is then an hyperbola.

3. If the plane makes with the axis an angle equal to the angle AVR of the cone, it will cut into one nappe of the cone indefinitely, but will not cut the other. If we imagine the plane EX to turn upon X until it assumes the position PX, the lower of the two tangent inscribed spheres will be moved off indefinitely. Hence the focus of the elliptic section of the cone will move off indefinitely, and the ellipse will become a parabola (§ 560). Hence:

834. *Corollary. The section of a cone by a plane parallel to an element is a parabola.*

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BOOK XI.

THE MEASUREMENT OF SOLIDS.

CHAPTER I.

SUPERFICIAL MEASUREMENT.

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835. *Def.* A **right** parallelopiped is a parallelopiped in which the lateral faces are at right angles to the base.

REMARK 1. Since any face may be taken as the base, any parallelopiped in which one face is at right angles to the four adjoining faces is a right parallelopiped.

REMARK 2. A right parallelopiped differs from a rectangular one (§ 685) in that its base need not be a rectangle.

836. *Def.* The lateral area of a prism or pyramid is the combined area of its lateral faces (§§ 672, 695).

837. *Def.* A prism is said to be inscribed in a cylinder when its bases coincide with the bases of the cylinder, and its lateral edges are elements of the cylinder.

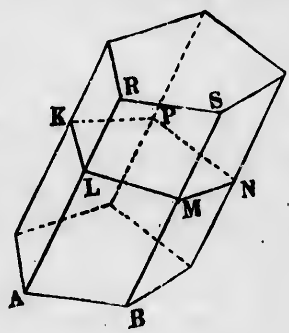
838. *Def.* A pyramid is said to be inscribed in a cone when its base is a polygon inscribed in the base of the cone, and its lateral edges are elements of the cone.

839. *Def.* A **regular pyramid** is a pyramid of which the base is a regular polygon, the perpendicular through whose centre passes through the vertex of the pyramid.

840. The **slant height** of a regular pyramid is the distance from the vertex to the middle point of any edge of the base.

THEOREM I.

841. *The lateral area of a prism is equal to the product of a lateral edge into the perimeter of a right section.*



Proof. Let $ABRS$ be any lateral face of a prism, and $KLMNP$ a right section (§ 674). Then—

1. Because the lines LM, MN, NP, etc., which make up the perimeter of the section are all in one plane perpendicular to the parallel edges AR, BS, etc., they are perpendicular to these edges.

2. Because $ABRS$ is a parallelogram of which LM may be taken as an altitude,

$$\text{Area } ABRS = LM.BS. \qquad (\S\ 295)$$

3. Because the lateral edges are all equal, we find, by taking the sum of those areas formed in the same way,

$$\text{Lateral surface} = (KL + LM + MN + NP + PK)BS.$$

Q.E.D.

Corollary. Because the base of a right prism is a right section, and its altitude a lateral edge:

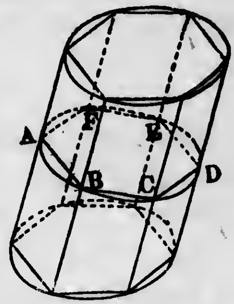
842. *The lateral area of a right prism is equal to its altitude into the perimeter of its base.*

THEOREM II.

843. *The area of a cylindrical surface is equal to the length of the generating line into the circumference of a right section.*

Proof. Let there be inscribed in the cylinder a regular prism of any number of sides, and let $ABCDEF$ be a right section of this prism. Then—

1. Because the prism is inscribed in the cylinder, each lateral edge will be equal in length to an element of the cylinder, and will lie in the cylindrical surface (§ 837).



2. Therefore the lateral surface of the prism will be equal to the perimeter of the cross-section AD into the length of each element.

Now suppose the number of sides of the prism to be increased without limit. Then—

3. The perimeter of the cross-section AD (which call P) will approach the circumference of the cross-section of the cylinder (which call C) as its limit (§ 482).

4. The lateral surface of the prism (which call S) will approach the surface of the cylinder (which call S') as its limit.

5. Because we continually have $S = P \times$ length of element, however great the number n of sides, and because P approaches C as its limit and S approaches S' as its limit, we have $S' = C \times$ length of element. Q.E.D.

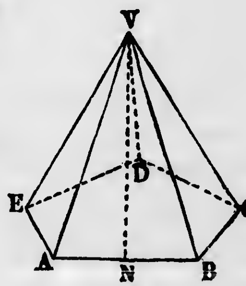
Theorem III.

844. *The lateral area of a regular pyramid is equal to half its slant height into the perimeter of its base.*

Proof. Let $V\text{-}ABCDE$ be a regular pyramid, and VN its slant height. Then—

1. Considering AB as the base of the triangle VAB, VN will be its altitude. Therefore

Area $VAB = \frac{1}{2} VN \,.\, AB$.



2. Because the slant heights are all equal to VN,

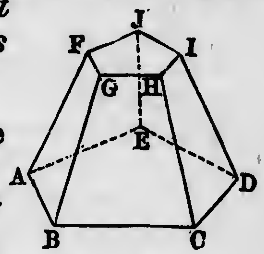
Area $VBC = \frac{1}{2} VN \,.\, BC$,
Area $VCD = \frac{1}{2} VN \,.\, CD$,
etc. etc.

3. Taking the sum of all these areas,

$$\text{Lateral area} = \tfrac{1}{2} VN (AB + BC + CD + \text{etc.}),$$
$$= \tfrac{1}{2} VN \,.\, \text{perimeter } ABCDE. \quad \text{Q.E.D.}$$

Theorem IV.

845. *The lateral area of a frustum of a regular pyramid is equal to its slant height into half the sum of the perimeters of its bases.*



Proof. Let $ABCDE\text{-}FGHIJ$ be the frustum. Then—

1. Because the lateral faces are trapezoids, having $FG \parallel AB$, $GH \parallel BC$, etc.,

$$\text{Area } ABFG = \tfrac{1}{2}(AB + FG) \times \text{slant height},$$
$$\text{Area } BCGH = \tfrac{1}{2}(BC + GH) \times \text{slant height},$$
$$\text{etc.} \qquad \text{etc.} \qquad (\S\ 305)$$

2. Taking the sum of all these areas,

$$\text{Lateral area} = \tfrac{1}{2}(AB + BC + \text{etc.} + FG + GH + \text{etc.}) \times \text{slant height}.$$

3. $AB + BC + \text{etc.} = \text{perimeter of lower base};$
$FG + GH + \text{etc.} = \text{perimeter of upper base}.$

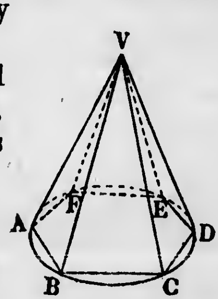
Therefore

$$\text{Lateral area} = \tfrac{1}{2} \text{ sum of perimeters} \times \text{slant height}. \quad \text{Q.E.D.}$$

Theorem V.

846. *The lateral area of a right cone is equal to half its slant height into the circumference of its base.*

Proof. Let $V\text{-}ABCDEF$ be the right cone. Inscribe in it a pyramid having a regular base of any number of sides. Then—



1. Because the edges of the pyramid coincide with the elements of the cone, they are all equal and the pyramid is regular. Therefore

$$\text{Lateral area of pyramid} = \tfrac{1}{2} \text{ slant height} \times \text{perimeter of base}.$$

Let the number of sides be indefinitely increased. Then—

2. The perimeter of the base of the pyramid will approach the circumference of the base of the cone as its limit.

3. Therefore the slant height of the pyramid will approach the slant height of the cone as its limit.

4. The surface of the pyramid will approach the surface of the cone as its limit. Therefore

Lateral area of cone $= \frac{1}{2}$ slant height $\times$ circumf. of base.

Q.E.D.

