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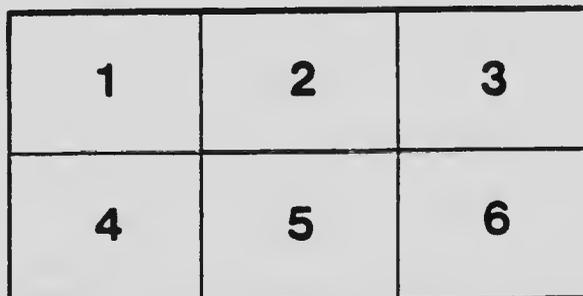
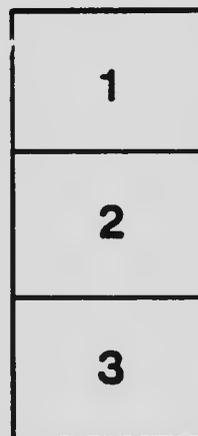
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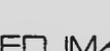
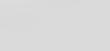
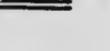
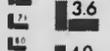
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PART I.—SYNTHETIC GEOMETRY

PART II.—ANALYTICAL GEOMETRY

BY

A. H. McDOUGALL, B.A.

PRINCIPAL OTTAWA COLLEGIATE INSTITUTE

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PREFACE

The object of this book is to supply the want that has been felt by the mathematical teachers of the Province of Ontario of a single volume that would cover both parts of the work in Geometry required for Entrance to the Faculties of Education and for the Honour Matriculation Examination of the Universities.

The principles involved in the different propositions are illustrated by carefully graded sets of exercises, while the miscellaneous sets give material for review purposes and also contain a considerable number of examples that correspond in difficulty to those given on the problem papers for scholarship candidates.

In the part on Analytical Geometry the formulæ which must be fixed in the memory are set forth in bold type.

Acknowledgments for valuable assistance received from him are due to Mr. T. Norris, B.A., Mathematical Master of the Ottawa Collegiate Institute.

OTTAWA, April, 1911.



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PART I.—SYNTHETIC GEOMETRY

SYNTHETIC GEOMETRY

CHAPTER I

THEOREMS OF MENELAUS AND CEVA

1. Menelaus' Theorem:—If a transversal cut the sides BC, CA, AB of the triangle ABC in the points D, E, F respectively, $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$.

(NOTE.—The transversal must cut two sides and the third side produced, or cut all three produced.)

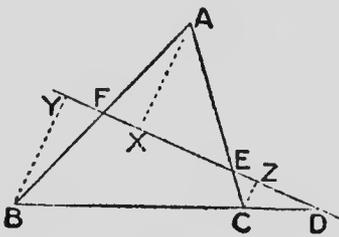


FIG. 1.

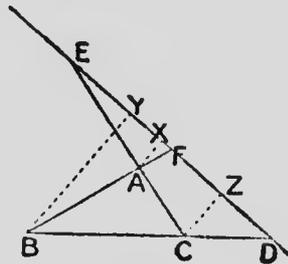


FIG. 2.

Draw $AX, BY, CZ \perp$ to the transversal.

From similar Δ s:—

$$\frac{AF}{FB} = \frac{AX}{BY},$$

$$\frac{BD}{DC} = \frac{BY}{CZ},$$

and $\frac{CE}{EA} = \frac{CZ}{AX}.$

By multiplication,

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1,$$

and $\therefore AF \cdot BD \cdot CE = FB \cdot DC \cdot EA.$

2. Converse of Menelaus' Theorem:—If, in $\triangle ABC$, on two of the sides BC , CA , AB and on the third produced, or if on all three produced, points D , E , F respectively be taken so that $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$, the points D , E , F are collinear.

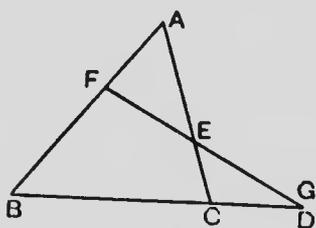


FIG. 3.

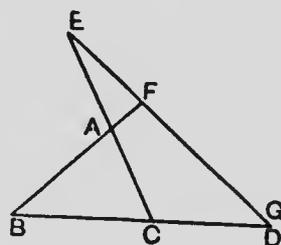


FIG. 4.

Join EF , and produce EF to cut BC at G .

$\therefore FEG$ is a st. line,

$$\therefore AF \cdot BG \cdot CE = FB \cdot GC \cdot EA. \quad (\S 1.)$$

But $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$, by hypothesis.

$$\therefore, \text{dividing, } \frac{BG}{BD} = \frac{GC}{DC};$$

$$\text{or, by alternation, } \frac{BG}{GC} = \frac{BD}{DC}.$$

$\therefore G$ coincides with D . (O. H. S. Geometry, § 121.)

$\therefore D, E, F$ are collinear.

3. Ceva's Theorem:—If from the vertices A, B, C , of $\triangle ABC$ concurrent straight lines AO, BO, CO be drawn to cut BC, CA, AB at D, E, F respectively, $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$.

(NOTE.— D, E and F must be on the three sides, or on one side and on the other two produced.)

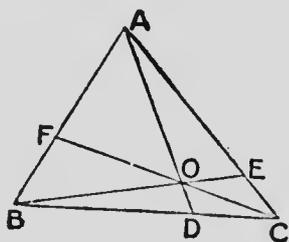


FIG. 5.

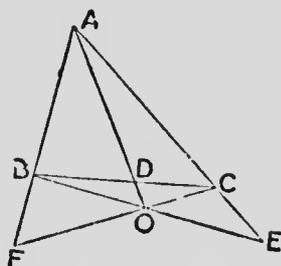


FIG. 6.

FOC is a transversal of $\triangle ABD$,

$$\therefore AF \cdot BC \cdot DO = FB \cdot CD \cdot OA. \quad (\S 1.)$$

BOE is a transversal of $\triangle ADC$,

$$\therefore AO \cdot DB \cdot CE = OD \cdot BC \cdot EA.$$

By multiplication, and division by DO, OA and BC ,

$$AF \cdot BD \cdot CE = FB \cdot DC \cdot EA.$$

(For another proof of this theorem see O. H. S. Geometry, § 122, Exercises 12 and 14.)

4. Converse of Ceva's Theorem:—If, in $\triangle ABC$, on the three sides BC , CA , AB , or if on one of these sides and on the other two produced, points D , E , F respectively be taken so that $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$, the lines AD , BE , CF are concurrent.

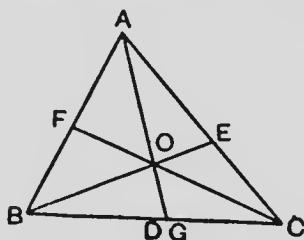


FIG. 7.

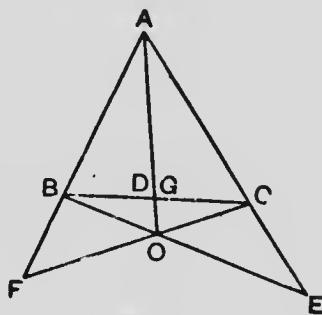


FIG. 8.

Draw BE , CF and let them cut at O . Join AO and let it cut BC at G .

$\therefore AG, BE, CF$ are concurrent,

$\therefore AF \cdot BG \cdot CE = FB \cdot GC \cdot EA$,

(§ 3.)

But $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$, by hypothesis.

\therefore , dividing, $\frac{BG}{BD} = \frac{GC}{DC}$;

or, by alternation, $\frac{BG}{GC} = \frac{BD}{DC}$.

$\therefore G$ coincides with D .

$\therefore AD, BE, CF$ are concurrent.

5. The perpendiculars from the vertices of a triangle to the opposite sides are concurrent.

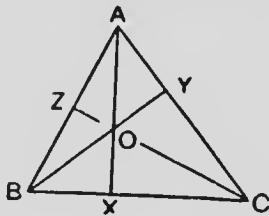


FIG. 9.

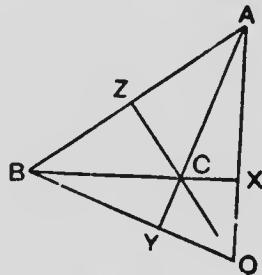


FIG. 10.

In $\triangle ABC$, draw $AX \perp BC$, $BY \perp CA$, $CZ \perp AB$.

To prove that AX , BY , CZ are concurrent.

$$\therefore \triangle AZC \parallel \triangle AYB,$$

$$\therefore \frac{AZ}{YA} = \frac{CA}{AB}.$$

$$\text{Similarly, } \frac{BX}{ZB} = \frac{AB}{BC},$$

$$\text{and } \frac{CY}{XC} = \frac{BC}{CA}.$$

$$\therefore \frac{AZ}{YA} \cdot \frac{BX}{ZB} \cdot \frac{CY}{XC} = \frac{CA}{AB} \cdot \frac{AB}{BC} \cdot \frac{BC}{CA} = 1.$$

$$\therefore AZ \cdot BX \cdot CY = ZB \cdot XC \cdot YA.$$

\therefore , by § 4, AX , BY , CZ are concurrent.

6. The point where the \perp s from the vertices of a \triangle to the opposite sides intersect is called the **orthocentre** of the \triangle . The \triangle formed by joining the feet of these \perp s, X , Y , Z in Figures 9 or 10, is called the **orthocentric**, or **pedal**, \triangle .

7.—Exercises

1. Show, from the converse of Ceva's Theorem, that the medians of a \triangle are concurrent.

2. In $\triangle ABC$, the bisectors of the \angle s A, B, C cut BC, CA, AB at D, E, F respectively. Show that $AF = \frac{bc}{a+b}$.

If $a = 25, b = 35, c = 20$, show that $AF \cdot BD \cdot CE = 2062 \frac{86}{207}$.

3. Show, from the converse of Ceva's Theorem, that the bisectors of the \angle s of a \triangle are concurrent.

4. In $\triangle ABC$, the bisector of the interior \angle at A and of the exterior \angle s at B, C cut BC, CA, AB at D, E, F respectively. Show that $AF = \frac{bc}{b-a}$.

If $a = 44, b = 33, c = 22$, show that $AF \cdot BD \cdot CE = 76665 \frac{3}{5}$.

5. Show, from the converse of Ceva's Theorem, that the bisector of the \angle at one vertex of a \triangle and the bisectors of the exterior \angle s at the other two vertices are concurrent.

6. In $\triangle ABC$, AX, BY, CZ the \perp s to BC, CA, AB intersect at O . Show that:—

(a) $\text{rect. } AO \cdot OX = \text{rect. } BO \cdot OY = \text{rect. } CO \cdot OZ$;

(b) $\text{rect. } AB \cdot AZ = \text{rect. } AO \cdot AX = \text{rect. } AC \cdot AY$;

(c) if AX meet the circumscribed circle of $\triangle ABC$ at K , $OX = XK$;

(d) if S be the centre of the circumscribed circle of $\triangle ABC$ and SD be $\perp BC$, $AO = \text{twice } SD$;

(e) \triangle s AYZ, BZX, CXY, ABC are similar;

(f) AX, BY, CZ bisect the \angle s of the pedal $\triangle XYZ$;

(g) of the four points A, B, C, O , each is the orthocentre of the \triangle of which the other three points are the vertices;

(h) if a $\triangle LMN$ be formed by drawing through A, B, C lines $MN, NL, LM \parallel BC, CA, AB$ respectively, O is the circumscribed centre of $\triangle LMN$;

(i) if S be the centre of the circumscribed circle of $\triangle ABC$, AS, BS, CS are respectively $\perp YZ, ZX, XY$ the sides of the orthocentric \triangle .

7. In $\triangle ABC$, the inscribed circle touches BC, CA, AB at D, E, F respectively, and $s =$ the semi-perimeter. Show that $AF = s - a$.

If $a = 43, b = 31, c = 26$, show that $AF \cdot BD \cdot CE = 3192$.

8. The st. lines joining the vertices of a \triangle to the points of contact of the opposite sides with the inscribed circle are concurrent.

9. In $\triangle ABC$, an escribed circle touches BC at D and AB, AC produced at F, E respectively. Show that $FB = s - c$.

If $a = 40, b = 30, c = 50$, show that $AF \cdot BD \cdot CE = 18000$.

10. The st. lines joining the vertices of a \triangle to the points of contact of the opposite sides with any one of the escribed circles are concurrent.

11. O is a point within the $\triangle ABC$ and AO, BO, CO produced cut BC, CA, AB at D, E, F respectively. The circle through D, E, F cuts BC, CA, AB again at P, Q, R . Show that AP, BQ, CR are concurrent.

12. The bisectors of $\angle s B, C$ of $\triangle ABC$ cut CA, AB at E, F respectively. FE, BC produced meet at D . Prove that AD bisects the exterior \angle at A .

13. The points where the bisectors of the exterior $\angle s$ at A, B, C of $\triangle ABC$ meet BC, CA, AB respectively are collinear.

14. AB, CD, EF are three \parallel st. lines. AC, BD meet at N ; CE, DF at L ; EA, FB at M . Prove that L, M, N are collinear.

15. If two Δ s are so situated that the st. lines joining their vertices in pairs are conenrrent, the intersections of pairs of corresponding sides are collinear.

State and prove the converse.

16. The inscribed circle of ΔABC touches the sides BC, CA, AB at D, E, F respectively; EF, FD, DE , produced meet BC, CA, AB respectively at L, M, N . Show that L, M, N are collinear.

17. Tangents to the circunneirele at A, B, C meet BC, CA, AB respectively in collinear points.

18. Given the base and vertical \angle of a Δ , find the locus of its orthocentre.

19. If the base BC and vertical $\angle A$ of a ΔABC be given, and the base be trisected at D, E , the locus of the centroid is an arc containing an \angle equal to $\angle A$, and having DE as its chord.

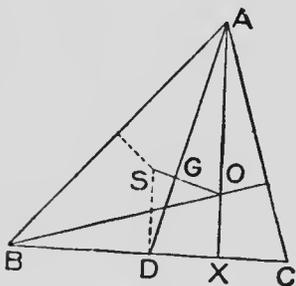


FIG. 11.

20. ABC is a Δ having $AX \perp BC$, AD a median, O the orthocentre, and S the circumscribed centre. Show that OS cuts AD at the centroid G . (Use Ex. 6, *d*.) Show also that G is a point of trisection in SO .

A general enunciation of these results may be given as follows:—

The Orthocentre, Centroid, and the centre of the Circumscribed Circle of a Δ are in the same st. line, and the Centroid is a point of trisection in the st. line joining the other two.

21. Where is the orthocentre of a rt.- \angle d \triangle ?
22. If one escribed circle of $\triangle ABC$ touch AC at F and BA produced at G , and another escribed circle touch AB at H and CA produced at K , FH , KG produced cut BC produced in points equidistant from the middle point of BC .
23. If O is the orthocentre, S the circumcentre and I the centre of the inscribed circle of $\triangle ABC$, prove that IA bisects $\angle OAS$
-

THE NINE-POINT CIRCLE

8. The three middle points of the sides of a triangle, the three projections of the vertices on the opposite sides, and the three middle points of the straight lines joining the vertices to the orthocentre are all concyclic.

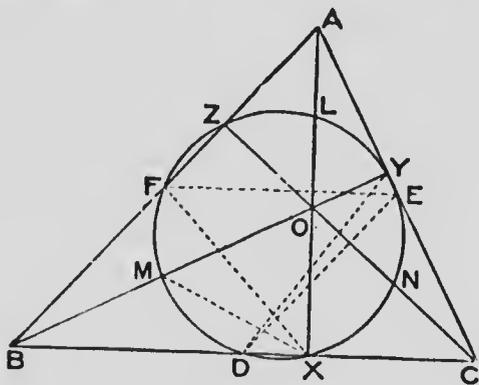


FIG. 12.

Let ABC be a \triangle , AX , BY , CZ the \perp s from A , B , C to BC , CA , AB respectively, O the orthocentre, L , M , N the middle points of AO , BO , CO respectively D , E , F the middle points of BC , CA , AB respectively.

It is required to show that the nine points X , Y , Z , L , M , N , D , E , F are concyclic.

Join DE , EF , FX .

$\therefore F$ is the centre of circle ABX ,

$\therefore \angle FXB = \angle FBX$.

$\therefore FBDE$ is a \parallel gm,

$\therefore \angle FBD = \angle FED$.

$\therefore \angle FXB = \angle FED$.

And \therefore D, E, F, X are concyclic.

Similarly it may be shown that the circle through D, E, F passes also through Y and Z.

Join MX, YD.

\therefore D is the centre of circle BYC,

$\therefore \angle$ DYB = \angle DBY.

Similarly \angle MXB = \angle MBX.

$\therefore \angle$ MXB = \angle MYD.

And \therefore Y, X, D, M are concyclic.

Similarly this circle passes through L and N.

\therefore The nine points are concyclic.

9. The centre of the N.-P. circle is at the middle point of the straight line joining the orthocentre to the circumcentre; and the diameter of the N.-P. circle equals the radius of the circumscribed circle.

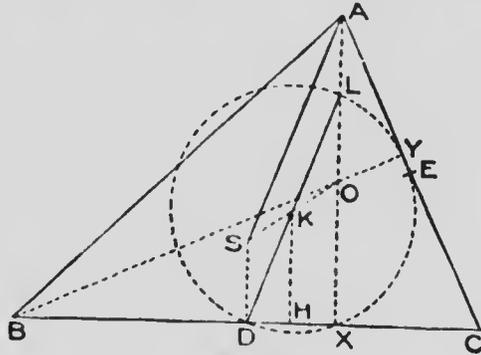


FIG. 13.

S, O are the circumcentre and orthocentre respectively of $\triangle ABC$.

DX is a chord of the N.-P. circle, hence the right bisector HK of DX must pass through the centre.

Similarly the right bisector of YE passes through the centre.

But the \perp s bisecting DX, EY both bisect SO .

$\therefore K$ the middle point of SO is the centre.

Join DL and SA .

DL is the diameter of the N.-P. circle since DXL is a rt. \angle , and SA is the radius of the circumscribed circle.

AL, SD are both $\perp BC$, $\therefore AL \parallel SD$; and $SD = AL$.
(§ 7, Ex. 6, *d.*)

Hence $LD = AS$.

SIMPSON'S LINE

10. If any point is taken on the circumference of the circumscribed circle of a triangle, the projections of this point on the three sides of the triangle are collinear.

Let P be any point on the circle ABC , X, Y, Z , the projections of P on BC, CA, AB respectively.

It is required to show that X, Y, Z are in the same st. line.

Join ZY, YX, PC, PA .

$\angle PYC = \angle PXC, \therefore P, Y, X, C$ are concyclic, and $\angle XYC = \angle XPC$.

$\angle AZP + \angle AYP = 2 \text{ rt. } \angle s, \therefore A, Z, P, Y$ are concyclic, and $\angle AYZ = \angle APZ$.

$APCB$ is a cyclic quadrilateral,

$\therefore \angle APC + \angle B = 2 \text{ rt. } \angle s$.

In quadrilateral $BZPX$,

$\angle s \text{ } BZP, BXP$ are rt. $\angle s$,

$\therefore \angle ZPX + \angle B = 2 \text{ rt. } \angle s$.

Hence $\angle ZPX = \angle APC$, and as the part APX is common to these $\angle s$,

$\angle APZ = \angle XPC$.

$\therefore \angle AYZ = \angle XYC$, and

$\angle XYC + \angle CYZ = \angle AYZ + \angle CYZ = 2 \text{ rt. } \angle s$;

$\therefore XY$ and YZ are in the same st. line.

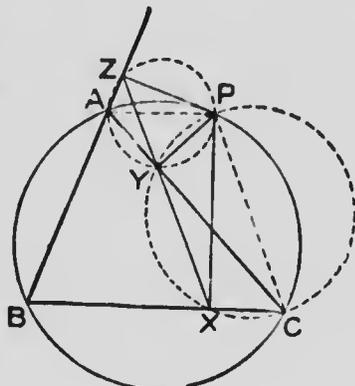


FIG. 14.

11.—Exercises

1. P is the orthocentre of $\triangle DEF$, and the $\parallel\text{gm EPFG}$ is completed. Show that DG is a diameter of the circle circumscribing DEF .
2. In $\triangle ABC$, I is the centre of the inscribed circle, L, M, N the centres of the escribed circles. Prove that the circumcircle of $\triangle ABC$ bisects IL and LM .
3. ABC is a \triangle ; L, M, N the centres of its escribed circles. Show that the circle circumscribed about ABC is the N.-P. circle of $\triangle LMN$.
4. O is the orthocentre of $\triangle ABC$. Prove that \triangle s OBC, ABC have the same N.-P. circle.
5. Given the base and vertical \angle of a \triangle , show that the locus of the centre of its N.-P. circle is a circle having its centre at the middle point of the base.
6. If the projections of a point on the sides of a \triangle are collinear, the point is on the circumcircle of the \triangle .
7. Find a point such that its projections on the four sides of a given quadrilateral are collinear.
8. The three circles which go through two vertices of a \triangle and its orthocentre are each equal to the circle circumscribed about the \triangle .
9. The \perp from the middle point of a side of a \triangle on the opposite side of the pedal \triangle bisects that side.
10. Construct a \triangle given a vertex, the circumcircle and the orthocentre.
11. DEF is a \triangle and O is its orthocentre. About DOF a circle is described and EO is produced to meet the circumference at P . Show that DF bisects EP .

12. I is the centre of the inscribed circle of $\triangle ABC$, and AI, BI, CI are produced to meet the circumcircle at L, M, N . Prove that I is the orthocentre of $\triangle LMN$.
13. I is the centre of the inscribed circle of $\triangle ABC$, and the circumcircle of $\triangle IBC$ cuts AB at D . Prove that $AD = AC$.
14. In $\triangle ABC$, the \perp s from A, B to the opposite sides meet the circumcircle at D, E . Show that arc $CD =$ arc CE .
15. XYZ is the pedal \triangle of $\triangle ABC$. Prove that A, B, C are the centres of the escribed circles of $\triangle XYZ$.
16. X, Y, Z are the projections of A, B, C on BC, CA, AB . Prove that
- (a) $YZ \cdot ZX = AZ \cdot ZB$;
 - (b) $YZ \cdot ZX \cdot XY = AZ \cdot BX \cdot CY$.
17. Construct a \triangle having given the base, the vertical \angle and the radius of an escribed circle. (Two cases.)
18. O is the orthocentre of $\triangle ABC$, and D, E, F are the centres of the circumcircles of \triangle s BOC, COA, AOB . Show that $\triangle DEF \cong \triangle ABC$.
19. In the $\triangle ABC$, the \perp from A to BC is produced to cut the circumcircle at P . Prove that the Simpson's Line of P is \parallel to the tangent to the circumcircle at A .
20. P is any point on the circumcircle of $\triangle ABC$. The \perp s from P to the sides of the \triangle meet the circle at D, E, F . Prove that $\triangle DEF \cong \triangle ABC$.
21. P is any point on the circumcircle of a $\triangle ABC$ of which O is the orthocentre and X the projection of A on BC ; AX produced cuts the circumcircle at D and PD cuts BC at E . Prove that the Simpson's Line of P bisects PE , is $\parallel OE$, and bisects OP .
-

AREAS OF RECTANGLES

12. If from the vertex of a triangle a straight line is drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circumcircle of the triangle.

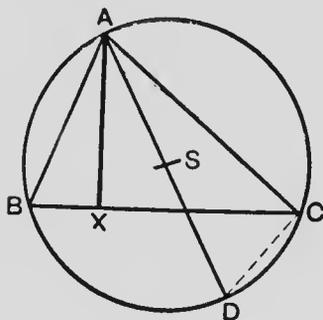


FIG. 15.

$AX \perp BC$ and AD is a diameter of the circumcircle of $\triangle ABC$.

To prove that

rect. $AB \cdot AC = \text{rect. } AX \cdot AD$.

Join DC .

$\therefore \angle AXB = \angle ACD$,
and $\angle ABX = \angle ADC$.

$\therefore \triangle AXB \parallel \triangle ACD$.

$\therefore \frac{AB}{AD} = \frac{AX}{AC}$.

$\therefore \text{rect. } AB \cdot AC = \text{rect. } AX \cdot AD$.

13. If the vertical angle of a triangle is bisected by a straight line which also cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base together with the square on the straight line which bisects the angle.

ABC is a \triangle and AD the bisector of $\angle A$.

It is required to show that
the rect. $AB \cdot AC = \text{rect. } BD \cdot DC + AD^2$.

Circumscribe a circle about the $\triangle ABC$. Produce AD to cut the circumference at E . Join EC .

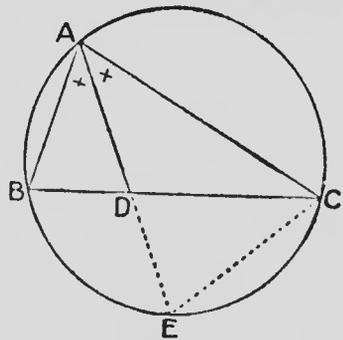


FIG. 16.

In \triangle s BAD, EAC $\angle BAD = \angle EAC, \angle ABD = \angle AEC,$
 $\therefore \angle ADB = \angle ACE$ and the \triangle s are similar;

hence
$$\frac{BA}{AD} = \frac{EA}{AC},$$

and $\therefore BA \cdot AC = AD \cdot EA.$

But $AD \cdot EA = AD (AD + DE)$
 $= AD^2 + AD \cdot DE$
 $= AD^2 + BD \cdot DC$

\therefore rect. $BA \cdot AC =$ rect. $BD \cdot DC + AD^2.$

14. The sum of the rectangles contained by the opposite sides of a quadrilateral is not less than the rectangle contained by the diagonals.

$ABCD$ is a quadrilateral,
 AC, BD its diagonals.

Required to show that
 $AB \cdot DC + AD \cdot BC$ is not
less than $AC \cdot BD.$

Make $\angle BAE = \angle CAD$
and $\angle ADE = \angle ACB.$
Join $EB.$

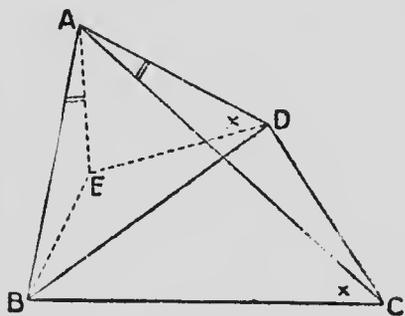


FIG. 17.

\triangle s BAC, EAD are similar,

$\therefore \frac{BA}{AE} = \frac{CA}{AD}$

and $\angle BAE = \angle CAD,$

$\therefore \triangle$ s BAE, CAD are also similar.

From the similar \triangle s BAE, CAD

$\frac{AB}{BE} = \frac{AC}{CD}$ and $\therefore AB \cdot CD = AC \cdot BE.$

From the similar \triangle s BAC , EAD ,

$$\frac{BC}{AC} = \frac{ED}{AD}, \text{ and } \therefore BC \cdot AD = AC \cdot ED.$$

Consequently $AB \cdot CD + BC \cdot AD = AC (BE + ED)$;

but $BE + ED$ is not $< BD$;

$\therefore AB \cdot CD + BC \cdot AD$ is not $< AC \cdot BD$.

15. Ptolemy's Theorem :—The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the rectangles contained by its opposite sides.

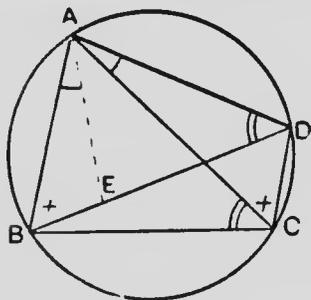


FIG. 18.

This theorem is a particular case of that of § 14.

$ABCD$ is a quadrilateral inscribed in a circle.

To prove that

$$AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

Make $\angle BAE = \angle CAD$, and produce AE to cut BD at E .

$$\therefore \triangle ABE \parallel \triangle ACD,$$

$$\therefore \frac{AB}{AC} = \frac{BE}{CD}.$$

$$\therefore AB \cdot CD = AC \cdot BE.$$

$$\therefore \triangle ADE \parallel \triangle ABC,$$

$$\therefore \frac{AD}{AC} = \frac{DE}{BC}.$$

$$\therefore AD \cdot BC = AC \cdot DE.$$

$$\begin{aligned} \therefore AB \cdot CD + AD \cdot BC &= AC \cdot BE + AC \cdot DE \\ &= AC (BE + ED) \\ &= AC \cdot BD \end{aligned}$$

16.—Exercises

1. If the exterior vertical $\angle A$ of $\triangle ABC$ be bisected by a line which cuts BC produced at D , rect. $AB \cdot AC =$ rect. $BD \cdot CD - AD^2$.

2. Draw $\triangle ABC$ having $a = 81$ mm., $b = 60$ mm., $c = 30$ mm. Bisect the interior and exterior $\angle s$ at A and produce the bisectors to meet BC and BC produced at D and E . Measure AD , AE ; and check your results by calculation.

3. If R is the radius of the circumcircle and Δ the area of $\triangle ABC$, prove that

$$R = \frac{abc}{4\Delta}.$$

If the sides of a \triangle are 39, 42, 45, show that $R = 24\frac{3}{8}$.

4. P is any point on the circumcircle of an equilateral $\triangle ABC$. Show that, of the three distances PA , PB , PC , one is the sum of the other two.

5. From any point P on a circle $\perp s$ are drawn to the four sides and to the diagonals of an inscribed quadrilateral. Prove that the rect. contained by the $\perp s$ on either pair of opposite sides is equal to the rect. contained by the $\perp s$ on the diagonals.

6. With given base and vertical \angle construct a \triangle having the rect. contained by its sides equal to the square on a given st. line.

7. A , B , C , D are given points on a circle. Find a point P on the circle such that $PA \cdot PC = PB \cdot PD$.

8. AB is the chord of contact of tangents drawn from a point P to a circle. PCD cuts the circle at C , D . Prove that $AB \cdot CD = 2 AC \cdot BD$.

RADICAL AXIS

17. The locus of the points from which tangents drawn to two circles are equal to each other is called the **radical axis** of the two circles.

18. If two circles cut each other, their common chord produced is the radical axis.

[Proof left for the pupil.]

19. A, B are the centres of two circles of radii R, r respectively.

To prove that the radical axis of the circles is a st. line \perp AB and cutting it at a point M such that

$$AM^2 - MB^2 = R^2 - r^2.$$

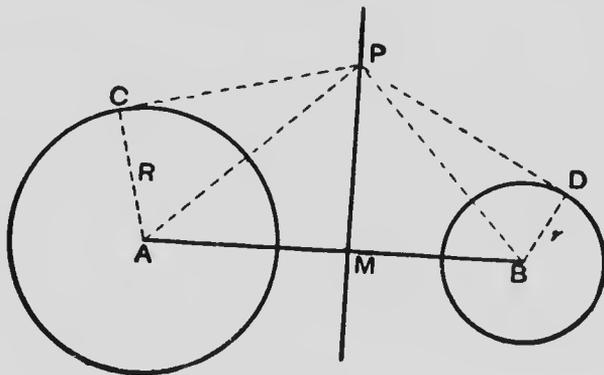


FIG. 19.

Let P be any point on the radical axis, and draw $PM \perp AB$.

Draw the tangents PC, PD to the circles, and join PA, PB, AC, BD.

$$PC^2 = PA^2 - R^2 = PM^2 + AM^2 - R^2.$$

$$PD^2 = PB^2 - r^2 = PM^2 + MB^2 - r^2.$$

But, by the definition of radical axis, $PC = PD$.

$$\therefore AM^2 - R^2 = MB^2 - r^2,$$

$$\text{or, } AM^2 - MB^2 = R^2 - r^2.$$

$$\therefore (AM + MB)(AM - MB) = R^2 - r^2.$$

But $AM + MB = AB$,

$$\therefore AM - (AB - AM) = \frac{R^2 - r^2}{AB}.$$

$$\therefore AM = \frac{1}{2} \left(AB + \frac{R^2 - r^2}{AB} \right).$$

Hence, AM is a constant distance for all positions of P on the radical axis, and

\therefore the $\perp PM$ is the radical axis.

Also it has been shown that

$$AM^2 - MB^2 = R^2 - r^2.$$

20. To draw the radical axis of two non-intersecting circles.

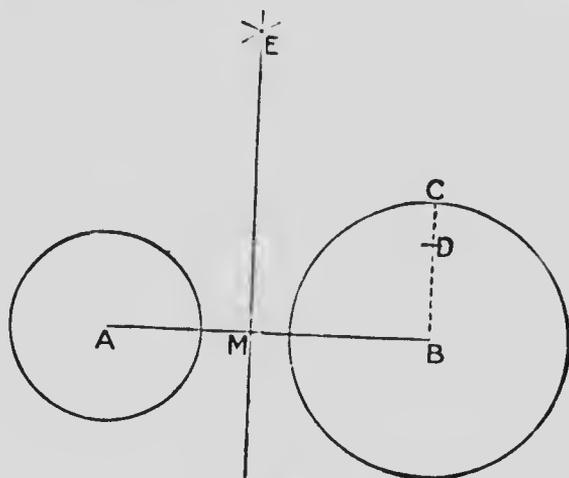


FIG. 20.

Let A , B be the centres of the two circles.

Join AB . Through B draw $BC \perp AB$ cutting the circle with centre B at C , and cut off BD equal to the radius of the other circle.

With centre A and radius AD describe an arc, and with centre B and radius AC describe another arc cutting the first at E . Draw $EM \perp AB$.

$$BM^2 = BE^2 - EM^2 = AB^2 + BC^2 - EM^2.$$

$$MA^2 = AE^2 - EM^2 = AB^2 + BD^2 - EM^2.$$

$$\therefore BM^2 - MA^2 = BC^2 - BD^2.$$

\therefore , by § 19, EM is the radical axis.

