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BY THE SAME AUTHOR.

ALGEBRA FOR COLLEGES.

KEY TO ABOVE.

ELEMENTS OF GEOMETRY.

ELEMENTS OF PLANE AND SPHERICAL TRIGONOMETRY.

ASTRONOMY. For Students and General Readers. By Simon Newcomb and Edward S. Holden.

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NEWCOMB'S MATHEMATICAL SERIES.

ALGEBRA

FOR

SCHOOLS AND COLLEGES

BY

SIMON NEWCOMB
Professor of Mathematics, United States Navy

THIRD EDITION, REVISED.

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PREFACE.

THE course of algebra embodied in the present work is substantially that pursued by students in our best preparatory and scientific schools and colleges, with such extensions as seemed necessary to afford an improved basis for more advanced studies. For the convenience of teachers the work is divided into two parts, the first adapted to well-prepared beginners and comprising about what is commonly required for admission to college; and the second designed for the more advanced general student. As the work deviates in several points from the models most familiar to our teachers, a statement of the principles on which it is constructed may be deemed appropriate.

One well-known principle underlying the acquisition of knowledge is that an idea cannot be fully grasped by the youthful mind unless it is presented under a concrete form. Whenever possible an abstract idea must be embodied in some visible representation, and all general theorems must be presented in a variety of special forms in which they may be seen inductively. In accordance with this principle, numerical examples of nearly all algebraic operations and theorems have been presented. For the purpose of illustration, numbers have been preferred to literal symbols when they would serve the purpose equally well. The relations of positive and negative algebraic quantities have been represented by lines and directions from the beginning in order that the pupil might be able to give, not only a numerical, but a visible, meaning to all algebraic quantities. Should it appear to any one that we thus detract from the generality of algebraic quantities, it is sufficient to reply that the system is the same which mathematicians use to assist their conceptions of advanced algebra, and without which they would never have been able to grasp the complicated relations of imaginary quantities. Algebraic operations with pure numbers are made to precede the use of symbols, and the latter are introduced only after the pupil has had a certain amount of familiarity with the distinction between algebraic and numerical operations.

Another, but, unfortunately, a less familiar fact is, that all mathematical conceptions require time to become engrafted upon the mind, and the more time the greater their abstruseness. It is, the author conceives, from a failure to take account of this fact, rather than from any inherent defect in the minds of our youth, that we are to attribute the backward state of mathematical instruction in this country, as compared with the continent of Europe. Let us take for instance the case of the student commencing the calculus. On the system which was almost universal among us a few years ago, and which is still widely prevalent, he is confronted at the outset with a number of entirely new conceptions, such as those of variables, functions, increments, infinitesimals and limits. In his first lesson he finds these all combined with a notation so entirely different from that to which he has been accustomed, that before the new ideas and forms of thought can take permanent root in his mind, he is through with the subject, and all that he has learned is apt to vanish from his memory in a few months.

The author conceives that the true method of meeting this difficulty is to adopt the French and German plan of teaching algebra in a broader way, and of introducing the more advanced conceptions at the earliest practicable period in the course. Accordingly, the attempt is made in the present work to introduce each advanced conception, disguised perhaps under some simple form, in advance of its general enunciation and at as early a period as the student can be expected to understand it. In doing this, logical order is frequently sacrificed to the exigencies of the case, because there are several subjects with which a certain amount of familiarity must be acquired before the pupil can even clearly comprehend general statements respecting them.

A third feature of the work is that of subdividing each subject as minutely as possible, and exercising the pupil on the details preparatory to combining them into a whole. To cite one or two instances: a difficulty which not only the beginner but the expert mathematician frequently meets is that of stating his conceptions in algebraic language. Exercises in such statements have therefore been made to precede any solution of problems. In general each principle which is to be presented or used is stated singly, and the pupil is practiced upon it before proceeding to another.

Subjects have for the most part been omitted which do not find application either in the work itself or in subsequent parts of the usual course of mathematics, or which do not conduce to a mathematical training. Sturm's Theorem has been entirely omitted, and a more simple process substituted. The subject of the greatest common divisor of two polynomials has been postponed to what the author considers its proper place, in the general theory of equations. It has, however, been presented in such a form that it can be taught to pupils preparing for colleges where it is still required for admission.

Thoroughness at each step has been aimed at rather than multiplicity of subjects. It is, the author conceives, a great and too common mistake to present a mathematical subject to the mind of the student without sufficient fulness of explanation and variety of illustration to enable him to comprehend and apply it. If he has not time to master a complete course, it is better to omit entirely what is least necessary than to gain time by going rapidly over a great number of things. Some hints to those who may not have time to master the whole work may therefore be acceptable.

Part I is essential to every one desiring to make use of algebra. Book VIII, especially the concluding sections on notation, is to be thoroughly mastered, before going farther, as forming the foundation of advanced algebra; and affording a very easy and valuable discipline in the language of mathematics. Afterward, a selection may be made according to circumstances. The student who is pursuing the subject for the sole purpose of liberal training, and without intending to advance beyond it, will find the theories of numbers and the combinatory analysis most worthy of study. The theory of probabilities and the method in which it is applied to such practical questions as those connected with insurance will be of especial value in training his judgment to the affairs of life.

The student who intends to take a full course of mathematics with a view of its application to physics, engineering, or other subjects, may, if necessary, omit the book on the theory of numbers, and portions of the chapter on the summation of series. Functions and the functional notation, the doctrine of limits, and the general theory of equations will claim his especial attention, while the theory of imaginary quantities will be studied mainly to secure thoroughness in subsequent parts of his course.

As it has frequently been a part of the author's duty to ascertain what is really left of a course of mathematical study in the minds of those who have been through college, some hints on the best methods of study in connection with the present work may be excused. If asked to point out the greatest error in our usual system of mathematical instruction from the common school upward, he would reply that it consisted in expending too much of the mental power of the student upon problems and exercises above his capacity. With the exception of the fundamental routine-operations, problems and exercises should be confined to insuring a proper understanding of the principles involved: this once ascertained, it is better that the student should go on rather than expend time in doing what it is certain he can do. Problems of some difficulty are found among the exercises of the present work; they are inserted rather to give the teacher a good choice from which to select than to require that any student should do them all.

It would, the author conceives, be found an improvement on our usual system of teaching algebra and geometry successively if the analytic and the geometric courses of mathematics were pursued simultaneously. The former would include algebra and the calculus, the latter elementary geometry, trigonometry, and analytic geometry. The analytic course would then furnish methods for the geometric one, and the latter would furnish applications and illustrations for the analytic one.

The Key to the work, which will be issued as soon as practicable, will contain not only the usual solutions, but the explanations and demonstrations of the less familiar theorems, and a number of additional problems.

The author desires, in conclusion, to express his obligation to the many friends who have given him suggestions respecting the work, and especially to Professor J. Howard Gore of the Columbian University who has furnished solutions to most of the problems, and given the benefit of his experience on many points of detail.

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FIRST PART.

ELEMENTARY COURSE.

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BOOK I.

THE ALGEBRAIC LANGUAGE.

CHAPTER I.

OF ALGEBRAIC NUMBERS AND OPERATIONS.

General Definitions.

1. *Definition.* **Mathematics** is the science which treats of the relations of magnitudes.

The magnitudes of mathematics are time, space, force, value, or other things which can be thought of as entirely made up of parts.

2. *Def.* A **Quantity** is a definite portion of any magnitude.

EXAMPLE. Any definite number of feet, miles, acres, bushels, years, pounds, or dollars, is a quantity.

3. *Def.* **Algebra** treats of those relations which are true of quantities of every kind of magnitude.

4. The relations treated of in Algebra are discovered by means of numbers.

To measure a quantity by number, we take a certain portion of the magnitude to be measured as a unit, and express how many of the units the quantity contains.

REMARK. It is obviously essential that the quantity and its unit shall be the same kind of magnitude.

5. *Def.* A **Concrete Number** is one in which the kind of quantity which it measures is expressed or understood; as 7 *miles*, 3 *days*, or 10 *pounds*.

6. *Def.* An **Abstract Number** is one in which no particular kind of unit is expressed; as 7, 3, or 10.

REMARK. An abstract number may be considered as a concrete one expressing a certain number of units, without respect to the kind of units. Thus, 7 means 7 *units.*

Algebraic Numbers.

7. In Arithmetic, the numbers begin at 0, and increase without limit, as 0, 1, 2, 3, 4, etc. But the quantities we usually measure by numbers, as time and space, do not really begin at any point, but extend without end in opposite directions.

For example, time has no beginning and no end. An epoch of time 1000 years from Christ may be either 1000 years after Christ, or 1000 years before Christ.

A heavy body tends to fall to the ground. A body which did not tend to move at all when unsupported would have no weight, or its weight would be 0. If it tended to rise upward, like a balloon, it would have the opposite of weight.

If we have to measure a distance from any point on a straight line, we may measure out in either direction on the line. If the one direction is east, the other will be west.

One who measures his wealth is poorer by all that he owes. If he owes more than he possesses, he is worth less than nothing, and there is no limit to the amount he may owe.

8. In order to measure such quantities on a uniform system, the numbers of Algebra are considered as increasing from 0 in two opposite directions. Those in one direction are called **Positive**; those in the other direction **Negative**.

9. Positive numbers are distinguished by the sign $+$, *plus*; negative ones by the sign $-$, *minus*.

If a positive number measures years after Christ, a negative one will mean years before Christ.

If a positive number is used to measure toward the right, a negative one will measure toward the left.

If a positive number measures weight, the negative one will imply levity, or tendency to rise from the earth.

If a positive number measures property, or credit, the negative one will imply debt.

10. The series of algebraic numbers will therefore be considered as arranged in the following way, the series going out to infinity in both directions.

NEGATIVE DIRECTION.	POSITIVE DIRECTION.
Before.	After.
Downward,	Upward.
Debt.	Credit.
etc.	etc.

etc. $-5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5,$ etc.

REM. It matters not which direction we take as the positive one, so long as we take the opposite one as negative.

If we take time before as positive, time after will be negative; if we take west as the positive direction, east will be negative; if we take debt as positive, credit will be negative.

11. Positive and negative numbers may be conceived as measuring distances from a fixed point on a straight line, extending indefinitely in both directions, the distances one way being positive, and the other way negative, as in the following scheme: *

etc. $-7, -6, -5, -4, -3, -2, -1, \quad 0, \quad +1, +2, +3, +4, +5, +6, +7,$ etc.

In this scale, the distance between any two consecutive numbers is considered a unit or unit step.

12. *Def.* The signs + and − are called the **Algebraic Signs**, because they mark the direction in which the numbers following them are to be taken.

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* The student should copy this scale of numbers, and have it before him in studying the present chapter.

The sign $+$ may be omitted before positive numbers, when no ambiguity is thus produced. The numbers 2, 5, 12, taken alone, signify $+2$, $+5$, $+12$. But the negative sign must always be written when a negative number is intended.

13. *Def.* One number is said to be **Algebraically Greater** than another when on the preceding scale it lies to the positive (right hand) side. Thus,

-2	is	algebraically	greater	than	-7;
0	"	"	"	"	-2;
5	"	"	"	"	-5.

Algebraic Addition.

14. *Def.* In Algebra, **Addition** means the combination of quantities according to their algebraic signs, the positive quantities being counted one way or added, and negative ones the opposite way or subtracted.

15. *Def.* The **Algebraic Sum** of several quantities is the surplus of the positive quantities over the negative ones, or of the negative quantities over the positive ones, according as the one or the other is the greater.

The sum has the same algebraic sign as the preponderating quantity.

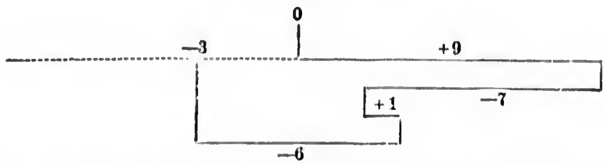
EXAMPLE. The sum of

$+7$	and	-7	is	0;
$+9$	"	-7	"	$+2$;
$+5$	"	-7	"	-2.

The sum of several positive numbers may be represented on the line of numbers, § 11, by the length of the line formed by placing the lengths represented by the several numbers end to end. The total length will be the sum of the partial lengths.

If any of the numbers are negative, the algebraic sum is represented by laying their lengths off in the opposite direction.

EXAMPLE 1. The algebraic sum of the four numbers 9, -7, 1, -6, would be represented thus:



Here, starting from 0, we measure 9 to the right, then 7 to the left, then 1 to the right, then 6 to the left. The result would be 3 steps to the left from 0, that is, -3. Thus, -3 is the algebraic sum of $+9$, -7, $+1$, and -6.

Ex. 2. If we imagine a person to walk back and forth along the line of numbers, his distance from the starting-point will always be the algebraic sum of the separate distances he has walked.

Ex. 3. A man's wealth is the algebraic sum of his possessions and credits, the debts which he owes being negative credits. If he has in money \$1000, due from A \$1200, due to X \$500, due to Y \$350, his possessions would, in the language of algebra, be summed up as follows:

Cash, ...	+ \$1000
Due *from* A, ...	+ 1200
Due *from* X, ...	− 500
Due *from* Y, ...	− 350
Sum total, ...	+ \$1350

[In the language of Algebra, the fact that he owes X \$500 may be expressed by saying that X owes him — \$500.]

16. *Def.* To distinguish between ordinary and algebraic addition, the former is called **Numerical** or **Arithmetical** addition.

Hence, the numerical sum of several numbers means their sum as in arithmetic, without regard to their signs.

17. Rem. In Algebra, whenever the word *sum* is used without an adjective, the *algebraic sum* is understood.

Algebraic Subtraction.

18. *Memorandum of arithmetical definitions and operations.*

The **Subtrahend** is the quantity to be subtracted.

The **Minuend** is the quantity from which the subtrahend is taken.

The **Remainder** or **Difference** is what is left.

If we subtract 4 from 7, the remainder 3 is the number of unit steps on the scale of numbers (§ 11) from $+4$ to $+7$. This is true of any arithmetical difference of numbers. In Algebra, the operation is generalized as follows:

19. *Def.* The **Algebraic Difference** of two numbers is represented by the distance from one to the other on the scale of numbers.

The number *from* which we measure is the **Subtrahend.**

That *to* which we measure is the **Minuend.**

If the minuend is algebraically the greater (§ 13), the difference is positive.

If the minuend is less than the subtrahend, the difference is negative.

In Arithmetic we cannot subtract a greater number from a less one. But there is no such restriction in Algebra, because algebraic subtraction does not mean taking away, but finding a difference. However the minuend and subtrahend may be situated on the scale, a certain number of spaces toward the right or toward the left will always carry us from the subtrahend to the minuend, and these spaces make up the difference of the two numbers.

20. The general rule for algebraic subtraction may be deduced as follows: It is evident that if we pass from the subtrahend to 0 on the scale, and then from 0 to the minuend, the algebraic sum of these two motions will be the entire space between the subtrahend and minuend, and will therefore be the remainder required. But the first motion will be equal to the subtrahend, but positive if that quantity is negative, and *vice versa,* and the second motion will be equal to the minuend.

Hence the remainder will be found by changing the algebraic sign of the subtrahend, and then adding it algebraically to the minuend.

EXAMPLES.

Subtracting	$+5$	from	$+8$,	the difference is	$8-5=3$.
"	$+8$	"	$+5$,	" " "	$5-8=-3$.
"	$+8$	"	-5,	" " "	$-5-8=-13$.
"	-8	"	5,	" " "	$5+8=+13$.
"	$+13$	"	0,	" " "	-13.
"	-13	"	0,	" " "	$+13$.

21. By comparing algebraic addition and subtraction, it will be seen that to subtract a positive number is the same thing as to add its negative, and *vice versa*. Thus,

To subtract 5 from 8 gives the same result as to add -5 to 8, namely 3.

To subtract -5 from 8 gives $8+5$, namely 13.

Hence, algebraic subtraction is equivalent to the algebraic addition of a number with the opposite algebraic sign. Algebraists, therefore, do not consider subtraction as an operation distinct from addition.

Algebraic Multiplication.

22. *Memorandum of arithmetical definitions.*

The **Multiplicand** is the quantity to be multiplied.

The **Multiplier** is the number by which it is multiplied.

The result is called the **Product**.

Factors of a number are the multiplicand and multiplier which produce it.

23. To multiply any algebraic quantity by a positive whole number means, as in Arithmetic, to take it a number of times equal to the multiplier.

Thus, $4\times 3 = 4+4+4=+12$;

$-4\times 3=-4-4-4=-12$.

The product of a negative multiplicand by a positive multiplier will therefore be negative.

24. If the multiplier is negative, the sign of the product will be the opposite of what it would be if the multiplier were positive.

Thus,
$$+4 \times -3 = -12;$$
$$-4 \times -3 = +12.$$

The product of two negative factors is therefore positive.

25. The most simple way of mastering the use of algebraic signs in multiplication is to think of the sign — as meaning **opposite** in direction. Thus, in § 11, -4 is opposite in direction to $+4$, the direction being that from 0. If we multiply this negative factor by a negative multiplier, the direction will be the *opposite* of negative, that is, it will be *positive*. A third negative factor will make the product negative again, a fourth one positive, and so on. For example,

$$-3 \times -4 = +12;$$
$$-2 \times -3 \times -4 = -2 \times +12 = -24;$$
$$-3 \times -2 \times -3 \times -4 = -3 \times -24 = +72;$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

Hence,

26. *Theorem.* The continued product of an even number of negative factors is positive; of an odd number, negative.

REM. Multiplying a number by -1 simply changes its sign.

Thus,
$$+4 \times -1 = -4;$$
$$-4 \times -1 = +4.$$

EXERCISES.

Find the algebraic sums of the following quantities:

1. $4 - 6 + 12 - 1 - 18$.
2. $-6 - 3 - 8$.
3. $-6 - 10 - 9 + 34$.
4. Subtract the sum in Ex. 3 from the sum in Ex. 2.
5. Subtract the sum $5 - 6 + 3 - 1 - 16$, from the sum $-2 - 7 - 4 + 8$.

6. Subtract the sum $5 - 6 + 3 - 1 - 16$, from the sum $7 - 3 - 8 + 4$.
7. Form the product -7×8.
8. Form the product -8×7.
9. Form the product $6 \times -5 \times 7 \times -4$.
10. Form the product $-6 \times -11 \times 8 \times -2$.
11. Form the product $-1 \times -1 \times -1 \times -1$.
12. Subtract the sum in Ex. 1 from the sum in Ex. 3, and multiply the remainder by the sum in Ex. 2.
13. Subtract 8 from -3, -3 from -1, -1 from 8, and find the sum of the three remainders.
14. Subtract 7 from -9 and the remainder from 2, and multiply the result by the product in Ex. 7.

Algebraic Division.

27. *Memorandum of arithmetical definitions.*

The **Dividend** is the quantity to be divided.
The **Divisor** is the number by which it is divided.
The **Quotient** is the result.

28. *Rule of Signs in Division.* The requirement of division in Algebra is the same as in Arithmetic; namely,

The product of the quotient by the divisor must be equal to the dividend.

In Algebra, two quantities are not equal unless they have the same algebraic sign. Therefore the product,

$$\textit{quotient} \times \textit{divisor}$$

must have the same algebraic sign as the dividend. From this we can deduce the rule of signs in division.

Let us divide 6 by 2, giving 6 and 2 both algebraic signs, and find the signs of the quotient 3:

$+3 \times +2 = +6$;	therefore,	$+6$	divided	by $+2$	gives	$+3$.
$+3 \times -2 = -6$;	"	-6	"	" -2	"	$+3$.
$-3 \times +2 = -6$;	"	-6	"	" $+2$	"	-3.
$-3 \times -2 = +6$;	"	$+6$	"	" -2	"	-3.

Hence, the rule of signs is the same in division as in multiplication, namely:

Like signs in dividend and divisor give $+$. *Unlike signs give* $-$.

EXERCISES.

Execute the following algebraic divisions, expressing each result as a whole number or vulgar fraction:

1. Dividend, $-7+10-11+25$; divisor, $20-3$.
2. Dividend, $12-3+15-10$; divisor, $3-10$.
3. Dividend, $25-36+6-20$; divisor, $-3+8$.
4. Dividend, -7×-8; divisor, $-8+4$.
5. Dividend, $56+8\times-3$; divisor, $-4-4$.
6. Dividend, -24×-1; divisor, -3×-3.
7. Dividend, $-13\times-10\times-8$; divisor, $-4\times5\times-6$.
8. Dividend, -1×-1; divisor, -3×-3.

CHAPTER II.

ALGEBRAIC SYMBOLS.

Symbols of Quantity.

29. Algebraic quantities may be represented by letters of the alphabet, or other characters.

The characters of Algebra are called **Symbols**.

30. *Def.* The **Value** of an algebraic symbol is the quantity which it represents or to which it is equal.

The value of a symbol may be any algebraic quantity whatever, positive or negative, which we choose to assign to the symbol.

31. The language of Algebra differs in one respect from ordinary language. In the latter, each special word or sign has a definite and invariable meaning, which every one who uses the language must learn once for all. But in Algebra a symbol may stand for any quantity which the writer or speaker chooses, and his results must be interpreted according to this meaning.

32. The same character may be used to represent several quantities by applying accents or attaching numbers to it to distinguish the different quantities. Thus, the four symbols, a, a', a'', a''', may represent four different quantities. The symbols a_1, a_2, a_3, a_4, a_5, etc., may be used to designate any number of quantities which are distinguished by the small number written after the letter a.

Signs of Operation.

33. In Algebra, the signs $+$, $-$, and $\times$ are used, as in Arithmetic, to represent addition, subtraction, and multiplication, these operations being algebraic, not numerical.

34. *Signs of Addition and Subtraction.* The combination $a+b$ means the algebraic sum of the quantities a and b, and $a-b$ means their algebraic difference.

EXAMPLES.

If $a = +4$ and $b = +3$, then $a+b = +7$, $a-b = +1$.
If $a = +5$ and $b = -7$, then $a+b = -2$, $a-b = +12$.
If $a = -6$ and $b = +3$, then $a+b = -3$, $a-b = -9$.
If $a = -6$ and $b = -3$, then $a+b = -9$, $a-b = -3$.

The signs of addition and subtraction are the same as those used to indicate positive and negative quantities, but the two applications may be made without confusion, because the opposite positive and negative directions correspond to the opposite operations of adding and subtracting.

35. *Sign of Multiplication.* The sign of multiplication, $\times$, is generally omitted in Algebra, and when different symbols are to be multiplied, the multiplier is written before the multiplicand without any sign between them.

Thus, $4a$ means $a \times 4$.
ax " $x \times a$.
$3abmy$ " $y \times m \times b \times a \times 3$.

If numbers are used instead of symbols, some sign of multiplication must be inserted between them to avoid confusion. Thus, 34 would be confounded with the number *thirty-four*. A simple dot is therefore inserted instead of the sign $\times$.

Thus,
$$3 \cdot 4 = 4 \times 3 = 12.$$
$$3 \cdot 12 \cdot 2 = 72.$$
$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120.$$
$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720.$$

The only reason why the point is used instead of $\times$, is that it is more easily written and takes up less space.

36. *Division* in Algebra is sometimes represented by the symbol $\div$, the dividend being placed to the left and the divisor to the right of this symbol.

Ex. $a \div b$ means the quotient of a divided by b.

But division is more generally represented by writing the dividend as the numerator and the divisor as the denominator of a fraction.

Ex. The quotient of a divided by b is written $\frac{a}{b}$.

It is shown in Arithmetic that a fraction is equal to the quotient of its numerator divided by its denominator; hence this expression for a quotient is a vulgar fraction.

37. *Powers and Exponents.* A **Power** of a quantity is the product obtained by taking that quantity a certain number of times as a factor.

Def. The **Degree** of the power means the number of times the quantity is taken as a factor.

If a quantity is to be raised to a power, the result may, in accordance with the rule for multiplication, be expressed by writing the quantity the required number of times.

EXAMPLES. The fifth power of a may be written

$$a \times a \times a \times a \times a \quad \text{or} \quad aaaaa;$$

and the fourth power of 7, $\quad 7 \cdot 7 \cdot 7 \cdot 7 = 2401.$

To save repetition, the symbol of which the power is to be expressed is written but once, and the number of times it is taken as a factor is written in small figures after and above it.

Thus,	$aaaaa$	is	written	a^5;
	$7 \cdot 7 \cdot 7 \cdot 7$	"	"	7^4;
	xxx	"	"	x^3.

Def. A figure written to indicate a power is called an **Exponent.**

Def. The operation of forming a power is called **Involution.**

38. *Roots.* A **Root** is one of the equal factors into which a number can be divided.

Def. The figure or letter showing the number of equal factors into which a quantity is to be divided is called the **Index** of the root.

The square root of a symbol is expressed by writing the sign $\sqrt{\ }$ (called *root*) before it.

Ex. 1. $\sqrt{49}$ means the square root of 49, that is, 7.

Ex. 2. $\sqrt{x}$ means the square root of x.

Any other root than the square is represented by writing its index before the sign of the root.

Ex. 1. $\sqrt[3]{x}$ means the cube root of x.

Ex. 2. $\sqrt[4]{x}$ means the fourth root of x.

Def. The operation of extracting a root is called **Evolution.**

39. The operations of Addition, Subtraction, Multiplication, Division, Involution, and Evolution, are the six fundamental operations of Algebra.

40. *Def.* An **Algebraic Expression** is any combination of algebraic symbols made in accordance with the foregoing principles.

EXERCISES.

In the following expressions, suppose $a = -7$, $b = -5$, $c = 0$, $m = 3$, $n = 4$, $p = 9$, and compute their numerical values.

1.	$a + b + m + p.$	2.	$a + m + n.$
3.	$m - n - a - b.$	4.	$n + p - m - a.$
5.	$3a - m + b - 2n.$	6.	$2a - 7p + 2b - m.$
7.	$3mnp.$	8.	$mncp.$
9.	$bmn.$	10.	$bnp.$
11.	$abmp.$	12.	$2^2abnp.$
13.	$am + bn.$	14.	$am - bn.$
15.	$bp - an.$	16.	$6p + an.$
17.	$n^2p + m^2b.$	18.	$m^3n - ap^2.$
19.	$a^2 + b^2.$	20.	$a^3 + b^3.$
21.	$a^3 - b^3.$	22.	$a^3m - b^3n.$
23.	$a^3b^2 - m^3n^2.$	24.	$a^2b^3 - b^2m^3.$
25.	$ab^2 + a^2b.$	26.	$ab^3 - a^3b.$
27.	$\dfrac{ab + mn}{ab - mn}.$	28.	$\dfrac{ac - bp}{bn - mp}.$
29.	$\dfrac{2m^2n^2 - 10m^3}{p - bcm}.$	30.	$\dfrac{ab - mp}{m - n}.$

In the following expressions, suppose $a = 8$, $b = -3$, and x to have in succession the fifteen values -7, -6, -5, etc., to $+7$, and compute the fifteen corresponding values of each expression:

31. $x^2 + bx + a.$ 32. $\dfrac{a + bx}{a - bx}.$

Arrange the results in a table, thus:

$x = -7$;	Expression 31	$= 78$;	Exp. 32 $= -\frac{29}{13}$.
$x = -6$;	"	" $= 62$;	etc.
$x = -5$;	"	" $= 48$.	
etc.	etc.	etc.	

CHAPTER III.

FORMATION OF COMPOUND EXPRESSIONS.

Fundamental Principles.

41. The following are two fundamental principles of the algebraic language:

First Principle. Every algebraic expression, however complex, represents a quantity, and may be operated upon as if it were a single symbol of that quantity.

Second Principle. A single symbol may be used to represent any algebraic expression whatever.

42. When an expression is to be operated upon as a single quantity, it is enclosed between parentheses, but the parentheses may be omitted, when no ambiguity or error will result from the omission.

EXAMPLE. Let us have to subtract b from a, and multiply the remainder by the factor m. The remainder will be expressed by $a - b$, and if we write the product of this quantity by m, in the way of § 35, the result will be

$$ma - b.$$

But this will mean b subtracted from ma, which is not what we want, because it is not a, but $a - b$ which is to be multiplied by m. To express the required operations, we enclose $a - b$ in brackets or parentheses, and write m outside, thus:

$$m(a - b).$$

NUMERICAL EXAMPLES.

$7(8 - 2) = 7 \cdot 6 = 42$; but $7 \cdot 8 - 2 = 56 - 2 = 54$.

$12(3 + 4) = 12 \cdot 7 = 84$.

$(6 + 3)(2 + 6) = 9 \cdot 8 = 72$.

$(7 - 4)(1 - 5)(2 + 7) = 3 \times -4 \cdot 9 = -108$.

EXAMPLE 2. Suppose that the expression $a - b + c$ is to be added to m, subtracted from m, multiplied by m, divided by m, raised to the third power, or have the cube root extracted. The results will be written:

Added to m,	$m + (a - b + c)$.
Subtracted from m,	$m - (a - b + c)$.
Multiplied by m,	$m(a - b + c)$.
Divided by m,	$\frac{(a - b + c)}{m}$.
Cubed,	$(a - b + c)^3$.
Cube root extracted,	$\sqrt[3]{(a - b + c)}$.

There are two of these six cases in which the parentheses are unnecessary, although they do no harm, namely, addition and division, because in the case of addition,

$$m + (a - b + c)$$

is the same as $$m + a - b + c.$$

[For example, $10 + (8 - 5 + 4) = 10 + 7 = 17$,

and $10 + 8 - 5 + 4 \quad = 17$ also.]

Again, in the case of the fraction, it will be seen that it has exactly the same meaning with or without the parentheses.

43. An algebraic expression having parentheses as a part of it may be itself enclosed in parentheses with other expressions, and this may be repeated to any extent. Each order of parentheses must then be made larger or thicker, or different in shape to distinguish it.

EXAMPLES. 1. Suppose that we have to subtract a from b, the remainder from c, that remainder from d, and so on. We shall have,

First remainder,	$b - a$.
Second,	$c - (b - a)$.
Third,	$d - [c - (b - a)]$.
Fourth,	$e - \{d - [c - (b - a)]\}$.
Fifth,	$f - \big[e - \{d - [c - (b - a)]\}\big]$.

2. Suppose that we have to multiply the difference of the quantities a and b by p and subtract the product from m. The result or remainder will be

$$m - p(a - b).$$

Suppose now that we have to multiply this result by $p+q$. We must enclose both factors in parentheses, and the result will then be written:

$$(p + q)[m - p(a - b)].$$

EXERCISES.

In the following expressions, suppose

$$a = -1, \quad b = 3, \quad m = 5. \quad x = -3, -1, +1, +3,$$

and calculate the four values of each expression which result from giving x the above four values in succession.

1. $\dfrac{x(x-a)(x-2a)(x-3a)}{1 \cdot 2 \cdot 3 \cdot 4}$.

2. $\dfrac{[a(b-x) - b(a-x)]^2}{m(b-x) + b(m-x)}$.

3. $[ax + b(x-a)^2 + m(x-a)^3]^3 \dfrac{x-m}{x+m}$.

4. $[\sqrt{(mx^2 + b)} - \sqrt{(mx^2 - b)}]\sqrt{(mb - a)}$.

NOTE. When the square root is not an integer, it will be sufficient to express it without computing it in full.

Thus, for $x = -3$, we shall have

$$\sqrt{(mx^2 + b)} - \sqrt{(mx^2 - b)} = \sqrt{48} - \sqrt{42}.$$

This is a sufficient answer without extracting the roots.

Definitions.

44. *Coefficient.* Any number which multiplies a quantity is called a **Coefficient** of that quantity. A coefficient is therefore a multiplier.

EXAMPLE. In the expression $4abx$,

4	is the	coefficient	of	abx,
$4a$	"	"	"	bx,
$4ab$	"	"	"	x.

Def. A **Numerical Coefficient** is a simple number, as 4, in the above example.

Def. A **Literal Coefficient** is one containing one or more letters used as algebraic symbols.

REM. Any quantity may be considered as having the coefficient 1, because $1x$ is the same as x.

Reciprocal. The **Reciprocal** of a number is unity divided by that number. In the language of Algebra,

$$\textit{Reciprocal of } N = \frac{1}{N}.$$

Formula. A **Formula** is an expression used to show how a quantity is to be expressed or calculated.

Term. When an expression is made up of several parts connected by the signs + or −, each of these parts is called a **Term.**

EXAMPLE.—In the expression,

$$a + bx + 3mx^2,$$

there are three terms, a, bx, and $3mx^2$.

When several terms are enclosed between parentheses, so as to be operated on as a single symbol, they form a single term.

Thus, the expression

$$\frac{(a + bx + 3mx^2)(a + b)}{(x + y)(x - y)}$$

forms but a single term, though both numerator and denominator are each a product of several terms. Such expressions may be called compound terms.

Aggregate. A sum of several terms enclosed between parentheses in order to be operated upon as a single quantity is called an **Aggregate.**

Algebraic expressions are divided into *monomials* and *polynomials*.

A **Monomial** consists of a single term.

A **Polynomial** consists of more than one term.

A **Binomial** is a *polynomial* of two terms.

A **Trinomial** is a *polynomial* of three terms.

NOTE. The last three words are commonly applied only to sums of simple terms, formed of single symbols or products of single symbols.

Entire. An **Entire Quantity** is one which is expressed without any denominator or divisor, as 2, 3, 4, etc.; a, b, x, etc.; $2ab$, $2mp$, $ab(x-y)$, etc.

A **Theorem** is the statement of any general truth.

45. *Other Algebraic Signs.* Besides the signs already defined, others are of occasional use in Algebra.

$>$, the **Sign of Inequality,** shows when placed between two quantities, that the one at the open end of the angle is the greater.

Ex. 1. $a > b$ means a is greater than b.

Ex. 2. $m < x < n$ means x is greater than m, but less than n.

$:$, another **Sign of Division,** is placed between two quantities to express their ratio.

Thus, $a : b$ means the ratio of a to b, or the quotient of a divided by b.

$\therefore$ means **Hence,** or **Consequently**; as,

$$a + 2 = 5; \qquad \therefore \quad a = 3.$$

∞ means a quantity infinitely great, or **Infinity**.

$\overline{}$, the **Vinculum,** is sometimes placed over an aggregate to include it in one mass, in lieu of parentheses.

Ex. $\overline{a-b}\ \overline{c-d}$ is the same as $(a-b)(c-d)$.

It is mostly used with the radical sign. We often write

$$\sqrt{a+b+c} \quad \text{instead of} \quad \sqrt{}(a+b+c).$$

CHAPTER IV.

CONSTRUCTION OF ALGEBRAIC EXPRESSIONS.

46. All operations upon algebraic quantities, however complex, consist in combinations of the elementary operations already described. The result of each single operation will be an aggregate, a product, a quotient, or a root, and every such result may, in subsequent operations, be operated upon as a single symbol. There are only three cases in which an expression needs any modification in order to be operated upon, namely:

CASE I. An aggregate must be enclosed in parentheses, if any other operations than addition or division are to be performed upon it. (§ 42.)

CASE II. When a product is to be raised to a power, or to have a root extracted, it may be enclosed in parentheses in order to show that the operation extends to all the factors.

If we take the product abc, and write an exponent, 2 for instance, after it thus, abc^2, it would apply only to c, and would mean $a \times b \times c^2$. So with the radical sign; $\surd abc$ might mean only $\surd a \times b \times c$. To indicate that the power or root is that of the product as a whole, we may enclose it in parentheses, thus:

$$\text{Square root of } abc = \surd(abc).$$
$$\text{Square of } abc = (abc)^2.$$

But a root sign is commonly made to include the whole product by simply extending a vinculum over all the factors of the product, thus: Square root of $abc = \sqrt{abc}$.

CASE III. If negative quantities are to be multiplied, merely writing them after each other would lead to mistakes. Thus, the product $a \times -b \times -c$, if written without the $\times$ sign, would be $a - b - c$, and would not mean a product at all. But, by enclosing $-b$ and $-c$ in parentheses, we have

$$a(-b)(-c),$$

which would correctly express the product required.

47. The following example will show how operations may be combined to any extent.

The quantity a is to be subtracted from b, and the difference multiplied by y, forming a product P.* The quotient of $p - r$ divided by q is to be multiplied by m, and the product subtracted from P. The difference is to form the numerator N of a fraction. To form the denominator, b is to be added to a and subtracted from it, and the product Q of the sum and difference formed. The quantity q is to be added to and subtracted from p, and the product R of the sum and difference formed. The quotient of Q divided by R is to form the denominator of the fraction of which the numerator is P.

The quantity b subtracted from a leaves $b - a$.

Multiplying it by y, the product P is $y\,(b - a)$.

Quotient of $p - r$ divided by q $\dfrac{p - r}{q}$.

Multiplying it by m, $m\dfrac{p - r}{q}$

[If instead of multiplying the fraction as a whole by m, we had multiplied its numerator, we should have had to enclose the $p - r$ in parentheses, thus: $\dfrac{m\,(p - r)}{q}$. But when the multiplier is written at the end of the line, between the terms of the fraction, as above, it indicates that the fraction, as a whole, is multiplied by m.]

Subtracting the last product from P, it is $y(b - a) - m\dfrac{p - r}{q}$.

Adding b to a, $a + b$.

Subtracting b from a, $a - b$.

The product Q of the sum and difference, $(a + b)\,(a - b)$.

The product R of $p + q$ by $p - q$, $(p + q)\,(p - q)$.

The quotient of Q divided by R, $\dfrac{(a + b)\,(a - b)}{(p + q)\,(p - q)}$.

* In mathematical language, when a substantive is followed by a symbol in this manner, the latter is used as a sort of proper name to designate the substantive, so that the latter can be afterward referred to by the letter without ambiguity.

In the present case, the capital letters are used in accordance with the second general principle, § 41.

The fraction having N for its numerator and this quotient for its denominator is

$$\frac{y(b-a)-m\dfrac{p-r}{q}}{\dfrac{(a+b)(a-b)}{(p+q)(p-q)}}$$

48. By the second general principle, § 41, a single symbol may be written in place of any algebraic expression whatever. When several symbols indicating such expressions are combined, the original expressions may be substituted for them, and be treated in accordance with the first principle.

EXAMPLES.

Suppose $P = a + bx;\quad Q = \dfrac{a-bx}{m};$

$T = x - y;\quad V = mpq.$

It is required to form the expression

$$\frac{PQ-TV}{PT-QV}.$$

The answer is

$$\frac{(a+bx)\dfrac{a-bx}{m}-(x-y)\,mpq}{(a+bx)(x-y)-\dfrac{a-bx}{m}\,mpq}.$$

EXERCISES.

Form the expressions:

1. $P - T.$
2. $T - P.$
3. $P - Q.$
4. $Q - V.$
5. $\sqrt{P}.$
6. $\sqrt{(P+T)}.$
7. $\sqrt{(P-T)}.$
8. $P^2T^2.$
9. $V^3.$
10. $T^3V^3.$
11. $\dfrac{VP-QT}{Q^2-T^2}.$
12. $\dfrac{PT}{QV}.$
13. $\dfrac{(P+T)(P-T)}{(Q+V)(Q-V)}.$
14. $\dfrac{(3P-2T)^2}{(4Q)^2}.$
15. $\dfrac{P^2-T}{\sqrt{(P-T)^2}}.$
16. $\dfrac{2(P+T)^2}{(2T-V)^2}.$

17. $\dfrac{P(T^2 - V)}{P^2 T^2}$.

18. $\dfrac{Q^3 - T^2}{PT + TQ}$.

19. $\dfrac{\sqrt{2P} + 2\sqrt{T}}{(P + T)^2}$.

20. $\dfrac{P^2 + Q^2}{(V - T)(V + T)}$.

21. $[(V + Q)^2 + P]\,T$.

22. $\dfrac{P\sqrt{Q^2 - T}}{(\sqrt{P} - Q)^2}$.

23. $\dfrac{V - (Q^2 - T)^2}{P - (Q + T)(Q - T)}$.

24. $\dfrac{V - Q^2 + T}{P - (Q + V)\,T}$.

EXERCISES IN ALGEBRAIC LANGUAGE.

The following questions are proposed to practice the student in expressing the relations of quantities in algebraic language. Should any of them offer difficulties, he is recommended to substitute numbers for the algebraic letters, examine the process by which he proceeds, and then apply the same process to the letters that he applied to the numbers. No solutions of equations are required.

1. How many cents are there in m dollars?

2. How many dollars in m cents?

3. A man had a dollars in one pocket, and b cents in the other; how many cents had he in all? How many dollars?

4. The sum of the quantities a and b is to be multiplied by m. Express the product, and its square.

5. A man having b dollars paid out m dollars to one person and n dollars to another. Express what he had left in two ways?

6. How many chickens at k cents a piece can be purchased for m dollars?

7. A man walked from home a distance of m miles at 4 miles an hour, and returned at the rate of 3 miles an hour. How long did it take him to go and come?

8. A man going to market bought tomatoes at h cents per peck and potatoes at k cents a peck, of each an equal number. They cost him m cents. How many pecks of each did he buy?

9. How many minutes will it require to go a miles, at the rate of b miles an hour?

10. A man bought from his grocer a pounds of tea at x cents a pound, b pounds of sugar at y cents a pound, and c pounds of coffee at z cents a pound. How many cents will the whole amount to? How many dollars? How many mills?

11. A man bought f pounds of flour at m cents a pound,

and handed the grocer an x-dollar bill to be changed? How many cents ought he to receive in change?

12. From two cities a miles apart two men started out at the same time to meet each other, one going m miles an hour and the other n miles an hour. How long before they will meet? How far will the first one have gone? How far will the second one have gone?

13. A man left his n children a bonds worth x dollars each, and b acres of land worth y dollars an acre; but he owed m dollars to each of q creditors. What was each child's share of the estate?

14. Two numbers, x and y, are to be added together, their sum multiplied by s, that product divided by $a+b$, and the quotient subtracted from k. Express the result.

15. The sum of the numbers p and q is to be divided by the sum of the numbers a and b, forming one quotient. The difference of the numbers p and q is to be divided by the difference of the numbers a and b, forming another quotient. The sum of the two quotients is to be multiplied by $r+s$. Express the product.

16. The quotient of x divided by a is to be subtracted from the quotient of y divided by b, and the remainder multiplied by the sum of x and y divided by the difference between x and y. Express the result.

17. The number x is to be increased by 6, the sum is to be multiplied by $a+b$, q is to be added to the product, and the sum is to be divided by $r-s$. Express the result.

18. A family of brothers a in number each had a house worth a thousand dollars each. What was the total value of all the houses in dollars? What was it in cents?

19. A grocer mixed a pounds of tea worth x cents a pound, and b pounds worth y cents a pound. How much a pound was the mixture worth?

20. $x+y$ houses each had $a+b$ rooms, and each room $m+n$ pieces of furniture. How many pieces of furniture were there in all?

21. In a library were $p+q$ volumes, each volume had $p+q$ pages, each page $p+q$ words, and each word on the average 8 letters. How many letters were there in all the books of the library?

22. A post-boy started out from a station, travelling k miles an hour. Three hours afterward, another one started after him, riding m miles an hour. How far was the first one

ahead of the second at the end of x hours after the second started?

23. Two men started to make the same journey of m miles, one going r miles an hour, and the other s miles an hour. How much sooner will the man going r miles an hour make his journey than the one going s miles an hour? How much sooner will the one going s miles an hour make his journey than the one going r miles an hour?

24. One train runs from Boston to New York in h hours, at the rate of n miles an hour. How long will it take another train running 5 miles an hour faster to perform the journey?

25. If a man bought h horses for t dollars, and n yoke of oxen for m dollars, how much more did one horse cost than one yoke of oxen? How much more did one yoke of oxen cost than one horse?

26. A train making a journey of $2m$ miles goes the first half of the way at the rate of r miles an hour, and the second half at the rate of s miles an hour. How long did it take it to go? What was the average speed for the journey?

27. Two men, A and B, started to walk from Hartford to New Haven and back, the distance between the two cities being a miles. A goes p miles an hour and B q miles an hour. How far will A have got on his return journey when B reaches Hartford?

28. A man having k dollars bought b books at \$6 each. How many books at \$4 each can he buy with the balance of his money?

29. A man going to his grocer with m dollars, bought s pounds of sugar at a cents a pound, and r pounds of coffee at b cents a pound. How many barrels of flour at q dollars a barrel can he buy with the balance of his money?

30. A man divided m dollars equally among a poor Chinese and n dollars equally among h orphans. Two of the Chinese and three of the orphans put their shares together and bought x Bibles for the heathen. How much did each Bible cost?

31. A pedestrian having agreed to walk the a miles from Boston to Natick in h hours, travels the first k hours at the rate of m miles an hour. At what rate must he travel the remainder of the time?

32. A train having to make a journey of x miles in h hours, ran for k hours at the rate of r miles an hour, and then made a stop of m minutes. How fast must it go during the remainder of its journey to arrive on time?

BOOK II.

ALGEBRAIC OPERATIONS.

General Remarks.

The algebraic expressions formed in accordance with the rules of the preceding book admit of being transformed and simplified in a variety of ways. This transformation is effected by operations which have some resemblance to the arithmetical operations of addition, subtraction, multiplication, and division, and which are therefore called by the same names.

In performing these algebraic operations, the student is not, as in Arithmetic, seeking for a result which can be written in only one way, but is selecting out of a great variety of forms of expression some one form which is the simplest or the best for certain purposes. Sometimes one form and sometimes another is the best for a particular problem. Hence, it is essential that the algebraist, in studying an expression, should be able to see the different ways in which it may be written.

Definitions.

49. *Function.* An algebraic expression containing any symbol is called a **Function** of the quantity represented by that symbol.

Ex. 1. The expression $3x^2$ is a function of x.

2. The expression $\dfrac{a+x}{a-x}$ is a function of x and also a function of a.

When an expression contains several symbols, we may select one of them for special consideration, and call the expression a function of that particular one. For instance, although the expressions,

$$a + bx^2 + cx^3,$$
$$m + n\sqrt{x},$$

contain other symbols besides x, they are both functions of x.

50. An **Entire Function** is one in which the quantity is used only in the operations of addition, subtraction and multiplication.

EXAMPLE. The expressions

$$ax + y,$$
$$(a^2 - y^2)x^3 - (b^2 + y)x^2 - x + d,$$

are entire functions of x. But the expressions

$$\frac{ax + y}{ax - y} \quad \text{and} \quad 3\sqrt{x}$$

are not entire functions of x, because in the one x appears as part of a divisor, and in the other its square root is extracted.

An entire function of x can always be expressed as a sum of terms, arranged according to the powers of x which they contain as factors. The form of the expression will then be

$$A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.},$$

where A, B, C, etc., may represent any algebraic expressions which do not contain x.

51. Like Terms are those which are formed of the same algebraic symbols, combined in the same way, and differ only in their numerical coefficients.

Ex. The terms ax, $2ax$, $-5ax$ are like terms.

52. The **Degree** of any term is the number of its literal factors.

EXAMPLES. The expression $abxy$ is of the fourth degree, because it contains four literal factors.

The expression x^3 is of the third degree, because the letter x is taken three times as a factor.

The expression ab^2x^3 is of the sixth degree, because it contains a once, b twice, and x three times as a factor.

When an expression consists of several terms, its degree is that of its highest term.

CHAPTER I.

ALGEBRAIC ADDITION AND SUBTRACTION.

Algebraic Addition.

53. By the language of Algebra, the sum of any number of quantities, positive or negative, may be expressed by writing them in a row, with the sign $+$ before all the positive quantities, and the sign $-$ before the negative ones.

Ex. $A+B-D-X+Y$, etc., is the algebraic sum of the several quantities A, B, $-D$, $-X$, Y, etc.

54. *To simplify an expression of the sum of several quantities.*

1. When dissimilar terms are to be added, no simplification can be effected.

Ex. If we require the sum of the five expressions, a, $-xy$, mp, nq, and $-bhs$, we can only write,

$$a - xy + mp + nq - bhs,$$

according to the language of Algebra, and cannot reduce the expression to a simpler form.

2. If mere numbers are among the quantities to be added, their algebraic sum may be formed.

Ex. The sum of the five quantities -8, ab, 5, mnp, -15, is found to be $-18 + ab + mnp$.

3. When several terms are similar, add the coefficients and affix the common symbol to the sum.

When no numerical coefficient is written, the coefficient $+1$ or -1 is understood. (§ 44.)

EXAMPLES.

$a + a = 2a$ [because $1 + 1 = 2$].
$2a - a = a$ [because $2 - 1 = 1$].

$3a + 4a - 7a = 0$ [because $3 + 4 - 7 = 0$].
$a + 2x - 3a - 5x = -2a - 3x$ [adding the a's and the x's].
$-3axy + 4bm - 2axy + bm = -5axy + 5bm$.

Add the expressions,

1. $7x + 5by^2,\ 2x - 3by^2,\ -4x - 5by^2,\ 5x - by^2,\ x - by^2$.

For convenience, the several terms may be written under each other, as in the margin. The coefficients of x are 7, 2, -4, 5, and 1, of which the algebraic sum is 11. The coefficients of y^2 are 5, -3, -5, -1, -1; the sum is -5. Hence the result.

WORK.

$$\begin{array}{rrr} & 7x & +\ 5by^2 \\ & 2x & -\ 3by^2 \\ & -\ 4x & -\ 5by^2 \\ & 5x & -\ \ by^2 \\ & x & -\ \ by^2 \\ \hline \textit{Sum,} & 11x & -\ 5by^2 \end{array}$$

2. $8ax - y - 2y + 5,\ 7ax - y - 9 + am,\ 2ax - y - 3 + 5p$.

Here $2x$, am, and p, all being different symbols, the terms containing them do not admit of simplification (§ 54, 1). The numbers 5, -9, -3, are added by the rule (§ 54, 2). The coefficients of ax^2 cancel each other ($8-7-1=0$).

WORK.

$$\begin{array}{rrrrrr} & 8ax^2 & -\ y & -\ 2x & +\ 5 & \\ & -\ 7ax^2 & -\ y & & -\ 9 & +\ am \\ & -\ \ ax^2 & -\ y & & -\ 3 & +\ 5p \\ \hline \textit{Sum,} & & -\ 3y & -\ 2x & -\ 7 & +\ am\ +\ 5p \end{array}$$

3. Add $6(x + y),\ 5(x + y) + a,\ 2(x + y) - 3a$.

Here the aggregate, $x+y$, enclosed in parentheses, is treated as a simple symbol.

NOTE. When the student can add the coefficients mentally, it is not necessary to write the expressions under each other. Nor is it necessary to repeat the symbol after each coefficient.

WORK.

$$\begin{array}{rlr} & 6\,(x + y) & \\ & 5 & +\ a \\ & 2 & -\ 3 \\ \hline \textit{Sum,} & 13\,(x + y) & -\ 2a \end{array}$$

EXERCISES.

1. $3a + 7b - 8c + d,\ 3a - 2b + c - e,\ -a - b - c - d$.
2. $7a - (x + y),\ 8a - (x + y),\ 3(x + y) - 16a$.
3. $7x^2 - 2x - 5,\ 2x^2 - 3x + 8,\ -9x^2 + 5x + 3$.
4. $x^2 + 2x - y,\ 4x^2 + 7x - 2y,\ -2x^2 + x - 9y,\ -3x^2 - x - y$.
5. $9(a + b)^2,\ 10(a + b)^2,\ (a+b)^2,\ 2(a+b)^2,\ -x - y - z$.
6. $2(m + n) + 3(a + b),\ (a + b) - (m + n),\ (a + b) - (m + n)$.
7. $7a^3 - 2a^2 + 3ax,\ -a^3 - a^2 - ax,\ -6a^3 + 3a^2 - 2ax$.

8. $(m+n)^2+x$, $2(m+n)^2-y$, $3(m+n)^2-2x$, $(m+n)^2-y$.

9. $(p+q)^2-6$, $(p+q)^2+a$, $(p+q)^2+b$, $(p+q)^2+c$.

10. $6a(x-y)$, $5a(x-y)$, $2a(x-y)$, $a(x-y)$.

11. $2(m-n)x+2$, $3(m+n)x-5$, $5(m+n)x-6$, $7(m+n)x-8$.

12. $3\frac{x}{a}$, $2\frac{x}{a}+3\frac{y}{b}$, $\frac{x}{a}-\frac{y}{b}$, $\frac{y}{b}-\frac{6}{7}$, $\frac{x}{a}-\frac{1}{7}$.

13. $\frac{x}{y}-\frac{m}{n}$, $2\frac{x}{y}-2\frac{m}{n}$, $3\frac{x}{y}-3\frac{m}{n}$, $4\frac{x}{y}-4\frac{m}{n}$.

14. $\frac{x+y}{m+n}+3\frac{x+y}{m+n}$, $5\frac{x+y}{m+n}+7\frac{x+y}{m+n}$.

15. Of two farmers, the first had $2x-3y$ acres, and the second had $x-y$ acres more than the first. How many acres had they both?

16. A had $2x$ dollars, B had y dollars less than A, and C had $2y$ dollars more than A and B together. How many had they all?

17. A father gave his eldest son x dollars, his second 5 dollars less than the first, his third 5 dollars less than his second, and his fourth 5 dollars less than his third. How much did he give them all?

55. *Addition with Literal Coefficients.* When different terms contain the same symbol, multiplied by different literal coefficients, these coefficients may be added and the common symbol be affixed to their aggregate.

EXAMPLES.

1. As we reduce the polynomial

$$6x+5x-2x$$

to the single term $(6+5-2)x=3x$,
so we may reduce the polynomial

$$ax+bx-cx$$

to the single term, $(a+b-c)x$.

2. The expression

$$mx+ny-bx+dy+a+b$$

may be expressed in the form

$$(m-b)x+(n+d)y+a+b.$$

EXERCISES.

Collect the coefficients of x and y in the following expressions:

1. $ax + by + mx + ny.$
2. $mnx + 2by + pqx - 4by.$
3. $3x - 2y + 6bx - 4y + 7ax + m + n.$
4. $8ax + 8bx + by + 7x - 5y + x - 5y.$
5. $ax + by + cz - mx - ny - pz.$
6. $2dx + 3ey + 4fz - 2fx - 3dy + 4ez.$
7. $\frac{2}{3}ay - 2x + \frac{3}{4}by + 6ax.$
8. $2ax - by - 3bx - 4ay.$
9. $\frac{1}{2}ax + \frac{2}{3}by - \frac{1}{6}mx + \frac{3}{4}ny.$
10. $4mx + 2y - 3ax - 6cx + ay - \frac{2}{3}mx + \frac{1}{2}dx.$
11. $5abx - 3mny - abx + 4cdy - dx.$
12. $3ay + 2bx - \frac{1}{4}dx + 2ay - 3bx.$
13. $\frac{1}{2}ay - 3x + 2y - \frac{3}{4}ay - 5x + y.$
14. $3mx - ax - \frac{1}{2}ay + x + dx - y.$
15. $3abx - my + 2c\sqrt{x} - dy + \sqrt{x}.$
16. $5m\sqrt{y} - 6x + 4\sqrt{y} - 3\sqrt{x} - y + \sqrt{y}.$
17. $4\sqrt{x} - 6y + a\sqrt{y} + cx - \sqrt{y} - 4a\sqrt{y} + \sqrt{x}.$

Algebraic Subtraction.

56. *Def.* **Algebraic Subtraction** consists in expressing the difference of two algebraic quantities.

Rule of Subtraction. It has been shown (§ 21) that to subtract a positive quantity, b, is the same as to add, algebraically, the negative quantity, $-b$. Also, that to subtract $-b$ is equivalent to adding $+b$. Hence the rule:

Change the algebraic sign of all the terms of the subtrahend, or conceive them to be changed, and then proceed as in addition.

NUMERICAL EXAMPLES.

Min.,	$10+6=16$	$10+\ 6=16$	$10+\ 6=16$	$10+\ 6=\ 16$
Subt.,	$9\ \ \ \ =\ 9$	$9-\ 4=\ 5$	$9-\ 8=\ 1$	$9-12=-3$
Rem.,	$1+6=\ 7$	$1+10=11$	$1+14=15$	$1+18=\ 19$

ALGEBRAIC EXERCISES.

1. From $3x-4ay+5b+c$,
   Subtract $x-7ay-8b+d$.

WORK.

$$\begin{array}{lr}
\text{Minuend,} & 3x-4ay+\ 5b+c \\
\text{Subtrahend with signs changed,} & -x+7ay+\ 8b-d \\ \hline
\text{Difference,} & 2x+3ay+13b+c-d
\end{array}$$

Next we may simply imagine the signs changed.

2. From $7x-\ 4bxy-12cy+\ 8b+3ac$
   Take $2x+\ 7bxy+\ 8cy-\ 5b-2d$
   Diff., $5x-11bxy-\ 4cy+13b+3ac+2d$

3. From $8a+9b-12c-18d-4x+3cy$
   Take $19a-7b-\ 8c-25d+3x-4y$

4. From $257z+201z^2+92y+35ax-\ 6$
   Take $140z-\ 82z^2+20y+92ax+14$

5. From $8a+14b$ subtract $6a+20b$.
6. From $a-b+c-d$ take $-a+b-c+d$.
7. From $8a-2b+3c$ subtract $4a-6b-c-2d$.
8. From $2x^2-8x-1$ subtract $5x^2-6x+3$.
9. From $4x^4-3x^3-2x^2-7x+9$ subtract $x^4-2x^3-2x^2+7x-9$.
10. From $2x^2-2ax+3a^2$ subtract x^2-ax+a^2.
11. From $a^3-3a^2b+3ab^2-b^3$ subtract $-a^3+3a^2b$.
12. From $7x^3-2x^2+2x+2$ subtract $4x^3-2x^2-2x-14$.
13. From $5(x-y)+7(x-z)+9(z-x)$ take $9(x-y)+7(x-z)+5(z-x)$.
14. From $12(a-b)-3(a+b)+7a-2b$ take $7(a-b)-5(a+b)$.
15. From $7\frac{x}{y}-11\frac{y}{z}-15\frac{z}{x}$ take $-5\frac{x}{y}+6\frac{y}{z}-7\frac{z}{x}+8\frac{a}{b}$.

Clearing of Parentheses.

57. In § 42, 2, it was shown that an aggregate of terms included between parentheses might be added or subtracted by simply writing + or — before the parentheses.

When an aggregate not multiplied by a factor is to be added or subtracted, the parentheses may be removed by the rules for addition and subtraction, as follows:

58. *Plus Sign before Parentheses.* If the parentheses are preceded by the sign +, they may be removed, and all the terms added without change.

EXAMPLE 1. $27+(8-5-4+7)=27+8-5-4+7=33.$

2. $m+(a-x-y+z)=m+a-x-y+z.$

3. $2x+(-3x-5y)+(3y-4a)+(2y-2a)$
$$=2x-3x-5y+3y-4a+2y-2a$$
$$=-x-6a.$$

The sign + which precedes the parentheses should also be considered as removed, but if the first term within the parenthesis has no sign, the sign + is understood, and must be written after removing the parentheses.

EXERCISES.

Clear of parentheses and simplify

1. $x-y+(x+y).$
2. $x+y+(y-x).$
3. $3ab-2mp+(ab-3x-2mp).$
4. $2ax-3by+(mx-2ax-pz+3by).$
5. $3\frac{a}{b}+\left(\frac{a}{b}-2\frac{m}{n}\right)+\left(\frac{a}{b}+2\frac{m}{n}\right).$

59. *Minus Sign before Parentheses.* If the parentheses are preceded by the sign —, they may be removed and the algebraic sign of each of the included terms changed, according to the rule for subtraction in § 56.

EXAMPLES.

1. $27-(8-5-4+7)=27-8+5+4-7=21;$ that is, $27-6=21.$

2. $m - (-a - p + y + x) = m + a + p - y - x.$

3. $3a + x - (2a - 5x) - (9x - a) = 3a + x - 2a + 5x - 9x + a.$

Simplifying as in § 54, this reduces to $2a - 3x$.

EXERCISES.

Clear the following expressions of parentheses and reduce the results to the simplest form by the method of § 54.

1. $ab - (m - 3ab + 2ax) - 7ab.$
2. $x - (a - x) + (x - a).$
3. $2b + (b - 2c) - (b + 2c).$
4. $4x - 3y + 2z - (-7x + 5y - 3z) - (x - y).$
5. $7ax - 2by - (8ax + 3by) - (8ax - 3by).$
6. $(a - x) - (a + x) + 2x.$
7. $-(a - b) - (b - c) - (c - a).$
8. $-(3m + 2n) - (3m - 2n) + 9m.$

60. We may reverse the process of clearing of parentheses by collecting several terms into a single aggregate, and changing their signs when we wish the parentheses to be preceded by the minus sign. The proof of the operation is to clear the parentheses introduced, and thus obtain the original expression.

EXERCISES.

Reduce the following expressions to the form

$$x - (\textit{an aggregate}).$$

1. $x - a - b.$ *Ans.* $x - (a + b).$
2. $x - m - n.$
3. $a + x - 3x + 2y.$ *Ans.* $x - (-a + 3x - 2y).$
4. $-3b + x + 2c + 5d.$
5. $2x - 2a + 2b.$ *Ans.* $x - (-x + 2a - 2b).$
6. $2x + a - b.$
7. $3x - 2m + 2n.$
8. $3x + ab - m - 3ab + 2m.$
9. $x - 2m - (3a - 2b).$ *Ans.* $x - (2m + 3a - 2b).$
10. $x + 3 - (a + b).$
11. $x + a - (b - c) + (m - n).$
12. $x - (am + b) - (p - q) - (am - n).$
13. $x - (a + b) - (p - q) - (m - n).$

Compound Parentheses.

61. When parentheses of addition or subtraction are enclosed between others, they may be separately removed by the preceding rules.

We may either begin with the outer ones and go inward, or begin with the inner ones and go outward.

It is common to begin with the inner ones.

EXAMPLES.

Clear of parentheses:

1. $f - [e - \{d - [c - (b - a)]\}]$.

Beginning with the inner parentheses, the expression takes, in succession, the following forms:

$$\begin{aligned} &f - [e - \{d - [c - b + a]\}] \\ =\ &f - [e - \{d - c + b - a\}] \\ =\ &f - [e - d + c - b + a] \\ =\ &f - e + d - c + b - a. \end{aligned}$$

2. $x - [-(a + b) + (m + n) - (x - y)]$.

Removing the inner parentheses, one by one, we have,

$$\begin{aligned} &x - [-a - b + m + n - x + y] \\ =\ &x + a + b - m - n + x - y. \end{aligned}$$

EXERCISES.

Remove the parentheses in the following expressions, and combine terms containing x and y, as in §§ 54 and 55.

1. $m + [-(p - q) + (a - b) + (-c + d)]$.
2. $m - [-(a - b) - (p + q) + (n - k)]$.
3. $7ax - [(2ax + by) - (3ax - by) + (-7ax + 2by)]$.
4. $a - [a - \{a - [a - (a - a)]\}]$.
5. $p - [a - b - (s + t + a) + (-m - n)]$.
6. $2ax - [3ax - by - (7ax + 2by) - (5ax - 3by)]$.
7. $ax + by + cz + [2ax - 3cz - (2cz + 5ax) - (7by - 3cz)]$.
8. $x - \{2x - y - [3x - 2y - (4x - 3y)]\}$.
9. $ax - bz - \{ax + bz - [ax - bz - (ax + bz)]\}$.
10. $my - \{x + 3y + [2my - 3(x - y) - 4ab] + 5\}$.

11. $ax + 4cx - (mx + cx - y) + [mx - (cx + y)]$.
12. $3ax - 3bx - (-3ay - 3az + 3by) - 3bz$.
13. $13ax + 2xy - d - [7ad + (xy + d)] - 4xy$.
14. $m + 4x - [-4y + 2x + (ay - x) + p]$.
15. $2a\sqrt{y} - 3m - [b\sqrt{x} - 6n + (\sqrt{y} - 2\sqrt{y})]$.

CHAPTER II.

MULTIPLICATION.

62. The product of several factors can always be expressed by writing them after each other, and enclosing those which are aggregates within parentheses.

EXAMPLES.

The product of $a + b$ by $c = c(a + b)$.

The product of $\frac{x + y}{2}$ by $x - y = (x - y)\frac{x + y}{2}$.

The product of $a + b$ by $c + d = (c + d)(a + b)$.

Such products may be transformed and simplified by the operation of algebraic multiplication.

General Laws of Multiplication.

63. *Law of Commutation.* Multiplier and multiplicand may be interchanged without altering the product.