847. *Corollary* 1. *The lateral area of the frustum of a right cone is equal to the product of its slant height into half the sum of the circumference of its bases.*

Cor. 2. If the frustrum of the pyramid be cut midway by a plane parallel to the base, the section of each face will be half the sum of the bottom and top edges (§ 170). Hence the perimeter of the section will be half the sum of the perimeter of its bases. The same will be true of the cone. Hence:

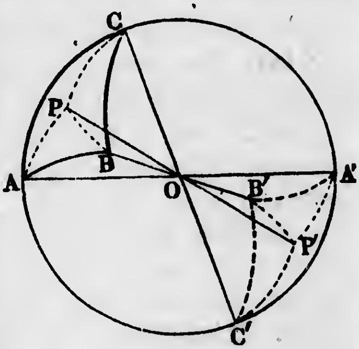
848. *The lateral area of a frustum of a cone is equal to its slant height into the circumference of its mid-section.*

Spherical Areas.

Theorem VI.

849. *Two symmetrical spherical triangles are equal in area.*

Proof. Let ABC and $A'B'C'$ be the two symmetrical triangles placed opposite each other (§786); P, the pole of the circle of the sphere passing through A, B, and C. From P draw the diameter POP'. Then—



1. Because P is the pole of the circle through A, B, and C,

Arc $PA =$ arc $PB =$ arc PC. (§ 745)

2. Because A', B', C', and P' are opposite points to A, B, C, and P, respectively,

$$P'A' = PA, \quad P'B' = PB, \quad P'C' = PC;$$

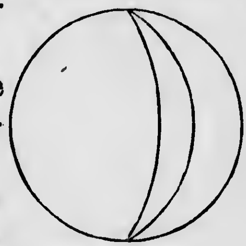
whence $$P'A' = P'B' = P'C'.$$

3. Therefore APC and $A'P'C'$, BPA and $B'P'A'$, CPB and $C'P'B'$, are respectively isosceles symmetrical triangles and identically equal (§§ 784, 792).

4. Because the sum of the three triangles APC, etc., makes up ABC, and the sum of the three equal triangles $A'P'B'$, etc., makes up $A'B'C'$, therefore

Area $A'B'C'$ = area ABC. Q.E.D.

850. *Def.* A **lune** is a portion of the surface of a sphere bounded by two great semicircles.


A lune.

851. *Def.* The angle of the lune is the angle between the great semicircles which bound it.

Corollary. The angle of a lune is equal to the dihedral angles between the planes of its bounding semicircles (§ 774).

Theorem VII.

852. *On the same sphere, or on equal spheres, lunes of equal angles are identically equal.*

Proof. Let the two spheres be applied so that their centres shall coincide. Then, because they are equal, their surfaces will also coincide.

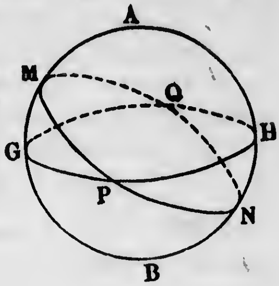
Turn one sphere round on its centre so that the vertex and one semicircle of its lune shall coincide with the corresponding vertex and semicircle of the other.

Because the angles are identically equal, the planes of the two semicircles will then coincide, and therefore the bounding semicircles will also coincide. The lunes are therefore identically equal, by definition. Q.E.D.

Theorem VIII.

853. *When three great circles mutually intersect, the areas of the two triangles on opposite sides of any vertex are together equal to the area of a lune having the angle formed at the vertex.*

Hypothesis. AB, GH, MN, any three great circles; P, any vertex where two of them cross.



Conclusion.

Areas $PGM + PHN =$ lune $PHQNP$.

Proof. 1. Because PMG and QNH are opposite triangles formed between the same great circles, they are symmetrical triangles (§ 790).

Therefore

Area PMG = area QNH.

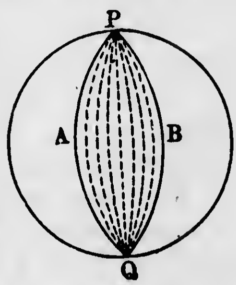
2. Therefore

Area PNH + area PMG = area $PNH + HNQ$,
= area of lune $PHQNP$. Q.E.D.

THEOREM IX.

854. *The area of a lune is to the whole surface of the sphere as the angle of the lune to a circumference.*

Proof. 1. Let $PAQB$ be a lune of which P and Q are the vertices.



From the vertex P to Q pass n semicircles making equal angles with each other. The whole surface of the sphere will then be divided into n equal lunes. (§ 852)

2. Let m be the number of these equal lunes contained in the given lune $PAQB$. The ratio of this lune to the surface of the sphere will then be $m : n$.

3. The angle APB is made up of m angles, of which n are required to make up the whole circumference around P.

Therefore the ratio of the angle APB to a circumference is $m : n$.

4. Because these ratios are equal however great the numbers m and n, they remain equal when the angle of the lune and the circumference are incommensurable. Therefore

Area of lune : surface of sphere :: angle of lune : circumf.

Q.E.D.

855. *Corollary.* 1. If there are lunes of angles A, B, etc., we have

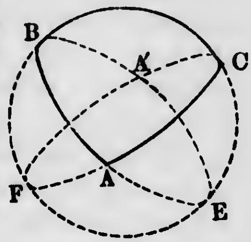
$$\text{Area (lune } A + \text{lune } B + \text{etc.)} = \text{area lune } (A + B + \text{etc.}).$$

856. *Cor.* 2. The area of a hemisphere is that of a lune of 180°.

Theorem X.

857. *The area of a spherical triangle is proportional to its spherical excess.*

Proof. Let ABC be the triangle. Continue any one side, as BC, so as to form the great circle $BCEF$. Continue the other two sides till they meet this circle in E and F. Then—



1. Area ABC + area ACE = lune $BAEC$ = lune of angle B.

Area ABC + area ABF = lune $CAFB$ = lune of angle C.

Area ABC + area AFE = lune $ACA'B$ = lune of angle A. (§ 853)

2. Because $BCEF$ is a great circle, the sum of the four areas ABC, ACE, ABF, and AFE is a hemisphere. Therefore, if we add the three equations (1), we have

2 area ABC + hemisphere = lune angle $(A+B+C)$. (§ 855)

3. Because the hemisphere is the same as a lune of 180° (§ 856), we may write the last equation

$$2 \text{ area } ABC + \text{lune } 180° = \text{lune angle } (A + B + C);$$

whence, by transposition,

$$2 \text{ area } ABC = \text{lune } (A + B + C - 180°)$$

and $$\text{Area } ABC = \tfrac{1}{2} \text{ lune } (A + B + C - 180°).$$

4. $A + B + C - 180°$ is, by definition, the spherical excess of the triangle ABC. Because the area of the lune is proportional to its angle, the area ABC is proportional to the same angle or to $A + B + C - 180°$. Q.E.D.

Corollary. If the three vertices of a triangle are on one great circle, its three sides will coincide with that circle, and each of its angles will be 180°. Its area will then be a hemisphere, and its spherical excess $3.180° - 180° = 360°$.

Doubling these quantities, we have the area of the whole sphere corresponding to a spherical excess of 720°. Hence

858. *The area of a spherical triangle is to that of the whole sphere as its spherical excess is to 720°.*

Scholium. Every spherical triangle divides the surface of the sphere into two parts, of which one may be considered within the triangle, and the other without it. We may consider either of those parts as the area of the triangle, if, in applying the preceding proposition, we measure the angles through that part of the spherical surface whose areas we are considering. If the inner angles are A, B, and C, the angles measured on the outer surface will be $360° - A$, $360° - B$, and $360° - C$ (cf. §§ 25–27). Subtracting 180° from the sum of these three angles gives $900° - (A + B + C)$ as the *outer* spherical excess. If the inner area becomes indefinitely small, $A + B + C$ will approach to 180°, and the outer spherical excess will approach to $900° - 180° = 720°$, which is therefore the spherical excess for the outer angles of the triangle whose outer area is that of the whole sphere.

859. *Def.* A **zone** is that portion of the surface of a sphere contained between two parallel circles of the sphere.