21. Give another method of drawing the radical axis, by describing a circle to cut the two given circles.

22.—Exercises

1. Draw two circles, radii 1 inch and 2 inches, with their centres 4 inches apart. Find a point whose tangents to the two circles are each $1\frac{1}{2}$ inches in length.

2. The radical axis of two circles bisects their common tangents.

3. Prove that the radical axis of any three circles taken two and two together meet in a point.

NOTE.—This point is called the **radical centre** of the three circles.

4. O is a fixed point outside a given circle; find a st. line such that the tangent drawn from any point P in the line to the given circle equals PO .

5. If the square on the distance between the centres of two circles equals the sum of the squares on their radii, the tangents to the circles at a common point are at right angles to each other.

NOTE.—Circles which cut each other so that the tangents at a common point are at right angles to each other are said to be **orthogonal**.

6. Two given circles intersect each other. Draw a system of circles coaxial with the given circles.

7. Draw two non-intersecting circles with centres A and B . Draw their radical axis PO cutting AB at O . From O draw a tangent OE to either circle. With centre O and radius OE describe a circle cutting AB at C and D .

The circle CED cuts the two given circles orthogonally.

Show that any circle which cuts one of the first circles orthogonally, and has its centre in PO , cuts the other orthogonally and passes through C and D .

On the circle CED take any point F , draw the tangent at F to the circle CED and let it cut AB at G . Show that the circle with centre G and radius GF is coaxial with the first two circles; and that, in this manner, a system of circles may be drawn coaxial with two given non-intersecting circles.

NOTE.—No circle of the coaxial system has its centre between C and D , and consequently these points are called the **limiting points** of the system.

8. If from any point P tangents be drawn to two circles, the difference between their squares equals twice the rectangle contained by the \perp from P on the radical axis of the two circles and the distance between their centres.
 9. The tangent drawn from a limiting point to any circle of a coaxial system is bisected by the radical axis.
 10. Find the locus of the centre of a circle the tangents to which from two given points are respectively equal to two given st. lines.
 11. If O be the orthocentre of $\triangle ABC$, the circles described on AB and CO as diameters are orthogonal.
 12. With a given radius describe a circle to cut two given circles orthogonally.
 13. If circles are described on the three sides of a \triangle as diameters, their radical centre is the orthocentre of the \triangle .
 14. XYZ is the pedal \triangle of $\triangle ABC$; YZ, BC meet in L ; ZX, CA meet in M ; XY, AB meet in N . Show that L, M, N , are on the radical axis of the circumscribed and N.-P. circles of $\triangle ABC$.
- (NOTE.— $\triangle s$ MAZ, MXC are easily shown to be similar.)

CHAPTER II
MEDIAL SECTION

23. When a straight line is divided into two parts such that the square on one part is equal to the rectangle contained by the given straight line and the other part, it is said to be divided in medial section.

24. To divide a given straight line in medial section.

Let AB be the given st. line.

Draw $AC \perp AB$ and $= AB$. Bisect AC at D . With centre D and radius DB describe an arc cutting CA produced at E . With centre A and radius AE describe an arc cutting AB at F .

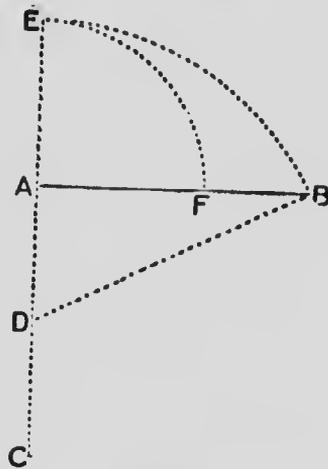


FIG. 21.

Then AB is divided in medial section at F .

$$DA^2 + 2 DA \cdot AE + AE^2 = DE^2.$$

$$= DB^2.$$

$$= DA^2 + AB^2.$$

But $AE = AF$, and $2 DA = AB$

$$\therefore AF^2 = AB^2 - AB \cdot AE.$$

$$= AB (AB - AF).$$

$$= AB \cdot FB.$$

25.—Exercises

1. If a st. line **AB** be divided at **F** so that $AF^2 = AB \cdot BF$, show that $AB : AF = AF : BF$.

Give a general statement of this result.

2. Draw a st. line **AB**. Through **A** draw $AC \perp AB$ and $AC = AB$. Bisect **AC** at **D**. With centre **D** and radius **DB** describe an arc cutting **AC** produced through **C** at **F**. With centre **A** and radius **AF** describe an arc cutting **BA** produced through **A** at **G**. Show that $AG^2 = AB \cdot BG$.

AB is said to be divided externally in medial section at **G**.

3. A given st. line **AB** is to be divided in medial section. Let **F** be the point of section, a the length of **AB**, x the length of **AF**.

Then, by the definition of medial section, $x^2 = a(a - x)$
or $x^2 + ax - a^2 = 0$.

Solving this quadratic equation $x = \frac{-a \pm a\sqrt{5}}{2}$.

Show that the construction in § 24 is suggested by the root $\frac{-a + a\sqrt{5}}{2}$; and the construction in Ex. 2 by the root

$\frac{-a - a\sqrt{5}}{2}$.

4. Divide a st. line 4 inches in length in medial section. Measure the length of each part, and test the results by calculation.

5. The difference of the squares on the parts of a st. line divided in medial section equals the rectangle contained by the parts.

6. If AB be divided at C so that $AC^2 = AB \cdot BC$, show that $AB^2 + BC^2 = 3AC^2$.

7. If the sides of a rt.- \angle triangle are in continued proportion, the \perp from the rt. \angle divides the hypotenuse in medial section.

8. Describe a rt.- \angle triangle whose sides are in geometrical progression.

26. To describe an isosceles triangle having each of the angles at the base double the vertical angle.

Draw a st. line AB and divide it at H so that $AH^2 = AB \cdot BH$. (§ 24.)

Describe arcs with centres A, B and radii AB, AH respectively and let them cut at C .

Join AC, BC .

ABC is the required triangle.

Join HC .

$\angle B$ is common to the Δ s ABC, CBH , and since

$$\frac{AB}{AH} = \frac{AH}{BH}, \therefore \frac{AB}{BC} = \frac{BC}{BH}$$

\therefore these Δ s are similar and $\angle BCH = \angle A$.

$$\text{Again } \frac{AC}{BC} = \frac{AB}{AH} = \frac{AH}{HB}$$

$\therefore CH$ bisects $\angle ACB$.

$\therefore \angle ACB = \text{twice } \angle BCH = \text{twice } \angle A$.

and also $\angle ABC = \text{twice } \angle A$,

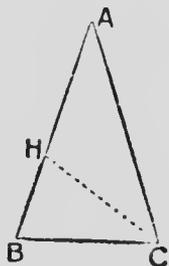


FIG. 22.

27.—Exercises

- Express the \angle s of the $\triangle ABC$ (Fig. 22) in degrees.
- Construct \angle s of 36° , 18° , 9° , 6° , 3° .
- Show that $\triangle AHC$ (Fig. 22) is an isosceles \triangle having the vertical \angle three times each of the base \angle s.

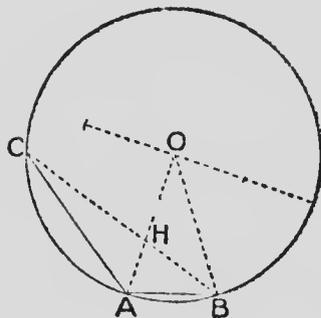


FIG. 23.

4. In a given circle CAB draw any radius OA . Divide AO at H so that $OH^2 = AO \cdot AH$. Place the chord $AB = HO$.

Join BH and produce BH to cut the circumference at C . Join AC .

Show that AB is a side of a regular decagon inscribed in the circle; and that AC is a side of a regular pentagon inscribed in the circle.

- Show that the side of a regular decagon inscribed in a circle of radius r is $\frac{r}{2}(\sqrt{5} - 1)$.
- The square on a side of a regular pentagon inscribed in a circle equals the sum of the squares on a side of the regular inscribed decagon and on the radius of the circle.
- Show that the side of a regular pentagon inscribed in a circle of radius r is $\frac{r}{2}\sqrt{10 - 2\sqrt{5}}$.
- In a circle of radius 2 inches inscribe a regular decagon by the method of Ex. 4. Measure a side of the decagon and check your result by calculation.
- In a circle of radius 3 inches inscribe a regular pentagon by the method of Ex. 4. Measure a side of the pentagon and check your result by calculation.

10. In a given circle draw two radii OA, OB at rt. \angle s to each other. Bisect OB at C . Join AC , and cut off $CD = CO$.

Show that AD is equal to a side of a regular decagon inscribed in the circle.

The regular inscribed pentagon may be drawn by joining alternate points obtained by placing successive chords each equal to AD .

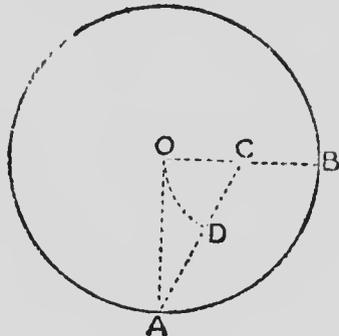


FIG. 24.

11. Draw a regular pentagon on a given st. line.
12. Circumscribe a regular pentagon about a given circle.
13. $ABCDE$ is a regular pentagon. Show that AD, BD trisect $\angle CDE$.
14. $ABCDE$ is a regular pentagon. Show that AC, BD divide each other in medial section.
15. Construct a regular 5-pointed star. What is the measure of the \angle at each vertex?
16. Construct a regular decagon by cutting off the corners of a regular pentagon.
17. On a st. line 2 inches in length describe a regular pentagon. Measure a diagonal of the pentagon and check your result by calculation.
18. If the circumference of a circle be divided into n equal arcs,
 - (a) The points of division are the vertices of a regular polygon of n sides inscribed in the circle;
 - (b) If tangents be drawn to the circle at these points, these tangents are the sides of a regular polygon of n sides circumscribed about the circle.

MISCELLANEOUS THEOREMS

28. ABC is a triangle and P is a point in BC such that $\frac{BP}{PC} = \frac{n}{m}$. It is required to show that

$$mAB^2 + nAC^2 = (m+n)AP^2 + mBP^2 + nCP^2.$$

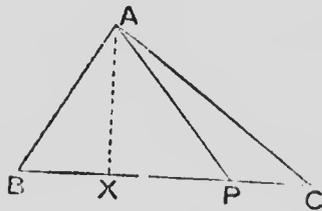


FIG. 25.

Draw $AX \perp BC$.

From $\triangle ABP$,

$$AB^2 = AP^2 + BP^2 - 2BP \cdot PX.$$

From $\triangle APC$,

$$AC^2 = AP^2 + CP^2 + 2CP \cdot PX.$$

Multiplying both sides of the first of these equations by m , both sides of the second by n , adding the results and using the condition $mBP = nPC$, we obtain

$$mAB^2 + nAC^2 = (m+n)AP^2 + mBP^2 + nCP^2.$$

29. What does the result in § 28 become when $m = n$?

In a $\triangle ABC$, $a = 77$ mm, $b = 90$ mm and $c = 123$ mm. Find the distances from C to the points of trisection AB .

30. In a right-angled triangle a rectilinear figure described on the hypotenuse equals the sum of the similar and similarly described figures on the other two sides.

ABC is a \triangle rt.- \angle d at C and having the similar and similarly described figures X, Y, Z on the sides.

It is required to show that $X + Y = Z$.

Similar figures are to each other as the squares on corresponding sides.

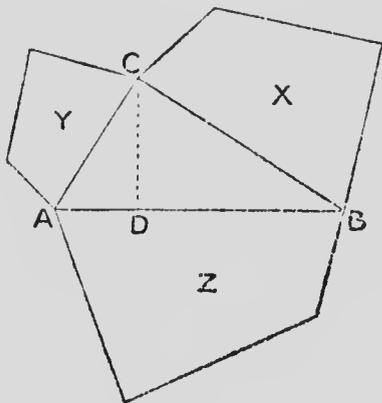


FIG. 26.

$$\therefore \frac{Y}{Z} = \frac{AC^2}{AB^2},$$

$$\frac{X}{Z} = \frac{BC^2}{AB^2}.$$

$$\therefore \frac{X+Y}{Z} = \frac{AC^2+BC^2}{AB^2}.$$

But $AC^2 + BC^2 = AB^2$.

$$\therefore X + Y = Z.$$

31. Prove the theorem of § 30 by drawing a \perp from C to AB and using the theorem:—If three st. lines are in continued proportion, as the first is to the third so is any polygon on the first to the similar and similarly described polygon on the second.

SIMILAR AND SIMILARLY SITUATED POLYGONS

32. Similar polygons are said to be **similarly situated** when their corresponding sides are parallel and drawn in the same direction from the corresponding vertices.

33. If two similar triangles have their corresponding sides parallel, the st. lines joining corresponding vertices are concurrent.

Let ABC , DEF be two similar Δ s having the sides BC , CA , AB respectively \parallel to the corresponding sides EF , FD , DE .

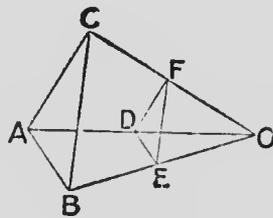


FIG. 27.

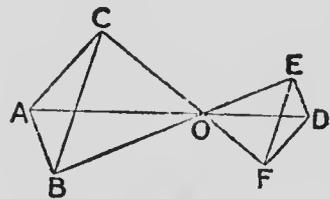


FIG. 28.

Prove AD , BE , CF concurrent.

34. When two similar polygons are so situated that their corresponding sides are parallel but drawn in opposite directions from the corresponding vertices, they are said to be **oppositely situated**.

In Fig. 28, the similar Δ s ABC , DEF are oppositely situated.

35. If two similar polygons have the sides of one respectively parallel to the corresponding sides of the other, the straight lines joining corresponding vertices are concurrent.

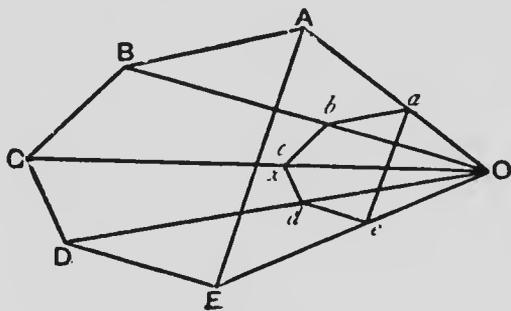


FIG. 29.

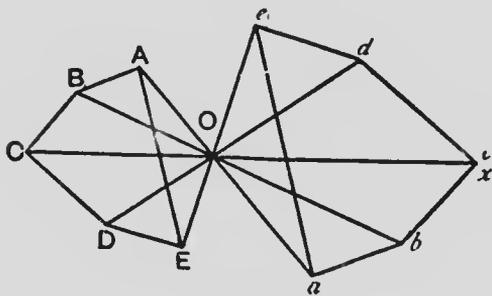


FIG. 30.

Let $ABCDE$, $abcde$ be two similar polygons, similarly situated in Fig. 29, oppositely situated in Fig. 30.

Join Aa , Bb and let the joining lines meet at O . Join CO cutting bc at x .

From similar Δ s,

$$\frac{ax}{AB} = \frac{Ob}{OB} = \frac{bx}{BC}.$$

But, by hypothesis,

$$\frac{ab}{AB} = \frac{bc}{BC}.$$

$\therefore br = bc$, and

$\therefore OC$ passes through c .

Similarly it may be shown that the st. lines joining the remaining pairs of corresponding vertices pass through O .

36.—Exercises

1. Inscribe a square in a given Δ . Show that there are three solutions.
2. In a given Δ inscribe a rectangle similar to a given rectangle. Show that there are six solutions.
3. In a given semi-circle inscribe a square.
4. In a given semi-circle inscribe a rectangle having its sides in a given ratio.
5. In a given Δ inscribe a Δ having its sides \parallel to three given st. lines.
6. The base of a square lies on one given st. line and one of its upper vertices lies on another given st. line. Find the locus of the other upper vertex.
7. In Figures 29 or 30, P is any point in AB , Q is any point in CD and p, q are the corresponding points in ab, cd respectively. Prove that $PQ \parallel pq$.
8. $\Delta ABC \sim \Delta abc$, but their corresponding sides are not \parallel . BC, bc meet at P . The circles BPb, CPc meet again at O . Prove that the Δabc may be rotated about O to a position where it is similarly situated to ΔABC .

CHAPTER III

HARMONIC RANGES AND PENCILS

37. A set of collinear points is called a **range**.

38. A set of concurrent straight lines is called a **pencil**.

The lines are called the **rays of the pencil**; and their common point is called the **vertex of the pencil**.

39. When three magnitudes are such that the first has the same ratio to the third that the difference between the first and second has to the difference between the second and third, the differences being taken in the same order, the magnitudes are said to be in **harmonic proportion**. (H. P.)

Thus, if a , b and c represent three numbers such that $a : c = b - a : c - b$, a , b and c are in H. P.

40. Take any point **C** in a st. line **AB**.

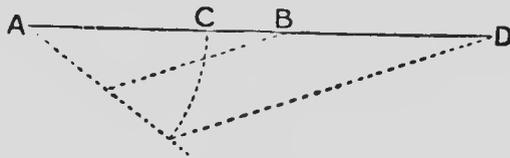


FIG. 31.

Find the point **D** in **AB** produced such that $AD : DB = AC : CB$.

(a) Then since $AC : CB = AD : DB$,

by alternation, $AC : AD = CB : DB$.

$\therefore AC : AD = AB - AC : AD - AB$.

\therefore by the definition of § 39, **AC**, **AB**, **AD** are in H. P.

(b) By inversion, $DB : AD = BC : AC$.

$\therefore DB : DA = DC - DB : DA - DC$.

\therefore , by § 39, **DB**, **DC**, **DA** are in H. P.

From (a) and (b), it follows that, if in a range of four points A, C, B, D the st. line AB is divided internally at C and externally at D in the same ratio, the distances from either end of the range to the other three points are in H. P.

State and prove a converse to this theorem.

41. When a range of four points A, C, B, D is such that $AC : CB = AD : DB$, it is called a **harmonic range**.

If any point P be joined to the four points of a harmonic range, the joining lines form a **harmonic pencil**.

42. If, in the harmonic pencil P (A, C, B, D), a straight line through B parallel to PA cut PC, PD at E, F respectively, $BE = BF$.

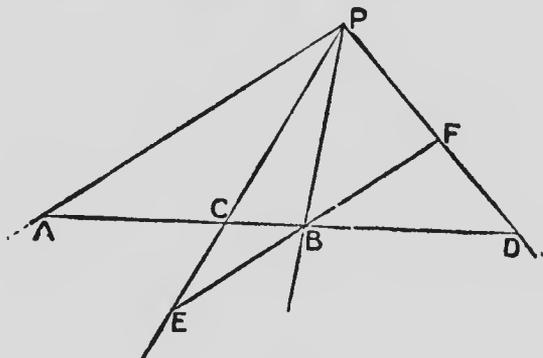


FIG. 32.

$$\therefore \triangle ACP \parallel \triangle BCE,$$

$$\therefore AP : EB = AC : CB.$$

$$\therefore \triangle ADP \parallel \triangle BDF,$$

$$\therefore AP : BF = AD : DB.$$

But, by hypothesis, $AC : CB = AD : DB$.

$$\therefore AP : EB = AP : BF.$$

$$\therefore EB = BF.$$

43. By a proof similar to that of § 42, the following converse to the theorem of that article may be shown to be true.

If, in the pencil $P(A, C, B, D)$, a straight line through B parallel to PA cut PC, PD at E, F respectively such that $BE = BF$, then $P(A, C, B, D)$ is a harmonic pencil.

44. Any transversal is cut harmonically by the rays of a harmonic pencil.

A transversal cuts the rays PA, PC, PB, PD of the harmonic pencil $P(A, C, B, D)$ at K, M, N, L respectively.

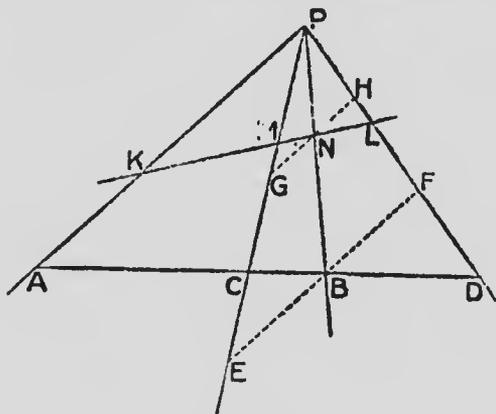


FIG. 33.

It is required to show that K, M, N, L is a harmonic range.

Through B, N respectively draw $EF, GH \parallel PA$.

$\therefore P(A, C, B, D)$ is a harmonic pencil, and $EBF \parallel AP$

\therefore , by § 42, $EB = BF$.

From similar Δ s, $\frac{GN}{NP} = \frac{EB}{BP}$ and $\frac{NP}{NH} = \frac{BP}{BF}$,

\therefore , by multiplication, $GN : NH = EB : BF$,

$\therefore GN = NH$; and \therefore , by § 43, K, M, N, L is a harmonic range.

45. If A, C, B, D is a harmonic range A and B are said to be **harmonic conjugates** with respect to C and D ; and C and D are said to be **harmonic conjugates** with respect to A and B .

46. Exercises

1. Show how to find the fourth ray of a harmonic pencil when three rays are given.

2. Prove the theorem of § 44 when the transversal cuts the rays produced through the vertex.

3. In the $\triangle ABC$ the bisectors of the interior and exterior \angle s at A cut BC and BC produced at D, E respectively. Show that B, D, C, E is a harmonic range.

4. A, C, B, D is a harmonic range and P is any point on the circle described on AB as diameter. Show that PA, PB respectively bisect the exterior and interior vertical \angle s of $\triangle CPD$.

(NOTE.— Draw $EBF \parallel AP$ cutting PC, PD at E, F respectively.)

5. A, C, B, D is a harmonic range and O is the middle point of AB . Show that $OB^2 = OC \cdot OD$.

State and prove a converse.

6. A, C, B, D is a harmonic range. Show that the circles described on AB, CD as diameters cut each other orthogonally.

7. The inscribed circle of $\triangle ABC$ touches BC, CA, AB at D, E, F respectively, and DF meets CA produced at P . Show that C, E, A, P is a harmonic range.

(NOTE.— Use Menelaus' Theorem.)

8. The diameter AB of a circle is \perp to a chord CD . P is any point on the circumference. PC, PD cut AB , or AB produced, at E, F . Show that A, E, B, F is a harmonic range.

9. Through E , the middle point of the side AC of the $\triangle ABC$, a transversal is drawn to cut AB at F , BC produced at D , and a line through $B \parallel CA$ at G . Show that G, F, E, D is a harmonic range.

10. A common tangent of two given circles is divided harmonically by any circle which is coaxial with the given circles.

11. In a circle AC, BD are two diameters at rt. \angle s to each other, and P is any point on the quadrant AD . Show that PA, PB, PC, PD constitute a harmonic pencil.

THE COMPLETE QUADRILATERAL

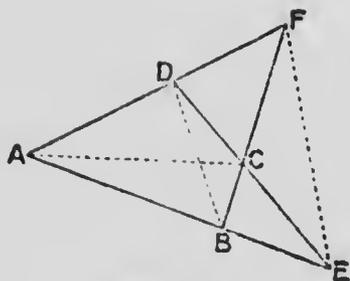


FIG. 34.

47. The figure formed by four straight lines which meet in pairs in six points is called a **complete quadrilateral**.

The figure **ABCDEF** is a complete quadrilateral, of which **AC**, **BD** and **EF** are the three diagonals.

48. In a complete quadrilateral each diagonal is divided harmonically by the two other diagonals, and the angular points through which it passes.

ABCDEF is a complete quadrilateral having the diagonal **AC** cut by **DB** at **P** and by **EF** at **Q**.

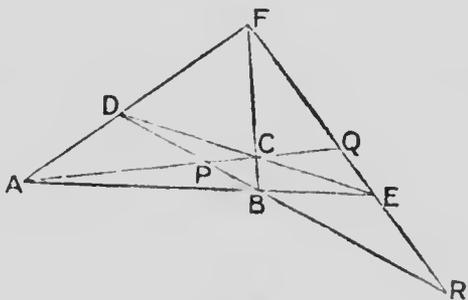


FIG. 35.

It is required to show that **A, P, C, Q** is a harmonic range.

In $\triangle ACF$, **AB**, **CD**, **FQ** are concurrent at **E** and \therefore , by Ceva's Theorem,

$$FD \cdot AQ \cdot CB = DA \cdot QC \cdot BF.$$

The transversal **DPB** cuts the sides of the \triangle **ACF** and
 \therefore , by Menelaus' Theorem,

$$\mathbf{FD} \cdot \mathbf{AP} \cdot \mathbf{CB} = \mathbf{DA} \cdot \mathbf{PC} \cdot \mathbf{BF}.$$

Hence, by division,

$$\frac{\mathbf{AQ}}{\mathbf{AP}} = \frac{\mathbf{QC}}{\mathbf{PC}}$$

or,
$$\frac{\mathbf{AP}}{\mathbf{PC}} = \frac{\mathbf{AQ}}{\mathbf{QC}}$$

and \therefore **A, P, C, Q** is a harmonic range.

From the above result it is seen that **F (A, P, C, Q)**
 is a harmonic pencil, and consequently, by § 44,
D, P, B, R is a harmonic range.

Show, in the same manner that **F, Q, E, R** is a
 harmonic range.



POLES AND POLARS

49. If through a fixed point a line be drawn to cut a given circle and at the points of intersection tangents be drawn, the locus of the intersection of the tangents is called the **polar** of the fixed point; and the fixed point is called the **pole** of the locus.

50. If C is the centre of a given circle and D is a fixed point, the polar of D with respect to the circle is a straight line which is perpendicular to CD and cuts it at a point E such that $CE \cdot CD$ equals the square on the radius.

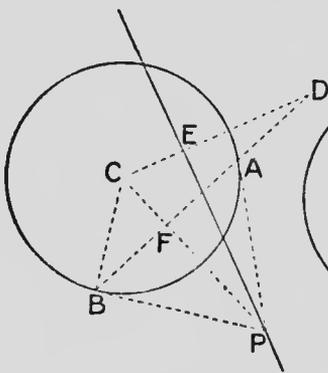


FIG. 36.

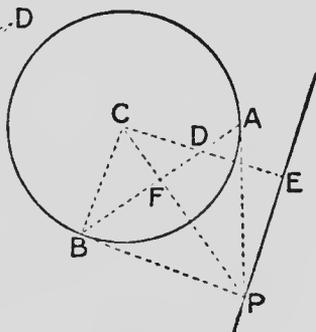


FIG. 37.

Through D draw any st. line cutting the circle at A and B . At A, B draw tangents to the circle intersecting at P .

P is a point on the polar of D .

Join CD and from P draw $PE \perp CD$.

Join CP cutting AB at F . Join CB .

$\therefore \angle s DEP, DFP$ are rt. $\angle s$,

$\therefore D, E, F, P$ are concyclic.

$\therefore CE \cdot CD = CF \cdot CP$.

But $CF \cdot CP = CB^2$.

$\therefore CE \cdot CD = CB^2$.

Then, since CD and CB are constants, CE must also be constant.

\therefore the polar of D must be the st. line $\perp CD$ through the fixed point E such that $CE \cdot CD = CB^2$.

51. If a point P lies on the polar of a point D with respect to a circle, then D lies on the polar of P .

In Figures 36 and 37 P lies on the polar of D .

To prove that D lies on the polar of P .

$\because CF \cdot CP = CB^2$, and $DF \perp CP$.

\therefore , by § 50, BFD is the polar of P .

$\therefore D$ lies on the polar of P .

52.—Exercises

1. P is a point at a distance of 4 c.m. from the centre of a circle of radius 6 c.m. Construct the polar of P .
2. P is a point at a distance of 7 cm. from the centre of a circle of radius 5 cm. Construct the polar of P .
3. Draw a st. line at a distance of 7 cm. from the centre of a circle of radius 4 cm. Construct the pole of the line.
4. When the point P is within the given circle, the polar of P falls without the circle; and when P is without the circle, the polar of P cuts the circle.
5. The polar of a point on the circumference is the tangent at that point.
6. P is a point without a given circle and the polar of P cuts the circle at A . Show that PA is a tangent to the circle.
Give a general statement of this theorem.
7. If any number of points are collinear, their polars with respect to any circle are concurrent.
8. Any number of lines pass through a given point; find the locus of their poles with respect to a given circle.

9. If **A** and **B** are two points such that **A** lies on the polar of **B** with respect to a circle, and consequently, **B** lies on the polar of **A**, and if **C** be the intersection of the polars of **A** and **B**, then the line joining **A** and **B** is the polar of **C**.

Definition.—A \triangle such that each side is the polar of the opposite vertex is said to be **self-conjugate**.

10. If a st. line **PAB** cut a circle at **A**, **B** and cut the polar of **P** at **C**, and if **D** be the middle point of **AB**,

$$PA \cdot PB = PC \cdot PD.$$

11. Two circles **ABC**, **ABD** cut orthogonally. Show that the polar of **D**, any point on the circle **ABD**, with respect to the circle **ABC** passes through **E**, the point diametrically opposite to **D**.

12. **A** is a given point and **B** any point on the polar of **A** with respect to a given circle. Show that the circle described on **AB** as diameter cuts the given circle orthogonally.

13. **ABC** is a \triangle inscribed in a circle, and a \parallel to **AC** through the pole of **AB** with respect to the circle meets **BC** at **D**. Show that **AD = CD**.

53. Any straight line which passes through a fixed point is cut harmonically by the point, any circle, and the polar of the point with respect to the circle.

P is the fixed point, **O** the centre of the circle, **PACB** any line through **P** cutting the circle at **A**, **B** and the polar **EC** of **P** with respect to the circle at **C**.

It is required to show that B, C, A, P is a harmonic range.

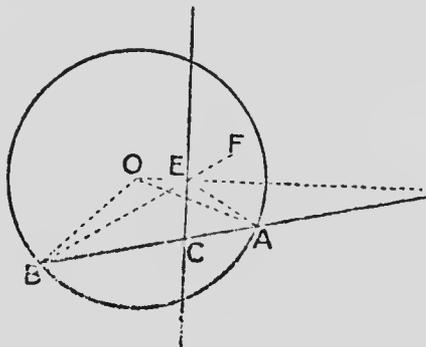


FIG. 38.

Join BO, OA, BE, EA and produce BE to F .

$PO \cdot OE = OA^2, \therefore PO : OA = OA : OE,$

and $\angle POA$ is common to the $\triangle s$ $POA, AOE,$

\therefore these $\triangle s$ are similar,

and consequently $\angle OEA = \angle OAP,$

$\therefore \angle PEA = \angle OAB = \angle OBA.$

Similarly, from $\triangle s$ $POB, BOE,$

$\angle OBA = \angle OEB = \angle FEP.$

$\therefore \angle FEP = \angle PEA;$

and since PEC is a rt. $\angle, \angle BEC = \angle CEA.$

Hence $\frac{BC}{CA} = \frac{BE}{EA} = \frac{BP}{PA};$

and B, C, A, P is a harmonic range.

Prove this theorem when the line PAB passes through the centre of the circle.

Prove this theorem when the point P is within the circle.

54. $ABCD$ is a quadrilateral inscribed in a circle. AB, DC are produced to meet at E ; BC, AD to meet at F , forming the complete quadrilateral $ABCDEF$.

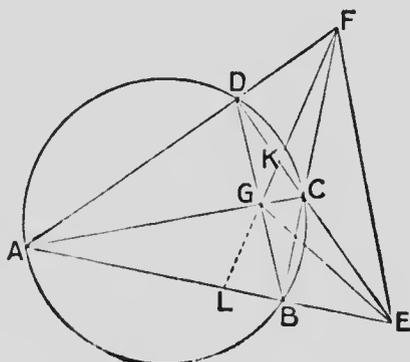


FIG. 39.

AC cuts BD at G , FG cuts AB at L .

From the complete quadrilateral $FDGCAB$, A, L, B, E
and D, K, C, E are harmonic ranges, (§ 48.)

$\therefore L$ and K are points on the polar of E ; (§ 52.)
that is, GF is the polar of E .

Similarly, GE is the polar of F .

Hence FE is the polar of G ; and the $\triangle EFG$ is self-conjugate with respect to the circle ABC .

55.—Exercises

1. Tangents AB, AC are drawn to a circle. The tangent at any point P cuts BC, CA, AB at X, Y, Z respectively. Show that X, Z, P, Y is a harmonic range.

2. If PM, QN be respectively drawn \perp to the polars of Q, P with respect to a circle whose centre is $O, PM : QN = OP : OQ$.

NOTE.—Draw $OK \perp PM, OL \perp QN$; and use similar Δs OPK, OQL .

3. A and B are two points such that the polar of either with respect to a circle, with centre O , passes through the other. Prove that the pole of AB is the orthocentre of the ΔAOB

4. D is the middle point of the hypotenuse BC of the rt.- $\angle d$ ΔABC . A circle is described to touch AD at A . Prove that the polar of either of the points B, C with respect to the circle passes through the other point.

5. If a quadrilateral be circumscribed about a circle, the st. lines joining the points of contact of opposite sides are concurrent with the two diagonals of the quadrilateral.

6. C is the middle point of a chord AB of a circle, and D, E are two points on the circumference such that CA bisects the $\angle DCE$. Prove that the tangents at D and E intersect on AB .

7. If, in figure 39, ABC is a fixed Δ and D is a variable point on the circle, prove that each side of the ΔEFG passes through a fixed point.