This law is proved for whole numbers in the following way. Form several rows of quantities, each represented by the letter a, with an equal number in each row, thus,

$$\begin{matrix} a & a & a & a & a & a \\ a & a & a & a & a & a \\ a & a & a & a & a & a \\ a & a & a & a & a & a \\ a & a & a & a & a & a \end{matrix}$$

Let m be the number of rows, and n the number of a's in each row. Then, counting by rows there will be

$$m \times n \text{ quantities.}$$

Counting by columns, there will be

$$n \times m \text{ quantities.}$$

Therefore, $$m \times n = n \times m,$$

or $$nm = mn.$$

64. *Law of Association.* When there are three factors, m, n, and a,

$$m\,(na) = (mn)\,a.$$

EXAMPLE. $$3 \times (5 \times 8) = 3 \times 40 = 120.$$
$$(3 \times 5) \times 8 = 15 + 8 = 120.$$

Proof for Whole Numbers. If a in the above scheme represents a number, the sum of each row will be na. Because there are m rows, the whole sum will be $m\,(na)$.

But the whole number of a's is mn. Therefore,

$$m\,(na) = (mn)\,a.$$

65. *The Distributive Law.* The product of an aggregate by a factor is equal to the sum of the products of each of the parts which form the aggregate, by the same factor. That is,

$$m\,(p + q + r) = mp + mq + mr. \qquad (1)$$

Proof for Whole Numbers. Let us write each of the quantities p, q, r, etc., m times in a horizontal line, thus,

$$\begin{array}{lll} p + p + p + \text{etc.,} & m \text{ times} & = mp. \\ q + q + q + \text{etc.,} & m \text{ times} & = mq. \\ r + r + r + \text{etc.,} & m \text{ times} & = mr. \\ \quad\text{etc.} & \quad\text{etc.} & \quad\text{etc.} \end{array}$$

If we add up each vertical column on the left-hand side, the sum of each will be $p + q + r +$ etc., the columns being all alike.

Therefore the sum of the m columns, or of all the quantities, will be

$$m\,(p + q + r, \text{ etc.}).$$

The first horizontal line of p's being mp, the second mq, etc., the sum of the right-hand column will be

$$mp + mq + mr, \text{ etc.}$$

Since these two expressions are the sums of the same quantities, they are equal, as asserted in the equation (1).

Multiplication of Positive Monomials.

66. *Rule of Exponents.* Let us form the product

$$x^m \times x^n.$$

By § 37, x^m means xxx, etc., taken m times as factor.
x^n means xxx, etc., taken n times as factor.

The product is $xxxxx$, etc., taken $(m+n)$ times as factor.

Therefore, $x^m \times x^n = x^{m+n}$.

Hence,

Theorem. The exponent of the product of like symbols is the sum of the exponents of the factors.

67. As a result of the laws of commutation and association, the factors of a product may be arranged and multiplied in such order as will give the product the simplest form.

68. Any product of monomials may be formed by combining these principles.

Example. Multiply $5\,m\,n^2\,x^3\,y^4$ by $7\,b\,n\,x^2\,y$.

By the rules of algebraic language, the product may be put into the form

$$5\,m\,n^2\,x^3\,y^4\,7\,b\,n\,x^2\,y.$$

By interchanging the factors so as to bring identical symbols together,

$$5 \cdot 7\,b\,m\,n^2\,n\,x^3\,x^2\,y^4\,y.$$

Multiplying the numerical factors and adding the exponents, the product becomes

$$35\,b\,m\,n^3\,x^5\,y^5.$$

69. We thus derive the following

RULE. *Multiply the numerical coefficients of the factors, affix all the literal parts of the factors, and give to each the sum of its exponents in the separate factors.*

EXERCISES.

1. Multiply xy by x^2y. *Ans.* x^3y^2.
2. Multiply $3ax$ by $2abx^2$.
3. Multiply $5m^2y$ by $3m^2x$.
4. Multiply $21my$ by $2a^2m$.
5. Multiply $2am$ by $2ma$.
6. Multiply $5x^7y^3z$ by x^2y^3z.
7. Multiply $3xyz$ by $3xyz$.
8. Multiply $2abm$ by $2mba$.
9. Multiply $3ab^2x^3$ by $3a^3b^2x$.
10. Multiply $2\cdot6mpqr$ by $2\cdot6pqrs$.
11. Multiply $12axy$ by $12xyz$.
12. Multiply $\frac{3}{2}m^6x^5$ by $\frac{2}{3}m^5y'$.
13. Multiply $\frac{3}{4}n^2k$ by $4mk$.
14. Multiply $\frac{7}{2}abcd$ by $4defg$.

70. When we have to find the product of three or more quantities, we multiply two of them, then that product by the third, that product again by the fourth, and so on.

Ex. $2ab \times 2a^2b \times 3ab^2 \times 3bmxy = 36a^4b^5mxy.$

EXERCISES. Multiply

15. $mx \times my \times mz.$
16. $ax \times bx \times cx \times dx.$
17. $3a^2m \times 4b^2n \times mn.$
18. $ab \times 2bc \times 7ca.$
19. $3mn^2 \times 5np^2 \times 9pm^2.$
20. $ab \times ac \times ad \times am3 \times y \times 2yz \times zx.$
21. $amx \times anx \times amxy \times anxy \times amxyz.$
22. $a^2x \times a^2y \times ax^2 \times ay^2 \times a^2x^2 \times a^2y^2 \times x^2y^2.$
23. $2am \times 3an \times a^2 \times m^2 \times 4mx \times 2nx.$

Rule of Signs in Multiplication.

71. It was shown in § 25 that a product of two factors is positive when the factors have like signs, and negative when they have unlike signs. Hence the rule of signs,

$$
\begin{array}{ccccc}
+ & \times & + & \text{makes} & +, \\
+ & \times & - & \text{“} & -, \\
- & \times & + & \text{“} & -, \\
- & \times & - & \text{“} & +.
\end{array}
$$

EXAMPLES. The quantity a

Multiplied by		3	makes	$+3a$.
"	"	2	"	$+2a$.
"	"	1	"	$+a$.
"	"	0	"	0.
"	"	-1	"	$-a$.
"	"	-2	"	$-2a$.

The quantity $-a$

Multiplied by		3	makes	$-3a$.
"	"	2	"	$-2a$.
"	"	1	"	$-a$.
"	"	0	"	0.
"	"	-1	"	$+a$.
"	"	-2	"	$+2a$.

72. *Geometrical Illustration of the Rule of Signs.* Suppose the quantity a to represent a length of one centimetre from the zero point toward the right on the scale of § 11.

Then we shall have

$a =$ this line 

The product of the line by the factors from $+3$ to -3 will be

$a \times 3,$

$a \times 2,$

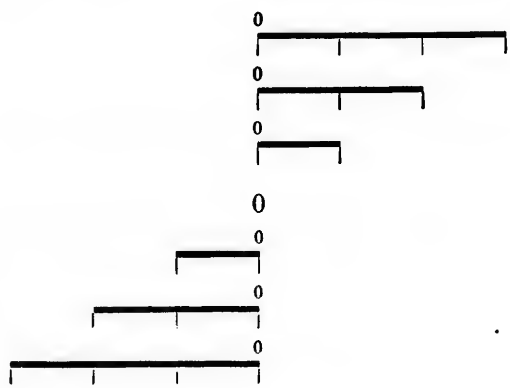
$a \times 1,$

$a \times 0,$

$a \times -1,$

$a \times -2,$

$a \times -3,$



We shall also have

$-a =$ this line 

The products by the same factors will be

$-a \times 3,$

$-a \times 2,$

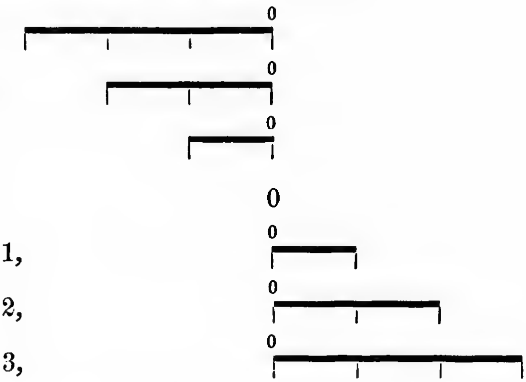
$-a \times 1,$

$-a \times 0,$

$-a \times -1,$

$-a \times -2,$

$-a \times -3,$



These results are embodied in the following two theorems:

1. Multiplying a magnitude by a negative factor, multiplies it by the factor and turns it in the opposite direction.

2. Multiplying by -1 turns it in the opposite direction without altering its length.

NOTE. When more than two factors enter a product, the sign may be determined by the theorem, § 26.

EXERCISES.

1. $am \times ab \times ac \times ad.$ 2. $ax \times -bx \times cx \times dx.$
3. $x \times -ax \times -abx \times -abcx.$
4. $3ax \times -2a^2b^3 \times -5a^3mx.$
5. $-7m^2y \times -3a^2y^2 \times 5ax.$
6. $-2nzn \times -5n^2x^m \times -n^3yz - x^n.$
7. $2m \times n \times -a \times -2b.$
8. $-3ax \times -2km \times -7x \times -4bmx.$
9. $-ny \times gy \times -2y \times 3bm.$
10. $xy \times 2y^2 \times y^2x \times 2ayx^2.$
11. $5y^2 \times -3gy \times -2z^2 \times -ax^2z.$
12. $5ax \times anx \times 3z \times b^2xy.$
13. $-4bz \times -xz \times -yz \times agz.$
14. $2c^2n \times 2x^2z \times -z^2 \times -bgz^2.$
15. $-c^2x \times 3x \times cb^2 \times ay.$

16. $-2e \times -2y \times a \times bx$.
17. $-4ax \times 3ay \times -2a^2y \times -xy$.
18. $a^2x \times -ay^2 \times ax^2 \times -x^{\frac{1}{2}}y$.
19. $ax^2 \times -y^2 \times -1 \times 3ax \times -a^2y$.
20. $m^2x \times -n^2x \times -mn^2 \times mx \times -m^3$.
21. $-abx \times -ay^2 \times ax \times a^2x^2$.
22. $px^2 \times qy^2 \times xy \times -ax$.
23. $abc \times -d^2 \times ax^2 \times -1 \times 3ax$.
24. $\frac{1}{4}ax \times 3cx \times -\frac{1}{2}mx \times -4y^2 \times 6m$.
25. $-6mx \times -2n^2x \times \frac{1}{6}ac \times -\frac{1}{5}m^2$.
26. $-a \times bc \times -1 \times \frac{1}{4} \times 3a^2 \times 4xy \times y$.
27. $-1 \times ax \times a^2x \times a^5x^2 \times bx \times d$.
28. $-an \times 2am^2 \times -3mn \times 5n^2y \times -m$.
29. $-mx \times nx \times -mn \times -xy \times -1$.
30. $-2px \times -3qx \times \frac{1}{6}m^2x \times \frac{1}{5}y^2 \times -1$.

Products of Polynomials by Monomials.

73. The rule for multiplying a polynomial is given by the distributive law (§ 65).

RULE. *Multiply each term of the polynomial by the monomial, and take the algebraic sum of the products.*

EXERCISES. Multiply

1. $3x^2 - 4xy - 5y^2$ by $-4ax$.
   Ans. $-12ax^3 + 16ax^2y + 20axy^2$.
2. $3x^2 - xy + y^2$ by $3x$.
3. $x^2 + xy + y^2$ by $3x$.
4. $ax + by + cz$ by $axyz$.
5. $3ax^3 - 5ay^2 - 7$ by $9abx$.
6. $4mp - 6nq$ by $-3mq$.
7. $5a^2y^3 - 7a^3y^2 - 7a^4y$ by $8ab$.

74. The products of aggregates by factors are formed in the same way, the parentheses being removed, and each term of the aggregate multiplied by the factor.

EXAMPLE. Clear the following expression of parentheses:

$$am(a-b+c) - p[a-(h-k)-m(a-b)].$$

By the rule of § 73, the first term will be reduced to

$$a^2m - amb + amc. \qquad (1)$$

The aggregate of the second term within the large parentheses will be

$$a-h+k-m(a-b)$$
$$= a-h+k-ma+mb, \qquad (2)$$

because, by the rule of signs in multiplication,

$$-m(a-b) = -m \times a - m \times -b = -ma + mb.$$

Multiplying the sum (2) by $-p$ and adding it to (1), we have for the result required:

$$a^2m - amb + amc - pa + ph - pk + pma - pmb.$$

EXERCISES.

Clear the following expressions of parentheses:

1. $p(a+m-p) + q(b-c) - r(b+c)$.
2. $(m-an)x - (m+an)y + (an-m)z$.
3. $a(x-y)c - b(x-y)d + f(x+y)cd$.

Here note that the coefficient of $x-y$ in the first term is ac.

4. $am[x-a(b-c)] - bn[ax+b(c+d)]$.
5. $p[-a(m+n)+b(m-n)] - q[b(m-n)-a(m+n)]$.
6. $3x(2q-nc) + 2y(5x-3c) - z(2m+7n)$.
7. $am[m(a-b)c - 3h(2k-4d) + 4n]$.
8. $2pq[3a-5b-6c-pq(2m-3n)]$.
9. $bn[-7a-7b(a-c)-(3-a-b)]$.
10. $p(q-r) + q(r-p) + r(p-q)$.

75. The reverse operation, of summing several terms into one or more aggregates, each multiplied by a factor, is of frequent application. Thus, in § 65, having given

$$mp + mq + mr,$$

we express the sum in the form

$$m(p+q+r).$$

The rule for the operation is

If the sum of several terms having a common factor is to be formed, the coefficients of this factor may be added, and their aggregate be multiplied by the factor.

NOTE. This operation is, in principle, identical with that of § 55.

EXAMPLES.

$$abx - bcx - ady + 3dy - 3bx + 4ady + my - amy - 3cmx + bmx.$$

Collecting the coefficients of x and y as directed, we have

$$(ab - bc - 3b - 3cm + bm)\,x + (-ad + 3d + 4ad + m - am)\,y.$$

Applying the same rule to the terms within the parentheses, we find

$$\begin{aligned} ab - bc - 3b &= b\,(a - c - 3). \\ -3cm + bm &= m\,(b - 3c). \\ -ad + 3d + 4ad &= 3ad + 3d \\ &= (3a + 3)\,d \\ &= 3\,(a + 1)\,d. \\ m - am &= m\,(1 - a). \end{aligned}$$

Substituting these expressions, the reduced expression becomes

$$[b\,(a - c - 3) + m\,(b - 3c)]\,x + [3\,(a + 1)\,d + m\,(1 - a)]\,y.$$

The student should now be able to reverse the process, and reduce this last expression to its original form by the method of § 74.

EXERCISES.

In the following exercises, the coefficients of y, z, and their products are to be aggregated, so that the results shall be expressed as entire functions of x, y, and z, as in § 55.

1. $ax + bx - 3ax + 3bx + 6x - 7x$.

   Ans. $(-2a + 4b - 1)\,x$.

2. $my + py - my - 2py - 3gy$.

3. $mx - ny + px - gy + rx - sy$.

   Ans. $(m + p + r)\,x - (n + g + s)\,y$.

4. $3az - y - 2az + z - az + y$.

5. $abxy - bcxy + bdxy.$
6. $36abxy - 24x - ax - 7xy.$
7. $ay - by - may - nby + 3x.$
8. $amy - bmy + any - bny.$
9. $prz - 2qrz - 4ppz + 8qhz.$
10. $cnx + bnx - amy - 2bny.$

76. An entire function of two quantities can be regarded as an entire function of either of them (§§ 49, 50), and when expressed as a function of one may be transformed into a function of the other.

EXAMPLE. The expression

$$(2a + 3)\,x^3 - (4a^2 - 2a)\,x^2 + (a^2 - 2a + 1)\,x - a^2$$

has the form of an entire function of x. It is required to express it as an entire function of a.

Clearing of parentheses, it becomes

$$2ax^3 + 3x^3 - 4a^2x^2 + 2ax^2 + a^2x - 2ax + x - a^2.$$

Now, collecting the coefficients of a^3, a^2, etc., separately, it becomes

$$(-\,4x^2 + x - 1)\,a^2 + (2x^3 + 2x^2 - 2x)\,a + 3x^3 + x,$$

which is the required form.

EXERCISES.

Express the following as entire functions of y:

1. $(3y^2-4y)\,x^3+(y^3-2y^2+1)\,x^2+(2y^3+5y^2-7)x-y^2-6.$
2. $(y^4 - y^2)\,x^2 + (y^3 - y)\,x + y^2 - 1.$
3. $(y^5 - 2y^3)\,x^3 + (y^4 - 2y^2)\,x^2 + (y^3 - 2y)\,x + y^2 - 2.$
4. $(y^5 + 3y^2)\,x^4 + (y^4 + 3y^3)\,x^3 + (y^3+3y)\,x^2 + (y^2+3)\,x.$

Multiplication of Polynomials by Polynomials.

77. Let us consider the product

$$(a + b)\,(p + q + r).$$

This is of the same form as equation (1) of § 65, $(a + b)$ taking the place of m. Therefore the product just written is equal to

$$(a + b)\,p + (a + b)\,q + (a + b)\,r.$$

But
$$(a + b)p = ap + bp.$$
$$(a + b)q = aq + bq.$$
$$(a + b)r = ar + br.$$

Therefore the product is

$$ap + bp + aq + bq + ar + br.$$

It would have been still shorter to first clear the parentheses from $(a + b)$, putting the product into the form

$$a(p + q + r) + b(p + q + r).$$

Clearing the parentheses again, we should get the same result as before.

We have therefore the following rule for multiplying aggregates:

78. RULE. *Multiply each term of the multiplicand by each term of the multiplier, and add the coefficients with their proper algebraic signs.*

EXERCISES.

1. $(a + b)(2a - bn^2 - 2bn^3)$.
2. $(a - b)(3m + 2n - 5abmn)$.
3. $(m^2 - n^2)(2mn + pm + qn)$.
4. $(p^2 + q^2 + r^2)(pq + qr + rp)$.
5. $(2a - 3b)(2a + 2b)$.
6. $(mx - ny)(mx + ny)$.

79. It is frequently necessary to multiply polynomials containing powers of the same letter. In this case the beginner may find it easier to arrange multiplicand, multiplier, and product under each other, as in arithmetical multiplication.

Ex. 1. Multiply $7x^3 - 6x^2 + 5x - 4$ by $3x^2 - 4x - 5$.

WORK.

$$\begin{array}{rrrrrr}
 & & 7x^3 & -6x^2 & +5x & -4 \\
 & & & 3x^2 & -4x & -5 \\
\hline
21x^5 & -18x^4 & +15x^3 & -12x^2 & & \\
 & -28x^4 & +24x^3 & -20x^2 & +16x & \\
 & & -35x^3 & +30x^2 & -25x & +20 \\
\hline
21x^5 & -46x^4 & +4x^3 & -2x^2 & -9x & +20
\end{array}$$

The first line under the multiplier contains the products of the several terms of the multiplicand by $3x^2$. The second contains the products by $-4x$, and the third by -5. Like terms are placed under each other to facilitate the addition.

Ex. 2. Multiply $m + nx + px^2$ by $a - bx$.

$$\begin{array}{l}
m + nx + px^2 \\
\underline{a - bx \qquad\qquad} \\
am + anx + apx^2 \\
\underline{\qquad - bmx - bnx^2 - bpx^3} \\
am + (an - bm)\,x + (ap - bn)\,x^2 - bpx^3
\end{array}$$

In the following exercises arrange the terms according to the powers and products of the leading letters, a, b, x, y, or z.

Multiply

1. $3a^2 + 5a + 7$ by $2a^2 - 3a + 4$.
2. $a^2 + ab + b^2$ by $a - b$.
3. $a^3 + a^2 + ax^2 + x^3$ by $a - x$.
4. $a^3 - a^2 + a - 1$ by $a^2 - a + 1$.
5. $x^4 + ax^3 + a^2x^2 + a^3x + a^4$ by $x - a$.
6. $a + bz + cz^2 + dz^3$ by $m - nz + pz^2$.
7. $3a^2 + 5a + 7$ by $2a^2 + 3a - 4$.
8. $a^2 - ab + b^2$ by $a + b$.
9. $a^3 + a^2x + ax^2 + x^3$ by $a - x$.
10. $a^3 - a^2 + a - 1$ by $a^2 + a - 1$.
11. $x^4 + ax^3 + a^2x^2 + a^3x + a^4$ by $x + a$.
12. $a + bz + cz^2 + dz^3$ by $m + nz - pz^2$.
13. $(a + bx)\,(m + nx)$.
14. $(a + bx + cx^2)\,(m + nx + px^2)$.
15. $(y^3 - 3y + 2)\,(y^2 - 2)$.
16. $(y^3 + y^2 + y + 1)\,(y^2 + y + 1)$
17. $(y^3 - 2y^2 + 3y - 4)\,(y^3 + 2y^2 + 3y + 4)$.
18. $3a^{2m}x - 3a^2y + 2a^{2n}$ by $a^m - a^n$
19. $a^2 + 6ab + \frac{1}{3}b$ by $a - \frac{1}{3}b$.
20. $(a + b) + (a - b)$ by $(a + b) - (a - b)$.
21. $a^2 - b^2 + (a - b)$ by $a^2 + b^2 + (a + b)$.
22. $a + b + c$ by $a - b + c$.
23. $a^2 + b^2 - (3a^2 + b^2)$ by $2a + 2b - 2\,(a - b)$.
24. $2\,(a - b) + x - y$ by $a + b - (x + y)$.
25. $ax^m + bx^n - abx$ by $ax^2 + bx^3$.
26. $a^m - b^n$ by $a^m + b^n$.

27. $-15x^2y + 3xy^2 - 12y^3$ by $-5xy$.

28. $\frac{2}{3}x^2 + 3ax - \frac{7}{5}a^2$ by $2x^3 - ax - \frac{1}{4}a^2$.

NOTE. Aggregates entering into either factor should be simplified before multiplying.

Special Forms of Multiplication.

80. 1. To find the square of a binomial, as $a + b$. We multiply $a + b$ by $a + b$.

$$\begin{aligned} a(a+b) &= a^2 + ab \\ b(a+b) &= \underline{\phantom{a^2 +{}} ab + b^2} \\ (a+b)(a+b) &= a^2 + 2ab + b^2 \end{aligned}$$

Hence, $$(a+b)^2 = a^2 + 2ab + b^2 \qquad (1)$$

2. We find, in the same way,

$$(a-b)^2 = a^2 - 2ab + b^2. \qquad (2)$$

These forms may be expressed in words thus:

Theorem. The square of a binomial is equal to the sum of the squares of its two terms, plus or minus twice their product.

3. To find the product of $a + b$ by $a - b$.

$$\begin{aligned} a(a+b) &= a^2 + ab \\ -b(a+b) &= \underline{ - ab - b^2} \end{aligned}$$

Adding, $$(a+b)(a-b) = a^2 - b^2. \qquad (3)$$

That is:

Theorem. The product of the sum and difference of two numbers is equal to the difference of their squares.

The forms (1), (2), and (3) should be memorized by the student, owing to their constant occurrence.

When $b = 1$, the form (3) becomes

$$(a+1)(a-1) = a^2 - 1.$$

The student should test these formulæ by examples like the following:

$$(9+4)^2 = 9^2 + 2\cdot9\cdot4 + 4^2 = 81 + 72 + 16 = 169.$$
$$(9-4)^2 = 9^2 - 2\cdot9\cdot4 + 4^2 = 81 - 72 + 16 = 25.$$

$(9+4)(9-4) = 9^2 - 4^2 = 65.$

Prove these three equations by computing the left-hand member directly.

EXERCISES.

Write on sight the values of

1. $(m+2n)^2$.
2. $(m-2n)^2$.
3. $(3a-2b)^2$.
4. $(4x-5y)^2$.
5. $(2x+y)(2x-y)$.
6. $(3x+1)(3x-1)$.
7. $(4x^2+1)(4x^2-1)$.
8. $(5x^3-3)(5x^3+3)$.

81. Because the product of two negative factors is positive, it follows that the square of a negative quantity is positive.

EXAMPLES. $(-a)^2 = a^2 = (+a)^2$.

$$(b-a)^2 = a^2 - 2ab + b^2 = (a-b)^2.$$

Hence,

The expression $a^2 - 2ab + b^2$ *is the square both of* $a-b$ *and of* $b-a$.

82. We have $\quad -a \times a = -a^2$.

Hence,

The product of equal factors with opposite signs is a negative square.

EXAMPLE. $\quad -(a-b)(a-b) = -a^2 + 2ab - b^2$,

which is the negative of (2). Because $-(a-b) = b-a$, this equation may be written in the form,

$$(b-a)(a-b) = -a^2 + 2ab - b^2,$$

which is readily obtained by direct multiplication.

EXERCISES.

Write on sight the values of

1. $-(a+b) \times -(a+b)$.
2. $(x-y)(y-x)$.
3. $(x+y)(-x-y)$.
4. $(2a-3b)(3b-2a)$.
5. $(3b-2a)(-3b+2a)$.
6. $(am-bn)(bn-am)$.
7. $(xy-2)(2-xy)$.

CHAPTER III.

DIVISION.

83. The problem of algebraic division is to find such an expression that, when multiplied by the divisor, the product shall be the dividend.

This expression is called the quotient.

In Algebra, the quotient of two quantities may always be indicated by a fraction, of which the numerator is the dividend and the denominator the divisor.

Sometimes the numerator cannot be exactly divided by the denominator. The expression must then be treated as a fraction, by methods to be explained in the next chapter.

Sometimes the divisor will exactly divide the dividend. Such cases form the subject of the present chapter.

Division of Monomials by Monomials.

84. In order that a dividend may be exactly divisible by a divisor, it is necessary that it shall contain the divisor as a factor.

Ex. 1. 15 is exactly divisible by 3, because $3 \cdot 5 = 15$.

2. The product ab^2c is exactly divisible by ac, because ac is a factor of it.

To divide one expression by another which is an exact divisor of it:

Rule. *Remove from the dividend those factors the product of which is equal to the divisor. The remaining factors will be the quotient.*

85. *Rule of Exponents.* If both dividend and divisor contain the same symbol, with different exponents, say m and n, then, because the dividend contains this symbol m times as a factor, and the divisor n times, the quotient will contain it $m - n$ times. Hence, *In dividing, exponents of like symbols are to be subtracted.*

EXERCISES.

1. Divide $26xy$ by $2y$. *Ans.* $13x$.
2. Divide $21a^2bc$ by $7bc$.
3. Divide x^3 by x^2. *Ans.* x.
4. Divide $18a^2$ by $6a$. *Ans.* $3a$.
5. Divide $15a^2m$ by $3a$. *Ans.* $5am$.
6. Divide $14a^3m^2$ by $7am$.
7. Divide $16a^5m^4$ by $8a^3m^2$.
8. Divide $36xy^2z^3$ by $6xyz$.
9. Divide $40a^2x^3z^5$ by $10a^2xz^4$.
10. Divide $35ab^5$ by $7ab$.

Rule of Signs in Division.

86. The rule of signs in division corresponds to that in multiplication, namely:

If dividend and divisor have the same sign, the quotient is positive.

If they have opposite signs, the quotient is negative.

Proof.

$$+mx \div (+m) = +x, \text{ because } +x \times (+m) = +mx.$$
$$+mx \div (-m) = -x, \quad \text{"} \quad -x \times (-m) = +mx.$$
$$-mx \div (+m) = -x, \quad \text{"} \quad -x \times (+m) = -mx.$$
$$-mx \div (-m) = +x, \quad \text{"} \quad +x \times (-m) = -mx.$$

The condition to be fulfilled in all four of these cases is that the product, *quotient* $\times$ *divisor*, shall have the same algebraic sign as the dividend.

EXERCISES.

Divide

1. $+a$ by $+a$. *Ans.* $+1$.
2. $+a$ by $-a$. *Ans.* -1.
3. $-a$ by $+a$. *Ans.* -1.
4. $-a$ by $-a$. *Ans.* $+1$.
5. $-33a^2mx$ by $11ax$. *Ans.* $-3am$.
6. $-24x^2yz$ by $12xyz$. *Ans.* $-2x$.
7. $21am^2x^m$ by $-7amx^n$. *Ans.* $-3mx^{m-n}$.

8. $-18a^m p^n$ by $-6a^n p$. *Ans.* $3a^{m-n} p^{n-1}$.
9. $-16a^2 x^m y^n$ by $4ax^2 y^n$.
10. $14b^s p^t$ by $-7b^r p^q$.
11. $-12b^m t^n k^n$ by $-4b^n t^n k^n$.
12. $12(a-b)^3 c^4$ by $3(a-b)^2 c$. *Ans.* $4(a-b)c^3$.
13. $42(x-y)^m$ by $-7(x-y)^n$.
14. $-44a^s(x-y)^t$ by $11a^t(x-y)^t$.
15. $-45b^m(a-b)^n$ by $9b^n(a-b)^s$.
16. $-48(m+n)^p$ by $-8(m+n)^q$.
17. $64(a+b)^n(x-y)^m$ by $4(a+b)(x-y)$.

Division of Polynomials by Monomials.

87. By the distributive law in multiplication, whatever quantities the symbols m, a, b, c, etc., may represent, we have:

$$(a + b + c + \text{etc.}) \times m = ma + mb + mc + \text{etc.}$$

Therefore, by the condition of division,

$$(ma + mb + mc + \text{etc.}) \div m = a + b + c + \text{etc.}$$

We therefore conclude,

1. In order that a polynomial may be exactly divisible by a monomial, each of its terms must be so divisible.

2. The quotient will be the algebraic sum of the separate quotients found by dividing the different terms of the polynomial.

EXERCISES.

Divide

1. $2a^2 + 6a^3x - 8a^5x^2$ by $2a^2$. *Ans.* $1 + 3ax - 4a^3x^2$.
2. $6m^2n - 12m^3n^2 - 18mn^5$ by $6mn$.
3. $8a^3b^5 - 16a^4b^4 + 8a^5b^3$ by $4a^3b^3$.
4. $4xy^5 - 8x^3y^3 + 4x^5y$ by $-4xy$.
5. $12abx - 24abx^2$ by $-12abx$.
6. $21am^2x^m - 14a^2m^4x^n + 28a^5m^2x^p$ by $-7amx^n$.
7. $72a^3x + 24ax + 48ax^2$ by $24ax$.
8. $a(b-c) + b(c-a) + c(a-b) + abc$ by abc.
9. $27(a-b)^5 - 18(a-b)^3 + 9(a-b)^2$ by $9(a-b)$.
10. $a^m(a-b)^n - a^n(a-b)^m$ by $a^n(a-b)^n$.

11. $(a+b)^p(a-b)^q + (a+b)^q(a-b)^p$ by $(a+b)(a-b)$.
12. $10(x+y)^m(x-y)^n - 5(x+y)^p(x-y)^q$ by $5(x+y)(x-y)$.
13. $(a+b)(a-b)$ by $a^2 - b^2$.

Factors and Multiples.

88. As in Arithmetic some numbers are composite and others prime, so in Algebra some expressions admit of being divided into algebraic factors, while others do not. The latter are by analogy called **Prime** and the former **Composite**.

A single symbol, as a or x, is necessarily prime.

A product of several symbols is of course composite, and can be divided into factors at sight.

A binomial or polynomial is sometimes prime and sometimes composite, but no universal rule can be given for distinguishing the two cases.

89. When the same symbol or expression is a factor of all the terms of a polynomial, the latter is divisible by it.

EXAMPLES.

1. $ax + abx^2 + a^2cx^3 = a(x + bx^2 + acx^3)$.
2. $a^2b^3x + a^3b^2x^2 = a^2b^2x(b + ax)$.
3. $a^{2n} + a^nx^n = a^n(a^n + x^n)$.

EXERCISES.

Factor

1. $ax^2 + a^2x$.
2. $a^3b^2cy + a^2bc^3y + ab^3c^2y$.
3. $a^{2n}b^n + a^nb^{2n}$.
4. $a^{3n}x^n - a^{2n}x^{2n} + a^nx^{3n}$.
5. $a^nb^{2n}c^{3n} + a^{2n}b^{3n}c^n + a^{3n}b^nc^{2n}$.

90. There are certain forms of composite expressions which should be memorized, so as to be easily recognized. The following are the inverse of those derived in § 80.

1. $a^2 + 2ab + b^2 = (a+b)^2$.
2. $a^2 - 2ab + b^2 = (a-b)^2$.
3. $a^2 - b^2 = (a+b)(a-b)$.

The form (3) can be applied to any difference of even powers; thus,

$$a^4 - b^4 = (a^2 + b^2)(a^2 - b^2);$$
$$a^6 - b^6 = (a^3 + b^3)(a^3 - b^3);$$

and, in general, $a^{2n} - b^{2n} = (a^n + b^n)(a^n - b^n)$.

If the exponent is a multiple of 4, the second factor can be again divided.

EXAMPLES.

$$a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (a^2+b^2)(a+b)(a-b).$$
$$a^8 - b^8 = (a^4 + b^4)(a^4 - b^4) = (a^4+b^4)(a^2+b^2)(a+b)(a-b).$$

When b is equal to 1 or 2, the forms become

$$a^2 - 1 = (a + 1)(a - 1).$$
$$a^2 - 4 = (a + 2)(a - 2).$$
$$a^2 + 2a + 1 = (a + 1)^2.$$
$$a^2 + 4a + 4 = (a + 2)^2.$$
$$a^2 - 2a + 1 = (a - 1)^2 = (1 - a)^2.$$
$$a^2 - 4a + 4 = (a - 2)^2 = (2 - a)^2.$$

By putting $2b$ for b, they give

$$a^2 - 4b^2 = (a + 2b)(a - 2b).$$
$$a^2 + 4ab + 4b^2 = (a + 2b)^2.$$

EXERCISES.

Divide the following expressions into as many factors as possible:

1. $x^4 - 16$. *Ans.* $(x^2 + 4)(x + 2)(x - 2)$.
2. $y^4 - 16x^4$.
3. $x^2 + 6x + 9$. *Ans.* $(x + 3)^2$.
4. $x^2 - 6x + 9$.
5. $4a^2x^2 - 9b^2y^2$.
6. $16a^4x^4 - 1$.
7. $9x^2 - 12xy + 4y^2$.
8. $a^2x^2 + 2axy + y^2$.
9. $4a^2x^2 + 4abxy + b^2y^2$.
10. $a^4 + 4a^2b^2 + 4b^2$.
11. $x^4 - 2x^2y^2 + y^4$.
12. $x^4 - 4x^2y^2 + 4y^4$.
13. $a^4 - 4a^2b^2 + 4b^4$.
14. $a^4 - a^2b^2$.
15. $a^{2n} - 2a^n + 1$.
16. $x^{2n} - 4ax^n + 4a^2$.
17. $1 - y^4$.
18. $x^6z + 2x^3y^3z + y^6z$.

Ans. $z(x^6 + 2x^3y^3 + y^6) = z(x^3 + y^3)^2$.

19. $a^3 - 4a^2b + 4ab^2$.
20. $a^{2m} - b^{4n}$.
21. $25x^4 - 40x^3y + 16x^2y^2$.
22. $4x^4y^4 - 9x^2y^2$.
23. $4x^4y^4 - 12x^3y^2 + 9x^2$.
24. $x^8 - x^2y^6$.
25. $x^{4m} - 2x^{2m}y^n + y^{2n}$.
26. $x^{4m} - 2x^{2m} + 1$.
27. $x^2 + x + \frac{1}{4}$.
28. $x^{2m} + x^m + \frac{1}{4}$.

91. By combining the preceding forms, yet other forms may be found.

For example, the factors

$$(a^2 + ab + b^2)(a^2 - ab + b^2), \qquad (1)$$

are respectively the sum and difference of the quantities

$$a^2 + b^2 \quad \text{and} \quad ab.$$

Hence the product (1) is equal to the difference of the squares of these quantities, or to

$$(a^2 + b^2)^2 - a^2b^2 = a^4 + a^2b^2 + b^4.$$

Hence the latter quantity can be factored as follows:

$$a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2).$$

EXERCISES.

Factor

1. $x^4 + x^2y^2 + y^4$.
2. $a^4 + 8a^2b^2 + 16b^4$.
3. $a^4 + 9a^2x^2 + 81x^4$.
4. $a^{4n} + a^{2n}b^{2n} + b^{4n}$.
5. $a^4x^2 + 4a^2b^2x^2 + 16b^4x^2$.
6. $a^6 + 8a^4b^2 + 16a^2b^4$.
7. $x^{5n} + x^{3n}y^{2n} + x^ny^{4n}$.
8. $m^2 - a^2 + 2ab - b^2$. *Ans.* $(m - a + b)(m + a - b)$.

Here the last three terms are a negative square. Compare § 82.

9. $a^2 - 4b^2 + 4bc - c^2$.
10. $a^3 - 4ab^2 + 4abc - ac^2$.

92. The following expression occurs in investigating the area of a triangle of which the sides are given:

$$(a + b + c)(a + b - c)(a - b + c)(a - b - c). \qquad (1)$$

By § 80, 3, the product of the first pair of factors is

$$(a + b)^2 - c^2 = a^2 + 2ab + b^2 - c^2;$$

and that of the second pair,

$$(a - b)^2 - c^2 = a^2 - 2ab + b^2 - c^2.$$

By the same principle, the product of these products is

$$(a^2 + b^2 - c^2)^2 - 4a^2b^2,$$

which we readily find to be

$$a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2. \qquad (2)$$

Hence this expression (2) can be divided into the four factors (1).

Factors of Binomials.

93. Let us multiply

$$x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1} \text{ by } x - a.$$

OPERATION.

$$\begin{array}{lllllll}
x^{n-1} & + ax^{n-2} & + a^2x^{n-3} & + a^3x^{n-4} & + \dots & + a^{n-2}x & + a^{n-1} \\
x - a & & & & & & \\
\hline
x^n & + ax^{n-1} & + a^2x^{n-2} & + a^3x^{n-3} & + \dots & + a^{n-1}x & \\
 & - ax^{n-1} & - a^2x^{n-2} & - a^3x^{n-3} & - \dots & - a^{n-1}x & - a^n \\
\hline
\text{Prod., } x^n & 0 & 0 & 0 & & 0 & - a^n
\end{array}$$

The intermediate terms all cancel each other in the product, leaving only the two extreme terms.

The product of the multiplicand by $x - a$ is therefore $x^n - a^n$. Hence, if we divide $x^n - a^n$ by $x - a$, the quotient will be the above expression. Hence the binomial $x^n - a^n$ may be factored as follows:

$$x^n - a^n = (x - a)\,(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1}).$$

Therefore we have,

Theorem. The difference of any power of two numbers is divisible by the difference of the numbers themselves.

ILLUSTRATION. The difference between any power of 7 and the same power of 2 is divisible by $7 - 2 = 5$. For instance,

$$\begin{aligned}
7^2 - 2^2 &= 45 = 5.9. \\
7^3 - 2^3 &= 335 = 5.67. \\
7^4 - 2^4 &= 2385 = 5.477. \\
\text{etc.} \quad & \text{etc.} \quad \text{etc}
\end{aligned}$$

94. Let us multiply

$$x^{n-1} - ax^{n-2} + a^2x^{n-3} - \ldots . + (-a)^{n-2}x + (-a)^{n-1}$$

by $x + a = x - (-a)$.

REM. This expression is exactly like the preceding, except that $-a$ is substituted for a. It will be noticed that the coefficients of the powers of x in the multiplicand are the powers of $-a$, because

$$\begin{aligned} (-a)^1 &= -a, \\ (-a)^2 &= +a^2, \\ (-a)^3 &= -a^3, \\ (-a)^4 &= +a^4, \\ \text{etc.} & \quad \text{etc.} \end{aligned}$$

The sign of the last term will be positive or negative, according as $n - 1$ is an even or odd number.

OPERATION.

$$\begin{array}{llllll} & x^{n-1} - ax^{n-2} + a^2x^{n-3} - a^3x^{n-4} + \ldots . + (-a)^{n-2}x + (-a)^{n-1} \\ & \underline{x + a = x - (-a)} \\ & x^n - ax^{n-1} + a^2x^{n-2} - a^3x^{n-3} \ldots + (-a)^{n-1} \\ & \quad + ax^{n-1} - a^2x^{n-2} + a^3x^{n-3} \ldots . - (-a)^{n-1}x - (-a)^n \\ \hline \text{Prod.,} & x^n \quad 0 \quad 0 \quad 0 \quad 0 \quad - (-a)^n \end{array}$$

The multiplier $x + a$ is the same as $x - (-a)$ (§ 59). In multiplying the first terms, we use $+a$, and in the last ones $-(-a)$, because the latter shows the form better.

Hence, reasoning as in (1), the expression $x^n - (-a)^n$ admits of being factored thus:

$$x^n - (-a)^n = (x + a)\,[x^{n-1} - ax^{n-2} + a^2x^{n-3} - \ldots . + (-a)^{n-2}x + (-a)^{n-1}].$$

If n is an even number, then $(-a)^n = a^n$, and

$$x^n - (-a)^n = x^n - a^n.$$

If n is an odd number, then $(-a^n) = -a^n$, and

$$x^n - (-a)^n = x^n + a^n.$$

Therefore,

Theorem 1. When n is odd, the binomial $x^n + a^n$ is divisible by $x + a$.

Theorem 2. When n is even, the binomial $x^n - a^n$ is divisible by $x+a$.

NOTE. These theorems could have been deduced immediately from that of § 93, by changing a into $-a$, because $x-a$ would then have been changed to $x+a$, and $x^n - a^n$ to $x^n + a^n$ or $x^n - a^n$, according as n was odd or even.

The forms of the factors in the two cases are:

When n is odd,

$$x^n + a^n = (x+a)(x^{n-1} - ax^{n-2} + a^2x^{n-3} - \dots + a^{n-1}).$$

When n is even,

$$x^n - a^n = (x+a)(x^{n-1} - ax^{n-2} + a^2x^{n-3} - \dots - a^{n-1}). \qquad (a)$$

In the latter case, the last factor can still be divided, because $x^n - a^n$ is divisible by $x-a$ as well as by $x+a$. We find, by multiplication,

$$(x-a)(x^{n-2} + a^2x^{n-4} + a^4x^{n-6} + \dots + a^{n-2})$$
$$= x^{n-1} - ax^{n-2} + a^2x^{n-3} - a^3x^{n-4} + \dots + a^{n-2}x - a^{n-1}.$$

Therefore, from the last equation (a) we have:

When n is even,

$$x^n - a^n = (x+a)(x-a)(x^{n-2} + a^2x^{n-4} + a^4x^{n-6} - \dots + a^{n-2}).$$

EXERCISES.

Factor the following expressions, and when they are purely numerical, prove the results.

1. $5^2 - 2^2$. *Ans.* $(5+2)(5-2)$.

[*Proof.* $5^2 - 2^2 = 25 - 4 = 21$;
$(5+2)(5-2) = 7 \cdot 3 = 21$.]

2. $5^3 - 2^3$.
3. $5^4 - 2^4$.
4. $5^5 - 2^5$.
5. $5^6 - 2^6$.
6. $7^3 + 2^3$.
7. $7^3 - 2^3$.
8. $7^4 - 2^4$.
9. $x^2 - a^2$.
10. $x^3 - a^3$.
11. $x^4 - a^4$.
12. $x^5 - a^5$.
13. $x^3 + a^3$.
14. $x^5 + a^5$.
15. $a^3 - 8b^3$.
16. $8a^3 - 27b^3$.
17. $16a^4 - b^4$.
18. $x^3 + 8y^3$.
19. $x^4 - 16y^4$.
20. $8a^3 + 27b^3$.
21. $x^6 - 64a^6$.

Least Common Multiple.

95. *Def.* A **Common Multiple** of several quantities is any expression of which all the quantities are factors.

EXAMPLE. The expression am^2n^3 is a common multiple of the quantities a, m, n, am, amn, am^2, m^2n^3, etc., and finally of the expression itself, am^2n^3. But it is not a multiple of a^2, nor of x, nor of any other symbol which does not enter into it as a factor.

Def. The **Least Common Multiple** of several quantities is the common multiple which is of lowest degree. It is written for shortness L. C. M.

RULE FOR FINDING THE L. C. M. *Factor the several quantities as far as possible.*

If the quantities have no common factor, the least common multiple is their product.

If several of the quantities have a common factor, the multiple required is the product of all the factors, each of them being raised to the highest power which it has in any of the given quantities.

Ex. 1. Let the given quantities be

$$2ab, \qquad 3b^3c, \qquad 6ac.$$

The factors are 2, 3, a, b, and c. The highest power of b is b^3, while a and c only enter to the first power. Hence,

$$\text{L. C. M.} = 6ab^3c.$$

Ex. 2. $a^2 - b^2$, $a^2 + 2ab + b^2$, $a^2 - 2ab + b^2$, $a^4 - b^4$.

Factoring, we find the expressions to be,

$$(a + b)(a - b), \quad (a + b)^2, \quad (a - b)^2, \quad (a^2 + b^2)(a + b)(a - b).$$

By the rule, the L. C. M. required is

$$(a + b)^2 (a - b)^2 (a^2 + b^2).$$

EXERCISES.

Find the L. C. M. of

1. $xy,\ xz,\ yz.$
2. $a^2b,\ b^2c,\ c^2d,\ d^2a.$
3. $a,\ ab,\ abc,\ abcd.$
4. $a^2,\ ab^3,\ bc^4.$
5. $x^2 - y^2,\ x + y,\ x - y.$
6. $x^4 - 4,\ x^2 - 4x + 4,\ x^2 + 4x + 4.$
7. $16a^2x^2 - 4m^2,\ 2ax + m,\ 2ax - m.$
8. $x^2 - 1,\ x^2 + 1,\ x^2 - 2x + 1,\ x^2 + 2x + 1.$
9. $4a\,(b + c),\ b\,(a - c),\ 2ab.$
10. $2\,(a - b)^2,\ 2\,(a + b)^2,\ 2\,(a - b)\,(a + b).$
11. $3\,(x + y),\ 3\,(x - y),\ 3\,(x^3 + y^3).$
12. $a - b,\ a^2 - b^2,\ a^3 - b^3,\ a^4 - b^4.$
13. $x + y,\ x - y,\ a + b,\ a - b.$
14. $x^4 - a^4,\ x^3 + a^3,\ x^2 - a^2,\ x + a.$
15. $x^6 - 64a^6,\ x^4 - 16a^4,\ x^2 - 4a^2.$
16. $a + b,\ a^2 + 2ab + b^2,\ a^4 - b^4.$

Division of one Polynomial by another.

If the dividend and divisor are both polynomials, and entire functions of the same symbol, and if the degree of the numerator is not less than that of the denominator, a division may be performed and a remainder obtained. The method of dividing is similar to long division in Arithmetic.

96. Case I. *When there is only one algebraic symbol in the dividend and divisor.*

Let us perform the division,

$$3x^4 - 4x^3 + 2x^2 + 3x - 1 \div x^2 - x + 1.$$

We first find the quotient of the highest term of the divisor x^2, into the highest term of the dividend $3x^4$, multiply the whole divisor by the quotient $3x^2$, and subtract the product from the dividend. We repeat the process on the remainder, and continue doing so until the remainder has no power of x so high as the highest term of the diviser. The work is most conveniently arranged as follows:

$$
\begin{array}{lr|ll}
 & \textbf{Dividend.} & \textbf{Divisor.} & \\
 & 3x^4 - 4x^3 + 2x^2 + 3x - 1 & x^2 - x + 1 & \\
\hline
3x^2 \times \text{Divisor,} & \underline{3x^4 - 3x^3 + 3x^2 \qquad\qquad} & 3x^2 - x - 2 & \text{Quotient.} \\
\text{First Remainder,} & - x^3 - x^2 + 3x - 1 & & \\
-x \times \text{Divisor,} & \underline{- x^3 + x^2 - x \quad} & & \\
\text{Second Remainder,} & - 2x^2 + 4x - 1 & & \\
-2 \times \text{Divisor,} & \underline{- 2x^2 + 2x - 2} & & \\
\text{Third and last Remainder,} & 2x + 1 & &
\end{array}
$$

The division can be carried no farther without fractions, because x^2 will not go into x. We now apply the same rule as in Arithmetic, by adding to the quotient a fraction of which the numerator is the remainder and the denominator the divisor. The result is,

$$\frac{3x^4 - 4x^3 + 2x^2 + 3x - 1}{x^2 - x + 1} = 3x^2 - x - 2 + \frac{2x + 1}{x^2 - x + 1}. \qquad (a)$$

This result may now be proved by multiplying the quotient by the divisor and adding the remainder.

There is an analogy between the result (a) and the corresponding one of Arithmetic. An algebraic fraction like (a), in which the degree of the numerator is greater than that of the denominator may be called an *improper fraction.* As in Arithmetic an improper fraction may be reduced to an *entire number* plus a proper fraction, so in Algebra an improper fraction may be reduced to an *entire function* of a symbol plus a proper fraction.

EXERCISES.

Execute the following divisions, and reduce the quotients to the form (a) when there is any remainder.

1. Divide $x^2 - 2x - 1$ by $x + 1$.
2. Divide $x^3 + 2x^2 - 2x - 1$ by $x - 1$.
3. Divide $x^3 - 3x^2 + 2x - 1$ by $x^2 - x$.
4. Reduce $\dfrac{2x^4 - 2x^3 + x^2 - x - 5}{x^2 - x - 1}$.
5. Divide $24a^3 - 38a^2 - 32a + 50$ by $2a - 3$.

Ans. Quot. $= 12a^2 - a - \dfrac{35}{2}$; Rem. $= -\dfrac{5}{2}$.

6. Divide $x^4 - 1$ by $x - 1$.

When terms are wanting in the dividend, they may be considered as zero. In this last exercise, the terms in x^3, x^2, and x are wanting. But the beginner may write the dividend and perform the operation thus:

$$
\begin{array}{l|l}
x^4 + 0x^3 + 0x^2 + 0x - 1 & x - 1 \\
\hline
x^4 - x^3 & x^3 + x^2 + x + 1 \\
\quad x^3 + 0x^2 & \\
\quad x^3 - x^2 & \\
\qquad x^2 + 0x & \\
\qquad x^2 - x & \\
\qquad\quad x - 1 & \\
\qquad\quad x - 1 & \\
\qquad\quad 0 \quad 0 &
\end{array}
$$

The operation is thus assimilated to that in which the expression is complete; but the actual writing of the zero terms in this way is unnecessary, and should be dispensed with as soon as the student is able to do it.

7. Divide $a^3 - 2a + 1$ by $a - 1$.
8. Divide $x^2 + 1$ by $x + 1$.
9. Divide $8a^3 + 125$ by $2a + 5$.
10. Divide $a^5 + 1$ by $a + 1$.
11. Divide $a^4 + 2a^2 + 9$ by $a^2 + 2a + 3$.
12. Divide $a^6 - 1$ by $a^3 + 2a^2 + 2a + 1$.
13. Divide $x^6 - 12x^4 + 36x^2 - 32$ by $x^2 - 2$.
14. Divide $(x^3 - 2x + 1)(x^3 - 12x - 16)$ by $x^2 - 16$.

For some purposes, we may equally well perform the operation by beginning with the term containing the lowest power of the quantity, or not containing it at all. Take, for instance, Example 9:

$$
\begin{array}{l|l}
125 + 8a^3 & 5 + 2a \\
\hline
125 + 50a & 25 - 10a + 4a^2 \\
\quad - 50a & \\
\quad - 50a - 20a^2 & \\
\qquad\quad 20a^2 + 8a^3 & \\
\qquad\quad 20a^2 + 8a^3 &
\end{array}
$$

15. Divide $1 + 3x + 3x^2 + x^3$ by $1 + x$.
16. Divide $1 - 4x + 4x^2 - x^3$ by $1 - x$.
17. Divide $15 + 2a - 3a^2 + a^3 + 2a^4 - a^5$ by $5 + 4a - a^3$.
18. Divide $1 - y^6$ by $1 + 2y + 2y^2 + y^3$.
19. Divide $64 - 64x + 16x^2 - 8x^3 + 4x^4 - x^6$ by $-4 + 2x + x^2$.
20. Divide $64 - 16x^2 + x^6$ by $4 - 4x + x^2$.

97. CASE II. *When there are several algebraic symbols in the divisor and dividend.*

Let us suppose the dividend and divisor arranged according to powers of some one of the symbols, which we may suppose to be x, as in § 76.

Let us call A the coefficient of the highest power of x in the dividend, and H the term independent of x, so that the dividend is of the form

$$Ax^n + (\text{terms with lower powers of } x) + H.$$

Let us call a the coefficient of the highest power of x in the divisor, and h the term of the divisor independent of x, so that the divisor is of the form

$$ax^m + (\text{terms with lower powers of } x) + h.$$

Then we have the following

Theorem. In order that the dividend may be exactly divisible by the divisor, it is necessary:

1. That the term containing the highest power of x in the dividend shall be exactly divisible by the corresponding term of the divisor.

2. That the term independent of x in the dividend shall be exactly divisible by the corresponding term of the divisor.

Reason. The reason of this theorem is that if we suppose the quotient also arranged according to the powers of x, then,

1. The highest term of the dividend, Ax^n, will be given by multiplying the highest term of the divisor, ax^m, by the highest term of the quotient. Hence we must have,

$$\text{Highest term of quotient} = \frac{Ax^n}{ax^m}.$$

2. The lowest term of the dividend will be given by multiplying the lowest term of the dividend by the lowest term of the quotient. Hence, we must have,

$$\text{Lowest term of quotient} = \frac{H}{h}.$$

REM. 1. Since we may arrange the dividend and divisor according to the powers of any one of the symbols, the above theorem must be true whatever symbol we take in the place of x.

REM. 2. It does not follow that when the conditions of the theorem are fulfilled, the division can always be performed. This question can be decided only by trial.

We now reach the following rule:

I. *Arrange both dividend and divisor according to the ascending or descending powers of some common symbol.*

II. *Form the first term of the quotient by dividing the first term of the dividend by the first term of the divisor.*

III. *Multiply the whole divisor by the term thus found, and subtract the product from the dividend.*

IV. *Treat the remainder as a new dividend in the same way, and repeat the process until a remainder is found which is not divisible by the quotient.*

Ex. 1. Divide $x^3 + 3ax^2 + 3a^2x + a^3$ by $x + a$.

OPERATION.

$$
\begin{array}{rrrr|l}
x^3 + 3ax^2 & + 3a^2x & + a^3 & & x + a \\
\underline{x^3 + ax^2} & & & & \overline{x^2 + 2ax + a^2} \\
2ax^2 & + 3a^2x & & & \\
\underline{2ax^2} & \underline{+ 2a^2x} & & & \\
& a^2x & + a^3 & & \\
& \underline{a^2x} & \underline{+ a^3} & & \\
& 0 & 0 & &
\end{array}
$$

Ex. 2. Divide $x^3 - ax^2 + a(b+c)x - abc - bx^2 - cx^2 + bcx$ by $x - a$.

Arranging according to § 76, we have the dividend as follows:

$$
\begin{array}{llll|l}
x^3 - (a+b+c)x^2 & + (ab+bc+ca)x & - abc & & x - a \\
\underline{x^3 - ax^2} & & & & \overline{x^2 - (b+c)x + bc} \\
 - (b+c)x^2 & + (ab+bc+ca)x & & & \\
\underline{ - (b+c)x^2} & \underline{+ (ab+ac)x} & & & \\
& bcx & - abc & & \\
& \underline{bcx} & \underline{- abc} & & \\
& 0 & 0 & &
\end{array}
$$

EXERCISES.

1. Divide the dividend of Ex. 2 above by $x - b$.
2. Divide the dividend of Ex. 2 above by $x - c$.
3. Divide $a^3 + b^3 - c^3 + 3abc$ by $a + b - c$.
4. Divide $a^3 + b^3 + 3ab - 1$ by $a + b - 1$.
5. Divide $a^2b^2 + 2abx^2 - (a^2 + b^2)x^2$ by $ab + (a - b)x$.
6. Divide $(a^2 - bc)^3 + 8b^3c^3$ by $a^2 + bc$.
7. Divide $(a + b + c)(ab + bc + ca) - abc$ by $a + b$.
8. Divide $(a + b - c)(b + c - a)(c + a - b)$ by $a^2 - b^2 - c^2 + 2bc$.
9. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.
10. Divide $x^4 + 4a^4$ by $x^2 - 2ax + 2a^2$.
11. Divide $a^2(b + x) - b^2(x - a) + (a - b)x^2 + abx$ by $x + a + b$.
12. Divide $x^3 - ax^2 - b^2x + ab^2$ by $(x - a)(x + b)$.
13. Divide $12a^4x^9 - 14a^5x^6 + 6a^6x^3 - a^7$ by $2a^2x^3 - a^3$.

CHAPTER IV.

OF ALGEBRAIC FRACTIONS.

98. *Def.* An **Algebraic Fraction** is the expression of an indicated quotient when the divisor will not exactly divide the dividend.

EXAMPLE. The quotient of $p \div q$ is the fraction $\frac{p}{q}$.

Def. The numerator and denominator of a fraction are called its two **Terms**.

Transformation of Single Fractions.

99. *Reduction to Lowest Terms.* If the two terms of a fraction are multiplied or divided by the same quantity, the value of the fraction will not be altered.

EXAMPLE. Consider the fraction $\frac{ax}{ay}$. If we divide both terms by a, the fraction will become $\frac{x}{y}$.

$$\frac{ax}{ay} = \frac{x}{y}.$$

Corollary. If the numerator and denominator contain common factors, they may be cancelled.

Def. When all the factors common to the two terms of a fraction are cancelled, the fraction is said to be reduced to its **Lowest Terms.**

To reduce a fraction to its lowest terms, factor both terms, when necessary, and cancel all the common factors.

Ex. 1. $\frac{abxy^2}{acny^2} = \frac{bx}{cn}$.

The factor ay^2 common to both terms is cancelled.

Ex. 2. $\frac{a^7b^2}{a^2b^7} = \frac{a^5}{b^5}$.

The factor a^2b^2 common to both terms is cancelled.

Ex. 3. Reduce $\frac{a^5x}{a^7x}$.

Here a^5x is a divisor of both terms of the fraction. Dividing by it, the result is $\frac{1}{a^2}$. Hence $\frac{a^5x}{a^7x} = \frac{1}{a^2}$.

Ex. 4. $\frac{a^2 + 2ab + b^2}{a^2 - b^2} = \frac{(a+b)^2}{(a+b)(a-b)} = \frac{a+b}{a-b}$.

Ex. 5. $\frac{mu - nu}{mx - nx} = \frac{(m-n)u}{(m-n)x} = \frac{u}{x}$.

EXERCISES.

Reduce the following fractions to their lowest terms:

1. $\frac{a^5b^2p^2}{a^2b^4p}$.
2. $\frac{am}{a^2mx}$.
3. $\frac{10pqr^2}{12p^2r^4}$.
4. $\frac{12axy}{15a^2x^2y^2}$.

5. $\dfrac{72\,(a-x)\,(b-c)}{36\,(a^2-2ax+x^2)}$.

6. $\dfrac{20\,(a+x)\,(m-n)}{24\,(a^2-2ax+x^2)(m-n)}$.

7. $\dfrac{ay-by}{ax-bx}$.

8. $\dfrac{a^2y^2-b^2y^2}{ay-by}$.

9. $\dfrac{a^2-b^2}{a^2-2ab+b^2}$.

10. $\dfrac{a^2+4ax+4x^2}{a^2-4x^2}$.

11. $\dfrac{x^3+y^3}{a\,(x+y)}$.

12. $\dfrac{a^3+8b^3}{ay+2by}$.

13. $\dfrac{a^4-b^4}{a^2-b^2}$.

14. $\dfrac{a^2+ab+b^2}{a^4+a^2b^2+b^4}$.

15. $\dfrac{x^2-y^2}{x^5-y^5}$.

16. $\dfrac{x^{2n}-y^{2n}}{x^n+y^n}$.

17. $\dfrac{axm-axn}{bym-byn}$.

18. $\dfrac{mx-nx}{(a+b)\,(m-n)}$.

100. *Rule of Signs in Fractions.* Since a fraction is an indicated quotient, the rule of signs corresponds to that for division. The following theorems follow from the laws of multiplication and division:

1. If the terms are of the same sign, the fraction is positive; if of opposite signs, it is negative.

2. Changing the sign of either term changes the sign of the fraction.

3. Changing the signs of both terms leaves the fraction with its original sign.

4. The sign of the fraction may be changed by changing the sign written before it.

5. To these may be added the general principle that an even number of changes of sign restores the fraction to its original sign.

Ex. 1. $\dfrac{a}{b}=\dfrac{-a}{-b}=-\dfrac{-a}{b}=-\dfrac{a}{-b}$.

Ex. 2. $-\dfrac{a}{b}=-\dfrac{-a}{-b}=\dfrac{-a}{b}=\dfrac{a}{-b}$.

Ex. 3. $\dfrac{a-b}{m-n}=\dfrac{b-a}{n-m}=-\dfrac{a-b}{n-m}=-\dfrac{b-a}{m-n}$.

EXERCISES.

Express the following fractions in four different ways with respect to signs:

1. $\frac{x-y}{a}$. 2. $\frac{x-y}{a-b}$.

3. $\frac{m}{p-q}$. 4. $\frac{a}{a-b+c}$.

5. $-\frac{m-n}{p+q-r}$. 6. $\frac{a+m-x}{a-m+x}$.

Write the following fractions so that the symbols x and y shall be positive in both terms:

7. $+\frac{x-b}{c-y}$. 8. $+\frac{m-x}{n-y}$.

9. $+\frac{a+x-b}{a-x+b}$. 10. $-\frac{a-x}{b-x}$.

11. $-\frac{x-a+b}{b-x}$. 12. $\frac{a+b-x}{a-b+y}$.

101. When the numerator is a product, any one or more of its factors can be removed from the numerator and made a multiplier.

Ex. $\frac{abmx}{p+q} = ab\frac{mx}{p+q} = abm\frac{x}{p+q} = abmx\frac{1}{p+q}$.

EXERCISES.

Express the following fractions in as many forms as possible with respect to factors:

1. $\frac{pqx}{mn}$. 2. $\frac{ab}{c}$. 3. $\frac{abc}{a+b}$.

4. $\frac{x^2-y^2}{a-b}$. 5. $\frac{a^4-b^4}{x}$. 6. $\frac{x^4-16a^4}{x+2a}$.

102. *Reduction to Given Denominator.* A quantity may be expressed as a fraction with any required denominator, D, by supposing it to have the denominator 1, and then multiplying both terms by D.

For, if we call a the quantity, we have $a = \frac{a}{1} = \frac{aD}{D}$.

Ex. If we wish to express the quantity ab as a fraction having xy for its denominator, we write

$$\frac{abxy}{xy}.$$

If the quantity is fractional, both terms of the fraction must be multiplied by that factor which will produce the required denominator.

Ex. To express $\frac{a}{b}$ with the denominator nb^3, we multiply both members by $nb^3 \div b = nb^2$. Thus,

$$\frac{a}{b} = \frac{anb^2}{nb^3}.$$

This process is the reverse of reduction to lowest terms.

EXERCISES.

Express the quantity

1.	a	with	the	denominator	b.
2.	ax	"	"	"	ax.
3.	ab	"	"	"	ab^n.
4.	$\frac{m}{n}$	"	"	"	$n(x-y)$.
5.	-1	"	"	"	x.
6.	$\frac{m(n-p)}{a+b}$	"	"	"	a^2-b^2.
7.	$\frac{x+y}{x-y}$	"	"	"	x^2-y^2.
8.	$\frac{x^2+1}{x+1}$	"	"	"	x^2+2x+1.
9.	$\frac{a+1}{a-1}$	"	"	"	a^4-1.

Negative Exponents.

103. By the principle of § 85, we have

$$\frac{a^n}{a^k} = a^{n-k}.$$

If we have $k > n$, the exponent of the second member of the equation will be negative, and the first member, by cancelling n factors from each term of the fraction, will become $\frac{1}{a^{k-n}}$. Hence $\frac{1}{a^{k-n}} = a^{n-k}$.

By putting for shortness $k - n = s$, the equation will be

$$\frac{1}{a^s} = a^{-s}.$$

Hence,

A negative exponent indicates the reciprocal of the corresponding quantity with a positive exponent.

If in the formula $a^{n-k} = \frac{a^n}{a^k}$ we suppose $k = n$, it will become $a^0 = \frac{a^n}{a^n}$, or $a^0 = 1$. Hence, because a may be any quantity whatever,

Any quantity with the exponent 0 *is equal to unity.*

This result may be made more clear by successive divisions of a power of a by a. Every time we effect this division, we diminish the exponent by 1, and we may suppose this diminution to continue algebraically to negative values of the exponent. On the left-hand side of the equations in the margin, the division is effected symbolically by diminishing the exponents; on the right the result is written out in the usual way.

$$\begin{aligned} a^3 &= aaa \\ a^2 &= aa \\ a^1 &= a \\ a^0 &= 1 \\ a^{-1} &= \frac{1}{a} \\ a^{-2} &= \frac{1}{aa} \\ \text{etc.} & \quad \text{etc.} \end{aligned}$$

EXERCISES.