860. *Def.* The **altitude** of a zone is the perpendicular distance between the planes of its bounding circles.

THEOREM XI.

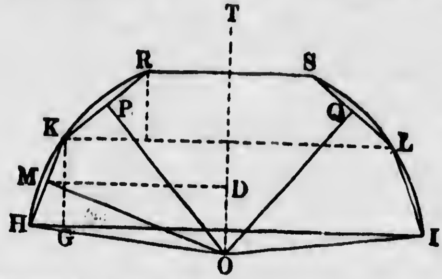
861. *The area of a zone is equal to the product of its altitude by the circumference of a great circle.*

Proof. Let $HKRSLI$ be a zone of a sphere whose centre is at O, and let the plane of the paper be a section of the sphere through the centre, perpendicular to the base HI of the zone.

The zone is then generated by the motion of the arc HKR around the axis OT, perpendicular to its base.

In the arc HKR inscribe the chords HK, KR. Let M be the middle point of HK. Join OM, and from M drop the perpendicular MD upon OT. Then—

1. By the revolution around the axis OT, the chord HK will describe the lateral surface of the frustrum of a cone of



which the area will be $HK \times$ circumference of a circle of which MD is the radius (§ 848). That is,

$$\text{Area of frustrum} = 2\pi MD.HK. \qquad (\S\, 484)$$

2. Because OMK is a right angle,

$$\text{Angle } OMD = \text{comp. } KMD = \text{comp. } KHI;$$

also,

$$\text{Angle } G = \text{angle } D = \text{right angle.}$$

Therefore the triangles OMD and HKG are similar, and

$$HK : KG :: MO : MD;$$

whence

$$MD.KH = MO.KG.$$

3. Hence, from (1),

$$\text{Area of frustrum} = 2\pi.OM.KG.$$

4. In the same way,

Area of frustrum $KRSL = 2\pi OP \times$ alt. of frustrum.

Inscribe in the arc HKR an indefinite number of equal chords and consider the frustra they describe. Then—

5. The perpendiculars OM, OP, etc., will approach the radius of the sphere as their limit.

6. The sum of the lateral surfaces of all the frustra will approach the surface of the zone as their limit.

7. Because the area of each frustrum approaches the limit

$$2\pi \times \text{radius of sphere} \times \text{alt. of frustrum},$$

the sum of all the areas will approach the limit

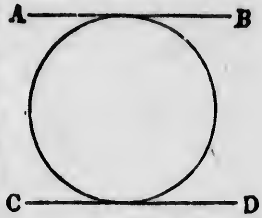
$$2\pi \text{ radius of sphere} \times \text{sum of altitudes of frustra}$$
$$= 2\pi \text{ radius of sphere} \times \text{alt. of zone},$$

which limit is the area of the zone.

8. But 2π radius of sphere = circumf. of great circle. Hence:

Area of zone = alt. of zone × circ. of great circle. Q.E.D.

Corollary 1. Let AB and CD be two parallel tangent planes to a sphere. Since the preceding theorem is true of zones of all altitudes, it will remain true how near soever we suppose the bases of the zones to the tangent planes. If, then, we suppose the bases of the zones to approach the tangent planes as their limit, the altitude of the zone will approach to the diameter of the sphere, and its surface to the surface of the sphere. Hence:



862. *The entire surface of a sphere is equal to the product of its diameter into its circumference.*

Cor. 2. If we put r for the radius of the sphere, we have

$$\text{Diameter} = 2r,$$
$$\text{Circumference} = 2\pi r;$$

whence

$$\text{Surface} = 4\pi r^2.$$

Now we have found the area of a circle of radius r to be πr^2 (§ 480). Hence:

863. *The area of the surface of a sphere is equal to the area of four great circles.*

The area of a hemisphere is equal to twice that of a great circle.

CHAPTER II.

VOLUMES OF SOLIDS.

864. *Def.* The **volume** of a solid is the measure of its magnitude.

865. *Def.* The **base** of a solid is that one of its faces which we select for distinction.

866. *Def.* The **altitude** of a solid is the perpendicular dropped from its highest point upon the plane of the base.

867. A **right parallelopiped** is a parallelopiped of which one pair of opposite faces are perpendicular to the other four faces.

REMARK. A right parallelopiped differs from a rectangular one (§ 685) in that two of the faces of the former may make any angle with each other, whereas the rectangular parallelopiped has all its faces perpendicular to each other.

868. *Def.* The **unit of volume** is the volume of a cube of which each edge is the unit of length.

Volumes of Polyhedrons.

THEOREM XII.

869. *Right prisms having equal altitudes and identically equal bases are identically equal.*

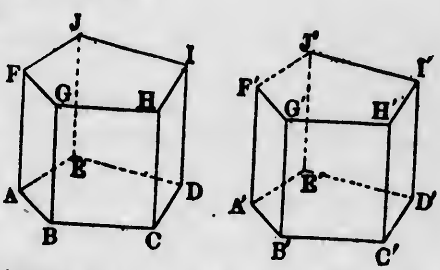
Hypothesis. $ABCDE\text{-}FGHIJ$ and $A'B'C'D'E'\text{-}F'G'H'I'J'$, two right prisms in which

$$\text{Base } ABCDE = \text{base } A'B'C'D'E' \text{ (identically)}$$

and

$$\text{Alt. } AF = \text{alt. } A'F'.$$

Conclusion. The two prisms are identically equal.



Proof. Apply the bases to each other so that they shall wholly coincide. Let $A' \equiv A$, $B' \equiv B$, etc. Then—

1. Because AF and $A'F'$ are each perpendicular to the base, they will coincide (§ 585).

2. Because $AF = A'F'$,

$$\text{Point } F \equiv \text{point } F'.$$

3. In the same way, every vertex of the one will coincide with a corresponding vertex of the other.

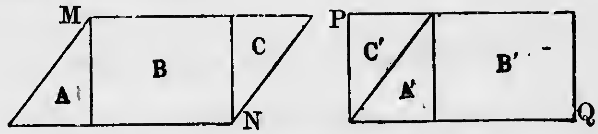
4. Therefore every edge of the one will coincide with a corresponding edge of the other.

5. Therefore every face of the one will coincide with the corresponding face of the other, whence the figures will be identically equal. Q.E.D.

THEOREM XIII.

870. ***Right prisms having equal bases and altitudes are equal in volume.***

Hypothesis. MN and PQ, two equal bases of right prisms having equal altitudes.



Conclusion. These prisms are equal in volume.

Proof. By the definition of equal magnitudes (§13) the hypothesis implies that the bases can be divided into parts such that each part of the one base shall be identically equal to a corresponding part of the other base.

Let A, B, and C be the parts of MN, and A', B', and C' the corresponding identically equal parts of PQ.

Let each prism be divided into smaller prisms by planes perpendicular to the base, and intersecting it along the bounding lines which divide the bases into the parts A, B, C, A', etc. Then, because the bases A and A' are identically equal and have equal altitudes,

Prism on base A = prism on base A' *identically*. (§869)

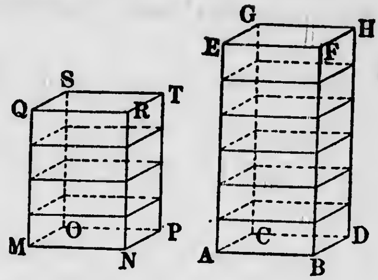
In the same way, each part of the one prism is identically equal to a corresponding part of the other.

Therefore the two prisms, being made up of these equal parts, are identically equal. Q.E.D.

THEOREM XIV.

871. ***The volumes of right prisms having equal bases are proportional to their altitudes.***

Hypothesis. AH, MT, two right prisms in which the base $ABCD$ is equal to $MNOP$, and the altitude AE is to the altitude MQ as $m : n$.



Conclusion.

Vol. AH : vol. MT :: m : n.