8. Using a ruler only, find the polar of a given point with respect to a given circle.

9. Using a ruler only, find the pole of a given st. line with respect to a given circle.

10. Using a ruler only, draw the tangents from a given external point to a given circle.

Miscellaneous Exercises

56.—EXERCISES ON LOCI

1. Construct the locus of a point such that the \perp s from it to two intersecting st. lines are in the ratio of two given st. lines.

2. A fixed point O is joined to any point A on a given st. line which does not pass through O . P is a point on OA such that the ratio of OP to OA is constant. Find the locus of P .

3. A fixed point O is joined to any point A on the circumference of a given circle, P is a point on OA such that the ratio of OP to OA is constant. Find the locus of P .

Find the locus when P is on AO produced.

4. A fixed point O is joined to any point A on a given st. line which does not pass through O . P is a point on OA such that the rect. $OP \cdot OA$ is constant. Show that the locus of P is a circle.

Find the locus when P is on AO produced.

5. Through a fixed point O within an $\angle YXZ$ draw a st. line MON , terminated in the arms of the \angle , and such that the rect. $OM \cdot ON$ has a given area.

6. Find the locus of a point such that the sum of the squares on its distances from the arms of a given rt. \angle is equal to the square on a given st. line.

7. The locus of a point, such that the sum of its distances from two given intersecting st. lines equals a given st. line, consists of the sides of a rectangle; and the locus of a point such that the difference of its distances from the intersecting st. lines equals the st. line, consists of the produced parts of the sides of the rectangle.

8. Given the base of a \triangle and the ratio of the other two sides, find the locus of the vertex.
9. AB is a fixed chord and AC a variable chord of a given circle; find the locus of the middle point of BC .
10. Find the locus of the points from which tangents drawn to two concentric circles are \perp to each other.
11. Construct the locus of the centre of the circle of given radius which intercepts a chord of fixed length on a given st. line.
12. Find the locus of the centre of a circle of radius a which cuts a given circle at an $\angle A$.
13. A circle rotates about a fixed point in its circumference. Show that the locus of the points of contact of tangents drawn \parallel to a fixed st. line consists of the circumferences of two circles.
14. In $\triangle ABC$, two circles touch AB at B and AC at C respectively and touch each other. Find the locus of their point of contact.
15. \triangle s are described on a given base and having a given vertical \angle . Find the loci of the middle points of their sides.
16. In a quadrilateral $ABCD$, AB is fixed in position, AC , BC and AD are given in length:—
- (a) Find the locus of the middle point of the other diagonal.
- (b) Find the locus of the middle point of the st. line joining the middle points of the two diagonals.
17. What is the locus of the point P when the st. line MN which joins the feet of the \perp s PM , PN drawn to two fixed lines OX , OY is of given length.

18. BAC is any chord passing through a fixed point A within a given circle with centre E . Circles described on BA , AC as chords touch the given circle internally at B , C respectively and cut each other at D . Show that the locus of D is a circle described on AE as diameter.
19. AB , CD are two chords of a circle, AB being fixed in position and CD of given length. Find the loci of the intersections of AD , BC and of AC , BD .
20. A and B are the centres of two circles which intersect at C ; through C a st. line is drawn terminated in the circumferences at D and E . DA , EB are produced to meet at P . Find the locus of P .
21. A transversal cuts the sides BC , CA , AB of a given $\triangle ABC$ at D , E , F respectively. The circumscribed circles of the \triangle s AFE , CED cut again at P . Find the locus of P .
22. From C , any point on the arc ACB , CD is drawn $\perp AB$; with centre C and radius CD a circle is described. Tangents from A and B to this circle are produced to meet at P . Find the locus of P .
23. Two similar \triangle s ABC , ABC' have a common vertex A , and the $\triangle ABC'$ rotates in the common plane about the point A . Show that the locus of the point of intersection of CC' and BB' is the circumscribed circle of $\triangle ABC$.
24. If a $\triangle ABC$ remains similar to itself while it turns in its plane about the fixed vertex A and the vertex B describes the circumference of a circle, find the locus of C .
25. OX , OY are two fixed st. lines and from them equal successive segments are cut off; AC , CE , etc. on OX ; BD , DF , etc., on OY . Show that the middle points of AB , CD , EF , etc., lie on a st. line \parallel to the bisector of the $\angle XOY$.

26. **AB** is the diameter of a given circle, **E** the centre and **C** any point on the circumference. Produce **BC** to **D** making $CD = BC$. Find the locus of the point of intersection of **AC** and **ED**.

27. A rectangle inscribed in a given \triangle **ABC** has one of its sides on **BC**. Show that the locus of the point of intersection of its diagonals is the line joining the middle point of **BC** to the middle point of the \perp from **A** to **BC**.

28. Any secant **ABD** is drawn from a given point **A** to cut a given circle at **B** and **D**. Through **A**, **B** and **A**, **D** respectively two circles are drawn to touch the given circle; find the locus of their second point of intersection.

29. Any chord **BAC** is drawn through a fixed point **A** within a circle. On **BC** as hypotenuse a rt.- \angle d \triangle **BPC** is described such that **A** is the projection of **P** on **BC**. Find the locus of **P**.

30. Any circle is drawn through the vertex of a given \angle . Find the loci of the ends of that diameter which is \parallel to the line joining the points where the circle cuts the arms of the \angle .

31. Through **C**, a point of intersection of two given circles, a st. line **ACB** is drawn terminated in the circumferences at **A** and **B**. Find the locus of the middle point of **AB**.

32. From a fixed point **P**, two st. lines **PA**, **PB**, at rt. \angle s to each other, are drawn to cut the circumference of a fixed circle at **A** and **B**. Find the locus of the middle point of **AB**.

33. A \parallel gm is inscribed in a given quadrilateral **ABCD** with sides \parallel **AC** and **BD**. The locus of the point of intersection of the diagonals of the \parallel gm is the st. line joining the middle points of the diagonals of the quadrilateral.

MAXIMA AND MINIMA

57. If a magnitude, such as the length of a st. line, an angle, or an area, varies, subject to given conditions, it is said to be a **maximum** when it has its greatest possible value; and a **minimum** when it has its least possible value.

58.—Exercises

1. **A** and **B** are two fixed points, and **CD** is a fixed st. line. Find the point **P** in **CD**, such that $PA + PB$ is a minimum.

(a) When **A** and **B** are on opposite sides of **CD**;

(b) When **A** and **B** are on the same side of **CD**.

2. **A** and **B** are two fixed points, and **CD** is a fixed st. line. Find the point **P** in **CD**, such that the difference between **PA** and **PB** is a minimum.

(a) When **A** and **B** are on the same side of **CD**;

(b) When **A** and **B** are on opposite sides of **CD**.

3. Two sides **AB**, **AC** of a \triangle are given in length. Find the \angle **A** for which the area of the \triangle is a max.

4. **A**, **B** are two fixed points. Find a point **P** such that $PA^2 + PB^2$ is a min.

5. **A**, **B**, **C** are three fixed points. Find a point **P** such that $PA^2 + PB^2 + PC^2$ is a min.

6. The rectangle contained by the two segments of a st. line is a max. when the st. line is bisected.

7. Through a given point within a given circle draw the chord of min. length.

8. A, B are two fixed points. Find a point P on a fixed circle such that $PA^2 + PB^2$ is a max. or min.
9. Of all Δ s of given base and given vertical \angle , the isosceles Δ has the greatest area.
10. Give examples showing that if a magnitude vary continuously, a maximum value is in a position where the magnitude is greater than in the positions close to it on either side; and a minimum value is in a position where the magnitude is less than in the positions close to it on either side.
11. Give examples showing that if a magnitude vary continuously, there must be between any two equal values of the magnitude at least one maximum or minimum value.
12. Find the max. and min. st. lines from a given point to a given circle.
13. Of all chords drawn through a given point within a circle, that which is bisected at the point cuts off the min. area.
14. Given two intersecting st. lines and a point within the \angle formed by them, of all st. lines drawn through the point and terminated in the st. lines that which is bisected by it cuts off the min. area.
15. Of all Δ s on the same base the isosceles Δ has—
(a) Min. perimeter when the area is given;
(b) Max. area when the perimeter is given.
16. Of all Δ s having a given area, the equilateral has min. perimeter.
17. Of all rt.- \angle d Δ s on the same hypotenuse the isosceles Δ has the max. perimeter.

18. Find a point on a given st. line such that the sum of the squares of its distances from two given points is a min.

19. **A** and **B** are two given points on the same side of a given st. line: find a point in the line at which **AB** subtends the ma.

20. Two towns are on opposite sides of a canal, unequally distant from it, and not opposite to each other. Where must a bridge be built across the sides of the canal, that the distance between the towns, by way of the bridge, may be a min.?

21. A \parallel gm is inscribed in a Δ by drawing from a point in the base st. lines to the sides. Find the position of the point for which the area of the \parallel gm is a max.

22. The max. rectangle inscribed in a given Δ equals half the Δ .

23. **A** is the centre of a given circle and **B** is a point without the circle. Draw a st. line **BCD** cutting the circle at **C** and **D** and such that the area of the Δ **ACD** is a max.

24. **A**, **B** are fixed points within a given circle. Find a point **P** on the circumference such that when **PA**, **PB** produced meet the circumference at **C**, **D** respectively, **CD** is a max.

25. Find the point in a given st. line from which the tangent drawn to a given circle is a min.

26. Through a point of intersection **A** of two circles draw the max. st. line terminated in the two circumferences.

(NOTE—Draw **CAD** \parallel the line of centres and any other st. line **EAF**. Join **C**, **D**, **E**, **F** to the other point of intersection and use similar Δ s.)

27. Of all Δ s inscribed in a given Δ , the orthocentric Δ has the least perimeter.
28. A, B are two points without a given circle. On the circle find a point P such that $\angle APB$ is a max. or a min.
29. A is a fixed point within the $\angle XOY$. In OX, OY find points C, D respectively, such that the perimeter of the ΔACD is a min.
30. The $\angle A$ of the ΔABC is fixed and the sum of AB, AC is constant. Prove that BC is a min. when $AB = AC$.
31. A is a fixed point within the $\angle XOY$. The st. line BAC cuts OX, OY at B, C . Prove that $BA \cdot AC$ is a min. when $OB = OC$.
32. If the area of the rectangle contained by two st. lines is given, the sum of the lines is a min. when they are equal.
33. If the sum of the squares on two lines is given, the sum of the lines is a max. when they are equal.
34. CAD is any st. line through a common point A of circles CAB, DAB . Prove that $CA \cdot AD$ is a max. when the tangents at C, D meet on BA produced.
35. Describe the maximum ΔDEF which is similar to a given ΔABC and has its sides EF, FD, DE passing respectively through fixed points P, Q, R which are not collinear.
36. A, B are fixed points on the same side of a fixed st. line XY . Place points P, Q on XY such that the distance PQ equals a given st. line and $AP + BQ$ is a min.
-

59.—THEOREMS

1. Concurrent st. lines drawn from the vertices of the $\triangle ABC$ cut the opposite sides BC, CA, AB respectively at D, E, F . Show that

$$\sin ACF \cdot \sin BAD \cdot \sin CBE = \sin FCB \cdot \sin DAC \cdot \sin EBA.$$
2. Concurrent st. lines drawn from the vertices of the $\triangle ABC$ cut the opposite sides BC, CA, AB respectively at D, E, F . Prove that the st. lines drawn through the middle points of BC, CA, AB respectively \parallel to AD, BE, CF are concurrent.
3. Show that the area of $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}$ where $2s = a + b + c$.
4. Find the area of the $\triangle ABC$ and also the radius of its circumcircle, given:—
 - (i) $a = 65$ mm., $b = 70$ mm., $c = 75$ mm.;
 - (ii) $a = 7$ cm., $b = 8$ cm., $c = 9$ cm.
5. If L, M, N , be the centres of the escribed circles of $\triangle ABC$, the circumscribed circle of $\triangle ABC$ is the N.-P. circle of $\triangle LMN$.
6. In Ex. 5 if I be the centre of the inscribed circle, P the point where the circumscribed circle cuts IL and $PH \perp AC$, AH equals half the sum and CH half the difference of b and c .
7. If O be the orthocentre of $\triangle ABC$, A, B, C, O are the centres of the circles which touch the sides of the pedal \triangle .
8. The Simpson's line of a point P bisects PO , where O is the orthocentre of $\triangle ABC$.
9. CA, CB are two tangents to a circle; E is the foot of the \perp from B on the diameter AD ; prove that CD bisects BE .

10. The \perp from the vertex of the rt. \angle on the hypotenuse of a rt.- \angle d \triangle is a harmonic mean between the segments of the hypotenuse made by the point of contact of the inscribed circle.

11. The side of a square inscribed in a \triangle is half the harmonic mean between the base and the \perp from the vertex to the base.

12. The circumscribed centre of a \triangle is the orthocentre of the \triangle formed by joining the middle points of its sides; and the two \triangle s have a common centroid.

13. ABC is a \triangle . Describe a circle to touch AC at C and pass through B . Describe another circle to touch BC at B and pass through A . Let P be the second point of intersection of these circles. Show that $\angle ACP = \angle CBP = \angle BAP$; and that the circumscribed circle of $\triangle APC$ touches BA at A . Find another point Q such that $\angle QBA = \angle QAC = \angle QCB$.

14. O is the orthocentre of $\triangle ABC$, AX , BY , CZ are the \perp s from A , B , C on the opposite sides, BD is a diameter of the circumscribed circle. Show that:—

(a) $DC = AO$;

(b) $AO^2 + BC^2 = BO^2 + CA^2 = CO^2 + AB^2 =$ the square on the diameter of the circumscribed circle.

15. If a \triangle be formed with its sides equal to AD , BE , CF , the medians of $\triangle ABC$, the medians of the new \triangle will be respectively three-fourths of the corresponding sides of the original \triangle .

16. The opposite sides of a quadrilateral inscribed in a circle are produced to meet; show that the bisectors of the two \angle s so formed are \perp to each other.

17. AG is a median of the $\triangle ABC$. $BDEF$ cuts AG , AC and the line through $A \parallel BC$ at D , E , F respectively. Show that B , D , E , F is a harmonic range.

18. If A , C , B , D be a harmonic range, show that

$$\frac{2}{AB} = \frac{1}{AC} + \frac{1}{AD}.$$

19. A , B are the centres of two circles. A common tangent, direct or transverse, cuts the line of centres at S . Show that the corresponding ends of two \parallel diameters of the circles are in the same st. line with S .

20. A , B are the centres of two circles of radii R , r . The transverse and direct common tangents intersect the line of centres at P , Q . Show that

$$\frac{AP}{PB} = \frac{AQ}{QB} = \frac{R}{r}.$$

NOTE.—If A , B be the centres of two circles, and points P , Q be found in AB and AB produced such that $\frac{AP}{PB} = \frac{AQ}{QB} = \frac{R}{r}$, the points P , Q are called the **centres of similitude** of the circles.

21. If a circle touch two fixed circles, the line joining the points of contact passes through a centre of similitude of the two circles.

22. In a system of coaxial circles the two limiting points and the points in which any one circle of the system cuts the line of centres form a harmonic range.

23. If two circles cut orthogonally, any diameter of one which cuts the other is cut harmonically by that other.

24. The six centres of similitude of three circles lie three by three on four st. lines.

25. The sides AB , BC , CD , DA of a quadrilateral touch a circle at E , F , G , H respectively. Show that the opposite vertices of $ABCD$, the intersection of the diagonals of $EFGH$, and the intersections of the opposite sides of $EFGH$ form two sets of collinear points.

26. A and B are the centres of two circles and C the point where their radical axis cuts AB . Show that the locus of the centres of circles which bisect the two circumferences is a st. line \parallel to the radical axis and cutting AB at D so that $AD = BC$.

27. The middle points of the diagonals of a complete quadrilateral are collinear.

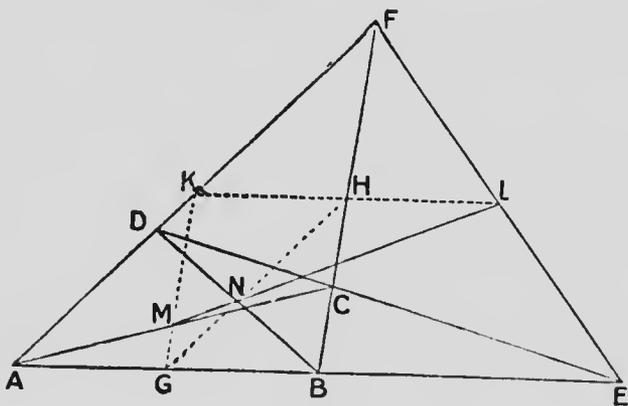


FIG. 40.

$ABCDEF$ is a complete quadrilateral; L , M , N the middle points of its diagonals.

Draw $LHK \parallel AE$, $KMG \parallel FB$ and join GH .

Prove L , M , N collinear.

28. $ABCD$ is a quadrilateral and O is a point within it such that $\triangle AOB + \triangle COD = \triangle BOC + \triangle AOD$; show that the locus of O is the st. line joining the middle points of the diagonals AC and BD .

29. If a quadrilateral be circumscribed about a circle, the centre of the circle is in the st. line joining the middle points of the diagonals.

30. G is any point in the base BC of the $\triangle ABC$ and O is a point within the \triangle such that $\triangle AOB + \triangle COG = \triangle AOC + \triangle BOG$; show that the locus of O is the st. line joining the middle points of BC and AG .

31. G is the point of contact of the inscribed circle of $\triangle ABC$ with BC . It is required to show that the centre of the circle is in the st. line joining the middle points of BC and AG .

32. A is a fixed point on a given circle and P is a variable point on the circle. Q is taken on AP produced so that $AQ : AP$ is constant. Show that the locus of Q is a circle which touches the given circle at A .

33. S is a centre of similitude of two circles $PQT, P'Q'T'$, and a variable line through S cuts the circles at the corresponding points $P, P'; Q, Q'$. Prove that, if STT' is a common tangent

$$SP \cdot SQ' = SP' \cdot SQ = ST \cdot ST'.$$

34. AD is a median of the $\triangle ABC$. A st. line CLM cuts AD at L and AB at M . Prove that $ML : LC = AM : AB$.

35. The locus of the point at which two given circles subtend equal \angle s is the circle described on the join of their centres of similitude as diameter.

36. Having the N.-P. circle and one vertex of a \triangle given, prove that the locus of its orthocentre is a circle.

37. AB is a chord of a circle and the tangents at A, B meet at C . From any point P on the circle \perp s PX, PY, PZ are drawn to BC, CA, AB respectively. Prove that $PX \cdot PY = PZ^2$.

38. Prove that the radical axis of the inscribed circle of $\triangle ABC$ and the escribed circle which touches BC and AB, AC produced bisects BC .

39. If a st. line is divided in medial section and from the greater segment a part is cut off equal to the less, show that the greater segment is divided in medial section.

40. If a st. line is divided in medial section, the rectangle contained by the sum and difference of the segments is equal to the rectangle contained by the segments.

41. In Figure 22, show that the centre of the circumcircle of $\triangle HBC$ lies on the circumcircle of $\triangle AHC$.

42. The st. line which joins any two points is the polar, with respect to a given circle, of the point of intersection of their polars.

43. The point of intersection of any two st. lines is the pole of the st. line which joins their poles.

44. Show that the radius of a circle inscribed in an equilateral \triangle is one-third of that of any one of the escribed circles.

45. The five diagonals of a regular pentagon intersect at five points within it. Show that the area of the pentagon with these points for vertices is $\frac{7-3\sqrt{5}}{2} A$, where A is the area of the given pentagon.

46. $ABCD$ is a rectangle. If A, P, C, Q and B, R, D, S are each harmonic ranges, show that P, Q, R, S are concyclic.

47. If one pair of opposite sides of a cyclic quadrilateral when produced intersect at a fixed point, prove that the other pair when produced intersect on a fixed st. line.

What is the connection between the fixed point and the fixed st. line ?

48. A, B, C, D and P, Q, R, S are harmonic ranges and AP, BQ, CR are concurrent at a point O . Prove that DS passes through O .

49. A, B, C, D and A, E, F, G are harmonic ranges on two st. lines AD, AG . Prove that BE, CF, DG are concurrent.

60.—PROBLEMS

1. Draw a st. line, terminated in the circumferences of two given circles, equal in length to a given st. line, and \parallel to a given st. line.

2. Through a given point on the circumference of a circle draw a chord which shall be bisected by a given chord.

3. In the hypotenuse of a rt.- \angle d \triangle find a point such that the sum of the \perp s on the arms of the rt. \angle equals a given st. line. What are the limits to the length of the given st. line?

4. In the hypotenuse of a rt.- \angle d \triangle find a point such that the difference of the \perp s on the arms of the rt. \angle equals a given st. line.

When will there be two, one or no solutions?

5. In the hypotenuse of a rt.- \angle d \triangle find a point such that the \perp s on the arms of the rt. \angle are in a given ratio.

6. Through a given point draw a st. line terminated in the circumferences of two given circles and divided at the given point in a given ratio.

7. In a given circle inscribe a rectangle having its perimeter equal to a given st. line.

8. In a given circle inscribe a rectangle having the difference between adjacent sides equal to a given st. line.

9. In a given circle inscribe a rectangle having its sides in a given ratio.
10. In a circle of radius 5 cm. inscribe a rectangle having its area 22 sq. cm.
11. **A** and **B** are fixed points on the circumference of a given circle. Find a point **C** on the circumference such that **CA**, **CB** intercept a given length on a fixed chord.
12. **A** and **B** are fixed points on a circumference. Find a point **C** on the circumference such that **CA**, **CB** cut a fixed diameter at points equally distant from the centre.
13. In a given \triangle find a point such that the \perp s from it to the sides are proportional to the lengths of the sides.
14. Two towns are on different sides of a straight canal, at unequal distances from it, and not opposite to each other. Where must a bridge be built \perp to the direction of the canal so that the towns may be equally distant from the bridge?
15. Divide a given st. line into two parts so that the squares on the two parts are in the ratio of two given st. lines.
16. Construct the locus of a point the difference of the squares of whose distances from two points 3 inches apart is 5 square inches.
17. Two points **A** and **B** are four inches apart. Construct the locus of the point the sum of the squares of whose distances from **A** and **B** is 20.5 square inches.
18. Divide a given st. line into two parts such that the sum of the squares on the whole st. line and on one part is twice the square on the other part.

19. Two non-intersecting circles have their centres at **A** and **B**, and **C** is a point in **AB**. Draw a circle through the point **C** and coaxial with the two given circles.

(NOTE.—Use Ex. 22, § 59.)

20. Construct a \triangle having one side and two medians equal to three given st. lines. (Two cases.)

21. Construct a \triangle having the three medians equal to three given st. lines.

22. Given the vertical \angle , the ratio of the sides containing it, and the diameter of the circumscribing circle; construct the \triangle .

23. Given the feet of the \perp s drawn from the vertices of a \triangle to the opposite sides; construct the \triangle .

24. Draw a circle to touch a given circle, and also to touch a given st. line at a given point.

25. Draw a circle to pass through two given points and touch a given circle.

26. Draw a circle to pass through a given point and touch two given intersecting st. lines.

27. **AB** is the chord of a given segment of a circle. Find a point **P** on the arc such that **AP** + **BP** is a maximum.

28. Find a point **O**, within a \triangle **ABC** such that:—

$$(1) \triangle AOB : \triangle BOC : \triangle COA = 1 : 2 : 3;$$

$$(2) \triangle AOB : \triangle BOC : \triangle COA = l . m : n.$$

29. Find a point such that its distances from the three sides of a \triangle may be proportional to three given st. lines.

30. Through a given point within a circle draw a chord which shall be divided in a given ratio at the given point.

31. A, B, C, D are points in a st. line. Find a point at which AB, BC, CD subtend equal \angle s.
32. Given a vertex, the orthocentre and the centre of the N.-P. circle of a \triangle , construct the \triangle .
33. Having divided a st. line internally in medial section, find the point of external division in medial section by ratio and proportion.
34. Describe an equilateral \triangle with one vertex at a given point and the other two vertices on two given \parallel st. lines.
35. Find the locus of the middle point of the chord of contact of tangents drawn from a point on a given st. line to a given circle.
-

PART II.—ANALYTICAL GEOMETRY

FORMULÆ

The following important results from algebra and trigonometry are frequently used in analytical geometry :—

1. The roots of the quadratic equations $ax^2 + 2bx + c = 0$ are

$$\frac{-b + \sqrt{b^2 - ac}}{a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - ac}}{a}.$$

These roots are

- | | |
|---|------------------------------------|
| real, | if $b^2 >$, or $=$, ac ; |
| imaginary, | if $b^2 < ac$; |
| equal to each other, | if $b^2 = ac$; |
| equal in magnitude but
opposite in sign, | if $b = 0$. |
| rational, | if $b^2 - ac$ is a perfect square. |
| One root = 0, | if $c = 0$; |
| both roots = 0, | if $b = c = 0$. |

The sum of the roots = $\frac{-2b}{a}$.

The product of the roots = $\frac{c}{a}$.

2. The fraction $\frac{a}{b} = \infty$, if $b = 0$ and a is not = 0.
3. The equation $ax + by + c = 0$ is the same as $px + qy + r = 0$,

$$\text{if } \frac{a}{p} = \frac{b}{q} = \frac{c}{r} .$$

4. $ax^2 + 2bx + c$ is a perfect square, if $b^2 = ac$.

5. If $a_1x + b_1y + c_1z = 0$
and $a_2x + b_2y + c_2z = 0$,

then $\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$.

iii

6. For all values of a ,

$$\sin^2 a + \cos^2 a = 1.$$

7. If $\tan a = k$, $a = \tan^{-1}k$.

$$8. \sin(A \pm B) = \sin A \cos B \pm \cos A \sin B.$$

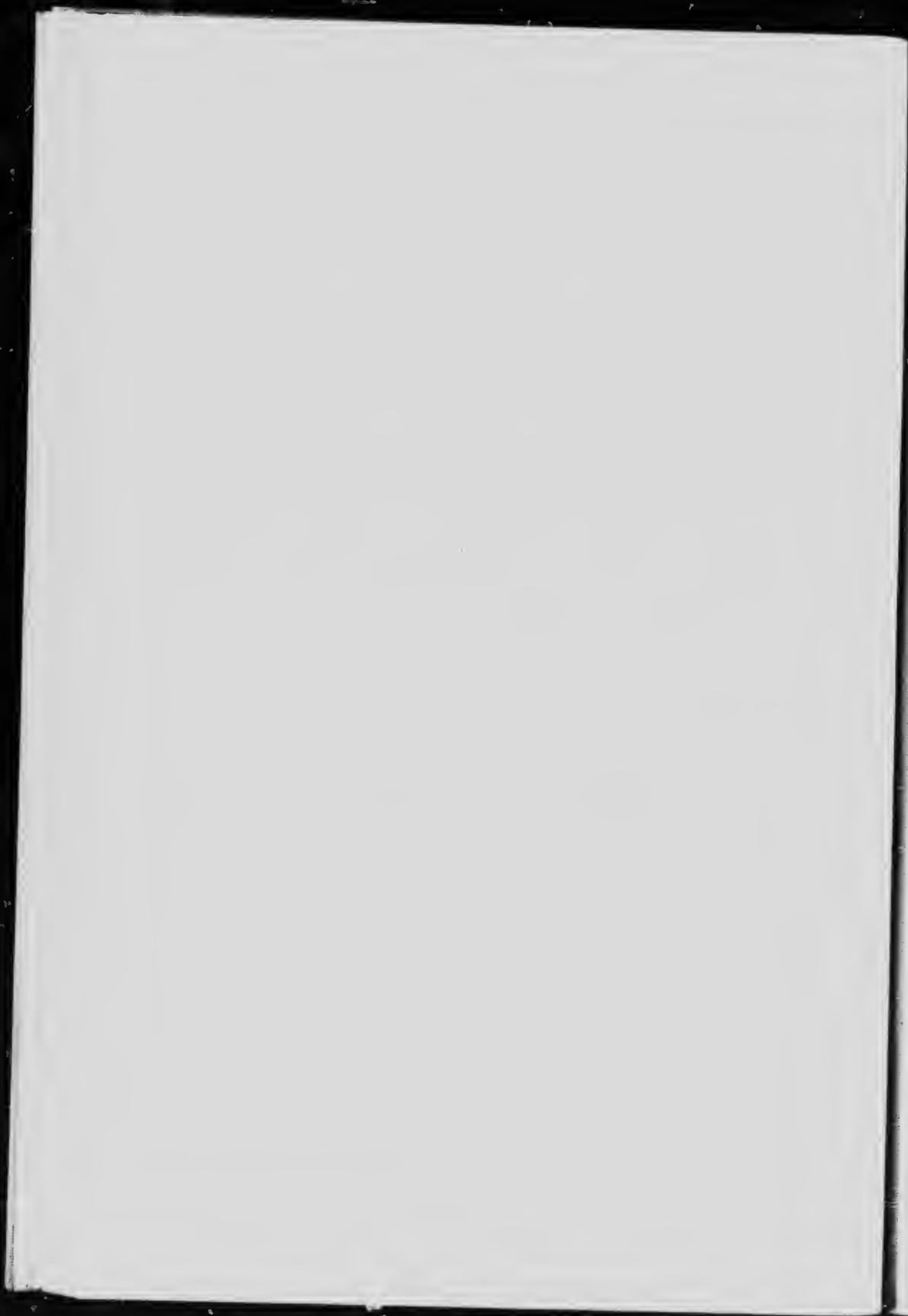
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

$$9. \sin A + \sin B = 2 \sin \frac{A+B}{2}, \cos \frac{A-B}{2}, \text{ etc.}$$

$$10. \tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$$

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ELEMENTARY ANALYTICAL GEOMETRY

CHAPTER I

CARTESIAN COORDINATES

1. **Analytical, or algebraic, geometry** was invented by Descartes in 1637, and this invention marks the beginning of the history of the modern period of mathematics. It differs from pure geometry in that it lays down a general method, in which, by a few simple rules, any property can be at once proved or disproved, while in the latter each problem requires a special method of its own.

2. **The Origin.** In plane analytical geometry the positions of all points in the plane are determined by their distances and directions as measured from a fixed point.

If the points are all in a st. line, the fixed point is most conveniently taken in that line.



FIG. 1.

Thus, if the distance and direction of each of the points A, B, C, D, E from the point O are given, the positions of these points are known.

The point O is called the **origin**, or **pole**.

3. **Use of plus and minus.** In algebra the signs plus and minus are used to indicate opposite qualities of the numbers to which they are prefixed; and in analytical geometry, as in trigonometry, these signs are used to show difference of direction. In a horizontal st. line distances measured from the origin to the right are taken to be positive, while those to the left are negative; and in a vertical st. line distances measured upward are positive, while those measured downward are negative. Thus, in Fig. 1, if $OA = 2$ cm., $OB = 3$ cm., $OC = 5$ cm., $OD = 1$ cm., and $OE = 3$ cm., the positions of these points are respectively represented by 2, 3, 5, -1 and -3 , the understood unit being one centimetre.

RECTANGULAR COORDINATES

4. **Coordinates.** When points are not in the same st. line, their positions are determined by their distances

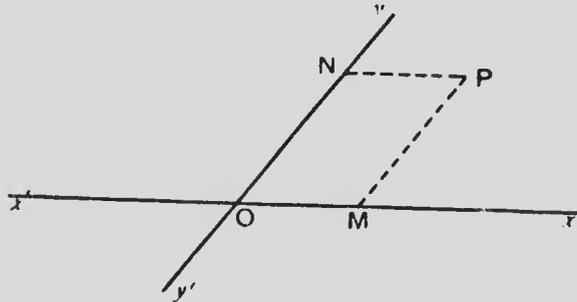


FIG. 2.

from two st. lines $x'Ox$ and $y'Oy$ drawn through the origin, the distances being measured in directions \parallel to the given st. lines.

These lines are called the **axes of coordinates**, or shortly, the **axes**.

$x'Ox$ is called the axis of x , and $y'Oy$ is called the axis of y .

From a point P draw $PM \parallel Oy$ and $PN \parallel Ox$, terminated in the axes.

PM is called the **ordinate** of P , and $PN (= OM)$, is called the **abscissa** of P . These two distances, the abscissa and ordinate, are called the **coordinates** of the point.

Sometimes, from the name of the inventor, they are spoken of as **cartesian coordinates**.

5. **Rectangular coordinates.** When the axes are at $rt. \angle$ s to each other, the distances of a point from the axes are called its **rectangular coordinates**.

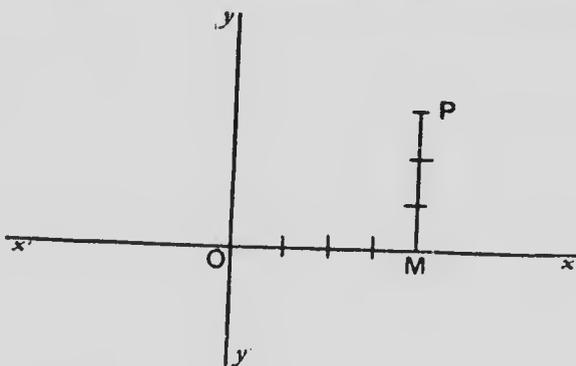


FIG. 3.

To locate the point of which the abscissa is 4 and the ordinate 3 when the coordinates are rectangular,

measure the distance $OM = 4$ units along Ox and at M erect the $\perp PM = 3$ units. P is the required point.