In the following exercises, write the quotients which are fractional both as fractions reduced to their lowest terms, and as entire quantities with negative exponents, on the principle,

$$\frac{a}{b} = ab^{-1}, \quad \frac{a^3}{b^2} = a^3b^{-2}, \quad \text{etc.}$$

Divide

1. x^2 by x. *Ans.* x.
2. x by x^2. *Ans.* $\frac{1}{x}$ or x^{-1}.
3. $-2b^3$ by b^5.
4. $4ab^2$ by $-2a^3b$. *Ans.* $-\frac{2b^2}{a^2}$ or $-2a^{-2}b^2$.

5. $-8a^2b$ by $4ab^2$. 6. $12a^3b^2xy$ by $4abx$.
7. $14a^4b^2c^3$ by $-7a^2b^4c^4$. 8. $24apqxy$ by $18abc$.
9. $-36a^3p^2x^2y$ by $-24c^3xy$.
10. $48a^2(x-y)^2$ by $36(x-y)$.
11. $42b^2\left(\dfrac{x+y}{x-y}\right)^3$ by $20\left(\dfrac{x+y}{x-y}\right)^2$.
12. $22(a-b)(m-n)$ by $15(a+b)(m+n)$.
13. $25(a^2-b^2)(m^2-n^2)$ by $15(a-b)(m+n)$.
14. $(x^4-1)(a^2-4b^2)$ by $(x^2-1)(a+2b)$.
15. x^6-1 by x^3+1.
16. $a^2b^3x^4y^5$ by $a^5b^4x^3y^2$.
17. $m^6n^4y^2z$ by $mn^2y^4z^6$.
18. $m(m+1)(m+2)(m+3)$ by $m(m-1)(m-2)(m-3)$.
19. a^m by a^n. 20. ab^mc^n by qb^nc^m.

Dissection of Fractions.

104. If the numerator is a polynomial, each of its terms may be divided separately by the denominator, and the several fractions connected by the signs + or −.

The principle is that on which the division of polynomials is founded (§ 87). The general form is

$$\frac{A+B+C+\text{etc.}}{m} = \frac{A}{m} + \frac{B}{m} + \frac{C}{m} + \text{etc.} \qquad (1)$$

The separate fractions may then be reduced to their lowest terms.

EXAMPLE. Dissect the fraction

$$\frac{32a^2b^2x - 18amy + 15bnz - 12b^2n^2u}{16abx}.$$

The general form (1) gives for the separate fractions,

$$\frac{32a^2b^2x}{16abx} - \frac{18amy}{16abx} + \frac{15bnz}{16abx} - \frac{12b^2n^2u}{16abx}.$$

Reducing each fraction to its lowest terms, the sum becomes

$$2ab - \frac{9my}{8bx} + \frac{15nz}{16ax} - \frac{3bn^2u}{4ax}.$$

EXERCISES.

Separate into sums of fractions,

1. $\dfrac{abc + bcd + cda + dab}{abcd}$.

2. $\dfrac{-xyzu + x^2yzu^2 + xy^2z^2u - x^2y^2z^2u^2}{x^2y^2z^2u^2}$.

3. $\dfrac{a^2 - b^2}{ab}$. 4. $\dfrac{a^2x - b^2y}{ax}$.

5. $\dfrac{(m-n)(n+q) - (m+n)(p-q)}{(m-n)(p-q)}$.

6. $\dfrac{(x-a)(y-b) + (x-y)(a-b) + (x-b)(y-a)}{x^2 - y^2}$.

7. $\dfrac{(a+b)(m-n) - (a-b)(m+n)}{a^2 - b^2}$.

Aggregation of Fractions.

105. When several fractions have equal denominators, their sum may be expressed as a single fraction by aggregating their numerators and writing the common denominator under them.

Ex. 1. $\dfrac{A}{m} - \dfrac{B}{m} + \dfrac{C}{m} = \dfrac{A - B + C}{m}$.

Ex. 2. $\dfrac{a-b}{x-y} + \dfrac{b-c}{y-x} + \dfrac{c-a}{x-y}$

$$= \frac{a-b}{x-y} + \frac{c-b}{x-y} + \frac{c-a}{x-y} = \frac{2c - 2b}{x-y} = \frac{2(c-b)}{x-y}.$$

REM. This process is the reverse of that of dissecting a fraction.

EXERCISES.

Aggregate

1. $\dfrac{a}{abc} - \dfrac{ab}{abc} + \dfrac{abc}{abc}$. 2. $\dfrac{a}{(a-b)^2} - \dfrac{b}{(a-b)^2}$.

3. $\dfrac{x-a}{a^2x} + \dfrac{y-b}{a^2x} + \dfrac{a+b}{a^2x} + \dfrac{x-y}{a^2x}$.

4. $\dfrac{a}{a-b} + \dfrac{b}{b-a} - \dfrac{c}{a-b} - \dfrac{d}{b-a}$.

5. $\dfrac{a-b}{m-n} - \dfrac{a-c}{m-n} - \dfrac{c-b}{n-m} + \dfrac{c+a}{n-m}$.

106. When all the fractions have not the same denominator, they must be reduced to a common denominator by the process of § 102.

Any common multiple of the denominators may be taken as the common denominator, but the least common multiple is the simplest.

To reduce to a Common Denominator. *Choose a common multiple of the denominators.*

Multiply both terms of each fraction by the multiplier necessary to change its denominator to the chosen multiple.

Note 1. The required multipliers will be the quotients of the chosen multiple by the denominator of each separate fraction.

Note 2. When the denominators have no common factors, the multiplier for each fraction will be the product of the denominators of all the other fractions.

Note 3. An entire quantity must be regarded as having the denominator 1. (§ 102.)

EXAMPLES.

1. Aggregate the sum

$$1 - \frac{1}{a} + \frac{1}{ab} - \frac{1}{abc} + \frac{1}{abcd}$$

in a single fraction.

The least common multiple of the denominators is $abcd$.

The separate multipliers necessary to reduce to this common denominator are

$$abcd, \quad bcd, \quad cd, \quad d, \quad 1.$$

The fractions reduced to the common denominator $abcd$ are

$$\frac{abcd}{abcd}, \quad \frac{-bcd}{abcd}, \quad \frac{+cd}{abcd}, \quad \frac{-d}{abcd}, \quad \frac{+1}{abcd}.$$

The sum is $\dfrac{abcd - bcd + cd - d + 1}{abcd}$.

By dissecting this fraction as in § 104, it may be reduced to its original form.

2. Reduce the sum

$$\frac{1}{a} - \frac{a}{b} + \frac{b}{c} - \frac{c}{d}$$

to a single fraction.

The multipliers are, by Note 2, bcd, acd, abd, abc.

Using these multipliers, the fractions become

$$\frac{bcd}{abcd}, \quad \frac{-a^2cd}{abcd}, \quad \frac{ab^2d}{abcd}, \quad \frac{-abc^2}{abcd},$$

from which the required sum is readily formed.

3. Reduce the sum

$$1 + \frac{1}{x-1} + \frac{x}{x+1} + \frac{x^2}{x^2-1}.$$

The least common multiple of the denominators is $x^2 - 1$.

The multipliers are, by Note 1,

$$x^2 - 1, \quad x + 1, \quad x - 1, \quad 1.$$

The sum of the fractions is found to be

$$\frac{x^2 - 1 + x + 1 + x^2 - x + x^2}{x^2 - 1} = \frac{3x^2}{x^2 - 1}.$$

EXERCISES.

Reduce to a single fraction the sums,

1. $1 + \dfrac{1}{x-1}.$

2. $1 - \dfrac{1}{x+1}.$

3. $\dfrac{1}{1-x} - \dfrac{1}{1+x}.$

4. $\dfrac{1}{1-x} + \dfrac{1}{1+x}.$

5. $x - \dfrac{ax}{a+x} - \dfrac{x^2}{a+x}.$

6. $\dfrac{a}{a-b} - \dfrac{b}{a+b}.$

7. $\dfrac{a}{x(a-x)} - \dfrac{x}{a(a-x)}.$

8. $\dfrac{2x-5}{4x^2-1} + \dfrac{5}{2x-1} - \dfrac{3}{x}.$

9. $\dfrac{1}{x+y} + \dfrac{2y}{x^2-y^2} - \dfrac{1}{x-y}.$

10. $\dfrac{1}{a-b} + \dfrac{1}{b-c} + \dfrac{1}{c-a}.$

11. $\dfrac{a}{x+y} + \dfrac{a}{x-y}.$

12. $\dfrac{a+b}{a-b} - \dfrac{a-b}{a+b}.$

13. $\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{b}{a-b}+\dfrac{a}{a+b}$.

14. $\dfrac{1}{2(x-1)}-\dfrac{1}{2(x+1)}-\dfrac{1}{x^2}$.

15. $\dfrac{a}{a-b}-\left(1-\dfrac{b}{a-b}\right)$.

16. $\dfrac{m+n}{m-n}-\dfrac{x-y}{x+y}$.

17. $\dfrac{y}{m^2}-\dfrac{m+y}{m(m-y)}$.

18. $1-\dfrac{a}{a-x}-\dfrac{x^2}{a^2-x^2}$.

19. $\dfrac{a-b}{a+b}+\dfrac{b-c}{b+c}+\dfrac{c-a}{c+a}+\dfrac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}$.

20. $\dfrac{a}{b}-\left(\dfrac{b}{a-b}+\dfrac{a}{b-a}\right)$.

21. $\dfrac{m-(x-a)}{x+y}-\dfrac{m-(x+a)}{x-y}$.

22. $\dfrac{c}{ab}+\dfrac{a}{bc}+\dfrac{b}{ac}$.

23. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-a)(b-c)}+\dfrac{c}{(c-a)(c-b)}$.

24. $\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}+4x$.

25. $\dfrac{ab}{a+b}-\dfrac{a^2}{a-b}+\dfrac{a(a^2+b^2)}{a^2-b^2}$.

26. $1-\dfrac{a}{x+a}-\dfrac{x}{x-a}$.

27. $1-\dfrac{x^2-2xy+y^2}{x^2+y^2}$.

28. $1-\dfrac{a^2+y^2-x^2}{2ay}$.

29. $\dfrac{1}{(a+b)^2}+\dfrac{1}{(a-b)^2}+\dfrac{1}{a^2-b^2}$.

30. $1+\dfrac{a^2-2ab+b^2}{4ab}$.

Factoring Fractions.

107. If several terms of the numerator contain a common factor, the coefficients of this factor may be added, and their aggregate multiplied by the factor for a new form of the numerator.

EXAMPLES.

1. $$\frac{ax - bx + cx + dx}{m} = \frac{(a - b + c + d)\,x}{m}$$
$$= (a - b + c + d)\frac{x}{m}. \quad (\S\ 101.)$$

2. $$\frac{abx + bcx + acy - aby}{abn} = \frac{(ab + bc)\,x}{abn} + \frac{(ac - ab)\,y}{abn}$$
$$= (a + c)\frac{x}{an} + (c - b)\frac{y}{bn}.$$

EXERCISES.

Reduce

1. $\dfrac{aby - bcy - acy}{abc}$.

2. $\dfrac{mnu + mpu + pnu}{mn}$.

3. $\dfrac{abq + bcq + abr + bcr}{abc}$.

4. $\dfrac{ax - by - 3bx - 4ay}{2ma}$.

5. $\dfrac{4mx + 2y - 3ax - 6cx + ay}{xyz}$.

6. $\dfrac{a^3 + 2a^2b + ab^2}{xy}$.

7. $\dfrac{a^2x - 4abc - (3y - 4c)\,a}{p + q}$.

8. $\dfrac{x^2y - [4x + x\,(2b - 4c) + 3ax]}{a + b}$.

9. $\dfrac{ax^2 - 4cx - 3\,[mx + m\,(a - x) - am]}{2a - 3b}$.

10. $\dfrac{4a\sqrt{x} - 2c\sqrt{x} + 2b\sqrt{x} - 2\,(mn\sqrt{x} - 4\sqrt{x})}{3a - 4b}$.

Multiplication and Division of Fractions.

108. *Fundamental Theorems in the Multiplication and Division of Fractions:*

Theorem I. A fraction may be multiplied by any quantity by either multiplying its numerator or dividing its denominator by that quantity.

Cor. 1. A fraction may be multiplied by its denominator by simply cancelling it.

Cor. 2. If the denominator of the fraction is a factor in the multiplier, cancel the denominator to multiply by this factor, and then multiply the numerator by the other factors.

Ex. $$\frac{m}{a(x-b)} \times a^2(x^2-b^2) = am(x+b),$$

because the multiplier $a^2(x^2-b^2) = a(x-b)a(x+b)$.

Theorem II. A fraction may be divided by either dividing its numerator or multiplying its denominator.

Theorem III. To multiply by a fraction, the multiplicand must be multiplied by the numerator of the fraction, and this product must be divided by its denominator.

Let us multiply $\frac{a}{b}$ by $\frac{m}{n}$

We multiply by m by multiplying the numerator (Th. I), and we divide by n by multiplying the denominator (Th. II).

Hence the product is $\frac{am}{bn}$.

That is, *the product of the numerators is the numerator of the required fraction, and the product of the denominators is its denominator.*

EXERCISES.

Multiply

1. $\frac{ab+y}{x-a}$ by $x-a$.
2. $\frac{ab}{x}$ by $\frac{x}{a}$.
3. $\frac{ab}{-x}$ by xy.
4. $\frac{ac}{x-a}$ by x^2-a^2.


5. $\dfrac{abm}{x^2y}$ by xy^2. 6. $\dfrac{m}{x^2}$ by $ax^3 + \dfrac{m-a}{x-m}$.

7. $\dfrac{a-b}{m}$ by $\dfrac{a+b}{m}$. 8. $a + \dfrac{m}{n}$ by $n + \dfrac{n}{m}$.

9. $ab - \dfrac{x}{y}$ by $ay + \dfrac{y-ab}{x}$. 10. $\dfrac{m+n}{m-n}$ by $\dfrac{n-m}{m+n}$.

11. Multiply $a + \dfrac{bx}{m}$ by $\dfrac{a}{b} + \dfrac{b}{x} + \dfrac{x}{a}$.

12. Reduce $\left(m + \dfrac{mn}{m-n}\right)\left(m - \dfrac{mn}{m+n}\right)$.

13. Reduce $\left(a - \dfrac{bx}{a}\right)\left(b - \dfrac{ax}{b}\right)$.

14. Multiply $b - \dfrac{bx}{a}$ by $\dfrac{a}{x}$.

15. Divide $\dfrac{m}{n}$ by p. *Ans.* $\dfrac{m}{np}$.

16. Divide $\dfrac{a}{a-b}$ by $a+b$.

17. Divide $\dfrac{x-a}{x+1}$ by $x-1$.

18. Divide $\dfrac{a+b}{x^2-1}$ by $1+x^2$.

19. Divide $\dfrac{-2a-3m}{a^n+b^n}$ by $b^n - a^n$.

109. *Reciprocal of a Fraction.* The reciprocal of a fraction is formed by simply inverting its terms.

For, let $\dfrac{a}{b}$ be the fraction. By definition, its reciprocal will be

$$\frac{1}{\frac{a}{b}}$$

Multiplying both terms by b, the numerator will be b and the denominator $\dfrac{a}{b} \times b$, that is, a.

Hence the reciprocal required will be $\dfrac{b}{a}$, or, in algebraic language,

$$\frac{1}{\frac{a}{b}} = \frac{b}{a}.$$

110. *Def.* A **Complex Fraction** is one of which either of the terms is itself fractional.

EXAMPLE. $$\frac{\frac{a}{b}}{m + \frac{x}{y}}$$

is a complex fraction, of which $\frac{a}{b}$ is the numerator, and $m + \frac{x}{y}$ the denominator.

The terms of the lesser fractions which enter into the numerator and denominator of the main fraction may be called **Minor Terms.**

Thus, b and y are minor denominators, and a and x are minor numerators.

To reduce a complex fraction to a simple one, multiply both terms by a multiple of the minor denominators.

EXAMPLE. Reduce $$\frac{\frac{am}{y^2}}{\frac{b}{y} + \frac{h}{x}}.$$

Multiplying both terms by xy^2, the result will be

$$\frac{amx}{bxy + hy^2},$$

which is a simple fraction.

EXERCISES.

Reduce to simple fractions :

1. $\dfrac{1 + \frac{x}{y}}{1 - \frac{x}{y}}.$

2. $\dfrac{a + \frac{b}{x}}{a - \frac{b}{x}}.$

3. $\dfrac{\frac{a - x}{a + x}}{\frac{a + x}{a - x}}.$

4. $\dfrac{\frac{ab}{mn}}{\frac{bd}{km}}.$

5. $\dfrac{1+\dfrac{n-1}{n+1}}{1-\dfrac{n-1}{n+1}}$.

6. $\dfrac{\dfrac{1+x}{1-x}+\dfrac{1-x}{1+x}}{\dfrac{1+x}{1-x}-\dfrac{1-x}{1+x}}$.

7. $\dfrac{am+\dfrac{b}{m}}{an-\dfrac{b}{n}}$.

8. $\dfrac{2x-\dfrac{3}{y}}{a+b-x}$.

9. $\dfrac{1+\dfrac{(a-b)^2}{4ab}}{1-\dfrac{b^2-a^2}{2ab}}$.

10. $\dfrac{\dfrac{1}{1+a}+\dfrac{a}{1-a}}{\dfrac{1}{1-a}-\dfrac{a}{1+a}}$.

11. $\dfrac{a^2+\dfrac{1}{a^2}+2}{\dfrac{1}{a}+a}$.

12. $\dfrac{\dfrac{a^2}{b^3}+\dfrac{1}{a}}{\dfrac{a}{b}-\dfrac{1}{b}+\dfrac{1}{a}}$.

13. $\dfrac{\dfrac{a+2b}{a+b}+\dfrac{a}{b}}{\dfrac{a+2b}{b}-\dfrac{a}{a+b}}$.

14. $\dfrac{\dfrac{x-y}{x+y}+\dfrac{y+x}{y^2-x^2}}{\dfrac{x+y}{x-y}-\dfrac{x^2-y^2}{x^4-y^4}}$.

Division of one Fraction by Another.

111. Let us divide $\frac{a}{b}$ by $\frac{m}{n}$. The result will be expressed by the complex fraction

$$\frac{\frac{a}{b}}{\frac{m}{n}}.$$

Reducing this fraction by the rule of § 110, it becomes

$$\frac{an}{bm},$$

which is equal to $\frac{a}{b}\times\frac{n}{m}$. That is,

To divide by a fraction, we have only to multiply by its reciprocal.

EXERCISES.

Divide

1. $\dfrac{ab}{a-b}$ by $\dfrac{a}{b}$.

2. $\dfrac{x+1}{8}$ by $\dfrac{2x}{9}$.

3. $\dfrac{x}{x-1}$ by $\dfrac{x}{2}$.

4. $\dfrac{a^4-b^4}{a^2-2ab+b^2}$ by $\dfrac{a^2+ab}{a-b}$.

5. $\dfrac{x+1}{x-1}$ by $\dfrac{x+1}{x^2-1}$.

6. $\dfrac{a}{b}+\dfrac{m}{n}$ by $\dfrac{b}{a}-\dfrac{n}{m}$.

7. $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}$ by $\dfrac{m}{x}+\dfrac{n}{y}+\dfrac{p}{z}$.

8. $\dfrac{a}{a-b}-\dfrac{b}{a+b}$ by $\dfrac{b}{a-b}+\dfrac{a}{a+b}$.

Reciprocal Relations of Multiplication and Division.

112. The fundamental principles of the operations upon fractions are included in the following summary, the understanding of which will afford the student a test of his grasp of the subject.

1. The reciprocal of the reciprocal of a number is equal to the number itself. In the language of Algebra,

$$\frac{1}{\frac{1}{a}} = a.$$

2. The reciprocal of a monomial may be expressed by changing the algebraic sign of its exponent.

3. To multiply by a number is equivalent to dividing by its reciprocal, and *vice versa*. That is,

$$N \div a \quad \text{or} \quad \frac{N}{\frac{1}{a}} = aN,$$

and *vice versa*, $$N \times \frac{1}{a} = \frac{N}{a}.$$

4. When the numerator or denominator of a fraction is a product of several factors, any of these factors may be transferred from one term of the fraction to the other by changing it to its reciprocal. That is,

$$\frac{abc}{pqr} = \frac{bc}{\frac{1}{a}pqr} = \frac{\frac{1}{p}abc}{qr}, \quad \text{etc.}$$

Or,
$$\frac{abc}{pqr} = \frac{bc}{a^{-1}pqr} = \frac{p^{-1}abc}{qr}, \quad \text{etc.}$$

5. *Multiplication* by a factor
greater than unity *increases*,
less than unity *diminishes*.

Division by a divisor
greater than unity *diminishes*,
less than unity *increases*.

6. (α) When a factor becomes zero, the product also becomes zero.

(β) When a denominator becomes zero, the product becomes infinite. That is,

$$0 \times a = a \times 0 = 0.$$

$$\frac{a}{0} = \text{infinity}.$$

NOTE. The following way of expressing what is meant by this last statement is less simple, but is logically more correct:

If a fraction has a fixed numerator, no matter how small, we can make the denominator so much smaller that the fraction shall be greater than any quantity we choose to assign.

EXERCISE.

If the numerator of a fraction is 2, how small must the denominator be in order that the fraction may exceed one thousand? That it may exceed one million? That it may exceed one thousand millions?

BOOK III.

OF EQUATIONS.

CHAPTER I.

THE REDUCTION OF EQUATIONS.

Definitions.

113. *Def.* An **Equation** is a statement, in the language of Algebra, that two expressions are equal.

114. *Def.* The two equal expressions are called **Members** of the equation.

115. *Def.* An **Identical Equation** is one which is true for all values of the algebraic symbols which enter into it, or which has numbers only for its members.

EXAMPLES. The equations

$$14 + 9 = 29 - 6,$$
$$5 + 13 - 3 \times 4 - 6 = 0,$$

which contain no algebraic symbols, are identical equations.

So also are the equations

$$x = x,$$
$$x - x = 0.$$
$$(x + a)(x - a) = x^2 - a^2,$$
$$(1 + y)(1 - y) - 1 + y^2 = 0,$$

because they are necessarily true, whatever values we assign to x, a, and y.

REM. All the equations used in the preceding two books to express the relations of algebraic quantities are identical ones, because they are true for all values of these quantities.

116. *Def.* An **Equation of Condition** is one which can be true only when the algebraic symbols are equal to certain quantities, or have certain relations among themselves.

EXAMPLES. The equation

$$x + 6 = 22$$

can be true only when x is equal to 16, and is therefore an equation of condition.

The equation

$$x + b = a$$

can be true only when x is equal to the difference of the two quantities a and b.

REM. In an equation of condition, some of the quantities may be supposed to be known and others to be unknown.

117. *Def.* To **Solve** an equation means to find some number or algebraic expression which, being substituted for the unknown quantity, will render the equation identically true.

This value of the unknown quantity is called **a Root** of the equation.

EXAMPLES.

1. The number 3 is a root of the equation

$$2x^2 - 18 = 0,$$

because when we put 3 in place of x, the equation is satisfied identically.

2. The expression $\dfrac{2a - b}{c}$ is a root of the equation

$$2cx - 4a + 2b = 0,$$

when x is the unknown quantity, because when we substitute this expression in place of x, we have

$$2c\left(\frac{2a - b}{c}\right) - 4a + 2b = 0,$$

or
$$4a - 2b - 4a + 2b = 0,$$

which is identically true.

REM. It is common in Elementary Algebra to represent unknown quantities by the last letters of the alphabet, and quantities supposed to be known by the first letters. But this is not at all necessary, and the student should accustom himself to regard any one symbol as an unknown quantity.

Axioms.

118. *Def.* An **Axiom** is a proposition which is taken for granted, without proof.

Equations are solved by operations founded upon the following axioms, which are self-evident, and so need no proof.

Ax. I. If equal quantities be added to the two members of an equation, the members will still be equal.

Ax. II. If equal quantities be subtracted from the two members of an equation, they will still be equal.

Ax. III. If the two members be multiplied by equal factors, they will still be equal.

Ax. IV. If the two members be divided by equal divisors (the divisors being different from zero), they will still be equal.

Ax. V. Similar roots of the two members are equal.

These axioms may be summed up in the single one,

Similar operations upon equal quantities give equal results.

119. An algebraic equation is solved by performing such similar operations upon its two members that the unknown quantity shall finally stand alone as one member of an equation.

Operations of Addition and Subtraction—Transposing Terms.

120. *Theorem.* Any term may be transposed from one member of an equation to the other member, if its sign be changed.

Proof. Let us put, in accordance with § 41, 2d Prin.,
t, any term of either member of the equation.
a, all the other terms of the same member.
b, the opposite member.
The equation is then

$$a + t = b.$$

Now subtract t from both sides (Axiom II),

$$a + t - t = b - t;$$

or by reduction, $a = b - t.$

This equation is the same as the one from which we started, except that t has been transposed to the second member, with its sign changed from + to −.

If the equation is

$$b - t = a,$$

we may add t to both members, which would give

$$b = a + t.$$

NUMERICAL EXAMPLE.

The learner will test each side of the following equations:

$$19+3-9+4 = 7+10.$$

Transposing 4, $19+3-9 = 7+10-4.$
" 9, $19+3 = 7+10-4+9.$
" 19, $3 = 7+10-4+9-19.$
" 3, $0 = 7+10-4+9-19-3.$

121. Rem. All the terms of either member of an equation may be transposed to the other member, leaving only 0 on one side.

Example. If in the equation

$$b = a + t,$$

we transpose b, we have $0 = a + t - b.$

By transposing a and t, we have

$$b - a - t = 0.$$

122. *Changing Signs of Members.* If we change the signs of all the terms in both members of an equation, it will still be true. The result will be the same as multiplying both members by -1, or transposing all the terms of each member to the other side, and then exchanging the terms.

EXAMPLE. The equation

$$17 + 8 = 11 + 14$$

may be transformed into $0 = 11 + 14 - 17 - 8,$

or, $\quad 0 = -11 - 14 + 17 + 8,$

or, $\quad -17 - 8 = -11 - 14.$

Operation of Multiplication.

123. *Clearing of Fractions.* The operation of multiplication is usually performed upon the two sides of an equation, in order to clear the equation of fractions.

To clear an equation of fractions:

FIRST METHOD. *Multiply its members by the least common multiple of all its denominators.*

SECOND METHOD. *Multiply its members by each of the denominators in succession.*

REM. 1. Sometimes the one and sometimes the other of these methods is the more convenient.

REM. 2. The operation of clearing of fractions is similar to that of reducing fractions to a common denominator.

EXAMPLE OF FIRST METHOD. Clear from fractions the equation

$$\frac{x}{4} + \frac{x}{6} + \frac{x}{8} = 26.$$

Here 24 is the least common multiple of the denominators. Multiplying each term by it, we have,

$$6x + 4x + 3x = 624,$$

or

$$13x = 624.$$

EXAMPLE OF SECOND METHOD. Clear the equation

$$\frac{a}{x - a} + \frac{a}{x + a} + \frac{c}{x} = 0.$$

Multiplying by $x - a$, we find

$$a + \frac{ax - a^2}{x + a} + \frac{cx - ca}{x} = 0.$$

Multiplying by $x + a$,

$$ax + a^2 + ax - a^2 + \frac{cx^2 - ca^2}{x} = 0.$$

Reducing and multiplying by x,

$$2ax^2 + cx^2 - ca^2 = 0.$$

EXERCISES.

Clear the following equations of fractions:

1. $\frac{2x}{9} - 6 = 0.$

2. $\frac{x}{5} - \frac{x}{7} = 70.$

3. $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} = 5.$

4. $\frac{x}{a} + \frac{x^2}{a^2} = \frac{b}{a}.$

5. $\frac{x}{ab} + \frac{y}{a} + \frac{7}{b} = \frac{c}{a^2b^2}.$

6. $\frac{a}{3} + \frac{b}{4} = \frac{x}{5}.$

7. $\frac{x}{x-a} - \frac{x}{x+a} = 1.$

8. $\frac{x}{x-a} = \frac{2x}{x+b}.$

9. $\frac{x+a}{x-a} = \frac{x^2+2ax}{x-a}.$

10. $\frac{x-2}{x-5} = \frac{x+2}{x+5}.$

11. $\frac{x}{y} - \frac{y}{x} = \frac{a}{b}.$

12. $\frac{x-a}{x+a} - \frac{x+a}{x-a} + \frac{x}{a} = 0.$

13. $\frac{x}{a-b} + \frac{y}{b-a} = z.$

Here the second term is the same as $\frac{-y}{a-b}$.

14. $\frac{x+a}{a-x} = \frac{x-b}{x-a}.$

Reduction to the Normal Form.

124. *Def.* An equation is in its **Normal Form** when its terms are reduced and arranged according to the powers of the unknown quantity.

In the normal form one member of the equation is expressed as an entire function of the unknown quantity, and the other is zero. (Compare §§ 50, 76.)

To reduce an equation to the normal form:

I. *Transpose all the terms to one member of the equation, so as to leave 0 as the other member.*

II. *Clear the equation of fractions.*

III. *Clear the equation of parentheses by performing all the operations indicated.*

IV. *Collect each set of terms containing like powers of the unknown quantity into a single one.*

V. *Divide by any common factor which does not contain the unknown quantity.*

REM. This order of operations may be deviated from according to circumstances. After a little practice, the student may take the shortest way of reaching the result, without respect to rules.

EXAMPLES.

1. Reduce to the normal form

$$\frac{(x-2)(x-3)}{x-5} = \frac{(x+2)(x+4)}{x+5}.$$

1. Clearing of fractions,

$$(x+5)(x-2)(x-3) = (x-5)(x+2)(x+4).$$

2. Performing the indicated operations,

$$x^3 - 19x + 30 = x^3 + x^2 - 22x - 40.$$

3. Transposing all the terms to the second member and reducing,

$$0 = x^2 - 3x - 70,$$

which is the normal form of the equation.

REM. Had we transposed the terms of the second member to the first one, the result would have been

$$-x^2 + 3x + 70 = 0.$$

Either form of the equation is correct, but, for the sake of uniformity, it is customary to transpose the terms so that the coefficient of the highest power of x shall be positive. If it comes out negative, it is only necessary to change the signs of all the terms of the equation.

Ex. 2. Reduce to the normal form,

$$\frac{5mx^2}{x-a} - \frac{2ax}{x+a} - \frac{3mx^3}{x^2-a^2} = 2mx - 5a.$$

1. Transposing to the first member,

$$\frac{5mx^2}{x-a} - \frac{2ax}{x+a} - \frac{3mx^3}{x^2-a^2} - 2mx + 5a = 0.$$

2. To clear of fractions, we notice that the least common multiple of the denominators is $x^2 - a^2$. Multiplying each term by this factor, we have,

$$5mx^2(x+a) - 2ax(x-a) - 3mx^3 - 2mx(x^2-a^2) + 5a(x^2-a^2) = 0.$$

3. Performing the indicated operations,

$$5mx^3 + 5amx^2 - 2ax^2 + 2a^2x - 3mx^3 - 2mx^3 + 2a^2mx + 5ax^2 - 5a^3 = 0.$$

4. Collecting like powers of x, as in § 76,

$$(3a + 5am)x^2 + (2a^2 + 2a^2m)x - 5a^3 = 0.$$

5. Every term of the equation contains the factor a. By Axiom IV, § 118, if both members of the equation be divided by a, the equation will still be true. The second member being zero, will remain zero when divided by a. Dividing both members, we have

$$(3 + 5m)x^2 + 2a(1 + m)x - 5a^2 = 0,$$

which is the normal form.

EXERCISES.

Reduce the following equations to the normal form, x, y, or z being the unknown quantity:

1. $\dfrac{3y^2 + 2y}{7} = \dfrac{y - 7}{2}$.
2. $\dfrac{x - a}{x + a} = \dfrac{x + a}{x}$.
3. $\dfrac{x - 7}{2x + 10} = \dfrac{2x + 6}{4x - 2}$.
4. $\dfrac{x^3 - 3a^2x + 2a^3}{2x + a} - x^2 - 5ax = \dfrac{7x^3 - 5ax^2}{2x - a}$.
5. $\dfrac{y}{a - y} + \dfrac{2y}{a + y} + \dfrac{3y}{a^2y^2} = 7$.
6. $\dfrac{z}{a + b} + \dfrac{a}{b + z} + \dfrac{b}{a + z} = 0$.
7. $\dfrac{z^2}{a - z} + \dfrac{z^3}{a^2 - x^2} = \dfrac{a^2z}{z^2 - a^2}$.

8. $7 + \dfrac{6}{y} + \dfrac{5}{y^2} + \dfrac{4}{y^3} = 0.$

9. $\dfrac{a}{x-a} + \dfrac{a^2}{x^2-a^2} + \dfrac{a^4}{x^4-a^4} = 1.$

10. $\dfrac{b}{c-z} + \dfrac{b^2}{c^2-z^2} + \dfrac{b^4}{c^4-z^4} = \dfrac{b^6}{c^6-z^6}.$

11. $\dfrac{a}{b-\dfrac{1}{x}} = \dfrac{b}{x-a}.$ 12. $\dfrac{m}{nx-\dfrac{n}{x}} = \dfrac{m}{x+\dfrac{1}{x}}.$

13. $\dfrac{a}{a-\dfrac{1}{x}} + \dfrac{a^2}{a^2-\dfrac{1}{x^2}} = \dfrac{a^3}{x^3}.$

14. $\dfrac{3z}{z+\dfrac{1}{2}} - \dfrac{5z^2}{3z-\dfrac{3}{z}} = \dfrac{1}{z}.$

15. $\dfrac{ax}{1-\dfrac{1}{x+a}} = \dfrac{bx}{1+\dfrac{1}{x-a}}.$

16. $\dfrac{\dfrac{a}{x}-\dfrac{b}{a-x}}{\dfrac{b}{x}} = \dfrac{a}{a-\dfrac{b}{x}}.$

Degree of Equations.

125. *Def.* An equation is said to be of the n^{th} degree when n is the highest power of the unknown quantity which appears in the equation after it is reduced to the normal form.

EXAMPLES.

The equation	$Ax + B = 0$	is of the	first degree.
	$Ax^2 + B = 0$	" "	second "
	$Ax^3 + Bx + C = 0$	" "	third "
	etc.		etc.

An equation of the second degree is also called a **Quadratic Equation.**

An equation of the third degree is also called a **Cubic Equation.**

EXAMPLE. The equation

$$ax^2 + bx^2y^2 + y^3 + a^2z = 0$$

is a quadratic equation in x, because x^2 is of the highest power of x which enters into it.

It is a cubic equation in y.

It is of the first degree in z.

CHAPTER II.

EQUATIONS OF THE FIRST DEGREE WITH ONE UNKNOWN QUANTITY.

126. REMARK. By the preceding definition of the degree of an equation, it will be seen that an equation of the first degree, with x as the quantity supposed to be unknown, is one which can be reduced to the form

$$Ax + B = 0, \qquad (a)$$

A and B being any numbers or algebraic expressions which do not contain x.

Such an equation is frequently called a **Simple Equation.**

Solution of Equations of the First Degree.

127. If, in the above equation, we transpose the term B to the second member, we have

$$Ax = -B.$$

If we divide both members by A (§ 118, Ax. IV), we have,

$$x = -\frac{B}{A}.$$

Here we have attained our object of so transforming the equation that one member shall consist of x alone, and the other member shall not contain x.

To prove that $-\frac{B}{A}$ is the required value of x, we substitute it for x in the equation (a). The equation then becomes,

$$-\frac{AB}{A} + B = 0;$$

or, by reducing, $$-B + B = 0,$$

an equation which is identically true. Therefore, $-\frac{B}{A}$ is the required root of the equation (a). (§ 117, *Def.*)

128. In an equation of the first degree, it will be unnecessary to reduce the equation entirely to the normal form by transposing all the terms to one member. It will generally be more convenient to place the terms which do not contain x in the opposite member from those which are multiplied by it.

EXAMPLE. Let the equation be

$$mx + a = nx + b. \qquad (1)$$

We may begin by transposing a to the second member and nx to the first, giving at once,

$$mx - nx = b - a,$$

or $$(m - n)x = b - a,$$

without reducing to the normal form. The final result is the same, whatever course we adopt, and the division of both members by $m - n$ gives

$$x = \frac{b - a}{m - n}.$$

129. The rule which may be followed in solving equations of the first degree with one unknown quantity is this:

I. *Clear the equation of fractions.*

II. *Transpose the terms which are multiplied by the unknown quantity to one member; those which do not contain it to the other.*

III. *Divide by the total coefficient of the unknown quantity.*

NOTE. Rules in Algebra are given only to enable the beginner to go to work in a way which will always be sure, though it may not always be the shortest. In solving equations, he should emancipate himself from the rules as soon as possible, and be prepared to solve each equation presented by such process as appears most concise and elegant. No operation upon the two members in accordance with the axioms (§ 118) can lead to incorrect results (provided that no quantity which becomes zero is used as a multiplier or divisor), and the student is therefore free to operate at his own pleasure on every equation presented.

EXAMPLES.

1. Given $$\frac{ax}{by} = 1.$$

It is required to find the value of each of the quantities a, b, x, and y, in terms of the others.

Clearing of fractions, we have

$$ax = by.$$

To find a, we divide by x, which gives

$$a = \frac{by}{x}.$$

To find b, we divide by y, which gives

$$\frac{ax}{y} = b.$$

To find x, we divide by a, which gives

$$x = \frac{by}{a}.$$

To find y, we divide by b, which gives

$$\frac{ax}{b} = y.$$

Thus, when any three of the four quantities a, b, x, and y, are given, the fourth can be found.

2. Let us take the equation,

$$\frac{x-7}{2x+10} = \frac{2x+6}{4x-2}.$$

Clearing of fractions, we have

$$4x^2 - 30x + 14 = 4x^2 + 32x + 60.$$

Transposing and reducing,

$$-62x = 46.$$

Dividing both members by -62,

$$x = \frac{46}{-62} = -\frac{46}{62} = -\frac{23}{31}.$$

This result should now be proved by computing the value of both members of the original equation when $-\frac{23}{31}$ is substituted for x.

3. $\frac{x}{m} + \frac{x}{n} = \frac{ax}{b} - \frac{1}{m}$.

Proceeding in the regular way, we clear of fractions by multiplying by mnb. This gives

$$nbx + mbx = amnx - nb.$$

Transposing and reducing,

$$(nb + mb - amn)\,x = -nb.$$

Dividing by the coefficient of x,

$$x = -\frac{nb}{nb + mb - amn} = \frac{nb}{amn - mb - nb}.$$

These two values are equivalent forms (§ 100).

But we can obtain a solution without clearing of fractions. Transposing $\frac{ax}{b}$, we have

$$\frac{x}{m} + \frac{x}{n} - \frac{ax}{b} = -\frac{1}{m},$$

which may be expressed in the form

$$\left(\frac{1}{m} + \frac{1}{n} - \frac{a}{b}\right)x = -\frac{1}{m}.$$

Dividing by the coefficient of x,

$$x = -\frac{\frac{1}{m}}{\frac{1}{m} + \frac{1}{n} - \frac{a}{b}}$$

This expression can be reduced to the other by § 110.

EXERCISES.

Find the values of x, y, or u in the following equations:

1. $\dfrac{5-3x}{2}=\dfrac{8x-9}{3}.$ 2. $-x=a.$

3. $\dfrac{x}{1}+\dfrac{x}{2}+\dfrac{x}{3}=22.$ 4. $\dfrac{x+23}{x-1}=9.$

5. $\dfrac{y}{a}+\dfrac{y}{b}-\dfrac{y}{c}=1.$ 6. $\dfrac{36}{u-5}=\dfrac{45}{u}.$

7. $\dfrac{u}{3}-\dfrac{u}{4}+\dfrac{u}{5}=u-26.$ 8. $a-bx=b+ax.$

9. $\dfrac{u}{a}+\dfrac{u}{b}=\dfrac{1}{a}+\dfrac{1}{b}.$ 10. $3x+\dfrac{3-x}{3}=x.$

11. $\dfrac{a}{c-x}=\dfrac{c}{a-x}.$

12. $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-5}{x-6}-\dfrac{x-6}{x-7}.$

13. $-y=a-b.$

14. $\dfrac{1}{x-2}-\dfrac{1}{x-4}=\dfrac{1}{x-6}-\dfrac{1}{x-8}.$

15. $\dfrac{1}{2}\left(x-\dfrac{a}{3}\right)-\dfrac{1}{3}\left(x-\dfrac{a}{4}\right)+\dfrac{1}{4}\left(x-\dfrac{a}{5}\right)=0.$

16. $\dfrac{u}{a}+\dfrac{u}{b-a}=\dfrac{a}{b+a}.$

17. $ax+b=\dfrac{x}{a}+\dfrac{1}{b}.$

18. $\dfrac{u-a}{b}+\dfrac{u-b}{c}+\dfrac{u-c}{a}=\dfrac{u-(a+b+c)}{abc}.$

19. $\dfrac{m\,(x+a)}{x+b}+\dfrac{n\,(x+b)}{x+a}=m+n.$

20. $(x-a)^3+(x-b)^3+(x-c)^3=3\,(x-a)\,(x-b)\,(x-c).$

Find the values of each of the four quantities, a, b, c, and d, in terms of the other three, from the equations

21. $\dfrac{a}{b-c}+\dfrac{d}{b-d}=0.$ 22. $\dfrac{ab}{cd}+1=0.$

Problems leading to Simple Equations.

130. The first difficulty which the beginner meets with in the solution of an algebraic problem is to state it in the form of an equation. This is a process in which the student must depend upon his own powers. The following is the general plan of proceeding:

1. Study the problem, to ascertain what quantities in it are unknown. There may be several such quantities, but the problems of the present chapter are such that all these quantities can be expressed in terms of some one of them. Select that one by which this can be most easily done as the unknown quantity.

2. Represent this unknown quantity by any algebraic symbol whatever.

It is common to select one of the last letters of the alphabet for the symbol, but the student should accustom himself to work equally well with any symbol.

3. Perform on and with these symbols the operations required by the problem. These operations are the same that would be necessary to verify the adopted value of the unknown quantity.

4. Express the conditions stated or implied in the problem by means of an equation.

5. The solution of this equation by the methods already explained will give the value of the unknown quantity. It is always best to verify the value found for the unknown quantity by operating upon it as described in the equation.

EXAMPLES.

1. A sum of 440 dollars is to be divided among three people so that the share of the second shall be 30 dollars more than that of the first, and the share of the third 80 dollars less than those of the first and second together. What is the share of each?

SOLUTION. 1. Here there are really three unknown quantities, but it is only necessary to represent the share of the first by an unknown symbol.

2 Therefore let us put

$$x = \text{share of the first.}$$

3. Then, by the terms of the statement, the share of the second will be

$$x + 30.$$

To find the share of the third we add these two together, which makes

$$2x + 30.$$

Subtracting 80, we have

$$2x - 50$$

as the share of the third.

We now add the three shares together, thus,

Share of first,	x
" " second,	$x + 30$
" " third,	$2x - 50$
Shares of all,	$4x - 20$

4. By the conditions of the problem, these three shares must together make up 440 dollars. Expressing this in the form of an equation, we have

$$4x - 20 = 440.$$

5. Solving, we find

$$x = 115 = \text{share of first.}$$

Whence, $115 + 30 = 145 = \text{share of second.}$

$$115 + 145 - 80 = 180 = \text{share of third.}$$

$$\text{Sum} = 440. \quad \textit{Proof.}$$

Ex. 2. Divide the number 90 into four parts, such that the first increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, shall all be equal to the same quantity.

Here there are really five unknown quantities, namely, the four parts and the quantity to which they are all to be equal when the operation of adding to, subtracting, etc., is performed upon them. It will be most convenient to take this last as the unknown quantity. Let us therefore put it equal to u. Then,

Since the first part increased by 2 must be equal to u, its value will be $u - 2$.

Since the second part diminished by 2 must be equal to u, its value will be $u + 2$.

Since the third part multiplied by 2 must be u, its value will be $\frac{u}{2}$.

Since the fourth part divided by 2 must make u, its value will be $2u$.

Adding these four parts up, their sum is found to be $\frac{9u}{2}$.

By the conditions of the problem, this sum must make up the number 90. Therefore we have

$$\frac{9u}{2} = 90.$$

Solving this equation, we find

$$u = 20.$$

Therefore

$$\begin{array}{llll} \text{1st part} & = u - 2 & = 18. \\ \text{2d} \quad \text{"} & = u + 2 & = 22. \\ \text{3d} \quad \text{"} & = u \div 2 & = 10. \\ \text{4th} \quad \text{"} & = 2u & = 40. \end{array}$$

The sum of the four equals 90 as required, and the first part increased by 2, the second diminished by 2, etc., all make the number 20, as required.

PROBLEMS FOR EXERCISE.

1. What number is that from which we obtain the same result whether we multiply it by 4 or subtract it from 100?

2. What number is that which gives the same result when we divide it by 8 as when we subtract it from 81?

3. Divide 284 dollars among two people so that the share of the first shall be three times that of the second and $16 more.

4. Find a number such that $\frac{1}{3}$ of it shall exceed $\frac{1}{4}$ of it by 12.

5. A shepherd describes the number of his sheep by saying that if he had 10 sheep more, and sold them for 5 dollars each, he would have 6 times as many dollars as he now has sheep. How many sheep has he?

6. An applewoman bought a number of apples, of which 60 proved to be rotten. She sold the remainder at the rate of 2 for 3 cents, and found that they averaged her one cent each for the whole. How many had she at first?

7. If you divide my age 10 years hence by my age 20 years ago, you will get the same quotient as if you should divide my present age by my age 26 years ago. What is my present age?

8. Divide $500 among A, B, and C, so that B shall have $20 less than A, and C $20 more than A and B together.

9. A father left \$10000 to be divided among his five children, directing that each should receive \$500 more than the next younger one. What was the share of each?

10. A man is 6 years older than his wife. After they have been married 12 years, 8 times her age would make 7 times his age. What was their age when married?

11. Of three brothers, the youngest is 8 years younger than the second, and the eldest is as old as the other two together. In 10 years the sum of their ages will be 120. What are their present ages?

12. The head of a fish is 9 inches long, the tail is as long as the head and half the body, and the body is as long as the head and tail together. What is the whole length of the fish?

13. In dividing a year's profits between three partners, A, B, and C, A got one-fourth and \$150 more, B got one-third and \$300 more, and C got one-fifth and \$60 more. What was the sum divided?

14. A traveller inquiring the distance to a city, was told that after he had gone one-third the distance and one-third the remaining distance, he would still have 36 miles more to go. What was the distance of the city?

15. In making a journey, a traveller went on the first day one-fifth of the distance and 8 miles more; on the second day he went one-fifth the distance that remained and 15 miles more; on the third day he went one-third the distance that remained and 12 miles more; on the fourth he went 35 miles and finished his journey. What was the whole distance travelled?

16. When two partners divided their profits, A had twice as much as B. If he paid B \$300, he would only have half as much again as B had. What was the share of each?

17. At noon a ship of war sees an enemy's merchant vessel 15 miles away sailing at the rate of 6 miles an hour. How fast must the ship of war sail in order to get within a mile of the vessel by 6 o'clock?

18. A train moves away from a station at the rate of h miles an hour. Half an hour afterward another train follows it, running m miles an hour. How long will it take the latter to overtake it?

19. What two numbers are they of which the difference is 9, and the difference of their squares 351?

20. A man bought 25 horses for \$2500, giving \$80 a piece

for poor horses and \$130 each for good ones. How many of each kind did he buy?

21. A man is 5 years older than his wife. In 15 years the sums of their ages will be three times the present age of the wife. What is the age of each?

22. How far can a person who has 8 hours to spare ride in a coach at the rate of 6 miles an hour, so that he can return at the rate of 4 miles an hour and arrive home in time?

23. A working alone can do a piece of work in 15 days, and B alone can perform it in 12 days. In what time can they perform it if both work together?

METHOD OF SOLUTION. In one day A can do $\frac{1}{12}$ of the whole work and B can do $\frac{1}{15}$. Hence, both together can do $(\frac{1}{12}+\frac{1}{15})$ of it.

If both together can do it in x days, then they can do $\frac{1}{x}$ of it in 1 day.

Hence, $$\frac{1}{x}=\frac{1}{12}+\frac{1}{15}$$

is the equation to be solved.

24. A cistern can be filled in 12 minutes by two pipes which run into it. One of them alone will fill it in 20 minutes. In what time would the other one alone fill it?

25. A cistern can be emptied by three pipes. The second pipe runs twice as much as the first, and the third as much as the first and second together. All three together can empty the cistern in one hour. In what time would each one separately empty it?

26. A marketwoman bought apples at the rate of 5 for two cents, and sold half of them at 2 for a cent and the other half at 3 for a cent. Her profits were 50 cents. How many did she buy?

27. A grocer having 50 pounds of tea worth 90 cents a pound, mixed with it so much tea at 60 cents a pound that the combined mixture was worth 70 cents. How much did he add?

28. A laborer was hired for 40 days, on the condition that every day he worked he should receive \$1.50, but should forfeit .50 cents for every day he was idle. At the end of the time \$52 were due him. How many days was he idle?

29. A father left an estate to his three children, on the condition that the eldest should be paid \$1200 and the second \$800 for services they had rendered. The remainder was to be equally divided among all three. Under this arrangement,

the youngest got one-fourth of the estate. What was the amount divided?

30. A person having a sum of money to divide among three people gave the first one-third and $20 more, the second one-third of what was left and $20 more, and the third one-third of what was then left and $20 more, which exhausted the amount. How much had they to divide?

31. One shepherd spent $720 in sheep, and another got the same number of sheep for $480, paying $2 a piece less. What price did each pay?

32. A crew which can pull at the rate of 9 miles an hour, finds that it takes twice as long to go up the river as to go down. At what rate does the river flow?

33. A person who possesses $12000 employs a portion of the money in building a house. Of the money which remains, he invests one-third at four per cent. and the other two-thirds at five per cent., and obtains from these two investments an annual income of $392. What was the cost of the house?

34. An income tax is levied on the condition that the first $600 of every income shall be untaxed, the next $3000 shall be taxed at two per cent., and all incomes in excess of $3600 shall be taxed three per cent. on the excess. A person finds that by a uniform tax of two per cent. on all incomes he would save $200. What was his income?

35. At what time between 3 and 4 o'clock is the minute-hand 5 minutes ahead of the hour hand?

36. One vase, holding a gallons, is full of water; a second, holding b gallons, is full of brandy. Find the capacity of a dipper such that whether it is filled from the first vase and the water removed replaced by brandy, or filled from the second vase and the latter then filled with water, the strength of the mixture will be the same.

37. Divide a number m into four such parts that the first part increased by a, the second diminished by a, the third multiplied by a, and the fourth divided by a shall all be equal.

38. Divide a dollars among five brothers, so that each shall have n dollars more than the next younger.

39. A courier starts out from his station riding 8 miles an hour. Four hours afterwards he is followed by another riding 10 miles an hour. How long will it require for the second to overtake the first, and what will be the distance travelled?

If x be the number of hours required, the second will have travelled x hours and the first $(x+4)$ hours when they meet. At this time they must have travelled equal distances.

Problem of the Couriers.

Let us generalize the preceding problem thus:

131. *A courier starts out from his station riding* c *miles an hour;* h *hours later, he is followed by another riding* a *miles an hour. How long will the latter be in overtaking the first, and what will be the distance from the point of departure.*

Let us put t for the time required. Then the first courier will have travelled $(t+h)$ hours, and the second t hours. Since the first travelled c miles an hour, his whole distance at the end of $t+h$ hours will be $(t+h)c$. In the same way, the distance travelled by the other will be at. When the latter overtakes the former, the distances will be equal; hence,

$$at = c(t+h). \qquad (1)$$

Solving this equation with respect to t, we find

$$t = \frac{ch}{a-c}. \qquad (2)$$

Multiplying by a gives us the whole distance travelled, which is

$$\text{Distance} = \frac{ach}{a-c}.$$

This equation solves every problem of this kind by substituting for a, c, and h their values in numbers supposed in the problem. For example, in Problem 39, we supposed $a = 10$, $c = 8$, $h = 4$. Substituting these values in equation (2), we find

$$t = 16,$$

which is the number of hours required.

To illustrate the generality of an algebraic problem, we shall now inquire what values t shall have when we make different suppositions respecting a, c, and h.

(1.) Let us suppose $a = c$, or $a - c = 0$, that is, the rates of travelling equal. Then equation (2) will become

$$t = \frac{ch}{0},$$

an expression for infinity (§ 112, 6), showing that the one courier would never overtake the other. This is plain enough. But,

(2.) Let us suppose that the second courier does not ride so fast as the first, that is, a less than c, and $a - c$ negative. Then the fraction $\frac{ch}{a-c}$ will not be infinite, but will be negative, because it has a positive numerator and a negative denominator. It is plain that the second courier would never overtake the first in this case either, because the latter would gain on him all the time; yet the fraction is not infinite.

What does this mean?

It means that the problem solved by Algebra is more general, that is, involves more particular problems than were implied in the statement. If we count the hours *after* the second courier set out as positive, then a negative time will mean so many hours before he set out, and this will bring out a time when, according to our idea of the problem, the horses were still in the stable.

The explanation of the difficulty is this. Suppose S to be the point from which the couriers started, and AB the road along which they travelled from S toward B. Suppose also that the first courier started out

from S at 8 o'clock and the second at 12 o'clock. By the rule of positive and negative quantities, distances towards A are negative. Now, because algebraic quantities do not commence at 0, but extend in both the negative and positive directions, the algebraic problem does not suppose the couriers to have really commenced their journey at S, but to have come from the direction of A, so that the first one passes S, without stopping, at 8 o'clock, and the second at 12. It is plain that if the first courier is travelling the faster, he must have passed the other before reaching S, that is, the time and distance are both negative, just as the problem gives them.

The general principle here involved may be expressed thus:

In Algebra, roads and journeys, like time, have no beginning and no end.

(3.) Let us suppose that the couriers start out at the same time and ride with the same speed. Then h and $a-c$ are both zero, and the expression for t assumes the form,

$$t = \frac{0}{0}.$$

This is an expression which may have one value as well as another, and is therefore indeterminate. The result is correct, because the couriers are always together, so that all values of t are equally correct.

The equation (1) can be used to solve the problem in other forms. In this equation are four quantities, a, c, h, and t, and when any three of these are given, the fourth can be found. There are therefore four problems, all of which can be solved from this equation.

First Problem, that already given, in which the time required for one courier to overtake the other is the unknown quantity.

Second Problem. *A courier sets out from a station, riding* c *miles an hour. After* h *hours another follows him from the same station, intending to overtake him in* t *hours. How fast must he ride?*

The problem can be put into the form of an equation in the same way as before, and we shall have the equation (1), only a will now be the unknown quantity. If we use the numbers of Prob. 39 instead of the letters, we shall have, instead of equation (1), the following:

$$16a = 8(16+4) = 8 \cdot 20 = 160,$$

whence $\qquad a = 10.$

If we use letters, we find from (1),

$$a = \frac{c(t+h)}{t},$$

and the problem is solved in either case.

Third Problem. *The second courier can ride just* a *miles an hour, and the first courier starts out* h *hours* *before him. How fast must the latter ride in order that the other may take* **t** *hours to overtake him?*

Here c, the rate of the first courier, is the unknown quantity, and by solving equation (1), we find

$$c = \frac{at}{t + h}.$$

FOURTH PROBLEM. *The swiftest of two couriers can ride* **a** *miles an hour, and the slower* **c** *miles an hour. How long a start must the latter have in order that the other may require* **t** *hours to overtake him?*

Here, in equation (1), h is the unknown quantity. By solving the equation with respect to h, we find,

$$h = \frac{at - ct}{c},$$

which solves the problem.

PROBLEMS OF CIRCULAR MOTION.

40. Two men start from the same point to run repeatedly round a circle one mile in circumference. If A runs 7 miles an hour and B 5, it is required to know:

1. At what intervals of time will A pass B?
2. At how many different points on the circle will they be together?

We reason thus: since A runs 2 miles an hour faster than B, he gets away from him at the rate of 2 miles an hour. When he overtakes him, he will have gained upon him one circumference, that is, 1 mile. This will require 30 minutes, which is therefore the required interval. In this interval A will have gone round $3\frac{1}{2}$ and B $2\frac{1}{2}$ times, so that they will be together at the point opposite that where they were together 30 minutes previous. Hence, they are together at two opposite points of the circle.

41. What would be the answer to the preceding question if A should run 8 miles an hour, and B 5?

42. Two race-horses run round and round a course, the one making the circuit in 30, the other in 35 seconds. If they start out together, how long before they will be together again?

NOTE. In x seconds one will make $\frac{x}{30}$ circuit and the other $\frac{x}{35}$.

43. If one planet revolves round the sun in T and the other in T'' years, what will be the interval between their conjunctions?

CHAPTER III.

EQUATIONS OF THE FIRST DEGREE WITH SEVERAL UNKNOWN QUANTITIES.

CASE I. ***Equations with Two Unknown Quantities.***

132. *Def.* An equation of the first degree with two unknown quantities is one which admits of being reduced to the form

$$ax + by = c,$$

in which x and y are the unknown quantities and a, b, and c represent any numbers or algebraic equations which do not contain either of the unknown quantities.

Def. A set of several equations containing the same unknown quantities is called a **System of Simultaneous Equations.**

Solution of a Pair of Simultaneous Equations containing Two Unknown Quantities.

133. To solve two or more simultaneous equations, it is necessary to combine them in such a way as to form an equation containing only one unknown quantity.

134. *Def.* The process of combining equations so that one or more of the unknown quantities shall disappear is called **Elimination**.

The term "elimination" is used because the unknown quantities which disappear are *eliminated.*

There are three methods of eliminating an unknown quantity from two simultaneous equations.

Elimination by Comparison.

135. Rule. *Solve each of the equations with respect to one of the unknown quantities and put the two values of the unknown quantity thus obtained equal to each other.*

This will give an equation with only one unknown quantity, of which the value can be found from the equation.

The value of the other unknown quantity is then found by substitution.

Example. Let the equations be

$$\left.\begin{aligned} ax + by &= c, \\ a'x + b'y &= c'. \end{aligned}\right\} \qquad (1)$$

From the first equation we obtain,

$$x = \frac{c - by}{a}. \qquad (2)$$

From the second we obtain,

$$x = \frac{c' - b'y}{a'}. \qquad (3)$$

Putting these two values equal, we have

$$\frac{c - by}{a} = \frac{c' - b'y}{a'}.$$

Reducing and solving this equation as in Chapter II, we find,

$$y = \frac{ac' - a'c}{ab' - a'b},$$

which is the required value of y. Substituting this value of y in either of the equations (1), (2), or (3), and solving, we shall find

$$x = \frac{b'c - bc'}{ab' - a'b}.$$

If the work is correct, the result will be the same in whichever of the equations we make the substitution.

NUMERICAL EXAMPLE. Let the equations be

$$\left.\begin{aligned} x + y &= 28, \\ 3x - 2y &= 29. \end{aligned}\right\} \qquad (4)$$

From the first equation we find

$$x = 28 - y,$$

and from the second $\qquad x = \dfrac{29 + 2y}{3},$

from which we have $\quad 28 - y = \dfrac{29 + 2y}{3},$

$$y = 11.$$

Substituting this value in the first equation in x, it becomes

$$x = 28 - 11 = 17.$$

If we substitute it in the second, it becomes

$$x = \frac{29 + 22}{3} = \frac{51}{3} = 17,$$

the same value, thus proving the correctness of the work.

Elimination by Substitution.

136. RULE. *Find the value of one of the unknown quantities in terms of the other from either equation, and substitute it in the other equation. The latter will have but one unknown quantity.*

EXAMPLE. Taking the same equations as before,

$$\begin{aligned} ax + by &= c, \\ a'x + b'y &= c', \end{aligned}$$

the first equation gives $\qquad x = \dfrac{c - by}{a}.$

Substituting this value instead of x in the second equation, it becomes

$$\frac{a'c - a'by}{a} + b'y = c'.$$

Solving this equation with respect to y, we get the same result as before.

NUMERICAL EXAMPLE. To solve in this way the last numerical example, we have from the first equation (4),

$$x = 28 - y.$$

Substituting this value in the second equation, it becomes

$$84 - 3y - 2y = 29,$$

from which we obtain as before,

$$y = \frac{84 - 29}{5} = 11.$$

This method may be applied to any pair of equations in four ways:

1. Find x from the first equation and substitute its value in the second.

2. Find x from the second equation and substitute its value in the first.

3. Find y from the first equation and substitute its value in the second.

4. Find y from the second equation and substitute its value in the first.

Elimination by Addition or Subtraction.

137. RULE. *Multiply each equation by such a factor that the coefficients of one of the unknown quantities shall become numerically equal in the two equations.*

Then, by adding or subtracting the equations, we shall have an equation with but one unknown quantity.

REM. We may always take for the factor of each equation the coefficient of the unknown quantity to be eliminated in the other equation.

EXAMPLE. Let us take once more the general equation

$$ax + by = c,$$
$$a'x + b'y = c'.$$

Multiplying the first equation by a', it becomes

$$aa'x + a'by = a'c.$$

Multiplying the second by a, it becomes

$$aa'x + ab'y = ac'.$$

The unknown quantity x has the same coefficient in the last two equations. Subtracting them from each other, we obtain

$$(a'b - ab')\,y = a'c - ac',$$

$$y = \frac{a'c - ac'}{a'b - ab'}.$$

REM. We shall always obtain the same result, whichever of the above three methods we use. But as a general rule the last method is the most simple and elegant.

Problem of the Sum and Difference.

The following simple problem is of such wide application that it should be well understood.

138. PROBLEM. *The sum and difference of two numbers being given, to find the numbers.*

Let the numbers be x and y.
Let s be their sum and d their difference.
Then, by the conditions of the problem,

$$x + y = s,$$
$$x - y = d.$$

Adding the two equations, we have

$$2x = s + d.$$

Subtracting the second from the first,

$$2y = s - d.$$

Dividing these equations by 2,

$$x = \frac{s+d}{2} = \frac{s}{2} + \frac{d}{2}$$

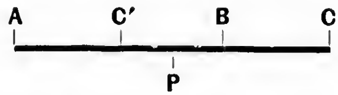
$$y = \frac{s-d}{2} = \frac{s}{2} - \frac{d}{2}.$$

We therefore conclude:

The greater number is found by adding half the difference to half the sum.

The lesser number is found by subtracting half the difference from half the sum.

This result can be illustrated geometrically. Let AB and BC be two lines placed end to end, so that AC is their sum. To find their difference, we cut off from AB a length $AC' = BC$; then $C'B$ is the difference of the two lines.



If P is half way between C′ and B, it is the middle point of the whole line, so that

$$AP = PC = \tfrac{1}{2}AC = \tfrac{1}{2} \text{ sum of lines.}$$
$$C'P = PB = \tfrac{1}{2}C'B = \tfrac{1}{2} \text{ difference of lines.}$$

If to the half sum AP we add the half difference PB, we have AB, the greater line.

If from the half sum AP we take the half difference C′P, we have left AC′, the lesser line.

EXERCISES.

Solve the following equations:

1. $3x - 2y = 33, \quad 2x - 3y = 18.$
2. $3x - 5y = 13, \quad 2x + 7y = 81.$
3. $7x + 6y = a, \quad 6x + 6y = b.$
4. $2x + 3y = m, \quad 2x - 3y = n.$
5. $ax + by = p, \quad ax - by = q.$
6. $\dfrac{x}{6} + \dfrac{y}{7} = 26, \quad \dfrac{x}{6} - \dfrac{y}{7} = 2.$
7. $\dfrac{x}{4} + \dfrac{y}{5} = 18, \quad \dfrac{x}{8} + \dfrac{y}{2} = 29.$
8. $\dfrac{x}{2} + \dfrac{y}{3} = a, \quad \dfrac{x}{2} - \dfrac{y}{3} = b.$
9. $7(x + y) + 3(x - y) = 102,$
   $7(x + y) - 3(x - y) = 66.$

NOTE. Solve this equation first as if $x+y$ and $x-y$ were single symbols, of which the values are to be found. Then find x and y by § 138 preceding.

10. $x + y + (x - y) = 14, \quad x + y - (x - y) = 10.$
11. $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{12}, \quad \dfrac{1}{x} - \dfrac{1}{y} = \dfrac{1}{12}.$

NOTE. Equations in this form can be best solved as if $\frac{1}{x}$ and $\frac{1}{y}$ were the unknown quantities. See next exercise.