Proof. Divide the prism AH into m parts of equal altitude by planes parallel to the base. Let prism MT be divided into n parts in the same way. Then—

1. Because the altitude AE is to MQ as $m : n$, and because AE is divided into m parts, and MQ into n parts, these parts are all equal (§ 337).

2. Because all the m parts of the prism AH and all the n parts of the prism MT are prisms having identically equal bases and altitudes, they are equal in volume (§ 869).

3. Because the volume MT is composed of n parts, of which m parts make up the volume AH, therefore

Volume AH : volume MT :: m : n. Q.E.D.

4. Because this is true how great soever the numbers m and n, it remains true for all cases (§ 359).

Theorem XV.

872. *In right prisms of equal altitudes the volumes are to each other as the areas of their bases.*

Proof. Let the bases be to each other as the number m to the number n. This means that if the one base be divided into m equal parts, n of these parts will make up the other base.

On each of these $m + n$ parts erect a prism equal in altitude to the given prisms.

These prisms will all be equal (§ 870).

One given prism will be made up of m of these equal prisms, and the other of n of them.

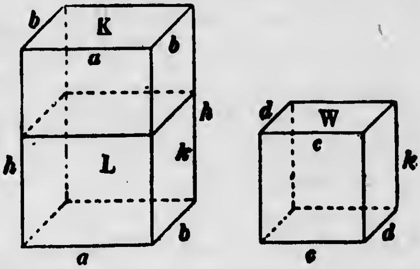
Therefore the volumes will be to each other as m to n; that is, the volumes will be as the areas of the bases. Q.E.D.

Corollary. Because a parallelopiped is a prism, we conclude:

873. *If a right prism and a right parallelopiped have equal altitudes, their volumes are as the areas of their bases.*

Theorem XVI.

874. *The volumes of rectangular parallelopipeds are proportional to the products of their three dimensions.*



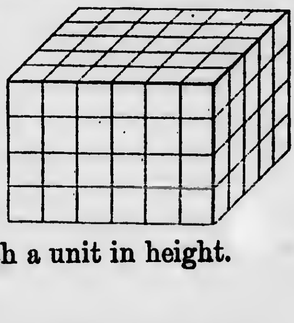
Proof. Let K be a parallelopiped of which the dimensions are a, b, and h; W, one of which the dimensions are c, d, and k.

Cut off from K a parallelopiped L of altitude k. Then—

1. Because L and W have equal altitudes,
   Vol. L : vol. W :: area $a.b$: area $c.d$, (§ 872)
   :: ab : cd. (§ 416)
2. Because K and L have equal bases,
   Vol. K : vol. L :: alt. h : alt. k. (§ 871)
3. Multiplying these ratios,
   Vol. K : vol. W :: abh : cdk. Q.E.D.

Corollary 1. If the dimensions c, d, and k of the volume W are each unity, the product cdk will be unity, and W will be the unit of volume (§ 868). The above conclusion will then become Vol. K : 1 :: abh : 1,
which gives Vol. $K = abh$;
that is:

875. *The volume of a rectangular parallelopiped is measured by the continued product of its three dimensions.*



Scholium. If the dimensions a, b, and g are all whole numbers, this result may be shown in the following simple way:

Being g units in height, it may be divided up into g layers each a unit in height.

Being b units in breadth, each of these g layers may be divided into b rows, each containing a units of volume.

Thus the total number of units of volume will be abg.

Cor. 2. Because the base of a rectangular parallelopiped is a rectangle, its area is equal to the product of its two dimensions. The third dimension is then the altitude of the parallelopiped. The preceding result may therefore be expressed in the form:

876. *The volume of a rectangular parallelopiped is expressed by the product of its base into its altitude.*

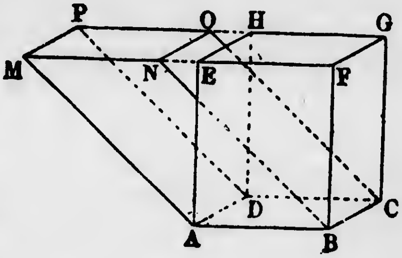
Cor. 3. Since a rectangular parallelopiped is a kind of right prism, every right prism is, by § 870, equal in volume to a rectangular parallelopiped having an equal base and altitude. Therefore we conclude:

877. *The volume of every right prism is expressed by the product of its base and altitude.*

Theorem XVII.

878. *All parallelopipeds having the same base and equal altitudes are equal in volume.*

Case I. *Hypothesis.* $ABCD$-$EFGH$ and $ABCD$-$MNOP$, two parallelopipeds having the same base $ABCD$ and equal altitudes.



In this case the edges FE and NM, also GH and OP, are supposed to lie in straight lines.

Proof. 1. Because EF and MN are each equal and parallel to AB, we have

$$MN = EF$$

and

$$ME = NF.$$

2. Considering the two triangular prisms AEM-DHP and BFN-CGO, it is proved, from the equality and parallelism of all their parts, that they are identically equal.

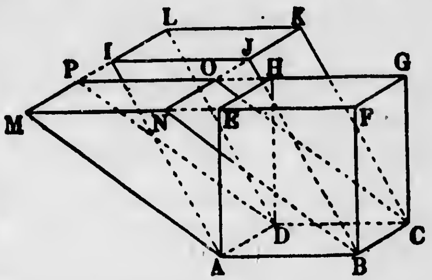
3. From the solid *ABFM-DCGP* take away the solid *AEM-DHP*, and we have left the parallelopiped *ABCD-FFGH.*

4. From the same solid *ABFM-DCGP* take away the equal solid *BFN-CGO*, and we have left the parallelopiped *ABCD-MNOP.*

5. Therefore

Volume *ABCD-EFGH* = volume *ABCD-MNOP.* Q.E.D.

Case II. Let the upper base be in any position, as *IJKL.*



Produce the parallel edges *FE* and *GH* to the points *M* and *P*, and produce *KJ* and *LI* to *N* and *M*, forming the parallelogram *MNOP*. Join *AM*, *BN*, *CO*, *DP*, forming the parallelopiped *ABCD-MNOP*. Then—

6. Because the parallelopiped *ABCD-IJKL* has the same base and altitude as *ABCD-MNOP*, and has the bounding edges *JK* and *IL* of its upper face in the same straight line with the edges *PM* and *ON*, we have, by Case I.,

Vol. *ABCD-MNOP* = vol. *ABCD-IJKL.*

7. We have, for the same reason,

Vol. *ABCD-EFGH* = vol. *ABCD-MNOP.*

8. Comparing with (6),

Vol. *ABCD-EFGH* = vol. *ABCD-IJKL.* Q.E.D.

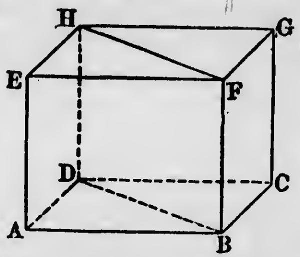
Corollary. Whatever be the oblique parallelopiped *ABCD-IJKL*, we may construct upon the same base a right parallelopiped *ABCD-EFGH* to which the above demonstration will apply. Therefore, from (§ 877):

879. *The volume of any parallelopiped is equal to the product of its base and altitude.*

Theorem XVIII.

880. *A diagonal plane divides any parallelopiped into two triangular prisms of equal volume.*

Case I. *Hypothesis.* *ABCD-EFGH, a right parallelopiped,* of which *BDHF* is a diagonal plane.



Conclusion.

Vol. *ABD-EFH* = vol. *DBC-HFG.*

Proof. 1. Because *ABCD* is a parallelogram, the diagonal *BD* divides its area into two equal parts.

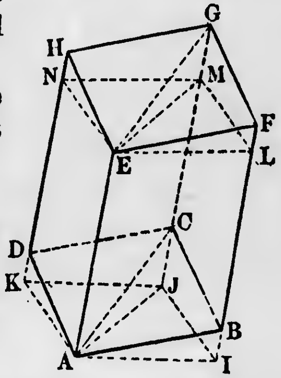
2. Therefore the two right prisms *ABD-EFH* and *CBD-GFH* have equal bases and the same altitude *BF.*

3. Therefore these prisms are equal (§ 870). Q.E.D.

Case II. *Hypothesis.* *ABCD-EFGH,* any parallelopiped of which *ACGE* is a diagonal plane.