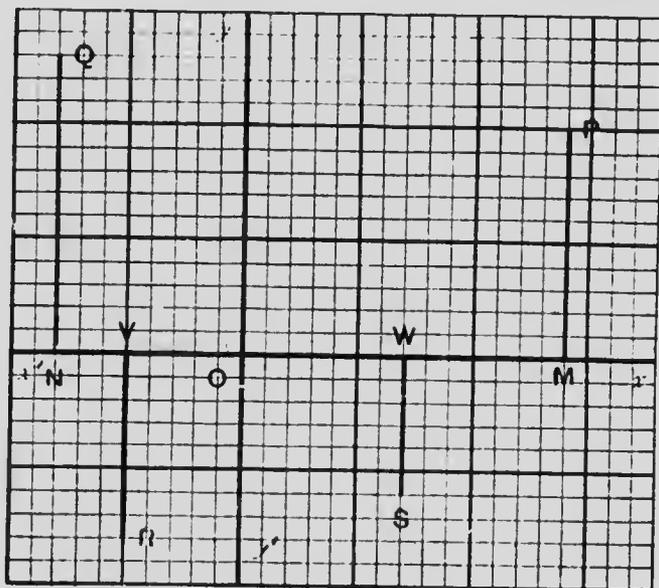


FIG. 4. (Unit = $\frac{1}{2}$ inch.)

In Fig. 4, the abscissa of $P = OM = 2.8$, the ordinate of $P = PM = 2$. The position of this point is then indicated by the notation $(2.8, 2)$. For Q , the abscissa = $ON = -1.6$, the ordinate = $QN = 2.6$ and the position of the point is indicated by $(-1.6, 2.6)$.

Similarly the position of R is $(-1, -1.6)$, and that of S is $(1.4, -1.2)$.

xOy , yOx' , $x'Oy'$ and $y'Or$ are respectively called the first, second, third and fourth quadrants; and we see from the diagram, that:—

for a point in the first quadrant both coordinates are positive;

for a point in the second quadrant the abscissa is negative and the ordinate is positive;

for a point in the third quadrant both are negative;

and for a point in the fourth the abscissa is positive and the ordinate is negative.

Thus the signs of the coordinates show at once in which quadrant the point is located.

6.—Exercises

1. Write down the coordinates of the points A, B, C, D, E, F, G, H and O in Fig. 5.

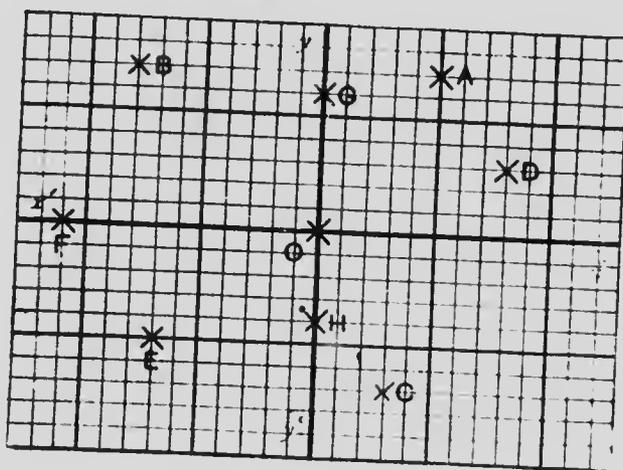


FIG. 5. (Unit = $\frac{1}{10}$ inch.)

2. Draw a diagram on squared paper, and mark on it the following points:—A (4, 3), B (4.6, 0), C (-2, -3), D (-4, 2), E (0, 2.8). Indicate the unit of measurement on the diagram.

3. Draw a diagram on squared paper and mark the following points:—(4, 3), (3, 4), (-3, -4), (-4, 3), (0, 5), (0, -5), (-5, 0). Describe a circle with centre O and

radius b . Should the circle pass through the seven points? Why?

4. The side of an equilateral $\triangle = 2a$. One vertex is at the origin, one side is on the axis of x and the \triangle is in the first quadrant. What are the coordinates of the three vertices?

5. One corner of a square is taken as origin and the axes coincide with two sides. The length of a side is b . What are the coordinates of the corners, the square being in the first quadrant?

THE DISTANCE BETWEEN TWO POINTS

7. In general, the abscissa of a point is represented by x , the ordinate by y .

8. To find the distance between a point $P (x_1, y_1)$ and the origin.

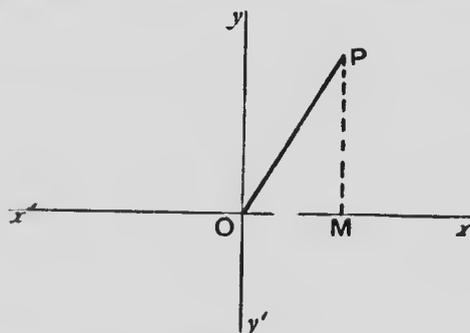


FIG. 6.

From P draw $PM \perp O.c.$

$\therefore PMO$ is a rt.- \angle d \triangle ,

$\therefore PO^2 = OM^2 + PM^2$

$$= x_1^2 + y_1^2.$$

$\therefore PO = \sqrt{x_1^2 + y_1^2}.$

9. To find the distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$.

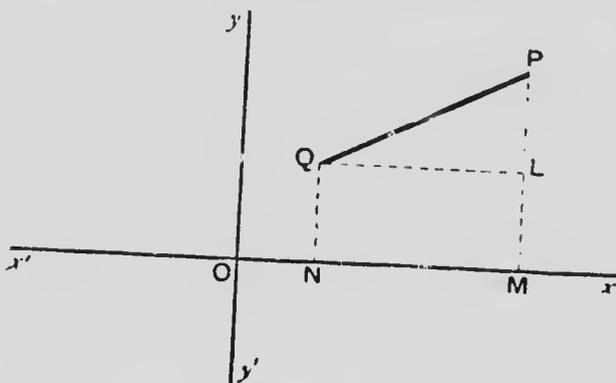


FIG. 7.

Draw PM and $QN \perp Ox$; $QL \perp PM$.

$$QL = NM = OM - ON = x_1 - x_2$$

$$PL = PM - LM = PM - QN = y_1 - y_2$$

$\therefore PLQ$ is a rt.- \angle d Δ ,

$$\therefore PQ^2 = QL^2 + PL^2$$

$$= (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$\therefore PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

10. If the point Q in § 9 coincides with the origin O , $x_2 = 0$ and $y_2 = 0$. Substituting these values, in the expression for PQ in that article we obtain

$$PO = \sqrt{x_1^2 + y_1^2}$$

This shows that the result in § 8 is a particular case of that in § 9

11. The result in § 9 holds good, *in the same form*, for any two points whether the coordinates are positive or negative.

For example—it is required to find the distance between $P (-3, 2)$ and $Q (5, -2)$.

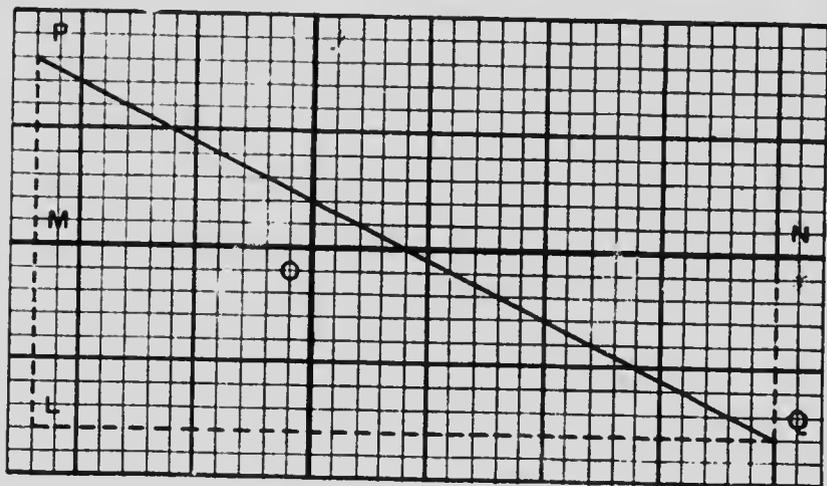


FIG. 8. (Unit = $\frac{1}{8}$ inch.)

Draw $PM, QN \perp Ox$; $QL \perp PM$.

The length of $ML =$ length of $NQ = 2$.

$$\therefore PL = PM + ML = 2 + 2 = 4.$$

The length of $QL = NM = 5 + 3 = 8$.

$$\begin{aligned} PQ^2 &= QL^2 + PL^2 \\ &= 64 + 16 = 80. \end{aligned}$$

$$\therefore PQ = 4\sqrt{5}.$$

If in the expression for PQ found in § 9, we substitute -3 for x_1 , 2 for y_1 , 5 for x_2 and -2 for y_2 , we obtain

$$\begin{aligned} PQ &= \sqrt{(-3 - 5)^2 + (2 + 2)^2} \\ &= 4\sqrt{5}, \end{aligned}$$

the same result.

12. The particular cases in §§ 10 and 11 illustrate what is known as the **continuity** of the formulæ in analytical geometry. Here continuity means, that *general* results which are obtained when the coordinates in the diagram used are all positive hold true in the same form for all points.

13. To find the coordinates of the middle point of the distance between two given points P (x_1, y_1) and Q (x_2, y_2).

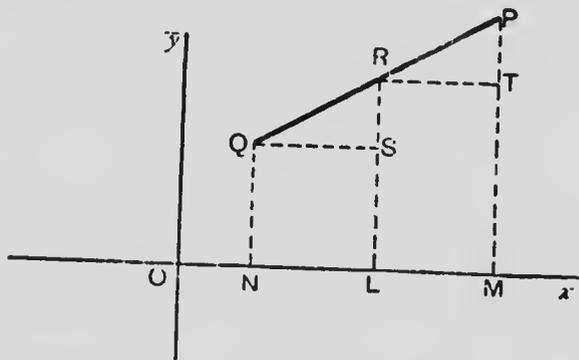


FIG. 9.

Let R (x, y) be the middle point of PQ.

Draw PM, QN, RL \perp Ox; QS \perp RL; RT \perp PM.

From the equality of Δ s PRT, RQS,

$$QS = RT \text{ and } RS = PT.$$

$$\therefore NL = LM,$$

$$\therefore x - x_2 = x_1 - x.$$

$$\therefore x = \frac{x_1 + x_2}{2}.$$

$$\therefore RS = PT,$$

$$\therefore y - y_2 = y_1 - y.$$

$$\therefore y = \frac{y_1 + y_2}{2}.$$

Thus the coordinates of R are

$$\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}$$

14. To find the coordinates of the point dividing the distance between P ($x_1 y_1$) and Q ($x_2 y_2$) in the ratio of m to n .

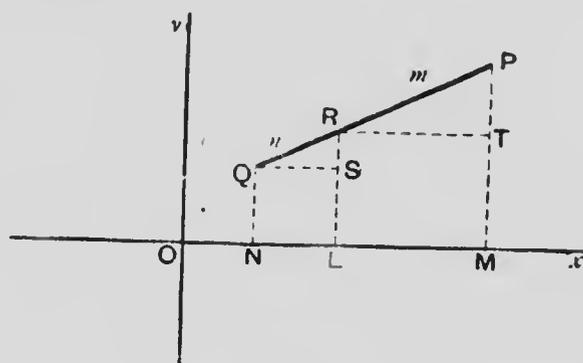


FIG. 10.

Let R (x, y) be the point dividing PQ such that

$$\frac{PR}{RQ} = \frac{m}{n}$$

Draw PM, QN, RL \perp O*x*; QS \perp RL; RT \perp PM.

From the similar Δ s PRT, RQS

$$\frac{RT}{QS} = \frac{PT}{RS} = \frac{PR}{RQ} = \frac{m}{n}$$

$$\therefore \frac{RT}{QS} = \frac{m}{n}$$

$$\therefore \frac{x_1 - x}{x - x_2} = \frac{m}{n}$$

$$\therefore mx - mx_2 = nx_1 - nx$$

$$\therefore x = \frac{nx_1 + mx_2}{m + n}.$$

$$\therefore \frac{PT}{RS} = \frac{m}{n},$$

$$\therefore \frac{y_1 - y}{y - y_2} = \frac{m}{n}.$$

$$\therefore ny - my_2 = ny_1 - ny.$$

$$\therefore y = \frac{ny_1 + my_2}{m + n}.$$

Thus the coordinates of **R** are

$$\frac{nx_1 + mx_2}{m + n}, \frac{ny_1 + my_2}{m + n}.$$

15. If the point **R** be taken in **PQ** produced such that $PR : RQ = m : n$, and the coordinates of **P**, **Q** be (x_1, y_1) , (x_2, y_2) it may be shown by a proof similar to that in the previous article that the coordinates of **R** are

$$\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}.$$

These results and also those of §§ 13 and 14 are the same for oblique and rectangular axes

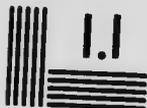
16.—Exercises

1. Find the distance between the points (6, 5) and (1, -7) and test your result by measurement on squared paper.
2. Find the distance between the points (2, -3) and (-1, 1) and test your result by measurement on squared paper.
3. Find the coordinates of the middle points of the st. lines joining the pairs of points in exercises 1 and 2 respectively and test the results by measurements on the diagrams.



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4. Find, to two decimal places, the distance between $(-3, 7)$ and $(4, -4)$.
5. The vertices of a \triangle are $(-2, 4)$, $(-8, -4)$ and $(7, 4)$. Find the lengths of its sides.
6. The vertices of a \triangle are $(-1, 5)$, $(-4, -2)$, $(5, -3)$. Find (a) the lengths of the sides; (b) the lengths of the medians.
7. The vertices of a quadrilateral are $(4, 3)$, $(-5, 2)$, $(-3, -4)$, $(6, -2)$. Find the lengths of its sides, and also of its diagonals.
8. Find the coordinates of the middle point of the st. line joining $(3, -2)$ and $(-3, 2)$.
9. Find the points of trisection of the st. line joining $(1, 3)$ and $(6, 1)$.
10. The st. line joining $P(-4, -3)$ and $Q(6, -1)$ is divided at $R(x, y)$ so that $PR : RQ = 5 : 2$. Show that $x = -2y$.
11. Find the length of the st. line joining the origin to $(a, -b)$.
12. The st. line joining the origin to $P(-4, 7)$ is divided at R, Q so that $OR : RQ : QP = 3 : 4 : 2$. Find the distance RQ .
13. The length of a st. line is 17 and the coordinates of one end are $(-5, -8)$. If the ordinate of the other end is 7, find its abscissa.
14. Find in its simplest form the equation which expresses the fact that (x, y) is equidistant from $(5, 2)$ and $(3, 7)$.
15. Find the centre and radius of the circle which passes through $(5, 2)$, $(3, 7)$ and $(-2, 4)$.

16. Find the points which are distant 15 from $(-2, -10)$ and 13 from $(2, 14)$.

17. Prove that the vertices of a rt.- \angle d Δ are equidistant from the middle point of the hypotenuse.

Suggestion:—Take the vertex of the rt. \angle for origin and the sides which contain the rt. \angle for axes.

18. In any ΔABC prove that

$$AB^2 + AC^2 = 2(AD^2 + DC^2),$$

where D is the middle point of BC.

Suggestion:—Take D as origin, DC as axis of x and the \perp to BC at D as axis of y. Let DC = a, and the coordinates of A be (x_1, y_1) .

19. If D is a point in the base BC of a ΔABC such that $BD : DC = m : n$, show that

$$n AB^2 + m AC^2 = (m + n) AD^2 + n BD^2 + m DC^2.$$

Suggestion:—Take D as origin, DC as axis of x and the \perp to BC at D as axis of y. Let $BD = -ma$, $DC = na$, and the coordinates of A be (x_1, y_1) .

20. The vertices of a Δ are the points (x_1, y_1) , (x_2, y_2) , (x_3, y_3) . Find the coordinates of its centroid.

21. The st. line joining A $(2, 1)$ to B $(5, 9)$ is produced to C so that $AC : BC = 7 : 2$. Find the coordinates of C.

22. The st. line joining A $(3, -2)$ to B $(-4, -6)$ is produced to C so that $AC : BC = 3 : 2$. Find the coordinates of C.

THE AREA OF A TRIANGLE

17. To find the area of the \triangle of which the vertices are A (x_1, y_1) , B (x_2, y_2) and C (x_3, y_3) .

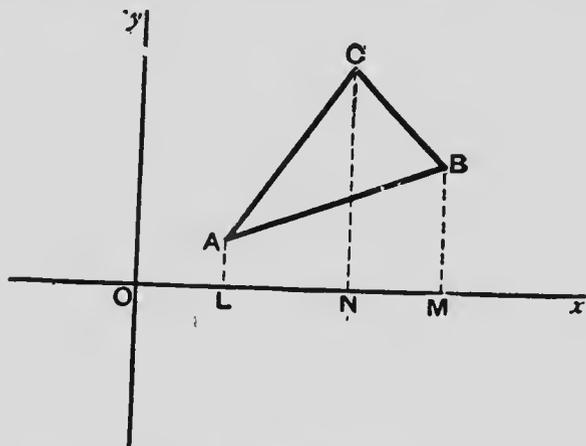


FIG. 11.

Draw the ordinates AL, BM, CN.

From the diagram,

$$\triangle ABC = \text{ALNC} + \text{CNMB} - \text{ALMB}.$$

The area of a quadrilateral of which two sides are \parallel = half the sum of the \parallel sides \times the distance between the \parallel sides.

$$\therefore \text{ALNC} = \frac{1}{2} (\text{AL} + \text{CN}) \times \text{LN} = \frac{1}{2} (y_1 + y_3) (x_3 - x_1),$$

$$\text{CNMB} = \frac{1}{2} (\text{CN} + \text{BM}) \times \text{NM} = \frac{1}{2} (y_3 + y_2) (x_2 - x_3),$$

$$\text{ALMB} = \frac{1}{2} (\text{AL} + \text{BM}) \times \text{LM} = \frac{1}{2} (y_1 + y_2) (x_2 - x_1).$$

$$\therefore \triangle ABC = \frac{1}{2} \{ (y_1 + y_3) (x_3 - x_1) + (y_3 + y_2) (x_2 - x_3) - (y_1 + y_2) (x_2 - x_1) \}.$$

Simplifying,

$$\triangle ABC = \frac{1}{2} \{ x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) \}.$$

NOTE.—The points have been taken in circular order about the \triangle in the opposite direction to that in which the hands of a clock rotate; if they are taken in the same direction as the hands rotate, the formula will give the same result only it will appear to be negative; but, of course the area of a \triangle must be positive.

18. To find the area of the \triangle of which the vertices are $(3, 2)$, $(-4, 3)$, $(-2, -4)$.

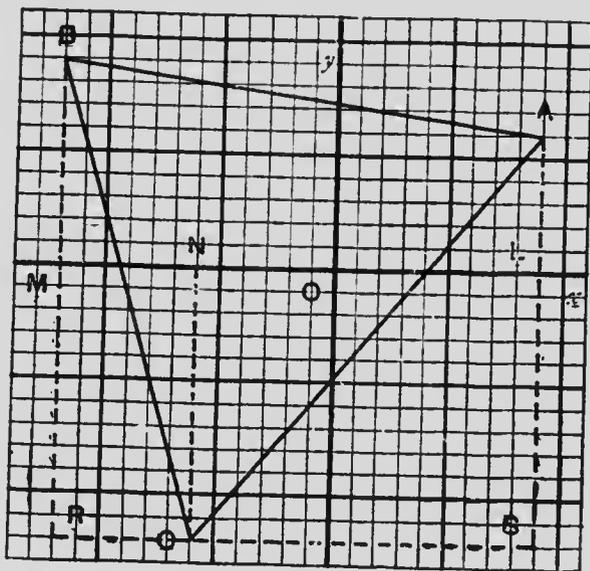


FIG. 12. (Unit = $\frac{1}{10}$ inch.)

Draw the diagram on squared paper. Draw the ordinates AL , BM . Through C draw $RCS \parallel Ox$ to meet AL , BM produced at S , R .

$$\triangle ABC = BRSA - \triangle BRC - \triangle ACS.$$

$$BRSA = \frac{1}{2} (BR + AS) RS = \frac{1}{2} (7 + 6) \times 7 = \frac{91}{2}.$$

$$\triangle BRC = \frac{1}{2} BR \times RC = \frac{1}{2} \times 7 \times 2 = \frac{14}{2}.$$

$$\triangle ASC = \frac{1}{2} AS \times SC = \frac{1}{2} \times 6 \times 5 = \frac{30}{2}.$$

$$\therefore \triangle ABC = \frac{91 - 14 - 30}{2} = \frac{47}{2}.$$

If we substitute the coordinates of **A**, **B** and **C** in the formula of § 17, we obtain

$$\begin{aligned} \triangle ABC &= \frac{1}{2} \{3(3+4) + (-4)(-4-2) + (-2)(2-3)\} \\ &= \frac{1}{2} (21 + 24 + 2) = \frac{47}{2}; \end{aligned}$$

the same result as before.

This illustrates the continuity of the symmetrical result found in § 17 for the area of a \triangle .

19.—Exercises

1. Find, from a diagram, the area of the \triangle of which the vertices are (o, o) , (a, b) , (c, d) . Check your result by using the formula of § 17.

2. Draw the following \triangle s on squared paper and find their areas; checking your results by using the formula of § 17:—

$$(a) (1, 4), (-2, 2), (5, -1);$$

$$(b) (4, -2), (-5, -1), (-2, -6);$$

$$(c) (0, 0), (3, 4.5), (-2.5, 4).$$

3. Find the area of the quadrilateral of which the vertices are $(3, 6)$, $(-2, 4)$, $(2, -2)$ and $(7, 3)$.

4. Find the area of the quadrilateral of which the vertices are $(0, 0)$, $(4, 0)$, $(3, 6)$ and $(-3, 3)$.

5. **D, E, F** are respectively the middle points of the sides **BC, CA, AB** of a \triangle . Prove by the formula of § 17, taking **B** as origin and **BC** as axis or x , that $\triangle \mathbf{ABC} = 4 \triangle \mathbf{DEF}$.
6. Find the area of the \triangle of which the vertices are (x, y) , $(3, 5)$, $(-2, 4)$; and thence show that if these points are in a st. line $5y - x = 22$.
7. Find the area of the $\triangle \mathbf{A}(-3, 2)$, $\mathbf{B}(7, 2)$, $\mathbf{C}(3, 10)$; and show that the \perp from **A** to **BC** = **BC**.
8. A man starts from **O** and goes to **A**, from **A** to **B**, **B** to **C**, **C** to **D**, **D** to **O**. If **O** be taken as the origin and the coordinates of **A, B, C, D** are $(0, -3)$, $(8, 3)$, $(-4, 8)$, $(-4, 3)$, find the distance he has travelled, the unit being one mile.
9. Show from the formula for the area of a \triangle that **A** $(3, -2)$, **B** $(19, 10)$ and **C** $(7, 1)$ are in the same st. line. Find the ratio of **AC** to **CB**.
10. Show that if the coordinates of the vertices taken in order of a quadrilateral are (x_1, y_1) , (x_2, y_2) , (x_3, y_3) and (x_4, y_4) , its area is
- $$\frac{1}{2} \{x_1(y_2 - y_4) + x_2(y_3 - y_1) + x_3(y_4 - y_2) + x_4(y_1 - y_3)\}.$$
11. In the $\triangle \mathbf{OAB}$, **P** is taken in **OA**, **Q** in **AB** and **R** in **BO** so that $\mathbf{OP} : \mathbf{PA} = \mathbf{AQ} : \mathbf{QB} = \mathbf{BR} : \mathbf{RO} = 3 : 1$. Show that $\triangle \mathbf{PQR} : \triangle \mathbf{OAB} = 7 : 16$.
-

LOCI

20. The definition of a locus (see Ontario H. S. Geometry, page 77) is:—

When a figure consisting of a line or lines contains all the points that satisfy a given condition, and no others, this figure is called the locus of these points.

The condition which the points satisfy may be expressed in the form of an equation involving the coordinates of the points. For example, take the locus of the points of which the ordinate is equal to 3. This condition, which is expressed by the equation $y = 3$ [*i.e.*: $-0x + y = 3$], is satisfied by an infinite number of points, as $(0, 3)$, $(1, 3)$, $(2, 3)$, $(7, 3)$, $(-4, 3)$, etc. All such points are on a st. line $AB \parallel$ to Ox and 3

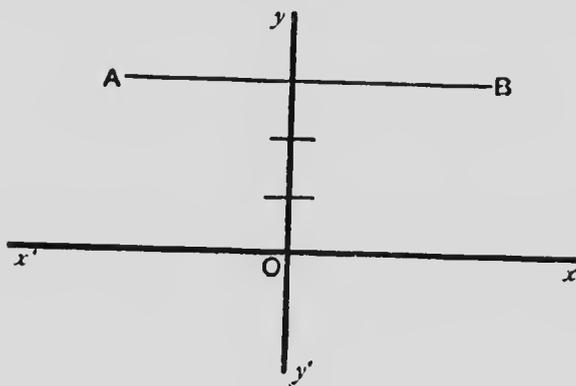
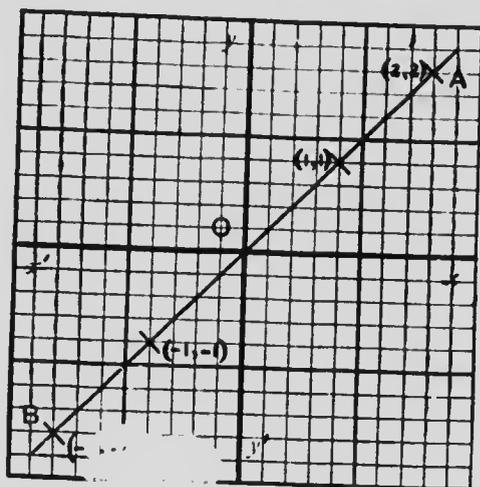


FIG. 13.

units above it; and this st. line contains no points which do not satisfy the condition. Thus the equation $y = 3$ represents the line AB .

Similarly the equation $y = -3$ represents a st. line $\parallel Ox$ and three units below it; $x = 3$ represents a st. line $\parallel Oy$ and three units to the right of the origin, and $x = -5$ a st. line $\parallel Oy$ and 5 units to the left of the origin.

For another example let us take the condition to be that the abscissa and ordinate of each point are equal. The points $(0, 0)$, $(1, 1)$, $(2, 2)$, $(4, 4)$, $(-1, -1)$, $(-5, -5)$, etc., satisfy this condition. It is expressed by the equation $y = x$. If we draw a diagram on



Unit = $\frac{1}{10}$ inch.)

squared paper, mark some of these points on it and join them we get a st. line AB bisecting the $\angle s xOy$ and $x'Oy'$ every point on which satisfies the given condition. Between O and $(1, 1)$ there are an infinite number of points, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{4}, \frac{1}{4})$, $(\frac{1}{10}, \frac{1}{10})$, $(\frac{9}{10}, \frac{9}{10})$, etc., which satisfy the condition, and so on continuously throughout the line. Thus the equation $y = x$ represents the line AB .

Again, we may consider the point which moves so that its distance from the origin is always 5. Its locus is plainly the circumference of a circle. Particular

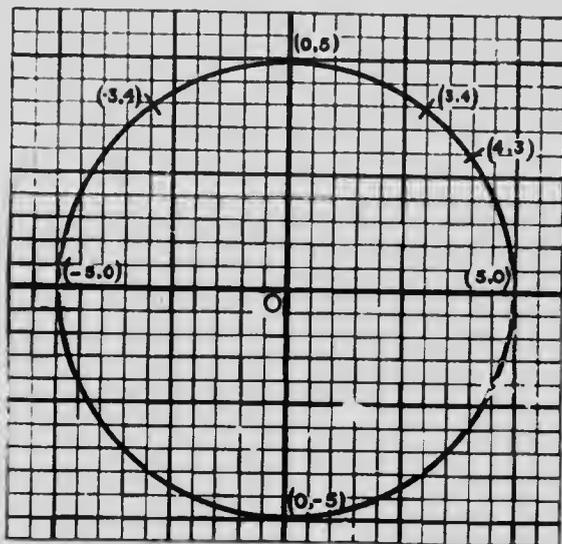


FIG. 15. (Unit = $\frac{1}{8}$ inch.)

points on this locus are $(5, 0)$, $(4, 3)$, $(3, 4)$, $(0, 5)$, $(-3, 4)$, etc., and its equation is $\sqrt{x^2 + y^2} = 5$, or $x^2 + y^2 = 25$.

21. In the equation of a locus the numbers that are the same for all points on the locus are called **constants**; while those that change in value continuously from point to point are called **variables**.

Thus, in the equation $x^2 + y^2 = 25$, x and y are variables and 25 is a constant.

22.—Exercises

1. Find four or five points on the locus represented by each of the following equations; and draw the locus on squared paper in each case:—

(a) $x = -4$; (b) $x + y = 0$; (c) $x - 2y = 0$;

(d) $3x + y = 0$; (e) $x = y + 4$; (f) $x^2 + y^2 = 169$.

2. A point moves so that its distance from the axis of x is 5 times its distance from the axis of y . Find the equation of its locus.

3. What locus is represented by the equation (a) $y = 0$; (b) $x = 0$?

4. A point moves so that it is equidistant from the origin and from $(8, 0)$. Find the equation of its locus.

5. A point moves so that it is equidistant from the origin and from $(3, -5)$. Find the equation of its locus, and draw the locus on squared paper.

6. A point is equidistant from $(1, -2)$ and $(-3, -4)$. Find the equation and draw the locus on squared paper.

7. A point moves so that its distance from $(4, 3)$ is always 5. Find the equation and show that the locus passes through the origin.

8. The coordinates of the ends of the base of a Δ are $(-2, -3)$ and $(4, -1)$, and the length of the median drawn to the base is 6. Find the equation of the locus of its vertex.

9. The coordinates of the ends of the base of a Δ are $(0, 0)$ and $(5, 0)$, and its area is 10. Show that the equation of the locus of its vertex is $y = 4$.

10. The coordinates of the ends of the base of a Δ are $(-1, -2)$ and $(5, 1)$ and its area is 9. Find the equation of the locus of its vertex.

23. An equation connecting two variables x and y has an infinite number of solutions. For example, in the equation $y = 3x + 7$, if any value is given to x , the corresponding value of y may then be determined. Thus, when

$$(a) \ x = 0, \ y = 7,$$

$$(b) \ x = 1, \ y = 10,$$

$$(c) \ x = 2, \ y = 13,$$

$$(d) \ x = -1, \ y = 4,$$

$$(e) \ x = -3, \ y = -2,$$

$$(f) \ x = \frac{1}{3}, \ y = 8,$$

etc.

The, in general, continuous line which passes through all the points (a) , (b) , (c) , etc., is the locus represented by this equation.

Another equation as $4x + 3y = 8$ has also an infinite number of solutions, and if these two equations are solved together, the common solution obtained, in this case $x = -1$, $y = 4$, gives the coordinates of the point of intersection of the loci represented by the equations.

Sets of solutions which satisfy the equation $4x + 3y = 8$ are given in the following table:—

x	y
$(d) \ -1$	4
$(g) \ 2$	0
$(h) \ 5$	-4

If we plot these two sets of results on squared

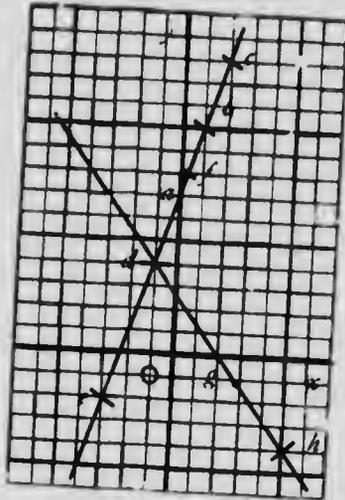


FIG. 16. (Unit = $\frac{1}{10}$ inch.)

paper, we see that the loci appear to be st. lines which intersect at the point (d) $(-1, 4)$.

24.—Exercises

1. Plot the following loci on squared paper and find the coordinates of their points of intersection:—

(a) $4x - y = 1$ and $x - 2y = -12$;

(b) $x + 2y = 7$ and $5x - 2y = 11$;

(c) $3x + 8y = -18$ and $4x + 3y = -1$;

(d) $3x + 4y = 0$ and $x^2 + y^2 = 100$;

(e) $3x - 5y + 45 = 0$ and $x^2 + y^2 = 169$.

2. Find the points where the locus $3x - 5y + 45 = 0$ cuts the axes.

3. Find the points where the locus $x^2 + y^2 = 6x$ cuts the axis of x .

4. Find the locus of a point such that the square of its distance from $(-a, 0)$ is greater than the square of its distance from $(a, 0)$ by $2a^2$.

5. Find the equation of the locus of a point such that the square of its distance from $(-2, -1)$ is greater than the square of its distance from $(5, 3)$ by 11.

6. **A** $(1, 0)$ and **B** $(9, 0)$ are two fixed points and **P** is a variable point such that $PB = 3PA$. Find the equation of the locus of **P**.

7. Plot the following loci and show that they are concurrent :—

$$3x + 4y = 10, \quad 5x - 2y = 8, \quad 4x + y = 9.$$

CHAPTER II

THE STRAIGHT LINE

25. To find the equation of a st. line in terms of the intercepts that it makes on the axes.

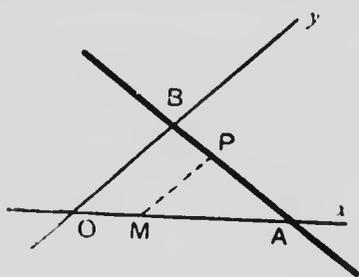


FIG. 17.

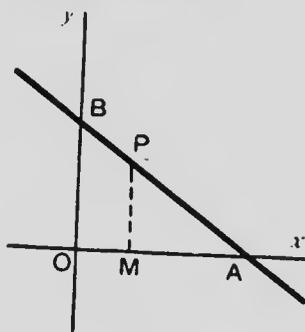


FIG. 18.