12. $\frac{3}{x} - \frac{2}{y} = \frac{11}{10}, \quad \frac{4}{x} + \frac{5}{y} = 3.$

SOLUTION. If we multiply the first equation by 4, and the second by 3, we have

$$\frac{12}{x} - \frac{8}{y} = \frac{44}{10} = \frac{22}{5},$$

$$\frac{12}{x} + \frac{15}{y} = 9 = \frac{45}{5}.$$

Subtracting the first from the second, we have

$$\frac{23}{y} = \frac{23}{5};$$

whence,

$$y = 5.$$

Again, to eliminate $\frac{1}{y}$, we multiply the first equation by 5 and the second by 2 and add. Thus,

$$\frac{15}{x} - \frac{10}{y} = \frac{11}{2},$$

$$\frac{8}{x} + \frac{10}{y} = 6 = \frac{12}{2},$$

$$\frac{23}{x} = \frac{23}{2};$$

whence,

$$x = 2.$$

13. $\frac{2}{x} + \frac{3}{y} = \frac{7}{12}, \quad \frac{2}{x} - \frac{3}{y} = -\frac{1}{12}$

14. $\frac{1}{x} + \frac{2}{y} = \frac{5}{12}, \quad \frac{2}{x} - \frac{1}{y} = \frac{5}{24}.$

15. $\frac{5}{x} - \frac{3}{y} = -\frac{1}{6}, \quad \frac{3}{x} - \frac{1}{y} = \frac{1}{30}.$

16. $\frac{5}{x+1} - \frac{3}{y-1} = -\frac{1}{6}, \quad \frac{3}{x+1} - \frac{1}{y-1} = \frac{1}{30}.$

17. $\frac{2}{x+2} + \frac{3}{y-3} = \frac{7}{12}, \quad \frac{2}{x+2} - \frac{3}{y-3} = -\frac{1}{12}.$

18. $\frac{a}{x} + \frac{b}{y} = c, \quad \frac{a}{x} - \frac{b}{y} = d.$

19. $\frac{x+y}{x-y} = 2, \quad \frac{2x+3y}{x+a} = b.$

20. $\frac{x}{a+b} + \frac{y}{a-b} = 2a, \quad \frac{x-y}{4ab} = 1.$

CASE II. ***Equations of the First Degree with Three or More Unknown Quantities.***

139. When the values of several unknown quantities are to be found, it is necessary to have as many equations as unknown quantities.

If there are more unknown quantities than equations, it will be impossible to determine the values of all of them from the equations. All that can be done is to determine the value of some in terms of the others.

If the number of equations exceeds that of unknown quantities, the excess of equations will be superfluous. If there are n unknown quantities, their values can be found from any n of the equations. If any selection of n equations we choose to make gives the same values of the unknown quantities, the equations, though superfluous, will be consistent. If different values are obtained, it will be impossible to satisfy them all.

Elimination.

140. When the number of unknown quantities exceeds two, the most convenient method of elimination is generally that by addition or subtraction. The unknown quantities are to be eliminated one at a time by the following method:

I. *Select an unknown quantity to be first eliminated. It is best to begin with the quantity which appears in the fewest equations or has the simplest coefficients.*

II. *Select one of the equations containing this unknown quantity as an eliminating equation.*

III. *Eliminate the quantity between this equation and each of the others in succession.*

We shall then have a second system of equations less by one in number than the original system and containing a number of unknown quantities one less.

IV. *Repeat the process on the new system of equations, and continue the repetition until only one equation with one unknown quantity is left.*

V. *Having found the value of this last unknown quantity, the values of the others can be found by successive substitution in one equation of each system.*

Example. Solve the equations

$$\left.\begin{array}{lr} (1) & 4x - 3y - z + u - 7 = 0, \\ (2) & x - y + 2z + 2u - 10 = 0, \\ (3) & 2x + 2y - z - 2u - 2 = 0, \\ (4) & x + 2y + z + u - 19 = 0. \end{array}\right\} \quad (a)$$

We shall select x as the first quantity to be eliminated, and take the last equation as the eliminating one. We first multiply this equation by three such factors that the coefficient of x shall become equal to the coefficient of x in each of the other equations. These factors are 4, 1, and 2. We write the products under each of the other equations, thus :

$$\begin{array}{lr} \text{Eq. (1),} & 4x - 3y - z + u - 7 = 0, \\ (4)\times 4, & \underline{4x + 8y + 4z + 4u - 76 = 0.} \\ \text{Eq. (2),} & x - y + 2z + 2u - 10 = 0, \\ (4)\times 1, & \underline{x + 2y + z + u - 19 = 0.} \\ \text{Eq. (3),} & 2x + 2y - z - 2u - 2 = 0, \\ (4)\times 2, & \underline{2x + 4y + 2z + 2u - 38 = 0.} \end{array}$$

By subtracting the one of each pair from the other, we obtain the equations,

$$\left.\begin{array}{r} 11y + 5z + 3u - 69 = 0, \\ 3y - z - u - 9 = 0, \\ 2y + 3z + 4u - 36 = 0. \end{array}\right\} \quad (b)$$

The unknown quantity x is here eliminated, and we have three equations with only three unknown quantities. Now eliminating y by means of the last equation, in the same way, and clearing of fractions, we find the two equations,

$$\left.\begin{array}{r} 23z + 38u - 258 = 0, \\ 11z + 14u - 90 = 0. \end{array}\right\} \quad (c)$$

The problem is now reduced to two equations with two unknown quantities, which we have already shown how to solve. We find by solving them,

$$z = -2,$$
$$u = 8.$$

We next find the value of y by substituting these values of z and u in either of the equations (b). The first of them thus becomes:

$$11y - 10 + 24 - 69 = 0,$$

from which we find,

$$y = 5.$$

We now substitute the values of y, z, and u in either of equations (a). The second of the latter becomes

$$x - 5 - 4 + 16 - 10 = 0,$$

and the fourth becomes,

$$x + 10 - 2 + 8 - 19 = 0,$$

either of which gives

$$x = 3.$$

We can now prove the results by substituting the values of x, y, z, and u in all four of equations (a), and seeing whether they are all satisfied.

EXERCISES.

1. One of the best exercises for the student will be that of resolving the previous equations (a) by taking the last equation as the eliminating one, and performing the elimination in different orders; that is, begin by eliminating u, then repeat the whole process beginning with z, etc. The final results will always be the same.

2. Find the values of x_1, x_2, x_3, and x_4, from the equations,

$$x_1 + x_2 + x_3 + x_4 = 64,$$
$$x_1 + x_2 - x_3 - x_4 = 34,$$
$$x_1 - x_2 + x_3 - x_4 = 6,$$
$$x_1 - x_2 - x_3 + x_4 = 4.$$

This example requires no multiplication, but only addition and subtraction of the different equations.

3. $$2x + 5y + 3z = 13,$$
$$2x + 2y - z = 12,$$
$$5x + 5y - 2z = 29.$$

4.
$$\begin{aligned} 3z + 2u - 5y &= 18, \\ 3x + y - 4u &= 9, \\ x + 7z - 6y &= 33, \\ 5z - 2x - 8y + 2u &= 15. \end{aligned}$$

5.
$$\begin{aligned} x + y + z &= a, \\ y + z + u &= b, \\ z + u + x &= c, \\ u + x + y &= d. \end{aligned}$$

6.
$$\begin{aligned} \frac{1}{x} - \frac{1}{y} &= m, \\ \frac{1}{y} - \frac{1}{z} &= n, \\ \frac{1}{z} + \frac{1}{x} &= p. \end{aligned}$$

PROBLEMS FOR SOLUTION.

1. A man had a saddle worth \$75 and two horses. If the saddle be put on horse A he will be double the value of B, but if it be put on B his value will be equal to that of A. What is the value of each horse?

2. What number of two digits is equal to 7 times the sum of its digits, and to 21 times the difference of its digits?

Let x be the first digit, or the number of tens, and y the units. Then the number itself will be $10x+y$. Seven times the sum of the digits are $7x+7y$, and 21 times the difference are $21x-21y$. Uniting and solving the equations, we find $x = 6$, $y = 3$; the number is therefore 63.

3. A number of two digits is equal to 6 times the sum of its digits, and if 9 be subtracted from the number the digits are reversed. What is the number?

4. Find a number of two digits such that it shall be equal to 6 times the sum of its digits increased by 1, while if 18 be subtracted from the number the digits will be reversed.

5. Find a number which is greater by 2 than 5 times the sum of its digits, and if 9 be added to it the digits will be reversed.

6. What number is that which is equal to 9 times the sum of its digits and is 4 greater than 11 times their difference?

7. What fraction is that which becomes equal to $\frac{2}{3}$ when the numerator is increased by 2, and equal to $\frac{4}{7}$ when the denominator is increased by 4.

8. Two drovers A and B went to market with cattle. A sold 50 and then had left half as many as B, who had sold none. Then B sold 54 and had remaining half as many as A. How many did each have?

9. A boy bought 42 apples for a dollar, giving 3 cents each for the good ones and 2 cents each for the poor ones. How many of each kind did he buy?

10. Find a fraction which becomes equal to $\frac{1}{3}$ when its denominator is increased by 13, and to $\frac{3}{5}$ when 4 is subtracted from its numerator.

11. Find a fraction which will become equal to $\frac{3}{5}$ by adding 2 to its numerator, or by adding to its denominator 3, will become $\frac{1}{3}$.

12. A huckster bought a certain number of chickens at 32 cents each and of turkeys at 75 cents each, paying $14 for the whole. He sold the chickens at 48 cents each, and the turkeys at $1 each, realizing $20 for the whole. How many chickens and how many turkeys had he?

13. An applewoman bought a lot of apples at 1 cent each, and a lot of pears at 2 cents each, paying $1.70 for the whole. 11 of the apples and 7 of the pears were bad, but she sold the good apples at 2 cents each and the good pears at 3 cents each, realizing $2.60. How many of each fruit did she buy?

14. When Mr. Smith was married he was $\frac{1}{3}$ older than his wife; twelve years afterward he was $\frac{1}{5}$ older. What were their ages when married?

15. A and B together can do a piece of work in 6 days, but A working alone can do it 9 days sooner than B working alone. In what time could each of them do it singly?

16. A husband being asked the age of himself and wife, replied: "If you divide my age 6 years hence by her age 6 years ago, the quotient will be 2. But if you divide her age 12 years hence by mine 21 years ago, the quotient will be 5.

17. The sum of two ages is 9 times their difference, but seven years ago it was only seven times their difference. What are the ages now?

18. Two trains set out at the same moment, the one to go from Boston to Springfield, the other from Springfield to Boston. The distance between the two cities is 98 miles. They meet each other at the end of 1 hr. 24 min., and the train from Boston travels as far in 4 hrs. as the other in 3. What was the speed of each train?

19. A grocer bought 50 lbs. of tea and 100 lbs. of coffee for $60. He sold the tea at an advance of $\frac{1}{4}$ on his price, and the coffee at an advance of $\frac{1}{3}$, realizing $77 from both. At what price per pound did he buy and sell each article?

NOTE. If x and y are the prices at which he bought, then $\frac{5}{4}x$ and $\frac{4}{3}y$ are the prices at which he sold.

20. For p dollars I can purchase either a pounds of tea and b pounds of coffee, or m pounds of tea and n pounds of coffee. What is the price per pound of each?

21. A goldsmith had two ingots. The first is composed of equal parts of gold and silver, while the second contains 5 parts of gold to 1 of silver. He wants to take from them a watch-case having 4 ounces of gold and 1 ounce of silver. How much must he take from each ingot?

22. A banker has two kinds of coin, such that a pieces of the first kind or b pieces of the second will make a dollar. If he wants to select c pieces which shall be worth a dollar, how many of each kind must he take?

23. A has a sum of money invested at a certain rate of interest. B has \$1000 more invested, at a rate 1 per cent. higher, and thus gains \$80 more interest than A. C has invested \$500 more than B, at a rate still higher by 1 per cent., and thus gains \$70 more than B. What is the amount each person has invested and the rate of interest?

24. A grocer had three casks of wine, containing in all 344 gallons. He sells 50 gallons from the first cask; then pours into the first one-third of what is in the second, and then into the second one-fifth of what is in the third, after which the first contains 10 gallons more than the second, and the second 10 more than the third. How much wine did each cask contain at first?

Equivalent and Inconsistent Equations.

141. It is not always the case that values of two unknown quantities can be found from two equations. If, for example, we have the equations

$$x + 2y = 3,$$
$$2x + 4y = 6,$$

we see that the second can be derived from the first by multiplying both members by 2. Hence every pair of values of x and y which satisfy the one will satisfy the other also, so that the two are equivalent to a single one.

If the equations were

$$x + 2y = 5,$$
$$2x + 4y = 6,$$

there would be no values of x and y which would satisfy both equations.

For, if we multiply the first by 2 and subtract the second from the product, we shall have,

$$\begin{array}{lr} \text{1st eq.} \times 2, & 2x + 4y = 10 \\ \text{2d eq.}, & \underline{2x + 4y = \ \ 6} \\ \qquad\text{Remainder,} & 0 = \ \ 4, \end{array}$$

an impossible result, which shows that the equations are inconsistent. This will be evident from the equations themselves, because every pair of values of x and y which gives

$$2x + 4y = 6,$$

must also give $\qquad x + 2y = 3,$

and therefore cannot give $x + 2y = 5.$

142. *Generalization of the preceding result.* If we take any two equations of the first degree between x and y which we may represent in the form

$$\left.\begin{aligned} ax + by &= c, \\ a'x + b'y &= c', \end{aligned}\right\} \qquad (1)$$

and eliminate x by addition or subtraction, as in § 137, we have for the equation in y,

$$(a'b - ab')\,y = a'c - ac'.$$

Now it may happen that we have,

$$a'b - ab' = 0 \textit{ identically.} \qquad (2)$$

In this case y will disappear as well as x, and the result will be

$$a'c - ac' = 0.$$

If this equation is identically true, the two equations (1) will be equivalent; if not true, they will be inconsistent. In neither case can we derive any value of y or x.

If we divide the above equation, (2), by aa' we shall have

$$\frac{b}{a} = \frac{b'}{a'}.$$

Hence,

Theorem. If the quoti ;nt of the coefficients of the unknown quantities is the same in the two equations, they will be either equivalent or inconsistent.

This theorem can be expressed in the following form:

If the terms containing the unknown quantity in the one equation can be multiplied by such a factor that they shall both become equal to the corresponding terms of the other equation, the two equations will be either equivalent or inconsistent.

Proof. If there be such a factor m that multiplying the first equation (1) by it, we shall have

$$ma = a',$$
$$mb = b'.$$

Eliminating m, we find

$$a'b - ab' = 0,$$

the criterion of inconsistency or equivalence.

143. When two equations are inconsistent, there are no values of the unknown quantities which will satisfy both equations.

When they are equivalent, it is the same as if we had a single equation; that is, we may assign any value we please to one of the unknown quantities, and find a corresponding value of the other.

CHAPTER IV.

OF INEQUALITIES.

144. *Def.* An **Inequality** is a statement, in the language of Algebra, that one quantity is algebraically greater or less than another.

Def. The quantities declared unequal are called **Members** of the inequality.

The statement that A is greater than B, or that $A - B$ is positive, is expressed by

$$A > B.$$

That A is less than B, or that $A - B$ is negative is expressed by

$$A < B.$$

The form $$A > B > C$$

indicates that the quantity B is less than A but greater than C.

The form $$A \geqq B$$

indicates that A may be either equal to or greater than B, but cannot be less than B.

Properties of Inequalities.

145. *Theorem I.* An inequality will still subsist after the same quantity has been added to or subtracted from each member.

Proof. If the inequality be $A > B$, $A - B$ must be positive. If we add the same quantity H to A and B, or subtract it from them, we shall have $A \pm H - (B \pm H)$, which is equal to $A - B$, and therefore positive. Hence, if

$$A > B,$$

then $$A \pm H > B \pm H.$$

Cor. If any term of an inequality be transposed and its sign changed, the inequality will remain true.

Theorem II. An inequality will still subsist after its members have been multiplied or divided by the same positive number.

Proof. If $A - B$ is positive, then (m or n being positive) $m(A - B)$ or $mA - mB$ will be positive, and so will

$$\frac{A - B}{n} \quad \text{or} \quad \frac{A}{n} - \frac{B}{n}.$$

Hence, if $$A > B,$$

then $$mA > mB,$$

and $$\frac{A}{n} > \frac{B}{n}.$$

It may be shown in the same way that if m or n is negative, $mA - mB$ or $\frac{A}{n} - \frac{B}{n}$ will be negative. Hence,

Theorem III. If both members of an inequality be multiplied or divided by the same negative number, the direction of the inequality will be reversed.

That is, if $$A > B,$$
then $$-mA < -mB,$$
and $$-\frac{A}{n} < -\frac{B}{n}.$$

Theorem IV. If the corresponding members of several inequalities be added, the sum of the greater members will exceed the sum of the lesser members.

Theorem V. If the members of one inequality be subtracted from the non-corresponding members of another, the inequality will still subsist in the direction of the latter.

That is, if $$A > B,$$
$$x > y,$$
then $$A - y > B - x.$$

The proof of the last three theorems is so simple that it may be supplied by the student.

Theorem VI. If two positive members of an inequality be raised to any power, the inequality will still subsist in the same direction.

Proof. Let the inequality be
$$A > B. \qquad (a)$$

Because A is positive, we shall have, by multiplying by A (Th. II),
$$A^2 > AB. \qquad (1)$$

Also, because B is positive, we have, by multiplying (a) by B,
$$AB > B^2. \qquad (2)$$

Therefore, from (1) and (2),

$$A^2 > B^2, \tag{3}$$

Multiplying the last inequality by A,

$$A^3 > AB^2. \tag{4}$$

Multiplying (2) by B,

$$AB^2 > B^3. \tag{5}$$

Whence, $A^3 > B^3$.

The process may be continued to any extent.

Examples of the Use of Inequalities.

146. Ex. 1. If a and b be two positive quantities, such that

$$a^2 + b^2 = 1,$$

we must have $a + b > 1$.

Proof. If $a + b \overline{\overline{<}} 1$,

we should have, by squaring the members (Th. VI),

$$a^2 + 2ab + b^2 \overline{\overline{<}} 1;$$

and by transposing the product $2ab$ (Th. I, Cor.),

$$a^2 + b^2 \overline{\overline{<}} 1 - 2ab.$$

Because a and b are positive, $2ab$ is positive, and

$$1 - 2ab < 1.$$

Therefore we should have

$$a^2 + b^2 < 1,$$

and could not have $a^2 + b^2 = 1$, as was originally supposed.

Ex. 2. If a, b, m, and n are positive quantities, such that

$$\frac{a}{b} > \frac{m}{n}, \tag{a}$$

then the value of the fraction $\dfrac{a+m}{a+n}$ will be contained between the values of $\dfrac{a}{b}$ and $\dfrac{m}{n}$; that is,

$$\frac{a}{b} > \frac{a+m}{b+n} > \frac{m}{n}. \tag{1}$$

To prove the first inequality, we must show that

$$\frac{a}{b} - \frac{a+m}{b+n} \tag{2}$$

is positive. Reducing this expression by § 106, it becomes

$$\frac{an - bm}{b(b+n)}. \tag{3}$$

From the original inequality (a) we have, by multiplying by the positive factor bn,

$$an > bm.$$

That is, $an - bm$ is positive; therefore the fraction (3) with this positive numerator is also positive, and (2) is positive as asserted.

The second inequality (1) may be proved in the same way.

EXERCISES.

1. Prove that if a and b be any quantities different from zero, and $1 > x > -1$, we must have

$$a^2 - 2abx + b^2 > 0.$$

2. Prove that $\left(\frac{a+b}{2}\right)^2 > ab.$

3. If $3x - 5 > 13$, then $x > 6$.

4. If $6x > \frac{3x}{2} + 18$, then $x > 4$.

5. If $\frac{7x}{5} - \frac{5x}{3} > \frac{x}{3} - 3$, then $x > 5$.

6. If $m - nx > p - qx$, then $x > \frac{p-m}{q-n}$.

7. If $\frac{x-y}{m} < 1 - \frac{x}{y}$, and m is positive, then $x < y$.

8. If $a^2 + b^2 + c^2 = 1$, and a, b, and c are not all equal, then $ab + bc + ca < 1$.

SUGGESTION. The squares of $a - b$, $b - c$, and $c - a$ cannot be negative.

BOOK IV.

RATIO AND PROPORTION.

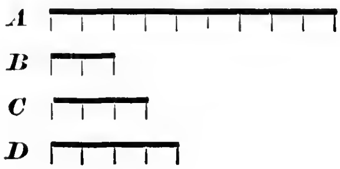
CHAPTER I.

NATURE OF A RATIO.

147. *Def.* The **Ratio** of a quantity A to another quantity B is a number expressing the value of A when compared with B as the standard or unit of measure.

EXAMPLES. Comparing the lengths A, B, C, D, it will be seen that

A is $2\frac{1}{4}$ times D;
B is $\frac{1}{2}$ of D;
C is $\frac{3}{4}$ of D.



We express this relation by saying,

$$\left.\begin{array}{l} \text{The ratio of } A \text{ to } D \text{ is } 2\tfrac{1}{4} \text{ or } \dfrac{9}{4}; \\ \quad\text{“} \quad\quad \text{“} \quad B \text{ to } D \text{ is } \dfrac{1}{2}; \\ \quad\text{“} \quad\quad \text{“} \quad C \text{ to } D \text{ is } \dfrac{3}{4}. \end{array}\right\} \qquad (1)$$

148. The ratio of one quantity to another is expressed by writing the unit of measure after the quantity measured, and inserting a colon between them.

The statements (1) will then be expressed thus:

$$A : D = 2\tfrac{1}{4} = \frac{9}{4}; \quad B : D = \frac{1}{2}; \quad C : D = \frac{3}{4}.$$

Def. The two quantities compared to form a ratio are called its **Terms**.

Def. The quantity measured, or the first term of the ratio, is called the **Antecedent**.

The unit of measure, or the second term of the ratio, is called the **Consequent**.

Rem. When the antecedent is greater than the consequent, the ratio is greater than unity.

When the antecedent is less than the consequent, the ratio is less than unity.

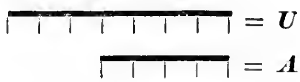
149. To find the ratio of a quantity A to a standard U, we imagine ourselves as measuring off the quantity A with U as a carpenter measures a board with his foot-rule.

There are then three cases to be considered, according to the way the measures come out.

Case I. We may find that, at the end, A comes out an exact number of times U. The ratio is then a whole number, and we say that U **exactly measures** A, or that A is a **multiple** of U.

Case II. We may find that, at the end, the measure does not come out exact, but a piece of A less than U is left over. Or, A may itself be less than U. We must then find what fraction of U the piece left over is equal to. This is done by dividing U up into such a number of equal parts that one of these parts shall exactly measure A or the piece of A which is left over. The ratio will then be a fraction of which the number of parts into which U is divided will be the denominator, and the number of these parts in A the numerator.

Example. If we find that by dividing U into 7 parts, 4 of these parts will exactly make A, then $A = \frac{4}{7}$ of U, and we have for the ratio of A to U,



$$A : U = \frac{4}{7}.$$

If we find that A contains U 3 times, and that there is then a piece equal to $\frac{4}{7}$ of U left over, we have

$$A : U = 3\tfrac{4}{7} = \frac{25}{7}.$$

The 3 U's are equal to $\frac{21}{7}$ of U, so that we may also say

$$A = \frac{25}{7} \text{ of } U, \quad \text{or} \quad A : U = \frac{25}{7}.$$

which is simply the result of reducing the ratio $3\frac{4}{7}$ to an improper fraction.

In general, if we find that by dividing U into n parts, A will be exactly m of these parts, then

$$A : U = \frac{m}{n},$$

whether m is greater or less than n.

When the magnitude of A measured by U can be exactly expressed by a vulgar fraction, A and U are said to be **commensurable**.

CASE III. It may happen that there is no number or fraction which will exactly express the ratio of the two magnitudes. The latter are then said to be **incommensurable**.

150. *Theorem.* The ratio of two incommensurable magnitudes may always be expressed as near the true value as we please by means of a fraction, if we only make the denominator large enough.

EXAMPLES. Let us divide the unit of measure into 20 parts, and suppose that the antecedent contains more than 28 but less than 29 of these parts. Then, by supposing it to contain 28 parts, the limit of error will be one part, or $\frac{1}{20}$ of the standard unit.

In general, if we wish to express the ratio within 1 n^{th} of the unit, we can certainly do it by dividing the unit into n or more parts, or by taking as the denominator of the fraction a number not less than n.

Illustration by Decimal Fractions. The square root of 2 cannot be rigorously expressed as a vulgar or decimal fraction. But, if we suppose

$\sqrt{2} = 1.4$	$= \frac{14}{10}$,	the error will be		$< \frac{1}{10}$;
$\sqrt{2} = 1.41$	$= \frac{141}{100}$,	"	"	$< \frac{1}{100}$;
$\sqrt{2} = 1.414$	$= \frac{1414}{1000}$,	"	"	$< \frac{1}{1000}$.
etc. etc.	etc.			etc.

Since the decimals may be continued without end, the square root of 2 can be expressed as a decimal fraction with an error less than any assignable quantity. This general fact is expressed by saying:

The limit of the error which we make by representing an incommensurable ratio as a fraction is zero.

151. *Ratio as a Quotient.* From Case II and the explanations which precede it we see that when we say

$$A : U = \frac{4}{7},$$

we mean the same thing as if we had said,

$$A \text{ is } \tfrac{4}{7} \text{ of } U, \quad \text{or} \quad A = \tfrac{4}{7} U.$$

If A and U are numbers, we may divide both sides of this equation by U, and obtain,

$$\frac{A}{U} = \frac{4}{7}.$$

We therefore conclude that when A and U are numbers,

$$A : U = \frac{A}{U}.$$

That is,

Theorem. The ratio of two numbers is equal to the quotient obtained by dividing the antecedent term by the consequent.

In the case of magnitudes, the relation of a ratio to a quotient may be shown thus:

Let us have two magnitudes M and V, such that M is 4 times V. Then we may write the relation,

$$M = 4V.$$

Dividing by 4, we have

$$\frac{M}{4} = V.$$

Since V is not a number, we cannot, strictly speaking, multiply or divide by it. But we may take the ratio of M to V without regard to number, and thus find,

$$M : V = 4.$$

REM. The theory of ratios the terms of which are magnitudes and not numbers, is treated in Geometry.

In Algebra we consider the ratios of numbers, or of magnitudes represented by numbers.

152. *Def.* If we interchange the terms of a ratio, the result is called the **Inverse** ratio.

That is, $U : A$ is the inverse of $A : U$.

If $$U : A = \frac{m}{n},$$

then $$U = \frac{m}{n} A,$$

and we have, by dividing by $\frac{m}{n}$,

$$A = \frac{n}{m} U,$$

or $$A : U = \frac{n}{m}.$$

Because $\frac{n}{m}$ is the reciprocal of $\frac{m}{n}$, we conclude:

Theorem. The inverse ratio is the reciprocal of the direct ratio.

Properties of Ratios.

153. *Theorem I.* If both terms of a ratio be multiplied by the same factor or divided by the same divisor, the ratio is not altered.

Proof. Ratio of B to $A = B : A = \frac{B}{A}$.

If m be the factor, then

$$\text{Ratio of } mB \text{ to } mA = mB : mA = \frac{mB}{mA} = \frac{B}{A},$$

the same as the ratio of B to A.

154. *Theorem II.* If both terms of a ratio be increased by the same quantity, the ratio will be increased if it is less than 1, and diminished if it is greater than 1; that is, it will be brought nearer to unity.

EXAMPLE. Let the original ratio be $2 : 5 = \frac{2}{5}$. If we repeatedly add 1 to both numerator and denominator of the fraction, we shall have the series of fractions,

$$\tfrac{2}{5},\ \tfrac{3}{6},\ \tfrac{4}{7},\ \tfrac{5}{8},\ \text{etc.},$$

each of which is greater than the preceding, because

$$\begin{array}{lll} \frac{3}{6} - \frac{2}{5} = \frac{3}{30}; & \text{whence,} & \frac{3}{6} > \frac{2}{5}. \\ \frac{4}{7} - \frac{3}{6} = \frac{3}{42}; & \text{whence,} & \frac{4}{7} > \frac{3}{6}. \\ \frac{5}{8} - \frac{4}{7} = \frac{3}{56}; & \text{whence,} & \frac{5}{8} > \frac{4}{7}. \\ \text{etc.} & & \text{etc.} \end{array}$$

General Proof. Let $a : b$ be the original ratio, and let both terms be increased by the quantity u, making the new ratio $a+u : b+u$. The new ratio *minus* the old one will be

$$\frac{(b-a)\,u}{b^2 + bu}.$$

If b is greater than a, this quantity will be positive, showing that the ratio is increased by adding u. If b is less than a, the quantity will be negative, showing that the ratio is diminished by adding u.

CHAPTER II.

PROPORTION.

155. *Def.* **Proportion** is an equality of two or more ratios.

Since each ratio has two terms, a proportion must have at least four terms.

Def. The terms which enter into two equal ratios are called **Terms** of the proportion.

If $a : b$ be one of the ratios, and $p : q$ the other, the proportion will be,

$$a : b = p : q. \qquad (1)$$

A proportion is sometimes written,

$$a : b :: p : q,$$

which is read, "As a is to b so is p to q." The first form is to be preferred, because no other sign than that of equality is necessary, but the equation may be read, "As a is to b so is p to q," whenever that expression is the clearer.

Def. The first and fourth terms of a proportion are called the **Extremes**, the second and third are called the **Means.**

Theorems of Proportion.

156. *Theorem I.* In a proportion the product of the extremes is equal to the product of the means.

Proof. Let us write the ratios in the proportion (1) in the form of fractions. It will give the equation,

$$\frac{a}{b} = \frac{p}{q}. \qquad (2)$$

Multiplying both sides of this equation by bq, we shall have

$$aq = bp. \qquad (3)$$

Cor. If there are two unknown terms in a proportion, they may be expressed by a single unknown symbol.

Example. If it be required that one quantity shall be to another as p to q, we may call the first px and the second qx, because

$$px : qx = p : q \text{ (identically).}$$

157. *Theorem II.* If the means in a proportion be interchanged, the proportion will still be true.

Proof. Divide the equation (3) by pq. We shall then have, instead of the proportion (1),

$$\frac{a}{p} = \frac{b}{q},$$

or

$$a : p = b : q.$$

Def. The proportion in which the means are interchanged is called the **Alternate** of the original proportion.

The following examples of alternate proportions should be studied, and the truth of the equations proved by calculation:

$$1 : 2 = 4 : 8; \quad \text{alternate,} \quad 1 : 4 = 2 : 8.$$
$$2 : 3 = 6 : 9; \quad \text{“} \quad 2 : 6 = 3 : 9.$$
$$5 : 2 = 25 : 10; \quad \text{“} \quad 5 : 25 = 2 : 10.$$

158. *Theorem III.* If, in a proportion, we increase or diminish each antecedent by its consequent, or each consequent by its own antecedent, the proportion will still be true.

EXAMPLE. In the proportion,

$$5 : 2 = 25 : 10,$$

the antecedents are 5 and 25, the consequents 2 and 10 (§ 148). Increasing each antecedent by its own consequent, the proportion will be

$$5+2 : 2 = 25+10 : 10, \quad \text{or} \quad 7 : 2 = 35 : 10.$$

Diminishing each antecedent by its consequent, the proportion will become,

$$5-2 : 2 = 25-10 : 10, \quad \text{or} \quad 3 : 2 = 15 : 10.$$

Increasing each consequent by its antecedent, the proportion will be

$$5 : 2+5 = 25 : 10+25, \quad \text{or} \quad 5 : 7 = 25 : 35.$$

These equations are all to be proved numerically.

General Proof. Let us put the proportion in the form

$$\frac{a}{b} = \frac{p}{q}. \tag{4}$$

If we add 1 to each side of this equation and reduce each side, it will give

$$\frac{a+b}{b} = \frac{p+q}{q};$$

that is,

$$a + b : b = p + q : q. \tag{5}$$

In the same way, by subtracting 1 from each side, it will be

$$a - b : b = p - q : q. \tag{6}$$

If we invert the fractions in equation (4), the latter will become

$$\frac{b}{a} = \frac{q}{p}.$$

By adding or subtracting 1 from each side of this equation, and then again inverting the terms of the reduced fractions, we shall find,

$$a : b + a = p : q + p;$$
$$a : b - a = p : q - p.$$

The form (5) was formerly designated as formed "by composition," and (6) as formed "by division." But these terms are now useless, because all the above forms are only special cases of a more general one to be now explained.

159. *Theorem IV.* If four quantities form the proportion

$$a : b = c : d, \qquad (a)$$

and if m, n, p, and q be any multipliers whatever, we shall have

$$ma + nb : pa + qb = mc + nd : pc + qd.$$

Proof. The proportion (a) gives the equation,

$$\frac{a}{b} = \frac{c}{d}.$$

Multiplying this equation by $\frac{p}{q}$ and adding 1 to each member,

$$\frac{pa}{qb} + 1 = \frac{pc}{qd} + 1.$$

Reducing each member to a fraction and inverting the terms,

$$\frac{qb}{pa + qb} = \frac{qd}{pc + qd}.$$

Dividing both members by q,

$$\frac{b}{pa + qb} = \frac{d}{pc + qd}. \qquad (7)$$

The original proportion (a) also gives, by inversion,

$$\frac{b}{a} = \frac{d}{c},$$

from which we obtain, by multiplying by $\frac{q}{p}$, adding 1, etc.,

$$\frac{qb + pa}{pa} = \frac{qd + pc}{pc}.$$

$$\frac{a}{pa + qb} = \frac{c}{pc + qd}. \qquad (8)$$

$(8) \times m + (7) \times n$ gives the equation,

$$\frac{ma + nb}{pa + qb} = \frac{mc + nd}{pc + qd},$$

or $$ma + nb : pa + qb = mc + nd : pc + qd, \qquad (9)$$

which is the result to be demonstrated.

160. *Theorem V.* If each term of a proportion be raised to the same power, the proportion will still subsist.

Proof. If $$a : b = p : q,$$

or $$\frac{a}{b} = \frac{p}{q},$$

then, by multiplying each member by itself repeatedly, we shall have

$$\frac{a^2}{b^2} = \frac{p^2}{q^2};$$

$$\frac{a^3}{b^3} = \frac{p^3}{q^3};$$

etc. etc.

Hence, in general,

$$a^n : b^n = p^n : q^n.$$

Cor. If $$a : b = p : q,$$

then $$a^n : a^n \pm b^n = p^n : p^n \pm q^n;$$

and $$a^n \pm b^n : b^n = p^n \pm q^n : q^n.$$

Theorem VI. When three terms of a proportion are given, the fourth can always be found from the theorem that the product of the means is equal to that of the extremes.

We have shown that whenever

$$a : b = p : q,$$

then $$aq = bp.$$

Considering the different terms in succession as unknown quantities, we find,

$$a = \frac{bp}{q},$$

$$b = \frac{aq}{p},$$

$$p = \frac{aq}{b},$$

$$q = \frac{bp}{a}.$$

Cor. 1. If, in the general equation of the first degree

$$ax + by = c,$$

the term c vanishes, the equation determines the ratio of the unknown quantities.

Proof. If $$ax + by = 0,$$

then $$ax = -by,$$

and $$\frac{x}{y} = -\frac{b}{a},$$

or $$x : y = -b : a.$$

Cor. 2. Conversely, if the ratio of two unknown quantities is given, the relation between them may be expressed by an equation of the first degree.

The Mean Proportional.

161. *Def.* When the middle terms of a proportion are equal, either of them is called the **Mean Proportional** between the extremes.

The fact that b is the mean proportional between a and c is expressed in the form,

$$a : b = b : c.$$

Theorem I then gives, $b^2 = ac$.

Extracting the square root of both members, we have

$$b = \sqrt{ac}.$$

Hence,

Theorem VII. The mean proportional of two quantities is equal to the square root of their product.

Multiple Proportions.

162. We may have any number of ratios equal to each other, as

$$a : b = c : d = e : f, \text{ etc.}$$

$$6 : 4 = 9 : 6 = 3 : 2 = 21 : 14. \qquad (a)$$

Such proportions are sometimes written in the form

$$6 : 9 : 3 : 21 = 4 : 6 : 2 : 14. \qquad (b)$$

In the form (b) the antecedents are all written on one side of the equation, and the consequents on the other. Any two numbers on one side then have the same ratio as the corresponding two on the other, and the proportions expressed by this equality of ratios are the alternates of the original proportions (a). For instance, in the proportion (b) we have,

$6 : 9 = 4 : 6$,	which is the alternate of	$6 : 4 = 9 : 6$.
$6 : 3 = 4 : 2$,	" " "	$6 : 4 = 3 : 2$.
$6 : 21 = 4 : 14$,	" " "	$6 : 4 = 21 : 14$.
$9 : 21 = 6 : 14$,	" " "	$9 : 6 = 21 : 14$.

163. A multiple proportion may also be expressed by a number of equations equal to that of the ratios. Since

$$a : b = c : d = e : f, \text{ etc.},$$

let us call r the common value of these ratios, so that

$$\frac{a}{b} = r, \quad \frac{c}{d} = r, \quad \text{etc.}$$

Then

$$\begin{aligned} a &= rb, \\ c &= rd, \\ e &= rf, \end{aligned} \qquad (c)$$

will express the same relations between the quantities a, b, c, d, e, f, etc., that is expressed by

$$a : b = c : d = e : f, \text{ etc.,} \qquad (a)$$

or

$$a : c : e : \text{etc.} = b : d : f : \text{etc.} \qquad (b)$$

It will be seen that where r enters in the form (c) there is one more equation than in the first form (a). [In this form each = represents an equation.] This is because the additional quantity r is introduced, by eliminating which we diminish the number of equations by one, as in eliminating an unknown quantity.

164. *Theorem.* In a multiple proportion, the sum of any number of the antecedents is to the sum of the corresponding consequents as any one antecedent is to its consequent.

Ex. We have $\frac{2}{5} = \frac{6}{15} = \frac{10}{25} = \frac{12}{30}$. Then

$$\frac{2+6+10+12}{5+15+25+30} = \frac{30}{75},$$

which has the same value as the other four functions.

General Proof. Let A, B, C, etc., be the antecedents, and a, b, c, etc., the corresponding consequents, so that

$$A : a = B : b = C : c, \text{ etc.} \qquad (1)$$

Let us call r the common ratio $A : a$, $B : b$, etc., so that

$$\begin{aligned} A &= ra, \\ B &= rb, \\ C &= rc. \\ \text{etc.} &\quad \text{etc.} \end{aligned}$$

Adding these equations, we have

$$A + B + C + \text{etc.} = r(a + b + c + \text{etc.}),$$

or

$$\frac{A + B + C + \text{etc.}}{a + b + c + \text{etc.}} = r;$$

that is, the ratio $A+B+C+\text{etc.} : a+b+c+\text{etc.}$ is equal to r, the common value of the ratios $A : a$, $B : b$, etc.

PROBLEMS.

1. A map of a country is made on a scale of 5 miles to 3 inches.

(1.) What will be the length of 8, 12, 17, 20, 33 miles on the map?

(2.) How many miles will be represented by 6, 8, 16, 20, 29 inches on the map?

REM. 1. If x, y, z, u, v be the required spaces on the map, we shall have

$$5 : 3 = 8 : x = 12 : y, \text{ etc.}$$

If a, b, c, etc., be the required number of miles, we shall have

$$3 : 5 = 6 : a = 8 : b = 16 : c, \text{ etc.}$$

REM. 2. When there are several ratios compared, as in this problem, it will be more convenient to take the inverse of the common ratio, and multiply the antecedent of each following ratio by it to obtain the consequent. In the first of the above proportions the inverse ratio is $\frac{3}{5}$, and

$$x = \tfrac{3}{5} \text{ of } 8, \quad y = \tfrac{3}{5} \text{ of } 12, \text{ etc.}$$

In the second, $\quad a = \frac{5}{3}$ of 6, $\quad b = \frac{5}{3}$ of 8, etc.

2. To divide a given quantity A into three parts which shall be proportional to the given quantities a, b, c, that is, into the parts x, y, and z, such that

$$x : a = y : b = z : c,$$

or

$$x : y : z = a : b : c.$$

SOLUTION. By Theorem IV,

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{x + y + z}{a + b + c} = \frac{A}{a + b + c}.$$

Therefore,

$$x = \frac{aA}{a + b + c}, \quad y = \frac{bA}{a + b + c}, \quad z = \frac{cA}{a + b + c}$$

3. Divide 102 into three parts which shall be proportional to the numbers 2, 4, 11.

4. Divide 1000 into five parts which shall be proportional to the numbers 1, 2, 3, 4, 5.

5. Find two fractions whose ratio shall be that of $a : b$, and whose sum shall be 1.

6. What two numbers are those whose ratio is that of $7 : 3$ and whose difference is 24.

7. What two numbers are those whose ratio is $m : n$, and whose difference is unity?

8. Find x and y from the conditions,

$$x : y = a : b,$$
$$ax - by = a + b.$$

9. Show that if $a : b = A : B,$
$c : d = C : D,$
we must also have $ac : bd = AC : BD.$

10. Having given $x = ay$, find the value of $\frac{x + 2y}{x - 2y}$.

11. Having given

$$\frac{x + 2y}{x - 2y} = 5,$$

find the value of $\frac{x + y}{x - y}$.

12. If $a : b = p : q,$

prove $a^2 + b^2 : \frac{a^3}{a + b} = p^2 + q^2 : \frac{p^3}{p + q},$

and $a^n + b^n : \frac{a^{n+1}}{a + b} = p^n + q^n : \frac{p^{n+1}}{p + q}.$

13. If $\frac{a + b + c + d}{a + b - c - d} = \frac{a - b + c - d}{a - b - c + d},$

show that $a : b = c : d.$

14. A year's profits were divided among three partners, A, B, and C, proportional to the numbers 2, 3, and 7. If C should pay B \$1256, their shares would be equal. What was the amount divided?

15. In a first year's partnership between A and B, A had 2 shares and B had 5. In the second year, A had 3 and B had 4. In the second year, A's profits were \$3200 greater and B's were \$1700 greater than they were the first. What was each year's profits?

16. In a poultry yard there are 7 chickens to every 2 ducks, and 3 ducks to every 2 geese. How many geese were there to every 42 chickens?

17. A drover started with a herd containing 4 horses to every 9 cattle. He sold 148 horses and 108 cattle, and then had 1 horse to every 3 cattle. How many horses and cattle had he at first?

18. If a bowl of punch contains a parts of water and b parts of wine, what is the ratio of the wine to the whole punch? What is the ratio of the water? What are the sums of these ratios?

19. One ingot consists of equal parts of gold and silver, while another has two parts of gold to one of silver. If I combine equal weights from these ingots, what proportion of the compound will be gold and what proportion silver?

20. What will be the proportions if, in the preceding problem, I combine one ounce from the first ingot with three from the second?

21. One cask contains a gallons of water and b gallons of alcohol, while another contains m gallons of water and n of alcohol. If I draw one gallon from each cask and mix them, what will be the quantities of alcohol and water?

22. What will be the ratio of the liquors in the last case, if I mix two parts from the first cask with one from the second?

23. What will it be if I mix p parts from the first with q parts from the second?

24. A goldsmith has two ingots, each consisting of an alloy of gold and silver. If he combines two parts from the first ingot with one from the second, he will have equal parts of gold and silver. If he combines one part from the first with two from the second, he will have 3 parts of gold to 5 of silver. What is the composition of each ingot?

SUGGESTION. Call r the ratio of the weight of gold in the first ingot to the whole weight of the ingot; then $1 - r$ will be the ratio of the silver in the first to the whole weight of the ingot. See the following question.

NOTE. Problems 18–24 form a graduated series, introductory to the processes of Problem 24.

25. Point out the mistake which would be made if the solution of the preceding problem were commenced in the following way:

If the first ingot contains p parts of gold to q parts of silver, and the second contains r parts of gold to s of silver, then

Two parts from the first ingot will have $2p$ of gold and $2q$ of silver.

One part from the second ingot will have r of gold and s of silver.

Therefore, the combination will contain $2p+r$ parts of gold, and $2q+s$ parts of silver.

Show also that if we subject p, q, r, and s to the condition

$$p + q = r + s,$$

the process would be correct.

26. Show that if the second term of a proportion be a mean proportional between the third and fourth, the third will be a mean proportional between the first and second.

BOOK V.

OF POWERS AND ROOTS.

CHAPTER I.

INVOLUTION.

CASE I. **Involution of Products and Quotients.**

165. *Def.* The result of taking a quantity, A, n times as a factor is called the **n^{th} power of A**, and as already known may be written either

$$AAA, \text{ etc., } n \text{ times,} \quad \text{or} \quad A^n.$$

Def. The number n is called the **Index** of the power.

Def. **Involution** is the operation of finding the powers of algebraic expressions.

The operation of involution may always be expressed by the application of the proper exponent, the expression to be involved being inclosed in parentheses.

EXAMPLE. The n^{th} power of $a + b$ is $(a + b)^n$.

The n^{th} power of abc is $(abc)^n$.

166. *Involution of Products.* The n^{th} power of the product of several factors a, b, c, may be expressed without exponents as follows:

$$abc\, abc\, abc, \text{ etc.,}$$

each factor being repeated n times.

Here there will be altogether n a's, n b's, and n c's, so that, using exponents, the whole power will be $a^nb^nc^n$ (§ 66, 67).

Hence, $$(abc)^n = a^nb^nc^n.$$

That is,

Theorem. The power of a product is equal to the product of the powers of the several factors.

167. *Involution of Quotients.* Applying the same methods to fractions, we find that the n^{th} power of $\frac{x}{y}$ is $\frac{x^n}{y^n}$. For

$$\left(\frac{x}{y}\right)^n = \frac{x}{y}\,\frac{x}{y}\,\frac{x}{y}, \text{ etc., } n \text{ times};$$

$$= \frac{xxx, \text{ etc., } n \text{ times}}{yyy, \text{ etc., } n \text{ times}} \quad (\S\ 109);$$

$$= \frac{x^n}{y^n}.$$

EXERCISES.

Express the cubes of

1. abc. 2. $\frac{ab}{c}$. 3. abc^{-1}.

4. $\frac{mn}{pq}$. 5. $\frac{a+b}{a-b}$. 6. $\frac{mn\,(a+b)}{pq\,(a-b)}$.

Express the n^{th} powers of the same quantities, the quantities between parentheses being treated as single symbols.

Case II. Involution of Powers.

168. Problem. It is required to raise the quantity a^m to the n^{th} power.

Solution. The n^{th} power of a^m is, by definition,

$$a^m \times a^m \times a^m, \text{ etc., } n \text{ times.}$$

By § 66, the exponents of a are all to be added, and as the exponent m is repeated n times, the sum

$$m + m + m + \text{etc., } n \text{ times,}$$

is mn. Hence the result is a^{mn}, or, in the language of Algebra,

$$(a^m)^n = a^{mn}.$$

Hence,

Theorem. If any power of a quantity is itself to be raised to a power, the indices of the powers must be multiplied together.

EXAMPLES.

$$(a^2)^3 = a^2a^2a^2 = a^6.$$
$$(3ab^2c^3)^4 = 81a^4b^8c^{12}.$$

NOTE. It will be seen that this theorem coincides with that of Case I when any of the factors have the exponent unity understood.

EXERCISES.

Write the cubes of the following quantities:

1. $3xy^2$. 2. $\frac{4a}{b}$. 3. a^m.

4. bx^4. 5. $2a^2m^n$. 6. $\frac{6a^m}{b}$.

Write the n^{th} powers of

7. a. 8. a^2b. 9. a^3b^2c.
10. a^ma^n. 11. $2p^mq^2$. 12. $(a+b)(c+d)$.
13. $(x+y)(x-y)$.
14. $7(a+b-c)(a-b)^p$.

Ans. $7^n(a+b-c)^n(a-b)^{np}$.

15. $\frac{a}{b}$. 16. $\frac{a^2}{b^2}$. 17. $\frac{x+y}{x-y}$.

18. $\frac{m^2ab^3}{xy^2}$. *Ans.* $\frac{m^{2n}a^nb^{3n}}{x^ny^{2n}}$.

19. $\frac{ab(c-d)^2}{(a-b)c^3}$.

Reduce:

20. $(2ab^2n^3)^3$. 21. $(-3mnx^2)^2$.
22. $2a(-3b^2mn^3)^3$. 23. $(7pq^2r^3)^4$.
24. $(ab^n)^4$. 25. $(2a^2x^3)^n$. 26. $(m^n)^n$.

NOTE 1. If the student find any of these exponential expressions difficult of expression, he may first express them by writing each quantity a number of times indicated by its exponent.

NOTE 2. The student is expected to treat the quantities in parentheses as single symbols.

REM. The preceding theorem finds a practical application when it is necessary to raise a small number to a high power. If, for example, we have to raise 2 to the 30th power, we should, without this theorem, have to multiply by 2 no less than 29 times. But we may also proceed thus:

$$\begin{aligned}
2^2 &= 4, \\
2^4 &= 2^2 \cdot 2^2 = 4 \cdot 4 = 16, \\
2^8 &= 2^4 \cdot 2^4 = 16 \cdot 16 = 256, \\
2^{16} &= 2^8 \cdot 2^8 = 256^2 = 65536, \\
2^{24} &= 2^{16} \cdot 2^8 = 2^{16} \cdot 256 = 16777216, \\
2^{30} &= 2^{24} \cdot 2^6 = 2^{24} \cdot 64 = 1073741824.
\end{aligned}$$

Case of Negative Exponents.

169. The preceding theorem may be applied to negative exponents. By the definition of such exponents,

$$\frac{a^p}{b^q} = a^p b^{-q}. \qquad (1)$$

Raising the first member to the n^{th} power, we have,

$$\left(\frac{a^p}{b^q}\right)^n = \frac{a^{np}}{b^{nq}} = a^{np} b^{-nq}.$$

This is the same result we should get by applying the theorem to the second member of (1), and proves the proposition.

EXERCISES.

Express the 6th powers of

1. ab^{-1}.
2. a^2b^{-2}.
3. amp^{-3}.
4. $a^{-m}b^{-n}$.
5. $(a+b)^3(a-b)^{-3}$.
6. $(x+y)^n(x+z)^{-n}$.
7. $\dfrac{a^{-p}}{b^{-q}}$.
8. $\dfrac{(a+b)^{-m}}{(a-b)^{-n}}$.

Reduce:

9. $[(a+b)^{-1}(a-b)]^n$.
10. $(ab^{-1}c^{-2})^5$.
11. $(ab^{-1}c^{-2})^{-5}$.
12. $(m^in^{-j})^{-i}$.
13. $(x^iy^{-i})^{-i}$.
14. $(a^0b^nc^{-n})^n$.

After forming the expressions, write them all with positive exponents, in the form of fractions.

Algebraic Signs of Powers.

170. Since the continued product of any number of positive factors is positive, all the powers of a positive quantity are positive.

By § 26, the product of an odd number of negative factors is negative, and the product of an even number is positive. Hence,

Theorem. The even powers of negative quantities are positive, and the odd powers are negative.

EXAMPLES.

$$(-a)^2 = a^2; \quad (-a)^3 = -a^3; \quad (-a)^4 = a^4, \quad \text{etc.}$$

EXERCISES.

Find the value of

1. $(-2)^2$.	2. $(-3)^3$.	3. 4^4.
4. $(-5)^2$.	5. $(-5)^3$.	6. $(-b)^7$.
7. $(-a-b)^3$.	8. $(-mn)^7$.	9. $(-pq)^6$.
10. $(-a)^{2n}$.	11. $(-b)^{2n+1}$.	12. $(-a-b)^{2n-1}$.
13. $(-1)^{2n}$.	14. $(-1)^{2n+1}$.	15. $(-1)^{2n-1}$.

Case III. ***Involution of Binomials—the Binomial Theorem.***

171. *It is required to find the n^{th} power of a binomial.*

1. Let $a + b$ be the binomial; its n^{th} power may be written

$$(a+b)^n.$$

Let us now transform this expression by dividing it by a^n, and then multiplying it by a^n, which will reduce it to its original value. We have (§ 167),

$$\frac{(a+b)^n}{a^n} = \left(\frac{a+b}{a}\right)^n = \left(1 + \frac{b}{a}\right)^n.$$

Multiplying this last expression by a^n, by writing this power outside the parentheses, it becomes

$$a^n\left(1 + \frac{b}{a}\right)^n, \qquad (1)$$

which is equal to $(a + b)^n$. Next let us put for shortness x to represent $\frac{b}{a}$, when the expression will become

$$(a + b)^n = a^n (1 + x)^n. \qquad (2)$$

2. Now let us form the successive powers of $(1 + x)^n$. We multiply according to the method of § 79:

$$\begin{array}{lrl}
 & (1 + x)^1 = & 1 + x \\
\text{Multiplier,} & & \underline{1 + x} \\
 & & 1 + x \\
 & & \underline{\quad + x + x^2} \\
 & (1 + x)^2 = & 1 + 2x + x^2 \\
\text{Multiplier,} & & \underline{\qquad 1 + x} \\
 & & 1 + 2x + x^2 \\
 & & \underline{\qquad x + 2x^2 + x^3} \\
 & (1 + x)^3 = & 1 + 3x + 3x^2 + x^3 \\
\text{Multiplier,} & & \underline{\qquad\qquad 1 + x} \\
 & & 1 + 3x + 3x^2 + x^3 \\
 & & \underline{\qquad x + 3x^2 + 3x^3 + x^4} \\
 & (1 + x)^4 = & 1 + 4x + 6x^2 + 4x^3 + x^4
\end{array}$$

It will be seen that whenever we multiply one of these powers by $1 + x$, the coefficients of x, x^2, etc., which we add to form the next higher power are the same as those of the given power, only those in the lower line go one place toward the right. Thus, to form $(1 + x)^4$, we took the coefficients of $(1 + x)^3$, and wrote and added them thus:

$$\begin{array}{lccccc}
\text{Coef. of } (1 + x)^3, & 1, & 3, & 3, & 1. & \\
 & & 1, & 3, & 3, & 1. \\
\hline
\text{Coef. of } (1 + x)^4, & 1, & 4, & 6, & 4, & 1.
\end{array}$$

It is not necessary to write the numbers under each other to add them in this way; we have only to add each number to the one on the left in the same line to form the corresponding number of the line below. Thus we can form the coefficients of the successive powers of x at sight as follows: The first figure in each line is 1; the next is the coefficient of x; the third the coefficient of x^2, etc.

First power,	$n = 1$,	coefficients,	1, 1.
Second "	$n = 2$,	"	1, 2, 1.
Third "	$n = 3$,	"	1, 3, 3, 1.
Fourth "	$n = 4$,	"	1, 4, 6, 4, 1.
Fifth "	$n = 5$,	"	1, 5, 10, 10, 5, 1.
Sixth "	$n = 6$,	"	1, 6, 15, 20, 15, 6, 1.
etc.	etc.		etc.

It is evident that the first quantity is always 1, and that the next coefficient in each line, or the coefficient of x, is n.

The third is not evident, but is really equal to

$$\frac{n(n-1)}{2}, \tag{b}$$

as will be readily found by trial; because, beginning with $n = 3$,

$$3 = \frac{3 \cdot 2}{2}, \quad 6 = \frac{4 \cdot 3}{2}, \quad 10 = \frac{5 \cdot 4}{2}, \quad \text{etc.}$$

The fourth number on each line is

$$\frac{n(n-1)(n-2)}{2 \cdot 3}.$$

Thus, beginning as before with the third line, where $n = 3$,

$$1 = \frac{3 \cdot 2 \cdot 1}{2 \cdot 3}, \quad 4 = \frac{4 \cdot 3 \cdot 2}{2 \cdot 3}, \quad 10 = \frac{5 \cdot 4 \cdot 3}{2 \cdot 3}, \quad \text{etc.} \tag{c}$$

3. These several quantities are called **Binomial Coefficients.** In writing them, we may multiply all the denominators by the factor 1 without changing them, so that there will be as many factors in the denominator as in the numerator. The fourth column of coefficients, or (c), will then be written,

$$\frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3}, \quad \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}, \quad \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}, \quad \text{etc.}$$

4. We can find all the binomial coefficients of any power when we know the value of n.

The numerator and denominator of the second coefficient will contain two factors, as in (b); of the third, three factors, as in (c); and of the s^{th}, s factors, whatever s may be.

In any coefficient, the first factor in the numerator is n, the second $n - 1$, etc., each factor being less by unity than the preceding one, until we come to the s^{th} or last, which will be $n - s + 1$.

Such a product is written,

$$n(n-1)(n-2)\ldots\ldots(n-s+1).$$

The dots stand for any number of omitted factors, because s may be any number. We have written 4 of the s factors, so that $s - 4$ are left to be represented by the dots.

The denominator of the fraction is the product of the s factors,

$$1 \cdot 2 \cdot 3 \ldots\ldots s,$$

each factor being greater by 1 than the preceding one, and the dots standing for any number of omitted factors, according to the value of s. Thus, the s^{th} coefficient in the n^{th} line will be

$$\frac{n(n-1)(n-2)\ldots\ldots(n-s+1)}{1 \cdot 2 \cdot 3 \ldots\ldots s}. \qquad (d)$$

If s is greater than $\frac{1}{2}n$, the last factors will cancel some of the preceding ones, so that as s increases from $\frac{1}{2}n$ to n, the values of the preceding coefficients will be repeated in the reverse order. Thus, suppose $n = 6$. Then, by cancelling common factors,

$$\frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{6 \cdot 5}{1 \cdot 2} = 15.$$

$$\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{6}{1} = 6.$$

$$\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = 1.$$

If we should add one more factor to the numerator, it would be 0, and the whole coefficient would be 0.

The conclusion we have reached is embodied in the following equation, which should be perfectly memorized:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \ldots\ldots + x^n.$$

EXERCISES.

1. Compute from the formula (d) all the binomial coefficients for $n = 6$, and from them express the development of $(1 + x)^6$.

2. Do the same thing for $n = 8$, and for $n = 10$.

172. To find the development of $(a + b)^n$, we replace x by $\frac{b}{a}$, and then multiply each term by a^n.

[See equations (1) and (2).] We thus have

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \text{etc. to } b^n.$$

The terms of the development are subject to the following rules:

I. *The exponents of **b**, or the second term of the binomial, are 0, 1, 2, etc., to **n**.*

Because b^0 is simply 1, a^n is the same as a^nb^0.

II. *The sum of the exponents of **a** and **b** is **n** in each term. Hence the exponents of **a** are*

$$n, \quad n-1, \quad n-2, \quad \text{etc., to} \quad 0.$$

III. *The coefficient of the first term is unity, and of the second **n**, the index of the power. Each following coefficient may be found from the next preceding one by multiplying by the successive factors,*

$$\frac{n-1}{2}, \quad \frac{n-2}{3}, \quad \frac{n-3}{4}, \quad \text{etc.}$$

IV. *If **b** or **a** is negative, the sign of its odd powers will be changed, but that of its even powers will remain the same.*

(Compare § 170.) Hence.

$$(a - b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 - \text{etc.},$$

the terms being alternately positive and negative.

EXERCISES—Continued.

3. Compute all the terms of $(a + b)^9$, using the binomial coefficients.

4. What is the coefficient of b^3 in the development of $(a + b)^{10}$.

5. What are the first four terms in the development of $(2am + 3n)^8$.

6. What are the first three terms in the development of $\left(1 + \frac{x}{y}\right)^{18}$? What are the last two terms?

7. What are the first three and the last three terms of $(a - x)^{13}$?

8. What is the development of $\left(a + \frac{1}{a}\right)^6$.

9. What are the first four terms in the development of the following binomials:

$$(1 + x^2)^n; \qquad (1 + 2x^2)^n; \qquad (1 - 2x^2)^n;$$

$$\left(\frac{1}{ax} + a\right)^8; \qquad \left(7\frac{y^2}{x^2} - 8\frac{x^2}{y^2}\right)^5; \qquad (3am^{\frac{1}{2}} - 5bn^{\frac{1}{2}})^{10}?$$

10. What are the sum and difference of the two developments, $(1 + x)^7$ and $(1 - x)^7$?

CASE IV. *Square of a Polynomial.*

173. 1. *Square of any Polynomial.* Let

$$a + b + c + d + \text{etc.},$$

be any polynomial. We may form its square thus:

$$\begin{array}{l}
a + b + c + d + \text{etc.} \\
\underline{a + b + c + d + \text{etc.}} \\
a^2 + ab + ac + ad + \text{etc.} \\
\qquad ab + b^2 + bc + bd + \text{etc.} \\
\qquad\qquad ac + bc + c^2 + cd + \text{etc.} \\
\qquad\qquad\qquad \underline{ad + bd + cd + d^2 + \text{etc.}} \\
a^2 + b^2 + c^2 + d^2 + \text{etc.} \\
\quad + 2ab + 2ac + 2ad + \text{etc.} \\
\qquad\qquad + 2bc + 2bd + \text{etc.} + 2cd + \text{etc.}
\end{array}$$

We thus reach the following conclusion:

Theorem. The square of a polynomial is equal to the sum of the squares of all its terms plus twice the product of every two terms.

2. *Square of an Entire Function.* Sometimes we wish to arrange the polynomial and its square as an entire function of some quantity, for example, of x.

Let us form the square of $a + bx + cx^2 + dx^3 +$ etc.

$$\begin{array}{l}
a + bx + cx^2 + dx^3 + \text{etc.} \\
\underline{a + bx + cx^2 + dx^3 + \text{etc.}} \\
a^2 + abx + acx^2 + adx^3 + \text{etc.} \\
\qquad abx + b^2x^2 + bcx^3 + bdx^4 + \text{etc.} \\
\qquad\qquad acx^2 + bcx^3 + c^2x^4 + \text{etc.} \\
\qquad\qquad\qquad adx^3 + bdx^4 + \text{etc.} \\
\hline
a^2 + 2abx + (2ac + b^2)\,x^2 + (2ad + 2bc)\,x^3 + \text{etc.}
\end{array}$$

We see that:

The coefficient of x^2 is $ac + bb + ca$.
" " " x^3 is $ad + bc + cb + da$.
" " " x^4 is $ae + bd + cc + db + ea$.
etc. etc.

The law of the products ae, bd, ce, etc., is that the first factor of each product is composed successively of all the coefficients in regular order up to that of the power of x to which the coefficient belongs, while the second factor is composed successively of the same coefficients in reverse order.

EXERCISES.

Form the squares of

1. $1 + 2x + 3x^2$.
2. $1 + 2x + 3x^2 + 4x^3$.
3. $1 + 2x + 3x^2 + 4x^3 + 5x^5$.
4. $1 + 2x + 3x^2 + 4x^3 + 5x^5 + 6x^6$.
5. $1 - 2x + 3x^2 - 4x^3$.
6. $a - b + c - d$.
7. $3a + 2b - c + d$.
8. $a + \frac{1}{a} - b - \frac{1}{b}$.

CHAPTER II.

EVOLUTION AND FRACTIONAL EXPONENTS.

174. *Def.* The n^{th} **Root** of a quantity q is such a number as, being raised to the n^{th} power, will produce q.

When $n = 2$, the root is called the **Square Root.**

When $n = 3$, the root is called the **Cube Root.**

EXAMPLES. 3 is the 4th root of 81, because

$$3 \cdot 3 \cdot 3 \cdot 3 = 3^4 = 81.$$

As the student already knows, we use the notation,

$$n^{th} \text{ root of } q = \sqrt[n]{q}.$$

There is another way of expressing roots which we shall now describe.

175. *Division of Exponents.* Let us extract the square root of a^6. We must find such a quantity as, being multiplied by itself, will produce a^6. It is evident that the required quantity is a^3, because, by the rule for multiplication (§§ 66, 166),

$$a^3 \times a^3 = a^6.$$

The square root of a^n will be $a^{\frac{n}{2}}$, because

$$a^{\frac{n}{2}} \times a^{\frac{n}{2}} = a^{\frac{n}{2}+\frac{n}{2}} = a^n.$$

In the same way, the cube root of a^n is $a^{\frac{n}{3}}$, because

$$a^{\frac{n}{3}} \times a^{\frac{n}{3}} \times a^{\frac{n}{3}} = a^n.$$

The following theorem will now be evident:

Theorem. The square root of a power may be expressed by dividing its exponent by 2, the cube root by dividing it by 3, and the n^{th} root by dividing it by n.

176. *Fractional Exponents.* Considering only the original definition of exponents, such an expression as $a^{\frac{3}{2}}$ would have no meaning, because we cannot write a $1\frac{1}{2}$ times. But by what has just been said, we see that $a^{\frac{3}{2}}$ may be interpreted to mean the square root of a^3, because

$$a^{\frac{3}{2}} \times a^{\frac{3}{2}} = a^3.$$

Hence,

A fractional exponent indicates the extraction of a root. If the denominator is 2, a square root is indicated; if 3, a cube root; if n, an n^{th} root.

A fractional exponent has therefore the same meaning as the radical sign $\sqrt{\ }$, and may be used in place of it.

EXERCISES.