Conclusion. Vol. *ACD-EHG* = vol. *ACB-EGF.*

Proof. Through the vertices *A* and *E* pass the planes *AIJK* and *ELMN* perpendicular to the parallel edges *AE, BF, CG,* and *DH.*



Let *I, J, K* and *L, M, N* be the points in which the cutting planes meet these edges, produced when necessary. Then—

1. Because the cutting planes are perpendicular to the same edges *AE,* etc., they are parallel. Therefore the solid *AIJK-ELMN* is a right parallelopiped, and

Vol. *AJK EMN* = vol. *AJI-EML.* (Case I.)

2. Because the edges of both parallelopipeds parallel to *AE* are also equal to it (§ 687, cor. 1),

$$HN = KD; \quad GM = CJ; \quad FL = BI;$$

also, by comparing the sides of the parallelograms,

$$EH = AD; \quad EN = AK; \quad EG = AC, \text{ etc.};$$

and because *EMG* and *AJC* are both right angles, by construction,

Angle *EMG* = angle *AJC.*

3. Therefore if the solid $E\text{-}MGHN$ be applied to the solid $A\text{-}JCDK$ so that the line EM shall fall on AJ, then

$$\begin{aligned} \text{Triangle } EMN &\equiv AJK, \\ MG &\equiv JC, \\ EG &\equiv AC, \\ NH &\equiv KD, \\ GH &\equiv CD. \end{aligned}$$

Therefore these two solids are identically equal.

4. If from the prism $ACD\text{-}EGH$ we take away the solid $E\text{-}MGHN$ and add the equal solid $A\text{-}JCDK$, we shall have the right prism $AJK\text{-}EMN$. Therefore

$$\text{Vol. } ACD\text{-}EGH = \text{vol. } AJK\text{-}EMN.$$

5. In the same way may be shown,

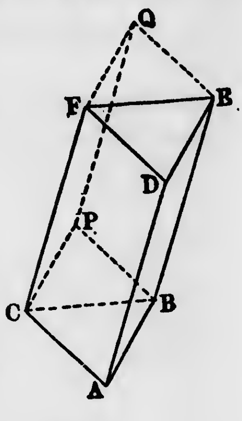
$$\text{Vol. } ABC\text{-}EFG = \text{vol. } AIJ\text{-}ELM.$$

6. Comparing with Case I.,

$$\text{Vol. } ACD\text{-}EGH = \text{vol. } ABC\text{-}EFG. \quad \text{Q.E.D.}$$

Theorem XIX.

881. *The volume of any prism is equal to the product of its base by its altitude.*



Case I. *A triangular prism.*

Proof. Let $ABC\text{-}DEF$ be any triangular prism. Draw

$$BP \parallel AC; \quad CP \parallel AB;$$
$$EQ \parallel DF; \quad FQ \parallel DE.$$

Then—

1. Because $ABPC$ and $DEQF$ are, by construction, equal parallelograms with the sides of the one parallel to the corresponding sides of the other, the solid $ABPC\text{-}DEQF$ is a parallelopiped. Therefore

$$\text{Vol. } ABPC\text{-}DEFQ = \text{base } ABPC \times \text{altitude}.$$

2. Because the bases are parallelograms,

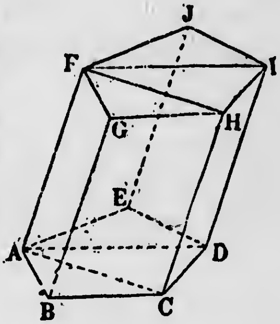
$$\text{Area } ABC = \tfrac{1}{2} \text{ base } ABPC.$$

3. Because $BCFE$ is a diagonal plane of the parallelopiped,

$$\begin{aligned} \text{Vol. } ABC\text{-}DEF &= \tfrac{1}{2} \text{ vol. } ABPC\text{-}DEQF, && (\S\ 880) \\ &= \tfrac{1}{2} \text{ base } ABPC \times \text{altitude}, && (1) \\ &= \text{area } ABC \times \text{altitude}. && (2) \end{aligned}$$

Q.E.D.

CASE II. *Any prism.*



Let $ABCDE\text{-}FGHIJ$ be any prism. Divide the prism into triangular prisms by passing planes through $ACHF$, $ADIF$, etc. These planes will divide the bases into triangles. Then—

1. Because $ABC\text{-}FGH$ is a triangular prism,

Vol. $ABC\text{-}FGH$ = base ABC × alt. of prism.

2. In the same way,

Vol. $ACD\text{-}FHI$ = base ACD × alt. of prism.
Vol. $ADE\text{-}FIJ$ = base ADE × alt. of prism.
etc. etc.

3. Adding these volumes, we have

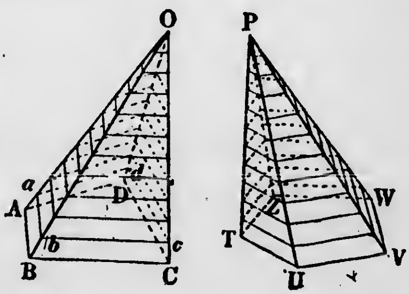
Sum of volumes = sum of bases × alt. of prism.

4. The sum of these volumes makes up the whole volume of the prism, and the sum of the triangular bases makes up the whole base of the prism.

Therefore volume of prism = base × altitude. Q.E.D.

THEOREM XX.

882. *All pyramids having equal bases and equal altitudes are equal in volume.*



Hypothesis. $O\text{-}ABCD$ and $P\text{-}TUVWZ$, two pyramids in which area $ABCD$ = area $TUVWZ$, and altitude of O = altitude of P.

Conclusion. The volumes of the two pyramids are equal.

Proof. Divide each pyramid into the same number n of slices by equidistant planes parallel to the base. Let us put

s, the thickness of each slice;
a, the common altitude of each pyramid;
b, the area of the base of each pyramid.

Then—

1. Because the altitude is divided into n parts, the thickness of each slice will be

$$s = \frac{a}{n}.$$

2. Because the number of slices is the same in each pyramid, the distances of corresponding slices from the vertex will be the same in the two pyramids. If we put

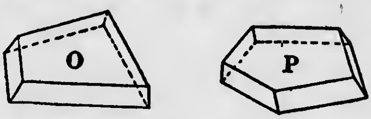
l, the distance of any section from the vertex,
r, the area of the section,

we shall have in each pyramid

$$r : b :: l^2 : a^2. \qquad (\S\ 700)$$

Also, the areas of corresponding sections are the same in the two pyramids (§ 701).

3. Let O and P be two corresponding slices from the same pyramid. Put



r', the area of each upper base;
r, the area of each lower base.

Because the lower base of each is greater than the upper base, each slice is greater than the prism of altitude s and base r', but less than the prism of altitude s and base r. Hence

$$s \times r' < \text{volume of each slice} < s \times r.$$

4. Hence the difference between the volumes of corresponding slices of the two pyramids must be less than $sr - sr'$; that is, less than $s(r - r')$.

5. Let us call the areas of the several sections from the vertex to the base r_1, r_2, r_3, etc. We then have

Difference of top slices	$< sr_1$.
Difference of second slices	$< s(r_2 - r_1)$.
Difference of third slices	$< s(r_3 - r_2)$.
etc.	etc.
Difference of bottom slices	$< s(r_n - r_{n-1})$.

Adding up all these differences, and noticing that $r_n = b$, we find

$$\text{Difference of volumes of pyramids} < sb.$$

That is:

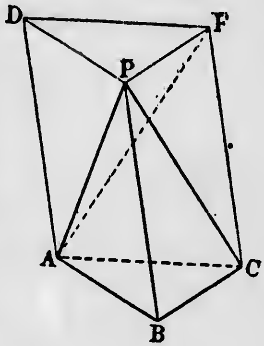
The difference of the volumes of the pyramids is less than the volume of a prism of equal base, and having for its altitude the thickness of a slice.