Let the st. line cut the axes at A, B so that $OA = a$, $OB = b$.

Take $P(x, y)$ any point on the line, and draw $PM \parallel Oy$ and terminated in Ox at M .

From the similar $\triangle s$ APM, ABO ,

$$\frac{PM}{BO} = \frac{AM}{AO}$$

$$\therefore \frac{y}{b} = \frac{a-x}{a};$$

$$\text{OR, } \frac{x}{a} + \frac{y}{b} = 1.$$

NOTE.—It is seen from the diagrams that both the proof and the form of the equation are the same for oblique and rectangular axes.

26. To find the equation of the st. line passing through A (x_1, y_1) and B (x_2, y_2) .

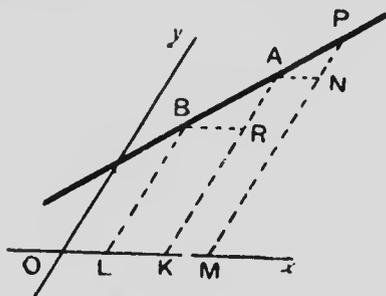


FIG. 19.

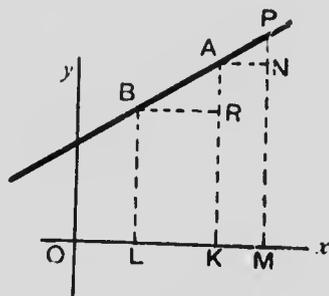


FIG. 20.

Take any point P (x, y) on the st. line.

Draw AK, BL, PM \parallel Oy and terminated in Ox at K, L, M; and AN, BR \parallel Ox and respectively terminated in PM at N and AK at R.

From the similar Δ s PNA, ARB,

$$\frac{AN}{BR} = \frac{PN}{AR}$$

$$AN = KM = OM - OK = x - x_1,$$

$$BR = LK = OK - OL = x_1 - x_2,$$

$$PN = PM - NM = PM - AK = y - y_1,$$

$$AR = AK - RK = AK - BL = y_1 - y_2$$

$$\therefore \frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}.$$

NOTE.—It is seen from the diagrams that both the proof and the form of the equation are the same for oblique and rectangular axes.

27.—Exercises

1. The equation of the st. line passing through (4, 3) and (-2, 7) is by the formula of § 26

$$\frac{x - 4}{4 + 2} = \frac{y - 3}{3 - 7},$$

$$\text{or, } 2x + 3y = 17.$$

To find the intercepts which this line makes on the axes, let $y = 0$ and $\therefore x = 8\frac{1}{2}$, let $x = 0$ and $\therefore y = 5\frac{2}{3}$. By § 25 the equation of the line may now be written

$$\frac{x}{8\frac{1}{2}} + \frac{y}{5\frac{2}{3}} = 1.$$

This is clearly the same as $2x + 3y = 17$.

2. Write down the equations of the st. lines which make the following intercepts on Ox , Oy respectively:—

(a) 5, 2; (b) -4, -6; (c) 3, -8.

3. Find the equations of the st. lines through the following pairs of points:—

(a) (6, 2), (3, 1); (b) (-1, 2), (-3, -7); (c) (4, -6), (-7, 2). Find the intercepts these st. lines make on the axes.

4. Find the point where the st. line which makes intercepts -3 and 5 on Ox and Oy respectively is cut by the st. line $x = -5$.

5. Find the point where the st. line making intercepts 7 and 2 on Ox and Oy respectively meets the st. line through (-2, 7) and (5, -3).

6. Find the point where the st. line through (3, 5) and (-7, -1) meets the st. line through (-8, 2) and (6, 5).

7. Prove that $(11, 4)$ lies on the st. line joining $(3, -2)$ and $(19, 10)$ and find the ratio of the segments into which the first point divides the join of the other two.

8. Find the equations of the sides of the \triangle of which the vertices are $(4, -2)$, $(-5, -1)$, and $(-2, -6)$. Find also the equations of the medians of the \triangle and the coordinates of its centroid.

9. The vertices of a quadrilateral are $(3, 6)$, $(-2, 4)$, $(2, -2)$ and $(7, 3)$. Find the equations of the four sides. Find also the equations of the three diagonals of the complete quadrilateral, and show that the middle points of the diagonals are collinear. Find the equation of the st. line passing through the middle points of the diagonals.

10. Find the vertices of the \triangle the sides of which are $11x - 3y = -45$, $5x - 11y = 47$ and $3x + 4y = 7$.

11. $P(x_1, y_1)$ is any point and $\frac{x}{a} + \frac{y}{b} = 1$ cuts Ox , Oy at A , B respectively. Show that the area of the $\triangle PAB = \frac{1}{2}(bx_1 + ay_1 - ab)$.

28. As explained in elementary algebra, the **degree of a term**, with respect to certain letters, is the number of such letters that occur as factors in the term.

$3x$, $-5y$, ax , by are terms of the first degree with respect to x and y .

$5x^2$, $3y^2$, $-2xy$, ax^2 are terms of the second degree with respect to x and y .

29. **Degree of an equation.** An equation is said to be of the first degree in x and y when it contains a term, or terms, of the first degree in x and y , but no term of a higher degree than the first.

The **general equation of the first degree** in x and y is

$$Ax + By + C = 0.$$

An equation is said to be of the second degree in x and y when it contains a term, or terms, of the second degree in x and y , but no term of a higher degree than the second.

The **general equation of the second degree** in x and y is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

or, in a more convenient form,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

30. To prove that an equation of the first degree always represents a st. line.

Let (x_1, y_1) , (x_2, y_2) , (x_3, y_3) be any three sets of simultaneous values of x and y which satisfy the equation $Ax + By + C = 0$.

$$\text{Then, (1) } Ax_1 + By_1 + C = 0.$$

$$(2) Ax_2 + By_2 + C = 0.$$

$$(3) Ax_3 + By_3 + C = 0.$$

From (1) and (2),

$$\frac{A}{y_1 - y_2} = \frac{B}{x_2 - x_1} = \frac{C}{x_1 y_2 - x_2 y_1}.$$

Dividing the three terms of (3) respectively by these equal fractions and 0 by any one of them,

$$x_3 (y_1 - y_2) + y_3 (x_2 - x_1) + x_1 y_2 - x_2 y_1 = 0.$$

Rearranging the terms, we get

$$(4) x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) = 0.$$

From § 17 the area of the Δ formed by joining (x_1, y_1) , (x_2, y_2) , (x_3, y_3) is

$$\frac{1}{2} \{x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)\}$$

and \therefore , from (4), in this case, the area of the Δ is zero.

This can only be so when the three points are in a st. line, and \therefore as any three points the coordinates of which satisfy

$$Ax + By + C = 0$$

are in a st. line, this equation must always represent a st. line.

31. The equation $Ax + By + C = 0$ may be changed to the form

$$\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1,$$

and by comparing this with the equation of § 25,

$$\frac{x}{a} + \frac{y}{b} = 1,$$

we see that the intercepts which the st. line $Ax + By + C = 0$ makes on the axes of x and y are respectively

$$-\frac{C}{A} \text{ and } -\frac{C}{B}.$$

The same results are obtained by alternately letting $y = 0$ and $x = 0$ in $Ax + By + C = 0$.

32. To obtain the result of § 26 from the general equation of the first degree.

Let (x_1, y_1) , (x_2, y_2) be fixed points on the st. line represented by the general equation, and we have

$$(1) Ax + By + C = 0,$$

$$(2) Ax_1 + By_1 + C = 0,$$

$$(3) Ax_2 + By_2 + C = 0.$$

From (2) and (3),

$$(4) \frac{A}{y_1 - y_2} = \frac{B}{x_2 - x_1} = \frac{C}{x_1y_2 - x_2y_1}.$$

\therefore , from (1) and (4),

$$x(y_1 - y_2) + y(x_2 - x_1) + x_1y_2 - x_2y_1 = 0.$$

This equation is seen to be the same as

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2},$$

when the latter is cleared of fractions and simplified.

33. To find the equation of a st. line in terms of its inclination to the axis of x and its intercept on the axis of y .

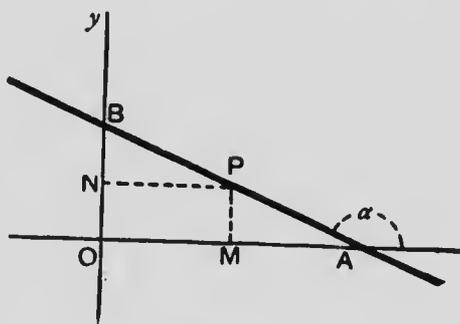


FIG. 21.

Let the st. line cut Ox , Oy at A , B respectively, $\angle BAx = \alpha$, and $OB = a$.

Take any point $P(x, y)$ in the line, and draw $PM \perp Ox$, $PN \perp Oy$.

$$\tan BPN = \frac{BN}{PN} = \frac{BO - PM}{OM} = \frac{a - y}{x}.$$

But, $\tan BPN = \tan PAM = -\tan \alpha$.

$$\therefore -\tan \alpha = \frac{a - y}{x},$$

$$\text{and } \therefore y = x \tan \alpha + a.$$

If we let $\tan \alpha = m$, the equation becomes

$$y = mx + a.$$

In this equation m is called the **slope** of the line, and the $\angle a$, or $\tan^{-1}m$, is always measured by a rotation in the positive direction from the positive direction of Ox , i.e., the \angle is traced out by a radius vector starting from the position Ax and rotating about A in the positive direction to the position AB .

NOTE.—For oblique axes the proof and result are different from those given above for rectangular axes.

34. The equation $Ax + By + C = 0$ may be changed to

$$y = -\frac{A}{B}x - \frac{C}{B},$$

from which by comparison with

$$y = mx + a,$$

it is seen that the slope of the st. line $Ax + By + C = 0$ is $-\frac{A}{B}$, and its intercept on the axis of y is $-\frac{C}{B}$.

35. To find the equation of a st. line in terms of the \perp on it from the origin and the \angle made by a positive rotation from Ox to this \perp .

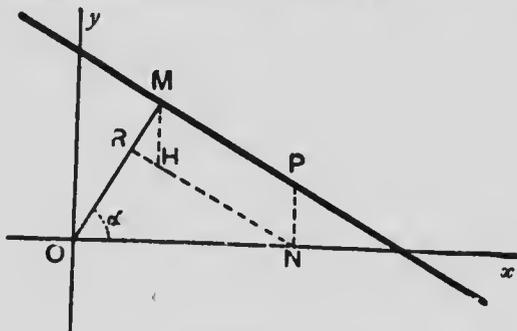


FIG. 22.

Let the \perp OM from O to the line $= p$, and $\angle xOM = a$.

Take any point $P(x, y)$ in the st. line.

Draw $PN \perp Ox$, $NR \perp OM$, $MH \parallel Oy$ to meet NR at H .

$$OR + RM = p.$$

$$OR = ON \cos RON = x \cos a.$$

$$RM = MH \cos RMH = PN \cos MOy = y \sin a.$$

$$\therefore x \cos a + y \sin a = p.$$

NOTE.—For oblique axes the proof and result are different from those given above for rectangular axes.

36. To reduce the equation $Ax + By + C = 0$ to the form $x \cos a + y \sin a = p$, where p is always a positive quantity.

The equations

$$x \cos a + y \sin a - p = 0$$

$$Ax + By + C = 0$$

will be identical if

$$\frac{\cos a}{A} = \frac{\sin a}{B} = \frac{-p}{C}.$$

If C is a positive quantity,

$$\frac{p}{C} = \frac{\cos a}{-A} = \frac{\sin a}{-B} = \frac{1}{\sqrt{A^2 + B^2}} = \frac{1}{\sqrt{A^2 + B^2}}$$

$$\therefore \cos a = \frac{-A}{\sqrt{A^2 + B^2}}, \sin a = \frac{-B}{\sqrt{A^2 + B^2}}, \text{ and } p = \frac{C}{\sqrt{A^2 + B^2}}$$

If C is a negative quantity, these results should be written

$$\cos a = \frac{A}{\sqrt{A^2 + B^2}}, \sin a = \frac{B}{\sqrt{A^2 + B^2}}, p = \frac{-C}{\sqrt{A^2 + B^2}}$$

Thus the equation is

$$\frac{\pm Ax}{\sqrt{A^2 + B^2}} + \frac{\mp By}{\sqrt{A^2 + B^2}} = \frac{\pm C}{\sqrt{A^2 + B^2}}$$

the upper signs being taken when C represents a positive quantity and the lower signs when C represents a negative quantity.

37. Ex. 1. Reduce the equation $3x + 4y - 12 = 0$ to the form $x \cos a + y \sin a = p$.

$$\text{Here } \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$$

Dividing the given equation by 5

$$\frac{3}{5}x + \frac{4}{5}y = \frac{12}{5}$$

$$\cos a = \frac{3}{5}, \sin a = \frac{4}{5}, \text{ and } \therefore a = \tan^{-1}\left(\frac{4}{3}\right) \text{ while the}$$

\perp from the origin on the line is $\frac{12}{5}$.

Ex. 2. Reduce the equation $x - y + 7 = 0$ to the form $x \cos a + y \sin a = p$.

Here $\sqrt{1^2 + 1^2} = \sqrt{2}$.

Dividing the given equation by $-\sqrt{2}$

$$-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \frac{7}{\sqrt{2}}$$

$$\text{or, } x \cos 135^\circ + y \sin 135^\circ = \frac{7}{\sqrt{2}}$$

i.e., $a = 135^\circ$, and the \perp from the origin on the st. line is $\frac{7}{\sqrt{2}}$.

38. To find the equation of a st. line in terms of the coordinates of a fixed point on the line and the \angle which the line makes with Ox .

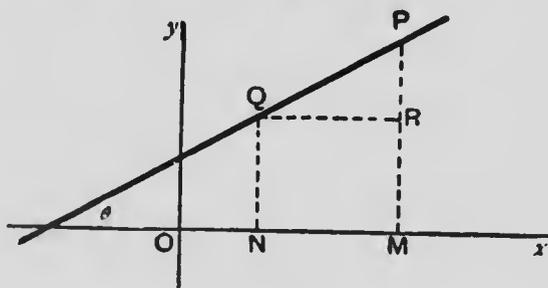


FIG. 23.

Let $Q(x_1, y_1)$ be the fixed point and θ the \angle .

Take any point $P(x, y)$ on the line and let $QP = r$. Draw $PM, QN \perp Ox$ and $QR \perp PM$.

$$QR = PQ \cos \angle PQR.$$

$$QR = NM = x - x_1, \text{ and } \angle PQR = \angle \theta.$$

$$\therefore \frac{x - x_1}{\cos \theta} = r.$$

$$PR = PQ \sin PQR.$$

$$PR = PM - RM = PM - QN = y - y_1.$$

$$\therefore \frac{y - y_1}{\sin \theta} = r.$$

$$\therefore \frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r.$$

This form will frequently be found useful in problems that involve the distance between two points on a st. line.

If $\cos \theta = l$ and $\sin \theta = m$, the equation becomes:—

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r.$$

l and m are called the **direction cosines** of the st. line, and $l^2 + m^2 = 1$.

29. For convenience of reference the different forms of the equation of the st. line are here collected:—

$$(1) Ax + By + C = 0.$$

$$(2) \frac{x}{a} + \frac{y}{b} = 1.$$

$$(3) \frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}.$$

$$(4) y = mx + a.$$

$$(5) x \cos a + y \sin a = p.$$

$$(6) \frac{x - x_1}{l} = \frac{y - y_1}{m} = r.$$

If the st. line passes through the origin (4) takes the convenient form:—

$$(7) y = mx.$$

If the st. line joins the origin to a fixed point (x_1, y_1) , we get, by letting $x_2 = y_2 = 0$ in (3):—

$$(8) \frac{x}{x_1} = \frac{y}{y_1}.$$

If the st. line passes through (x_1, y_1) and its slope is m , the equation is

$$(9) y - y_1 = m(x - x_1).$$

40.—Exercises

1. Name the constants and variables in each of the nine equations of § 39. Explain the meaning of each constant. Which of these equations are of the same form for rectangular and oblique axes?

2. Draw the following st. lines on squared paper:—

(a) $x + 2y = 8$; (b) $3x - 7y = -5$; (c) $\frac{x}{3} + \frac{y}{5} = -1$;
 (d) $2x + 3y = 13$; (e) $3y = 4x$.

3. Find the equation of the st. line.

(a) through the origin and making an \angle of 30° with Ox ;

(b) through the origin and making an \angle of 120° with Ox ;

(c) through $(0, 5)$ and making the $\angle \tan^{-1} \frac{5}{2}$ with Ox ;

(d) through $(0, -3)$ and making the $\angle \cos^{-1} \frac{3}{5}$ with Ox ;

(e) through $(-3, -4)$ and making the $\angle 45^\circ$ with Ox .

4. In the Δ of which the vertices are $(-2, 5)$, $(3, -7)$, $(1, 2)$

(a) find the slope of each side:

(b) show that the medians are concurrent and find the centroid.

5. $O(0, 0)$, $A(6, 0)$, $B(1, 6)$, $C(2, 8)$ are the vertices of a quadrilateral. Show that the st. lines joining the

middle points of OA , BC , of AB , CO and of OB , AC are concurrent, and find the coordinates of their common point.

6. Find the equation to the st. line through $(-4, 3)$ that cuts off equal intercepts from the axes.

7. Find the length of the \perp from the origin to the line $3x + 7y = 10$; find the \angle which this \perp makes with Ox .

8. What is the condition that the st. line $Ax + By + C = 0$ may

(a) pass through the origin;

(b) be $\parallel Ox$;

(c) be $\parallel Oy$;

(d) cut off equal intercepts from the axes;

(e) make $\angle 45^\circ$ with Ox ;

9. What must be the value of m if the line $y = mx + 7$ passes through $(-2, 5)$?

10. Find the values of m and a , if the st. line $y = mx + a$ passes through $(-2, 3)$ and $(7, 2)$.

11. Find the values of a and b , if the st. line $\frac{x}{a} + \frac{y}{b} = 1$ passes through $(-2, -5)$ and $(4, -2)$.

12. Show that the points $(4a, -3b)$, $(2a, 0)$, $(0, 3b)$ are in a st. line.

13. Show that the intercept made on the line $x = k$ by the lines $Ax + By + C = 0$ and $Ax + By + C' = 0$ is the same for all values of k .

14. $P(x_1, y_1)$ is any point and the lines $Ax + By + C = 0$ cuts Ox , Oy at N , R respectively. Show that $\triangle PNR =$

$$\frac{C}{2AB} (Ax_1 + By_1 + C).$$

THE ANGLE BETWEEN TWO STRAIGHT LINES

41. To find the \angle between two st. lines whose equations are given.

(i) Let the given equations be $y = m_1x + a_1$ and $y = m_2x + a_2$.

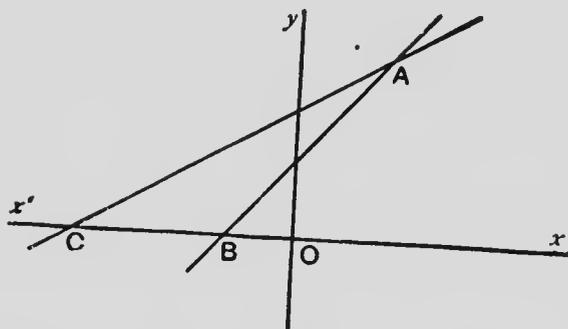


FIG. 24.

Let **AB** be the line $y = m_1x + a_1$ and **AC** be the line $y = m_2x + a_2$ when **B**, **C** are on the axis of x . Let $\angle \text{BAC} = \theta$.

Then $m_1 = \tan \text{AB}x$, $m_2 = \tan \text{AC}x$.

$$\angle \theta = \angle \text{AB}x - \angle \text{AC}x.$$

$$\therefore \tan \theta = \frac{\tan \text{AB}x - \tan \text{AC}x}{1 + \tan \text{AB}x \cdot \tan \text{AC}x} = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

$$\therefore \theta = \tan^{-1} \frac{m_1 - m_2}{1 + m_1 m_2}.$$

(ii) Let the given equations be $Ax + By + C = 0$ and $A_1x + B_1y + C_1 = 0$.

These equations may be changed to

$$y = -\frac{A}{B}x - \frac{C}{B} \quad \text{and} \quad y = -\frac{A_1}{B_1}x - \frac{C_1}{B_1}.$$

\therefore , writing $-\frac{A}{B}$ for m_1 and $-\frac{A_1}{B_1}$ for m_2 in the above result, the \angle between the lines

$$= \tan^{-1} \frac{-\frac{A}{B} + \frac{A_1}{B_1}}{1 + \frac{AA_1}{BB_1}} = \tan^{-1} \frac{A_1B - AB_1}{AA_1 + BB_1}$$

42. Condition of Parallelism. If two st. lines are \parallel , they make equal \angle s with the axis of x , *i.e.*, their slopes are the same.

\therefore , if their equations are $y = m_1x + a_1$ and $y = m_2x + a_2$, the condition is

$$m_1 = m_2.$$

If their equations are $Ax + By + C = 0$ and $A_1x + B_1y + C_1 = 0$, the condition is

$$-\frac{A}{B} = -\frac{A_1}{B_1},$$

$$\text{or, } AB_1 - A_1B = 0.$$

This may also be written $\frac{A}{A_1} = \frac{B}{B_1}$; and we see that the equation $ax + by = k$ can be made to represent an infinite number of st. lines by giving different values to k ; as:— $ax + by = k_1$, $ax + by = k_2$, etc.

43. Condition of Perpendicularity. If the st. lines $y = m_1x + a_1$, $y = m_2x + a_2$ are \perp ,

$$\tan^{-1} \frac{m_1 - m_2}{1 + m_1m_2} = \frac{\pi}{2}.$$

$$\therefore \frac{m_1 - m_2}{1 + m_1m_2} = \infty.$$

This will be true if $1 + m_1 m_2 = 0$, and \therefore the required condition is

$$m_1 m_2 = -1.$$

Similarly, if $Ax + By + C = 0$ and $A_1x + B_1y + C_1 = 0$ are \perp .

$$AA_1 + BB_1 = 0.$$

The st. lines

$$Ax + By + C = 0$$

$$Bx - Ay + C_1 = 0$$

satisfy the above condition and \therefore are \perp to each other.

44.—Exercises

1. Find the \angle between the st. lines

(a) $2x - 3y = 9$ and $x + 5y = 11$;

(b) $3x + 5y = 12$ and $(17\sqrt{3} + 30)x + 33y = 19$;

(c) $5y = 3x + 12$ and $5x + 3y = 17$;

(d) $4x + 7y = 13$ and $3x - y = 6$.

2. Find the equation of the st. line \parallel to $6x - 7y = 13$ and passing through $(-2, -5)$.

Solution:—The required equation is

$$6(x + 2) - 7(y + 5) = 0;$$

$$\text{i.e., } 6x - 7y = 23.$$

3. Find the equation of a st. line through $(-3, -5)$ and \parallel to $9x + 4y = 18$.

4. Find the equation of the st. line drawn through $(-2, -5)$ and \perp to $6x - 7y = 13$.

Solution:—The required equation is

$$7(x + 2) + 6(y + 5) = 0;$$

$$\text{i.e., } 7x + 6y + 44 = 0.$$

5. Find the equation of the st. line drawn through $(4, 2)$ and \perp $3x - 2y = 1$.

6. Find the equation of the st. line passing through $(-3, 5)$ and \parallel to the st. line joining $(2, 6)$ and $(7, -1)$.
7. Find the equation of the st. line passing through $(2, 6)$ and \perp the st. line joining $(-3, 5)$ and $(7, -1)$.
8. Find the equations of st. lines drawn through $(5, 7)$ which make \angle s 45° and 135° with Ox .
9. Find the equations of st. lines drawn through $(-5, -3)$ which make \angle s 30° and 150° with Ox .
10. Show that the \perp s from the vertices of the $\triangle (1, -3), (-5, -2), (4, 7)$ to the opposite sides are concurrent; and find the coordinates of the orthocentre.
11. Show that the \perp s from the vertices of the $\triangle (0, 0), (a, c), (b, c)$ to the opposite sides are concurrent; and find the orthocentre.
12. Find the ratio into which the \perp from the origin on the st. line joining $(2, 6)$ and $(5, 1)$ divides the distance between these points.
13. Find the equations of the st. lines which pass through (h, k) and form with $y = mx + a$ an isosceles \triangle of which the vertex is at the given point and each base $\angle = \alpha$.

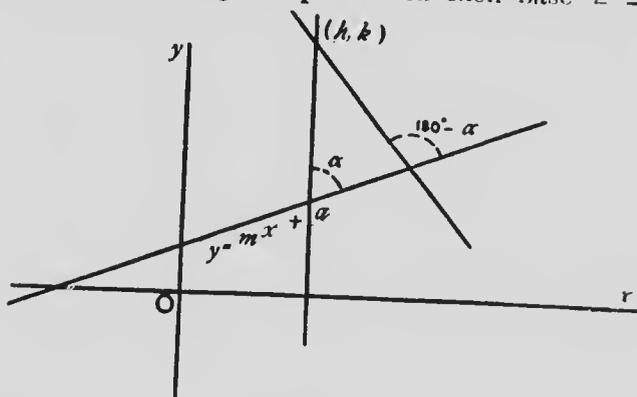


FIG. 25.

Solution :—Let $y - k = M$ h represent one side of the \triangle
 where the value of M is to find.

$$\tan a = \frac{m - M}{1 + m M}$$

$$\therefore M = \frac{m - \tan a}{1 + m \tan a}$$

\therefore the equation of this side is

$$y - k = \frac{m - \tan a}{1 + m \tan a} (x - h).$$

If for a we substitute $180^\circ - a$, the equation of the other side is found to be

$$y - k = \frac{m + \tan a}{1 - m \tan a} (x - h).$$

14. Find the equations of the st. lines passing through (2, 8) and making an \angle of 30° with $3x - 12y = 7$.

15. Find the equations of the st. lines passing through (-1, -2) and making an \angle of 45° with $\frac{x}{7} + \frac{y}{5} = 1$.

16. Show that the equation of the st. line through (a, b) and making an \angle of 60° with $x \cos a + y \sin a = p$ is $y - b = (x - a) \tan (a \pm 30^\circ)$.

17. Show that the right bisectors of the sides of the \triangle (0, 0), (a, 0), (b, c) are concurrent; and find their point of intersection.

18. Find the equation of a st. line \perp to $Ax + By + C = 0$, and at a distance p from the origin.

PERPENDICULARS

45. To find the length of the \perp from P (x_1, y_1) to $Ax + By + C = 0$.

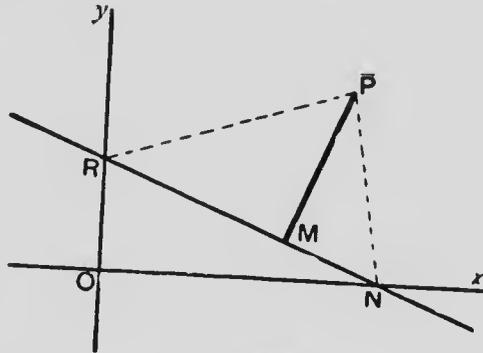


FIG. 26.

Draw $PM \perp$ the given st. line. Join P to N, R the points where the given st. line cuts Ox, Oy .

$$\triangle PRN = \frac{1}{2} PM \cdot RN.$$

$$ON = -\frac{C}{A} \text{ and } OR = -\frac{C}{B}$$

$$\therefore RN = \sqrt{\frac{C^2}{B^2} + \frac{C^2}{A^2}} = \frac{C}{AB} \sqrt{A^2 + B^2}.$$

By the formula of § 17,

$$\begin{aligned} \triangle PRN &= \frac{1}{2} \left\{ x_1 \left(-\frac{C}{B} \right) - \frac{C}{A} \left(y_1 + \frac{C}{B} \right) \right\} \\ &= -\frac{C}{2AB} (Ax_1 + By_1 + C). \end{aligned}$$

$$\therefore \frac{1}{2} PM \cdot \frac{C}{AB} \sqrt{A^2 + B^2} = -\frac{C}{2AB} (Ax_1 + By_1 + C).$$

$$\therefore PM = -\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}.$$

The length of the \perp is \therefore

$$\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}.$$

In the diagram **AB** is the line represented by the equation $4x + 5y - 20 = 0$.

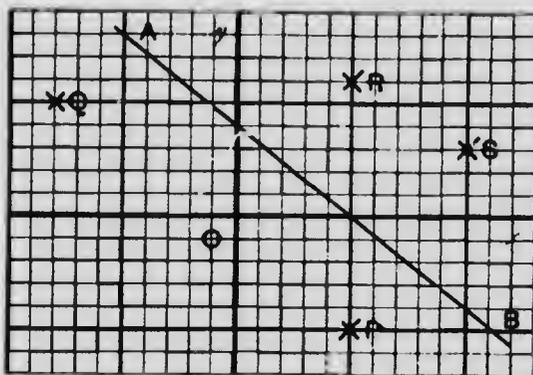


FIG. 27. (Unit = $\frac{1}{10}$ inch.)

If in the expression $4x + 5y - 20$ we substitute the coordinates of points **O**, **P**, **Q**, **R**, **S** which are not in the line, the following results are obtained.

$$\text{For } \mathbf{O} (0, 0), \quad 4x + 5y - 20 = -20.$$

$$\text{" } \mathbf{P} (5, -5), \quad 4x + 5y - 20 = -25.$$

$$\text{" } \mathbf{Q} (-8, 5), \quad 4x + 5y - 20 = -27.$$

$$\text{" } \mathbf{R} (5, 6), \quad 4x + 5y - 20 = +30.$$

$$\text{" } \mathbf{S} (10, 3), \quad 4x + 5y - 20 = +35.$$

In these results it will be observed that:—

For the origin the sign of the value of the expression is the same as the sign of the absolute term.

For other points that lie on the same side of the given st. line as the origin the signs of the values of the expression are the same as the sign of the result for the origin; while, for points on the side remote from the origin the signs of the values of the expression are different from the sign of the result for the origin.

A formal proof of these properties is given in the next article.

46. To prove that the sign of the expression $Ax + By + C$ is different for points on opposite sides of the line $Ax + By + C = 0$.

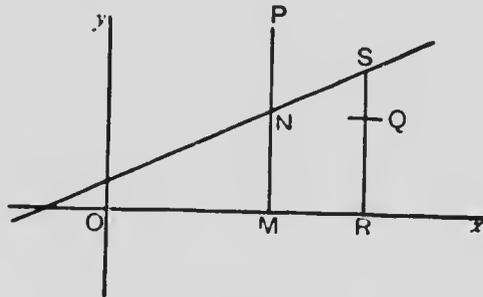


FIG. 28.

$P(x_1, y_1)$, $Q(x_2, y_2)$ are any points on opposite sides of $Ax + By + C = 0$.

Draw $PM, QR \perp Ox$ and let them cut the given line at N, S .

$$y_1 = PM = PN + NM; \quad y_2 = QR = SR - SQ.$$

$$\therefore Ax_1 + By_1 + C = Ax_1 + B.PN + B.NM + C,$$

and $Ax_2 + By_2 + C = Ax_2 + B.SR - B.SQ + C.$

But, $\because N$ and S are both on the given line,

$$Ax_1 + B.NM + C = 0,$$

$$\text{and } Ax_2 + B.SR + C = 0.$$

$$\therefore Ax_1 + By_1 + C = B.PN,$$

$$\text{and } Ax_2 + By_2 + C = -B.SQ.$$

\therefore , since PN, SQ are both taken as positive quantities, $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ have opposite signs.

When $x = 0$ and $y = 0$ the expression $Ax + By + C$ becomes C , \therefore a point whose coordinates when substituted in $Ax + By + C$ gives the same sign as C is on the same side of the st. line $Ax + By + C = 0$ as the origin.

47. **Sign of the Perpendicular.** It follows from the preceding article that, if the positive sign is always taken for $\sqrt{A^2 + B^2}$, when the sign of

$$\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}$$

is the same as the sign of C , the point (x_1, y_1) and the origin are on the same side of the line $Ax + By + C = 0$; and when the sign of this fraction is different from that of C , the point (x_1, y_1) and the origin are on opposite sides of $Ax + By + C = 0$.

48. To find the equations of the bisectors of the

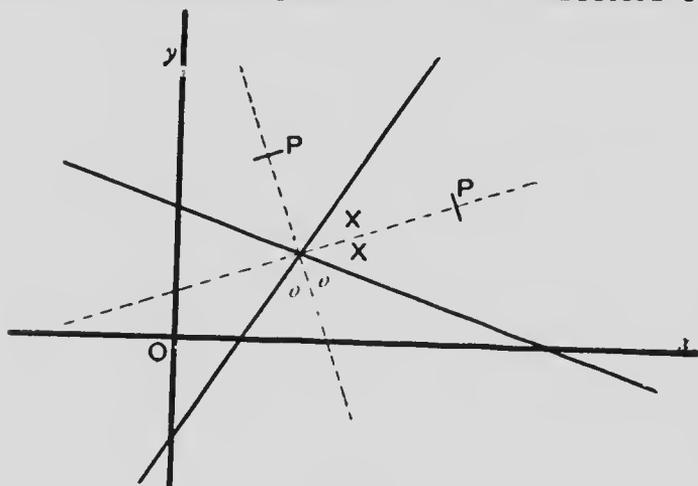


FIG. 29.

\angle s between the lines $Ax + By + C = 0$ and $A_1x + B_1y + C_1 = 0$.