Express the following roots by exponents only:

1.	$\sqrt{m}$.	2.	$\sqrt{(m+n)}$.	3.	$\sqrt{(a+b)^3}$.
4.	$\sqrt[3]{(a+b)^2}$.	5.	$\sqrt[4]{m^3}$.	6.	$\sqrt[5]{a^n}$.
7.	$\sqrt[n]{a^5}$.	8.	$\sqrt[m]{(a+b)^n}$.	9.	$\sqrt[n]{(a+b)^m}$.

177. Since the even powers of negative quantities are positive, it follows that an even root of a positive quantity may be either positive or negative.

This is expressed by the double sign $\pm$.

EXERCISES.

Express the square roots and also the cube roots and the n^{th} roots of the following:

1.	$(a+b)^3$.	2.	$(a+b)^2$.	3.	$a+b$.
4.	$(x+y)^{\frac{3}{2}}$.	5.	$(x+y)^{\frac{1}{2}}$.	6.	$(x+y)^{\frac{1}{n}}$.

178. If the quantity of which the root is to be extracted is a product of several factors, we extract the root of each factor, and take the product of these roots.

EXAMPLE. The n^{th} root of am^2p is $a^{\frac{1}{n}}m^{\frac{2}{n}}p^{\frac{1}{n}}$, because

$$\left(a^{\frac{1}{n}}m^{\frac{2}{n}}p^{\frac{1}{n}}\right)^n = am^2p, \text{ by §§ 168 and 176.}$$

If the quantity is a fraction, we extract the root of both members.

Proof. $\left(\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}\right)^n = \dfrac{a}{b}\cdot$ (§§ 167, 168.)

Because $\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$ taken n times as a factor makes $\dfrac{a}{b}$, therefore, by definition, it is the n^{th} root of $\dfrac{a}{b}\cdot$

EXERCISES.

Express the square roots of

1. $4x^2.$ 2. $\dfrac{9a^2x^2}{49m}\cdot$ 3. $\dfrac{64ab^2c^3}{81mp^2q^3}$

Express the cube roots of

4. $27 \cdot 64.$ 5. $27a^3.$ 6. $64 \cdot 27a^3b^6.$

7. $ab^2c^3d^4.$ 8. $\dfrac{8a^m}{125xy^n}\cdot$

Express the n^{th} roots of

9. $7.$ 10. $4 \cdot 7.$ 11. $4 \cdot 7 \cdot 10.$

12. $\dfrac{5ab^n}{6mp^n}\cdot$ 13. $6a^nb^{2n}.$ 14 $\dfrac{6a^2b^{\frac{n}{2}}}{c^md^{\frac{m}{n}}}\cdot$

15. $\dfrac{x^{m+1}\,y^n\,z^{m-2}}{a^{mn}\,b^{\frac{n}{m}}}\cdot$

16. $3^{5n}\,a^{-2n}\,(a+b)^{4n}\,(x-y)^n\,4^n\,(b-c+d)^{-4n}.$

Reduce to exponential expressions:

17. $\sqrt[n]{a\,(b-c)^m}.$ 18. $\sqrt[m]{ab^2c^3}.$

19. $\sqrt[n]{a^pb^q}.$ 20. $\sqrt[n]{\dfrac{a}{b}}\cdot$

21. $\sqrt[m]{\dfrac{(a+b)^n}{(a-b)^n}}\cdot$

Powers of Expressions with Fractional Exponents.

179. *Theorem.* The p^{th} power of the n^{th} root is equal to the n^{th} root of the p^{th} power.

In algebraic language,

$$(\sqrt[p]{a})^n = \sqrt[p]{a^n}.$$

or

$$(a^{\frac{1}{p}})^n = (a^n)^{\frac{1}{p}},$$

EXAMPLE. $$(\sqrt[3]{8})^2 = 2^2 = 4,$$

$$\sqrt[3]{8^2} = \sqrt[3]{64} = 4;$$

or, in words, the square of the cube root of 8 (that is, the square of 2) is the cube root of the square of 8 (that is, of 64).

General Proof. Let us put $x =$ the n^{th} root of a, so that

$$x^n = a. \tag{1}$$

The p^{th} power of this root x will then be x^p. (2)

Raising both sides of the equation (1) to the p^{th} power, we have

$$x^{np} = a^p = p^{th} \text{ power of } a.$$

The n^{th} root of the first member is found by dividing the exponent by n, which gives

$$n^{th} \text{ root of } p^{th} \text{ power} = x^p,$$

the same expression (2) just found for the p^{th} power of the n^{th} root.

This theorem leads to the following conclusion:

1. The expression

$$a^{\frac{p}{n}}$$

may mean indifferently the p^{th} power of $a^{\frac{1}{n}}$, or the nth root of a^p, these quantities being identical.

2. The powers of expressions having fractional exponents may be formed by multiplying the exponents by the index of the power.

EXERCISES.

Express the squares, the cubes, and the n^{th} powers of the following expressions:

1. $a^{\frac{1}{2}}$. 2. $a^{\frac{1}{3}}$. 3. $a^{\frac{2}{3}}$.

4. $a^{\frac{1}{n}}$. 5. $ab^{\frac{1}{n}}$. 6. $ab^{\frac{1}{3}}c^{\frac{n}{2}}$.

7. $a^{\frac{m}{2}}b^{\frac{m}{3}}$.
8. $a^{\frac{p}{q}}b^{-\frac{q}{p}}$.
9. $(a+b)^{\frac{m}{n}}(a-b)^{-n}$.
10. $a^{-n}b^{n}$.
11. $a^{-\frac{1}{n}}b^{\frac{1}{n}}$.
12. $\dfrac{(x+y)^{-\frac{1}{p}}}{(x-y)^{-\frac{m}{q}}}$.

Reduce to simple products and fractions:

13. $\left(x^{\frac{m}{n}}y^{-\frac{m}{n}}\right)^{p}$.
14. $\left(a^{\frac{2}{3}}b^{\frac{1}{3}}c^{-\frac{5}{3}}\right)^{\frac{m}{n}}$.
15. $\left(a^{\frac{1}{2}}b^{\frac{3}{4}}\right)^{-q}$.
16. $\left(a^{-\frac{m}{n}}\right)^{\frac{p}{q}}$.
17. $\left(\dfrac{x^{-\frac{1}{2}}}{y^{-\frac{2}{3}}}\right)^{\frac{3}{2}}$.
18. $\dfrac{a^{m-\frac{1}{2}}}{b^{m+\frac{1}{2}}} : \dfrac{a^{m+\frac{1}{2}}}{b^{m-\frac{1}{2}}}$.

CHAPTER III.

REDUCTION OF IRRATIONAL EXPRESSIONS.

Definitions.

180. *Def.* A **Rational Expression** is one in which the symbols are only added, subtracted, multiplied, or divided.

All the operations we have hitherto considered, except the extraction of roots, have led to rational expressions.

Def. An expression which involves the extraction of a root is called **Irrational.**

Example. Irrational expressions are

$$\sqrt{a}, \qquad \sqrt[3]{(a+b)}, \qquad \sqrt{27};$$

or, in the language of exponents,

$$a^{\frac{1}{2}}, \qquad (a+b)^{\frac{1}{3}}, \qquad 27^{\frac{1}{2}}.$$

In order that expressions may be really irrational, they must be **Irreducible,** that is, incapable of being expressed without the radical sign.

EXAMPLE. The expressions

$$\sqrt{a^2 + 2ab + b^2}, \qquad \sqrt{36},$$

are not properly irrational, because they are equal to $a + b$ and 6 respectively, which are rational.

Def. A **Surd** is the root which enters into an irrational expression.

EXAMPLE. The expression $a + b\sqrt{x}$ is irrational, and the surd is $\sqrt{x}$.

Def. Irrational terms are **Similar** when they contain the same surds.

EXAMPLES. The terms $\sqrt{30}$, $7\sqrt{30}$, $(x + y)\sqrt{30}$, are similar, because the quantity under the radical sign is 30 in each.

The terms $(a + b)\sqrt{x + y}$, $3\sqrt{x + y}$, $m\sqrt{x + y}$ are similar.

Aggregation of Similar Terms.

181. Irrational terms may be aggregated by the rules of §§ 54–56, the surds being treated as if they were single symbols. Hence:

When similar irrational terms are connected by the signs + or —, the coefficients of the similar surds may be added, and the surd itself affixed to their sum.

EXAMPLE. The sum

$$a\sqrt{(x + y)} - b\sqrt{(x + y)} + 3\sqrt{(x + y)}$$

may be transformed into $(a - b + 3)\sqrt{(x + y)}$.

EXERCISES.

Reduce the following expressions to the smallest number of terms:

1. $7\sqrt{3} - 5\sqrt{2} + 6\sqrt{3} + 7ab\sqrt{2}$.

2. $6\sqrt{(x+y)} + a\sqrt{(x-y)} + 2(a+b)\sqrt{(x+y)}$
$- 3(a+b)\sqrt{(x-y)}$.

3. $\frac{a+b}{2}\sqrt{7} + \frac{a-b}{2}\sqrt{7}$.

4. $(a+b)\sqrt{xy} + (a-b)\sqrt{xy}$.

5. $\sqrt{x}(a-b) + (b-c)\sqrt{x} + (c-a)\sqrt{x}$.

6. $a\sqrt{x} - \sqrt{x} + 2a\sqrt{x} - (a+b)\sqrt{x}$.

7. $\frac{3}{4}\sqrt{x} - a\sqrt{x} + 6\sqrt{x} - c\sqrt{x} + \frac{1}{3}\sqrt{x}$.

8. $\frac{a+b}{2}\sqrt{x} - 6c\sqrt{x} - \frac{a+b}{3}\sqrt{x} + \sqrt{x}$.

9. $\frac{3}{4}\sqrt{x} - \sqrt{x} + (a-b)\sqrt{x} + \frac{2(a-b)}{3}\sqrt{x}$.

10. $\sqrt{a} - b\sqrt{a} - \sqrt{x} + \frac{6(a-b)}{4}\sqrt{a} - \frac{1}{2}\sqrt{a}$.

11. $\frac{3}{4}\sqrt{x} - \sqrt{x} + \frac{2(a-b)}{3}\sqrt{x}$.

12. $4\sqrt{x} - \frac{1}{3}\sqrt{x} + (a-b)\sqrt{x}$.

Factoring Surds.

182. Irrational expressions may sometimes be transformed so as to have different expressions under the radical sign, by the method of § 178, applying the following theorem:

Theorem. A root of the product of several factors is equal to the product of their roots.

In the language of Algebra,

$$\sqrt[n]{abcd}, \text{ etc.} = \sqrt[n]{a}\,\sqrt[n]{b}\,\sqrt[n]{c}\,\sqrt[n]{d}, \text{ etc.}$$
$$= a^{\frac{1}{n}}\,b^{\frac{1}{n}}\,c^{\frac{1}{n}}\,d^{\frac{1}{n}}, \text{ etc.}$$

Proof. By raising the members of this equation to the n^{th} power, we shall get the same result, namely,

$$a \times b \times c \times d, \text{ etc.}$$

EXAMPLE. $\sqrt{30} = \sqrt{6}\,\sqrt{5}$.

EXERCISES.

Prove the following equations by computing both sides:

$$\sqrt{4}\,\sqrt{49} = \sqrt{4\cdot 49} = \sqrt{196}.$$

Proof. $\sqrt{4}\,\sqrt{49} = 2\cdot 7 = 14$, and $\sqrt{196} = 14$.

$$\sqrt{4}\,\sqrt{9} = \sqrt{36}.$$
$$\sqrt{4}\,\sqrt{25} = \sqrt{4\cdot 25}.$$
$$\sqrt{9}\,\sqrt{16} = \sqrt{9\cdot 16}.$$
$$\sqrt{25}\,\sqrt{36} = \sqrt{25\cdot 36}.$$

Express with a single surd the products:

1. $\sqrt{(a+b)}\,\sqrt{(a-b)}$.

SOLUTION. $\sqrt{(a+b)}\,\sqrt{(a-b)} = \sqrt{(a+b)(a-b)}$
$= \sqrt{(a^2-b^2)}$.

2. $\sqrt{7}\,\sqrt{5}$.
3. $\sqrt{7}\,\sqrt{a}$.
4. $\sqrt{a}\,\sqrt{(a+y)}$.
5. $\sqrt{a}\,\sqrt{b}\,\sqrt{(a+b)}$.
6. $\sqrt{(x+1)}\,\sqrt{(x-1)}$.
7. $\sqrt{(x^2+1)}\,\sqrt{(x+1)}\,\sqrt{(x-1)}$.
8. $\left[(a+b)^{\frac{1}{4}}(a-b)^{\frac{1}{4}}\right]^2$.
9. $\left[(x^2+1)^{\frac{1}{3}}(x+1)^{\frac{1}{3}}(x-1)^{\frac{1}{3}}\right]^2$.

183. If we can separate the quantity under the radical sign into two factors, one of which is a perfect square, we may extract its root and affix the surd root of the remaining factor to it.

EXAMPLES.

$$\sqrt{a^2b} = \sqrt{a^2}\,\sqrt{b} = a\sqrt{b}.$$
$$\sqrt{ab}\,\sqrt{ac} = \sqrt{a^2bc} = a\sqrt{bc}.$$
$$\sqrt{12}\,\sqrt{6} = \sqrt{72} = \sqrt{36}\,\sqrt{2} = 6\sqrt{2}.$$
$$\sqrt{(4a^3+8a^2b-16a^3c)} = \sqrt{4a^2(a+2b-4ac)}$$
$$= 2a\sqrt{(a+2b-4ac)}.$$
$$(x^3-4x^2y+4xy^2)^{\frac{1}{2}} = (x-2y)\,x^{\frac{1}{2}}.$$

EXERCISES.

Reduce, when possible:

1.	$\sqrt{8}$.	2.	$\sqrt{32}$.
3.	$\sqrt{128}$.	4.	$\sqrt{3}\sqrt{27}$.
5.	$\sqrt{ab}\sqrt{ca}\sqrt{bc}$.	6.	$\sqrt{2}\sqrt{72}$.
7.	$\sqrt{4}\sqrt{72}$.	8.	$\sqrt{(x+1)}\sqrt{(x+1)}$.
9.	$\sqrt{175}$.	10.	$\sqrt{150}$.
11.	$\sqrt{108}$.	12.	$\sqrt{x^2(a+b)}$.
13.	$\sqrt{(a^2x + 2abx + b^2x)}$.		

Here the quantity under the radical sign is equal to

$$(a^2 + 2ab + b^2)x = (a+b)^2 x.$$

In questions of this class, the beginner is apt to divide an expression like $\sqrt{a+b+c}$ into $\sqrt{a} + \sqrt{b} + \sqrt{c}$, which is wrong. The square root of the sum of several quantities cannot be reduced in this way.

14.	$\sqrt{a^2y + 4ay + 4y}$.	15.	$\sqrt{4m^2z + 8mz + 4z}$.

Reduce and add the following surds:

16.	$4\sqrt{2} - 6\sqrt{8} + 10\sqrt{32}$.	17.	$\sqrt{12} + \sqrt{27} + \sqrt{75}$.
18.	$\sqrt{4a} - 2\sqrt{a}$.	19.	$125^{\frac{1}{2}} - 45^{\frac{1}{2}} - 80^{\frac{1}{2}}$.
20.	$\sqrt[3]{81} - \sqrt[3]{192}$.	21.	$(a^2b^3)^{\frac{1}{3}} + (a^2c^6)^{\frac{1}{3}}$.

Multiplication of Irrational Expressions.

184. Irrational polynomials may be multiplied by combining the foregoing principles with the rule of § 78.

The following are the forms:

To multiply $a + b\sqrt{x}$ by $m + n\sqrt{y}$.

$$a(m + n\sqrt{y}) = am \times an\sqrt{y}.$$
$$b\sqrt{x}(m + n\sqrt{y}) = bm\sqrt{x} + bn\sqrt{xy}.$$

The product is $\qquad am + an\sqrt{y} + bm\sqrt{x} + bn\sqrt{xy}.$

EXERCISES.

Perform the following multiplications and reduce the results to the simplest form (compare § 80):

1.	$(2 + 3\sqrt{5})(5 - 3\sqrt{2})$.	2.	$(7 + 2\sqrt{32})(9 - 5\sqrt{2})$.

3. $(a + \sqrt{b})(a - \sqrt{b})$.   4. $(\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d})^2$.

5. $(m + n^{\frac{1}{3}})(m + 2n^{\frac{1}{3}})$.   6. $(a^{\frac{1}{2}} - a^{\frac{1}{3}})(a^{\frac{1}{2}} + a^{\frac{1}{3}})$.

7. $(a + a^{-1})^2$.   8. $(a^{\frac{1}{2}} - a^{-\frac{1}{2}})^4$.

9. $[a + b\sqrt{(x + y)}]\,[a - b\sqrt{(x + y)}]$.

10. $[m + n\sqrt{(a + b)}]\,[m - n\sqrt{(a - b)}]$.

11. $[x + \sqrt{(x^2 - 1)}]\,[x - \sqrt{(x^2 - 1)}]$.

12. $[(b^2 + 1)^{\frac{1}{2}} + b]\,[(b^2 + 1)^{\frac{1}{2}} - b]$.

Expressions may often be transformed and factored by combining the foregoing processes.

EXAMPLE. To factor $ax^{\frac{7}{2}} + bx^{\frac{5}{2}} + cx^{\frac{3}{2}} + dx^{\frac{1}{2}}$, we notice that

$$x^{\frac{7}{2}} = x^{\frac{1}{2}}x^3, \qquad x^{\frac{5}{2}} = x^{\frac{1}{2}}x^2, \quad \text{etc.}$$

so that the expression may be written,

$$ax^3x^{\frac{1}{2}} + bx^2x^{\frac{1}{2}} + cxx^{\frac{1}{2}} + dx^{\frac{1}{2}} = (ax^3 + bx^2 + cx + d)\,x^{\frac{1}{2}}.$$

EXERCISES.

Reduce the following expressions to products:

13. $2 + \sqrt{2}$.   14. $3^{\frac{3}{2}} + 2 \cdot 3^{\frac{1}{2}}$.

15. $(a + b)^{\frac{3}{2}}$.   16. $\sqrt{y + ay^3 - by^5}$.

17. $x - y - \sqrt{x - y}$.

Reduce to the lowest terms:

18. $\dfrac{2}{\sqrt{2}}$.   19. $\dfrac{\sqrt{a + b}}{a + b}$.   20. $\dfrac{ax^{\frac{3}{2}} + bx^{\frac{1}{2}}}{ax^{\frac{3}{2}} - bx^{\frac{1}{2}}}$.

21. $\dfrac{a - x + \sqrt{a - x}}{a - x - \sqrt{a - x}}$.   22. $\dfrac{\sqrt{a^2 - b^2}}{a + b}$.

185. *Rationalizing Fractions.* The quotient of two surds may be expressed as a fraction with a rational numerator or a rational denominator, by multiplying both terms by the proper multiplier.

EXAMPLE. Consider the fraction $\dfrac{\sqrt{5}}{\sqrt{7}}$.

Multiplying both terms by $\sqrt{7}$, the fraction becomes $\frac{\sqrt{35}}{7}$, and has the rational denominator 7.

Multiplying by $\sqrt{5}$, it becomes $\frac{5}{\sqrt{35}}$, and has the rational numerator 5.

The numerator or denominator may also be made rational when they both consist of two terms, one or both of which are irrational.

Let us have a fraction of the form

$$\frac{A + D\sqrt{B}}{P + Q\sqrt{R}},$$

in which the letters A, D, P, Q, and R stand for any algebraic or numerical expressions whatever. If we multiply both numerator and denominator by $P - Q\sqrt{R}$, the denominator will become

$$P^2 - Q^2R.$$

The numerator will become

$$AP + PD\sqrt{B} - AQ\sqrt{R} - DQ\sqrt{BR}.$$

so that the value of the fraction is

$$\frac{AP + PD\sqrt{B} - AQ\sqrt{R} - DQ\sqrt{BR}}{P^2 - Q^2R}.$$

EXERCISES.

Reduce the following fractions to others having rational denominators:

1. $\sqrt{\left(\frac{a+b}{a-b}\right)}$. 2. $\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}$. 3. $\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$.

4. $\frac{7\sqrt{3}}{9\sqrt{5}}$. 5. $\frac{2\sqrt{18}}{3\sqrt{6}}$. 6. $\frac{5\sqrt{24}}{2\sqrt{2}}$.

7. $\frac{a+\sqrt{b}}{a-\sqrt{b}}$. 8. $\frac{a-\sqrt{x}}{a+\sqrt{x}}$. 9. $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$.

10. $\dfrac{a + 2\sqrt{(x + y)}}{a + \sqrt{(x + y)}}$.   11. $\dfrac{2\sqrt{3} + 7\sqrt{5}}{\sqrt{5} - \sqrt{3}}$.

12. $\dfrac{\sqrt{x} - \sqrt{(x + y)}}{\sqrt{x} + \sqrt{(x + y)}}$.   13. $\dfrac{1}{x - \sqrt{x^2 - a^2}}$.

14. $\dfrac{1}{a^{\frac{1}{2}} + (a + 1)^{\frac{1}{2}}}$.   15. $\dfrac{\sqrt{x + a} + \sqrt{x - a}}{\sqrt{x + a} - \sqrt{x - a}}$.

Perfect Squares.

186. *Def.* **A Perfect Square** is an expression of which the square root can be formed without any surds, except such as are already found in the expression.

EXAMPLES. $4m^4$, $4a^2 + 4a + 1$ are perfect squares, because their square roots are $2m^2$, $2a + 1$, expressions without the radical sign.

The expression $a + 2\sqrt{ab} + b$, of which the root is

$$\sqrt{a} + \sqrt{b},$$

may also be regarded as a perfect square, because the surds $\sqrt{a} + \sqrt{b}$ are in the product $2\sqrt{ab}$.

Criterion of a Perfect Square. The question whether a trinomial is a perfect square can always be decided by comparing it with the forms of § 80, namely:

$$a^2 + 2ab + b^2 = (a + b)^2,$$

or $$a^2 - 2ab + b^2 = (a - b)^2.$$

We see that to be a perfect square a trinomial must fulfil the following conditions:

(1.) Two of its three terms must be perfect squares.

(2.) The remaining term must be equal to twice the product of the square roots of the other two terms.

When these conditions are fulfilled, the square root of the trinomial will be the sum or difference of the square roots of the terms, according as the product is positive or negative.

The root may have either sign, because the squares of positive and negative quantities have the same sign.

If the terms which are perfect squares are *both* negative, the trinomial will be the negative of a perfect square.

EXAMPLES.

$$\sqrt{a^2 + 2ab + b^2} = a + b \text{ or } -(a + b).$$
$$\sqrt{a^2 - 2ab + b^2} = a - b \text{ or } b - a.$$
$$-a^2 + 2ab - b^2 = -(a - b)^2 = -(b - a)^2.$$

EXERCISES.

Find which of the following expressions are perfect squares, and extract their square roots:

1. $9 + 12 + 4$.
2. $x^2 + 4x + 4$.
3. $4x^4 + 2x^2 + \frac{1}{4}$.
4. $a^2 + ab - b^2$.
5. $4a^{2n} + 12a^nb^n + 9b^{2n}$.
6. $a^2 + 2ab - b^2$.
7. $x^6 - ax^3y + \frac{1}{4}a^2y^2$.
8. $a^2b^2 - 2abcd + c^2d^2$.
9. $m + 2m^{\frac{1}{2}}n^{\frac{1}{2}} + n$.
10. $a^2 - 2ax + y^2$.
11. $a + 4a^{\frac{1}{2}}b^{\frac{1}{2}} + 4b$.
12. $a - 2 + a^{-1}$.
13. $25p^4 + 9q^2 - 30p^2q$.
14. $6hm^{2n} + h^2 + 9m^{4n}$.
15. $49x^2y^2 + 9z^2 - 42xyz$.
16. $9m^{8n} - 2m^{4n}pq + \frac{p^2q^2}{9}$.

To Complete the Square.

187. If one term of a binomial is a perfect square, such a term can always be added to the binomial that the trinomial thus formed shall be a perfect square.

This operation is called **Completing the Square.**

Proof. Call a the root of the term which is a perfect square, which term we suppose the *first*, and call m the other term, so that the given binomial shall be

$$a^2 + m.$$

Add to this binomial the term $\frac{m^2}{4a^2}$, and it will become

$$a^2 + m + \frac{m^2}{4a^2}.$$

This is a perfect square, namely, the square of

$$a + \frac{m}{2a};$$

that is,

$$a^2 + m + \frac{m^2}{4a^2} = \left(a + \frac{m}{2a}\right)^2.$$

Hence the following

RULE. *Add to the binomial the square of the second term divided by four times the first term.*

EXAMPLE. What term must be added to the expression

$$x^2 - 4ax \qquad (1)$$

to make it a perfect square?

The rule gives for the term to be added,

$$\frac{(-4ax)^2}{4x^2} = 4a^2.$$

Therefore the required perfect square is

$$x^2 - 4ax + 4a^2 = (x - 2a)^2.$$

We may now transpose $4a^2$, so that the left-hand member of the equation shall be the original binomial (1). Thus,

$$x^2 - 4ax = (x - 2a)^2 - 4a^2.$$

The original binomial is now expressed as the difference of two squares. Therefore, the above process is a solution of the problem: *Having a binomial of which one term is a perfect square, to express it as a difference of two squares.*

EXERCISES.

Express the following binomials as differences of two squares:

1. $x^2 + 2xy$.
2. $x^2 + 4xy$.
3. $x^2 + 6ax$.
4. $4x^2 + 4xy$.
5. $4x^2 + 4xy$.
6. $9x^2 + ax$.
7. $16x^2 + 8mx$.
8. $x^2 + 4x$.
9. $a^2x^2 + 2a^2x$.
10. $b^2x^2 + 2$.
11. $m^2x^2 + 1$.
12. $9p^2x^2 + bx$.
13. $\frac{1}{4x^2} + 1$.
14. $\frac{1}{9a^2x^2} - 6a^2$.

Irrational Factors.

188. When we introduce surds, many expressions can be factored which have no rational factors. The following theorem may be applied for this purpose:

Theorem. The difference of any two quantities is equal to the product of sum and difference of their square roots.

In the language of algebra, if a and b be the quantities, we shall have

$$a - b = (a^{\frac{1}{2}} - b^{\frac{1}{2}})(a^{\frac{1}{2}} + b^{\frac{1}{2}}),$$

which can be proved by multiplying and by § 80, (3).

EXERCISES.

Factor

1. $m - n$.
2. $m - 1$.
3. $am - bn$.
4. $4a^2m - 9$.
5. $x^2 - m$.
6. $x^2 - (m + n)$.
7. $(x - a)^2 - \frac{1}{4}(m - n)$.
8. $x^2 - (m - n)$.
9. $(a + b)^2 - (4p^2 - q)$.
10. $x^2 + 2xy + y^2 - (m + n)^{\frac{1}{2}}$.

Find the irrational square roots of the following expressions by the principles of § 186:

11. $a - 2 + a^{-1}$. *Ans.* $a^{\frac{1}{2}} - a^{-\frac{1}{2}}$.
12. $x - 2\sqrt{xy} + y$.
13. $4 + 4\sqrt{3} + 3$.
14. $9 + 5 - 6\sqrt{5}$.
15. $4a + b - 4a^{\frac{1}{2}}b^{\frac{1}{2}}$.
16. $a + b + 2(a + b)^{\frac{1}{2}}x + x^2$.
17. $3 + 2\sqrt{15} + 5$.
18. $3 + 5 - 2\sqrt{15}$.
19. $\frac{x}{4} + \frac{y}{4} - \frac{\sqrt{xy}}{2}$.
20. $a - 2\sqrt{a} + 1$.
21. $a - 2a^{\frac{5}{6}} + a^{\frac{2}{3}}$.
22. $a + 2a^{\frac{1}{3}} + \frac{1}{a^{\frac{1}{3}}}$.
23. $a^{\frac{7}{4}} - a + \frac{a^{\frac{1}{4}}}{4}$.
24. $\frac{a}{4} + \frac{a}{3} + \frac{a}{9}$.
25. $\frac{a}{16} + \frac{1}{4} + \frac{a^{\frac{1}{2}}}{4}$.
26. $a^5 + 2 + a^{-5}$.
27. $4x^3 - 8 + 4x^{-3}$.
28. $a + b - 4 + \frac{4}{a + b}$.

BOOK VI.

EQUATIONS REQUIRING IRRATIONAL OPERATIONS.

CHAPTER I.

EQUATIONS WITH TWO TERMS ONLY.

189. In the present chapter we consider equations which contain only a single power or root of the unknown quantity.

Such an equation, when reduced to the normal form, will be of the form

$$Ax^n + B = 0.$$

By transposing B, dividing by A, and putting

$$a = -\frac{B}{A},$$

the equation may be written,

$$x^n - a = 0.$$

or $$x^n = a, \qquad (1)$$

Here n may be an integer, or it may represent some fraction.

Such an equation is called a **Binomial Equation,** because the expression $x^n - a$ is a binomial.

Solution of a Binomial Equation.

190. 1. *When the exponent of x is a whole number.* If we extract the n^{th} root of both members of the equation (1), these roots will, by Axiom V, still be equal. The n^{th} root of x^n being x, and that of a being $a^{\frac{1}{n}}$, we have

$$x = a^{\frac{1}{n}},$$

and the equation is solved.

2. *When the exponent is fractional.* Let the equation be

$$x^{\frac{m}{n}} = a.$$

Raising both members to the n^{th} power, we have

$$x^m = a^n.$$

Extracting the m^{th} root,

$$x = a^{\frac{n}{m}}.$$

If the numerator of the exponent is unity, we only have to suppose $m = 1$, which will give

$$x = a^n.$$

Hence the binomial equation always admits of solution by forming powers, extracting roots, or both.

Special Forms of Binomial Equations.

Def. When the exponent n is an integer, the equation is called a **Pure Equation** of the degree n.

When $n = 2$, the equation is a **Pure Quadratic Equation.**

When $n = 3$, the equation is a **Pure Cubic Equation.**

EXERCISES.

Find the values of x in the following equations:

1. $\dfrac{p}{x^{\frac{1}{2}}} = q.$ *Ans.* $x = \dfrac{p^2}{q^2}.$

2. $\dfrac{a+b}{x^{\frac{1}{3}}} = c.$

3. $\dfrac{a}{x^{\frac{1}{3}} - b} = \dfrac{b}{x^{\frac{1}{3}} - a}.$

4. $\dfrac{9}{x} = \dfrac{x^2}{24}.$

5. $\dfrac{x - 2a}{x - a} = \dfrac{2x - b}{x - b}.$

6. $\dfrac{x^2 - na}{x^2 - a} = \dfrac{nx^2 - b}{x^2 - b}.$

7. $\dfrac{a+b}{x^{\frac{m}{n}}} = \dfrac{x^{\frac{p}{q}}}{a-b}.$

8. $\dfrac{x^{\frac{1}{3}}}{y^{\frac{2}{3}}} = \dfrac{y^{\frac{1}{3}}}{x^{\frac{2}{3}}}.$

9. $\dfrac{\sqrt{x + a^2}}{a + b} = \dfrac{b - a}{\sqrt{x - a^2}}.$

In the last example, clearing the equation of fractions, we shall have

$$\sqrt{x^2 - a^4} = b^2 - a^2,$$

or

$$(x^2 - a^4)^{\frac{1}{2}} = b^2 - a^2.$$

We square both sides of this equation, which gives another in which x^2 only appears.

10. $(x - a)^{\frac{1}{3}} = b^{\frac{2}{3}}$. 11. $(x^2 - a^2)^{\frac{1}{2}} = mx$.

12. $(\sqrt{x} - \sqrt{b})^{\frac{1}{2}} = nx^{\frac{1}{4}}$.

Positive and Negative Roots.

191. Since the square root of a quantity may be either positive or negative, it follows that when we have an equation such as

$$x^2 = a,$$

and extract the square root, we may have either

$$x = +a^{\frac{1}{2}},$$

or

$$x = -a^{\frac{1}{2}}.$$

Hence there are two roots to every such equation, the one positive and the other negative. We express this pair of roots by writing

$$x = \pm a^{\frac{1}{2}},$$

the expression $\pm a^{\frac{1}{2}}$ meaning either $+a^{\frac{1}{2}}$ or $-a^{\frac{1}{2}}$.

It might seem that since the square root of x^2 is either $+x$ or $-x$, we should write

$$\pm x = \pm a^{\frac{1}{2}},$$

having the four equations,

$$x = a^{\frac{1}{2}},$$
$$x = -a^{\frac{1}{2}},$$
$$-x = +a^{\frac{1}{2}},$$
$$-x = -a^{\frac{1}{2}}.$$

But the first and fourth of these equations give identical values of x by simply changing the sign, and so do the second and third.

PROBLEMS LEADING TO PURE EQUATIONS.

1. Find three numbers, such that the second shall be double the first, the third one-third the second, and the sum of their squares 196.

2. The sum of the squares of two numbers is 369, and the difference of their squares 81. What are the numbers?

3. A lot of land contains 1645 square feet, and its length exceeds its breadth by 12 feet. What are the length and breadth?

To solve this equation as a binomial, take the mean of the length and breadth as the unknown quantity, so that the length shall be as much greater than the mean as the breadth is less.

4. Find a number such that if 9 be added to and subtracted from it, the product of the sum and difference shall be 175.

5. Find a number such that if a be added to it and subtracted from it the product of the sum and difference shall be $2a + 1$.

6. One number is double another, and the difference of their squares is 192. What are the numbers?

7. One number is 8 times another, and the sum of their cube roots is 12. What are the numbers?

8. Find two numbers of which one is 3 times the other, and the square root of their sum, multiplied by the lesser, is equal to 128.

9. What two numbers are to each other as $2:3$, and the sum of their squares $= 325$?

NOTE. If we represent one of the numbers by $2x$, the other will be $3x$.

10. What two numbers are to each other as $m:n$, and the square of their difference equal to their sum?

11. What two numbers are to each other as 9 to 7, and the cube root of their difference multiplied by the square root of their sum equal to 16?

12. Find x and y from the equations

$$ax^2 + by^2 = c,$$
$$a'x^2 + b'y^2 = c'.$$

13. The hypothenuse of a right-angled triangle is 26 feet in length, and the sum of the sides is 34 feet. Find each side.

NOTE. It is shown in Geometry that the square of the hypothenuse of a right-angled triangle is equal to the sum of the squares of the other two sides. In the present problem, take for the unknown quantity the amount by which each unknown side differs from half their sum.

14. Two points start out together from the vertex of a right angle along its respective sides, the one moving m feet per second and the other n feet per second. How long will they require to be c feet apart?

15. By the law of falling bodies, the distance fallen is proportional to the square of the time, and a body falls 16 feet the first second. How long will it require to fall h feet?

CHAPTER II.

QUADRATIC EQUATIONS.

192. *Def.* A **Quadratic Equation** is one which, when reduced to the normal form, contains the second and no higher power of the unknown quantity.

A quadratic equation is the same as an equation of the second degree.

Def. A **Pure** quadratic equation is one which contains the second power only of the unknown quantity.

The treatment of a pure quadratic equation is given in the preceding chapter.

Def. A **Complete** quadratic equation is one which contains both the first and second powers of the unknown quantity.

The normal form of a complete quadratic equation is

$$ax^2 + bx + c = 0. \qquad (1)$$

If we divide this equation by a, we obtain

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \qquad (2)$$

Putting, for brevity, $\frac{b}{a} = p,$

$$\frac{c}{a} = q,$$

the equation will be written in the form,

$$x^2 + px + q = 0. \qquad (3)$$

Def. The equation

$$x^2 + px + q = 0$$

is called the **General Equation of the Second Degree,** or the **General Quadratic Equation,** because it is the form to which all such equations can be reduced.

Solution of a Complete Quadratic Equation.

193. *A quadratic equation is solved by adding such a quantity to its two members that the member containing the unknown quantity shall be a perfect square.* (§ 187.)

We first transpose q in the general equation, obtaining

$$x^2 + px = -q.$$

We then add $\frac{p^2}{4}$ to both members, making

$$x^2 + px + \frac{p^2}{4} = \frac{p^2}{4} - q.$$

The first member of the equation is now a perfect square. Extracting the square roots of both sides, we have

$$x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} - q}.$$

From this equation we obtain a value of x which may be put in either of the several forms,

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

$$x = -\frac{p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}.$$

$$x = \frac{1}{2}(-p \pm \sqrt{p^2 - 4q}).$$

If instead of substituting p and q, we treat the equation in the form (2) precisely as we have treated it in the form (3), we shall obtain the several results,

$$x^2 + \frac{b}{a}x + \frac{1}{4}\frac{b^2}{a^2} = \frac{1}{4}\frac{b^2}{a^2} - \frac{c}{a},$$

and

$$x = -\frac{b}{2a} \pm \sqrt{\left(\frac{b^2}{4a^2} - \frac{c}{a}\right)}$$

$$= \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

194. The equation in the normal form, (1), may also be solved by the following process, which is sometimes more convenient. Transposing c, and multiplying the equation by a, we obtain the result

$$a^2x^2 + abx = -ac.$$

To make the first member a perfect square, we add $\frac{b^2}{4}$ to each member, giving

$$a^2x^2 + abx + \frac{b^2}{4} = \frac{b^2}{4} - ac.$$

Extracting the square root of both sides, we have

$$ax + \frac{b}{2} = \frac{1}{2}\sqrt{(b^2 - 4ac)},$$

from which we obtain the same value of x as before.

195. Since the square root in the expression for x may be either positive or negative, there will be two roots to every quadratic equation, the one formed from the positive and the other from the negative surds. If we distinguish these roots with x_1 and x_2, their values will be

$$\left.\begin{aligned} x_1 &= \frac{-b + \sqrt{(b^2 - 4ac)}}{2a}, \\ x_2 &= \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}. \end{aligned}\right\} \qquad (4)$$

We can always find the roots of a given quadratic equation by substituting the coefficients in the preceding expression for x. But the student is advised to solve each separate equation by the process just given, which is embodied in the following rule:

I. *Reduce the equation to its normal or its general form, as may be most convenient.*

II. *Transpose the terms which do not contain x to the second member.*

III. *If the coefficient of x^2 is unity, add one-fourth the square of the coefficient of x to both members of the equation and extract the square root.*

IV. *If the coefficient of x^2 is not unity, either divide by it so as to reduce it to unity, or multiply all the terms* *by such a factor that it shall become a perfect square, and complete the square by the rule of* § 187.

EXAMPLE.

Solve the equation

$$\frac{x-1}{x-20} = 2x.$$

Clearing of fractions and transposing, we find the equation to become

$$2x^2 - 41x + 1 = 0,$$

$$x^2 - \frac{41x}{2} = -\frac{1}{2}.$$

Adding $\frac{1}{4}$ the square of the coefficient of x to each side, we have

$$x^2 - \frac{41}{2}x + \frac{1681}{16} = \frac{1681}{16} - \frac{1}{2} = \frac{1673}{16}.$$

Extracting the square root and reducing, we find the values of x to be

$$x_1 = \frac{1}{4}(41 + \sqrt{1673}),$$

and

$$x_2 = \frac{1}{4}(41 - \sqrt{1673}).$$

Using the other method, in order to avoid fractions, we multiply the equation (5) by 2, making the equation,

$$4x^2 - 82x = -2.$$

Adding $\frac{41^2}{4} = \frac{1681}{4}$ to each side of the equation, we have

$$4x^2 - 82x + \frac{41^2}{4} = \frac{1681}{4} - 2 = \frac{1673}{4}.$$

Extracting the square root,

$$2x - \frac{41}{2} = \sqrt{\frac{1673}{4}} = \frac{\sqrt{1673}}{2};$$

whence we find

$$x = \frac{-41 \pm \sqrt{1673}}{4},$$

the same result as before.

EXERCISES.

Reduce and solve the following equations

1. $\dfrac{x+2}{x-2} - \dfrac{x-2}{x+2} = \dfrac{5}{6}.$

2. $\dfrac{y+4}{y-4} + \dfrac{y-4}{y+4} = \dfrac{10}{3}.$

3. $\dfrac{1}{x-1} + \dfrac{2}{x-2} = \dfrac{4}{3}$.

4. $y^2 - 2ay + a^2 - b^2 = 0$.

5. $\dfrac{1}{a+b+x} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{x}$.

6. $\dfrac{a^2}{x^2 - a^2} + \dfrac{b}{x+a} - \dfrac{b}{x-a} = 0$.

7. $\dfrac{1 + \dfrac{x+a}{x-a}}{1 - \dfrac{x-a}{x+a}} = 3$.

8. $\dfrac{2}{2+y} - \dfrac{y}{y^2-4} + \dfrac{2}{2-y} = 4$.

9. $\dfrac{y+a}{y-a} - \dfrac{y-a}{y+a} = \dfrac{1}{y-a} - \dfrac{1}{y^2-a^2} + \dfrac{1}{y-a}$.

10. $\dfrac{x}{a+x} - \dfrac{x}{a-x} + 3 = 0$.

PROBLEMS.

1. Find two numbers such that their difference shall be 6 and their product 567.

2. The difference of two numbers is 6, and the difference of their cubes is 936. What are the numbers?

3. Divide the number 34 into two such parts that the sum of their squares shall be double their product?

4. The sum of two numbers is 60, and the sum of their squares 1872. What are the numbers?

5. Find three numbers such that the second shall be 5 greater than the first, the third double the second, and the sum of their squares 1225.

6. Find four numbers such that each shall be 4 greater than the one next smaller, and the product of the two lesser ones added to the product of the two greater shall be 312.

7. A shoe dealer bought a box of boots for \$210. If there had been 5 pair of boots less in the box, they would have cost him \$1 per pair more, if he had still paid \$210 for the whole. How many pair of boots were in the box?

REM. If we call x the number of pairs, the price paid for each pair must have been $\dfrac{210}{x}$.

8. A huckster bought a certain number of chickens for \$10, and a number of turkeys for \$15.75. There were 4 more chickens than turkeys, but they each cost him 35 cents a piece less. How many of each did he buy?

9. A farmer sold a certain number of sheep for \$240. If he had sold a number of sheep 3 greater for the same sum, he would have received \$4 a piece less. How many sheep did he sell?

10. A party having dined together at a hotel, found the bill to be \$9.60. Two of the number having left before paying, each of the remainder had to pay 24 cents more to make up the loss. What was the number of the party?

11. A pedler bought \$10 worth of apples. 30 of them proved to be rotten, but he sold the remainder at an advance of 2 cents each, and made a profit of \$3.20. How many did he buy?

12. In a certain number of hours a man traveled 48 miles; if he had traveled one mile more per hour, it would have taken him 4 hours less to perform his journey; how many miles did he travel per hour?

13. The perimeter of a rectangular field is 160 metres, and its area is 1575 square metres. What are its length and breadth?

14. The length of a lot of land exceeds its breadth by a feet, and it contains m^2 square feet. What are its dimensions?

15. A stage leaves town A for town B, driving 8 miles an hour. Three hours afterward a stage leaves B for A at such a speed as to reach A in 18 hours. They meet when the second has driven as many hours as it drives miles per hour. What is the distance between A and B?

NOTE. The solution is very simple when the proper quantity is taken as unknown.

Equations which may be Reduced to Quadratics.

196. Whenever an equation contains only two powers of the unknown quantity, and the index of one power is double that of the other, the equation can be solved as a quadratic.

Special Example. Let us take the equation

$$x^6 + bx^3 + c = 0. \qquad (1)$$

Transposing c and adding $\frac{1}{4}b^2$ to each side of the equation, it becomes

$$x^6 + bx^3 + \frac{1}{4}b^2 = \frac{1}{4}b^2 - c.$$

The first member of this equation is a perfect square, namely, the square of $x^3 + \frac{1}{2}b$. Extracting the square roots of both members, we have

$$x^3 + \frac{1}{2}b = \sqrt{\left(\frac{1}{4}b^2 - c\right)} = \pm\frac{1}{2}\sqrt{(b^2 - 4c)}.$$

Hence, $$x^3 = \frac{1}{2}[-b \pm \sqrt{(b^2 - 4c)}].$$

Extracting the cube root, we have

$$x = \frac{1}{2^{\frac{1}{3}}}[-b \pm \sqrt{(b^2 - 4c)}]^{\frac{1}{3}}.$$

General Form. We now generalize this solution in the following way. Suppose we can reduce an equation to the form

$$ax^{2n} + bx^n + c = 0,$$

in which the exponent n may be any quantity whatever, entire or fractional. By dividing by a, transposing, and adding $\frac{1}{4}\frac{b^2}{a^2}$ to both sides of the equation, we find

$$x^{2n} + \frac{b}{a}x^n + \frac{1}{4}\frac{b^2}{a^2} = \frac{1}{4}\frac{b^2}{a^2} - \frac{c}{a}.$$

The first side of this equation is the square of

$$x^n + \frac{1}{2}\frac{b}{a}.$$

Hence, by extracting the square root, and reducing as in the general equation, we find

$$x^n = \frac{1}{2a}[-b \pm \sqrt{(b^2 - 4ac)}].$$

Extracting the n^{th} root of both sides, we have

$$x = \frac{1}{2^{\frac{1}{n}} a^{\frac{1}{n}}} [-b \pm \sqrt{(b^2 - 4ac)}]^{\frac{1}{n}}.$$

$$= \left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)^{\frac{1}{n}}.$$

If the exponent n is a fraction, the same course may be followed.

Suppose, for example,

$$ax^{\frac{4}{3}} + bx^{\frac{2}{3}} + c = 0.$$

Dividing by a and transposing, we have

$$x^{\frac{4}{3}} + \frac{b}{a}x^{\frac{2}{3}} = -\frac{c}{a}.$$

Adding $\frac{b^2}{4a^2}$ to both sides,

$$x^{\frac{4}{3}} + \frac{b}{a}x^{\frac{2}{3}} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}.$$

The left-hand member of this equation is the square of

$$x^{\frac{2}{3}} + \frac{b}{2a}.$$

Extracting the square root of both members,

$$x^{\frac{2}{3}} + \frac{b}{2a} = \left(\frac{b^2}{4a^2} - \frac{c}{a}\right)^{\frac{1}{2}} = \frac{(b^2 - 4ac)^{\frac{1}{2}}}{2a};$$

whence, $$x^{\frac{2}{3}} = \frac{-b \pm (b^2 - 4ac)^{\frac{1}{2}}}{2a}.$$

Raising both sides of this equation to the $\frac{3}{2}$ power, we have

$$x = \left[\frac{-b \pm (b^2 - 4ac)^{\frac{1}{2}}}{2a}\right]^{\frac{3}{2}}.$$

EXERCISES.

1. Find a number which, added to twice its square root, will make 99.

2. What number will leave a remainder of 99 when twice its square root is subtracted from it.

3. One-fifth of a certain number exceeds its square root by 30. What is the number?

4. What number added to its square root makes 306?

5. If from 3 times a certain number we subtract 10 times its square root and 96 more, and divide the remainder by the number, the quotient will be 2. What is the number?

Solve the equations:

6. $\frac{1}{3}y^4 - 2y^2 = 15.$ 7. $3y^4 - 7y^2 = 25.$

8. $5y^{\frac{1}{2}} - 8y^{\frac{1}{4}} = 13.$

9. $(x^2 + a^2)^{\frac{m}{n}} - 4\,(x^2 + a^2)^{\frac{m}{2n}} = a^2 - 2 + \frac{1}{a^2}.$

197. When the unknown quantity appears in the form $x^2 + \frac{1}{x^2}$, the square may be completed by simply adding 2 to this expression, because $x^2 + 2 + \frac{1}{x^2}$ is a perfect square, namely, the square of $x + \frac{1}{x}$. The value of x may then be deduced from it by solving another quadratic equation.

EXAMPLE. $3x^2 + \frac{3}{x^2} = 22.$

We first divide by 3 and add 2 to each side of the equation, obtaining

$$x^2 + 2 + \frac{1}{x^2} = \frac{22}{3} + 2 = \frac{28}{3}.$$

Extracting the square root of both sides,

$$x + \frac{1}{x} = \frac{2\sqrt{7}}{\sqrt{3}} = \frac{2\sqrt{21}}{3} = \frac{2}{3}\sqrt{21}.$$

By multiplying by x, this equation becomes a quadratic, and can be solved in the usual way.

Let us now take this equation in the more general form,

$$x + \frac{1}{x} = c, \qquad (a)$$

which reduces to the foregoing by putting $c = \frac{2}{3}\sqrt{21}$. Clearing of fractions and transposing,

$$x^2 - cx + 1 = 0;$$

which being solved in the usual way, gives

$$x = \frac{e \pm \sqrt{(e^2 - 4)}}{2}.$$

The two roots are therefore

$$x_1 = \frac{e + \sqrt{(e^2 - 4)}}{2},$$

$$x_2 = \frac{e - \sqrt{(e^2 - 4)}}{2}.$$

If in the first of these equations we rationalize the numerator by multiplying it by $e - \sqrt{(e^2 - 4)}$ (§ 185), we shall find it to reduce to $\dfrac{2}{e - \sqrt{(e^2 - 4)}}$, that is, to $\dfrac{1}{x_2}$. Therefore,

$$x_1 = \frac{1}{x_2} \textit{ identically.}$$

Vice versa, x_2 is identically the same as $\dfrac{1}{x_1}$.

This must be the case whenever we solve an equation of the form (a), that is, one in which the value of $x + \dfrac{1}{x}$ is given. Let us suppose first that $e = \dfrac{50}{7}$, so that the equation is

$$x + \frac{1}{x} = \frac{50}{7}.$$

It is evident that $x = 7$ is a root of this equation, because when we put 7 for x, the left-hand member becomes $7 + \dfrac{1}{7}$, which is equal to $\dfrac{50}{7}$. If we put $\dfrac{1}{7}$ for x, the left-hand member will become

$$\frac{1}{7} + \frac{1}{\frac{1}{7}} = \frac{1}{7} + 7.$$

Hence x and $\dfrac{1}{x}$ exchange values by putting $\dfrac{1}{7}$ instead of 7, so that their sum $x + \dfrac{1}{x}$ remains unaltered by the change.

The general result may be expressed thus:

Because the value of the expression $x + \frac{1}{x}$ remains unaltered when we change x into $\frac{1}{x}$, therefore the reciprocal of any root of the equation

$$x + \frac{1}{x} = c$$

is also a root of the same equation.

EXERCISES.

Find all the roots of the following equations without clearing the given equations from denominators:

1. $x^2 + \frac{1}{x^2} = \frac{17}{4}$.
2. $a^2x^2 + \frac{1}{a^2x^2} = m^2 - 2$.
3. $16y^2 + \frac{1}{y^2} = 28$.
4. $\frac{m^4}{y^2} + y^2 = 2m^2$.
5. Show, without solving, that if r be any root of the equation

$$x^2 + \frac{1}{x^2} = a,$$

then $-r$, $\frac{1}{r}$, and $-\frac{1}{r}$ will also be roots.

Factoring a Quadratic Equation.

198. 1. *Special Case.* Let us consider the equation

$$x^2 - 2x - 15 = 0,$$

or

$$x^2 - 2x + 1 - 16 = 0,$$

or

$$(x - 1)^2 - 4^2 = 0.$$

Factoring, it becomes (§ 90),

$$(x - 1 + 4)(x - 1 - 4) = 0,$$

or

$$(x + 3)(x - 5) = 0.$$

Therefore the original equation can be transformed into

$$(x + 3)(x - 5) = 0,$$

a result which can be proved by simply performing the multiplications.

This last equation may be satisfied by putting either of its factors equal to zero; that is, by supposing

$$x + 3 = 0, \quad \text{whence} \quad x = -3;$$

or

$$x - 5 = 0, \quad \text{whence} \quad x = +5.$$

These are the same roots which we should obtain by solving the original equation.

2. *Factoring the General Quadratic Equation.* Let us consider the general quadratic equation,

$$x^2 + px + q = 0. \qquad (a)$$

Now, instead of thinking of x as a root of this equation, let us suppose x to have any value whatever, and let us consider the expression

$$x^2 + px + q, \qquad (1)$$

which for shortness we shall call X. Let us also inquire how it can be transformed without changing its value.

First we add and subtract $\frac{1}{4}p^2$, so as to make part of it a perfect square. It thus becomes,

$$X = x^2 + px + \frac{1}{4}p^2 - \frac{1}{4}p^2 + q;$$

or, which is the same thing,

$$X = \left(x + \frac{1}{2}p\right)^2 - \left(\frac{1}{4}p^2 - q\right).$$

Factoring this expression as in § 188, it becomes

$$X = \left[x + \frac{1}{2}p + \left(\frac{1}{4}p^2 - q\right)^{\frac{1}{2}}\right]\left[x + \frac{1}{2}p - \left(\frac{1}{4}p^2 - q\right)^{\frac{1}{2}}\right].$$

The student should now prove that this expression is really equal to $x^2 + px + q$, by performing the multiplication.

Let us next put, for brevity,

$$\left.\begin{aligned} \alpha &= -\frac{1}{2}p - \left(\frac{1}{4}p^2 - q\right)^{\frac{1}{2}}, \\ \beta &= -\frac{1}{2}p + \left(\frac{1}{4}p^2 - q\right)^{\frac{1}{2}}. \end{aligned}\right\} \qquad (2)$$

The preceding value of X will then become,

$$X = (x - \alpha)(x - \beta), \qquad (3)$$

an expression identically equal to (1), when we put for α and β their values in (2).

Let us return to the supposition that this expression is to be equal to zero, and that x is a root of the equation.

The equation (a) will then be

$$(x - \alpha)(x - \beta) = 0. \qquad (4)$$

But no product can be equal to zero unless one of the factors is zero. Hence we must have either

$$x - \alpha = 0, \quad \text{whence} \quad x = \alpha;$$

or

$$x - \beta = 0, \quad \text{whence} \quad x = \beta.$$

Hence, α and β are the two roots of the equation (a).

The above is another way of solving the quadratic equation.

To compare the expressions (1) and (2), let us perform the multiplication in the latter. It will become,

$$X = x^2 - (\alpha + \beta)x + \alpha\beta.$$

Since this expression is identically the same as x^2+px+q, the coefficients of the like powers of x must be the same. That is,

$$\left.\begin{aligned} \alpha + \beta &= -p, \\ \alpha\beta &= q, \end{aligned}\right\} \qquad (5)$$

which can be readily proved by adding and multiplying the equations (2).

This result may be expressed as follows:

Theorem. When a quadratic equation is reduced to the general form

$$x^2 + px + q = 0,$$

the coefficient of x will be equal to the sum of the roots with the sign changed.

The term independent of x will be equal to the product of the roots.

The student may ask why can we not determine the roots of the quadratic equation from equations (5), regarding α and β as the unknown quantities?

We can do so, but let us see what the result will be. We eliminate either α or β by substitution or by comparison.

From the second equation (5) we have,

$$\beta = \frac{q}{\alpha}.$$

Substituting this in the first equation, we have

$$\alpha + \frac{q}{\alpha} = -p.$$

Clearing of fractions and transposing,

$$\alpha^2 + p\alpha + q = 0.$$

We have now the same equation with which we started, only α takes the place of x. If we had eliminated α, we should have had the same equation in β, namely,

$$\beta^2 + p\beta + q = 0.$$

So the equations (5), when we try to solve them, only lead us to the original equation.

199. *To form a Quadratic Equation when the Roots are given.* The foregoing principles will enable us to form a quadratic equation which shall have any given roots. We have only to substitute the roots for α and β in equation (4), and perform the multiplications.

EXERCISES.

Form equations of which the roots shall be:

1. $+1$ and -1.
2. 3 and 2.
3. -3 and -2.
4. $3+2\sqrt{10}$ and $3-2\sqrt{10}$.
5. $7+2\sqrt{3}$ and $7-2\sqrt{3}$.
6. $+1$ and $+2$.
7. -1 and $+2$.
8. -1 and -2.
9. $+1$ and -2.
10. $2+\sqrt{5}$ and $2-\sqrt{5}$.
11. $\frac{3}{4}$ and $\frac{4}{5}$.
12. $\frac{7}{2}$ and $\frac{9}{2}$.
13. $2+\sqrt{2}$ and $2-\sqrt{2}$.
14. $9+2\sqrt{2}$ and $9-2\sqrt{2}$.
15. $5+7\sqrt{5}$ and $5-7\sqrt{5}$.
16. $a+b$ and $a-b$.
17. $a+\sqrt{a^2-b^2}$ and $a-\sqrt{a^2-b^2}$.

Equations having Imaginary Roots.

200. When we complete the square in order to solve a quadratic equation, the quantity on the right-hand side of the equation to which that square is equal must be positive, else there can be no real root. For if we square either a positive or negative quantity, the result will be positive. Hence, if the square of the first member comes out equal to a negative quantity, there is no answer, either positive or negative, which will fulfil the conditions. Such a result shows that impossible conditions have been introduced into the problem.

EXAMPLES.

1. To divide the number 10 into two such parts that their product shall be 34.

If we proceed with this equation in the usual way, we shall have, on completing the square,

$$x^2 - 10x + 25 = -9,$$

or

$$(x - 5)^2 = -9.$$

The square being negative, there is no answer. On considering the question, we shall see that the greatest possible product which the two parts of 10 can have is when they are each 5. It is therefore impossible to divide the number 10 into two parts of which the product shall be more than 25; and because the question supposes the product to be 34, it is impossible in ordinary numbers.

2. Suppose a person to travel on the surface of the earth to any distance; how far must he go in order that the straight line through the round earth from the point whence he started to the point at which he arrives shall be 8000 miles?

It is evident that the greatest possible length of this line is a diameter of the earth, namely, 7,912 miles. Hence he can never get 8,000 miles away, and the answer is impossible.

In such cases the square root of the negative quantity is considered to be part of a root of the equation, and because it is not equal to any positive or negative algebraic quantity, it is called an *imaginary root*. The theory of such roots will be explained in a subsequent book.

CHAPTER III.

REDUCTION OF IRRATIONAL EQUATIONS TO THE NORMAL FORM.

201. An **Irrational Equation** is one in which the unknown quantity appears under the radical sign.

An irrational equation may be cleared of fractions in the same way as if it were rational.

Example. Clear from fractions the equation

$$\frac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=\frac{2a}{\sqrt{x^2-a^2}}.$$

Multiplying both members by $\sqrt{x^2-a^2}=\sqrt{x+a}\,\sqrt{x-a}$, we have

$$\frac{(x+a)\sqrt{x-a}+(x-a)\sqrt{x+a}}{\sqrt{x+a}-\sqrt{x-a}}=2a.$$

Next, multiplying by $\sqrt{x+a}-\sqrt{x-a}$, we have

$$(x+a)\sqrt{x-a}+(x-a)\sqrt{x+a}=2a\sqrt{x+a}-2a\sqrt{x-a}.$$

Transposing and reducing, we have

$$(x+3a)\sqrt{x-a}+(x-3a)\sqrt{x+a}=0,$$

and the equation is cleared of denominators.

Clearing of Surds.

202. In order that an irrational equation may be solved, it must also be cleared of surds which contain the unknown quantity. In showing how this is done, we shall suppose the equation to be cleared of denominators, and to be composed of terms some or all of which are multiplied by the square roots of given functions of x.

Let us take, as a first example, the equation just found. Since a surd may be either positive or negative, the equation in question may mean any one of the following four:

$$(x+3a)\sqrt{x-a}+(x-3a)\sqrt{x+a}=0, \quad (1)$$
$$(x+3a)\sqrt{x-a}-(x-3a)\sqrt{x+a}=0, \quad (2)$$
$$-(x+3a)\sqrt{x-a}+(x-3a)\sqrt{x+a}=0, \quad (3)$$
$$-(x+3a)\sqrt{x-a}-(x-3a)\sqrt{x+a}=0. \quad (4)$$

But the third equation is merely the negative of the second, and the fourth the negative of the first, so that only two have different roots. Let us put, for brevity,

$$\left.\begin{aligned} P &= (x+3a)\sqrt{x-a}+(x-3a)\sqrt{x+a}, \\ Q &= (x+3a)\sqrt{x-a}-(x-3a)\sqrt{x+a}, \end{aligned}\right\} \quad (5)$$

and let us consider the equation,

$$PQ = 0. \quad (6)$$

Since this equation is satisfied when, and only when, we have either $P=0$ or $Q=0$, it follows that every value of x which satisfies either of the equations (1) or (2) will satisfy (6). Also, every root of (6) must be a root either of (1) or (2).

If we substitute in (6) the values of P and Q in (5), we shall then have

$$(x+3a)^2(x-a)-(x-3a)^2(x+a)=0,$$

which reduces to $\quad 5x^2-9a^2=0,$

and gives $\quad x = \pm\dfrac{3a}{\sqrt{5}}.$

It will be remarked that the process by which we free the equation from surds is similar to that for rationalizing the terms of a fraction employed in § 185.

As a second example, let us take the equation,

$$\sqrt{x+11}+\sqrt{x-4}-5=0. \quad (a)$$

We write the three additional equations formed by combining the positive and negative values of the surds in every way:

$$-\sqrt{x+11}+\sqrt{x-4}-5=0,$$
$$\sqrt{x+11}-\sqrt{x-4}-5=0,$$
$$-\sqrt{x+11}-\sqrt{x-4}-5=0.$$

The product of the first two equations is

$$(\sqrt{x-4}-5)^2-(x+11)=0,$$

or $$10-10\sqrt{x-4}=0. \qquad (1)$$

The product of the last two is

$$10+10\sqrt{x-4}=0. \qquad (2)$$

The product of these two products is

$$100-100(x-4)=0,$$

which gives $$x=5.$$

It will be remarked that (2) differs from (1) only in having the sign of the surd different. This must be the case, because the second pair of equations formed from (a) differ from the first pair only in having the sign of the surd $\sqrt{x-4}$ different. Hence it is not necessary to write more than one pair of the equations at each step. The general process is as follows:

I. *Change the sign of one of the surds in the given equation, and multiply the equation thus formed by the original equation.*

II. *Reduce this product, in it change the sign of another of the surds, and form a new product of the two equations thus formed.*

III. *Continue the process until an equation without surds is reached.*

Example. Solve

$$\sqrt{8x+9}+\sqrt{2x+6}+\sqrt{x+4}=0.$$

Changing the sign of $\sqrt{x+4}$,

$$\sqrt{8x+9}+\sqrt{2x+6}-\sqrt{x+4}=0.$$

The product is

$$(\sqrt{8x+9}+\sqrt{2x+6})^2-(x+4)=0,$$

or, after reduction,

$$9x+11+2\sqrt{8x+9}\sqrt{2x+6}=0.$$

Changing the sign of $\sqrt{2x+6}$, we have

$$9x+11-2\sqrt{8x+9}\sqrt{2x+6}=0.$$

The product of the last two equations reduces to

$$17x^2 - 66x - 95 = 0,$$

which being solved gives $$x = \frac{33 \pm 52}{17}.$$

REMARK. Equations containing surds may often reduce to the form treated in § 196. In this case, the methods of that section may be followed.

EXERCISES.

Solve the equations:

1. $$\frac{1}{\sqrt{x}+\sqrt{a}} + \frac{1}{\sqrt{x}-\sqrt{a}} = \frac{2\sqrt{a}-2\sqrt{x}}{x-a}.$$

2. $$\frac{\sqrt{x^2+a}}{\sqrt{a^2-x}} = \frac{x}{a}.$$

3. $$\sqrt{x+3} - \sqrt{x-4} = 1.$$

4. $$\sqrt{x+14} + \sqrt{x-14} = 14.$$

5. $$(3-x)^{\frac{1}{2}} - (3+x^2)^{\frac{1}{4}} = 0.$$

6. $$\sqrt{a+\sqrt{x}} + \sqrt{a-\sqrt{x}} = 2\sqrt{x+\frac{a}{2}}.$$

7. $$\frac{1}{\sqrt{x}+2} + \frac{\sqrt{x}}{x-4} - \frac{1}{\sqrt{x}-2} = 0.$$

8. $$\frac{5x-9}{\sqrt{5x}+3} - 1 = \frac{\sqrt{5x}-3}{2}.$$

9. $$\sqrt{a^2-2x} + \frac{x}{\sqrt{a^2-2x}} = b.$$

10. $$\frac{x+\sqrt{x}}{x-\sqrt{x}} = \frac{x(x-1)}{4}.$$

11. $$\frac{\sqrt{1+a}}{\sqrt{x-a}+\sqrt{ax-1}} = \frac{1}{\sqrt{x-1}}.$$

CHAPTER IV.

SIMULTANEOUS QUADRATIC EQUATIONS.

Between a pair of simultaneous general quadratic equations one of the unknown quantities can always be eliminated. The resulting equation, when reduced, will be of the fourth degree with respect to the other unknown quantity, and cannot be solved like a quadratic equation.

But there are several cases in which a solution of two equations, one of which is of the second or some higher degree, may be effected, owing to some of the terms being wanting in one or both equations.