6. But we may take the slices so thin that this volume sb shall be less than any assignable quantity. Therefore the volumes of the pyramids differ by less than any assignable quantity; that is, they do not differ at all. Q.E.D.

Theorem XXI.

883. *The volume of a pyramid is one third the volume of a prism having the same base and altitude.*

Case I. Let $P\text{-}ABC$ be a triangular pyramid.



Through AC pass a plane $ACFD$ parallel to the opposite edge BP.

Complete the triangular prism $ABC\text{-}DPF$ by drawing the edges PD, PF, DF, AD and CF parallel to BA, BC, AC, BP.

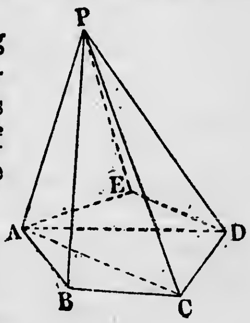
Divide the quadrangular pyramid $P\text{-}ACFD$ into two triangular prisms by the plane PAF. Then—

1. Because $ACFD$ is a parallelogram, the areas ADF and ACF are equal. Therefore

$$\text{Vol. } P\text{-}ADF = \text{vol. } P\text{-}ACF \text{ (§ 882).}$$

2. In the same way, considering PBC and PFC as the equal bases of two triangular pyramids having their vertices at A,

$$\text{Vol. } A\text{-}PBC = \text{vol. } A\text{-}PFC.$$



3. Comparing (1) and (2), and noting $P\text{-}ACF$ and $A\text{-}PFC$ are the same pyramid, we see that the prism $ABC\text{-}DEF$ is divided into three equal pyramids, of which one is the original pyramid. Hence

$$\text{Vol. } P\text{-}ABC = \tfrac{1}{3} \text{ vol. } ABC\text{-}DEF.$$

Q.E.D.

Case II. $P\text{-}ABCDE$, any pyramid.

Through P pass the planes PAC,

PAD, etc., dividing the pyramid into the triangular pyramids *P-ABC*, *P-ACD*, etc.

Let a be the altitude of the pyramid. Then, by Case I.,

$$\text{Vol. } P\text{-}ABC = \tfrac{1}{3} \text{ prism } ABC\text{-alt. } a.$$
$$\text{Vol. } P\text{-}ACD = \tfrac{1}{3} \text{ prism } ACD\text{-alt. } a.$$
$$\text{Vol. } P\text{-}ADE = \tfrac{1}{3} \text{ prism } ADE\text{-alt. } a.$$

The sum of these pyramids makes up the given pyramid, and the sum of the prisms is a prism having the base *ABCDE* and the altitude a. Therefore, adding,

$$\text{Vol. } P\text{-}ABCDE = \tfrac{1}{3} \text{ prism } ABCDE\text{-alt. } a.$$

Q.E.D.

Corollary. Because the volume of a prism is equal to the product of its base by its altitude, we conclude:

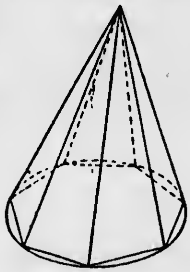
884. *The volume of a pyramid is one third the product of its base by its altitude.*

Volumes of Round Bodies.

Theorem XXII.

885. *The volume of a cone is equal to one third the product of its base by its altitude.*

Proof. In the base of the cone inscribe a regular polygon of any number of sides, and upon it erect a pyramid of which the vertex shall be in the vertex of the cone. Then—



1. Because the angles of the base of the pyramid are on the surface of the cone, and its vertex in the vertex of the cone, the lateral edges of the pyramid will lie on the conical surface, and its altitude will be equal to the altitude of the cone.

Let us call a the common altitude of cone and pyramid. Then—

2. Volume of pyramid $= \frac{1}{3} a \times$ base of pyramid.

3. Let the number of sides of the base of the pyramid be indefinitely increased. Then the base of the pyramid will approach the base of the cone as its limit, and its volume will approach the volume of the cone as its limit. Therefore

Volume of cone $= \frac{1}{3} a \times$ base of cone. Q.E.D.

THEOREM XXIII.

886. *The volume of a cylinder is equal to the product of its base by its altitude.*

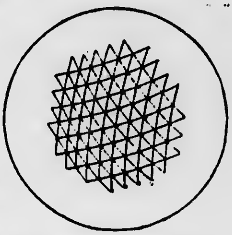
Proof. Inscribe in the cylinder a prism of which the number of sides may be increased without limit. Then the base of the prism will approach the base of the cylinder as its limit, and the volume of the prism will approach the volume of the cylinder as its limit.

Because the volume of the prism is continually equal to the product of its base by its altitude, the volume of the cylinder must also be equal to the product of its base by its altitude. Q.E.D.

THEOREM XXIV.

887. *The volume of a sphere is equal to one third its radius into the area of its surface.*

Proof. Make a great number of points on the surface of the sphere, and join them by arcs of great circles so as to divide the whole surface into spherical triangles.



The planes of these arcs will form the lateral faces of triangular pyramids having their vertices in the centre of the sphere, and the angles of their bases resting upon the surface.

Because the volume of each pyramid is $\frac{1}{3}$ base $\times$ altitude, the combined volume of all is

$$\tfrac{1}{3} \text{ sum of bases} \times \text{altitude}.$$

Let the number of spherical triangles be indefinitely increased. Then the sum of the bases of all the pyramids will approach the surface of the sphere as its limit, and the altitudes will all approach the radius of the sphere as their limit. Therefore

$$\text{Volume of sphere} = \tfrac{1}{3} \text{ radius} \times \text{surface of sphere}. \quad \text{Q.E.D.}$$

Corollary. We have found (§ 862) for a sphere of radius r,

$$\text{Surface of sphere} = 4\pi r^2.$$

Multiplying this by $\frac{1}{3}r$, we have

$$\text{Volume of sphere} = \tfrac{4}{3}\pi r^3.$$

This result admits of being memorized in the following way. Suppose a cube circumscribed about the sphere. Because each of its edges is $2r$, its volume will be $8r^3$. Comparing with the expression for the volume of the sphere, we find

$$\text{Vol. sphere} : \text{vol. cube} :: \tfrac{1}{3}\pi : 2.$$

Now if π were exactly 3, this ratio would be $1 : 2$; that is, the volume of the sphere would be one half that of the cube. And in reality the sphere is greater than half the cube in the same ratio that π is greater than 3, which is nearly $\frac{1}{21}$ part (§ 484).

Therefore if we fit a sphere into a cubical box, it will occupy a little more than half the volume of the box.

PROBLEMS OF COMPUTATION.

1. The altitude of a right cone is 4 metres, and the diameter of its base 6 metres. Compute its slant height, lateral surface, area of base and volume.

Ans. Slant height, 5 m.; lateral surface, $\frac{45\pi}{2}$; area of base, 9π; volume, 12π.

2. The lateral area A of a right cone being given, what relation must subsist between its altitude a and the diameter D of its base?

$$\text{Ans.}\quad \tfrac{1}{2}\pi D\sqrt{(a^2+\tfrac{1}{4}D^2)} = A.$$

3. The lateral surface of a right cone is double the area of its base. What is the ratio of its altitude to the radius of its base? What must be the ratio in order that the lateral surface may be n times the area of the base?

$$\text{Ans.}\quad \sqrt{3} \text{ and } \sqrt{n^2-1}.$$

4. Find the ratio of the volume of a sphere to that of its right circumscribed cylinder.

Ans. Vol. of sphere $= \frac{2}{3}$ vol. of cylinder.

NOTE. The circumscribed cylinder is that whose bases and elements are all tangents to the sphere.

5. The slant height of a right cone = diameter of its base $= 2a$. Express its altitude, lateral area, and volume, and the radius, surface, and volume of its inscribed and circumscribed spheres.