The \perp s to the st. lines from any point $P(x, y)$ on either bisector are equal to each other,

\therefore the required equations are

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = \pm \frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}}.$$

If the equations are so written that C and C_1 have the same sign and P is on the bisector of the \angle that contains the origin, the \perp s from P have the same sign as the \perp s from the origin on the lines, and the equation of the bisector is

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = \frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}}.$$

If P is on the bisector of the \angle which does not contain the origin, the \perp s from P have opposite signs and the equation of the bisector is

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = - \frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}}.$$

49. To find the distance from (a, b) to $Ax + By + C = 0$ in the direction whose direction cosines are l, m .

The equation of the st. line passing through (a, b) in the given direction is, by § 38,

$$\frac{x - a}{l} = \frac{y - b}{m} = r.$$

$$\therefore x = a + lr, \quad y = b + mr.$$

Substituting these values for x and y in $Ax + By + C = 0$,

$$Aa + Alr + Bb + Bmr + C = 0.$$

$$\therefore r = -\frac{Aa + Bb + C}{Al + Bm}.$$

50. The length of the \perp from (a, b) to $Ax + By + C = 0$ may be deduced from the result of § 49.

For, if $\frac{x - a}{l} = \frac{y - b}{m}$ and $Ax + By + C = 0$ are \perp to each other,

$$\frac{A}{l} = \frac{B}{m} = \frac{Al + Bm}{l^2 + m^2} = Al + Bm,$$

since $l^2 + m^2 = 1$.

Also each of these fractions

$$= \frac{A^2 + B^2}{\sqrt{l^2 + m^2}} = \sqrt{A^2 + B^2}.$$

$$\therefore Al + Bm = \sqrt{A^2 + B^2},$$

and the length of the \perp is

$$\frac{Aa + Bb + C}{\sqrt{A^2 + B^2}}.$$

51. To find the equation of a line passing through the intersection of two loci.

The equation

$$k(Ax + By + C) + l(A_1x + B_1y + C_1) = 0, \quad (1)$$

being of the first degree in x and y represents a st. line.

If (x_1, y_1) is the point of intersection of

$$Ax + By + C = 0 \quad (2)$$

$$\text{and } A_1x + B_1y + C_1 = 0, \quad (3)$$

the values x_1, y_1 substituted for x, y will plainly satisfy equation (1), and \therefore the st. line (1) must pass through the point of intersection of the st. lines (2) and (3).

From the same reasoning the following more general theorem is seen to be true:—

If two equations are multiplied by any numbers and the results either added or subtracted, the resulting equation represents a locus that passes through the point (or points) of intersection of the loci represented by the first two.

52. *Example*—Find the equation of the st. line passing through the intersections of $17x - 7y = 9$, $3x + 19y = 34$ and \perp to $11x - 4y = 13$.

$$17x - 7y - 9 + l(3x + 19y - 34) = 0$$

is a st. line passing through the intersection of the first two lines.

This equation may be written

$$(3l + 17)x + (19l - 7)y - 34l - 9 = 0.$$

If this line is \perp to $11x - 4y - 13$,

$$11(3l + 17) - 4(19l - 7) = 0$$

$$\therefore l = 5,$$

and the required equation is found to be

$$32x + 88y = 179.$$

53. To find the condition that the three st. lines

$$a_1 x + b_1 y + c_1 = 0 \dots\dots\dots(1)$$

$$a_2 x + b_2 y + c_2 = 0 \dots\dots\dots(2)$$

$$a_3 x + b_3 y + c_3 = 0 \dots\dots\dots(3)$$

may be concurrent.

If the three st. lines are concurrent, the coordinates of the common point satisfy the three equations.

For that point, from (1) and (2),

$$\frac{x}{b_1 c_2 - b_2 c_1} = \frac{y}{c_1 a_2 - c_2 a_1} = \frac{1}{a_1 b_2 - a_2 b_1}.$$

Dividing the terms of (3) respectively by these equal fractions,

$$a_3 (b_1 c_2 - b_2 c_1) + b_3 (c_1 a_2 - c_2 a_1) + c_3 (a_1 b_2 - a_2 b_1) = 0.$$

This is the relationship that must hold among the constants in order that the lines may be concurrent.

54. Exercises

- Find the length of the \perp
 - from $(-4, 7)$ to $5x - 2y = 1$;
 - from $(-4, -3)$ to $\frac{x}{2} + \frac{y}{3} = 1$;
 - from $(3, -2)$ to $y = 7x + 1$;
 - from $(-2, -7)$ to the st. line joining $(5, 3)$ and $(-3, -7)$;
 - from the origin to the st. line joining $(7, 0)$ and $(0, -5)$.
- Find the distance between the \parallel lines $4x - 3y = 9$, $4x - 3y = 2$.
- Find the distance between the \parallel lines $ax + by + c_1 = 0$, $ax + by + c_2 = 0$.
- Find the point in the line $\frac{x}{2} + \frac{y}{5} = -1$ such that its \perp distance from the st. line joining $(2, 7)$, $(5, 3)$ is 8.
- Find the equation of the st. line through the intersection of $3x - 2y = 12$, $5x + 4y = 9$ and \parallel to $\frac{x}{3} + \frac{y}{4} = 1$. (See §§ 51 and 52).
- Find the equation of the st. line joining the origin to the intersection of $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$.
- Find the distance from the point of intersection of $7x - 5y = 13$, $4x + 9y = 43$ to the line $12x = 5y$.
- Find the equation of the st. line passing through the intersection of $y = mx + c$, $y = m_1x + c_1$ and \perp to $\frac{x}{a} + \frac{y}{b} = 1$.

9. Show that the st. lines $x + 2y = 5$, $2x + 3y = 8$, $3x + y = 5$, $x + y = 3$ and $2x - y = 0$ are concurrent.
10. Find the condition that the lines $ax + by + g = 0$, $hx + by + f = 0$, $gx + fy + c = 0$ are concurrent.
11. Find the equation of the st. line passing through the intersection of $y = mx + c$, $y = m_1x + c_1$ and also through (a, b) .
12. Find the equation of the st. line joining the origin to the point of intersection of $Ax + By + C = 0$ and $A_1x + B_1y + C_1 = 0$.
13. Find the distance from the orthocentre of the \triangle $O(0, 0)$, $A(8, 0)$, $B(3, 5)$ to the st. line AB .
14. Plot the lines $2x - 3y = 1$, $3x + y = 7$ on squared paper and find the intercepts that the bisectors of the \angle s between them make on the axis of y .
15. Find the equations of the bisectors of the \angle s between $5x - 12y = 17$ and $8x + 15y = 31$.
16. Show that the bisectors of the supplementary \angle s between $y = mx + a$ and $y = m_1x + a_1$ are \perp to each other.
17. Find the equations of the bisectors of the \angle s between the st. lines joining $(4, 5)$ and $(-5, 2)$ respectively to $(3, -7)$.
18. Show that the st. lines $x + 6y = 15$, $2x - 5y + 4 = 0$ and $9x + y = 29$ are concurrent.
19. The sides of a \triangle are $3x + 4y = 15$, $12x - 5y = 17$, $24x + 7y = 30$. Plot the lines on squared paper and find the point where the bisectors of the interior \angle s of the \triangle intersect.

20. Find the equation of the st. line passing through the intersection of the lines $2x - 3y = 6$ and $3x + 4y = 18$, and also through the middle point of the st. line joining $(-1, 2)$ and $(3, 4)$.

21. Find the distance from $(5, 3)$ in the direction in which the slope is $\frac{1}{\sqrt{3}}$ to the line $7x - 11y = 13$.

22. Find the distance from $(-4, 6)$ in the direction of which the slope is 1 to the line $\frac{x}{2} + \frac{y}{3} = 1$.

Draw the diagram on squared paper.

23. The sum of the distances from a point to the lines $x + 2y = 7$, $5x - 2y = 11$ is 7. Show that the locus of the point is a st. line which makes equal \angle s with the given st. lines.

24. Find the distance between the \parallel lines $\frac{x}{a} + \frac{y}{b} = c$, $\frac{x}{a} + \frac{y}{b} = d$.

25. Find the equation of the line passing through $P(2, 5)$ and cutting Ox at A , Oy at B so that $AP:PB = 7:3$.

26. Find the equations of the st. lines passing through $(4, 7)$ and making an \angle of 45° with $3x - 10y = 8$.

27. Find the equations of the st. lines passing through $(-4, -7)$ and forming an equilateral \triangle with $3x - 2y = 7$.

28. Find the equations of the st. lines drawn \parallel to $5x - 12y = 9$ and at a distance 5 from it.

29. Find the equations of the two st. lines which pass through $(4, 7)$ and are equally distant from $A(7, 3)$, $B(3, -1)$. Find also the distances from A and B to these lines.

30. Find the point in $4x - 3y = 12$ which is equally distant from $(2, 7)$ and $(4, -1)$.
31. Having given the length of the base and the difference of the squares of the other two sides of a \triangle , prove that the locus of its vertex is a st. line \perp the base.
32. Find the equations of the st. lines which are at a distance 2 from the origin and which pass through the intersection of $x - 7y + 11 = 0$ and $3x + 4y - 17 = 0$.
33. Find the equation of the st. line \parallel to $Ax + By + C = 0$ and at a distance p from the origin.
34. Find the equation of the st. line passing through (h, k) and \perp to $Ax + By + C = 0$.
35. The equations of the sides of a \triangle are $5x + 3y - 15 = 0$, $2x - y + 4 = 0$, $3x - 7y - 21 = 0$. (a) Show that the \perp s from the vertices to the opposite sides are concurrent and find the coordinates of the orthocentre. (b) Show that the right bisectors of the sides are concurrent and find the coordinates of the circumcentre. (c) Show that the centroid is at a point of trisection of the st. line joining the orthocentre to the circumcentre.
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CHAPTER III

THE STRAIGHT LINE CONTINUED. TRANSFORMATION OF COORDINATES

55. An equation of the second degree may represent two st. lines.

For example, $2x^2 - 5xy + 3y^2 = 0$ is the same as $(x - y)(2x - 3y) = 0$, and will be true for all values of x and y which make either of the factors $x - y$ or $2x - 3y$ equal to zero, and \therefore all points on the st. lines $x - y = 0$, $2x - 3y = 0$ are on the locus represented by $2x^2 - 5xy + 3y^2 = 0$.

Similarly, an equation of the third degree may represent three st. lines, one of the fourth degree may represent four st. lines, etc.

56. The general equation $ax^2 + 2hxy + by^2 = 0$ represents two st. lines passing through the origin.

Solving as a quadratic in x

$$x = \frac{-h \pm \sqrt{h^2 - ab}}{a} y,$$

from which it is seen that the given equation is equivalent to

$$\{ax + hy + y\sqrt{h^2 - ab}\} \{ax + hy - y\sqrt{h^2 - ab}\} = 0,$$

and \therefore represents the two st. lines

$$ax + hy + y\sqrt{h^2 - ab} = 0$$

$$ax + hy - y\sqrt{h^2 - ab} = 0,$$

both of which pass through the origin.

If $h^2 < ab$, the lines are both real.

If $h^2 = ab$, the lines are coincident.

If $h^2 > ab$, the lines are imaginary, and we have two imaginary st. lines passing through the real point $(0, 0)$.

57. To find the \angle between the two st. lines represented by $ax^2 + 2hxy + by^2 = 0$.

The given equation may be written

$$y^2 + 2 \frac{h}{b} xy + \frac{a}{b} x^2 = 0.$$

If $y - m_1x$ and $y - m_2x$ are the factors of the expression on the left hand side of this equation,

$$m_1 + m_2 = -\frac{2h}{b}, \quad m_1m_2 = \frac{a}{b}.$$

$$\therefore m_1^2 + 2m_1m_2 + m_2^2 = \frac{4h^2}{b^2}.$$

$$4m_1m_2 = \frac{4a}{b}.$$

$$\therefore (m_1 - m_2)^2 = \frac{4(h^2 - ab)}{b^2},$$

$$\text{and } m_1 - m_2 = \frac{2\sqrt{h^2 - ab}}{b}.$$

If then θ is the \angle between the st. lines, by § 41,

$$\begin{aligned} \tan \theta &= \frac{m_1 - m_2}{1 + m_1m_2} = \frac{2\sqrt{h^2 - ab}}{b} \times \frac{b}{a + b} \\ &= \frac{2\sqrt{h^2 - ab}}{a + b}. \end{aligned}$$

$$\therefore \theta = \tan^{-1} \frac{2\sqrt{h^2 - ab}}{a + b}.$$

Condition of perpendicularity. If $\theta = 90^\circ$,

$\tan \theta = \infty$. This will be the case if

$$a + b = 0.$$

58. To find the equation of the st. lines which bisect the \angle s between the st. lines represented by $ax^2 + 2hxy + by^2 = 0$.

Let the given equation represent the st. lines

$y - m_1x = 0$, $y - m_2x = 0$, so that

$$m_1 + m_2 = -\frac{2h}{b}, \quad m_1m_2 = \frac{a}{b}.$$

The equations of the bisectors of the \angle s between these lines are

$$\frac{y - m_1x}{\sqrt{1 + m_1^2}} + \frac{y - m_2x}{\sqrt{1 + m_2^2}} = 0 \quad \text{and} \quad \frac{y - m_1x}{\sqrt{1 + m_1^2}} - \frac{y - m_2x}{\sqrt{1 + m_2^2}} = 0.$$

These equations may be combined into

$$\frac{(y - m_1x)^2}{1 + m_1^2} - \frac{(y - m_2x)^2}{1 + m_2^2} = 0.$$

Simplifying and dividing by $m_2 - m_1$,

$$(m_1 + m_2)y^2 - 2(m_1m_2 - 1)xy - (m_1 + m_2)x^2 = 0.$$

Substituting and multiplying by b .

$$h(x^2 - y^2) - (a - b)xy = 0.$$

59. To find the relationship that must connect the constants in the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

in order that this equation may represent two st. lines.

If the given equation represents two st. lines, it must be equivalent to two equations of the form $y - m_1x - a_1 = 0$, $y - m_2x + a_2 = 0$, from either of which y can be expressed in terms of the first degree of x .

Solving the given equation for y , we obtain

$$y = \frac{-(hx + f) \pm \sqrt{(hx + f)^2 - b(ax^2 + 2gx + c)}}{b}.$$

In order that these values of y may be in terms of the first degree of x , the expression under the radical sign must be a perfect square; *i.e.*,

$$(hx - ab)x^2 + 2(hf - bg)x + f^2 - bc$$

is a perfect square for all values of x .

$$\therefore (hf - bg)^2 = (h^2 - ab)(f^2 - bc).$$

Simplifying, we get the condition in the form

$$2fgh - af^2 - bg^2 - ch^2 + abc = 0.$$

60.—Exercises

1. Show that the following equations represent two st. lines and find the separate equations of the lines:—

(a) $x^2 - (a + b)x - ab = 0$; (b) $x^2 - y^2 = 0$;

(c) $x^2 - 3xy = 0$; (d) $8x^2 + 3y^2 = 10xy$;

(e) $xy + bx = ay + ab$; (f) $3x^2 - 10xy + 3y^2 - 11x - 7y - 20 = 0$.

2. Show that $2x^2 - 7xy + 6y^2 + 2x - 5y - 4 = 0$ represents two st. lines and find the slope of each.
 3. Interpret the locus represented by $xy = 0$.
 4. Find the \angle s between the st. lines in 1. (*d*), (*e*) and (*f*).
 5. Find the condition that $axy + bx + cy + d = 0$ may represent two st. lines.
 6. Find the value of **B** for which the equation $3x^2 - 10xy + By^2 - 2x - 2y = 21$ will represent two st. lines.
 7. Find the single equation which represents the two st. lines passing through (5, 3) and making an equilateral \triangle with the axis of x .
 8. Prove that $y^2 - 2xy \sec a + x^2 = 0$ represents two st. lines through the origin and inclined to each other at an $\angle = a$. Show also that one of these lines makes the same \angle with the axis of x that the other makes with the axis of y .
-

TRANSFORMATION OF COORDINATES

61. It is often necessary to change the coordinates involved in a problem into a different set which are referred to axes drawn

(a) from a new origin, or

(b) in directions different from the original axes.

62. To change from a pair of axes to another pair which are \parallel to the former, but have a different origin.

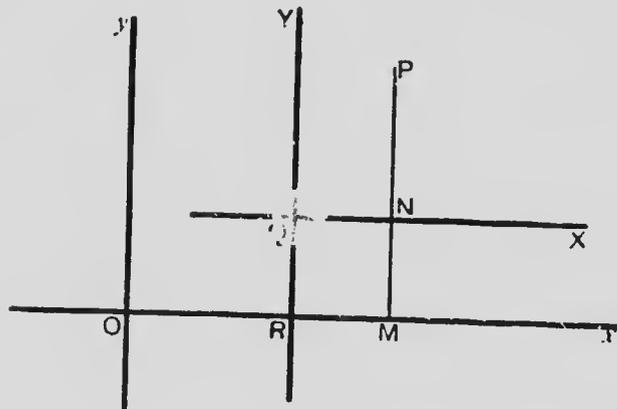


FIG. 30.

$Q(h, k)$ is the new origin.

Let $P(x, y)$ be any point referred to Ox and Oy and X, Y the coordinates of the same point referred to the new axes QX and QY .

Draw $PNM \perp$ to Ox and QX , and let YQ cut Ox at R .

$$x = OM = OR + QN = h + X.$$

$$y = PM = QR + PN = k + Y.$$

Thus, if for x, y respectively we substitute $h + X, k + Y$ in any equation the origin is changed to the point (h, k) .

To return to the original origin the substitutions would be $X = x - h, Y = y - k$.

63. To change the direction of the axes, without changing the origin, the axes being rectangular.

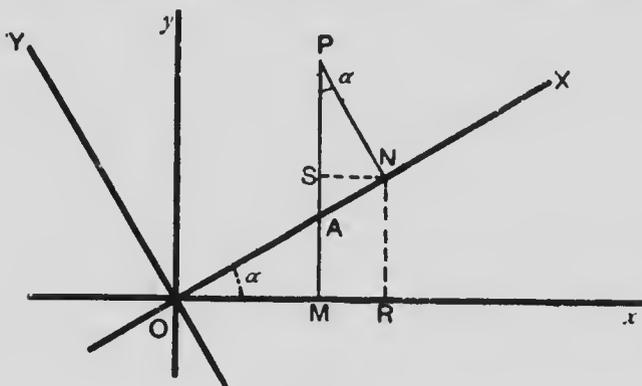


FIG. 31.

Let $P(x, y)$ be any point referred to Ox, Oy ; and X, Y the coordinates of the same point referred to axes OX, OY such that $\angle XOx = \alpha$.

Draw $PM \perp Ox, PN \perp OX, NR \perp Ox, NS \perp PM$.

$$\angle NPS = 90 - \angle NAP = 90 - \angle MAO = \alpha.$$

$$x = OM = OR - NS = X \cos \alpha - Y \sin \alpha.$$

$$y = PM = NR + PS = X \sin \alpha + Y \cos \alpha.$$

Thus, if for x, y we substitute respectively $X \cos \alpha - Y \sin \alpha, X \sin \alpha + Y \cos \alpha$, the axes are rotated in the positive direction through an $\angle \alpha$.

64. By § 35, in an equation of the form $x \cos a + y \sin a = p$, the length of the \perp from the origin on the st. line is the absolute term p .

If, without changing the direction of the axes, the origin be transferred to (x_1, y_1) the equation becomes

$$(x + x_1) \cos a + (y + y_1) \sin a = p,$$

$$\text{i.e., } x \cos a + y \sin a = p - x_1 \cos a - y_1 \sin a.$$

The new equation is of the same form as the old one except that the absolute term is now $p - x_1 \cos a - y_1 \sin a$.

This absolute term is then the length of the \perp from the new origin to the st. line; or, reverting to the original origin, the length of the \perp from (x_1, y_1) to the line $x \cos a + y \sin a = p$ is $p - x_1 \cos a - y_1 \sin a$.

This is the same as the result that would be obtained by using the formula of § 45.

65.—Exercises

1. What does the equation $2x^2 - 11xy + 12y^2 + 7x - 13y + 3 = 0$ become when the origin is changed to the point $(1, 1)$ the directions of the axes being unchanged?

2. Transform the equation $x^2 + xy - 7x - 4y + 12 = 0$ to \parallel axes through $(4, -1)$.

3. Find the point that must be taken as origin, the directions of the axes being unchanged, in order that the terms of the first degree in x and y may vanish from the equation $x^2 + y^2 + 5x - 9y + 17 = 0$. Find also what the equation becomes.

4. Show that the terms of the first degree in x and y will vanish from the expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, if the origin be changed to $\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right)$, the directions of the axes being unchanged.

5. Transform the equation $Ax + By + C = 0$ by rotating the axes through an \angle of 30° .

6. Find what the equation $x^2 - y^2 = a^2$ becomes when the axes are turned through an \angle of 45° , the origin remaining the same.

7. Show that the equation $x^2 + y^2 = a^2$ is not changed when the axes are turned through any \angle α , the origin remaining the same.

8. Find what the equation $33x^2 - 34xy + 3y^2 = 0$ becomes when the axes are turned through an \angle of 60° , the origin remaining the same.

9. Find the smallest positive \angle through which the axes must be turned in order that the coefficient of xy in the equation $59x^2 + 24xy + 66y^2 = 250$ may vanish; and also find what the equation becomes.

10. Show that the term involving xy in the expression $ax^2 + 2hxy + by^2$ will vanish, if the axes are turned through the \angle .

$$\frac{1}{2} \tan^{-1} \frac{2h}{a-b}$$

66.—Review Exercises

1. Find the distances between the following pairs of points :—

$$(a) (-2, 7), (6, -2);$$

$$(b) (2a + b, a - 2b), (a - b, 3a + b);$$

$$(c) (a \cos a, a \sin a), (-b \cos a, -b \sin a).$$

Verify the result in (b), on squared paper, when $a = 1$, $b = -2$.

2. **A** $(-5, -1)$, **B** $(4, 6)$ are two given points, **P** is taken in **AB** and **Q** in **AB** produced such that **AP** : **PB** = **AQ** : **QB** = 5 : 3. Find the coordinates of **P** and **Q**.

3. Find the area of the Δ of which the vertices are $(3a, 2b)$, $(2a, 3b)$ and (a, b) .

4. Find the area of the Δ contained by the lines $2x + 11y + 43 = 0$, $9x + 8y - 14 = 0$ and $7x - 3y + 26 = 0$.

5. Find the \perp distance from $(-2, 3)$ to the line $\frac{2x}{3} - \frac{y}{2} = 6$.

Should the result be considered positive or negative and why?

6. Find the condition that the three points (x_1, y_1) , (x_2, y_2) , (x_3, y_3) may lie in a st. line.

7. Find the locus of a point such that the square of its distance from $(-3, -7)$ exceeds the square of its distance from $(5, 0)$ by 43.

8. Prove that the equation $Ax + By + C = 0$ represents a st. line.

9. Find the equation of the st. line which is equidistant from the \parallel lines $ax + by = c$, $ax + by = d$.

10. Find the equation of a st. line which makes an $\angle a$ with Oy and cuts off an intercept b from Ox .

11. Show that the st. lines $Ax + By + C = 0$, $A_1x + B_1y + C_1 = 0$ are \parallel , if $AB_1 = A_1B$.

12. Explain the meaning of the constants in the equations

$$\frac{x - h}{\cos \theta} = \frac{y - k}{\sin \theta} = r.$$

13. Show that the line $y = x \tan a$ passes through the point $(a \cos a, a \sin a)$, and find the equation of the \perp to the line at that point.

14. Find the \angle between the st. line joining $(-4, 5)$, $(5, 1)$ and the st. line joining $(3, 7)$, $(-6, -3)$.

15. Find the values of m and a such that the line $y = mx + a$ will pass through $(3, -2)$ and $(-1, -5)$.

16. Find the equation of the st. line which passes through $(2, -2)$, and makes an \angle of 150° with Ox .

17. Find the length of the st. line drawn from (h, k) , in the direction inclined at $\angle a$ to Ox , and terminated in the line $y = mx + a$.

18. Find the equation of the st. line through (h, k) , and \parallel to $\frac{x}{a} + \frac{y}{b} = 1$.

19. Show that the st. lines $Ax + By + C = 0$, $A_1x + B_1y + C_1 = 0$ are \perp to each other, if $AA_1 + BB_1 = 0$.

20. Write the equation of the st. line which is \perp to $ax - by = c$, and cuts off an intercept $= d$ from Oy .

21. Find which of the following points are on the origin side of $\frac{x}{3} - \frac{y}{5} = 1$:— $(5, 3)$, $(-2, -8)$, $(2, -2)$, $(-6, -14)$, $(-7, -17)$. Illustrate by a diagram on squared paper.

22. Show that the points $(2, 6)$, $(1, 11)$, $(-4, 7)$, $(-3, 3)$ are in the four different angular spaces made by the lines

$$\frac{x}{3} + \frac{y}{5} = 1 \text{ and } \frac{x}{5} - \frac{y}{9} = -1.$$

Illustrate by a diagram on squared paper.

23. Find the values of a for which the lines $2x - ay + 1 = 0$, $ax - 6y - 1 = 0$, $18x - ay - 7 = 0$ are concurrent; and find also the coordinates of the respective points of intersection.

24. Show that the condition that the lines $ax + by = 1$, $cx + dy = 1$, $hx + ky = 1$ are concurrent is the same as the condition that the the points (a, b) , (c, d) , (h, k) are collinear.

25. Find the \angle contained by the lines $4x - 7y + a = 0$, $3x + 11y + b = 0$.

26. Find the equation of the st. line passing through the intersection of $4x - 7y + a = 0$ and $3x + 11y + b = 0$ and making an \angle of 45° with the axis of x .

27. Find the equation of the st. line passing through the intersection of $\frac{x}{a} + \frac{y}{b} = 1$, $y = mx + c$ and also through $(d, 0)$.

28. Show that the equation of the st. line joining the intersection of $x \cos a + y \sin a = p$, $x \cos \beta + y \sin \beta = p$ to the origin is $y = x \tan \frac{a + \beta}{2}$.

29. Find the length of the \perp from (a, b) to $\frac{x}{a} + \frac{y}{b} = 1$.

30. Find the equation of the st. line through $(-5, 1)$ and \parallel to $3x + 12y = 17$.

31. Find the equation of the st. line through $(8, -2)$ and \perp to $7x = y + 4$.

32. Find the coordinates of the four points each of which is equally distant from the three lines $\frac{x}{4} - \frac{y}{3} = 1$, $\frac{x}{12} + \frac{y}{5} = 1$, $\frac{x}{24} - \frac{y}{7} = -1$.

33. Find the coordinates of the foot of the perpendicular from (3, 5) to the st. line joining (-1, -2) and (8, 1).

34. Find the separate equations of the st. lines represented by $3x^2 + 14xy - 2y^2 = 0$.

35. Find the product of the \perp s drawn from (3, -2) to the st. lines represented by $5x^2 + 12xy + 2y^2 = 0$.

36. Show that the \angle between the lines $y = mx + c$, $y = nx + b$ is $\tan^{-1} \frac{m - n}{1 + mn}$.

37. Find the tangent of the \angle between the st. lines represented by $5x^2 - 8xy - y^2 = 0$.

38. Find the equations of the st. lines which pass through (3, 6), and are inclined at an \angle of 45° to $\frac{x}{5} + \frac{y}{7} = 1$.

39. Show that the st. line joining the point (1, 1) to the intersection of $\frac{x}{a} + \frac{y}{b} = 1$ with $\frac{x}{b} + \frac{y}{a} = 1$ passes through the origin.

40. Show that the equation of the st. line passing through the intersection of $x \cos \alpha + y \sin \alpha = p$, $x \cos \beta + y \sin \beta = q$, and \parallel to $x + y = k$ is

$$(x + y) \sin(\alpha - \beta) + p(\sin \beta - \cos \beta) + q(\cos \alpha - \sin \alpha) = 0.$$

41. Find the equation of the st. line passing through the intersection of $5x - 7y - 16$, $2x - 3y = 7$ and \perp to $6x - 4y = 19$.

42. Find the equations of the st. lines drawn through the vertices and \parallel to the opposite sides of the \triangle of which the equations of the sides are $3x + 11y = 23$, $4x - 9y = 11$, $7x - 2y = -31$.

43. Show that the lines $5x + y = 4$, $2x + y = 2$, $3x + 3y = 4$ are concurrent; and find the coordinates of their common point.

44. Find the equation of the st. line passing through the intersection of $ax + by + c = 0$, $fx + gy + h = 0$, and (a) also through the origin; (b) \perp to $x + y = k$.

45. Find the equations of the st. lines which bisect the \angle s between the lines $12x - 5y = 17$, $8x + 15y = 13$.

46. Find the equation of the st. line which passes through the point of intersection of the lines $5x + y = 4$, $4x - 9y = 11$, and is \perp to the former.

47. Show that the points $(4, 2)$, $(6, 2)$, $(5, 2 + \sqrt{3})$ are the vertices of an equilateral \triangle .

48. Find the locus of a point which moves so that the sum of its distances from the axes is 10. Trace the locus on squared paper.

49. Find the locus of a point which moves so that the difference of its distances from the axes is 10. Trace the locus on squared paper.

50. Find the equations of the st. lines each of which passes through $(-5, -3)$ and is such that the part of it between the axes is divided at the given point in the ratio 7 : 3.

51. Find the equation of the st. line which passes through $(3, -2)$, and is \perp to $4x + y + 12 = 0$.

52. Find the equation of the right bisector of the st. line joining (a, b) and (h, k) .

53. Two st. lines are drawn through $(0, -3)$ such that the \perp s on them from $(-6, -6)$ are each of length 3. Find the equation of the st. line joining the feet of the \perp s.

54. Find the \angle of inclination of the lines $ax + by = c$, $(a + b)x - (a - b)y = d$.

55. A st. line is drawn through $(2, -4)$ and \perp to $7x - 3y = 11$. Find the equations of the bisectors of the \angle s between the \perp and the given st. line.

56. Find the equation of the st. lines which bisect the \angle s between the lines represented by $x^2 + 2xy \sec \theta + y^2 = 0$.

57. Find the value of h for which the equation $3x^2 + hxy - 10y^2 + x + 29y - 10 = 0$ will represent two st. lines.

58. Show that, if the axes are rotated through an \angle of 45° , the term containing xy vanishes from the equation $x^2 + 2xy \sec \theta + y^2 = 0$; and the separate equations of the two st. lines become $x = \pm y \tan \frac{\theta}{2}$.

59. Three vertices of a \parallel gm are $(3, 4)$, $(-3, 1)$, $(5, -2)$. Find the coordinates of the fourth vertex.

60. Prove that the two st. lines which join the middle points of the opposite sides of any quadrilateral mutually bisect each other.

61. What must be the value of m , if the line $y = mx - 5$ passes through the intersection of $7x - 11y = 14$ and $5x + 2y = -11$.

62. Find the area of the \triangle contained by the lines $x + y = 12$, $2x - y = 12$, $x - 2y = -12$.

CHAPTER IV

THE CIRCLE

67. A **circle** is the locus of the points that lie at a fixed distance from a fixed point.

The fixed point is the **centre** and the fixed distance is the **radius** of the circle.

68. To find the equation of a circle having its centre at the origin.

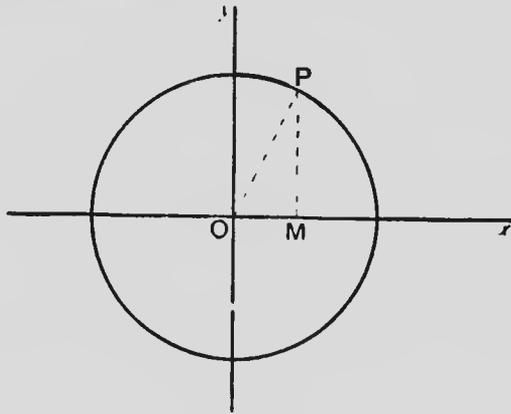


FIG. 32.

Let $P(x, y)$ be any point on the circle of which the centre is O . Let the radius $= a$.

Draw $PM \perp Ox$. Join PO .

$\therefore OPM$ is a rt.- \angle d Δ ,

$\therefore OM^2 + PM^2 = OP^2$.

$\therefore x^2 + y^2 = a^2$.

This being the relation which holds between the coordinates of any point on the circle and the given radius is the required equation.

69. To find the equation of a circle, the centre being at any fixed point (h, k) and the radius equal to a .

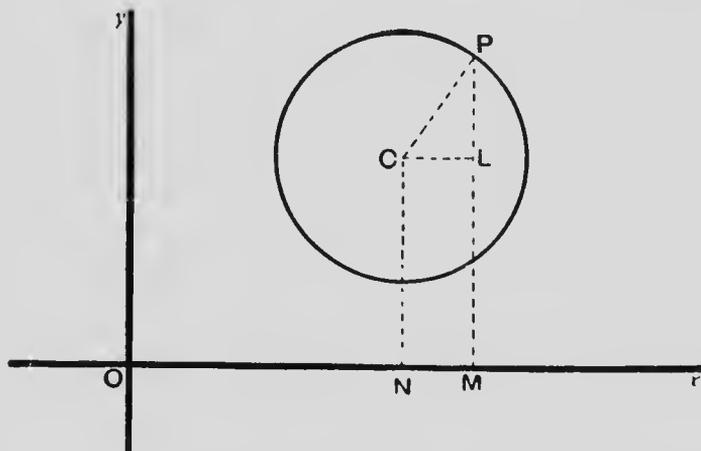


FIG. 33.

$C(h, k)$ is the centre: and $P(x, y)$ is any point on the circle.