203. Case I. ***When one of the equations is of the first degree only.***

This case may be solved thus:

Rule. *Find the value of one of the unknown quantities in terms of the other from the equation of the first degree. This value being substituted in the other equation, we shall have a quadratic equation from which the other unknown quantity may be found.*

Example. Solve

$$\left.\begin{aligned} 2x^2 + 3xy - 5y^2 - x - 5y &= 26, \\ 2x - 3y &= 5. \end{aligned}\right\} \qquad (a)$$

From the second equation we find

$$x = \frac{3y + 5}{2}. \qquad (b)$$

Whence, $$x^2 = \frac{9y^2 + 30y + 25}{4}.$$

Substituting this value in the first equation and reducing, we find

$$4y^2 + 16y + 10 = 26.$$

Solving this quadratic equation,

$$y = -2 \pm \sqrt{8} = -2 \pm 2\sqrt{2}.$$

This value of y being substituted in the equation (b) gives,

$$x = \frac{-1 \pm 3\sqrt{8}}{2} = \frac{-1 \pm 6\sqrt{2}}{2}.$$

The same problem may be solved in the reverse order by eliminating y instead of x. The second equation (a) gives

$$y = \frac{2x - 5}{3}.$$

If we substitute this value of y in the first equation, we shall have a quadratic equation in x, from which the value of the latter quantity can be found.

EXERCISES.

Solve

1. $$x^2 - 2xy + 4y^2 = 21.$$ $$2x + y = 12.$$

2. $$3x^2 - 2y^2 + 5x - 2y = 28.$$ $$x + y + 4 = 0.$$

3. $$5xy + 7y^2 - x - y = 72,$$ $$x + 2y = 0.$$

4. $$3x^2 + 2y^2 = 813,$$ $$7x - 4y = 17.$$

5. $$x + y = 7,$$ $$\frac{x}{y} - \frac{y}{x} = \frac{7}{12}.$$

204. Case II. ***When each equation contains only one term of the second degree, and that term has the same product or square of the unknown quantities in the two equations.***

Such equations are

$$\left.\begin{aligned} ax^2 + dx + ey + f &= 0, \\ a'x^2 + d'x + e'y + f' &= 0, \end{aligned}\right\} \qquad (a)$$

where the only term of the second degree is that in x^2.

If we eliminate x^2 from these equations by multiplying the first by a' and the second by a, and subtracting, we have

$$(a'd - ad')\,x + (a'e - ae')\,y + a'f - af' = 0.$$

Solving this equation with respect to x, we find

$$x = \frac{(ae' - a'e)\,y + af' - a'f}{a'd - ad'}. \qquad (b)$$

By substituting this value of x in either of the equations (a), we shall have a quadratic equation in y. Solving the latter, we shall obtain two values of y. Substituting these in (b), we shall have the two corresponding values of x, and the solution will be complete. Hence the rule,

Eliminate the term of the second degree by addition or subtraction, and use the resulting equation of the first degree with either of the original equations, as in Case I.

EXAMPLE. Solve

$$\left.\begin{aligned} 2xy - 4x + 5y &= 23, \\ 3xy + 7x + y &= 41. \end{aligned}\right\} \qquad (a)$$

Multiplying the first equation by 3 and the second by 2, and subtracting, we have

$$-\,26x + 13y = -\,13\,; \qquad (b)$$

whence,

$$x = \frac{1}{2}y + \frac{1}{2}. \qquad (c)$$

Substituting this value in the first equation, we find a quadratic equation, which, being solved, gives

$$y = -\,2 \pm \sqrt{29}.$$

Substituting these values in (c), the result is

$$x = -\,\frac{1}{2} \pm \frac{1}{2}\sqrt{29}.$$

The two sets of values of the unknown quantities are therefore

$$x_1 = -\,\frac{1}{2} + \frac{1}{2}\sqrt{29}, \qquad x_2 = -\,\frac{1}{2} - \frac{1}{2}\sqrt{29}.$$

$$y_1 = -\,2 + \sqrt{29}, \qquad y_2 = -\,2 - \sqrt{29}.$$

We might have obtained the same result by solving the equation (c) with respect to y, and substituting in (a). The student should practice both methods.

EXERCISES.

1. $$6x^2 - 3x - 4y = 25,$$
$$x^2 + 2x - 3y = 18.$$

2. $$2y^2 + y = 28.$$
$$y^2 + 3x - 4y = 18$$

3. $$xy + 6x + 7y = 66,$$
$$3xy + 2x + 5y = 70.$$

205. CASE III. ***When neither equation contains a term of the first degree in x or y.***

RULE. *Eliminate the constant terms by multiplying each equation by the constant term of the other, and adding or subtracting the two products. The result will be a quadratic equation, from which either unknown quantity can be determined in terms of the other. Then substitute as in Case I.*

EXAMPLE. Solve
$$\left.\begin{aligned} x^2 + xy - y^2 &= 5, \\ 2x^2 - 3xy + 2y^2 &= 14. \end{aligned}\right\} \qquad (1)$$

$14 \times$ 1st eq., $$14x^2 + 14xy - 14y^2 = 70.$$
$5 \times$ 2d eq., $$10x^2 - 15xy + 10y^2 = 70.$$
Subtracting, $$4x^2 + 29xy - 24y^2 = 0.$$

This is a quadratic equation, by which one unknown quantity can be expressed in terms of the other without the latter being under the radical sign.

Transposing, $$4x^2 + 29xy = 24y^2. \qquad (2)$$

Completing square, $$4x^2 + 29xy + \frac{841}{16}y^2 = \frac{1225}{16}y^2.$$

Extracting root, $$2x + \frac{29}{4}y = \pm\frac{35}{4}y.$$

Whence, $$x = \frac{-29 \pm 35}{8}y = \frac{3}{4}y \text{ or } -8y.$$

Substituting the first of these values of x in either of the original equations, we shall have
$$y^2 = 16;$$

whence, $\qquad y = \pm 4; \qquad x = \pm 3.$

Substituting the second value of x, we have

$$y^2 = \frac{1}{11};$$

whence, $\qquad y = \pm \frac{1}{\sqrt{11}}; \qquad x = \mp \frac{8}{\sqrt{11}}.$

Therefore the four possible values of the unknown quantities are,

$$x = +3, \quad -3, \quad +\frac{8}{\sqrt{11}}, \quad -\frac{8}{\sqrt{11}}.$$
$$y = +4, \quad -4, \quad -\frac{1}{\sqrt{11}}, \quad +\frac{1}{\sqrt{11}}.$$

Each of these four pairs of values satisfies the original equation.

A slight change in the mode of proceeding is to divide the equation (2) by either x^2 or y^2, and to find the value of the quotient. Dividing by y^2 and putting

$$u = \frac{x}{y},$$

the equation will become

$$4u^2 + 29u - 24 = 0.$$

This quadratic equation, being solved, gives

$$u = \frac{-29 \pm 35}{8} = \frac{3}{4} \text{ or } -8,$$

Putting $\frac{x}{y}$ for u, and multiplying by y,

$$x = \frac{3}{4}y \text{ or } -8y, \text{ as before.}$$

EXERCISES.

Solve

1. $$x^2 - xy + y^2 - 3 = 0,$$
$$x^2 - 2xy + 4y^2 - 4 = 0.$$

2. $$2x^2 + 3xy - y^2 - 2 = 0,$$
$$x^2 + 3xy - 4y^2 + 1 = 0.$$

206. Case IV. ***When the expressions containing the unknown quantities in the two equations have common factors.***

Rule. *Divide one of the equations which can be factored by the other, and cancel the common factors. Then clear of fractions, if necessary, and we shall have an equation of a lower degree.*

EXAMPLES.

1. $x^3 + y^3 = 91, \quad x + y = 7.$

We have seen (§ 94, Th. 1) that x^3+y^3 is divisible by $x+y$. So dividing the first equation by the second, we have

$$x^2 - xy + y^2 = 13.$$

This is an equation of the second degree only, and when combined with the second of the original equations, the solution may be effected by Case I. The result is,

$$x = 3 \text{ or } 4, \qquad y = 4 \text{ or } 3.$$

2. $xy + y^2 = 133, \quad x^2 - y^2 = 95.$

Factoring the first member of each equation, the equations become

$$y(x + y) = 133, \qquad (x + y)(x - y) = 95.$$

Dividing one equation by the other, and clearing of fractions,

$$12y = 7x, \quad \text{or} \quad y = \frac{7}{12}x.$$

The problem is now reduced to Case I, this value of y being combined with either of the original equations.

207. There are many other devices by which simultaneous equations may be solved or brought under one of the above cases, for which no general rule can be given, and in which the solution must be left to the ingenuity of the student. Sometimes, also, an equation which comes under one of the cases can be solved much more expeditiously than by the rule.

Let us take, for instance, the equations,

$$x^2 + y^2 = 65, \qquad xy = 28.$$

These equations can be solved by Case III, but the work would be long and cumbrous. We see that by adding and subtracting twice the second equation to and from the first, we can form two perfect squares. Extracting the roots of these squares, we shall have two simple equations, which shall give the solution at once. Each unknown quantity will have four values, namely, $\pm 7 \pm 4$.

PROBLEMS AND EXERCISES.

The following equations can all be solved by some short and expeditious combination of the equations, or by factoring, without going through the complex process of Case III. The student is recommended not to work upon the equations at random, but to study each pair until he sees how it can be reduced to a simpler equation by addition, multiplication, or factoring, and then to go through the operations thus suggested.

1. $y^2 + xy = 14, \quad x^2 + xy = 35.$
2. $4x^2 - 2xy = 208, \quad 2xy - y^2 = 39.$
3. $x^2 + y = 4x, \quad y^2 + x = 4y.$

If we subtract one of these equations from the other, the difference will be divisible by $x - y$.

4. $x^3 + y^3 + 3x + 3y = 378, \quad x^3 + y^3 - 3x - 3y = 324.$
5. $x^2 + y^2 = 74, \quad x + y = 12.$
6. $x^2 + xy = 63, \quad x^2 - y^2 = 77.$
7. $\dfrac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}} = 4, \quad x^2 - y^2 = 544.$
8. $x^2 + xy = a, \quad y^2 + xy = b.$
9. $x^3 + xy^2 = 10, \quad y^3 + x^2y = 5.$
10. $x = a\sqrt{x + y}, \quad y = b\sqrt{x + y}.$
11. $x\sqrt{x + y} = 12. \quad y\sqrt{x + y} = 15.$
12. $2x^2 + 2y^2 = x + y, \quad x^2 + y^2 = x - y.$
13. $5x^2 - 5y^2 = x + y, \quad 3x^2 - 3y^2 = x - y.$
14. $x^2 + y^2 + z^2 = 30, \quad xy + yz + zx = 17, \quad x - y - z = 2.$
15. $\sqrt{\dfrac{6y}{x - y}} - 3\sqrt{\dfrac{x - y}{6y}} = 2,$

$$x + y - 2\sqrt{\frac{x + y}{x - y}} = \frac{8}{x - y}.$$

16. A principal of \$5000 amounts, with simple interest, to \$7100 after a certain number of years. Had the rate of interest been 1 per cent. higher and the time 1 year longer, it would have amounted to \$7800. What was the time and rate?

17. A courier left a station riding at a uniform rate. Five hours afterward, a second followed him, riding 3 miles an hour faster. Two hours after the second, a third started at the rate of 10 miles an hour. They all reach their destination at the same time. What was its distance and the rate of riding?

18. In a right-angled triangle there is given the hypothenuse $= a$, and the area $= b^2$; find the sides.

19. Find two numbers such that their product, sum, and difference of squares shall be equal to each other.

20. Find two numbers whose product is 216; and if the greater be diminished by 4, and the less increased by 3, the product of this sum and difference may be 240.

21. There are two numbers whose sum is 74, and the sum of their square roots is 12. What are the numbers?

22. Find two numbers whose sum is 72, and the sum of their cube roots 6.

23. The sides of a given rectangle are m and n. Find the sides of another which shall have twice the perimeter and twice the area of the given one.

24. A certain number of workmen require 3 days to complete a work. A number 4 less, working 3 hours less per day, will do it in 6 days. A number 6 greater than the original number, working 6 hours less per day, will complete the work in 4 days. What was the original number of workmen, and how long did they work per day?

25. Find two numbers whose sum is 18 and the sum of their fourth powers 14096.

NOTE. Since the sum of the two numbers is 18, it is evident that the one must be as much less than 9 as the other is greater. The equations will assume the simplest form when we take, as the unknown quantity, the common amount by which the numbers differ from 9.

26. Find two numbers, x and y, such that

$$x^3 + y^3 : x^3 - y^3 :: 35 : 19,$$

$$xy = 24.$$

27. Find two numbers whose sum is 14 and the sum of their fifth powers 161294.

BOOK VII.

PROGRESSIONS.

CHAPTER I.

ARITHMETICAL PROGRESSION.

208. *Def.* When we have a series of numbers each of which is greater or less than the preceding by a constant quantity, the series is said to form an **Arithmetical Progression.**

EXAMPLE. The series

$$7,\quad 12,\quad 17,\quad 22,\quad 27,\quad 32,\quad \text{etc.};$$
$$7,\quad 5,\quad 3,\quad 1,\quad -1,\quad -3,\quad \text{etc.};$$
$$a+b,\quad a,\quad a-b,\quad a-2b,\quad a-3b,\quad \text{etc.},$$

are each in arithmetical progression, because, in the first, each number is greater than the preceding by 5; in the second, each is less than the preceding by 2; in the third, each is less than the preceding by b.

Def. The amount by which each term of an arithmetical progression is greater than the preceding one is called the **Common Difference.**

Def. The **Arithmetical Mean** of two quantities is half their sum.

All the terms of an arithmetical progression except the first and last are called so many arithmetical means between the first and last as extremes.

EXAMPLE. The four numbers, 5, 8, 11, 14, form the four arithmetical means between 2 and 17.

EXERCISES.

1. Form four terms of the arithmetical progression of which the first term is 7 and common difference 3.

2. Write the first seven terms of the progression of which the first term is 11 and the common difference -3.

3. Write five terms of the progression of which the first term is $a-4n$ and the common difference $2n$.

Problems in Progression.

209. Let us put

a, the first term of a progression.
d, the common difference.
n, the number of terms.
l, the last term.
Σ, the sum of all the terms.

The series is then

$$a, \quad a+d, \quad a+2d, \quad . \; . \; . \; . \; l.$$

Any three of the above five quantities being given, the other two may be found.

PROBLEM I. *Given the first term, the common difference, and the number of terms, to find the last term.*

The 1st term is here a,
2d " " $a+d$,
3d " " $a+2d$.

The coefficient of d is, in each case, 1 less than the number of the term. Since this coefficient increases by unity for every term we add, it must remain less by unity than the number of the term. Hence,

$$\text{The } i^{th} \text{ term is } a + (i-1)\,d,$$

whatever be i. Hence, when $i = n$,

$$l = a + (n-1)\,d. \qquad (1)$$

From this equation we can solve the further problems:

PROBLEM II. *Given the last term* l, *the common difference* d, *and the number of terms* n, *to find the first term.*

The solution is found by solving (1) with respect to a, which gives

$$a = l - (n - 1)\,d. \tag{2}$$

Problem III. *Given the first and last terms, **a** and **l**, and the number of terms **n**, to find the common difference.*

Solution from (1), d being the unknown quantity,

$$d = \frac{l - a}{n - 1}. \tag{3}$$

Problem IV. *Given the first and last terms and the common difference, to find the number of terms.*

Solution, also from (1),

$$n = \frac{l - a}{d} + 1 = \frac{l - a + d}{d}. \tag{4}$$

Problem V. *To find the sum of all the terms of an arithmetical progression.*

We have, by the definition of Σ,

$$\Sigma = a + (a + d) + (a + 2d) + \ldots . (l - d) + l,$$

the parentheses being used only to distinguish the terms.

Now let us write the terms in reverse order. The term before the last is $l - d$, the second one before it $l - 2d$, etc.

We therefore have,

$$\Sigma = l + (l - d) + (l - 2d) \ldots . + (a + d) + a.$$

Adding these two values of Σ together, term by term, we find

$$2\Sigma = (a+l) + (a+l) + (a+l) + \ldots . + (a+l) + (a+l),$$

the quantity $(a+l)$ being written as often as there are terms, that is, n times. Hence,

$$2\Sigma = n\,(a + l),$$

$$\Sigma = n\,\frac{a + l}{2}. \tag{5}$$

Remark. The expression $\frac{a + l}{2}$, that is, half the sum of the extreme terms, is the *mean value* of all the terms. The sum of the n terms is therefore the same as if each of them had this value.

210. In the equation (5) we are supposed to know the first and last terms and the number of terms. If other quantities are taken as the known ones, we have to substitute for some one of the quantities in (5) its expression in one of the equations (1), (2), (3), or (4). Suppose, for example, that we have given only the last term, the common difference, and the number of terms, that is, l, d, and n. We must then in (5) substitute for a its value in (2). This will give,

$$\Sigma = n\left(l - \frac{n-1}{2}d\right) = nl - \frac{n(n-1)}{2}d. \qquad (6)$$

EXERCISES.

In arithmetical progression there are

1. Given, common difference, $+3$; third term $= 10$.
   Find first term. *Ans.* First term $= 4$.
2. Given 4th term $= b$, common difference $= -c$.
   Find first 7 terms, their sum and product.
3. Given 3d term $= a + b$, 4th term $= a + 2b$.
   Find first 5 terms.
4. Given 1st term $= a - b$, 9th term $= 9a + 7b$.
   Find 2d term and common difference.
5. Given, sum of 9 terms $= 108$.
   Find middle term and sum of 1st and 9th terms.
6. Given 5th term $= 7x - 5y$, 7th term $= 9x - 9y$.
   Find first 7 terms and common difference.
7. Given 1st term $= 12$, 50th term $= 551$.
   Find sum of all 50 terms.
8. To find the sum of the first 100 numbers, namely,

$$1 + 2 + 3 \ldots\ldots + 99 + 100.$$

Here the first term a is 1, the last term l 100, and the number of terms 100. The solution is by Problem V.

9. Find the sum of the first n entire numbers, namely,

$$1 + 2 + 3 \ldots\ldots + n.$$

10. Find the sum of the first n odd numbers, namely,

$$1 + 3 + 5 \ldots\ldots + 2n - 1.$$

Here the number of terms is n.

11. Find the sum of the first n even numbers, namely,

$$2 + 4 + 6 \ldots\ldots + 2n.$$

12. In a school of m scholars, the highest received 134 merit marks, and each succeeding one 6 less than the one next above him. How many did the lowest scholar receive? How many did they all receive?

13. The first term of a series is m, the last term $2m$, and the common difference d. What is the number of terms?

14. The first term is k, the last term $10k - 1$, and the number of terms 9. What is the common difference?

15. The middle term of a progression is s, the number of terms 5, and the common difference $-h$. What are the first and last terms and the sum of the 5 terms?

16. The sum of 5 numbers in arithmetical progression is 20 and the sum of their squares 120. What are the numbers?

NOTE. In questions like this it is better to take the middle term for one of the unknown quantities. The other unknown quantity will be the common difference.

17. Find a number consisting of three digits in arithmetical progression, of which the sum is 15. If the number be diminished by 792, the digits will be reversed.

18. The continued product of three numbers in arithmetical progression is 640, and the third is four times the first. What are the numbers?

19. A traveller has a journey of 132 miles to perform. He goes 27 miles the first day, 24 the second, and so on, travelling 3 miles less each day than the day before. In how many days will he complete the journey?

Here we have given the first term 27, the common difference -3, and the sum of the terms 132. To solve this, we take equation (5), and substitute for l its value in (1). This makes (5) reduced to

$$\Sigma = n\frac{a + a + (n-1)\,d}{2} = na + \frac{n\,(n-1)\,d}{2}.$$

Σ, a, and d are given by the problem, and n is the unknown quantity. Substituting the numerical value of the unknown quantities, the equation becomes

$$132 = 27n - 3\frac{n(n-1)}{2}.$$

This reduced to a quadratic equation in n, the solution of which gives two values of n. The student should explain this double answer by continuing the progression to 11 terms, and showing what the negative terms indicate.

20. Taking the same question as the last, only suppose the distance to be 140 miles instead of 132. Show that the answer will be imaginary, and explain this result.

21. A debtor owing \$160 arranged to pay 25 dollars the first month, 23 the second, and so on, 2 dollars less each month, until his debt should be discharged. How many payments must he make, and what is the explanation of the two answers?

22. A hogshead holding 135 gallons has 3 gallons poured into it the first day, 6 the second, and so on, 3 gallons more every day. How long before it will be filled?

23. The continued product of 5 consecutive terms is 12320 and their sum 40. What is the progression?

24. Show that the condition that three numbers, p, q, and r, are in arithmetical progression may be expressed in the form

$$\frac{q-p}{q-r} = -1.$$

25. In a progression consisting of 10 terms, the sum of the 1st, 3d, 5th, 7th, and 9th terms is 90, and the sum of the remaining terms is 110. What is the progression?

26. In a progression of an odd number of terms there is given the sum of the odd terms (the first, third, fifth, etc.), and the sum of the even terms (the second, fourth, etc.). Show that we can find the middle term and the number of terms, but not the common difference.

27. In a progression of an even number of terms is given the sum of the even terms $= 105$, the sum of the odd terms $= 119$, and the excess of the last term over the first $= 26$. What is the progression?

28. Given a and l, the first and last terms, it is required to insert i arithmetical means between them. Find the expression for the i terms required.

CHAPTER II.

GEOMETRICAL PROGRESSION.

211. *Def.* A **Geometrical Progression** consists of a series of terms of which each is formed by multiplying the term preceding by a constant factor.

An arithmetical progression is formed by continual addition or subtraction; a geometrical progression by repeated multiplication or division.

Def. The factor by which each term is multiplied to form the next one is called the **Common Ratio.**

The common ratio is analogous to the common difference in an arithmetical progression.

In other respects the same definitions apply to both.

EXAMPLES.

$$2, \quad 6, \quad 18, \quad 54, \quad \text{etc.},$$

is a progression in which the first term is 2 and the common ratio 3.

$$2, \quad 1, \quad \frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \quad \text{etc.},$$

is a progression in which the ratio is $\frac{1}{2}$.

$$+3, \quad -6, \quad +12, \quad -24, \quad \text{etc.},$$

is a progression in which the ratio is -2.

NOTE. A progression like the second one above, formed by dividing each term by the same divisor to obtain the next term, is included in the general definition, because dividing by any number is the same as multiplying by the reciprocal. Geometrical progressions may therefore be divided into two classes, increasing and decreasing. In the increasing progression the common ratio is greater than 1 and the terms go on increasing; in a diminishing progression the ratio is less than unity and the terms go on diminishing.

REM. In a progression in which the ratio is negative, the terms will be alternately positive and negative.

Def. A **Geometrical Mean** between two quantities is the square root of their product.

EXERCISES.

Form five terms of each of the following geometrical progressions:

1. First term, 1; common ratio, 5.
2. First term, 7; common ratio, -3.
3. First term, 1; common ratio, -1.
4. First term, $\frac{2}{3}$; common ratio, $\frac{3}{4}$.
5. First term, $\frac{4}{5}$; common ratio, $\frac{1}{2}$.

Problems of Geometrical Progression.

212. In a geometrical progression, as in an arithmetical one, there are five quantities, any three of which determine the progression, and enable the other two to be found. They are

a, the first term.
r, the common ratio.
n, the number of terms.
l, the last term.
Σ, the sum of the n terms.

The general expression for the geometrical progression will be

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad \text{etc.},$$

because each of these terms is formed by multiplying the preceding one by r.

The same problems present themselves in the two progressions. Those for the geometrical one are as follows:

PROBLEM I. *Given the first term, the common ratio, and the number of terms, to find the last term.*

The progression will be

$$a, \quad ar, \quad ar^2, \quad \text{etc.}$$

We see that the exponent of r is less by 1 than the number of the term, and since it increases by 1 for each term added, it must remain less by 1, how many terms so ever we take. Hence the n^{th} term is

$$l = ar^{n-1}. \qquad (1)$$

PROBLEM II. *Given the last term, the common ratio, and the number of terms, to find the first term.*

The solution is found by dividing both members of (1) by r^{n-1}, which gives

$$a = \frac{l}{r^{n-1}}. \qquad (2)$$

PROBLEM III. *Given the first term, the last term, and the number of terms, to find the common ratio.*

From (1) we find $\quad r^{n-1} = \frac{l}{a}.$

Extracting the $(n-1)^{th}$ root of each member, we have

$$r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}.$$

[The solution of Problem IV requires us to find n from equation (1), and belongs to a higher department of Algebra.]

PROBLEM V. *To find the sum of all **n** terms of a geometrical progression.*

We have $\quad \Sigma = a + ar + ar^2 + \text{etc.} + ar^{n-1}.$

Multiply both sides of this equation by r. We then have

$$r\Sigma = ar + ar^2 + ar^3 + \text{etc.} \ldots\ldots + ar^n.$$

Now subtract the first of these equations from the second. It is evident that, in the second equation, each term of the second member is equal to the term of the second member of the first equation which is one place farther to the right. Hence, when we subtract, all the terms will cancel each other except the first of the first equation and the last of the second.

ILLUSTRATION. The following is a case in which $a = 2$, $r = 3$, $n = 6$:

$$\Sigma = 2 + 6 + 18 + 54 + 162 + 486.$$
$$3\Sigma = 6 + 18 + 54 + 162 + 486 + 1458.$$

Subtracting, $3\Sigma - \Sigma = 1458 - 2 = 1456,$

or $2\Sigma = 1456,$ and $\Sigma = 728.$

Returning to the general problem, we have

$$(r-1)\Sigma = ar^n - a = a(r^n - 1);$$

whence,
$$\Sigma = a\frac{r^n - 1}{r - 1} = a\frac{1 - r^n}{1 - r}. \qquad (4)$$

It will be most convenient to use the first form when $r > 1$, and the second when $r < 1$.

By this formula we are enabled to compute the sum of the terms of a geometrical progression without actually forming all the terms and adding them.

EXERCISES

1. Given 3d term $= 9$, common ratio $= \frac{3}{2}$.
   Find first 5 terms.
2. Given 5th term $= \frac{32}{27}$, common ratio $= -\frac{2}{3}$.
   Find first 5 terms.
3. Given 5th term $= x^4y^7$, 1st term $= y^4$.
   Find common ratio.
4. Given 1st term $= 1$, 4th term $= a^3$.
   Find common ratio and first 3 terms.
5. Given 2d term $= m$, common ratio $= -m$.
   Find first 4 terms.

6. A farrier having told a coachman that he would charge him \$3 for shoeing his horse, the latter objected to the price. The farrier then offered to take 1 cent for the first nail, 2 for the second, 4 for the third, and so on, doubling the amount for each nail, which offer the coachman accepted. There were 32 nails. Find how much the coachman had to pay for the last nail, and how much in all. (Compare § 168, Rem.)

7. Find the sum of 11 terms of the series

$$2 + 6 + 18 + \text{etc.},$$

in which the first term is 2 and the common ratio 3.

8. If the common ratio of a progression is r, what will be the common ratio of the progression formed by taking

I. Every alternate term of the given progression?

II. Every n^{th} term?

9. The same thing being supposed, what will be the common ratio of the progression of which every alternate term is equal to every third term of the given progression?

10. Show that if, in a geometrical progression, each term be added to or subtracted from that next following, the sums or remainders will form a geometrical progression.

11. Show that if the arithmetical and geometrical means of two quantities be given, the quantities themselves may be found, and give the expressions for them.

12. The sum of the first and fourth terms of a progression is to the sum of the second and third as $21:5$. What is the common ratio?

13. Express the continued product of all the terms of a geometrical progression in terms of a, r, and n?

Limit of the Sum of a Progression.

213. *Theorem.* If the common ratio in a geometrical progression is less than unity (more exactly, if it is contained between the limits -1 and $+1$), then there will be a certain quantity which the sum of all the terms can never exceed, no matter how many terms we take.

For example, the sum of the progression

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\text{etc.},$$

in which the common ratio is $\frac{1}{2}$, can never amount to 1, no matter how many terms we take. To show this, suppose that one person owed another a dollar, and proceeded to pay him a series of fractions of a dollar in geometrical progression, namely,

$$\frac{1}{2},\quad \frac{1}{4},\quad \frac{1}{8},\quad \frac{1}{16},\quad \text{etc.}$$

When he paid him the $\frac{1}{2}$ he would still owe another $\frac{1}{2}$, when he paid the $\frac{1}{4}$ he would still owe another $\frac{1}{4}$, and so on.

That is, at every payment he would discharge one-half the remaining debt. Now there are two propositions to be understood in reference to this subject.

I. *The entire debt can never be discharged by such payments.*

For, since the debt is *halved* at every payment, if there was any payment which discharged the whole remaining debt, the half of a thing would be equal to the whole of it, which is impossible.

II. *The debt can be reduced below any assignable limit by continuing to pay half of it.*

For, however small the debt may be made, another payment will make it smaller by one-half; hence there is no smallest amount below which it cannot be reduced.

These two propositions, which seem to oppose each other, hold the truth between them, as it were. They constantly enter into the higher mathematics, and should be well understood. We therefore present another illustration of the same subject.

Suppose AB to be a line of given length. Let us go one-half the distance from A to B at one step, one-fourth at the second, one-eighth at the third, etc. It is evident that, at each step, we go half the distance which remains. Hence the two principles just cited apply to this case. That is,

1. We can never reach B by a series of such steps, because we shall always have a distance equal to the last step left.

2. But we can come as near B as we please, because every step carries us over half the remaining distance.

This result is often expressed by saying that we should reach B by taking an infinite number of steps. This is a convenient form of expression, and we may sometimes use it, but it is not logically exact, because no conceivable number can be really infinite. The assumption that infinity is an algebraic quantity often leads to ambiguities and difficulties in the application of mathematics.

Def. The **Limit** of the sum Σ of a geometrical progression is a quantity which Σ may approach so that its difference shall be less than any quantity we choose to assign, but which Σ can never reach.

EXAMPLES.

1. Unity is the limit of the sum

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\text{etc.}$$

2. The point B in the preceding figure is the limit of all the steps that can be taken in the manner described.

The following principle will enable us to find the limit of the sum of a progression:

214. *Principle.* If $r<1$, the power r^n can be made as small as we please by increasing the value of n, but can never be made equal to 0.

Suppose, for instance, that

$$r=\frac{3}{4}=1-\frac{1}{4}.$$

Then every time we multiply by r we diminish r^n by $\frac{1}{4}$ of its former value; that is,

$$\begin{aligned} r^2 &= \frac{3}{4}r = \left(1-\frac{1}{4}\right)r = r-\frac{1}{4}r, \\ r^3 &= \frac{3}{4}r^2 = r^2-\frac{1}{4}r^2, \\ r^4 &= \frac{3}{4}r^3 = r^3-\frac{1}{4}r^3, \\ &\text{etc.} \qquad \text{etc.} \qquad \text{etc.} \end{aligned}$$

Now let us again take the expression for the sum of a series of n terms, namely,

$$\Sigma = a\frac{1-r^n}{1-r},$$

which we may put into the form

$$\Sigma = \frac{a}{1-r}-\frac{a}{1-r}r^n$$

If r is less than unity, we can, by the principle just cited, make the quantity r^n as small as we please by increasing n indefinitely. From this it follows that we can also make the term $\frac{a}{1-r} r^n$ as small as we please.

Proof. Let us put, for brevity,

$$k = \frac{a}{1-r},$$

so that the term under consideration is

$$kr^n.$$

If we cannot make kr^n as small as we please, suppose s to be its smallest possible value. Let us divide s by k, and put

$$t = \frac{s}{k}.$$

No matter how small s may be, and how large k may be, $\frac{s}{k}$, or t, will always be greater than zero. Hence, by the preceding principle, we can find a value of n so great that r^n shall be less than t. That is,

$$r^n < \frac{s}{k}.$$

Multiplying both sides of this inequality by k,

$$kr^n < s.$$

That is, however small we take s, we can take n so large that kr^n shall be less than s, and therefore s cannot be the smallest value.

Since
$$\Sigma = \frac{a}{1-r} - kr^n,$$

and since we can make kr^n as small as we please, it follows that

$$\text{Limit of } \Sigma = \frac{a}{1-r}.$$

This is sometimes expressed by saying that when $r < 1$,

$$a + ar + ar^2 + ar^3 + \text{etc., } \textit{ad infinitum} = \frac{a}{1-r},$$

and this is a convenient form of expression, which will not lead us into error in this case.

EXERCISES.

Having given the progression

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \text{etc.},$$

of which the limit is 1, find how many terms we must take in order that the sum may differ from 1 by less than the following quantities, namely:

Firstly, .001; secondly, .000 001; thirdly, .000 000 001.

To do this, we must find what power of $\frac{1}{2}$ will be less than .001, what power less than .000 001, etc.

What are the limits of the sums of the following series:

1. $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \text{etc.},$ *ad infinitum.*

2. $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \text{etc.},$ *ad infinitum.*

3. $\frac{1}{9} - \frac{1}{9^2} + \frac{1}{9^3} - \text{etc.},$ *ad infinitum.*

4. $\frac{4}{9} + \frac{4^2}{9^2} + \frac{4^3}{9^3} + \text{etc.},$ *ad infinitum.*

5. $\frac{1}{1+b} + \frac{1}{(1+b)^2} + \frac{1}{(1+b)^3} + \text{etc.},$ *ad infinitum.*

6. $\frac{a}{b-1} - \frac{a}{(b-1)^2} + \frac{a}{(b-1)^3} - \text{etc.},$ *ad infinitum.*

7. $1 - \frac{2}{m} + \frac{1}{m^2} - \frac{2}{m^3} + \frac{1}{m^4} - \text{etc.},$ *ad infinitum.*

8. What is that progression of which the first term is 12 and the limit of the sum 8.

9. On the line AB a man starts from A and goes to the point c, half way to B; then he returns to d, half way back to A; then turns again and goes half way to c, then back half way to d, and so on, going at each turn half way to the point from which he last set out. To what point on the line will he continually approach?

215. As an interesting application of the preceding theory, we may examine the problem of finding the value of a circulating decimal. Such a decimal is always equal to a vulgar fraction, which is obtained as in the following examples:

1. What is the value of the decimal

$$.373737\ldots?$$

We find the figures which form the period to be 37. Dividing the decimal into periods of these figures, its value is

$$\frac{37}{100}+\frac{37}{100^2}+\frac{37}{100^3}+\text{etc.}$$

$$=37\left(\frac{1}{100}+\frac{1}{100^2}+\frac{1}{100^3}+\text{etc.}\right).$$

The quantity in the parenthesis is a geometrical progression, in which $a=\frac{1}{100}$, $r=\frac{1}{100}$. The limit of its sum is therefore $\frac{1}{99}$. Therefore the value of the decimal is $\frac{37}{99}$.

This result can be proved by changing this vulgar fraction to a decimal.

2. In the case of a decimal which has one or more figures before the period commences, we cut these figures off, and find the value of them and of the circulating part separately. Thus,

$$56363\text{ etc.}=\frac{5}{10}+\frac{63}{1000}+\frac{63}{100000}+\text{etc.}$$

$$=\frac{5}{10}+\frac{63}{1000}\left(1+\frac{1}{100}+\frac{1}{100^2}+\text{etc.}\right)$$

$$=\frac{5}{10}+\frac{63}{1000}\cdot\frac{100}{99}=\frac{5}{10}+\frac{63}{990}=\frac{558}{990}=\frac{31}{55}.$$

EXERCISES.

To what vulgar fractions are the following circulating decimals equal:

1. .111111 . . . . ?
2. .2222 . . . . ?
3. .9999 . . . . ?
4. .09999 . . . . ?
5. .454545 . . . . ?
6. .2454545 . . . . ?
7. .108108 . . . . ?
8. 72454545 . . . . ?

Compound Interest.

216. When one loans or invests money, collects the interest at stated intervals, and again loans or invests this interest, and so on, he gains compound interest.

Compound interest can always be gained by one who constantly invests all his income derived from interest, provided that he always collects the interest when due, and is able to loan or invest it at the same rate as he loaned his principal.

PROBLEM I. *To find the amount of* **p** *dollars for* **n** *years, at* **c** *per cent. compound interest.*

SOLUTION. At the end of one year the interest will be $\frac{pc}{100}$, which added to the principal will make $p\left(1+\frac{c}{100}\right)$.

If we put $\rho = \frac{c}{100} =$ the rate of annual gain,

the amount at the end of the year will be $p(1+\rho)$.

Now suppose this whole amount is put out for another year at the same rate. The interest will be $p(1+\rho)\rho$, which added to the new principal $p(1+\rho)$ will make $p(1+\rho)^2$.

It is evident that, in general, supposing the whole sum kept at interest, the total amount of the investment will be multiplied by $1+\rho$ each year. Hence the amount at the ends of successive years will be

$$p(1+\rho),\quad p(1+\rho)^2,\quad p(1+\rho)^3,\quad \text{etc.}$$

At the end of n years the amount will be

$$p(1+\rho)^n.$$

PROBLEM II. *A person puts out* **p** *dollars every year, letting the whole sum constantly accumulate at compound interest. What will the amount be at the end of* **n** *years?*

SOLUTION. The first investment will have been out at interest n years, the second $n-1$ years, the third $n-2$ years, and so on to the n^{th}, which will have been out 1 year. Hence, from the last formula, the amounts will be:

Amount of	1st	payment,	$p(1+\rho)^n$.
"	" 2d	"	$p(1+\rho)^{n-1}$.
"	" 3d	"	$p(1+\rho)^{n-2}$.
"	" 4th	"	$p(1+\rho)^{n-3}$.
"	" 5th	"	$p(1+\rho)^{n-4}$.
	etc.		etc.

The sum of the amounts is:

$$p(1+\rho) + p(1+\rho)^2 + p(1+\rho)^3 + \ldots . p(1+\rho)^n.$$

This is a geometrical progression, of which the first term is $p(1+\rho)$, the common ratio $1+\rho$, and the number of terms n. So in the formula (4), § 212, we put $p(1+\rho)$ for a, $1+\rho$ for r, and thus find,

$$\Sigma = p(1+\rho)\frac{(1+\rho)^n - 1}{1+\rho-1} = p\frac{(1+\rho)^{n+1} - (1+\rho)}{\rho}.$$

EXERCISES.

1. A man insures his life for \$5000 at the age of 30, pays for his insurance a premium of 80 dollars a year for 32 years, and dies at the age of 62, immediately before the 33d payment would have been due. If the company gains 4 per cent. interest on all its money, how much does it gain or lose by the insurance?

NOTE. Computations of this class can be made with great facility by the aid of a table of logarithms.

2. What is the present value of a dollars due n years hence, interest being reckoned at c per cent.?

NOTE. If p be the present value, Problem I gives the equation,

$$p\left(1+\frac{c}{100}\right)^n = a.$$

3. What is the present value of 3 annual payments, of a dollars each, to be made in one, two, and three years, interest being reckoned at 5 per cent.?

4. What is the present value of n annual payments, of a dollars each, the first being due in one year, if the rate of interest is c per cent.? What would it be if the first payment were due immediately?

SECOND PART.

ADVANCED COURSE.

BOOK VIII.

RELATIONS BETWEEN ALGEBRAIC QUANTITIES.

Of Algebraic Functions.

217. *Def.* When one quantity depends upon another in such a way that a change in the value of the one produces a change in the value of the other, the latter is called a **Function** of the former.

This is a more general definition of the word "function" than that given in § 49.

EXAMPLES. The time required to perform a journey is a function of the distance because, other things being equal, it varies with the distance.

The cost of a package of tea is a function of its weight, because the greater the weight the greater the cost.

An algebraic expression containing any symbol is a function of that symbol, because by giving different values to the symbol we shall obtain different values for the expression.

Def. An **Algebraic Function** is one in which the relations of the quantities is expressed by means of an algebraic equation.

EXAMPLE. If in a journey we call t the time, s the average speed, and d the distance to be travelled, the relation between these quantities may be expressed by the equation,

$$d = st.$$

Any one of these quantities is a function of the other two, defined by means of this equation.

An algebraic function generally contains more than one letter, and therefore depends upon several quantities. But we may consider it a function of any one of these quantities, selected at pleasure, by supposing all the other quantities to remain constant and only this one to vary. For example, the time required for a train to run between two points is a function not only of their distance apart, but of the speed of the train. The speed being supposed constant, the time will be greater the greater the distance. The distance being constant, the time will be greater the less the speed.

Def. The quantities between which the relation expressed by a function exists are called **Variables.**

This term is used because such quantities may vary in value, as in the preceding examples.

Def. An **Independent Variable** is one to which we may assign values at pleasure.

The function is a dependent variable, the value of which is determined by the value assigned to the independent variable.

Def. A **Constant** is a quantity which we suppose not to vary.

REM. This division of quantities into constant and variable is merely a supposed, not a real one; we can, in an algebraic expression, suppose any quantities we please to remain constant and any we please to vary. The former are then, for the time being, constants, and the latter variables.

ILLUSTRATION. If we put

d, the distance from New York to Chicago;
s, the average speed of a train between the two cities;
t, the time required for the train to perform the journey,

then, if a manager computes the different values of the time t corresponding to all values of the speed s, he regards d as a constant, s as an independent variable, and t as a function of s.

If he computes how fast the train must run to perform the journey in different given times, he regards t as the independent variable, and s as a function of t.

When we have any equation between two variables, we may regard either of them as an independent variable and the other as a function.

EXAMPLE. From the equation

$$ax + by = c,$$

we derive

$$x = -\frac{by}{a} + \frac{c}{a},$$

$$y = -\frac{ax}{b} + \frac{c}{b},$$

in one of which x is expressed as a function of y, and in the other y as a function of x.

218. Names are given to particular classes of functions, among which the following are the most common.

1. *Def.* A **Linear Function** of several variables is one in which each term contains one of the variables, and one only, as a simple factor.

EXAMPLE. The expression

$$Ax + By + Cz$$

is a linear function of x, y, and z, when A, B, and C are quantities which do not contain these variables.

A linear function differs from a function of the first degree (§ 52) in having no term not multiplied by one of the variables. For example, the expression

$$Ax + By + C$$

is a function of x and y of the first degree, but not a linear function.

The fundamental property of a linear function is this:

If all the variables be multiplied by a common factor, the function will be multiplied by the same factor.

Proof. Let $Ax + By + Cz$ be the linear function, and r the factor. Multiplying each of the variables x, y, and z by this factor, the function will become

$$Arx + Bry + Crz,$$

which is equal to $\quad r(Ax + By + Cz).$

Moreover, *a linear function is the only one which possesses this property.*

2. *Def.* A **Homogeneous Function** of several variables is one in which each term is of the same degree in the variables. (Compare § 52.)

EXAMPLE. The expression $ax^3+bx^2y+cy^2z+dz^3$ is homogeneous and of the third degree in the variables x, y, and z.

REM. A linear function is a homogeneous function of the first degree.

FUNDAMENTAL PROPERTY OF HOMOGENEOUS FUNCTIONS. *If all the variables be multiplied by a common factor, any homogeneous function of the n^{th} degree in those variables will be multiplied by the n^{th} power of that factor.*

Proof. If we take a homogeneous function and put rx for x, ry for y, rz for z, etc., then, because each term contains x, y, or z, etc., n times in all as a factor, it will contain r n times after the substitution is made, and so will be multiplied by r^n.

3. *Def.* A **Rational Fraction** is the quotient of two entire functions of the same variable.

A rational fraction is of the form,

$$\frac{a+bx+cx^2+\text{etc.}}{m+nx+px^2+\text{etc.}}.$$

Any rational function of a variable may be expressed as a rational fraction. Compare § 180.

Equations of the First Degree between Two Variables.

219. Since we may assign to an independent variable any values we please, we may suppose it to increase or decrease by regular steps. The difference between two values is then called an increment. That is,

Def. An **Increment** is a quantity added to one value of a variable to obtain another value.

REM. If we diminish the variable, the increment is negative.

Theorem. In a function of the first degree, equal increments of the independent variable cause equal increments of the function.

EXAMPLE. Let x be an independent variable, and call u the function $\frac{3}{2}x + 11$, so that we have

$$u = \frac{3}{2}x + 11.$$

If we give x the successive values -2, -1, 0, 1, 2, etc., and find the corresponding values of the function u, they will be

Values of x,	-2,	-1,	0,	1,	2,	3,	4,	etc.
" " u,	8,	$9\frac{1}{2}$,	11,	$12\frac{1}{2}$,	14,	$15\frac{1}{2}$,	17,	etc.

We see that, the increments of x being all unity, those of y are all $1\frac{1}{2}$.

General Proof. Let $au + bx = c$ be any equation of the first degree between the variable x and the function u. Solving this equation we shall have

$$u = \frac{c - bx}{a} = \frac{c}{a} - \frac{b}{a}x.$$

Let us assign to x the successive values,

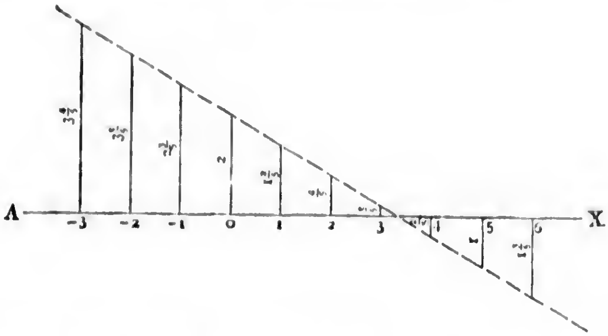
$$r, \quad r + h, \quad r + 2h, \quad \text{etc.},$$

the increment being h in each case. The corresponding values of the function u will be

$$\frac{c}{a} - \frac{b}{a}r, \quad \frac{c}{a} - \frac{b}{a}r - \frac{b}{a}h, \quad \frac{c}{a} - \frac{b}{a}r - \frac{2b}{a}h, \quad \text{etc.},$$

of which each is less than the preceding by the same amount, $\frac{b}{a}h$. Hence the increment of u is always $-\frac{b}{a}h$, which proves the theorem.

220. *Geometric Construction of a Relation of the First Degree.* The relation between a variable x and a function u of this variable may be shown to the eye in the following way:



Take a base line AX, mark a zero point upon it, and from this zero point lay off any values of x we please. Then at each point of the line corresponding to a value of x erect a vertical line equal to the corresponding value of u. If u is positive, the value is measured upward; if negative, downward. The line drawn through the ends of these values of u will show, by the distance of each of its points from the base line AX, the values of u corresponding to all values of x.

Let us take, as an example, the equation

$$5u + 3x = 10,$$

the solution of which gives $\quad u = 2 - \frac{3}{5}x.$

Computing the values of u corresponding to values of x from -3 to $+6$, we find:

$$x = -3,\quad -2,\quad -1,\quad 0,\quad +1,\quad +2,\quad +3,\quad +4,\quad +5,\quad +6.$$
$$u = +3\tfrac{4}{5},\quad +3\tfrac{1}{5},\quad 2\tfrac{3}{5},\quad 2,\quad 1\tfrac{2}{5},\quad \tfrac{4}{5},\quad \tfrac{1}{5},\quad -\tfrac{2}{5},\quad -1,\quad -1\tfrac{3}{5}.$$

Laying off these values in the way just described, we have the above figure. Wherever we choose to erect a value of u, it will end in the dotted line.

We note that by the property of functions of the first degree just proved, each value of u is less (shorter) than the preceding one by the same amount; in the present case by $\frac{3}{5}$. It is known from geometry that in this case the dotted line through the ends of u will be a straight line.

We call this line through the ends of y the **equation line.**

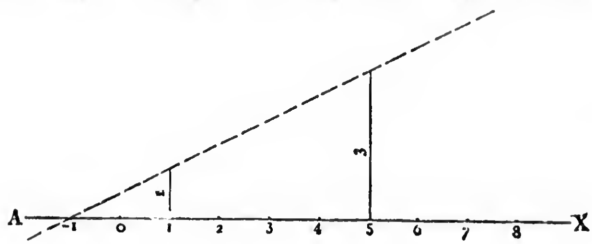
221. When we can once draw this straight line, we can find the value of y corresponding to every value of x without using the equation. We have only to take the point in the base line corresponding to any value of x, and by measuring the distance to the line, we shall have the corresponding value of u.

Now it is an axiom of geometry that one straight line, and only one, can be drawn between any two points. Therefore, to form any relation of the first degree we please between x and u, we may take any two values of x, assign to them any two values of u we please, plot these two pair of values of u in a diagram, draw the equation line through them, and then measure off, by this line, as many more values of y as we please.

Example. Let it be required that for $x = +1$ we shall have $u = +1$, and for $x = +5$, $u = +3$. What will be the values of y corresponding to $x = -3, -2, -1, 0$, etc.

Drawing the base line AX below, we lay off from 1 the vertical line $+1$ in length, and from the point 5 the vertical line $+2$. Then drawing the dotted line through the ends, we measure off different values of u, as follows:

$$x = -3,\ -2,\ -1,\quad 0,\ +1,\ +2,\quad +3,\ +4,\quad +5,\ +6,\ \text{etc.}$$
$$u = -1,\ -\tfrac{1}{2},\quad 0,\ +\tfrac{1}{2},\quad 1,\ +1\tfrac{1}{2},\ +2,\ +2\tfrac{1}{2},\ +3,\ +3\tfrac{1}{2},\ \text{etc.}$$



EXERCISES.

1. Plot the equation $2u + 3x = 6$.
2. Plot a line such that

   for $x = -6$ we shall have $u = +4$,
   for $x = +6$ " " $u = -4$,

and find the values of u for $x = 1, 2, 3, 4$, and 5.

222. The algebraic problem corresponding to the construction of § 220 is the following:

Having given two values of y corresponding to two given values of x, it is required to construct an equation of the first degree such that these two pairs of values shall satisfy it.

Example of Solution. Let the requirement be that of the equation plotted in the preceding example, namely,

for $x = 1$ we must have $u = 1$,
for $x = 5$ " " $u = 3$.

The problem then is to find such values of a, b, and c, that in the equation

$$ax + bu = c, \tag{1}$$

we shall have $u = 1$ for $x = 1$, and $u = 3$ for $x = 5$. Substituting these two pairs of values, we find that we must have

$$a \times 1 + b \times 1 = c,$$
$$a \times 5 + b \times 3 = c;$$

or

$$a + b = c,$$
$$5a + 3b = c.$$

Here a, b, and c are the unknown quantities whose values are to be found, and as we have only two equations, we cannot find them all. Let us therefore find a and b in terms of c.

Multiplying the first equation by 3, and subtracting the product from the second, we have

$$2a = -2c \quad \text{or} \quad a = -c.$$

Multiplying the first equation by 5, and subtracting the second from the product, we have

$$2b = 4c \quad \text{or} \quad b = 2c.$$

Substituting these values of a and b in (1), we find the required equation to be

$$2cu - cx = c.$$

We may divide all the terms of this equation by c (§ 120, Ax. III), giving

$$2u - x = 1,$$

thus showing that there is no need of using c. The solution of this equation gives

$$u = \frac{1 + x}{2},$$

from which, for $x = -3, -2, -1$, etc., we shall find the same values of u which we found from the diagram.

EXERCISES.

Write equations between x and y which shall be satisfied by the following pairs of values of x and y.

1. For $x = 2$, $y = 1$; and for $x = 5$, $y = -1$.
2. For $x = -2$, $y = -1$; and for $x = +2$, $y = +1$.
3. For $x = -5$, $y = +2$; and for $x = +5$, $y = -2$.
4. For $x = 0$, $y = -7$; and for $x = 15$, $y = 0$.
5. For $x = 25$, $y = 2$; and for $x = 30$, $y = 3$.

223. *Geometric Solution of Two Equations with Two Unknown Quantities.* The solution of two equations with two unknown quantities consists in finding that one pair of values which will satisfy both equations. If we lay off on the base line the required value of x, the two values of y corresponding to this value of x in the two equations must be the same; that is, *the two equation lines must cross each other at the point thus found.* Hence the following geometric solution:

I. *Plot the two equations from the same base line and zero point.*

II. *Continue the equation lines, if necessary, until they intersect.*

III. *The distance of the point of intersection from the base line is the value of y which satisfies both equations.*

IV. *The distance of the foot of the y line from the zero point is the required value of x.*

EXERCISES.

Solve the following equations by geometric construction:

1. $x - 2u = 3, \quad 2x + u = 5.$
2. $2u + 7x = 4, \quad 3u + x = 1.$

224. *Geometric Explanation of Equivalent and Inconsistent Equations.* If we have two equivalent equations (§ 200), each value of x will give the same value of the other quantity u or y. Hence the two lines representing the equation will coincide and no definite point of intersection can be fixed.

If the two equations

$$\begin{aligned} au + bx &= c, \\ a'u + b'x &= c', \end{aligned}$$

are inconsistent we shall have (§ 142),

$$\frac{b}{a} = \frac{b'}{a'}.$$

If b be any increment of x, the increments of u in the two equations (§ 219) will be $-\frac{b}{a}$ and $-\frac{b'}{a'}$. Therefore these increments will be equal, and the two equation lines will be parallel. Hence,

To inconsistent equations correspond parallel lines, which have no point of intersection.

If the two equations are equivalent (§ 141, 143), their lines will coincide.

Notation of Functions.

225. In Algebra we use symbols to express any numbers whatever. In the higher Algebra, this system is extended thus:

We may use any symbol, having a letter attached to it, to express a function of the quantity represented by that letter.

Example. If we have an algebraic expression containing a quantity x, which we consider as a function of x, but do not wish to write in full, we may call it

$$F(x), \quad \text{or} \quad \phi(x), \quad \text{or} \quad [x], \quad \text{or} \quad A_x,$$

or, in fine, any expression we please which shall contain the symbol x, and shall not be mistaken for any other expression.

In the first two of the above expressions, the letter x is enclosed in parentheses, in order that the expression may not be mistaken for x multiplied by F, or ϕ. The parentheses may be omitted when the reader knows that multiplication is not meant.

The fundamental principle of the functional notation is this:

When a symbol with a letter attached represents a function, then, if we substitute any other quantity for the letter attached, the combination will represent the function found by substituting that other quantity.

EXAMPLE. Let us consider the expression $ax^2 + b$ as a function of x, and let us call it $\phi(x)$, so that

$$\phi(x) = ax^2 + b.$$

Then, to form $\phi(y)$, we write y in place of x, obtaining

$$\phi(y) = ay^2 + b.$$

To form $\phi(x + y)$, we write $x+y$ in place of x, obtaining

$$\phi(x + y) = a(x + y)^2 + b.$$

To form $\phi(a)$, we write a instead of x, obtaining

$$\phi(a) = a^3 + b.$$

To form $\phi(ay^3)$, we put ay^3 in place of x, obtaining

$$\phi(ay^3) = a(ay^3)^2 + b = a^3y^6 + b.$$

The equation $\phi(z) = 0$ will mean

$$az^2 + b = 0.$$

EXERCISES.

Suppose $\phi(x) = ax^2 - a^2x$, and thence form the values of

1. $\phi(y)$.	2. $\phi(z)$.	3. $\phi(by)$.
4. $\phi(x + y)$.	5. $\phi(x + a)$.	6. $\phi(x - a)$.
7. $\phi(x + ay)$.	8. $\phi(x - ay)$.	9. $\phi(x^2)$.

Suppose $F(x) = xa^x$, and thence form the values of

10. $F(y)$.	11. $F(2y)$.	12. $F(3y)$.
13. $F(x + y)$.	14. $F(x - y)$.	15. $F(1)$.

Suppose $f(x) = x^2$, and thence form the values of

16. $f(1)$.	17. $f(x^2)$.	18. $f(x^3)$.
19. $f(x^4)$.	20. $f(x^5)$.	21. $f(x^n)$.

22. Prove that if we put $\phi(x) = a^x$, we shall have

$$\phi(x+y) = \phi(x) \times \phi(y), \qquad \phi(xy) = [\phi(x)]^y = [\phi(y)]^x.$$

Let us put $\phi(m) = m(m-1)(m-2)(m-3)$; thence form the values of

23. $\phi(6)$.	24. $\phi(5)$.	25. $\phi(4)$.
26. $\phi(3)$.	27. $\phi(2)$.	28. $\phi(1)$.
29. $\phi(0)$.	30. $\phi(-1)$.	31. $\phi(-2)$.

Functions of Several Variables.

226. An algebraic expression containing several quantities may be represented by any symbol having the letters which represent the quantities attached.

EXAMPLES. We may put

$$\phi(x, y) = ax - by,$$

the comma being inserted between x and y, so that their product shall not be understood. We shall then have,

$$\phi(m, n) = am - bn.$$
$$\phi(y, x) = ay - bx,$$

the letters being simply interchanged.

$$\begin{aligned}
\phi(x+y,\ x-y) &= a(x+y) - b(x-y) \\
&= (a-b)x + (a+b)y. \\
\phi(a, b) &= a^2 - b^2. \\
\phi(b, a) &= ab - ba = 0. \\
\phi(a+b,\ ab) &= a(a+b) - ab^2. \\
\phi(a, a) &= a^2 - ba. \\
\text{etc.} \quad & \quad \text{etc.}
\end{aligned}$$

If we put $\phi(a, b, c) = 2a + 3b - 5c$, we shall have

$$\begin{aligned}
\phi(x, z, y) &= 2x + 3z - 5y. \\
\phi(z, y, x) &= 2z + 3y - 5x. \\
\phi(m, m, m) &= 2m + 3m + 5m = 10m. \\
\phi(3, 8, 6) &= 2 \cdot 3 + 3 \cdot 8 - 5 \cdot 6 = 0.
\end{aligned}$$

EXERCISES.

Let us put

$$\begin{aligned}
\phi(x, y) &= 3x - 4y, \\
f(x, y) &= ax + by, \\
f(x, y, z) &= ax + by - abz.
\end{aligned}$$

Thence form the expressions:

1. $\phi(y, x)$.	2. $\phi(a, b)$.	3. $\phi(3, 4)$.
4. $\phi(4, 3)$.	5. $\phi(10, 1)$.	6. $f(a, b)$.
7. $f(b, a)$.	8. $f(y, x)$.	9. $f(7, -3)$.
10. $f(q, -p)$.	11. $f(z, x, y)$.	12. $f(b, a, 2)$.
13. $f(a, b, c)$.	14. $f(a^2, b^2, c^2)$.	
15. $f(-a, -b, -ab)$.		

Let us put $\quad (m, n) = \dfrac{m(m-1)(m-2)}{n(n-1)(n-2)}$.

Find the values of

16. $(3, 3)$.	17. $(4, 3)$.	18. $(5, 3)$.
19. $(6, 3)$.	20. $(7, 3)$.	21. $(8, 3)$.
22. $(2, -1)$.	23. $(3, -2)$.	24. $(4, -2)$.

Use of Indices.

226a. Any number of different quantities may be represented by a common symbol, the distinction being made by attaching numbers or accents to the symbol.

EXAMPLES.

1. Any n different quantities may be represented by the symbols, p_1, p_2, p_3, ... p_n.

2. A producer desires to have an algebraic symbol for the amount of money which he earns on each day of the year. If he calls q what he earns in a day he may put:

q_1	for the	amount	earned on	January	1,
q_2	"	"	"	"	2,
etc.	"	"	"	"	etc.,
q_{31}	"	"	"	"	31,
q_{32}	"	"	"	February	1;

and so on to the end of the year, when

q_{365} will be the amount for December 31.

Def. The distinguishing numbers 1, 2, 3, etc., are here called **Indices**.

A symbol with an index attached may represent a function of the index, as in the functional notation.

EXERCISES.

Let us put $a_t = t(t+1)$. Then find the value of

1. $a_0 + a_1 + a_2 + \ldots . + a_{10}$.

2. Prove the following equations by computing both members:

$$a_1 + a_2 = \frac{4}{3} a_2.$$

$$a_1 + a_2 + a_3 = \frac{5}{3} a_3.$$

$$a_1 + a_2 + a_3 + a_4 = \frac{6}{3} a_4.$$

If we put $S_i = 1 + 2 + 3 \ldots . + i$, we shall have

$$S_1 = 1.$$
$$S_2 = 1 + 2 = 3.$$
$$S_3 = 1 + 2 + 3 = 6, \text{ etc., etc.}$$

Using the preceding notation, find the values of the expressions:

3. $S_4 + S_5 + S_6 + S_7$.

4. $a_4 + a_5 + a_6 + a_7$.

5. $2S_5 - a_5$.

6. $2S_6 - a_6$.

227. Sometimes the relations between quantities distinguished by indices are represented by equations of the first degree. The following are examples:

Let us have a series of quantities,

$$A_0, \quad A_1, \quad A_2, \quad A_3, \quad A_4, \quad \text{etc.},$$

connected by the general relation,

$$A_{i+1} = A_i + A_{i-1}. \qquad (a)$$

It is required to express them in terms of A_0 and A_1.

We put, in succession, $i = 1$, $i = 2$, $i = 3$, etc. Then, when $i = 1$, we have from (a),

$$A_2 = A_1 + A_0.$$

When $i = 2$, $\quad A_3 = A_2 + A_1 = 2A_1 + A_0$.

$i = 3$, $\quad A_4 = A_3 + A_2 = 3A_1 + 2A_0$.

$i = 4$, $\quad A_5 = A_4 + A_3 = 5A_1 + 3A_0$.

$i = 5$, $\quad A_6 = A_5 + A_4 = 8A_1 + 5A_0$,

and so on indefinitely.

EXERCISES.

1. If $$A_{i+1} = A_i - A_{i-1},$$
what will be the values of $A_2 \ldots . A_{10}$, and in what way may all subsequent values be determined?

2. If $$A_{i+1} = 2A_i - A_0,$$
find A_2 to A_5 in terms of A_0 and A_1.

3. If $$A_{i+1} = iA_i + A_{i-1},\quad \text{find } A_2 \text{ to } A_5.$$

4. If $$A_i = A_{i-1} + h,$$
find the sum $A_0 + A_1 + A_2 + \ldots . + A_n$, in terms of A_0, h and n. (Comp. § 209, Prob. V.)

5. If $$A_{i+1} = rA_i,$$
find $A_1 + A_2 + A_3 + \ldots . + A_n$, in terms of A_0 and r.

6. If $$A_{i+1} = ikA_i + A_{i-1},$$
find $A_2, A_3, \ldots . A_6$, in terms of A_0 and A_1.

Miscellaneous Functions of Numbers.

228. We present, as interesting exercises, certain elementary forms of algebraic notation much used in Mathematics, and which will be employed in the present work.

1. When we have a series of symbols the number of which is either indeterminate or too great to be all written out, we may write only the first two or three and the last, the omitted ones being represented by a row of dots.

EXAMPLES.
$$a,\ b,\ c,\ \ldots .\ t,$$
$$1,\ 2,\ 3,\ \ldots .\ 25,$$
$$1,\ 2,\ \ldots .\ n,$$
n being in the last case any number greater than 2.

The number of omitted symbols is entirely arbitrary.

EXERCISES.

How many omitted expressions are represented by the dots in the following series:

1. $1, 2, 3, \ldots . n.$ 2. $1, 2, 3, \ldots . n - 2.$

3. $1, 2, 3, \ldots . n+2.$
4. $n, n-1, n-2, \ldots . n-s.$
5. $n, n-1, n-2, \ldots . n-s-1.$
6. $n, n-1, n-2, \ldots . n-s+1.$

What will be the last term in the series:

7. 2, 3, 4, etc., to n terms.
8. n, $n-1$, $n-2$, etc., to s terms.
9. 2, 4, 6, etc., to k terms.

2. *Product of the First n Numbers.* The symbol

$$n\,!$$

is used to express the product of the first n numbers,

$$1 \cdot 2 \cdot 3 \ldots . n.$$

Thus,

$$1! = 1.$$
$$2! = 1 \cdot 2 = 2.$$
$$3! = 1 \cdot 2 \cdot 3 = 6.$$
$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24.$$
$$\text{etc.} \qquad \text{etc.}$$

It will be seen that $2! = 2 \cdot 1!$

$$3! = 3 \cdot 2!$$

And, in general, $n! = n(n-1)!$

whatever number n may represent.

EXERCISES.

Compute the values of

1. $5!$ 2. $6!$ 3. $8!$

4. $\dfrac{7!}{3!\,4!}$ 5. $\dfrac{8!}{3!\,5!}$

6. Prove the equation $2 \cdot 4 \cdot 6 \cdot 8 \ldots . 2n = 2^n n!$
7. Prove that, when n is even,

$$\frac{n}{2}! = \frac{n(n-2)(n-4) \ldots . 4 \cdot 2}{2^{\frac{n}{2}}}.$$

3. *Binomial Coefficients.* The binomial coefficient

$$\frac{n(n-1)(n-2) \ldots . \text{ to } s \text{ terms}}{1 \cdot 2 \cdot 3 \ldots . s}$$

is expressed in the abbreviated form,

$$\binom{n}{s},$$

the parentheses being used to show that what is meant is not the fraction $\frac{n}{s}$.