Ans. Alt. of cone, $\sqrt{3}a$; lateral area, $2\pi a^2$; volume, $\frac{\pi a^3}{\sqrt{3}}$;

Rad. of insc. sphere, $\frac{a}{\sqrt{3}}$; surface, $\frac{4\pi a^2}{3}$; volume, $\frac{4\pi a^3}{9\sqrt{3}}$;

Rad. of circ. sphere, $\frac{2a}{\sqrt{3}}$; surface, $\frac{16\pi a^2}{3}$; volume, $\frac{32\pi a^3}{9\sqrt{3}}$.

6. The radius of a sphere is bisected at right angles by a plane. What is the ratio of the two parts into which the plane divides the spherical surface?

Ans. 3 : 1.

7. If a plane cut a cylinder at an angle of 45° with the elements, what will be the ratio of the axes of the ellipse of intersection?

Ans. $\sqrt{2} : 1$.

THEOREMS FOR EXERCISE

IN

GEOMETRY OF THREE DIMENSIONS.

BOOK VIII.

1. A line parallel to each of two intersecting planes is parallel to their line of intersection.

2. Two lines, one perpendicular to one plane and one to another plane, form equal angles with the planes to which they are not perpendicular.

3. If a straight line be perpendicular to a plane, every line perpendicular to that line is parallel to the plane.

4. The supplement of any face angle of a trihedral angle is less than the sum, but greater than the difference of the supplements of the two other face angles.

5. If, on a line intersecting a plane perpendicularly, two points, A and B, equally distant from the plane be taken, and these points be joined to three or more points of the plane, the joining lines will form the edges of two symmetric polyhedral angles having their vertices at A and B.

6. If a plane be passed through one of the diagonals of a parallelogram, the perpendiculars upon it from the extremities of the other diagonal are equal.

7. If the intersections of several planes are parallel, all perpendiculars upon these planes from the same point in space lie in one plane.

8. If any number of planes are respectively perpendicular to as many lines, and these lines all lie in one plane, or in parallel planes, the lines of intersection of the planes are all parallel to each other.

9. All points whose projections upon a plane lie in a straight line are themselves in one plane. How is this plane defined?

10. If two straight lines are on opposite sides of a plane, parallel to it, and equally distant from it (but not parallel to

each other), the plane will bisect every line from any point of the one line to any point of the other.

11. If any two straight lines A and B are parallel to a plane P, all lines joining a point of A to a point of B are cut by the plane P, internally or externally, into segments having the same ratio.

12. *Corollary.* If through the ends of a harmonically divided line two planes be passed perpendicular to the line, and through the harmonic points of division two lines A and B be drawn, each parallel to the planes, but not in one plane, then every line joining a point of A to a point of B is cut harmonically by the two planes.

13. A plane parallel to two sides of a quadrilateral in space divides the other two sides similarly.

BOOK IX.

1. If any two non-parallel diagonal planes of a prism are perpendicular to the base, the prism is a right prism.

2. If the four diagonals of a quadrangular prism pass through a point, the prism is a parallelopiped.

3. A plane passing through a triangular pyramid, parallel to one side of the base and to the opposite lateral edge, intersects its faces in a parallelogram.

4. The four middle points of two pairs of opposite edges of a triangular pyramid are in one plane, and at the vertices of a parallelogram.

NOTE. The six edges of a triangular pyramid may be divided into three pairs, such that the two edges of a pair do not meet each other. Since each edge meets two other edges at one vertex, and two yet other edges at the adjoining vertex, there is but one edge left to pair with it. The pair is called a pair of opposite edges.

5. The three lines joining the middle points of the three pairs of opposite edges of a triangular pyramid intersect in a point which bisects them all.

6. The four lines joining the vertices of a triangular pyramid to the centres of the opposite faces intersect in a point which divides each of them in the ratio 1 : 3.

NOTE. The centre of a triangle is the point of intersection of its three medial lines (§§ 168, 169).

7. The middle points of the edges of a regular tetrahedron are at the vertices of a regular octahedron.

8. The eight vertices of a cube are cut off by eight planes, each passing through the middle points of the three edges which diverge from each vertex. Explain the structure of the polyhedron thus formed, giving the number, form, and relation of its faces, edges, and vertices.

Describe its sympolar polyhedron, showing that each face is a rhombus, and explain the number and form of its edges and vertices.

NOTE. The sympolar of any polyhedron may be formed by drawing an edge across each edge of the given polyhedron, and uniting all the edges crossing the sides of each face into a single vertex.

BOOK X.

1. If lines be drawn from any point of a spherical surface to the ends of a diameter, they will form a right angle.

2. *Conversely,* the locus of the point from which a finite straight line subtends a right angle is a spherical surface having the line for a diameter.

3. If any number of lines in space pass through a point, the feet of the perpendiculars from another point upon these lines lie upon a spherical surface.

4. If any number of lines in a plane pass through a point, the feet of the perpendiculars upon these lines from any point not in the plane lie on a circle.

5. If the axis of an oblique circular cone is equal to the radius of the base, every plane passing through the axis of the cone intersects the conical surface in lines forming a right angle at the vertex. When the axis of the cone is less than the radius of the base, all the angles thus formed are obtuse, and when greater they are acute.

6. All parallel lines tangent to the same sphere intersect any plane in an ellipse.

BOOK XI.

1. The surface of a sphere is equal to the lateral surface of its circumscribed cylinder.

NOTE. See Problem 4, p. 389.

2. If the slant height of a right cone is equal to the diameter of its base, its lateral area is double the area of its base.

3. The lateral area of a pyramid is greater than the area of its base.

4. The volume of a triangular prism is equal to the area of any lateral face into half the perpendicular from the opposite edge upon that face.

5. Any plane passing through the middle points of a pair of opposite edges of a triangular pyramid bisects its volume.

6. If the three face angles of a triangular pyramid around the vertex are all right angles, the square of the area of the base is equal to the sum of the squares of the areas of the three lateral faces.

7. The bisecting plane of any edge angle of a triangular pyramid divides the opposite edge into segments proportional to the areas of the adjacent faces.

8. Equidistant parallel planes intercept equal areas of a spherical surface.

LOCI.

1. Find the locus of the point in space whose distances from two fixed points are in a given ratio.

2. Find the locus of the point equally distant from two parallel lines.

3. Find the locus of the point equally distant from two intersecting straight lines.

4. Find the locus of the point equally distant from three given points.

5. Find the locus of the point equally distant from the sides of a triangle.

6. Find the locus of the point equally distant from the three edges of a trihedral angle.

7. Two given lines being on opposite sides of a plane parallel to it, and equidistant from it, find the locus of the point in the plane which is equally distant from the two lines.

8. Find the locus of the point from which two adjacent segments of the same straight line subtend equal angles.

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APPENDIX.

NOTES ON THE FUNDAMENTAL CONCEPTS OF GEOMETRY.

THE true basis and form of the fundamental axioms and definitions of Geometry have been the subject of extended discussion in recent times, especially among German mathematicians. The following summary of conclusions is given partly to show the direction towards which these discussions tend, and partly to explain the reasons for the particular forms of definitions and axioms adopted in the present work. Although the writer conceives that these views concur with the general conclusions of those who have investigated the subject, no one but himself is to be considered responsible for the form in which they are stated.

I. Geometry has its foundation in observation. Clear conceptions of lines, as straight or curved; and, in general, the idea of relative positions in space, could never be acquired except through the eye and touch. The ancient axioms of Geometry proper, such as the impossibility of two straight lines inclosing a space, the equality of all right angles, and the necessity of two non-parallel lines in the same plane ultimately meeting if sufficiently produced, are not to be regarded, as they once were, as necessary conclusions apart from all observation, but only as necessary results of certain conceptions derived from observation. It has in fact been shown that a perfectly consistent Geometry can be constructed in which the axioms relating to straight lines are not fulfilled.