Draw $PM, CN \perp Ox, CL \perp PM$. Join CP .

$$CL = NM = OM - ON = x - h;$$

$$PL = PM - LM = PM - CN = y - k.$$

$\therefore CPL$ is a rt.- \angle d Δ ,

$$CL^2 + PL^2 = CP^2.$$

$$\therefore (x - h)^2 + (y - k)^2 = a^2.$$

This is the required equation.

70. If we expand the equation found in § 69, we obtain:—

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - a^2 = 0.$$

Comparing this result with the general equation of the second degree:—

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

we see that the conditions that the latter should represent a circle are that the coefficients of x^2 and y^2 should be equal and that the coefficient of xy should be zero.

Thus the equation

$$ax^2 + ay^2 + 2gx + 2fy + c = 0$$

may be changed to

$$\left(x + \frac{g}{a}\right)^2 + \left(y + \frac{f}{a}\right)^2 = \frac{g^2 + f^2 - ac}{a^2},$$

from which, by comparison with the formula of § 69, we see that it represents a circle having its centre at the point $\left(-\frac{g}{a}, -\frac{f}{a}\right)$ and its radius = $\frac{\sqrt{g^2 + f^2 - ac}}{a}$.

71. The general equation of the circle to rectangular axes is commonly written:—

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

When the circle passes through the origin and its centre is on the axis of x , the equation of § 69 becomes

$$(x - a)^2 + y^2 = a^2,$$

$$\text{or, } x^2 + y^2 = 2ax.$$

72.—Exercises

1. Write the equation of the circle with centre (0, 0) and radius = $\sqrt{5}$.

2. Write the equation of the circle with centre (6, 2) and radius = 3.

3. Write the equation of the circle with centre (-5, -1) and radius = $\sqrt{26}$. Show that this circle passes through the origin.

4. Write the equation of the circle with centre (-a, -b) and radius = c. Find the condition that this circle passes through the origin.

5. Find the coordinates of the centre and the radii of the following circles:—

(a), $x^2 + y^2 - 6x - 2y = 15$; (b), $4x^2 + 4y^2 + 7x + 5y = 16$; (c), $x^2 + y^2 = 14x$; (d), $x^2 + y^2 + 2by = c^2$.

6. Draw, on squared paper, the circles of which the equations are:—

(a), $x^2 + y^2 = 9$; (b), $x^2 + y^2 = 8x$; (c), $x^2 + y^2 + 6y = 7$.

7. Find the centre and radius of the circle which passes through the origin and cuts off intercepts = a and b from Ox and Oy respectively.

Solution.—Since the circle passes through the origin its equation must be satisfied by $x = 0$, $y = 0$, and \therefore the absolute term must be zero. Thus the equation may be written

$$x^2 + y^2 + 2gx + 2fy = 0.$$

Substituting in this equation the coordinates of the points (a, 0), (0, b) the two equations

$$a^2 + 2ga = 0$$

$$b^2 + 2fb = 0$$

are obtained from which $g = -\frac{a}{2}$, $f = -\frac{b}{2}$.

\therefore the equation of the circle is

$$x^2 + y^2 - ax - by = 0,$$

$$\text{or, } \left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2 + b^2}{4}.$$

\therefore the centre is $\left(\frac{a}{2}, \frac{b}{2}\right)$ and the radius = $\frac{\sqrt{a^2 + b^2}}{2}$.

8. Find the equation of the circle which passes through the origin and also through $(4, 3)$ and $(-2, 6)$.
9. Find the equation of the circle which has its centre on the axis of x and which passes through the points $(5, 3)$ and $(-3, 1)$.
10. Show from the general equation of §71 that three conditions are necessary and sufficient to determine a circle.
11. Find the condition that the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ may have its centre (a) on the axis of x ; (b) on the axis of y .
12. Find the equation of the circle which passes through $(3, 1)$ and $(5, -3)$ and has its centre on the line $x - y = 4$.
13. Find the equation of the circle having the st. line joining $(7, -5)$ and $(-3, -1)$ as a diameter.
14. A $(a, 0)$ is a fixed point and P (x, y) is a variable point such that $PO : PA = p : q$. Show that the locus of P is a circle having its centre on O*x*, and dividing OA internally and externally in the ratio $p : q$.
15. Find the equation of the circumcircle of the \triangle whose vertices are $(3, 4)$, $(-2, 3)$, $(-5, -7)$.
16. Find the length of the chord of the circle $x^2 + y^2 = 25$ cut off by the line $3x + y = 15$.
17. Find the length of the chord of the circle $x^2 + y^2 - 6x + 14y = 42$ cut off by the line $x - y = 8$.
18. Show that the locus of the centres of all circles which pass through two given points (p, q) , (r, s) is the right bisector of the st. line joining the given points.
19. Through the given point P (h, k) a st. line is drawn cutting the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at A and B. Prove that PA . PB is constant for all directions of the st. line.

Solution:—Take $\frac{x-h}{\cos \theta} = \frac{y-k}{\sin \theta} = r$ as the equation of the st. line.

Then

$$x = h + r \cos \theta, \quad y = k + r \sin \theta.$$

Substituting these values in the equation of the circle, and simplifying

$$r^2 + 2 \{ (h+g) \cos \theta + (k+f) \sin \theta \} r + h^2 + k^2 + 2gh + 2fk + c = 0.$$

The value of $\mathbf{PA} \cdot \mathbf{PB}$ = the product of the two values of r in this equation

$$= h^2 + k^2 + 2gh + 2fk + c,$$

an expression which does not contain θ and which is \therefore independent of the direction of the line.

20. Find the equation of that chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which is bisected at the point (h, k) .

Solution:—(First Method). As in Ex. 19, if we take for the equation of the chord $\frac{x-h}{\cos \theta} = \frac{y-k}{\sin \theta} = r$, we get

$$r^2 + 2 \{ (h+g) \cos \theta + (k+f) \sin \theta \} r + h^2 + k^2 + 2gh + 2fk + c = 0.$$

If the chord is bisected at (h, k) the two values of r got from this equation are equal in value but opposite in sign, and

$$\therefore (h+g) \cos \theta + (k+f) \sin \theta = 0.$$

Multiplying the terms of this equation by the equal fractions $\frac{x-h}{\cos \theta}, \frac{y-k}{\sin \theta}$, the required equation is found to be

$$(h+g)(x-h) + (k+f)(y-k) = 0.$$

(Second Method). Let the equation of the chord be

$$y - k = m(x - h).$$

The centre of the circle is $(-g, -f)$, and the equation of the \perp from the centre to the chord is

$$m(y+f) + x + g = 0.$$

The \perp from the centre bisects the chord, and, \therefore passes through (h, k) .

$$\therefore m(k+f) + h + g = 0.$$

$$\therefore m = -\frac{h+g}{k+f}.$$

\therefore the required equation is

$$(h+g)(x-h) + (k+f)(y-k) = 0.$$

21. Find the equation of the chord of the circle $x^2 + y^2 - 6x - 8y = 24$ which passes through $(5, -1)$ and is bisected at that point.

22. From the point $P(-3, -5)$ a st. line is drawn to cut the circle $x^2 + y^2 - 4x - 10y = 17$ at A and B . Find the area of the rectangle $PA \cdot PB$.

23. Find the equation of the common chord of the circles

$$x^2 + y^2 - 2x - 4y = 39,$$

$$x^2 + y^2 + 6x - 2y = 56.$$

24. Find the condition that the line $ax + by + c = 0$ is a chord of the circles

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

$$x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

passes through the origin.

25. Find the equation of the circle which passes through the origin and also through (h, k) and (k, h) .

TANGENTS

73. Let APQ be a secant cutting a curve at P and Q .

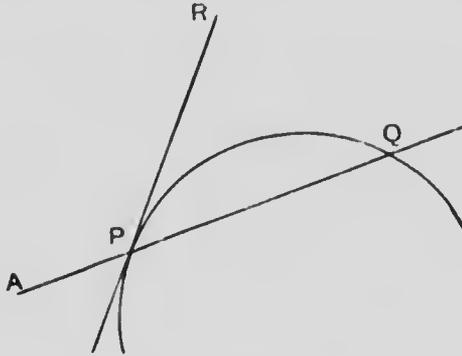


FIG. 34.

If the secant rotate about the point P until the second point Q approaches indefinitely near to P , the limiting position PR of the chord is called a **tangent** to the curve at the point P .

The point P is called the **point of contact** of the tangent PR .

74. To find the equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point $P(x_1, y_1)$ on the circle.

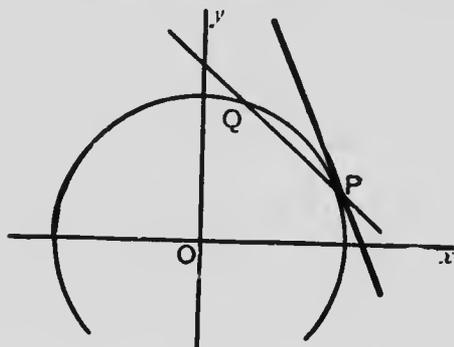


FIG. 35.

Let $Q(x_2, y_2)$ be another point on the circle.

Then the equation of PQ is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}. \quad (1)$$

\therefore P and Q are both on the circle,

$$x_1^2 + y_1^2 = a^2,$$

$$\text{and, } x_2^2 + y_2^2 = a^2.$$

\therefore , subtracting, $x_1^2 - x_2^2 + y_1^2 - y_2^2 = 0$.

$$\therefore, (x_1 - x_2)(x_1 + x_2) + (y_1 - y_2)(y_1 + y_2) = 0 \quad (2)$$

Multiplying the terms of (2) by the equal fractions in (1)

$$(x - x_1)(x_1 + x_2) + (y - y_1)(y_1 + y_2) = 0. \quad (3)$$

If, now, PQ rotates about P until Q coincides with P , $x_2 = x_1$ and $y_2 = y_1$.

Thus equation (3) becomes

$$2(x - x_1)x_1 + 2(y - y_1)y_1 = 0,$$

$$\text{or, } xx_1 + yy_1 = x_1^2 + y_1^2.$$

$$\text{But } x_1^2 + y_1^2 = a^2.$$

$$\therefore xx_1 + yy_1 = a^2.$$

This is the required equation.

75. Alternative Method of finding the equation of the tangent at the point $P(x_1, y_1)$ on the circle $x^2 + y^2 = a^2$.

Using the figure of § 74 let the equation of PQ be

$$\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r, \quad (1)$$

$$\text{and } \therefore x = x_1 + r \cos \theta, y = y_1 + r \sin \theta.$$

Substituting these values of x, y in the equation of the circle, and expanding

$$r^2 + 2(x_1 \cos \theta + y_1 \sin \theta)r + x_1^2 + y_1^2 = a^2 \quad (2)$$

Since P is on the circle, $x_1^2 + y_1^2 = a^2$.

\therefore one value of r is zero, and equation (2) becomes

$$r + 2(x_1 \cos \theta + y_1 \sin \theta) = 0. \quad (3)$$

If, now, PQ rotates about P until Q coincides with P , the other value of r also becomes zero, and,

$$\therefore x_1 \cos \theta + y_1 \sin \theta = 0. \quad (4)$$

Multiplying the terms in (4) by the equal quantities in (1),

$$x_1(x - x_1) + y_1(y - y_1) = 0.$$

$$\therefore xx_1 + yy_1 = x_1^2 + y_1^2 = a^2.$$

\therefore the required equation is

$$xx_1 + yy_1 = a^2.$$

76. The equation of OP (Fig. 34) is $\frac{x}{x_1} = \frac{y}{y_1}$, and by the condition of perpendicularity, the line represented by this equation is \perp to the line represented by $xx_1 + yy_1 = a^2$.

\therefore the radius of a circle drawn to the point of contact of a tangent is \perp to the tangent.

77. In any curve, the st. line drawn through the point of contact of a tangent and \perp to the tangent is called a **normal** to the curve at that point.

78. To find the equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at a point $P(x_1, y_1)$ on the circle.

Let the equation of a chord PQ be

$$\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r, \text{ ————— (1)}$$

and $\therefore x = x_1 + r \cos \theta, y = y_1 + r \sin \theta$.

Substituting these values of x, y in the equation of the circle and simplifying,

$$r^2 + 2 \{ (x_1 + g) \cos \theta + (y_1 + f) \sin \theta \} r + x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0.$$

Since $P(x_1, y_1)$ is a point on the circle, this equation reduces to

$$r + 2 \{ (x_1 + g) \cos \theta + (y_1 + f) \sin \theta \} = 0.$$

If now the secant PQ rotates about P until Q coincides with P , the second value of r becomes zero, and \therefore

$$(x_1 + g) \cos \theta + (y_1 + f) \sin \theta = 0. \quad (2)$$

Multiplying the terms in (2) by the equal quantities in (1),

$$(x - x_1)(x_1 + g) + (y - y_1)(y_1 + f) = 0.$$

$$\therefore x(x_1 + g) + y(y_1 + f) - x_1^2 - y_1^2 - gx_1 - fy_1 = 0.$$

But, $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0.$

\therefore , adding,

$$x(x_1 + g) + y(y_1 + f) + gx_1 + fy_1 + c = 0.$$

$$\therefore \mathbf{xx}_1 + \mathbf{yy}_1 + \mathbf{g(x + x}_1) + \mathbf{f(y + y}_1) + \mathbf{c} = 0.$$

79. By comparing the equation of the tangent

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

with that of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

the following rule is obtained for writing the equation of the tangent at a point (x_1, y_1) on the circle:—

In the equation of the circle change

$$x^2 \text{ into } xx_1, \quad y^2 \text{ into } yy_1,$$

$$2x \quad " \quad x + x_1, \quad 2y \quad " \quad y + y_1.$$

80. To find the equation of the tangent to the circle $x^2 + y^2 = a^2$ in terms of m , the slope of the tangent.

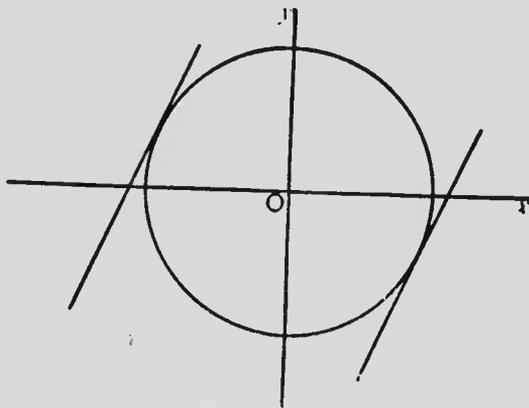


FIG. 36.

To find the abscissae of the points where the line $y = mx + k$ cuts the circle, eliminate y by substitution, and

$$x^2 + (mx + k)^2 = a^2,$$

$$\text{i.e., } (1 + m^2)x^2 + 2mkx + k^2 - a^2 = 0.$$

If the line is a tangent, the values of x from this equation are equal to each other, and

$$\therefore m^2k^2 = (1 + m^2)(k^2 - a^2),$$

$$\therefore k^2 = a^2(1 + m^2),$$

$$\text{and } k = \pm a\sqrt{1 + m^2}.$$

Thus the equation of the tangent is

$$y = mx \pm a\sqrt{1 + m^2}.$$

The double sign corresponds to the two tangents that have the same slope, as indicated in the diagram.

81.—Exercises

1. Find the equation of the tangent to the circle
 - (a) $x^2 + y^2 = 34$, at the point $(3, 5)$;
 - (b) $x^2 + y^2 - 10x + 12y = 39$ at $(-1, 2)$;
 - (c) $x^2 + y^2 + 18x - 14y = 39$ at $(3, 12)$;
 - (d) $x^2 + y^2 + 2gx + 2fy = 0$ at the origin.
2. Find the equations of the st. lines touching the circle $x^2 + y^2 = 35$ and making an \angle of 45° with the axis of x .
3. Find the equations of the st. lines which touch $x^2 + y^2 = r^2$ and are (a) \parallel to, (b) \perp to the line $\mathbf{A}x + \mathbf{B}y + \mathbf{C} = 0$
4. Prove that $x + 2y = 10$ is a tangent to the circle $x^2 + y^2 = 20$; and find the point of contact
5. Prove that $x - 2y = 4$ is a tangent to the circle $x^2 + y^2 - 8x - 10y + 21 = 0$; and find the point of contact
6. Find the condition that $\mathbf{A}x + \mathbf{B}y + \mathbf{C} = 0$ may touch
 - (a) $x^2 + y^2 = r^2$;
 - (b) $x^2 + y^2 + 2gx + 2fy + c = 0$.
7. Find the condition that $\frac{x}{a} + \frac{y}{b} = 1$ may touch $x^2 + y^2 = r^2$.
8. Find the condition that the axis of x may touch $x^2 + y^2 + 2gx + 2fy + c = 0$.
9. Find the equations of the circles passing through $(5, 2)$ and touching the axes of x and y .
10. Find the equations of the tangents to the circle $x^2 + y^2 - 2x + 2y = 10$ which make an $\angle = 30^\circ$ with the axis of x .

11. Find the length of the part of the line $2x - 5y + 10 = 0$ intercepted by the circle $x^2 + y^2 + 8x - 6y = 24$.

12. Show that the tangent to the circle $(x - a)^2 + (y - b)^2 = r^2$ at the point (x_1, y_1) on this circle is $(x - a)(x_1 - a) + (y - b)(y_1 - b) = r^2$.

Solution.—Transform the origin to the point (a, b) without changing the direction of the axes. The transforming relations are $x = X + a$, $y = Y + b$, $x_1 = X_1 + a$, $y_1 = Y_1 + b$.

The equation of the circle becomes

$$X^2 + Y^2 = r^2,$$

and, by § 74, the tangent at (X_1, Y_1) is

$$XX_1 + YY_1 = r^2.$$

Transforming back to the original origin, the equation of the tangent becomes

$$(x - a)(x_1 - a) + (y - b)(y_1 - b) = r^2.$$

13 Show that the point $(g + r \cos a, f + r \sin a)$ is on the circle $(x - g)^2 + (y - f)^2 = r^2$; and find the equation of the tangent at that point.

14. Show that the circles

$$x^2 + y^2 + 6x + 16y + 24 = 0,$$

$$x^2 + y^2 - 10x + 4y + 20 = 0$$

touch each other externally. Find the coordinates of the point of contact; and the equation of the common tangent at that point.

15. Show that, if the circles

$$(x - h)^2 + (y - k)^2 = r^2,$$

$$(x - m)^2 + (y - n)^2 = s^2$$

touch each other,

$$(h - m)^2 + (k - n)^2 = (r \pm s)^2.$$

16. Find the equation of the common chord; and the coordinates of the points of intersection of the circles

$$x^2 + y^2 - 2x - 6y = 14,$$

$$x^2 + y^2 - 5x + 3y = 5.$$

17. Find the equation of the circle whose centre is at the origin and which touches the line $Ax + By + C = 0$.

18. Find the equation of the circle whose centre is at $(7, 2)$ and which touches $3x - 5y = 4$.

19. Find the centres of similitude and the equations of the transverse and direct common tangents of the circles

$$x^2 + y^2 - 6x + 2y + 6 = 0,$$

$$x^2 + y^2 + 8x - 10y + 32 = 0.$$

20. Find the equations of the common tangents of the circles

$$x^2 + y^2 - 4x - 8y - 5 = 0,$$

$$x^2 + y^2 - 10x - 6y - 2 = 0.$$

Find also the coordinates of the points of contact of the tangents.

21. Find the equations of the tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which are \parallel to $x + 3y = 9$.

22. Find the equations of the two tangents to the circle $x^2 + y^2 = 25$ which make an \angle of 30° with the axis of x .

POLES AND POLARS

82. To find the equation of the chord of contact of tangents drawn from an outside point to the circle $x^2 + y^2 = a^2$.

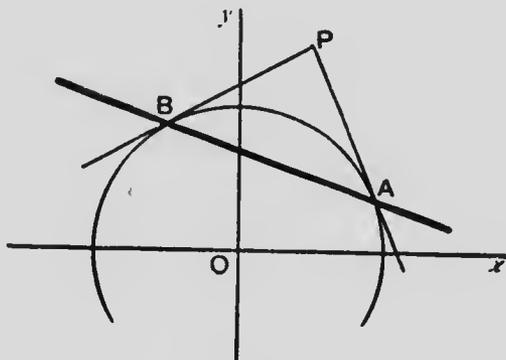


FIG. 37.

Let $P(x_1, y_1)$ be the given point; PA and PB the tangents.

It is required to find the equation of AB .

Take (x', y') , (x'', y'') to represent the coordinates of A , B respectively.

The equation of AP is, by § 74,

$$xx' + yy' = a^2;$$

and since the coordinates of P must satisfy this equation,

$$x_1x' + y_1y' = a^2. \quad (1)$$

Similarly,

$$x_1x'' + y_1y'' = a^2. \quad (2)$$

From these results it is seen that

$$xx_1 + yy_1 = a^2$$

is the equation of **AB**; for:—

since it is of the first degree it represents a st. line;

by (1), **A** is a point on the line;

by (2), **B** is a point on the line;

∴ the required equation is

$$xx_1 + yy_1 = a^2.$$

83. The equation of the chord of contact of tangents drawn from an outside point to a circle is of the same form as the equation of the tangent at a point on the circle.

This is in agreement with the fact that, if the point **P** approach the circle and ultimately fall on it, the chord of contact becomes the tangent at **P**, or the tangent at **P** is the final position of the chord of contact when **P** approaches the circle.

84. To find the equation of the polar of $P(x_1, y_1)$ with respect to the circle $x^2 + y^2 = r^2$.

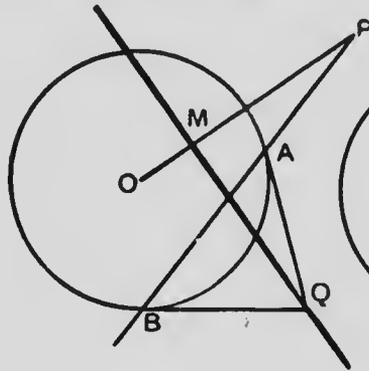


FIG. 38.

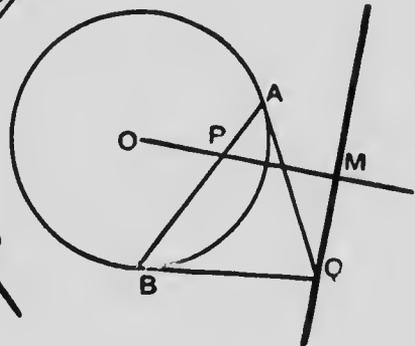


FIG. 39.

Through P draw any st. line cutting the circle at A, B . Draw tangents AQ, BQ intersecting at $Q(X, Y)$.

It is required to find the locus of Q .

By § 82, the equation of AB is

$$xX + yY = a^2;$$

and as the coordinates of P must satisfy this equation

$$Xx_1 + Yy_1 = a^2.$$

\therefore , as X, Y are the coordinates of any point on the polar of P , the required equation is

$$xx_1 + yy_1 = a^2.$$

85. The equation of OP is $xy_1 - yx_1 = 0$, and, by the condition for perpendicularity, this line is \perp to that represented by $xx_1 + yy_1 = a^2$.

\therefore the polar of P is a st. line which cuts OP at rt. \angle s

86. If the polar $xx_1 + yy_1 = a^2$ cuts OP at M , the

$$\text{length of } OM = \frac{a^2}{\sqrt{x_1^2 + y_1^2}} = \frac{a^2}{OP}.$$

$$\therefore OM \cdot OP = a^2.$$

87. The equation of the polar of the point $P(x_1, y_1)$ without the circle $x^2 + y^2 = a^2$ is the same as that of the chord of contact of tangents drawn from P to the circle. This shows that, when the point is without the circle, its polar is the chord of contact produced, or, that **tangents drawn from P touch the circle at the points where it is cut by the polar of P .**

88. The equation of the polar of any point $P(x_1, y_1)$ is of the same form as the equation of the tangent at a point on the circle.

This is in agreement with the fact that, if the point P approaches and ultimately coincides with the circle, OP becomes equal to a , and \therefore , by § 86, OM becomes equal to a , and the polar becomes the tangent at the point P .

89. If the polar of P passes through Q, the polar of Q passes through P.

Let (x_1, y_1) , (x_2, y_2) be the coordinates of P, Q respectively.

The polar of P with respect to $x^2 + y^2 = a^2$ is

$$xx_1 + yy_1 = a^2.$$

Since this line passes through Q,

$$x_2x_1 + y_2y_1 = a^2.$$

This proves that (x_1, y_1) is on the line

$$xx_2 + yy_2 = a^2;$$

and \therefore P is on the polar of Q.

Cor. If the point Q moves along the polar of P, the polar of Q changes its position, but always passes through P.

\therefore , if the pole moves along a st. line, its polar turns about the pole of that line.

90. To find the pole of the st. line $Ax + By + C = 0$ with respect to the circle $x^2 + y^2 = a^2$.

Let (x_1, y_1) be the coordinates of the pole.

The equation of the polar of (x_1, y_1) is

$$xx_1 + yy_1 - a^2 = 0.$$

This equation must be the same as

$$Ax + By + C = 0.$$

$$\therefore \quad \frac{x_1}{A} = \frac{y_1}{B} = \frac{-a^2}{C}$$

$$\therefore \quad x_1 = -\frac{a^2A}{C}, \quad y_1 = -\frac{a^2B}{C}.$$

91. To find the polar of $P(x_1, y_1)$ with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

The equation of the circle may be written

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c.$$

Transforming the origin to the point $(-g, -f)$, the transforming relations are $x = X - g$, $y = Y - f$, $x_1 = X_1 - g$, $y_1 = Y_1 - f$, and the equation of the circle becomes

$$X^2 + Y^2 = g^2 + f^2 - c.$$

The equation of the polar of $P(X_1, Y_1)$ with respect to this circle is, by § 84,

$$XX_1 + YY_1 = g^2 + f^2 - c.$$

Transforming back to the original origin, the equation becomes

$$(x + g)(x_1 + g) + (y + f)(y_1 + f) = g^2 + f^2 - c.$$

$$\therefore xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

NOTE.—As an exercise, the student should obtain the above result directly from the definition of poles and polars, by the method used in § 84.

92.—Exercises

1. Find the polar of the point
 - (a) (3, 5) with respect to $x^2 + y^2 = 30$;
 - (b) (a , 0) " " " $x^2 + y^2 = r^2$;
 - (c) (-2, 4) " " " $x^2 + y^2 - 4x - 8y = 5$;
 - (d) (-5, -1) " " " $x^2 + y^2 - 10x + 6y = 15$;
 - (e) (0, 0) " " " $(x - h)^2 + (y - k)^2 = r^2$.
2. Find the pole of the st. line
 - (a) $2x - 7y = 17$ with respect to $x^2 + y^2 = 17$;
 - (b) $x - 2y + 12 = 0$ " " " $x^2 + y^2 = 23$;
 - (c) $4x - y = 1$ with respect to $x^2 + y^2 - 2x - 4y = 4$;
 - (d) $4x + 5y = 5$ " " " $x^2 + y^2 - 8x - 10y = -5$.
3. (a) Show that $x^2 + y^2 = 25$ is the equation of a circle.
 (b) Show that (-3, 4) is on the circle.
 (c) Write the equation of the tangent to the circle at this point.
 (d) Show that the point (9, 13) is on this tangent.
 (e) Find the equation of the other tangent from (9, 13).
 (f) Write the equation of the polar of (9, 13).
 (g) Find the equation of the st. line through (9, 13) \perp to the polar, commenting on the form of the result.
 Draw the diagram on squared paper.
4. Find the pole of $\frac{x}{a} + \frac{y}{b} = 1$ with respect to $x^2 + y^2 = c^2$.
5. Find the pole of $lx + my = 1$ with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.
6. Prove that the polar of (-2, 5) with respect to $x^2 + y^2 = 18$ touches $x^2 + y^2 - 6x - 2y = 19$; and find the coordinates of the point of contact.

TANGENTS FROM AN OUTSIDE POINT

93. To find the length of the tangent PA from the point P (x_1, y_1) to a given circle.

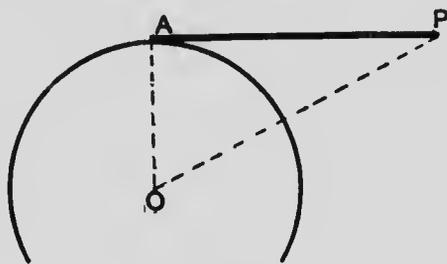


FIG. 40.

(1) Let the equation of the circle be

$$x^2 + y^2 = a^2.$$

Join OA, OP.

\therefore AOP is a rt.- \angle d Δ ,

$$\begin{aligned} \mathbf{AP}^2 &= \mathbf{OP}^2 - \mathbf{AO}^2 \\ &= x_1^2 + y_1^2 - a^2. \end{aligned}$$

$$\therefore \mathbf{AP} = \sqrt{x_1^2 + y_1^2 - a^2}.$$

(2) Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

This equation may be written

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c,$$

from which it is seen that the centre is $(-g, -f)$

and the radius $= \sqrt{g^2 + f^2 - c}$.

With the diagram and construction of Fig. 40,

$$\begin{aligned} \mathbf{AP}^2 &= \mathbf{OP}^2 - \mathbf{AO}^2 \\ &= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c) \\ &= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c. \end{aligned}$$

$$\therefore \mathbf{AP} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}.$$

94. To find the equation of the tangents from (x_1, y_1) to the circle $x^2 + y^2 = a^2$.

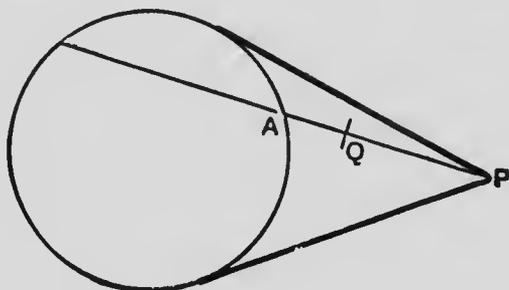


FIG. 41.

Let a secant drawn from $P(x_1, y_1)$ cut the circle at A ; and let $Q(x, y)$ be any point on the secant.

If $PA : QA = k : 1$, the coordinates of A are

$\left(\frac{kx - x_1}{k - 1}, \frac{ky - y_1}{k - 1}\right)$, and \therefore , since A is on the circle

$$(kx - x_1)^2 + (ky - y_1)^2 = (k - 1)^2 a^2,$$

$$\text{or, } (x^2 + y^2 - a^2)k^2 - 2(x_1x + y_1y - a^2)k + x_1^2 + y_1^2 - a^2 = 0.$$

If, now, the secant turn about P until it coincides with either of the tangents from P , the two values of k found from this equation, and which correspond to the two points where the secant cuts the circle, are equal to each other.

$$\therefore (xx_1 + yy_1 - a^2)^2 = (x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2).$$

This is the required equation.

95.—Exercises

1. Find the length of the tangent from

(a), (7, 3) to $x^2 + y^2 = 22$;

(b), (3, -5) to $x^2 + y^2 - 3x + 7y + 35 = 0$;

(c), (-2, -6) to $x^2 + y^2 = 12x$;

(d), (0, 0) to $x^2 + y^2 + 2gx + 2fy + c = 0$;

(e), (4, 2) to $x^2 + y^2 - 6x + 2y - 39 = 0$.

Explain the imaginary result in (e).

2. The length of the tangent drawn from a point to $x^2 + y^2 - 10x - 4y + 9 = 0$ is always 4. Find the locus of the point. Plot the diagram on squared paper.

3. The length of the tangent from P to $x^2 + y^2 = 4$ is twice the distance from P to (6, 0). Find the locus of P.

4. Find the equations of the tangents from (7, -1) to $x^2 + y^2 = 25$.

5. Show, by the method of § 94, that the equation of the tangents from (x_1, y_1) to $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$\{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c\}^2 \\ = (x^2 + y^2 + 2gx + 2fy + c)(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c).$$

RADICAL AXIS

96. To find the radical axis of the circles

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

$$x^2 + y^2 + 2g'x + 2f'y + c' = 0.$$

Since the tangents to the circles from any point on their radical axis are equal to each other, if (x, y) is any point on the locus, by § 93,

$$x^2 + y^2 + 2gx + 2fy + c = x^2 + y^2 + 2g'x + 2f'y + c'$$

∴ the required equation is

$$2(g - g')x + 2(f - f')y + c - c' = 0.$$

97. The centres of the circles in the last article are $(-g, -f)$ and $(-g', -f')$.

\therefore the st. line joining the centres is

$$\frac{x + g}{g - g'} = \frac{y + f}{f - f'}$$

$$\text{or, } (f - f')(x + g) - (g - g')(y + f) = 0.$$

By the condition of perpendicularity this line is \perp to $2(g - g')x + 2(f - f')y + c - c' = 0$.

\therefore the radical axis is \perp to the line of centres.

98.—Exercises

1. Find the radical axis of the circles

$$x^2 + y^2 - 4x - 6y + 9 = 0,$$

$$x^2 + y^2 - 16x - 14y + 104 = 0.$$

Draw the diagram on squared paper.

2. Find the radical axis of the circles

$$2x^2 + 2^2y + 9x - 8y - 3 = 0,$$

$$x^2 + y^2 = 9.$$

3. Show that the radical axes of the circles

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$

$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

taken two and two are concurrent. (The point of concurrence is the radical centre.)

4. Find the radical centre of the circles

$$x^2 + y^2 - 3x + 7y + 35 = 0,$$

$$x^2 + y^2 - 7x + 5y - 31 = 0,$$

$$x^2 + y^2 - 6x + 2y - 39 = 0.$$

5. Show that the circles

$$x^2 + y^2 - 3x + 5y - 9 = 0,$$

$$x^2 + y^2 + 7x + y - 11 = 0,$$

$$x^2 + y^2 + 2x + 3y - 10 = 0,$$

have a common radical axis. Show also that their centres are in a st. line which is \perp to the common radical axis.