EXAMPLES.

$$\binom{3}{1} = \frac{3}{1} = 3.$$

$$\binom{7}{5} = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4\cdot 5} = 21.$$

$$\binom{n}{1} = \frac{n}{1} = n.$$

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}.$$

$$\binom{n}{n} = \frac{n(n-1)\ldots\ldots 2\cdot 1}{1\cdot 2\cdot 3\ldots\ldots n} = 1.$$

$$\binom{n+4}{3} = \frac{(n+4)(n+3)(n+2)}{1\cdot 2\cdot 3}.$$

EXERCISES.

Compute the values of the expressions:

1. $\binom{8}{1} + \binom{8}{2} + \binom{8}{3} + \binom{8}{4} + \binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8}.$
2. $\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3} + \binom{7}{3}.$

Prove the formulæ:

3. $\binom{5}{2} = \frac{5!}{2!\,3!}$
4. $\binom{n}{s} = \frac{n!}{s!\,(n-s)!}$
5. $\binom{n+1}{s+1} = \frac{n+1}{s+1}\binom{n}{s}.$
6. $\binom{n}{1} + \binom{n}{2} = \binom{n+1}{2}.$
7. $\binom{n}{2} + \binom{n}{3} = \binom{n+1}{3}.$
8. $\binom{n}{3} + \binom{n}{4} = \binom{n+1}{4}.$

BOOK IX.

THE THEORY OF NUMBERS.

CHAPTER I.

THE DIVISIBILITY OF NUMBERS.

229. *Def.* The **Theory of Numbers** is a branch of mathematics which treats of the properties of integers.

Def. An **Integer** is any whole number, positive or negative.

In the theory of numbers the word *number* is used to express an integer.

Def. A **Prime Number** is one which has no divisor except itself and unity.

The series of prime numbers are

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, etc.

Def. A **Composite Number** is one which may be expressed as a product of two or more factors, all greater than unity.

Rem. Every number greater than 1 must be either prime or composite.

Def. Two numbers are **prime to each other** when they have no common divisor greater than unity.

Example. The numbers 24 and 35 are prime to each other, though neither of them is a prime number.

Rem. A vulgar fraction is reduced to its lowest terms when numerator and denominator are prime to each other.

Division into Prime Factors.

230. Every composite number may by definition be divided into two or more factors. If any of these factors are composite, they may be again divided into other factors. When none of the factors can be further divided, they will all be prime. Hence,

THEOREM. *Every composite number may be divided into prime factors.*

EXAMPLE.

$$180 = 9 \cdot 20,$$
$$9 = 3 \cdot 3,$$
$$20 = 4 \cdot 5 = 2 \cdot 2 \cdot 5.$$

Whence,

$$180 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 2^2 \cdot 3^2 \cdot 5.$$

Cor. 1. Because every number not prime is composite, and because every composite number may be divided into prime factors, we conclude: *Every number is either prime or divisible by a prime.*

Cor. 2. Every number, prime or composite, may be expressed in the form

$$p^\alpha q^\beta r^\gamma \text{ etc.,} \qquad (a)$$

where p, q, r, etc., are different prime numbers;

α, β, γ, etc., the exponents, are positive integers.

REM. If the number is prime there will be but one factor, namely, the number itself, and the exponent will be unity.

EXERCISES.

Divide the following numbers or products into their prime factors, if any, and thus express the numbers in the form (a):

1. 24. 2. 72. 3. 260. 4. 169. 5. 225.
6. 256. 7. 91. 8. 143. 9. 360. 10. 217.
11. 3072. 12. $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9$.

REM. In seeking for the prime factors of a number, it is never necessary to try divisors greater than its square root, for if a number is divisible into two factors, one of these factors will necessarily not exceed such root.

Common Divisors of Two Numbers.

231. THEOREM I. *If two numbers have a common factor, their sum will have that same factor.*

Proof. Let a be the common factor;
m, the product of all the other factors in the one number;
n, the corresponding product in the other number.

Then the two numbers will be

$$am \quad \text{and} \quad an.$$

Their sum will be $a(m + n)$.

Because m and n are whole numbers, $m+n$ will also be a whole number. Therefore a will be a factor of $am+an$.

THEOREM II. *If two numbers have a common factor, their difference will have the same factor.*

Proof. Almost the same as in the last theorem.

Cor. If a number is divisible by a factor, all multiples will be divisible by that factor.

REM. The preceding theorems may be expressed as follows:

If two numbers are divisible by the same divisor, their sum, difference, and multiples are all divisible by that divisor.

REM. If one number is not exactly divisible by another, a remainder less than the divisor will be left over. If we put

D, the dividend;
d, the divisor;
q, the quotient;
r, the remainder;

we shall have, $D = dq + r$,

or $D - dq = r$.

EXAMPLE. 7 goes into 66 9 times and 3 over. Hence this means

$$66 = 7 \cdot 9 + 3, \quad \text{or} \quad 66 - 7 \cdot 9 = 3.$$

232. Problem. *To find the greatest common divisor of two numbers.*

Let m and n be the numbers, and let m be the greater.

1. Divide m by n. If the remainder is zero, n will be the divisor required, because every number divides itself. If there is a remainder, let q be the quotient and r the remainder.

Then $$m - nq = r.$$

Let d be the common divisor required.

Because m and n are both divisible by d, $m - nq$ must also be divisible by d (Theorem II). Therefore,

$$r \text{ is divisible by } d.$$

Hence every common divisor of m and n is also a common divisor of n and r. Conversely, because

$$m = nq + r,$$

every common divisor of n and r is also a divisor of m. Therefore, the greatest common divisor of m and n is the same as the greatest common divisor of n and r, and we proceed with these last two numbers as we did with m and n.

2. Let r go into n q' times with the remainder r'.

Then $$n = rq' + r',$$

or $$n - rq' = r'.$$

Then it can be shown as before that d is a divisor of r', and therefore the greatest common divisor of r and r'.

3. Dividing r by r', and continuing the process, one of two results must follow. Either,

α We at length reach a remainder 1, in which case the two numbers are prime; or,

β. We have a remainder which exactly divides the preceding divisor, in which case this remainder is the divisor required.

To clearly exhibit the process, we express the numbers m, n, and the successive remainders in the following form:

$$
\begin{array}{lll}
m = n \cdot q + r, & & (r < n); \\
n = r \cdot q' + r', & & (r' < r); \\
r = r' \cdot q'' + r'', & & (r'' < r'); \\
r' = r'' \cdot q''' + r''', & & (r''' < r''); \\
\text{etc.} & \text{etc.} & \text{etc.,}
\end{array}
$$

until we reach a remainder equal to 1 or 0, when the series terminates.

EXERCISES.

1. Find the G. C. D.* of 240 and 155.

Dividend.		Div. Quo.		Rem.
240	=	155·1	+	85.
155	=	85·1	+	70.
85	=	70·1	+	15.
70	=	15·4	+	10.
15	=	10·1	+	5.
10	=	5·2.		

Therefore 5 is the greatest common divisor.

NOTE. Let the student arrange all the following exercises in the above form, first dividing in the usual way, if he finds it necessary.

Find the greatest common divisor of

2. 399 and 427.
3. 91 and 131.
4. 8 and 13.
5. 1000 and 212.
6. 799 and 1232.
7. 800 and 1729.
8. 250 and 625.
9. 1000 and 370.

10. If p be a number less than n and prime to n, show that $n - p$ is also prime to n.

11. If p be any number less than n, the greatest common divisor between n and p is the same as that between n and $n - p$.

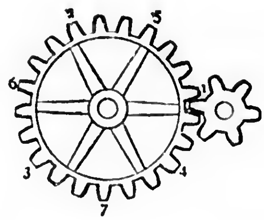
12. If n is any odd number, $\frac{n+1}{2}$ and $\frac{n-1}{2}$ are both prime to it.

Corollaries. 1. When two numbers are divided by their greatest common divisor, their quotients will be prime to each other.

* The letters G. C. D. are an abbreviation for Greatest Common Divisor.

2. Conversely, if two numbers, n and n', prime to each other, are each multiplied by any number d, then d will be the G. C. D. of dn and dn'.

233. *Gearing of Wheels.* An interesting problem connected with the greatest common divisor is afforded by a common pair of gear wheels. Let there be two wheels, the one having m teeth and the other n teeth, gearing into each other. If we start the wheels with a certain tooth of the one against a certain tooth of the other, then we have the questions:



(1.) How many revolutions must each wheel make before the same teeth will again come together?

(2.) With how many teeth of the one will each tooth of the other have geared?

Let q be the required number of turns of the first wheel, having m teeth.

Let p be the required number of turns of the second, having n teeth.

Then, because the first wheel has m teeth, qm teeth will have geared into the other wheel during the q turns. In the same way, pn teeth of the second wheel will have geared into the first. But these numbers must be equal. Therefore, when the two teeth again meet,

$$pn = qm.$$

Conversely, for every pair of numbers of revolutions p and q, which fulfil the conditions,

$$pn = qm,$$

the same teeth will come together, because each wheel will have made an entire number of revolutions. This equation gives

$$\frac{p}{q} = \frac{m}{n}.$$

Hence, if we reduce the fraction $\frac{m}{n}$ to its lowest terms, we shall have the smallest number of revolutions of the respective wheels which will bring the teeth together again.

To answer the second question:

After the first wheel has made q revolutions, qm of its teeth have passed a fixed point. Any one tooth of the other wheel gears into every n^{th} passing tooth of the first wheel. Therefore any such tooth has geared into $\frac{qm}{n}$ teeth of the first wheel, that is, into p teeth, because, from the last equation,

$$\frac{qm}{n} = p.$$

If d be the G. C. D. of m and n, then

$$m = dp,$$
$$n = dq;$$

or

$$p = \frac{m}{d},$$
$$q = \frac{n}{d}.$$

Therefore each tooth of the one wheel has geared into only every d^{th} tooth of the other.

In the figure on the preceding page, $n = 21$ and $n = 6$. Hence, $d = 3$, and each tooth of the one will gear into every third tooth of the other. The numbers on the large wheel show the order in which the gearing occurs.

How long soever the wheels run, the same contacts will be repeated in regular order. Hence, *if each tooth of the one wheel must gear with every tooth of the other, the numbers m and n must be prime to each other.*

EXERCISES.

1. If one wheel has 40 teeth and the other 10, show how they will run together.

Show the same thing for the following cases:

2. $m = 72,\ n = 15.$ 3. $m = 24,\ n = 18.$

4. $m = 36,\ n = 25.$ 5. $m = 24,\ n = 7.$

Relations of Numbers to their Digits.

234. In our ordinary method of expressing numbers, the second digit toward the right expresses 10's, the third 100's, etc. That is, each digit expresses a power of 10 corresponding to its position.

Def. The number 10 is the **Base** of our scale of numeration.

NOTE. The base 10 is entirely arbitrary, and is supposed to have originated from the number of the thumbs and fingers, these being used by primitive people in counting.

Any other number might equally well have been chosen as a base, but in any case we should need a number of separate characters (digits) equal to the base, and no more.

Had 8 been the base, we should have needed only the digits 0, 1, 2, etc., to 7, and different combinations of the digits would have represented numbers as follows:

$$\begin{aligned} 1 &= 1, \\ 7 &= 7, \\ 10 &= 1 \cdot 8 + 0 = \text{eight}. \\ 17 &= 1 \cdot 8 + 7 = \text{fifteen}. \\ 20 &= 2 \cdot 8 + 0 = \text{sixteen}. \\ 56 &= 5 \cdot 8 + 6 = \text{forty-six}. \\ 234 &= 2 \cdot 8^2 + 3 \cdot 8 + 4 = \text{one hundred fifty-six}, \end{aligned}$$

etc.

Let us take the arbitrary number z as the base of the scale. As in our scale of 10's we have

$$234 = 2 \cdot 10^2 + 3 \cdot 10 + 4,$$

so in the scale of z's the digits 234 would mean

$$2z^2 + 3z + 4.$$

In general, the combination of digits $abcd$ would mean

$$az^3 + bz^2 + cz + d.$$

Divisibility of Numbers and their Digits.

235. THEOREM. *If the sum of the digits of any number be subtracted from the number itself, the remainder will be divisible by* $z - 1$.

Proof. Let the digits be a, b, c, d. The number expressed will be

$$
\begin{array}{ll}
 & az^3 + bz^2 + cz + d \\
\text{Sum of digits} = & a \quad + b \quad + c \ + d \\
\hline
\text{Subtracting, rem.} = & a(z^3-1) + b(z^2-1) + c(z-1).
\end{array}
$$

The factors $z^3 - 1$, $z^2 - 1$, and $z - 1$ are all divisible by $z - 1$ (§ 93). Hence the theorem is proved. (§ 231.)

THEOREM. *In any scale having z as its base, the sum of the digits of any number, when divided by $z - 1$, will leave the same remainder as will the number itself when so divided.*

If we put: n, the number; s, the sum of the digits;
r, r', the remainders from dividing by $z - 1$;
q, q', the quotients; we shall have,

$$
\begin{array}{lrl}
\text{Number,} & n = & q(z-1) + r \\
\text{Sum of digits,} & s = & q'(z-1) + r' \\
\hline
\text{Remainder,} & & (q-q')(z-1) + r - r'.
\end{array}
$$

Because $n - s$ and $(q - q')(z - 1)$ are both divisible by $z - 1$, their difference $r - r'$ must be so divisible. Since r and r' are both less than $z - 1$, this remainder can be divided by $z - 1$ only when $r = r'$, which proves the theorem.

Zero is considered divisible by all numbers, because a remainder 0 is always left.

If a be any factor of $z - 1$, the same reasoning will apply to it, and therefore the theorem will be true of it.

In our system of notation, where $z = 10$, the above theorems may be put in the following well-known form:

If the sum of the digits of any number be divisible by 3 or 9, the number itself will be so divisible.

These are the only numbers of which the theorem is true, because 3 is the only divisor of 9.

THEOREM. *If from any number we subtract the digits of the even powers of z, and add those of the alternate powers, the result will be divisible by $z + 1$.*

$$
\begin{array}{lll}
\textit{Proof.} & \text{To} & az^3 + bz^2 + cz + d \\
 & \text{Add} & a \quad - b \quad + c \ - d \\
\hline
 & \text{Result,} & a(z^3+1) + b(z^2-1) + c(z+1).
\end{array}
$$

The factors of a, b, and c are all divisible by $z+1$ (§§ 93, 94), whence the result itself is so divisible.

Applying this result to the case of $z = 10$, we conclude:

If on subtracting the sum of the digits in the place of units, hundreds, tens of thousands, etc., from the sum of the alternate ones, the remainder is divisible by 11, the number itself is divisible by 11.

If m be any factor of z, it will divide all the terms of the number

$$az^3 + bz^2 + cz + d,$$

except the last. Hence, if it divide this last also, it will divide the number itself. Applying this result to the case of $z = 10$, we conclude:

If the last digit of any number is divisible by a factor of 10, the number itself is divisible by that factor.

The factors of 10 being 2 and 5, this rule is true of these numbers only.

It will be remarked that if the base of the system had been an odd number, we could not have distinguished even and odd numbers by their last figure, as we habitually do.

For example, if the base had been 9, the figures 72 would have represented what we call sixty-five, which is odd, and 73 would have represented what we call sixty-six, which is even.

The use of the base 10 makes it easy to detect when a number is divisible by either of the first three prime numbers, 2, 3, and 5. If the last figure is divisible by 2 or 5, the whole number is so divisible. To ascertain whether 3 is a factor, we find whether the sum of the digits is divisible by 3.

In taking the sum, it is not necessary to include all the digits, but in adding we may omit all 3's and 9's, and drop 3, 6, or 9 from the sum as often as convenient. Thus, if the number were

921642712,

we should perform the operation mentally, thus:

Drop 9; $2 + 1 = 3$, which drop; 6, drop; $4 + 2 = 6$, which drop; $7+1 = 8+2 = 10$, which leaves a remainder 1.

EXERCISES.

1. Prove that if an even number leaves a remainder 1 when divided by 3, its half will leave a remainder 2 when so divided.

2. If from any number we subtract the sum of units' digit plus the product of the tens' digit by i, plus the product of the hundreds' digit by i^2, etc., the remainder will be divisible by $10 - i$. (i may be any integer, positive or negative.)

NOTE. When $i = 1$, this gives the rule of 9's and when $i = -1$, the rule of 11's.

Prime Factors of Numbers.

236. FIRST FUNDAMENTAL THEOREM. *A product cannot be divided by a prime number unless one of the factors is divisible by that prime number.*

NOTE. This theorem is not true of composite divisors. For example, neither 8 nor 9 is divisible by 6, but the product $8 \cdot 9 = 72$ is so divisible. But if we take as many numbers as we please not divisible by 7, we shall always find their product to leave a remainder when we try to divide it by 7.

To make the demonstration better understood, we shall first take a special case:

The product $66a$ is not divisible by 7, unless a is divisible by 7.

Proof. Suppose	$66a$ div. by 7
7 goes into 66 9 times and 3 over, because $7 \cdot 9 = 63$,	$63a$ div. by 7
Therefore, by Theorem II, § 231, ...	$3a$ div. by 7
	2
3 goes into 7 2 times and 1 over. Multiply by 2,	$6a$ div. by 7
Subtracting, ...	$7a$ div. by 7
We have left, ...	a div. by 7

Hence, *if $66a$ is divisible by 7, then a is divisible by 7.*

Gauss's Demonstration. If it be possible, let am be the smallest multiple of m which is divisible by p, when neither a nor m is so divisible. If a is greater than p, then let p go into a b times and r over, so that

$$a = bp + r,$$

or

$$a - bp = r.$$

Then,	am	div. by p.
Subtract	bpm	" "
Remainder,	$(a - bp)m$	" "
Or	rm	" "

That is, if am is divisible by p, so is rm, where r is less than p.

Therefore the smallest multiple of m which fulfils the conditions must be less than pm.

Therefore, let $a < p$. Let a go into p c times and s over, so that

$$p = ca + s,$$

or

$$p - ca = s.$$

Then	pm	div.	by p.	
	cam	"	"	(by hypothesis).
Subtracting,	$(p - ca)\,m$	"	"	
Or,	sm	"	"	

Therefore, s being less than a, a is not the smallest multiple; whence the hypothesis that a is the smallest is impossible.

General Demonstration. Suppose

p, a prime number;
a, number not divisible by p;
am, a product divisible by p.

We have to prove that m must be divisible by p.

Let p go into a q times. Because a is not divisible by p, a remainder r will be left. That is,

$$a = pq + r, \quad \text{or} \quad a - pq = r.$$

Let r go into p q' times and leave a remainder r'. Then,

$$p = q'd + r',$$

and because pm and $q'rm$ are both divisible by p, rm is so divisible.

In the same way, if r' goes into p q'' times, and leave the remainder r'', $r''m$ will be divisible by p. Since each of the remainders r, r', r'', etc., must be less than the preceding, we shall at length reach a remainder 1, which will give

am	div.	by p.
pqm	"	"
rm	"	"
$q'rm$	"	"
pm	"	"
$r'm$	"	"
$q''r'm$	"	"
pm	"	"
$r''m$	"	"

m divisible by p. Q. E. D.

Extension to Several Factors. If m is a product $b \times n$, and b is not divisible by p, then we may show in the same way that n must be so divisible. If $n = cs$, and c is not divisible, then s must be divisible, and so on to any number of factors.

Hence,

THEOREM. *If a product of any number of factors is divisible by a prime number, then one of the factors must be divisible by the same prime.*

This theorem is the logical equivalent of the one just enunciated as the first fundamental theorem.

NOTE. The student will remark why the preceding demonstration applies only when the divisor p is a prime number. If it were composite, we might reach a remainder which would exactly divide it, and then the conclusion would not follow.

237. SECOND FUNDAMENTAL THEOREM. *A number can be divided into prime factors in only one way.*

For, suppose we could express the number N in the two ways (§ 204, Cor. 2),

$$N = p^{\alpha} q^{\beta} r^{\gamma},$$
$$N = a^{\mu} b^{\nu} c^{\pi},$$

where p, q, r, etc., a, b, c, etc., are all prime numbers. Then

$$p^{\alpha} q^{\beta} r^{\gamma} = a^{\mu} b^{\nu} c^{\pi}.$$

If common prime factors appeared on both sides of this equation, we could divide them out, leaving an equation in which the prime factors p, q, r, etc., are all different from a, b, c, etc.

Then, because a, b, c, etc., are all prime, none of them are divisible by p. Therefore, by the first fundamental theorem, their products cannot be so divisible. But the left-hand member of the equation is divisible by p, because p is one of its factors. Therefore the equation is impossible.

REM. This theorem forms the basis of the theory of the divisibility of numbers.

The preceding theorems enable us to place the definition of numbers prime to each other in a new shape.

Two numbers are said to be **prime to each other** when they have no common prime factors.

EXAMPLE. If one number is $p^\alpha q^\beta r^\gamma$, and the other is $a^\mu b^\nu c^\pi$ (p, q, r, etc., and a, b, c, etc., being prime numbers), then, if p, q, r, etc., are all different from a, b, c, etc., the two numbers will be prime to each other.

Elementary Theorems.

238. The following general theorems follow from the two preceding fundamental theorems, and their demonstration is in part left as an exercise for the student.

I. *No power of an irreducible vulgar fraction can be a whole number.*

NOTE. An irreducible vulgar fraction is one which is reduced to its lowest terms.

II. COROLLARY. *No root of a whole number can be a vulgar fraction.*

III. *If a number is divisible by several divisors, all prime to each other, it is also divisible by their product.*

Cor. To prove that a number N is divisible by a number $B = p^\alpha q^\beta r^\gamma$, it is sufficient to prove that it is divisible separately by p^α, by q^β, by r^γ, etc.

EXAMPLE. If a number is divisible separately by 5, 8, and 9, it is divisible by $5 \cdot 8 \cdot 9 = 360$. Hence, to prove that a number is divisible by 360, it is sufficient to show that 5, 8, and 9 are all factors of it.

IV. *If the numerator and denominator of a vulgar fraction have no common prime factors, it is reduced to its lowest terms.*

Binomial Coefficients.

239. *Theorem.* The product of any n consecutive numbers is divisible by the product of the numbers $1 \cdot 2 \cdot 3 \dots n$, or $n\,!$

REM. The theorem implies that all binomial coefficients are whole numbers, because they are quotients formed by dividing the product of n consecutive numbers by $n!$

Proof. 1. We have first to find the prime factors of the product

$$1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 \ldots . n = n!$$

beginning with the factor 2.

I. The numbers divisible by 2 are the even numbers 2, 4, 6, etc., to n or $n-1$, the number of which is $\left[\frac{n}{2}\right]$.

NOTE. The expression $\left[\frac{n}{2}\right]$ here means the *greatest whole number* in $\frac{n}{2}$, which is $\frac{n}{2}$ itself when n is even, and $\frac{n-1}{2}$ when n is odd.

The quotients of the division are

$$1,\ 2,\ 3,\ 4,\ \ldots . \left[\frac{n}{2}\right].$$

Of these quotients, $\left[\frac{n}{4}\right]$ are divisible by 2, leaving the second set of quotients,

$$1,\ 2,\ 3,\ \ldots . \left[\frac{n}{4}\right].$$

The next set of quotients will be

$$1,\ 2,\ \ldots . \left[\frac{n}{8}\right].$$

The process is to be continued until we have no even numbers left.

Therefore, if we put α for the number of times that the factor 2 enters into $n!$ we have,

$$\alpha = \left[\frac{n}{2}\right] + \left[\frac{n}{4}\right] + \left[\frac{n}{8}\right] + \text{etc.}$$

II. The numbers in the series $n!$ containing 3 as a factor are

$$3,\ 6,\ 9,\ 12,\ \text{etc.},$$

of which the number is $\left[\frac{n}{3}\right]$. The quotients obtained by dividing them by 3 are

$$1,\ 2,\ 3,\ \ldots\ldots \left[\frac{n}{3}\right].$$

Of these quotients, $\left[\frac{n}{9}\right]$ are again divisible by 3, and so on as before. Hence, if we put β for the number of times $n!$ contains 3 as a factor, we have

$$\beta = \left[\frac{n}{3}\right] + \left[\frac{n}{9}\right] + \left[\frac{n}{27}\right] + \text{etc.}$$

In the same way, if k be any prime number, $n!$ will contain k as a factor

$$\left[\frac{n}{k}\right] + \left[\frac{n}{k^2}\right] + \left[\frac{n}{k^3}\right] + \text{etc. times.}$$

NOTE. This elegant process enables us to find all the prime factors of $n!$ without actually computing it, and thus to exhibit $n!$ as a product of prime factors. If we suppose $n = 12$, we shall find,

$$12! = 1 \cdot 2 \cdot 3 \ldots\ldots 12 = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11.$$

2. Next let us find the prime factors of the product

$$(a+1)(a+2)\ldots\ldots(a+n),$$

which contains n factors. Dividing successively by 2, 3, 5, 7, etc., it is shown in the same way as before that the prime factor p is contained in the product *at least*

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \text{etc. times,}$$

whatever prime factor p may be. Therefore the numerator $(a+1)(a+2)\ldots\ldots(a+n)$ contains all the prime factors found in $n!$ to at least the same power with which they enter $n!$ Hence (§ 238, III), the numerator is divisible by $n!$

Cor. If the factor $a+n$ in the numerator is a prime number, that prime cannot be contained in $n!$ because it is greater than n. Hence the binomial factor will be divisible by it.

EXAMPLE. $\frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3}$ is divisible by 7.

We may show in the same way that the binomial coefficient is divisible by all the prime numbers in its numerator which exceed n.

Divisors of a Number.

240. *Def.* The expression

$$\phi\,(m)$$

is used to express how many numbers not greater than m are prime to m.

EXAMPLE. Let us find the value of $\phi\,(9)$.

1 is prime to 9, because their G. C. D. is 1.
2 " " " " " "
3 is not prime to 9, because their G. C. D. is 3.
4 is prime to 9.
5 " "
6 is not, because 6 and 9 have the G. C. D. 3.
7 is.
8 is.
9 is not.

Therefore, the numbers less than 9 and prime to it are

1, 2, 4, 5, 7, 8,

which are six in number. Hence,

$$\phi\,(9) = 6.$$

The numbers less than 12 and prime to 12 are 1, 5, 7, 11. Hence,

$$\phi\,(12) = 4.$$

We find in this way,

$$\phi\,(1) = 1, \qquad \phi\,(2) = 1, \qquad \phi\,(3) = 2,$$
$$\phi\,(4) = 2, \qquad \phi\,(5) = 4, \qquad \phi\,(6) = 2,$$
$$\phi\,(7) = 6, \qquad \text{etc.}, \qquad \text{etc.}$$

Cor. 1. The number 1 is prime to itself, but no other number is prime to itself.

Cor. 2. If m be a prime number, then

$$\phi(m) = m - 1,$$

because the numbers $1, 2, 3, \ldots . m - 1$ are then all prime to m.

The following remarkable theorem is associated with the functions $\phi(m)$.

241. *Theorem.* If N be any number, and d_1, d_2, d_3, etc., all its divisors, unity and n included, then

$$\phi(d_1) + \phi(d_2) + \phi(d_3) + \text{etc.} = N.$$

Example. Let the number be 18.

The divisors are 1, 2, 3, 6, 9, 18. We find, by counting,

$$\begin{aligned} \phi(1) &= 1 \\ \phi(2) &= 1 \\ \phi(3) &= 2 \\ \phi(6) &= 2 \\ \phi(9) &= 6 \\ \phi(18) &= \underline{6} \\ \text{Sum,}\ & \ 18. \end{aligned}$$

To show how this comes about, write down the numbers 1 to 18, and under each write the greatest common divisor of that number and 18. Thus,

Num.,	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18.
G.C.D.,	1	2	3	2	1	6	1	2	9	2	1	6	1	2	3	2	1	18.

Necessarily the numbers in the second line are all divisors of 18 as well as of the numbers over them.

The divisor 1 is under all the numbers prime to 18, so that there are

$$\phi(18) = \text{divisors } 1.$$

If n be any number over the divisor 2, then $\frac{n}{2}$ and $\frac{18}{2}$, or 9, must be prime to each other. (§ 232, Cor. 1.) That is, the numbers r are all those which, when divided by 2, are prime to 9. So there are

$$\phi(9) \text{ divisors } 2.$$

The divisor 3 marks all numbers which, when divided by 3, are prime to $\frac{18}{3} = 6$. Hence, there are

$$\phi(6) \text{ divisors } 3.$$

In the same way there are $\phi(3)$ divisors 6, $\phi(2)$ divisors 9, and $\phi(1)$ divisor 18.

The total number of these divisors is both 18 and $\phi(18) + \phi(9) + \text{etc.}$ Hence,

$$\phi(18) + \phi(9) + \phi(6) + \phi(3) + \phi(2) + \phi(1) = 18.$$

General Proof. Let m be the given number;

$$d_1,\ d_2,\ d_3,\ \text{etc., its divisors;}$$

$$q_1,\ q_2,\ q_3, \text{ the quotients } \frac{m}{d_1},\ \frac{m}{d_2}, \text{ etc.}$$

The quotients q_1, q_2, etc., will be the same numbers as d_1, d_2, etc., only in reverse order. The smallest of each row will be 1 and the greatest m. We shall then have

$$m = d_1 q_1 = d_2 q_2 = d_3 q_3, \quad \text{etc.}$$

From the list of numbers 1, 2, 3, ... m, select all those which have d_1 (unity) as the greatest common divisor with m, then those which have d_2 as such common divisor, then those which have d_3, etc., till we reach the last divisor, which will be m itself, and which will correspond to m.

The numbers having unity as G. C. D. will be those prime to m, by definition. Their number is $\phi(m)$.

Those having d_2 as G. C. D. with m will, when divided by d_2, give quotients prime to $\frac{m}{d_2}$ or to q_2. Moreover, such quotients will include all the numbers not greater than q_2 and prime to it, because each of these numbers, when multiplied by d_2, will give a number not greater than m, and having d_2 as its G. C. D. with m. Hence the number of numbers not greater than m, and having d_2 as its G. C. D. with m will be $\phi(q_2)$.

Continuing the process, we shall reach the divisor m, which will have m itself as G. C. D., and which will count as the number corresponding to $\phi(1) = 1$ in the list.

The m numbers $1, 2, 3, \ldots . m$ are therefore equal in number to

$$\phi(m) + \phi(q_2) + \phi(q_3) + \ldots . + \phi(1);$$

or, since the quotients and divisors are the same, only in reverse order, we shall have

$$\phi(1) + \phi(d_2) + \phi(d_3) + \ldots . + \phi(m) = m.$$

242. FERMAT'S THEOREM. *If p be any prime number, and a be a number prime to p, then $a^{p-1} - 1$ will be divisible by p.*

EXAMPLES. $a^4 - 1$ is divisible by 5; $a^6 - 1$ is divisible by 7.

Proof. Develop a^p in the following way by the binomial theorem,

$$\begin{aligned} a^p &= [1 + (a-1)]^p \\ &= 1 + p(a-1) + \binom{p}{2}(a-1)^2 + \ldots . + (a-1)^p. \end{aligned}$$

Because p is prime, all the binomial coefficients,

$$p, \quad \binom{p}{2}, \quad \text{etc., to} \quad \binom{p}{p-1},$$

are divisible by p (§ 239, *Cor.*). Transposing the terms of the last member of the equation which are not divisible by p, we find

$$a^p - (a-1)^p - 1 = \text{a multiple of } p.$$

or $\quad a^p - a - [(a-1)^p - (a-1)] = \text{a multiple of } p.$

Supposing $a = 2$, this equation shows that $2^p - 2$ is a multiple of p; then, supposing $a = 3$, we show by § 231, Th. II, that $3^p - 3$ is such a multiple, and so on, indefinitely.

Hence, $\quad a^p - a = \text{a multiple of } p,$

whatever be a. But $a^p - a = (a^{p-1} - 1)a$, and because this product is divisible by p, one of its factors must be so divisible (§ 236). Hence, if a is prime to p, $a^{p-1} - 1$ is divisible by p.

CHAPTER II.

OF CONTINUED FRACTIONS.

243. Any proper fraction may be represented in the form $\frac{1}{x_1}$, where x_1 is greater than unity, but is not necessarily a whole number. If a_1 be the greatest whole number in x_1, we can put

$$x_1 = a_1 + \frac{1}{x_2},$$

where x_2 will be greater than unity. In the same way we may put

$$x_2 = a_2 + \frac{1}{x_3},$$

$$x_3 = a_3 + \frac{1}{x_4},$$

$$\text{etc.} \qquad \text{etc.}$$

If for each x we substitute its expression, the fraction $\frac{1}{x_1}$ will take the form

$$\frac{1}{x_1} = \cfrac{1}{a_1 + \cfrac{1}{x_2}} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{x_3}}}, \text{ etc., etc.}$$

If the substitutions are continued indefinitely, the form will be

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5} \cdots}}}}$$

Such an expression is called a continued fraction.

Def. A **Continued Fraction** is one of which the denominator is a whole number plus a fraction; the denominator of this last fraction a whole number plus a fraction, etc.

A continued fraction may either terminate with one of its denominators or it may extend indefinitely.

Def. When the number of quotients a is finite, the fraction is said to be **Terminating.**

244. Problem. *To find the value of a continued fraction.*

We first find the value when we stop at the first denominator, then at the second, then at the third, etc.

Using only two denominators, the fraction will be

$$F = \frac{1}{x_1} = \frac{1}{a_1 + \frac{1}{x_2}} = \frac{x_2}{a_1 x_2 + 1},$$

F being put for the true value of the fraction.

To find the expression with three terms, we put, in the preceding expression, $a_2 + \frac{1}{x_3}$ in place of x_2. This gives

$$F = \frac{a_2 + \frac{1}{x_3}}{a_1 a_2 + \frac{a_1}{x_3} + 1} = \frac{a_2 x_3 + 1}{(a_1 a_2 + 1) x_3 + a_1}.$$

To find the result with the fourth denominator, we substitute $x_3 = a_3 + \frac{1}{x_4}$. The fraction becomes:

$$F = \frac{(a_2 a_3 + 1) x_4 + a_2}{[(a_1 a_2 + 1) a_3 + a_1] x_4 + a_1 a_2 + 1}. \qquad (a)$$

To investigate the general law according to which the successive expressions proceed, we put

P, the coefficient of x in any numerator;
P', the coefficient of x in the denominator;
Q, the terms not multiplied by x in the numerator;
Q', the terms not multiplied by x in the denominator;

and we distinguish the various expressions by giving each P and Q the same index as the x to which it belongs.

Then we may represent each value of F in the form,

$$F = \frac{P_i x_i + Q_i}{P'_i x_i + Q'_i}, \qquad (b)$$

where i may take any value necessary to distinguish the fraction. Comparing with the fractions as written, we see that:

$$\begin{array}{llll} P_1 = 0, & Q_1 = 1, & P'_1 = 1, & Q'_1 = 0; \\ P_2 = 1, & Q_2 = 0, & P'_2 = a_1, & Q'_2 = 1; \qquad (c) \\ P_3 = a_2, & Q_3 = 1, & P'_3 = a_1 a_2 + 1, & Q'_3 = a_1; \\ P_4 = a_2 a_3 + 1, & Q_4 = a_2, & P'_4 = a_3 P'_3 + a_1, & Q'_4 = a_1 a_2 + 1. \end{array}$$

To show that this form will continue, how far soever we carry the computation, we put in the expression (b) the general value of x_i,

$$x_i = a_i + \frac{1}{x_{i+1}},$$

which gives,
$$F = \frac{(a_i P_i + Q_i)\, x_{i+1} + P_i}{(a'_i P'_i + Q'_i)\, x_{i+1} + P'_i}. \qquad (d)$$

To show the general law of succession of the terms, let us compare the general equation (b) with (d). Putting $i+1$ for i in (b), it becomes,

$$F = \frac{P_{i+1} x_{i+1} + Q_{i+1}}{P'_{i+1} x_{i+1} + Q'_{i+1}}. \qquad (e)$$

Comparing this with (d), we find

$$P_{i+1} = a_i P_i + Q_i,$$
$$Q_{i+1} = P_i;$$

whence,
$$Q_i = P_{i-1}.$$

Substituting this value of Q_i in the equation previous, it becomes

$$P_{i+1} = a_i P_i + P_{i-1}. \qquad (f)$$

Working in the same way with the denominators, we find

$$P'_{i+1} = a_i P'_i + P'_{i-1}. \qquad (g)$$
$$Q'_{i+1} = P'_i.$$

By supposing i to take in succession the values 1, 2, 3, etc., these formulæ show that the successive values of P may be computed thus:

$$\left.\begin{array}{l} P_1 = 0, \\ P_2 = 1, \end{array}\right\} \text{ (from } c\text{);}$$

$$\begin{aligned} P_3 &= a_2 P_2 + P_1 = a_2, \\ P_4 &= a_3 P_3 + P_2, \\ P_5 &= a_4 P_4 + P_3, \\ P_6 &= a_5 P_5 + P_4, \end{aligned}$$

etc., to any extent.

Also,
$$\begin{aligned} P'_1 &= 1, \\ P'_2 &= a_1, \\ P'_3 &= a_2 P'_2 + P'_1, \\ P'_4 &= a_3 P'_3 + P'_2, \\ P'_5 &= a_4 P'_4 + P'_3, \\ \text{etc.} &\qquad \text{etc.} \end{aligned}$$

Since each value of Q is equal to the value of P having the next smaller index, it is not necessary to compute the Q's separately.

If the fraction terminates at the n^{th} value of a, we shall have

$$x_n = a_n, \text{ exactly.}$$

If it does not terminate, we have to neglect all the denominators after a certain point; and calling the last denominator we use the n^{th}, we must suppose

$$x_n = a_n.$$

In either case, the expression (b) will give the value of the fraction with which we stop by putting $i = n$ and $x_n = a_n$.

Therefore,
$$F = \frac{a_n P_n + Q_n}{a_n P'_n + Q'_n},$$

or, substituting for Q_n and Q'_n their values in (g),

$$F = \frac{a_n P_n + P_{n-1}}{a_n P'_n + P'_{n-1}}.$$

But the general expressions (f) and (g) give

$$a_n P_n + P_{n-1} = P_{n+1},$$
$$a_n P'_n + P'_{n-1} = P'_{n+1}.$$

Therefore,
$$F = \frac{P_{n+1}}{P'_{n+1}}.$$

Therefore, *to find the value of the fraction to the n^{th} term, we have only to compute the values of P_{n+1} and P'_{n+1}, without taking any account of Q.*

EXAMPLE. Take the fraction,

$$\cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5 \text{ etc.}}}}}}$$

Here, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, ... $a_i = i$.

We now have, by continuing the formulæ (e) and (f), and using those values of a_1, a_2, etc.:

$$\begin{aligned}
P_1 &= 0,\\
P_2 &= 1,\\
P_3 &= a_2 P_2 + P_1 = a_2 = 2,\\
P_4 &= a_3 P_3 + P_2 = 3 \cdot 2 + 1 = 7,\\
P_5 &= a_4 P_4 + P_2 = 4 \cdot 7 + 2 = 30,\\
P_6 &= a_5 P_5 + P_4 = 5 \cdot 30 + 7 = 157,\\
&\text{etc.} \qquad \text{etc.} \qquad \text{etc.}
\end{aligned}$$

$$\begin{aligned}
P'_1 &= 1,\\
P'_2 &= a_1 = 1,\\
P'_3 &= a_2 P'_2 + P'_1 = 2 \cdot 1 + 1 = 3,\\
P'_4 &= a_3 P'_3 + P'_2 = 3 \cdot 3 + 1 = 10,\\
P'_5 &= a_4 P'_1 + P'_3 = 4 \cdot 10 + 3 = 43,\\
P'_6 &= a_5 P'_5 + P'_4 = 5 \cdot 43 + 10 = 225.
\end{aligned}$$

Therefore, supposing in succession, $n = 1$, $n = 2$, $n = 3$, etc., we have, for the successive approximate values of the fraction, For $n = 1$, $\qquad F_1 = \dfrac{P_2}{P'_2} = 1.$

For $n = 2$, $\qquad F_2 = \dfrac{P_3}{P'_3} = \dfrac{2}{3}.$

For $n = 5$, $\qquad F_5 = \dfrac{P_6}{P'_6} = \dfrac{157}{225}.$

These successive approximate values of the continued fraction are called **Converging Fractions,** or **Convergents.**

245. The forms (f) and (g) may be expressed in words as follows:

The numerator of each convergent is formed by multiplying the preceding numerator by the corresponding a, and adding the second numerator preceding to the product.

The successive denominators are formed in the same way.

EXAMPLE. The ratio of the motions of the sun and moon relative to the moon's node is given by the continued fraction:

$$\cfrac{1}{12 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{3 + \text{etc.}}}}}}}$$

Let us find the successive convergents. We put the denominators $a_1 = 12$, $a_2 = 1$, etc., in a line, thus:

$$\begin{array}{lccccccc} a & = & 12, & 1, & 2, & 1, & 4, & 3. \\ \dfrac{P}{P'} & = & \dfrac{0}{1}, & \dfrac{1}{12}, & \dfrac{1}{13}, & \dfrac{3}{38}, & \dfrac{4}{51}, & \dfrac{19}{242}, & \dfrac{61}{777}. \end{array}$$

Under a_1 we write the fraction $\dfrac{0}{1}$, which is always the one with which to start, because $P_1 = 0$ and $P'_1 = 1$ (§ 244, e). Next to the right is $\dfrac{1}{a_1}$, because $P_2 = 1$ and $P'_2 = a$. After this, we multiply each term by the multiplier a above it, and add the term to the left to obtain the term on the right. Thus, $2\cdot 1+1=3$, $2\cdot 13+12=38$, etc.

Ex. 2. To compute the convergents of

$$\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{4 \text{ etc.}}}}}$$

a =	2,	4,	2,	4,	2,	4,	etc.
Numerators, Denominators,	$\frac{0}{1}$,	$\frac{1}{2}$,	$\frac{4}{9}$,	$\frac{9}{20}$,	$\frac{40}{89}$,	$\frac{89}{198}$,	etc.

EXERCISES.

Reduce the following continued fractions to vulgar fractions:

1. $$\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{16}}}.$$

2. $$\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{3}}}}.$$

3. $$\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{3}}}}}.$$

4. $$\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{x}}}.$$

5. $$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}}.$$

246. Problem. *To express a fractional quantity as a continued fraction.*

Let R be the given fraction, less than unity. Compute x_1 from the formula,

$$x_1 = \frac{1}{R}.$$

Let a_1 be the whole number and R' the fraction of x_1. Then compute

$$x_2 = \frac{1}{R'}.$$

Let a_2 be the whole number and R'' the fraction of x_2.

We continue this process to any extent, unless some value of x comes out a whole number, when we stop.

EXAMPLE. Express $\frac{26}{73}$ as a continued fraction.

$$x_1 = \frac{1}{R} = \frac{73}{26} = 2 + \frac{21}{26}; \qquad \therefore a_1 = 2; \qquad R' = \frac{21}{26}.$$

$$x_2 = \frac{1}{R'} = \frac{26}{21} = 1 + \frac{5}{21}; \qquad \therefore a_2 = 1; \qquad R'' = \frac{5}{21}.$$

$$x_3 = \frac{1}{R''} = \frac{21}{5} = 4 + \frac{1}{5}; \qquad \therefore a_3 = 4; \qquad R''' = \frac{1}{5}.$$

$$x_4 = \frac{1}{R'''} = \frac{5}{1} = 5; \qquad \therefore a_4 = 5; \qquad R^{\text{iv}} = 0.$$

So the continued fraction is

$$\cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{5}}}}.$$

It will be seen that the process is the same as that of finding the greatest common divisor of two numbers.

EXERCISES.

Develop the following quotients as continued fractions:

1. $\frac{113}{355}$. 2. $\frac{1049}{3326}$. 3. $\frac{628}{925}$.

247. The most simple continued fraction is that arising from the geometric problem of cutting a line in extreme and mean ratio. The corresponding numerical problem is:

To divide unity into two such fractions that the less shall be to the greater as the greater is to unity.

Let r be the greater fraction. Then $1 - r$ will be the lesser one. We must then have

$$1 - r : r :: r : 1,$$

which gives $$r^2 = 1 - r,$$
or $$r^2 + r = 1,$$
or $$r(r + 1) = 1,$$
or $$r = \frac{1}{1 + r}.$$

Now, let us put for r in the last denominator the expression $\frac{1}{1 + r}$, and repeat the process indefinitely. We shall have,

$$r = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 \text{ etc., } \textit{ad infinitum.}}}}}$$

Now we may form the successive convergents which approximate to the true value by the rule. As all the denominators a are 1, we have no multiplying, but only add each term to the preceding one to obtain the following one. Thus we find:

$$\frac{0}{1}, \ \frac{1}{1}, \ \frac{1}{2}, \ \frac{2}{3}, \ \frac{3}{5}, \ \frac{5}{8}, \ \frac{8}{13}, \ \frac{13}{21}, \ \frac{21}{34}, \ \frac{34}{55}, \ \text{etc.}$$

The true value of r may be found by solving the quadratic,

$$r^2 + r = 1,$$

which gives $$r = \frac{-1 \pm \sqrt{5}}{2}.$$

The positive root, with which alone we are concerned, is

$$r = \frac{-1 + \sqrt{5}}{2} = 0.61803399.$$

The values of the first nine convergents, with their errors, are:

$1 : 1$	$= 1.0,$	error $=$	$+ 0.382.$
$1 : 2$	$= 0.5,$	"	$- 0.118.$
$2 : 3$	$= 0.666....,$	"	$+ 0.0486.$
$3 : 5$	$= 0.600,$	"	$- 0.0180.$
$5 : 8$	$= 0.625,$	"	$+ 0.00697.$

$8 : 13 = 0.61538\ldots,$	error $=$	$-\ 0.00265.$
$13 : 21 = 0.61904\ldots,$	"	$+\ 0.00101.$
$21 : 34 = 0.617647\ldots,$	"	$-\ 0.000397.$
$34 : 55 = 0.618182\ldots,$	"	$+\ 0.000148.$
etc. etc.		etc.

Relations of Successive Convergents.

248. Theorem I. *The successive convergents are alternately too large and too small.*

Proof. The first convergent is $\frac{1}{a_1}\cdot$ The true denominator being $a_1 + \frac{1}{x_2}$, the denominator a_1 is too small, and therefore the fraction is too large.

In forming the second fraction, we put $\frac{1}{a_2}$ instead of $\frac{1}{x_2}\cdot$ Because $a_2 < x_2$, this fraction is too large, which makes the denominator $a_1 + \frac{1}{a_2}$ too small.

The third denominator a_3 is too small, which will make the preceding one too large, the next preceding too small, and so on alternately.

Theorem II. *If $\frac{m}{n}$ and $\frac{m'}{n'}$ be any two consecutive convergents, then*

$$mn' - m'n = \pm 1.$$

Proof. We show:

(α) That the theorem is true of the first pair of convergents.

(β) That if true of any pair, it will be true of the pair next following.

(α) The first pair of convergents are

$$\frac{1}{a_1}, \qquad \frac{1}{a_1 + \frac{1}{a_2}} = \frac{a_2}{a_1 a_2 + 1},$$

which gives $mn' - m'n = 1$, thus proving (α).

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(β) Let $$\frac{m}{n}, \quad \frac{m'}{n'}, \quad \frac{m''}{n''},$$
be three consecutive convergents, in which
$$mn' - m'n = \pm 1. \tag{1}$$
By (f) and (g) we shall have
$$m'' = am' + m,$$
$$n'' = an' + n.$$

Multiplying the second equation by m' and subtracting the product of the first by n', we have
$$m'n'' - m''n' = m'n - mn',$$
which is the negative of (1), showing that the result is ∓ 1.

The theorem being true of the first and second fractions, must therefore be true of the second and third; therefore of the third and fourth, and so on indefinitely.

Corollaries. Dividing (1) by nn', we have
$$\frac{m}{n} - \frac{m'}{n'} = \pm \frac{1}{nn'}. \quad \text{Hence,}$$

I. *The difference between the two successive convergents is equal to unity divided by the product of the denominators.*

Because the denominator of each fraction is greater than that of the preceding one, we conclude:

II. *The difference between two consecutive convergents constantly diminishes.*

Combining these conclusions with Th. I, we conclude:

III. *Each value of a convergent always lies between the values of the two preceding convergents.*

For if R_4, R_5, R_6 be three such fractions, and if R_5 is greater than R_4, then R_6 will be less than R_5. But it must be greater than R_4, else we should not have $R_5 - R_6$ numerically less than $R_4 - R_5$. Hence, if we arrange the successive convergents in a line in the order of magnitude, their order will be as follows:
$$R_4, \; R_6, \; R_8, \; \ldots \; R_9, \; R_7, \; R_5,$$
each convergent coming nearer a true central value. Hence,

IV. *The true value of the continued fraction always lies between the values of two consecutive convergents.*

Comparing with (I), we conclude:

V. *The error which we make by stopping at any convergent can never be greater than unity divided by the product of the denominators of that convergent and the one next following.*

EXAMPLE.

Referring to the table of values of $\frac{1}{2}(\sqrt{5}-1)$ in § 247, we see that:

$$\text{Error of } \quad 2:3 < \frac{1}{3\cdot 5}; \qquad \left(\text{for } .0486 < \frac{1}{15}\right).$$

$$\text{Error of } \quad 3:5 < \frac{1}{5\cdot 8}; \qquad \left(\text{for } .018 < \frac{1}{40}\right).$$

etc. etc.

Hence, in general, continued fractions give a very rapid approximation to the true value of a quantity. Their principal use arises from their giving approximate values of irrational numbers by vulgar fractions with the smallest terms.

EXAMPLE.

Develop the fractional part of $\sqrt{2}$ as a continued fraction, and find the values of eight convergents.

Because 1 is the greatest whole number in $\sqrt{2}$, we put

$$\sqrt{2} = 1 + \frac{1}{x}; \tag{1}$$

whence,

$$x = \frac{1}{\sqrt{2}-1}.$$

Rationalizing the denominator, § 185,

$$x = \sqrt{2} + 1.$$

Substituting for $\sqrt{2}$ its value in (1),

$$x = 2 + \frac{1}{x}.$$

Putting this value of x in (1) and again in the denominator, and repeating the substitution indefinitely, we find

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}} \text{ etc.}$$

Forming the convergents, we find them to be

$$\frac{1}{2},\quad \frac{2}{5},\quad \frac{5}{12},\quad \frac{12}{29},\quad \frac{29}{70},\quad \frac{70}{169},\quad \frac{169}{408},\quad \frac{408}{985},\quad \text{etc.}$$

Adding unity to each of them, we find the approximate values of $\sqrt{2}$:

$$\frac{3}{2},\quad \frac{7}{5},\quad \frac{17}{12},\quad \frac{41}{29},\quad \frac{99}{70},\quad \frac{239}{169},\quad \frac{577}{408},\quad \frac{1393}{985},\quad \text{etc.}$$

Rem. The square root of 2 may be employed in finding a right angle, because a right angle (by Geometry) can be formed by three pieces of lengths proportional to 1, 1, $\sqrt{2}$. If we make the lengths 12, 12, 17, the error will, by Cor. V, be less than $\frac{1}{12.29}$, or less than $\frac{1}{348}$ of the whole length.

EXERCISES.

Develop the following square roots as continued fractions, and find six or more of the partial fractions approximating to each:

1. $\sqrt{3}$. 2. $\sqrt{5}$. 3. $\sqrt{6}$. 4. $\sqrt{10}$.

5. Develop a root of the quadratic equation

$$x^2 - ax - 1 = 0,$$

commencing the operation by dividing the equation by x.

Periodic Continued Fractions.

249. *Def.* A **Periodic** continued fraction is one in which the denominators repeat themselves in regular order.

EXAMPLE. A continued fraction in which the successive denominators are

2, 3, 5, 2, 3, 5, 2, 3, 5, etc., *ad infinitum*,

is periodic.

A periodic continued fraction can be expressed as the root of a quadratic equation.

EXAMPLES.

1.
$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\text{etc.}}}}}$$

If we put x for the value of this fraction, we have

$$x = \cfrac{1}{1+\cfrac{1}{2+x}}.$$

We find the value thus:

$$1, \qquad 2+x.$$

$$\frac{0}{1}, \qquad \frac{1}{1}, \qquad \frac{2+x}{3+x}.$$

Because this expression is x itself, we have

$$x = \frac{2+x}{3+x},$$

which reduces to the quadratic equation

$$x^2 + 2x = 2.$$

2. Let us take the fraction of which the successive denominators are 2, 3, 5, 2, 3, 5, etc., namely,

$$x = \cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{2+\cfrac{1}{3+\text{etc.},}}}}}$$

or,
$$x = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{5 + x}}}.$$

We compute thus:

$$\begin{array}{cccc} 2, & 3, & x + 5. & \\ \frac{0}{1}, & \frac{1}{2}, & \frac{3}{7}, & \frac{3x + 16}{7x + 37}. \end{array}$$

Hence we have, to determine x, the quadratic equation,

$$x = \frac{3x + 16}{7x + 37}, \quad \text{or} \quad 7x^2 + 34x = 16.$$

250. *Development of the Root of a Quadratic Equation.* A root of a quadratic equation may be developed in a continued fraction by the following process. Let the equation in its normal form be (§ 192),

$$mx^2 + nx + p = 0, \qquad (1)$$

m, n, and p being whole numbers. We shall then have

$$x = \frac{-n \pm \sqrt{n^2 - 4mp}}{2m}.$$

Let a be the greatest whole number in x, which we may find either by trial in (1) or by this value of x. Then assume

$$x = a + \frac{1}{x_1},$$

and substitute this value of x in the original equation. Then, regarding x_1 as the unknown quantity, we reduce to the normal form, which gives

$$(ma^2 + na + p)x_1^2 + (2ma + n)x_1 + m = 0. \qquad (2)$$

If a_1 is the greatest whole number in x_1, we put

$$x_1 = a_1 + \frac{1}{x_2},$$

and by substituting this value of x_1 in (2), we form an equation in x_2. Continuing the transformations, we find the greatest whole number in x_2, which will be called a_2, and so on.

The root will then be expressed as a whole number a plus the continued fraction of which the denominators are a_1, a_2, a_3, etc.

BOOK X.

THE COMBINATORY ANALYSIS.

CHAPTER I.

PERMUTATIONS.

251. *Def.* The different orders in which a number of things can be arranged are called their **Permutations.**

Examples. The permutations of the letters a, b, are

$$ab, \quad ba.$$

The permutations of the numbers 1, 2, and 3 are

$$123, \quad 132, \quad 213, \quad 231, \quad 312, \quad 321.$$

Problem. *To find how many permutations of any given number of things are possible.*

Let us put

P_i, the number of permutations of i things.

It is evident from the first of the above examples that there are two permutations of two things. Hence,

$$P_2 = 2.$$

To find the permutations of three letters, a, b, c, we form three sets of permutations, the first beginning with a, the second with b, and the third with c.

In each set the first letter is to be followed by all possible permutations of the remaining letters, namely:

In	1st	set,	after	a	write	bc, cb,	making	abc,	acb.
"	2d	"	"	b	"	ac, ca,	"	bac,	bca.
"	3d	"	"	c	"	ab, ba,	"	cab,	cba.

Hence, $P_3 = 3 \cdot 2 = 6.$

The permutations of n things can be divided into sets. The first set begins with the first thing, followed by all possible permutations of the remaining $n - 1$ things, of which the number is P_{n-1}. The second set begins with the second thing, followed by all possible permutations of the remaining $n - 1$ things, of which the number is also P_{n-1}, and so with all n sets. Hence, whatever be n, there will be n sets of P_{n-1} permutations in each set. Therefore,

$$P_n = nP_{n-1}.$$

This equation enables us to find P_n whenever we know P_{n-1}, and thus to form all possible values of P_n, as follows:

It is evident that	$P_1 = 1.$	
We have found	$P_2 = 2 \cdot 1 = 2!$	
" "	$P_3 = 3 \cdot 2 \cdot 1 = 3! = 6.$	
Putting $n = 4$, we have	$P_4 = 4P_3 = 4! = 24.$	
" $n = 5$, " "	$P_5 = 5P_4 = 5! = 120.$	
etc.	etc.	etc.

It is evident that the number of permutations of n things is equal to the continued product

$$1 \cdot 2 \cdot 3 \cdot 4 \ldots . n,$$

which we have represented by the symbol $n!$ so that

$$P_n = n!$$

EXERCISES.*

1. Write all the permutations of the following letters:

bcd, *acd*, *abd*, *abcd*.

2. What proportion of the possible permutations of the letters a, e, m, t, make well-known English words?

3. Write all the numbers of four digits each of which can be formed by permuting the four digits 1, 2, 3, 4.

4. How many numbers is it possible to form by permuting the six figures 1, 2, 3, 4, 5, 6.

---

* If the student finds any difficulty in reasoning out these exercises, he is recommended to try similar cases in which few symbols are involved by actually forming the permutations, until he clearly sees the general principles involved.

5. At a dinner party a row of 6 plates is set for the host and 5 guests. In how many ways may they be seated, subject to the condition that the host must have Mr. Brown on his right and Mr. Hamilton on his left?

6. Of all numbers that can be formed by permuting the seven digits, 1, 2 . . . . 7:

(*a*) How many will be even and how many odd?

(*b*) In how many will the seven digits be alternately even and odd?

(*c*) In how many will the three even digits all be together?

(*d*) In how many will the four odd digits all be together?

7. In how many permutations of the 8 letters, *a, b, c, d, e, f, g, h,* will the letters *d, e, f,* stand together in alphabetical order?

8. In how many of the above permutations will the word *deaf* be found?

9. In how many of the permutations of the first 9 letters will the words *age* and *bid* be both found?

10. A party of 5 gentlemen and 5 ladies agree with a mathematician to dance a set for every way in which he can divide them into couples. How many sets can he make them dance?

11. In how many of the permutations of the letters *a, b, c, d, e,* will *d* and no other letter be found between *a* and *e*.

12. In how many of the permutations of the six symbols, *a, b, c, d, e, f,* will the letters *abc* be found together in one group, and the letters *def* in another?

13. How many permutations of the seven symbols, *a, b, c, d, e, f, g,* are possible, subject to the condition that some permutation of the letters *abc* must come first?

14. The same seven symbols being taken, how many permutations can be formed in which the letters *abc* shall remain together?

Permutations of Sets.

252. *Def.* When permutations are formed of only s things out of a whole number n, they are called **Permutations of n things taken s at a time.**

EXAMPLE. The permutations of the three letters a, b, c, taken two at a time, are

$$ab, \quad ba, \quad ac, \quad ca, \quad bc, \quad cb.$$

The permutations of 1, 2, 3, 4, taken two at a time, are

$$12,\ 13,\ 14,\ 21,\ 23,\ 24,\ 31,\ 32,\ 34,\ 41,\ 42,\ 43.$$

PROBLEM. *To find the number of permutations of n things taken s at a time.*

Suppose, first, that we take two things at a time, as in the above examples. We may choose any one of the n things as the first in order. Which one soever we take, we shall have $n - 1$ left, any one of which may be taken as the second in order. Hence, the permutations taken 2 at a time will be

$$n(n-1).$$

[Compare with the last example, where $n = 4$.]

To form the permutations 3 at a time, we add to each permutation by 2's any one of the $n - 2$ things which are left. Hence, the number of permutations 3 things at a time is

$$n(n-1)(n-2).$$

In general, the permutations of n things taken s at a time will be equal to the continued product of the s factors,

$$n(n-1)(n-2)\ldots\ldots(n-s+1),$$

which is equal to the quotient $\dfrac{n!}{s!}$

It will be remarked that when $s = n$, we shall have the case already considered of the permutations of all n things.

EXERCISES.

1. Write all the numbers of two figures each which can be formed from the four digits, 3, 5, 7, 9.

2. Write all the numbers of three figures, beginning with 1, which can be formed from the five digits, 1, 2, 3, 4, 5.

3. How many different numbers of four figures each can be formed with the digits 1, 2, 3, 4, 5, 6, no figure being repeated in any number?

4. Explain how all the numbers in the preceding exercise may be written, showing how many numbers begin with 1, how many with 2, etc.

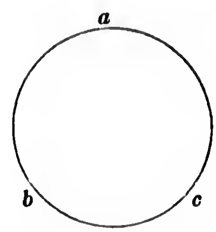
5. In how many ways can 3 gentlemen select their partners from 5 ladies?

6. How many even numbers of 3 different digits each can be formed from the seven digits, 1, 2, ... 7?

7. How many of these numbers will consist of an odd digit between two even ones?

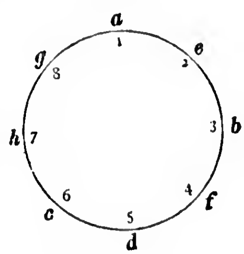
Circular Permutations.

253. If we have the three letters $a, b, c,$ arranged in a circle, as in the adjoining figure, then, however we arrange them, we shall find them in alphabetical order by beginning with a and reading them in the suitable direction. Hence, there are only two different circular arrangements of three letters instead of six, namely, the two directions in which they may be in alphabetical order.



Next suppose any number of symbols, say $a, b, c, d, e, f, g, h,$ and let there be an equal number of positions around the circle in which they may be placed. These positions are numbered 1, 2, 3, 4, 5, 6, 7, 8.

For every arrangement of the symbols we may turn them round in a body without changing the arrangement. Each symbol will then pass through all eight positions in succession.



By performing this operation with every arrangement, we shall have all possible permutations of the eight things among the eight positions, the number of which is 8!, which are therefore eight times as many as the circular arrangements.

Hence the number of different circular arrangements is $\frac{8!}{8}$, which is the same as $7!$

In general, if we represent the number of circular arrangements of n things by C_n, we shall have

$$C_n = (n-1)!$$

The same result may be reached by the following reasoning. To form a circular arrangement, we may take any one symbol, a for example, put it into a fixed position, say (1), and leave it there.

All possible arrangements of the symbols will then be formed by permuting the remaining symbols among the remaining positions. Hence,

$$C_n = P_{n-1} = (n-1)!$$

as before.

EXERCISES.

1. In how many orders can a party of 7 persons take their places at a round table?

2. In how many orders can a host and 7 guests sit at a round table in order that the host may have the guest of highest rank upon his right and the next in rank on his left?

3. Five works of four volumes each are to be arranged on a circular shelf. How many arrangements are possible which will keep the volumes of each set together and in proper order, it being indifferent in which direction the numbers of the volumes read.

4. In how many circular arrangements of the 5 letters a, b, c, d, e, will a stand between b and d?

5. If the 10 digits are to be arranged in a circle, in how many ways can it be done, subject to the condition that even and odd digits must alternate? (Note that 0 is even.)

6. The same thing being supposed, how many arrangements are possible, subject to the condition that the even digits must be all together?

7. In how many circular arrangements of the first six letters will the word *deaf* be found? What will be the difference of the results if you are allowed to spell it in either direction?

Permutations when Several of the Things are Identical.

254. If the same thing appears several times among the things to be permuted, the number of different permutations will be less than when the things are all different.

EXAMPLE. The permutations of *aabb* are

$$aabb, \quad abab, \quad abba, \quad baab, \quad baba, \quad bbaa, \tag{1}$$

which are only six in number.

PROBLEM. *To find the number of permutations when several of the things are identical.*

Let us first examine how all 24 permutations of 4 things may be formed from the above 6 permutations of *aabb*. Let us distinguish the two a's and the two b's by accenting one of each. Then, from each permutation as written, four may be formed by permuting the similar letters among themselves. For example, taking *abba*, and writing it $abb'a'$, we have, by permuting the similar letters,

$$ab'ba', \quad a'b'ba, \quad abb'a', \quad a'bb'a. \tag{2}$$

In the same way four permutations, differing only in the arrangement of the accents, may be formed from each of the 6 permutations (1), making 24 in all, as there ought to be. (§ 251.)

Generalizing the preceding operation, we reach the following solution of our problem. Let the symbols to be permuted be a, b, c, etc.

Suppose that	a	is repeated	r	times,
"	"	b "	"	s	"
"	"	c "	"	t	"
etc.		etc.		etc.	

and let the whole number of symbols, counting repetitions, be n, so that

$$n = r + s + t + \text{etc.}$$

[In the preceding example (1), $n = 4$, $r = 2$, $s = 2$.]

Also put X_n, the required number of different permutations of the n symbols.

Suppose these X_n different permutations all written out, and suppose the symbols which are repeated to be distinguished by accents. Then:

From each of the X_n permutations may be formed $P_r = r!$ permutations by permuting the a's among themselves, as in (2). We shall then have $r!\,X_n$ permutations.

From each of the latter may be formed $s!$ permutations by permuting the b's among themselves. We shall then have $s!\,r! \times X_n$ permutations.

From each of these may be found $t!$ permutations by interchanging the c's among themselves.