II. The general concepts of Geometry—points, lines, surfaces, and solids—are to be regarded as attaching to material bodies rather than as formed of mere space. A geometric solid, for instance, is an imaginary material body from which all qualities except those of extension and mobility are abstracted. The quality of impenetrability being abstracted, any two bodies may occupy the same space and may be brought into absolute coincidence if they are identically equal in their outlines. Surfaces, again, should rather be considered as extensions from which the idea of thickness is abstracted than as extensions absolutely without thickness. Similarly, a line need not be regarded as having no thickness, but may simply be considered as having the idea of thickness abstracted. A point is an object the magnitude of which we take no account of.

This slight change of conception may perhaps be regarded as having little more than metaphysical interest. But it has a certain amount of practical value in releasing the young mind from a seeming necessity of conceiving portions of pure space as bodies and magnitudes with only one or two dimensions. In fact, it may be doubted whether any definitions of lines, points, and surfaces, in general, are of value to a young beginner. He naturally falls into the habit of applying the terms the right way.

III. The following considerations have led to certain of the primary definitions adopted in the present work.

1. It may be doubted whether a straight line admits of any definition in the proper sense of the term. A student who does not know what a straight line is before it is defined will not know in consequence of the definition. The author therefore lays no stress upon the definition he has adopted, which is perhaps objectionable, but which has been chosen because most readily understood by a beginner.

2. A great majority of our writers upon Elementary Geometry make the mistake of trying to include the mode in which the angle is measured in the definition of it. The system of enunciating separate definitions of the angle and the method of measuring it has been adopted from Chauvenet, and its advantages are so obvious that they need not be pointed out.

3. That identically equal magnitudes are those which coincide is properly not an axiom, as used in the older geometry, but a definition of the word "equal" and its derivatives. This will be obvious upon reflecting that the word must have some definition, and that all we can mean by it is that the two objects to which the term is applied coincide when brought together, or are made up of coincident parts. Had all bodies been immovable we should never have had the idea of equality.

4. A statement of what shall be meant by the sum of two magnitudes, and especially of two angles, is absolutely necessary. The want of such a statement is one of the most serious defects in the Geometry of Euclid. Had Euclid enunciated a general definition of the sum of two angles, and adhered to it, his thirteenth proposition, that the angles which one straight line makes with another upon one side of it are together equal to two right angles, would have been unnecessary.

5. That a straight line is the shortest distance between any two of its points is here considered an axiom rather than a definition.

The reason of placing it in this category is simply that the idea of a straight line may be derived independently of any comparison of general measures of distance between the same two points.

6. Plane figures are defined after the modern instead of the ancient conceptions. As this will at first sight strike the teacher of the Euclidean Geometry as one of the most radical changes in the work, a comparison of the ideas on which the two systems of definitions are founded may be of interest.

The ancient geometry was primarily a science of mere magnitude. Solids were bodies, and plane figures were pieces of a plane. Of course other conceptions had to be brought in, but they were regarded as subsidiary.

In modern geometry form and position are of equal importance with magnitude, and in order that all the conceptions associated with a figure may come in on terms of equality as it were, it is necessary to confine the definition of a figure to what is really necessary to its formation. A flexibility and generality is thus given to the definitions which they cannot have under the older form. It is not, indeed, claimed that, for the purpose of elementary instruction, one of these systems of definitions has any great advantage over the other. But it is important that the definitions should accord with the conceptions naturally formed; with the language of every-day life, and with that of the higher modern geometry; and these considerations all point to the new system of definition. Let us take circles and polygons as examples.

In the older geometry a circle is a round piece of a plane or what, in ordinary language, is called a *circular disk.* In our usual language a circle is a curved line which the pupil can draw with a pencil. The ancient geometry, instead of using this term, calls this curved line the *circumference* of the circle, although the word circumference is applied to far wider uses. But this nomenclature is changed as soon as the student reaches Analytic Geometry or the Conic Sections, where he finds the circle and ellipse treated as curves. When the eccentricity vanishes the ellipse is said to become, not the circumference of a circle, but the circle itself. The equation of the bounding curve of the circular disk is also called the equation of the circle. In a word, the old definition entirely vanishes, and a new conception is attached to the word.

A polygon, again, is completely determined by its bounding lines. Hence the definition need not involve anything but these lines. Indeed in the higher modern geometry a polygon is considered merely as a collection of lines standing in certain relations to each other, and the more strongly a definition inconsistent with this is impressed on the mind of the beginner, the more difficult he finds it to make the necessary change in the conceptions attached to the word. It may seem that the ancient form has some advantages, especially in considering areas, and it will also be remarked that the word circumference is used in the present work in its ordinary acceptation. A glance at the true relations between geometric figures and words will make the state of the case quite clear, and will show a perfect analogy between the system of notation here adopted and the language of ordinary life.

A geometric figure is to be regarded as combining a great number of associated conceptions, of which a certain number necessarily involve the others, and may therefore be regarded as essential. The essential conceptions are those which suffice to determine the figure. Since the figure may be determined in various ways, the only rule that can be followed is to choose for a definition the most simple and easily understood set of conceptions. Let us consider a polygon, for example. We have in the polygon a collection of associated conceptions: a certain number of sides, a certain number of interior angles, an equal number of exterior angles, a form, an area, and a perimeter—the latter being the sum of all the sides. No one of these has any special claim over the others to be considered as the measure of the figure. Two different polygons equal in area have no more right to be considered equal than two other polygons of different areas but equal perimeters. Nor have the concepts by which the figure is defined, however they may be chosen, any right to be considered as the whole of the polygon because the associated concepts equally belong to it. Hence the proper course is to take the simplest defining conception; namely, the lines which the pupil draws when he constructs the figure, as being, not necessarily the polygon itself, but the things which *determine* or *form* it. Then its area, its angles, its perimeter, its centre (if it has one), its form, and any other concepts associated with it may be separately considered at pleasure. Again, with the circle we associate a circumference, an area, a centre, any number of radii, and any number of tangents we choose to draw. The word "circle" is properly applied only to the whole assemblage of concepts; but since the circumscribing line is the fundamental determining thing, the word can be more properly applied to it than to any other of the associated concepts. When, however, the length of this line comes into consideration, or when the line is to be considered in antithesis to some other conception, the centre for instance, then the word circumference is used.

The accordance of this mode of language with that of ordinary life will be seen by comparing it with the ideas which we attach to the word "house." We may equally define a house as a space surrounded by walls or as walls enclosing a space. We habitually use the word in both senses without any ambiguity or confusion. We speak of building a house when we really mean building the walls, and of living in the house when we mean living in the interior.

V. The greatest improvement in the modern over the ancient geometry is made in the extension of the idea of angular magnitude. In Euclid only angles less than 180° are considered as having any actual existence. Angular measures equal to or exceeding this limit are considered merely as sums of angles to which no visible geometric meaning is attached, and which are in fact treated as purely symbolic entities, like the imaginary quantities of modern mathematics. Some moderns have followed in his footsteps so slavishly as to actually apprise the pupil that an angle of 180° is not an angle! lest the pupil might be led into the mistake of considering the sum of two right angles as having some conceivable meaning!

We have already mentioned the failure of Euclid to give any definition of the sum of two angles. Without such a definition we do not know what the sum of two angles is. With such a definition the sum of two right angles becomes the angle formed by two straight lines extending from the same vertex in opposite directions.

In modern geometry angular measure is unlimited, and a given angle may have any number of measures differing from each other by any entire number of circumferences. It is not, however, advisable to burden the beginner by attempting to impress this idea upon his mind, but he should be led up to it gradually. Hence in commencing to write the present work, the author started out by confining angular measures to the limit of 180°. He soon found, however, that confusion would result from attempting to keep within this limit, especially in considering the relation of angles inscribed in a circle. He therefore adopted the plan of extending angular measures to one circumference, and explaining in the beginning the two measures of the angle. He finds by experience that there is no difficulty in making this double measure clear to a very young beginner.