Miscellaneous Exercises

1. Find the equation of the st. line passing through the intersection of $x - 2y = 5$, $x + 3y = 10$ and \parallel to $3x + 4y = 11$.

2. Find the equation of the st. line passing through the intersection of $8x + y = 7$, $11x + 2y = 28$ and \perp to the latter line.

3. Plot the quadrilateral $(4, 2)$, $(-5, 6)$, $(-9, -6)$, $(7, -4)$; and find its area.

4. Plot the lines $2x + 5y = 29$, $12x + y = 29$, $5x - 2y = 29$; and find the area contained by them.

5. Find the equation of the st. line passing through (h, k) and such that the portion of it between the axes is bisected at the given point.

6. Find the equation of the st. line passing through (h, k) and (a) \parallel to, (b) \perp to the st. line joining (x_1, y_1) , (x_2, y_2) .

7. Show that, if the lines $ax + by + c = 0$, $bx + cy + a = 0$, $cx + ay + b = 0$ are concurrent but not coincident, then $a + b + c = 0$.

8. Find the ratio in which the st. line joining $(-5, 3)$, $(6, -1)$ is divided by $x - 11y + 3 = 0$.

9. Find the centre of the inscribed circle of the Δ formed by the lines $4x - 3y = 18$, $5x + 12y = 9$, $24x + 7y = 30$.

10. Find the area of the Δ contained by $y = 3x$, $y = 5x$ and $x + 2y = 77$.

11. Show that the area of the Δ contained by $y = m_1x$, $y = m_2x$ and $Ax + By + C = 0$

$$= \frac{(m_1 - m_2) C^2}{2(A + m_1B)(A + m_2B)}$$

12. Find the locus of a point such that the square of its distance from $(6, 0)$ is three times the square of its distance from $(2, 0)$.

13. One vertex of a $\parallel\text{gm}$ is at the origin and the two adjacent vertices are at (a, b) , (c, d) . Find the fourth vertex.

14. Show that, if the two circles

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

$$x^2 + y^2 - 2fx - 2gy + c = 0$$

touch each other, then $(g - f)^2 = 2c$.

15. Give the geometrical interpretation of the equation $x^2 + y^2 + 2ax \cos a + 2ay \sin a + a^2 = 0$.

16. Find the locus of the intersection of the st. lines which pass through $(6, 0)$ and $(0, 3)$ respectively and cut each other at $\angle c$.

17. Find the equations of the tangents to the circle $x^2 + y^2 = 2kx$ which are \parallel to $3x - y = 0$.

18. Find the orthocentre of the Δ whose sides are $8x - 5y = 16$, $2x - 3y = 10$ and $x + 2y = 6$.

19. Prove that the radical axis of the circles

$$x^2 + y^2 = a^2,$$

$$x^2 + y^2 - 2a(x \cos a + y \sin a) = 0$$

bisects the st. line joining their centres.

20. Chords of the circle $x^2 + y^2 = a^2$ pass through the fixed point (h, o) . Find the locus of their middle points.

21. Find the equation of the circle which passes through the origin and makes intercepts a and b on Ox, Oy respectively.

22. Find the equation of the circle described on the st. line joining the origin to (y, f) as diameter.

23. Find the coordinates of a point such that the st. line joining it to $(4, -3)$ is bisected at rt. \angle s by $2x - 3y = 7$.

24. Find the locus of the points from which tangents drawn to $x^2 + y^2 = 13$ and $x^2 + y^2 - 2x + 6y + 1 = 0$ are as 5 is to 3.

25. Find the distances from the point $(2, 4)$ in the direction having the direction cosines $-\frac{3}{5}, -\frac{4}{5}$ to the curve whose equation is

$$3x^2 - 6xy + 5y^2 - 32 = 0.$$

26. Find the equation of the locus of a point P such that $PA : PB = k : 1$ where $A (x_1, y_1), B (x_2, y_2)$ are fixed points.

Show that the locus is a circle and find the relation of its centre to A and B .

27. (a) Find the coordinates of the point C which divides the st. line joining $A (3, -2), B (19, 10)$ in the ratio $AC : CB = 1 : 3$.

(b) Prove that $D (11, 4)$ lies on the st. line AB given above; and by computing the lengths of AD and BD , find the ratio in which D divides AB .

28. (a) Find the area of a triangle the coordinates of whose angular points are $(x_1, y_1), (x_2, y_2), (x_3, y_3)$.

(b) From the result of (a) deduce the equation of a st. line in terms of the coordinates of two given points through which it passes.

29. Show that $y = mx + a \sqrt{1 + m^2}$ is always a tangent to the circle $x^2 + y^2 = a^2$.

30. The equation $3x^2 + 3y^2 - 12x - 6y + 4 = 0$ can be reduced to one containing terms in x and y of the second degree only, by transforming to \parallel axes through a properly chosen point. What are the coordinates of the point?

31. Find the distance of the point of intersection of the lines $3x + 2y + 4 = 0$ and $2x + 5y + 8 = 0$ from the line $5x - 12y + 6 = 0$.

32. Find the equation of the circle whose centre is (h, k) and which passes through (a, b) .

33. Find the locus of the points from which tangents drawn to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

are at rt. \angle s to each other.

34. If the tangents at the points (x_1, y_1) , (x_2, y_2) on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are at rt. \angle s to each other, show that

$$x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + g^2 + f^2 = 0.$$

35. Find the equation of the st. line joining $(ab^2, 2ab)$ and $(ac^2, 2ac)$.

36. Show that the equation of the \perp to $\frac{\cos a}{a}x + \frac{\sin a}{b}y = 1$ at the point $(a \cos a, b \sin a)$ is $\frac{a}{\cos a}x - \frac{b}{\sin a}y = a^2 - b^2$.

37. Find the product of the \perp s from $(-7, -4)$ to the lines $3x^2 - 12xy + 11y^2 = 0$.

38. Show that the product of the \perp s from (c, d) to the lines $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{ac^2 + 2hcd + bd^2}{\sqrt{(a-b)^2 + 4h^2}}$$

39. Find the equation of the st. lines which join the origin to the points of intersection of

$$ax + by = k \tag{1}$$

$$\text{and } x^2 + y^2 + 2gx + 2fy + c = 0. \tag{2}$$

Solution :—

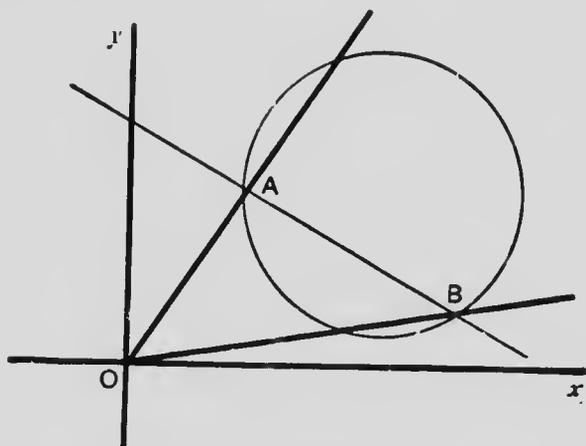


FIG. 42.

Let the line (1) cut the circle (2) at **A, B**.

It is required to find the equation representing **OA** and **OB**.

From (1) $k^2 = k(ax + by) = (ax + by)^2$.

$$\therefore k^2(x^2 + y^2) + 2k(ax + by)(gx + fy) + c(ax + by)^2 = 0. \tag{3}$$

Equation (3) has all its terms of the second degree in x and y , and \therefore , by § 56, it represents two st. lines passing through the origin.

Again, equation (3) is satisfied by the values of x and y which satisfy both (1) and (2);

\therefore the lines represented by (3) pass through **A** and **B**.

\therefore equation (3) represents **OA** and **OB**.

40. Find the equation of the st. lines joining the origin to the points of intersection of $2x - 3y = 1$ and $x^2 + y^2 = 5$.

41. Find the equation of the st. lines joining the origin to the points of intersection of $x + 2y = 2a$ and $5(x^2 + y^2) + 5ax + 10ay = 18a^2$, and show that they are \perp to each other.

42. Find the \angle between the st. lines which join the origin to the points of intersection of $\frac{x}{2} - \frac{y}{3} = 1$ and $x^2 + y^2 - 2x + 6y + 1 = 0$.

43. Find the equations of the st. lines passing through the intersection of $3x + 2y = 7$ and $x + 5y = 11$ and such that the \perp on each of them from $(4, 7)$ is equal to 5.

44. A, B are points on Oa, Oa' respectively and on OA, OB squares OACD, OBEF are described. EF produced cuts AC at G. Prove that OG, BC, ED are concurrent.

45. If $\frac{1}{a} + \frac{1}{b}$ is constant, show that the variable line $\frac{x}{a} + \frac{y}{b} = 1$ passes through a fixed point.

46. A st. line moves so that the sum of the \perp s to it from (a, b) , (c, d) is equal to the \perp to it from (g, h) . Show that the st. line passes through a fixed point and find the coordinates of the point.

47. Prove that the difference of the squares of the tangents from (x_1, y_1) to the circles

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$

$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

is equal to twice the rectangle contained by the distance between the centres of the circles and the length of the \perp from (x_1, y_1) to their radical axis.

48. Three circles touch each other at a common point. Prove that the polars of a fixed point (x_1, y_1) with respect to these circles are concurrent.

49. Find the equations of the st. lines which divide the \angle s between the lines $4x - 3y + 7 = 0$, $5x + 12y - 19 = 0$ into parts whose sines are as 5 to 7.

50. Show that the equation of the st. line joining $\{a \cos(a + \beta), b \sin(a + \beta)\}$ and $\{a \cos(a - \beta), b \sin(a - \beta)\}$ is $\frac{x}{a} \cos a + \frac{y}{b} \sin a = \cos \beta$.

51. Show that the bisectors of the interior \angle s of a \triangle are concurrent.

NOTE.—Take the origin within the \triangle , and let the equations of the sides be

$$\begin{aligned} x \cos a_1 + y \sin a_1 &= p_1 \\ x \cos a_2 + y \sin a_2 &= p_2 \\ x \cos a_3 + y \sin a_3 &= p_3 \end{aligned}$$

52. If the chord of the circle $x^2 + y^2 = a^2$ whose equation is $px + qy = 1$ subtends an \angle of 45° at the origin, then $a^2 (p^2 + q^2) = 4 = 2 \sqrt{2}$.

53. A st. line moves so that the sum, or the difference, of the intercepts cut off from the axes varies as the area of the \triangle contained by the st. line and the axes. Prove that the st. line passes, in either case, through a fixed point.

54. Show that the area of the \triangle contained by the lines $ax^2 + 2hxy + by^2 = 0$ and $Ax + By + C = 0$ is

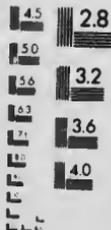
$$\frac{C^2 + h^2 - ab}{A^2b - 2ABh + B^2a}$$

55. OACB is a \square , P is a point in OA, Q is a point in OB; PS drawn \parallel OB meets BC at S; QR drawn \parallel OA meets AC at R. Show that PR, QS, OC are concurrent.



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56. P is a point such that the sum of the \perp s from P on Ox and on $x - by = 0$ is constant. Prove that the locus of P is the base of an isosceles \triangle of which O is the vertex and $y = 0$, $x - by = 0$ are the sides.

57. Given the base of a \triangle in magnitude and position and the magnitude of its vertical \angle ; prove that the locus of its vertex is a circle.

58. Prove that, if (x_1, y_1) , (x_2, y_2) are the extremities of the diameter of a circle, the equation of the circle may be written

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

59. If (h, k) is a point in the first quadrant, show that the equation of the st. line which passes through (h, k) and makes with the axes in that quadrant the \triangle of minimum area is $\frac{x}{h} + \frac{y}{k} = 2$.

60. Show that, if the chord of contact of tangents drawn from the point (h, k) to the circle $x^2 + y^2 = r^2$ subtends a rt. \angle at the centre, then $h^2 + k^2 = 2r^2$.

61. P, Q are two points and O is the centre of a circle. PM is \perp to the polar of Q with respect to the circle, and QN is \perp to the polar of P . Show that $PM : QN = OP : OQ$.

62. Tangents PA, PB are drawn from the point $P(h, k)$ to the circle $x^2 + y^2 = r^2$. Prove that

$$\triangle PAB = \frac{r(h^2 + k^2 - r^2)^{\frac{3}{2}}}{h^2 + k^2}.$$

63. Prove that the polar of (a, b) with respect to $x^2 + y^2 = c^2$ is a tangent to $(x - h)^2 + (y - k)^2 = r^2$, if $(ah + bk - c^2)^2 = (a^2 + b^2)r^2$.

64. ABC is a \triangle in which a variable line DE drawn \parallel to BC cuts AB at D and AC at E . Show that the locus of the intersection of BE , CD is the st. line joining A to the middle point of BC .

65. Show that the equation of the system of circles which pass through (h, k) and touch $Ax + By + C = 0$ may be written

$$(Ah + Bk + C) \{(Ax + By + C)^2 + (Bx - Ay + l)^2\} \\ = (Ax + By + C) \{(Ah + Bk + C)^2 + (Bh - Ak + l)^2\},$$

where l is an arbitrary constant.

66. The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts off from Ox , Oy chords of which the lengths are respectively a and b . Show that $4g^2 - a^2 = 4f^2 - b^2 = 4c$.

67. Find the locus of the middle points of the chords of the circle $x^2 + y^2 = a^2$ which pass through the fixed point (h, k) .

68. O is the centre of a fixed circle, A is a fixed point, Q is any point on the circle. The bisector of $\angle AOQ$ meets AQ at P . Show that the locus of P is a circle having its centre in AO .



ANSWERS

§ 6. (Page 5.)

4. $(0, 0)$, $(2a, 0)$, $(a, a\sqrt{3})$. 5. $(0, 0)$, $(b, 0)$, (b, b) , $(0, b)$.

§ 16. (Page 11.)

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|---|--|
| 4. 13.04. | 12. $\frac{4}{3}\sqrt{65}$. |
| 5. 10, 17, 9. | 13. 3 or -13. |
| 6. (a) $\sqrt{58}$, $\sqrt{82}$, 10; | 14. $4x - 10y + 29 = 0$. |
| (b) $\frac{1}{2}\sqrt{234}$, $3\sqrt{5}$, $\frac{1}{2}\sqrt{306}$. | 15. $(\frac{19}{3}, \frac{2}{3})$; 3.7 nearly. |
| 7. $\sqrt{82}$, $2\sqrt{10}$, $\sqrt{85}$, $\sqrt{29}$; | 16. $(7, 2)$, $(-\frac{2}{3}, \frac{1}{3})$. |
| $7\sqrt{2}$, $\sqrt{137}$. | 20. $(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3})$. |
| 8. $(0, 0)$ - the origin. | 21. $(\frac{3}{5}, \frac{4}{5})$. |
| 9. $(\frac{8}{3}, \frac{7}{3})$ and $(\frac{1}{3}, \frac{5}{3})$. | 22. $(-18, -14)$. |
| 11. $\sqrt{a^2 + b^2}$. | |

§ 19. (Page 16.)

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|----------|--------------|
| 3. 36.5. | 8. 36 miles. |
| 4. 25.5. | 9. 1 : 3. |

§ 22. (Page 21.)

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|----------------------------------|---------------------------------|
| 2. $y = 5x$. | 6. $2x + y + 5 = 0$. |
| 3. (a) The axis of x ; (b) The | 7. $x^2 + y^2 - 8x - 6y = 0$. |
| axis of y . | 8. $x^2 + y^2 - 2x + 4y = 31$. |
| 4. $x = 4$. | 10. $x - 2y + 3 = 0$. |
| 5. $3x - 5y = 17$. | |

§ 24. (Page 23.)

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|--------------------------------------|----------------------------|
| 1. (a) $(2, 7)$; (b) $(3, 2)$; (c) | 3. $(0, 0)$ and $(6, 0)$. |
| $(2, -3)$; (d) $(8, -6)$ and | 4. $2x = a$. |
| $(-8, 6)$; (e) $(5, 12)$ and | 5. $7x + 4y = 20$. |
| $(-\frac{2}{17}, \frac{2}{17})$. | 6. $x^2 + y^2 = 9$. |
| 2. $(0, 9)$ and $(-15, 0)$. | |

§ 27. (Page 27.)

2. (a) $\frac{x}{5} + \frac{y}{2} = 1$; (b) $\frac{x}{4} + \frac{y}{6} = -1$; (c) $\frac{x}{3} - \frac{y}{8} = 1$.
3. (a) $x - 3y = 0$; (b) $9x - 2y + 13 = 0$; (c) $8x + 11y + 34 = 0$.
Intercepts:—(a) 0, 0; (b) $-\frac{13}{9}, \frac{13}{2}$; (c) $-\frac{34}{8}, -\frac{34}{11}$.
4. $(-5, -3\frac{1}{2})$.
5. $(1\frac{7}{8}, 1\frac{1}{2})$.
6. $(1\frac{1}{3}, 4)$.
7. Ratio of equality.
8. Sides, $x + 9y + 14 = 0$; $5 + 3y + 28 = 0$, $2x - 3y - 14 = 0$;
Medians, $x - 5y = 14$, $x + 2y = -7$, $3x - y = 0$;
Centroid $(-1, -3)$.
9. Sides, $2x - 5y = -24$, $3x + 2y = 2$, $x - y = 4$, $3x + 4y = 33$;
Diagonals, $8x - y = 18$, $x + 9y = 34$, $87x + 438y = 5948$;
Line through middle points of diagonals, $2x = 5$.
10. $(-6, -7)$, $(5, -2)$, $(-3, 4)$.

§ 40. (Page 38.)

3. (a) $x = \sqrt{3}y$; (b) $y + x\sqrt{3} = 0$;
(c) $5x - 7y + 35 = 0$; (d) $3y \pm 4x + 9 = 0$; (e) $x - y = 1$.
4. (a) $9, -\frac{1}{2}, -\frac{1}{5}$; (b) $(\frac{5}{3}, 0)$.
5. $(3, 3\frac{1}{2})$.
6. $x + y + 1 = 0$.
7. $\frac{5}{29} \sqrt{58}$; $\tan^{-1} \frac{7}{3}$.
8. (a) $C = 0$; (b) $A = 0$; (c) $B = 0$; (d) $A = B$; (e) $A + B = 0$.
9. $m = 1$.
10. $m = -\frac{1}{9}$, $a = 27$.
11. $a = 8$, $b = -4$.
13. Intercept $= \frac{C' - C}{B}$.

§ 44. (Page 42.)

1. (a) 45° ; (b) 30° ; (c) 90° ;
(d) $\tan^{-1} 5$.
3. $9x + 4y + 47 = 0$.
5. $2x + 3y = 14$.
6. $7x + 5y = 4$.
7. $5x - 3y + 8 = 0$.
8. $x - y + 2 = 0$ and $x + y - 12 = 0$.
9. $x - \sqrt{3}y = 3\sqrt{3} - 5$ and $x + \sqrt{3}y = -3\sqrt{3} - 5$.
10. $(2\frac{1}{7}, -4\frac{1}{7})$.
11. $(b, \frac{ab - b^2}{c})$.
12. $12 : 5$.
14. $(17\sqrt{3} - 16)x + 47y = 344 + 34\sqrt{3}$; and $(17\sqrt{3} + 16)x - 47y = 34\sqrt{3} - 344$.
15. $6x + y + 8 = 0$; and $x - 6y = 11$.
17. $(\frac{a}{2}, \frac{b^2 + c^2 - ab}{2c})$.
18. $3x - Ay + p\sqrt{A^2 + B^2} = 0$.

§ 54. (Page 53.)

1. (a) $\frac{38}{\sqrt{29}}$; (b) $\frac{24}{\sqrt{13}}$; (c) $\frac{12\sqrt{2}}{5}$; 19. $(\frac{341}{233}, \frac{433}{233})$.
 (d) $\frac{5}{\sqrt{41}}$; (e) $\frac{35}{\sqrt{74}}$. 20. $33x + 61y = 216$.
 2. $1\frac{2}{3}$. 21. $11\sqrt{\frac{71}{13}(3+11)}$.
 3. $\frac{c_2 - c_1}{\sqrt{a^2 + b^2}}$. 22. $\frac{6}{5}\sqrt{2}$.
 4. $(-24, 55)$. 24. $\frac{ab(c-d)}{\sqrt{a^2 + b^2}}$.
 5. $8x + 6y = 15$. 25. $15x + 14y = 100$.
 6. $x - y = 0$. 26. $13x - 7y = 3$ and $7x + 13y = 119$.
 7. $2\frac{7}{13}$. 27. $(24 + 13\sqrt{3})x + 23y + 52\sqrt{3} + 257 = 0$ and $(24 - 13\sqrt{3})x + 23y - 52\sqrt{3} + 257 = 0$.
 8. $(m_1 - m)(ax - by) + b(m_1c - mc_1) + a(c_1 - c) = 0$. 28. $5x - 12y + 56 = 0$ and $5x - 12y - 74 = 0$.
 10. $abc - af^2 - bg^2 - ch^2 + 2fgh = 0$. 29. $6x + y = 31$, $x - y + 3 = 0$;
 11. $(b - m_1a - c_1)(y - m_1x - c) = (b - ma - c)(y - m_1x - c_1)$.
 12. $(\mathbf{AC}_1 - \mathbf{A}_1\mathbf{C})x + (\mathbf{BC}_1 - \mathbf{B}_1\mathbf{C})y = 0$.
 13. $\sqrt{2}$. 30. $(5\frac{1}{3}, 3\frac{2}{3})$.
 14. $\frac{\pm 2\sqrt{130 - 11}}{7}$. 32. $12x - 5y = 26$, $y = 2$.
 15. $x + 21y = 6$ and $189x - 9y = 692$. 33. $\mathbf{Ax} + \mathbf{By} + p\sqrt{\mathbf{A}^2 + \mathbf{B}^2} = 0$.
 17. $3x + y = 2$ and $x - 3y = 24$. 34. $\mathbf{Bx} - \mathbf{Ay} + \mathbf{Ak} - \mathbf{Bh} = 0$.
 35. (a) $(\frac{306}{121}, -\frac{87}{121})$; (b) $(-\frac{175}{121}, -\frac{61}{121})$.

§ 60 (Page 60.)

1. (a) $x = a$, $x = b$; 2. $\frac{2}{3}$ and $\frac{1}{2}$.
 (b) $x - y = 0$, $x + y = 0$; 3. The axes of coordinates.
 (c) $x = 0$, $x = 3y$; 4. (d) $\tan^{-1} \frac{2}{11}$; (e) 90° ; (f) $\tan^{-1} \frac{4}{3}$.
 (d) $2x - y = 0$, $4x - 3y = 0$;
 (e) $x = a$, $y = -b$; 5. $bc = ad$.
 (f) $3x - y = -4$, $x - 3y = 5$. 6. 8.
 7. $3x^2 - y^2 - 30x + 6y + 66 = 0$.

§ 65. (Page 64.)

1. $2x^2 - 11xy + 12y^2 = 0$.
2. $x^2 + xy = 0$.
3. $(-\frac{5}{2}, \frac{3}{2})$; $2(x^2 + y^2) = 19$.
5. $(A\sqrt{3} + B)x + (B\sqrt{3} - A)y + 2C = 0$.
6. $2xy + a^2 = 0$.
8. $9x^2 - 25y^2 = 0$.
9. $\tan^{-1} \frac{4}{3}$; $3x^2 + 2y^2 = 10$.

§ 66. (Page 66.)

1. (a) $\sqrt{145}$;
(b) $\sqrt{5a^2 + 16ab + 13b^2}$;
(c) $a + b$.
2. $P(\frac{2}{3}, 3\frac{2}{3})$; $Q(17\frac{1}{2}, 16\frac{1}{2})$.
3. $\frac{3}{2}ab$.
4. $41\frac{1}{2}$.
5. $10\frac{2}{3}$.
6. $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$.
7. $8x + 7y = 5$.
9. $2ax + 2by = c + d$.
10. $x = y \tan a + b$.
13. $x \cos a + y \sin a = a$.
14. $\tan^{-1} \frac{12^{\frac{3}{4}}}{41^{\frac{3}{4}}}$.
15. $m = \frac{3}{4}$, $a = -4\frac{1}{4}$.
16. $x + y\sqrt{3} + 2(\sqrt{3} - 1) = 0$.
17. $\frac{mh - k + a}{\sin a - m \cos a}$.
18. $\frac{x}{a} + \frac{y}{b} = \frac{h}{a} + \frac{k}{b}$.
20. $bx + ay = ad$.
21. $(-2, -8)$ and $(-6, -14)$.
23. $a = 6$ or -4 ; $(\frac{1}{2}, \frac{1}{3})$, $(\frac{1}{2}, -\frac{1}{2})$.
25. 45° .
26. $65x - 65y + 14a + 3b = 0$.
27. $b(am + c)x + (bd - ab + adm + ac)y - bd(am + c) = 0$.
29. $\frac{ab}{\sqrt{a^2 + b^2}}$.
30. $x + 4y + 1 = 0$.
31. $x + 7y + 6 = 0$.
32. $(6\frac{7}{18}, 51\frac{2}{18})$; $(-3\frac{9}{77}, \frac{6}{77})$;
 $(26\frac{2}{115}, 4\frac{67}{115})$; $(0, 57)$.
33. $(4\frac{5}{10}, -\frac{1}{10})$.
34. $2y - 7x \pm x\sqrt{55} = 0$.
35. $-\frac{19}{\sqrt{153}}$.
37. $\frac{\sqrt{21}}{2}$.
38. $x + 6y = 39$ and $6x - y = 12$.
41. $2x + 3y + 11 = 0$.
42. $165x + 605y + 3114 = 0$; $332x - 747y + 3466 = 0$; $497x - 142y - 2178 = 0$.
43. $(\frac{2}{3}, \frac{2}{3})$.
44. (a) $(ah - cf)x + (bh - cg)y = 0$;
(b) $(ay - bf)(x - y) + c(f + y) - h(a + b) = 0$.
45. $5x - 14y = 6$ and $154x + 55y = 229$.
46. $49x - 245y = 242$.
48. $x + y = 10$.
49. $x - y = 10$, or $y - x = 10$.
50. $7x + 5y + 50 = 0$, and $9x + 35y + 150 = 0$.
51. $x - 4y = 11$.
52. $2(a - h)x + 2(b - k)y = a^2 + b^2 - h^2 - k^2$.
53. $2x + y + 15 = 0$.
54. 45° .
55. $10x + 4y + 11 = 0$, and $4x - 10y - 33 = 0$.
56. $x^2 - y^2 = 0$.
57. -13 or $10\frac{1}{15}$.
59. $(11, 1)$, $(-1, -5)$, or $(-5, 7)$.
61. $-\frac{6}{31}$.
62. 24.

§ 72. (Page 75.)

1. $x^2 + y^2 = 5$.
2. $(x-6)^2 + (y-2)^2 = 9$.
3. $(x+5)^2 + (y+1)^2 = 26$.
4. $a^2 + b^2 = c^2$.
5. (a) (3, 1), 5; (b) $(-\frac{7}{2}, -\frac{5}{2})$, $\sqrt{\frac{330}{8}}$; (c) (7, 0), 7; (d) (0, -b), $\sqrt{b^2 + c^2}$.
6. $x^2 + y^2 - x - 7y = 0$.
7. $x^2 + y^2 - 3x = 19$.
8. (a) $f = 0$; (b) $g = 0$.
9. $x^2 + y^2 - 4x + 4y = 2$.
10. $x^2 + y^2 - 4x + 6y = 16$.
11. $47(x^2 + y^2) - 181x + 341y - 1996 = 0$.
12. $\sqrt{10}$.
13. 14, 12.
14. $2x - 5y = 15$.
15. 123.
16. $14(x+y) - 17$.
17. $c = c'$.
18. $(h+k)(x^2 + y^2) - (h^2 + k^2)(x+y) = 0$.

§ 81. (Page 85.)

1. (a) $3x + 5y = 34$; (b) $3x - 4y + 11 = 0$; (c) $12x + 5y = 96$; (d) $gx + fy = 0$.
2. $y = x \pm \sqrt{70}$.
3. (a) $Ax + By = \pm r \sqrt{A^2 + B^2}$; (b) $Bx - Ay = \pm r \sqrt{A^2 + B^2}$.
4. (2, 4).
5. (6, 1).
6. (a) $C^2 = r^2(A^2 + B^2) + B^2f^2 - C^2 = (y^2 + f^2 - c)$.
7. $a^2b^2 = r^2(a^2 + b^2)$.
8. $c = g^2$.
9. $x^2 + y^2 - 2(7 \pm 2\sqrt{5})(x+y) + 69 \pm 28\sqrt{5} = 0$.
10. $x - \sqrt{3}y + 3\sqrt{3} - 1 = 0$, and $x - \sqrt{3}y - 5\sqrt{3} - 1 = 0$.
11. $4\sqrt{\frac{313}{29}}$.
12. $(x-y) \cos a + (y-f) \sin a = r$.
13. $(\frac{1}{3}, -\frac{1}{3})$; $4x + 3y + 1 = 0$.
14. $x - 3y = 3$; $(\frac{21 \pm 3\sqrt{119}}{10}, \frac{-3 \pm \sqrt{119}}{10})$.
15. $(A^2 + B^2)(x^2 + y^2) = C^2$.
16. $34(x^2 + y^2) - 476x - 136y + 1753 = 0$.
17. $(\frac{3}{5}, \frac{7}{5})$; (17, -13).
transverse, $12(5y-7) = (-21 \pm 5\sqrt{15})(5x-1)$;
direct, $24(y+13) = (-21 \pm \sqrt{21})(x-17)$.
18. direct, $y = 9$ and $3x + 4y + 3 = 0$;
transverse are imaginary.
(2, 9) and (-1, 0) on $x^2 + y^2 - 4x - 8y - 5 = 0$;
(5, 9) and $(\frac{7}{5}, -\frac{9}{5})$ on $x^2 + y^2 - 10x - 6y - 2 = 0$.
19. $x + 3y + g + 3f \pm \sqrt{10(y^2 + f^2 - c)} = 0$.
20. $x - \sqrt{3}y \pm 10 = 0$.

§ 92. (Page 94.)

1. (a) $3x + 5y = 30$; (b) $ax = r^2$;
(c) $4x + 17 = 0$; (d) $10x - 2y = 7$; (e) $hx + ky = h^2 + k^2 - r^2$.
2. (a) $(2, -7)$; (b) $(-\frac{7}{12}, \frac{2}{3})$;
(c) $(-35, 11)$; (d) $(0, 0)$.
3. (c) $3x - 4y + 25 = 0$; (e) $24x - 7y = 125$; (f) $9x + 13y = 25$; (g) $13x - 9y = 0$.
4. $(\frac{c^2}{a}, \frac{c^2}{b})$.
5. $(\frac{f^2l - g - lc - fgm}{1 + fm + gl}, \frac{g^2m - f - mc - fgl}{1 + fm + gl})$.
6. $(1, 4)$.

§ 95. (Page 97.)

1. (a) 6; (b) 5; (c) 8; (d) \sqrt{c} .
2. $x^2 + y^2 - 10x - 4y = 7$.
3. $x^2 + y^2 - 16x + 51 = 0$.
4. $4x + 3y = 25$ and $3x - 4y = 25$.

§ 98. (Page 98.)

1. $12x + 8y = 95$.
2. $9x - 8y + 15 = 0$.
4. $(-13, -7)$.

MISCELLANEOUS EXERCISES. (Page 99.)

1. $3x + 4y = 25$.
2. $2x - 11y + 329 = 0$.
3. 113.
4. 29.
5. $\frac{x}{h} + \frac{y}{k} = 2$.
6. (a) $\frac{x-h}{x_1-x_2} = \frac{y-k}{y_1-y_2}$; (b) $(x_1 - x_2)(x-h) + (y_1 - y_2)(y-k) = 0$.
8. 7 : 4.
9. $(\frac{288}{113}, -\frac{111}{113})$.
10. 77.
12. $x^2 + y^2 = 12$.
13. $(a+c, b+d)$.
15. The point $(-a \cos a, -a \sin a)$.
16. $x^2 + y^2 - 6x - 3y = 0$.
17. $3x - y = 3k \pm k\sqrt{10}$.
18. $(\frac{382}{147}, \frac{302}{147})$.
20. $x^2 + y^2 = hx$.
21. $x^2 + y^2 - ax - by = 0$.
22. $x^2 + y^2 - gx - fy = 0$.
23. $(\frac{1}{3}, \frac{2}{3})$.
24. $8(x^2 + y^2) - 25x + 75y + 71 = 0$.
25. 10 or $\frac{6}{7}$.
26. Centro divides **AB** externally in ratio $k^2 : 1$.
27. (a) $(7, 1)$; (b) **AD** = **DB**.
30. $(2, 1)$.
31. $\frac{238}{143}$.
32. $x^2 + y^2 - 2h(x-a) - 2k(y-b) = a^2 + b^2$.

MISCELLANEOUS EXERCISES—Continued.

33. $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0.$
35. $2x - (b+c)y + 2abc = 0.$
37. $-\frac{\sqrt{13}}{4}.$
40. $19x^2 - 60xy + 44y^2 = 0.$
41. $3x^2 - 8xy - 3y^2 = 0.$
42. $\tan^{-1} \frac{8\sqrt{243}}{25}.$
43. $y = 2$ and $10x + 8y = 31.$
46. $(a+c-g, b+d-h).$
49. $163x + 9y + 54 = 0$ and $239x - 573y + 1112 = 0.$
67. $x^2 + y^2 - (hx + ky) = 0.$
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