Proceeding in the same way, we shall have

$$X_n \times r! \times s! \times t! \times \text{etc.}$$

possible permutations of all n things. But this number has been shown to be $n!$ Therefore,

$$X_n \times r! \times s! \times t! \times \text{etc.} = n!$$

By division, $$X_n = \frac{n!}{r!\,s!\,t!\,\text{etc.}}, \qquad (3)$$

which is the required expression.

We remark that if any symbols are not repeated, the formula (3) will still be true by supposing the number corresponding to r, s, or t to be 1.

EXAMPLES.

1. The number of possible permutations of *aabb* are

$$\frac{4!}{2!\,2!} = \frac{24}{2 \cdot 2} = 6, \text{ as already found.}$$

2. The possible permutations of *aaabbcd* are

$$\frac{7!}{3!\,2!} = \frac{5040}{6 \cdot 2} = 420.$$

EXERCISES.

Write all the permutations of the letters:

1. *aaab*. 2. *aabc*. 3. *aaabc*.

4. How many different numbers of seven digits each can be formed by permuting the figures 1112225 ?

5. If every different permutation of letters made a word, how many words of 13 letters each could be formed from the word *Massachusetts*.

The Two Classes of Permutations.

255. The $n!$ possible permutations of n things are divisible into two classes, commonly distinguished as **even** permutations and **odd** permutations in the following way:

We suppose the n things first arranged in alphabetical or numerical order,

$$a, b, c, d, \dots \quad \text{or} \quad 1, 2, 3, 4, \dots n,$$

and we call this arrangement an *even permutation*.

Then, having any other permutation, we count for each thing how many other things of lower order come after it, and take the sum.

If this sum is even, the permutation is an even one; if odd, an odd one.

EXAMPLES.

1. Consider the permutation 265143.

Here	2	is followed by	1	number of	lower	order,	namely,	1.
"	6	" "	4	"	"	"	"	5, 1, 4, 3.
"	5	" "	3	"	"	"	"	1, 4, 3.
"	1	" "	0	"	"	"		
"	4	" "	1	"	"	"	"	3.

Then $1+4+3+0+1=9$. Hence the permutation is odd.

2. Consider *cdbea*.

Here	c	is followed by	2	letters	before it	in order,	namely,	*ba*.
"	d	" "	2	"	"	"	"	*ba*.
"	b	" "	1	"	"	"	"	*a*.
"	e	" "	1	"	"	"	"	*a*.

Then $2+2+1+1=6$. Hence the permutation is even.

Def. The total number of times which a thing less in order follows one greater in order is called the **Number of Inversions** in a permutation.

EXAMPLE. In the preceding permutation, 265143, the number of inversions is 9. In *cdbed* it is 6.

REM. It will be seen that the class of a permutation is even or odd, according as the number of inversions is even or odd.

THEOREM I. *If, in a permutation, two things are interchanged, the class will be changed from even to odd, or from odd to even.*

Proof. Consider first the case in which a pair of adjoining things are interchanged. Let us call:

ik, the two things interchanged.
A, the collection of things which precede i and k.
C, the collection of things which follow them.

The first permutation will then be

$$AikC.^* \qquad (a)$$

After interchanging i and k, it will be

$$AkiC. \qquad (b)$$

Because the order of things in A remains undisturbed, each thing in A is followed by the same things as before. In the same way, each thing in C is preceded by the same things as before.

Hence, the number of times that each thing in A or C is followed by a thing less in order remains unchanged, and, leaving out the pair of things, i, k, the number of inversions is unchanged.

But, by interchanging i and k, the new inversion ki is introduced. Therefore the number of inversions is increased by 1.

---

* This form of algebraic notation differs from those already used in that the symbols A and C do not stand for quantities, but mere collections of letters. It is an application of the general principle that a single symbol may be used to represent any set of symbols, but must represent the same set throughout the same question. A and C are here used to show to the eye that in forming the permutations of (b) from (a), all the letters on each side of ik preserve their relative positions unchanged.

If the first arrangement is ki, this one inversion is removed. Hence, in either case the number of inversions is changed by 1, and is therefore changed from odd to even, or *vice versa.*

Illustration. In the permutation

$$265143,$$

the inversions, as already found, are the following nine:

$$21, \quad 65, \quad 61, \quad 64, \quad 63, \quad 51, \quad 54, \quad 53, \quad 43.$$

Let us now interchange 5 and 1, making the permutation

$$261543.$$

The inversions now are

$$21, \quad 61, \quad 65, \quad 64, \quad 63, \quad 54, \quad 53, \quad 43,$$

the same as before, except that 51 has been removed.

Next consider the case in which the things interchanged do not adjoin each other. Suppose that in the permutation

$$b\ a\ d\ e\ h\ c\ f$$

we interchange a and h. We may do this by successively interchanging a with d, with e, and with h, making three interchanges, producing

$$b\ d\ e\ h\ a\ c\ f.$$

Then we interchange h with e and with d, making two interchanges, and producing

$$b\ h\ d\ e\ a\ c\ f,$$

which effects the required interchange of a with h.

The number of the neighboring interchanges is $3 + 2 = 5$, an odd number. Because the number of inversions is changed from odd to even this same odd number of times, it will end in the opposite class with which it commenced.

Theorem II. *The possible permutations of n things are one-half even and one-half odd.*

Proof. Write the $n!$ possible permutations of the n things. Then interchange some one pair of things (*e.g.*, the first two things) in each permutation. We shall have the same permutations as before, only differently arranged.

By the change, every even permutation will be changed to odd, and every odd one to even.

Because every odd one thus corresponds to an even one, and *vice versa*, their numbers must be equal.

Illustration. The permutations in the second column following are formed from those in the first by interchanging the first two figures:

1 2 3	even,	2 1 3	odd.
1 3 2	odd,	3 1 2	even.
2 1 3	odd,	1 2 3	even.
2 3 1	even,	3 2 1	odd.
3 1 2	even,	1 3 2	odd.
3 2 1	odd,	2 3 1	even.

EXERCISES.

Count the number of inversions in each of the following permutations:

1. *bcdagef.* 2. *bcagdef.* 3. 325941.
4. 5432. 5. 82917364. 6. 82971364.

256. *Symmetric Functions.* An important application of the laws of permutation occurs in the problem, how many different values a function may acquire by permuting the letters which enter into it. We readily find that the expression a^2bc takes only the three values a^2bc, b^2ac, and c^2ab by permutation. Other expressions do not change at all by permuting their symbols.

Def. A **Symmetric Function** is one which is not changed by permuting the symbols which enter into it.

EXERCISES.

Show that the following functions are symmetric:

1. $a + b + c$. 2. abc.
3. $a(b + c) + b(c + a) + c(a + b)$.
4. $a^2(b - c) + b^2(c - a) + c^2(a - b)$.

CHAPTER II.

COMBINATIONS.

257. *Def.* The number of ways in which it is possible to select a set of s things out of a collection of n things is called the **Number of Combinations of** s **things in** n.

Ex. 1. From the three symbols a, b, c, may be formed the couplets,

$$ab, \qquad ac, \qquad bc.$$

Hence there are three combinations of 2 things in 3.

Ex. 2. From a stud of four horses may be formed six different span. If we call the horses A, B, C, D, the different span will be

AB, AC, AD, BC, BD, CD.

Rem. 1. A set is regarded as different when any one of its separate things is different.

Rem. 2. Combinations differ from permutations in that, in forming a combination, no account is taken of the order of arrangement of things in a set. For instance, ab and ba are the same combination. Hence, we may always suppose the letters or numbers of a combination to be written in alphabetical or numerical order.

Notation. The number of combinations of s things in n is sometimes designated by the symbol,

$$C_s^n.$$

Problem. *To find the number of combinations of s things in n.*

If we form every possible set of s things out of n things, and then permute the s things of each set in every possible way, we shall have all the permutations of n things taken s at a time (§ 252). That is,

$$C_s^n \times P_s$$

express the number of permutations of n things taken s at a time. But we have found this number to be

$$n(n-1)(n-2)\ldots\ldots(n-s+1).$$

We have also found

$$P_s = s! = 1 \cdot 2 \cdot 3 \cdot 4 \ldots\ldots s.$$

Hence, $\quad C_s^n \times s! = n(n-1)(n-2)\ldots\ldots(n-s+1),$

and
$$C_s^n = \frac{n(n-1)(n-2)\ldots\ldots(n-s+1)}{1 \cdot 2 \cdot 3 \cdot 4 \ldots\ldots s}$$
$$= \binom{n}{s} \quad (\S\ 228,\ 3);$$

or
$$C_s^n = \frac{n!}{s!\ (n-s)!},$$

which is the required expression.

Rem. For every combination of s things which we can take from n things, a combination of $n-s$ things will be left.

Hence, $\quad C_s^n = C_{n-s}^n.$

This formulæ may be readily derived from the expression for the number of combinations. For, if we take the equation

$$C_s^n = \frac{n!}{s!\ (n-s)!},$$

this formula remains unaltered when we substitute $n-s$ for s, and therefore also represents the combinations of $n-s$ things in n.

Def. Two combinations which together contain all the things to be combined are called two **Complementary** combinations.

EXERCISES.

1. Write all combinations of two symbols in the five symbols, a, b, c, d, e.

2. Write all combinations of three symbols in the same letters, and show why the number is the same as in Ex. 1.

3. A span of horses being different when either horse is changed, how many different span may be formed from a stud of 3? Of 7? Of 9?

4. If four points are marked on a piece of paper, how many distinct lines can be formed by joining them, two and two? How many in the case of n points?

From each one of the points can be drawn $n - 1$ lines to other points; then why are there not $n(n - 1)$ lines?

5. If five lines, no two of which are parallel, intersect each other, how many points of intersection will there be? How many in the case of n lines?

6. If n straight lines all intersect each other, how many different triangles can be found in the figure?

7. In how many different ways may a set of four things be divided into two pairs?

8. In how many ways can a party of four form partners at whist?

9. In how many ways can the following numbers be thrown with three dice:

(a) 1, 1, 1; (b) 1, 2, 2; (c) 1, 2, 3.

10. A school of 15 young ladies have the privilege of sending a party of 5 every day to a picture gallery, provided they do not send the same party twice. How many visits can they make?

Combinations with Repetition.

258. Sometimes combinations are formed with the liberty to repeat the same symbol as often as we please in any set.

Example. From the three things a, b, c, are formed the six combinations of two things with repetition,

$$aa, \quad ab, \quad ac, \quad bb, \quad bc, \quad cc.$$

Problem. *To find the number of combinations of s things in n, when repetition is allowed.*

Solution. Let the n things be the first n numbers,

$$1, 2, 3, 4, \ldots . n.$$

Form all possible sets of s of these numbers with repetition, the numbers of each set being arranged in numerical order.

Let R_s be the required number of sets. Then, in each set,

Let the first number stand unchanged.
Increase the 2d number by 1.
" " 3d " " 2.
" " 4th " " 3.
⋮ ⋮
" " s^{th} " " $s-1$

We shall then have R_s sets of s numbers, each without repetition.

Example. From the numbers 1, 2, 3 are formed with repetition,

11, 12, 13, 22, 23, 33.

Then, increasing the second numbers by 1, we have

12, 13, 14, 23, 24, 34.

The greatest possible number in any set after the increase will be $n + s - 1$, because the greatest number from which the selection is made is n, and the greatest quantity added is $s - 1$. Hence all the new sets will consist of combinations of s numbers each from the $n + s - 1$ numbers,

$$1, 2, 3, 4, \ldots . n \ldots . n + s - 1. \qquad (a)$$

No two of these combinations can be the same, because then two of the original combinations would have to be the same. Hence the new sets are all different combinations of s numbers from the $n + s - 1$ numbers (a). Therefore the number of combinations cannot exceed the quantity C_s^n.

Conversely, if we take all possible combinations of s different numbers in $n + s - 1$, arrange each in numerical order, and subtract 1 from the second, 2 from the third, etc., we shall have different combinations from the first n numbers with repetitions. Hence the number of combinations in the second class cannot exceed those of the first class.

Hence we conclude that the number of combinations of s things in n with repetition is the same as the combinations of s things in $n + s - 1$ without repetition, or

$$R_s^n = C_s^{n+s-1} = \binom{n+s-1}{s}$$
$$= \frac{n(n+1)(n+2)\ldots\ldots(n+s-1)}{1\cdot2\cdot3\cdot4\ldots\ldots s}.$$

EXERCISES.

1. Write all possible combinations of 3 numbers with repetition out of the three numbers 1, 2, 3; then increase the second of each combination by 1 and the third by 2, and show that we have all the combinations of three different numbers out of 1, 2, 3, 4, 5.

2. How many combinations of 4 things in 4 with repetition? Of n things in n?

In the last question and in the following, reduce the result to its lowest terms.

3. How many combinations of $n+1$ things in $n-1$ with repetition?

Special Cases of Combinations.

259. It is plain that

$$C_1^n = n,$$

because each of these combinations consist simply of one of the n things. Hence, also,

$$C_{n-1}^n = n,$$

because in every such combination one letter is omitted.

It is also plain that

$$C_n^n = 1,$$

because the only combination of n letters is that comprising the n letters themselves. Hence we write, by analogy,

$$C_0^n = 1,$$

although a combination of nothing does not fall within the original definition of a combination.

260. The formulæ of combinations sometimes enable us to discover curious relations of numbers.

1. Let us inquire how we may form the combinations of

$s+1$ things when we have those of s things. Let the n things from which the combinations are to be formed be the letters

$$a,\ b,\ c,\ d,\ e,\ f,\ g, \text{ etc. . . . . } (n \text{ in number}).$$

Let all the combinations of $s+1$ of these n letters be written in alphabetical order. Then:

1. In the combinations beginning with a, the letter a will be followed by all possible combinations of s letters out of the $n-1$ letters b, c, d, etc., of which the number is C_s^{n-1}.

2. In the combinations beginning with b, the letter b is followed by all combinations of s letters out of the $n-2$ letters c, d, e, f, etc. Therefore there are C_s^{n-2} combinations beginning with b.

3. In the same way it may be shown that there are C_s^{n-3} combinations beginning with c, C_s^{n-4} beginning with d, etc. The series will terminate with a single combination of the last $s+1$ letters.

Since we thus have all combinations of $s+1$ letters, we find, by summing up those beginning with the several letters a, b, c, etc.,

$$C_s^{n-1} + C_s^{n-2} + C_s^{n-3} + \ldots . + C_s^{s} = C_{s+1}^{n}. \qquad (a)$$

Substituting for the combinations their values, we find

$$\left(\frac{n-1}{s}\right) + \left(\frac{n-2}{s}\right) + \left(\frac{n-3}{s}\right) + \ldots . + \left(\frac{s}{s}\right) = \left(\frac{n}{s+1}\right).$$

By the notation (§ 228, 3), all the terms of the first member have the common denominator $s!$, while the numerators are each composed of the factors of s consecutive numbers. Multiplying both sides by $s!$ and reversing the order of terms in the first member, we have

$$\left.\begin{array}{l} 1\cdot2\cdot3 \ldots . s + 2\cdot3\cdot4 \ldots . s+1 + \text{etc.} \\ \qquad \text{etc.} \qquad\qquad\qquad \text{etc.} \\ \quad + (n-s-1) \ldots . (n-3)(n-2) \\ \quad + (n-s) \ldots . (n-2)(n-1) \end{array}\right\}$$

$$= \frac{(n-s) \ldots . (n-2)(n-1)\,n}{s+1}.$$

The student is now recommended to go over the preceding process with special simple numerical values of n and s which he may select for himself.

EXAMPLES.

If $n = 5$ and $s = 2$, we have

$$1\cdot2 + 2\cdot3 + 3\cdot4 = \frac{3\cdot4\cdot5}{3}.$$

If $n = 7$ and $s = 3$.

$$1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + 4\cdot5\cdot6 = \frac{4\cdot5\cdot6\cdot7}{4}.$$

If $n = 7$ and $s = 4$,

$$1\cdot2\cdot3\cdot4 + 2\cdot3\cdot4\cdot5 + 3\cdot4\cdot5\cdot6 = \frac{3\cdot4\cdot5\cdot6\cdot7}{5}.$$

If $n = 9$ and $s = 3$,

$$1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+4\cdot5\cdot6+5\cdot6\cdot7+6\cdot7\cdot8 = \frac{6\cdot7\cdot8\cdot9}{4}.$$

Prove these equations by computing both members.

261. Another curious example is the following:

Let us have $p + q$ things divided into two sets, the one containing p and the other q things. Then, to form all possible combinations of s things out of the whole $p + q$, we may take:

Any s things in set p;

Or any combination of $s - 1$ things in set p with any one thing of set q;

Or any combination of $s - 2$ things in set p with any combination of 2 things in q;

Or any combination of $s - 3$ things in p with any 3 out of q, etc.

We shall at length come to the combinations of all s things out of q alone. Adding up these separate classes, we shall have:

$$C^p_s + C^p_{s-1}\,C^q_1 + C^p_{s-2}\,C^q_2 + \ldots + C^p_1\,C^q_{s-1} + C^q_s.$$

This sum makes up all combinations of s things in the whole $p+q$, and is therefore equal to C^{p+q}_s. Putting the numerical expressions for the combinations, we have the theorem:

$$\binom{p+q}{s} = \binom{p}{s} + \binom{p}{s-1}\binom{q}{1} + \binom{p}{s-2}\binom{q}{2} + \dots$$
$$+ p\binom{q}{s-1} + \binom{q}{s}.$$

If, as an example, we put $s = 3$, $p = 4$, $q = 5$, this theorem will give

$$\frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} + \frac{4 \cdot 3}{1 \cdot 2} \cdot \frac{5}{1} + \frac{4}{1} \cdot \frac{5 \cdot 4}{1 \cdot 2} + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3},$$

the correctness of which is easily proved by computation.

EXERCISES.

1. Write all the combinations of three letters out of the five, a, b, c, d, e, and show that C^4_2 of them begin with a, C^3_2 with b, and C^2_2 with c, according to the reasoning of § 260.

2. Prove that $\quad C^5_3 = C^4_3 + C^4_2,$

$$C^6_4 = C^5_4 + C^5_3,$$

and in general, $\quad C^{n+1}_s = C^n_s + C^n_{s-1}.$

In the following two ways:

(1.) Let all combinations of s letters in the n letters

$$a, b, c, \dots n$$

be formed, their number being C^n_s. Then suppose one letter added, making the number $n + 1$. The combinations of s letters out of these $n + 1$ will include the C^n_s formed from the n letters, plus each combination of the additional $(n+1)^{st}$ letter with the combinations of $s - 1$ out of the first n letters.

(2.) Prove the same general result from the formula,

$$C^n_s = \binom{n}{s}.$$

3. If we form all combinations of 3 things out of 7, how many of these combinations will contain a 7, and how many will not?

4. If we form all the combinations of s letters out of the n letters

$$a, b, c, \dots n,$$

how many of these combinations will contain a, and how many will not?

5. In the preceding case, how many of the combinations will contain the three letters a, b, c?

262. THEOREM I. *The total number of combinations which can be formed from n things, including 1 zero combination, is 2^n.*

In the language of Algebra,

$$C_0^n + C_1^n + C_2^n + \dots + C_{n-1}^n + C_n^n = 2^n.$$

Proof. Let us begin with 3 things, a, b, c, and let us call the formal zero combination, $1 = C_0^n$. Then we have

C_0^3,	blank,	Number $= 1$
C_1^3,	a, b, c.	" $= 3$
C_2^3,	ab, ac, bc,	" $= 3$
C_3^3,	abc,	" $= 1$
		Sum $= 8 = 2^3$.

Now introduce a fourth letter d. The combinations out of the four things, a, b, c, d, will consist of the above 8, plus the 8 additional ones formed by writing d after each of the above eight. Their number will therefore be 16.

In the same way, it may be shown that we double the possible number of combinations for every thing we add to the set from which they are taken. We have found, for

$n = 3$,	Sum of combinations $=$	$8 = 2^3$;
$n = 4$,	" "	$= 2 \cdot 8 = 2^4$;
$n = 5$,	" "	$= 2 \cdot 2^4 = 2^5$;
etc.		etc.

which shows the theorem to be general.

THEOREM II. *If the signs of the alternate combinations of n things be changed, the algebraic sum will be zero.*

In algebraic language,

$$C_0^n - C_1^n + C_2^n - C_3^n + \text{etc.} \pm C_n^n = 0. \qquad (a)$$

Proof. If in the formula of § 261, Ex. 2, namely,

$$C_s^{n+1} = C_s^n + C_{s-1}^n,$$

we put $n - 1$ for n, it becomes

$$C_s^n = C_s^{n-1} + C_{s-1}^{n-1}.$$

Putting s successively equal to 0, 1, 2, ... n, we have

$$\begin{aligned}
C_0^n &= C_n^n = 1;\\
C_1^n &= C_0^{n-1} + C_1^{n-1} = 1 + C_1^{n-1};\\
C_2^n &= C_1^{n-1} + C_2^{n-1};\\
C_3^n &= C_2^{n-1} + C_3^{n-1};\\
&\vdots\\
C_{n-1}^n &= C_{n-2}^{n-1} + C_{n-1}^{n-1} = C_{n-2}^{n-1} + 1.
\end{aligned}$$

Substituting these values in the expression (a), it becomes

$$1 - (1 + C_2^{n-1}) + (C_1^{n-1} + C_2^{n-1}) - (C_2^{n-1} + C_3^{n-1}) + \ldots.$$
$$= 1 - 1 - C_1^{n-1} + C_1^{n-1} + C_2^{n-1} - C_2^{n-1} - C_3^{n-1} + \text{etc.}$$

How far soever we carry this process, all the terms cancel each other except the last. Therefore, if we continue the additions and subtractions until we come to C_{n-1}^n, the sum will be

$$C_0^n - C_1^n + C_1^n - \text{etc.} \ldots . \pm C_{n-1}^n = \pm C_{n-1}^{n-1} = \pm 1.$$

The last term will be $\mp C_n^n = \mp 1$, and will therefore just cancel the sum of the preceding terms.

NOTE. Theorem I may be demonstrated by these same formulæ, since the sum of all the terms taken positively will be duplicated every time we increase n by 1.

263. *Independent Combinations.* There is a system of combinations formed in the following way:

It is required to form a combination of s things, by taking one out of each of s different collections. How many combinations can be formed?

Let the	1st	collection	contain	a	things,
"	2d	"	"	b	"
"	3d	"	"	c	"
	etc.			etc.	

Then we may take any one of a things from the first collection.

With each of these we may combine any one of the b things in the second collection.

With each of these we may combine any one of the c things of the third collection.

Continuing the reasoning, we see that the total number of combinations is the continued product

$$abc \ldots \text{ to } s \text{ factors.}$$

If the number in each collection is equal, and we call it a, the number of combinations will be a^s.

This form of combinations is that which corresponds most nearly to the events of life, and is applicable to many questions concerning probabilities. For example, if any one of five different events might occur to a person every day, the number of different ways in which his history during a year might turn out is 5^{365}, a number so enormous that 255 digits would be required to express it.

EXERCISES.

1. A man driving a span of horses can choose one from a stud of 10 horses, and the other from a stud of 12. How many different span can he form?

2. It is said that in a general examination of the public schools of a county, the pupils spelt the word *scholar* in 230 different ways. If in spelling they might replace

   ch by *c* or *k*;
   o by *au*, *aw*, or *oo*;
   l by *ll*;
   a by *e*, *o*, *u*, or *ou*;
   r by *re*;

   in how many different ways might the word be spelt?

3. If a coin is thrown n times in succession, in how many different ways may the throws turn out?

4. If there are three routes between each successive two of the five cities, Boston, New York, Philadelphia, Baltimore, Washington, by how many routes could we travel from Boston to Washington?

The Binomial Theorem when the Power is a Whole Number.

264. The binomial theorem (§ 172), when the power is a positive integer, can be demonstrated by the doctrine of combinations, as follows:

Let it first be required to form the product of the n binomial factors,

$$(a_1 + x_1)(a_2 + x_2)(a_3 + x_3) \dots (a_n + x_n). \qquad (a)$$

To understand the form of the product, let us first study the special case when $n = 3$. Performing the multiplication of the first three factors, the product will consist of eight terms:

$$\left.\begin{array}{r} a_1a_2a_3 + a_1a_2x_3 + a_1a_3x_2 + a_2a_3x_1 + a_1x_2x_3 \\ + a_2x_1x_3 + a_3x_1x_2 + x_1x_2x_3. \end{array}\right\} \qquad (a')$$

This product is the expression (a) developed when $n = 3$.

We conclude, by induction, that the entire product (a) when developed in this same way, will be composed of a sum of terms, each term being a product of several literal factors.

When (a) is thus multiplied out, we shall call the result the *developed expression.*

The developed expression has the following properties:

I. *Each term contains n literal factors, a's and x's, and no more.*

For, suppose $x_1 = a_1$, $x_2 = a_2$, to $x_n = a_n$. Then the expression (a) will reduce to

$$2^n a_1a_2a_3 \dots a_n, \qquad (b)$$

and the developed expression must assume the same value; that is, it must consist of terms each of which reduces to the expression

$$a_1a_2a_3 \dots a_n, \qquad (c)$$

when we change x into a. Now if it contained any term with either more or less than n factors, it could not assume this form.

II. *The factors of each term have all the n indices*

$$1,\ 2,\ 3,\ \ldots\ n.$$

For, the index figure of no term is altered by changing x into a, as in I. Hence, if in any term any index figure were missing or repeated, that term would not reduce to the form (c), whence there can be neither omission nor repetition of any index.

III. *Because each term has n factors, it must either have*

n factors a;

$n-1$ factors a and one factor x;

$n-2$ factors a and two factors x;

In general, a term may have the factor a repeated $n-i$ times, and x repeated i times.

IV. In a term which contains i factors x, these i factors must be affected with some combination of i indices out of the whole number $1, 2, 3, \ldots n$; and the $n-i$ a's must be affected by the complementary combination of $n-i$ indices. We next inquire whether there is a term corresponding to every such combination. Let

$$1,\ 3,\ 4,\ 7,\ \ldots$$

be any combination of i indices, and

$$2,\ 5,\ 6,\ 8,\ \ldots$$

the complementary combination of $n-i$ indices.

Since the developed expression must be true for all values of a and x, let us put in (a),

$$\begin{array}{ll} a_1 = 0, & x_2 = 0; \\ a_3 = 0, & x_5 = 0; \\ a_4 = 0, & x_6 = 0; \\ a_7 = 0, & x_8 = 0; \\ \text{etc.} & \text{etc.} \end{array} \qquad (d)$$

The product (a) will then reduce to the single term,

$$x_1 a_2 x_3 x_4 a_5 a_6 x_7 a_8 \ldots \qquad (e)$$

By the same change the developed expression must reduce to this same value, and it cannot do this unless the expression (e) is one of its terms.

Hence the developed expression must contain a term corresponding to every combination.

V. Since every combination of i figures out of $1, 2, 3, \ldots . n$ will, in this way, give rise to a term like (e), containing the symbol a i times, and the symbol x $n-i$ times, there will be C_i^n such terms.

Now suppose $a_1 = a_2 = a_3 = \ldots . a_n = a.$

$$x_1 = x_2 = x_3 = \ldots . x_n = x.$$

The expression (a) will then reduce to $(a+x)^n$.

In the developed expression, all the C_i^n terms containing x i times and a $n-i$ times will now be equal and their sum will reduce to $C_i^n a^{n-i} x^i$.

Hence, putting in succession $i=0$, $i=1$, etc., to $i=n$, we shall have

$$(a+x)^n = a^n + C_1^n a^{n-1} x + C_2^n a^{n-2} x^2 + \ldots . + C_{n-1}^n a x^{n-1} + x^n.$$

Substituting for C_i^n its value, we shall have

$$(a+x)^n = a^n + na^{n-1}x + \binom{n}{2} a^{n-2}x^2 + \ldots . + \binom{n}{n-1} ax^{n-1} + \binom{n}{n} x^n,$$

which is the *Binomial Theorem*, enunciated, but not demonstrated, in Book V, Chapter I.

NOTE. If the student has any difficulty in understanding the steps of the preceding demonstration, he should suppose $n=3$, and refer the demonstration to the developed expression (a').

CHAPTER III.

THEORY OF PROBABILITIES.

265. *Def.* The **Theory of Probabilities** treats of the chances of the occurrence of events which cannot be foreseen with certainty.

Notation. Let a bag contain 4 balls, of which 1 is white and 3 black. If a ball be drawn at random from the bag, we should, in ordinary language, say that the chances were 1 to 3 in favor of the ball being white, or 3 to 1 in favor of its being black.

In the language of probabilities we say that the probability of a white ball is $\frac{1}{4}$, and that of a black one $\frac{3}{4}$.

In general, if there are m chances in favor of an event, and n chances against it, its probability is $\frac{m}{m+n}$. Hence,

Def. The **Probability** of an event is the ratio of the chances which favor it to the whole number of chances for and against it.

Illustrations. If an event is certain, its probability is 1.

If the chances for and against an event are even, its probability is $\frac{1}{2}$.

If an event is impossible, its probability is 0.

Cor. 1. If the probability that an event will occur is p, the probability that it will fail is $1-p$.

Cor. 2. A probability is always a positive fraction, greater than 0 and less than 1.

266. *Method of Probabilities.* To find the probability of an event, we must be able to do two things:

1. *Enumerate all possible ways in which the event may occur or fail, it being supposed that these ways are all equally probable.*

2. *Determine how many of these ways will lead to the event.*

If n be the total number of ways, and m the number which lead to the event, the probability required is $\frac{m}{n}$.

EXERCISES.

1. A die has 2 white and 4 black sides. What is the probability of throwing a white side?

2. A bag contains n balls numbered from 1 to n, the even numbers being white and the odd ones black. What is the probability of drawing a black ball when n is an odd number? What, when n is an even number?

3. A bag contains $3n+2$ balls, of which numbers 1, 4, 7, etc., are white; 2, 5, 8, etc., are red; 3, 6, 9, etc., are black. What are the respective probabilities of drawing a white, red, and black ball?

REM. In the last example the probabilities are all less than $\frac{1}{2}$; therefore, should one attempt to guess the color of the ball to be drawn, he would be more likely to be wrong than right, no matter what color he guessed. This exemplifies a lesson in practical judgment to be drawn from the theory of probabilities. If there are three or more possible results of any cause, it may happen that the best judgment would be more likely to be wrong than right in attempting to predict the result. Thus, if there are three presidential candidates with nearly equal chances, the chances would be against the election of any one that might be named.

Gamblers of the turf are nearly always found betting odds against every horse that may be entered for a race, though it is certain that one of them will win.

Hence, if a natural event may arise from a number of causes with nearly equal facility, it is unphilosophical to have any theory whatever of the cause, because the chances may be against the most probable cause being the true one.

Probabilities depending upon Combinations.

267. *Problem* I. Two coins are thrown. What are the respective probabilities that the result will be: Both heads? head and tail? both tails?

At first sight it might appear that the chances in favor of these three results were equal, and that therefore the probability of each was $\frac{1}{3}$. But this would be a mistake. To find the probabilities, we must combine the possible throws of the first coin (which call A) with the possible throws of the second (which call B), thus:

A, head;	B, head.
A, head;	B, tail.
A, tail;	B, head.
A, tail;	B, tail.

These combinations are all equally probable, and while there are only one each for both heads and both tails, there are two for head and tail. Hence the probabilities are $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$.

The sum of these three probabilities is 1, as it ought always to be when all possible results are considered.

Prob. 2. Five coins are thrown. What are the respective probabilities:

0 heads,	5 tails?
1 head,	4 tails?
2 heads,	3 tails?
etc.	etc.

Let the several coins be marked a, b, c, d, e. Coin a may be either head or tail, making two cases. Each of these two cases of coin a may be combined with either case of b (as in the last example), making 4 cases.

Each of these 4 cases may be combined with either case of coin c, making 8 cases.

Continuing the process, the total number of cases for five coins is $2^5 = 32$.

Of these 32 cases, only one gives no head and 5 tails.

There are 5 cases of 1 head, namely: a alone head, b alone head, etc., to e.

2 heads may be thrown by coins a, b; a, c, etc.; b, c; b, d, etc.; c, d, etc.; that is, by any combination of two letters out of the five, a, b, c, d, e. Hence the number of cases is

$$C_2^5 = 10.$$

In the same way the number of cases corresponding to 3, 4, and 5 heads are, respectively,

$$C_3^5 = 10, \qquad C_4^5 = 5, \qquad C_5^5 = 1.$$

Dividing by the whole number of cases, we find the respective probabilities to be

$$\frac{1}{32}; \qquad \frac{5}{32}, \ \frac{10}{32}; \qquad \frac{10}{32}, \ \frac{5}{32}, \ \frac{1}{32}.$$

The following general proposition is now to be proved by the student:

THEOREM. *If there are n coins, the probability of throwing s heads and $n - s$ tails is*

$$\frac{C_s^n}{2^n}.$$

From this result we may prove the theorem in combinations of § 262. If we suppose, in succession, $s = 0$, $s = 1$, $s = 2$, etc., to $s = n$, the respective probabilities of 0 head, 1 head, 2 heads, etc., will be

$$\frac{C_0^n}{2^n}, \ \frac{C_1^n}{2^n}, \ \frac{C_2^n}{2^n}, \ \text{etc., to} \ \frac{C_5^n}{2^n}.$$

Because the sum of all these probabilities must be unity, we find

$$C_0^n + C_1^n + C_2^n + \ldots . + C_n^n = 2^n.$$

Prob. 3. Two dice are thrown at backgammon. What are the respective probabilities of throwing 5 and 6 and two 6's?

If we call the dice a and b, any number from 1 to 6 on a may be combined with any number from 1 to 6 on b. Therefore, there are in all 36 possible combinations.

In order to throw two 6's, a must come 6 and b also. Therefore there is only one case for this result, so that its probability is $\frac{1}{36}$.

To bring 5 and 6, a may be 5 and b 6, or b 5 and a 6. So there are two cases leading to this result, and its probability is

$$\frac{2}{36} = \frac{1}{18}.$$

NOTE. That 5 and 6 are twice as probable as a double 6 may be clearly seen by supposing that the two dice are thrown in succession. If the first throw is either 5 or 6, there is a chance for the combination 5, 6, but there is no chance for a double 6 unless the first throw is 6.

Prob. 4. If three dice are thrown, what are the respective probabilities that the numbers will be:

1, 1, 1? 1, 1, 2? 1, 2, 3?

The solution of this case is left as an exercise for the student.

Prob. 5. From a bag containing 3 white and 2 black balls, 2 balls are drawn. What are the respective probabilities of

Both balls white?
1 white and 1 black?
Both black?

Since any 2 balls out of 5 may be drawn, the total number of cases is C^5_2.

Only one of these combinations consists of two white balls.

C^3_2 of the cases bring both balls black.

A white and black are formed by combining any one of the three white with any one of the two black.

The respective probabilities can now be deduced by the student.

EXERCISES.

1. It takes two keys to unlock a safe. They are on a bunch with two others. The clerk takes three keys at random from the bunch. What is the probability that he has both the safe keys?

2. A party of three persons, of whom two are brothers, seat themselves at random on a bench. What are the probabilities (*a*) that the brothers will sit together, (*b*) that they will have the third man between them?

3. If two dice are thrown at backgammon, what are the probabilities

(*a*) Of two aces?
(*b*) Of one ace and no more?

4. In order that a player at backgammon may strike a cer-

tain point, the sum of the numbers thrown must be 8. What are his chances of succeeding in one throw of his two dice?

5. A party of 13 persons sit at a round table. What is the probability that Mr. Taylor and Mr. Williams will be next to each other? (See § 253.)

6. An illiterate servant puts two works of 2 volumes each upon a shelf at random. What is the probability that both pair of companion volumes are together?

7. A gentleman having three pair of boots in a closet, sent a blind valet to bring him a pair. The valet took two boots at random. What are the chances that one was right and the other left? What is the probability that they were one pair?

8. If the volumes of a 3-volume book are placed at random on a shelf, what is the probability that they will be in regular order in either direction?

9. A man wants a particular span of horses from a stud of 8. His groom brings him 5 horses taken at random. What is the probability that both horses of the span are amongst them?

10. From a box containing 5 tickets, numbered 1 to 5, 3 are drawn at random. What is the probability that numbers 2 and 5 are both amongst them?

11. The same thing being supposed, what is the probability that the sum of the two numbers remaining in the box is 6?

12. Of two purses, one contains 5 eagles and another 10 dollar-pieces. If one of the purses is selected at random, and a coin taken from it, what is the probability that it is an eagle?

13. From a bag containing 3 white and 4 black balls 2 balls are drawn. What is the probability that they are of the same color?

14. The better of two chess players is twice as likely to win as to be beaten in any one game. What chance has his weaker opponent of winning 2 games in a match of 3?

15. From a bag containing m white and n black balls, two balls are drawn at random. What is the probability that one is white and the other black?

16. From a bag containing 1 white, 2 red, and 3 black balls, 3 balls are drawn. What is the probability that they are all of different colors?

17. If n coins are thrown, what is the chance that there will be one head and no more?

18. From a Congressional committee of 6 Republicans and 5 Democrats, a sub-committee of 3 is chosen by lot. What is the probability that it will be composed of two Republicans and one Democrat?

Compound Events.

268. THEOREM I. *The probability that two independent events will both happen is equal to the product of their separate probabilities.*

Proof. For the first event let there be m cases, of which p are favorable; and for the second n cases, of which q are favorable. Then, by definition, the respective probabilities will be $\frac{p}{m}$ and $\frac{q}{n}$.

When both events are tried, any one of the m cases may be combined with any one of the n cases, making in all $m \times n$ combinations of equal probability.

The combinations favorable to both events will be those only in which one of the p cases favorable to the first is combined with one of the q cases favorable to the second. The number of these combinations is $p \times q$.

Therefore the probability that both events will happen is

$$\frac{p \times q}{m \times n} = \frac{p}{m} \times \frac{q}{n},$$

which is the product of the individual probabilities.

If there are three events of which the probabilities are p, q, and r, and we wish to find the probability that all three will happen, we may by what precedes regard the concurring of the first two events as a single event, of which the probability is pq. Then the probability that the third event will also concur is the product of this probability into r, or

$$pqr.$$

Proceeding in the same way with 4, 5, 6, ... events, we reach the general

THEOREM II. *The probability that any number of independent events will all occur is equal to the continued product of their individual probabilities.*

REM. This theorem is of great practical use as a guide to our expectations. It teaches that if success in an enterprise requires the concurrence of a great number of favorable circumstances, the chances may be greatly against it, although each circumstance is more likely than not to occur.

This is illustrated by the following

EXAMPLE 1. A traveller on a journey by rail has 8 connections to make, in order that he may go through on time. There are two chances to one in favor of each connection. What is the probability of his keeping on time?

The probability of each connection being $\frac{2}{3}$, the probability of successfully making the first two connections will, by the preceding theorems, be $\left(\frac{2}{3}\right)^2$, the first three $\left(\frac{2}{3}\right)^3$, and all eight

$$\left(\frac{2}{3}\right)^8 = \frac{2^8}{3^8} = \frac{256}{6561} = \frac{1}{26}, \text{ nearly.}$$

Therefore there are 25 chances to 1 against his going through on time.

On the other hand, if, instead of any one accident being fatal to success, success can be prevented only by the concurrence of a series of accidents, the probability of failure may become very small.

EX. 2. A ship starts on a voyage. It is an even chance that she will encounter a heavy gale. The probability that she will not spring a leak in the gale is $\frac{9}{10}$. If a leak occurs, there is a probability of $\frac{9}{10}$ that the engine will be able to pump her out. If they fail, the probability is $\frac{3}{4}$ that the compartments will keep the ship afloat. If she sinks, it is an even chance that any one passenger will be saved by the boats What is the probability that any individual passenger will be lost at sea?

The probability that

the ship will meet a heavy gale is . . . . $\frac{1}{2}$

the ship will spring a leak in the gale is . . . . $\frac{1}{10}$

the engines cannot pump her out is . . . . $\frac{1}{10}$

the compartments cannot keep her afloat is . . . . . . $\frac{1}{4}$

the boats cannot save the passenger is . . . . $\frac{1}{2}$

The continued product of these probabilities is $\frac{1}{1600}$, which is the probability that the passenger will be lost.

269. The preceding theorem as enunciated supposes that the several events are *independent*, that is, that the probability of the occurrence of any one is not affected by the occurrence or non-occurrence of the others. To investigate what modification is required when the occurrence of one of the events alters the probability of another of the events, let us distinguish the two events as the *first* and *second.* We then reason thus:

Let the total number of equally possible cases be m, and let p of these cases favor the first event. Its probability will then be $\frac{p}{m}$.

It is certain that the events cannot both happen unless the first one happens. Hence the cases which favor both events can be found only among the p cases which favor the first. Let q of these p cases favor the second event. Then the probability of both events will be $\frac{q}{m}$.

In case the first event happens, one of the p cases which favor it must occur, and the probability of the second event will then be $\frac{q}{p}$. Then

$$\text{Probability of both events} = \frac{q}{m} = \frac{p}{m} \times \frac{q}{p}. \quad \text{Hence,}$$

THEOREM. *The probability that two events will both occur is equal to the probability of the first event multiplied by the probability of the second, in case the first occurs.*

By continuing the reasoning to more events, we reach the general

THEOREM. *The probability that a number of events will all occur is equal to the product*

$$\text{Prob. of first} \begin{cases} \times \text{ Prob. of second in case first occurs.} \\ \times \text{ Prob. of third in case first two occur.} \\ \times \text{ Prob. of fourth in case first three occur.} \\ \quad\text{etc.} \qquad \text{etc.} \qquad \text{etc.} \end{cases}$$

EXAMPLE. From a bag containing 2 white and 3 black balls, 2 balls are drawn. What are the probabilities (1) that both balls are white, (2) that both are black?

This problem has already been solved, but we are now to see how the answers may be reached by the last theorem. It is evident that we may suppose the two balls drawn out one after the other, and the probabilities of their being white or black will be the same as if both were drawn together.

I. *Both balls white.* The probability that the first ball drawn is white is $\frac{2}{5}$. If it really proves to be white, there will be left 1 white and 3 black balls. In this event, the probability that the second also will be white is $\frac{1}{4}$

Hence the probability that both are white is

$$\frac{2}{5} \times \frac{1}{4} = \frac{1}{10}.$$

II. *Both balls black.* Applying the same reasoning, we find for the probability of this case,

$$\frac{3}{5} \times \frac{1}{2} = \frac{3}{10}.$$

EXERCISES.

1. Two men embark in separate commercial enterprises. The odds in favor of one are 3 to 2; in favor of the other, 2 to 1. What are the probabilities (1) that both will succeed? (2) that both will fail?

2. The probability that a man will die within ten years is $\frac{1}{8}$, and that his wife will die is $\frac{1}{10}$. What are the respective probabilities that at the end of ten years,

(α) Both are living?
(β) Both are dead?
(γ) Husband living, but wife dead?
(δ) Husband dead, but wife living?

3. The probability that a certain door is locked is $\frac{2}{3}$. The key is on a bunch of 4. A man takes 2 of the four keys, and goes to the door. What are the chances that he will be able or unable to go through it?

4. Two bags contain each 4 black and 3 white balls. A person draws a ball at random from the first bag, and if it be white he puts it into the second bag, mixes the balls, and then draws a ball at random. What is the probability of drawing a white ball from each of the bags?

5. If a Senate consists of m Democrats and n Republicans, what is the probability that a committee of three will include 2 Democrats and 1 Republican?

6. A bag contains 2 white balls and 5 black ones. Six people, A, B, C, D, E, F, are allowed to go to the bag in alphabetical order and each take one ball out and keep it. The first one who draws a white ball is to receive a prize. What are their respective chances of winning?

NOTE. A's chance is easily calculated, because he has the draw from all 7 balls.

In order that B may win, A must first fail. Therefore, to find B's probability we find (1) the probability that A fails, (2) the probability that if A fails then B will win. We then take the product of these probabilities.

In order that C may gain the prize, (1) A must fail, (2) B must fail, (3) C himself must gain. So we find the successive probabilities of these occurrences.

Continuing to F, we find that he cannot win unless the 5 men before him all miss. He is then certain to gain, because only the two white balls would be left.

7. Two men have one throw each of a coin. X offers a prize if A throws head, and if he fails, but not otherwise, B may try for the prize. If both fail, X keeps the prize himself. What are the respective chances of the three men having the prize?

8. A and B are alternately to throw a coin until one of them throws a head and becomes the winner. If A has the first throw, what are their respective chances of winning?

9. A crowd of n men are allowed to throw in the same way for a prize, in alphabetical order, the game ceasing as soon as a head is thrown. What are the respective chances of the contestants?

10. Three men take turns in throwing a die, and he who first throws a 6 wins. What are their respective chances?

11. If 4 cards are drawn from a pack of 52, show that the probability that there will be one of each of the four suits is

$$\frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49}.$$

12. One purse contains 5 dimes and 1 dollar, and another contains 6 dimes. 5 pieces are taken from the first purse and put into the second, and after being mixed 5 are taken from the second and put into the first. Which purse is now most likely to contain the dollar?

13. Of two purses, one contains 4 eagles and 2 dollars, the other 4 eagles and 6 dollars. One being taken at random, and a coin drawn from it, what are the respective probabilities that it is an eagle or a dollar?

Cases of Unequal Probability.

270. *Def.* If two or more possible events are so related that only one of them can happen, they are called **Mutually Exclusive Events.**

THEOREM. *The probability that some one of several exclusive events, we care not which, will occur, is equal to the sum of their separate probabilities.*

Proof. Let there be m possible and equally probable cases in all; let p of these cases be favorable to one event, q to the second, r to the third, etc., so that $\frac{p}{m}$, $\frac{q}{m}$, $\frac{r}{m}$, are the respective probabilities.

Since only one of the events is possible, the p cases which favor one must be entirely different from the q cases which favor the second, and these cases $p+q$ must be entirely different from the r which favor the third, etc.

Hence there will be $p+q+r+$etc., cases which favor some one or another of the events. Hence the probability that some one of these events will occur is

$$\frac{p+q+r+\text{etc.}}{m},$$

which is equal to the sum of the probabilities,

$$\frac{p}{m}+\frac{q}{m}+\frac{r}{m}+\text{etc.}$$

REM. If the concurrence of some two events, say the first and second, had been possible, some one or more of the p cases which favor the first would have been found among the q cases which favor the second. Then the whole number of cases which favored either event would have been less than $p+q$, and the probability that one of the two events would happen less than the sum of their respective probabilities.

271. GENERAL PROBLEM. *To find the probability that an event of which the probability on any one trial is p, will happen exactly s times in n trials.*

This problem is at the basis of some of the widest applications of the theory of probability to practical questions, especially those associated with life and fire insurance. The conditions which it implies are therefore to be fully comprehended.

We may conceive a *trial* to mean *giving the event an opportunity to happen.* The simplest kind of trial is that of throwing a coin or die. At each throw, any side has an opportunity to come up. Then, if we throw 50 pieces, or which amounts to the same thing, throw the same piece 50 times, there will be 50 trials; and we may inquire into the probability that a given side will be thrown exactly 9 times in these trials.

The same conception occurs in another form if we have 50 men, each of whom has an equal chance of dying within 5 years. Waiting to see if any one man will die in the course of the 5 years is a *trial,* so that there are 50 trials in all, and we may inquire into the probability that 9 of the men will die during the trials, just as in the case of 50 throws of a die.

Let us distinguish the several trials by the letters

$$a, b, c, d, e, \ldots . n,$$

which must be n in number.

1. In order that the event may not happen at all, it must fail on every one of the n trials. The probability of this (§ 268, Th. II) is $(1-p)^n$ This is therefore the probability that it will not happen at all.

Because the probability of the event happening on any one trial is p, the probability of its failing is $1-p$. We now compare the possible results.

2. The event may happen once on any one of the n trials, a, b, c, etc. In order that it may happen only once, it must fail on the other $n-1$ trials. The probability that it will happen on any one trial, say e, and also fail on the remaining $n-1$ trials is, by the same theorem,

$$p\,(1-p)^{n-1}.$$

Because there are n trials on which it may equally happen, the probability that it will happen once and only once is

$$np\,(1-p)^{n-1}.$$

3. The event may happen twice on any two trials out of the n trials. In order that it may happen twice only, it must fail on the other $n-2$ trials. Taking any one combination, say

Happen on b, d;

Fail on $a, c, e, \dots n$,

the probability is $$p^2(1-p)^{n-2}.$$

But it may happen twice on any combination of two trials out of the n trials, $a, b, c, \dots n$. Because these combinations are mutually exclusive (§ 270), the total probability of happening twice is

$$C_2^n p^2 (1-p)^{n-2}.$$

4. In general, in order that the event may happen just s times, it must happen on some combination of s trials, and fail on the complementary combination of $n-s$ trials. The probability o[illegible] one combination is $p^s(1-p)^{n-s}$ and there are C_s^n such combinations. Hence the general probability of happening s times is

$$C_s^n p^s (1-p)^{n-s}. \qquad (a)$$

If there is on each trial an equal chance for and against the event, then $p = \frac{1}{2}$ and $1-p = \frac{1}{2}$. The probability of the event happening s times then becomes

$$\frac{C_s^n}{2^n}.$$

This case corresponds to that already treated in § 267, Problem 2, and the result is the same there found.

EXERCISES.

1. A die having two sides white and four sides black is thrown 5 times. What are the respective probabilities of a white side being thrown 1, 2, 3, 4, and 5 times?

NOTE. Here p, the probability of a white side on one throw, is $\frac{1}{3}$, and $1-p=\frac{2}{3}$. The number n of trials is 5.

2. Of 6 healthy men aged 50, the probability that any one will live to 80 is $\frac{1}{4}$. What is the probability that three or more of them will live to that age?

3. A chess-player whose chances of winning any one game from his opponent are as 2 to 1, undertakes to win 3 games out of 4. What is the probability that he will be able to do it?

NOTE. It would be a fallacy to suppose that the probability required is that of winning exactly 3 games, because he will equally win if he wins all four games.

272. *Events of Maximum Probability.* Returning to the general expression (a), let us inquire what number of times the event is most likely to occur on n trials. The required number is that value of s for which the probability

$$C^n_s p^s (1-p)^{n-s}$$

is the greatest.

If we call P_s the probability that the event will happen exactly s times, and if s is to be the number for which the probability is greatest, we must have

$$P_s > P_{s-1},$$
$$P_s > P_{s+1}.$$

Substituting for these quantities the corresponding forms of the expression (a), which is equal to P_s, we have

$$\left.\begin{aligned} C^n_s p^s (1-p)^{n-s} &> C^n_{s-1} p^{s-1} (1-p)^{n-s+1}, \\ C^n_s p^s (1-p)^{n-s} &> C^n_{s+1} p^{s+1} (1-p)^{n-s-1}, \end{aligned}\right\} \qquad (b)$$

The general formula for C^n_s in § 257 gives

$$\left.\begin{aligned} C^n_s &= \frac{n-s+1}{s} C^n_{s-1}, \\ C^n_{s+1} &= \frac{n-s}{s+1} C^n_s. \end{aligned}\right\} \qquad (c)$$

Hence we have, by dividing both terms of the first inequality (b) by $C^n_{s-1} p^{s-1} (1-p)^{n-s}$,

$$\frac{n-s+1}{s} p > 1-p.$$

Multiplying by s, this becomes

$$np - sp + p > s - sp.$$

Interchanging the members and reducing, we have

$$s < p\,(n + 1). \qquad (d)$$

Now divide the second inequality (b) by $C^n_s\, p^s\, (1 - p)^{n-s-1}$, and reducing by the second equation (c), we have

$$1 - p > \frac{n - s}{s + 1}\, p.$$

Multiplying by $s + 1$ and reducing, we find

$$s > p\,(n + 1) - 1. \qquad (e)$$

Comparing the inequalities (d) and (e), we see that s lies between the two quantities $p\,(n + 1)$ and $p\,(n + 1) - 1$; that is,

s is the greatest whole number in $p\,(n + 1)$.

If the number of trials n is a large number, and p is a small fraction, $p\,(n + 1)$ and pn will differ only by the fraction p. We shall then have, very nearly,

$$s = pn.$$

That is:

THEOREM I. *The most probable number of times that an event will happen on a great number of trials is the product of the number of trials by the probability on each trial.*

EXAMPLE. If a life insurance company has 6000 members, and the probability that each member will live one year is on the average $\frac{1}{60}$, then the most probable number of deaths during the year is 100.

REM. It must not be supposed that in this case the number of deaths is likely to be exactly 100, but only that they will fall somewhere near it.

There is a practical rule for determining what deviation must be guarded against, the demonstration of which requires more advanced mathematical methods than those employed in this chapter. It is:

THEOREM II. *Deviations from the most probable number of deaths, equal to the square root of that number, will be of frequent occurrence.*

Deviations much greater than this square root will be of infrequent occurrence, and deviations more than twice as great will be rare.

EXAMPLES. In a company of which the probable annual number of deaths is 10, the actual number will commonly fall between $10 - \sqrt{10}$ and $10 + \sqrt{10}$, or between 7 and 13. It will very rarely happen that the number of deaths is as small as 4 or as large as 16.

If the company is so large that the most probable number of deaths is 100, the actual number will commonly fall between $100 - \sqrt{100}$ and $100 + \sqrt{100}$, or between 90 and 110.

If the most probable number of deaths is 1000, the actual number will commonly range between 968 and 1032.

We now see the following result of this theorem:

The greater the number of deaths to be expected, the greater will be the probable deviation, but the less will be the ratio of this deviation to the whole number of deaths.

EXAMPLES. The reductions of the cases just cited are shown as follows:

Expected number of deaths.	Probable deviation.	Ratio of deviation to expected number.
10	3	0.33
100	10	0.10
1000	32	0.03

Application to Life Insurance.

273. At each age of human life there is a certain probability that a person will live one year. This probability diminishes as the person advances in age.

It is learned from observation, on the principle described in the preceding section, that events in a vast number of trials are likely to happen a number of times equal to the product of their probability on each trial, multiplied by the number of trials.

Therefore, by dividing the whole number of times the event has happened by the whole number of trials, the quotient is the most probable value of the probability on one trial.

EXAMPLE. If we take 50,000 people at the age of 25, and record how many of them are alive at the end of one year, this is making 50,000 trials whether a person of that age will live one year.

If 49,650 of them are alive at the end of the year, and 350 are dead, we would conclude:

Probability of living one year, ... 0.993
Probability of dying within the year, . . 0.007

The probability for all ages may be determined by taking a great number of infants, say 100,000, and counting how many die in each year until all are dead. If n are living at the age y, and n' at the age $y + 1$, then the probability of dying within one year after the age y will be $\frac{n - n'}{n}$, and that of living will be $\frac{n'}{n}$.

It is not, however, necessary to wait through a lifetime to reach this conclusion. It is sufficient to find from observation what proportion of the people of each age die during any one year. Suppose, for instance, that the census of a city is taken, and it is found that there are 2500 persons aged 30, and 2000 aged 50. At the end of a year another inquiry is made to ascertain how many are dead. It is found that 20 of the 30 year old people, and 30 of the 50 year old people have died. This would show:

At age 30, probability of dying within 1 year = 0.008.
" 50, " " " " = 0.015.

This same probability being obtained for every year of life, the probability of living 1 year at all ages would be known. Then a table of mortality could be formed.

A table of mortality starts out with any arbitrary number of people, generally 100,000, at a certain age, frequently 10 years. It then shows how many of these people will be living at the end of each subsequent year until all are dead. The following is a specimen of such a table.

Table of Mortality.

Ages.	Living.	Dying.	Prob. of surviving a year.	Prob. of dying within the year.	Ages.	Living.	Dying.	Prob. of surviving a year.	Prob of dying within the year.
10	100000	442	.99558	.00442	60	58373	1677	.97127	.02872
11	99558	407	.99591	.00408	61	56696	1760	.96895	.03104
12	99151	385	.99611	.00388	62	54936	1849	.96634	.03365
13	98766	376	.99619	.00380	63	53087	1936	.96353	.03646
14	98390	379	.99614	.00385	64	51151	2014	.96062	.03937
15	98011	396	.99595	.00404	65	49137	2080	.95766	.04233
16	97615	426	.99563	.00436	66	47057	2138	.95456	.04543
17	97189	469	.99517	.00482	67	44919	2186	.95133	.04866
18	96720	525	.99457	.00542	68	42733	2224	.94795	.05204
19	96195	581	.99396	.00603	69	40509	2268	.94401	.05598
20	95614	621	.99350	.00649	70	38241	2331	.93904	.06095
21	94993	645	.99321	.00679	71	35910	2401	.93313	.06686
22	94348	653	.99307	.00692	72	33509	2469	.92631	.07368
23	93695	651	.99305	.00694	73	31040	2531	.91846	.08154
24	93044	647	.99304	.00695	74	28509	2567	.90995	.09004
25	92397	647	.99299	.00700	75	25942	2542	.90201	.09798
26	91750	651	.99290	.00709	76	23400	2476	.89418	.10581
27	91099	668	.99266	.00733	77	20924	2369	.88678	.11321
28	90431	686	.99241	.00753	78	18555	2247	.87890	.12109
29	89745	703	.99216	.00783	79	16308	2110	.87061	.12938
30	89042	718	.99193	.00806	80	14198	1969	.86131	.13868
31	88324	726	.99178	.00821	81	12229	1823	.85092	.14907
32	87598	733	.99163	.00836	82	10406	1672	.83932	.16067
33	86865	743	.99144	.00855	83	8734	1522	.82573	.17426
34	86122	754	.99124	.00875	84	7212	1360	.81142	.18857
35	85368	768	.99100	.00899	85	5852	1186	.79733	.20266
36	84600	789	.99067	.00932	86	4666	1014	.78268	.21731
37	83811	811	.99032	.00967	87	3652	849	.76752	.23247
38	83000	830	.99000	.01000	88	2803	689	.75419	.24580
39	82170	844	.98972	.01027	89	2114	548	.74077	.25922
40	81326	854	.98949	.01050	90	1566	435	.72222	.27777
41	80472	860	.98931	.01068	91	1131	336	.70291	.29708
42	79612	869	.98908	.01091	92	795	247	.68930	.31069
43	78743	888	.98872	.01127	93	548	181	.66970	.33029
44	77855	913	.98827	.01172	94	367	131	.64305	.35694
45	76942	948	.98767	.01232	95	236	86	.63559	.36440
46	75994	989	.98698	.01301	96	150	56	.62666	.37333
47	75005	1029	.98628	.01371	97	94	44	.53191	.46808
48	73976	1067	.98557	.01442	98	50	33	.34000	.66000
49	72909	1102	.98488	.01511	99	17	11	⅓	⅔
50	71807	1133	.98422	.01577	100	6	4	⅓	⅔
51	70674	1167	.98348	.01651	101	2	2	.....	.....
52	69507	1204	.98267	.01732	102	0	....	.....	.....
53	68303	1251	.98168	.01831					
54	67052	1304	.98055	.01944					
55	65748	1358	.97934	.02065					
56	64390	1414	.97804	.02195					
57	62976	1471	.97664	.02335					
58	61505	1531	.97510	.02489					
59	59974	1601	.97330	.02669					

NOTE. The above table is that of the English Institute of Actuaries, prepared between 1862 and 1869, from the continued experience of twenty leading life insurance companies.

Problem. To find the probability that a person of age a will live to age y.

Solution. We take from the table the number living at age y, and divide it by the number living at age a. The quotient is the probability.

274. The principle on which the value of a contingent payment is determined is the following:

Theorem. *The value of a probable payment is equal to the sum to be paid, multiplied by the probability that it will be paid.*

Proof. Let there be n men, for each of whom there is a probability p that he will receive the sum s. Then by § 272, Th. I, pn of the men will probably receive the payment, so that the total sum which all will receive will probably be pns. Now, before they know who is to get the money, the value of each one's share is equal. Therefore, to find this value, we divide the whole amount to be received, namely, pns, by the number of men, n. This gives ps as the value of each one's chance, which proves the theorem.

Note. In this proof it is tacitly supposed that the pns dollars are as valuable divided among the pn men as divided among all n men. But this, though supposed in mathematical theory, is not morally true. Morally, the money will do more good when divided among all the men than when divided among a portion selected by chance. All gambling, whether by lotteries or games of chance, is in its total effects upon the pecuniary interests of all parties a source of positive disadvantage. This disadvantage is treated mathematically by more advanced methods in special treatises.

EXERCISES.

1. Find from the table the probabilities that a person

a.	Aged	30	will	live to	70.
b.	"	30	"	"	80.
c.	"	50	"	"	60.
d.	"	60	"	"	70.

e.	Aged	70	will	live to	80.
f.	"	80	"	"	90.
g.	"	90	"	"	95.
h.	"	95	"	"	100.

2. What age is that at which it is an even chance whether a person aged 40 will be living or dead?

3. Show that the probability that a person aged 30 will live to 70 is equal to the product of the probability that he will live to 60 multiplied by the probability that a man aged 60 will live to 70. (Apply the theorem of § 269.)

4. What premium ought a man of 65 to pay for insuring his life for $7000 for 1 year?

5. Ten young men of 25 form a club. What is the probability that it will be unbroken by death for ten years?

6. The probability that a planing mill will burn down within any one year is $\frac{1}{3}$. What ought an insurance company to charge to insure it to the amount of $3000 for 1 year, for 2 years, for 3 years, and for 4 years, respectively?

7. If the probability that a house will burn down in any one year is p, what ought to be the premium for insuring it for s years to the amount of a dollars?

NOTE. In cases like the last two, it is assumed that only one loss will be paid for.

8. What is the probability that if a man aged 25 marry a wife of 20, they will live to celebrate their golden wedding?

9. A company insures the joint lives of a husband aged 70 and a wife aged 50 for $5000 for 5 years, the stipulation being that if either of them die within that time the other shall be paid the money. What ought to be the premium, no allowance being made for interest?

10. A man aged 50 insures the life of his wife, aged 35, for $10,000 for 20 years, with the promise that the money is not to be paid unless he himself lives to the age of 70. What ought to be the premium?

NOTE. In computations relating to the management of life insurance, it is always necessary to allow compound interest on all payments. But the above exercises are intended only to illustrate the application of the theory of probabilities to the subject, and therefore no allowance for interest is expected to be made in the answers.

BOOK XI.

OF SERIES AND THE DOCTRINE OF LIMITS.

CHAPTER I.

NATURE OF A SERIES.

275. *Def.* A **Series** is a succession of terms following each other according to some general law.

Examples. An arithmetical progression is a series determined by the law that each term shall be greater than the preceding one by the same amount.

A geometrical progression is a series subject to the law that the ratio of every two consecutive terms is the same.

These two progressions are the simplest form of series.

A series may terminate at some term, or it may continue indefinitely.

Def. A series which continues indefinitely is called an **Infinite Series.**

Def. The **Sum** of a series is the algebraic sum of all its terms. Hence the sum of an infinite series will consist of the sum of an infinite number of terms.

276. The law of a series is generally such that the n^{th} term may be expressed as a function of n.

For example, in the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{etc.},$$

the n^{th} term is $$\frac{1}{n+1}.$$

In the series $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \text{etc.},$

the n^{th} term is $\frac{1}{n\,(n+1)}$.

Def. The expression for the n^{th} term of a series as a function of n is called the **General Term** of the series.

EXERCISES.

Express the n^{th} term of each of the following series:

1. $\frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \text{etc.}$

2. $1\cdot 2 + 3\cdot 4 + 5\cdot 6 + \text{etc.}$

3. $1 + \frac{x}{1\cdot 2} + \frac{x^2}{1\cdot 2\cdot 3} + \text{etc.}$

4. $\frac{a}{2\cdot 2} + \frac{a^2}{3\cdot 2^2} + \frac{a^4}{4\cdot 2^3} + \frac{a^8}{5\cdot 2^4} + \text{etc.}$

Write four terms of each of the series having the following general terms:

5. The n^{th} term to be $\frac{4n^2 - 1}{4n^2 + 1}$.

6. The i^{th} term to be $i\,(i+1)\,(i+2)\,x^i$.

7. The $(n+1)^{st}$ term to be $\frac{(n+3)\,(n+4)\,x^{n+1}}{(n+5)\,(n+6)}$.

8. The $(n-1)^{st}$ term to be $\frac{n^3 - 1}{1\cdot 2\,\ldots\,n}$.

277. The most common use of a series is to enable us to compute, by approximation, the values of expressions which it is difficult or impossible to compute directly. Suppose, for example, that we have to compute the value of $\frac{1+x}{1-x}$ when x is a small fraction, say $\frac{1}{50}$, and to have the result accurate to eight decimals. We shall see hereafter that when x is less than 1, we have

$$\frac{1+x}{1-x} = 1 + 2x + 2x^2 + 2x^3 + \text{etc.}, \textit{ ad infinitum.}$$

Suppose $x = \frac{1}{50} = .02$. We compute this series thus:

	1
$2 \times .02 =$	.04
Multiplying by .02,	.0008
" "	.000016
" "	.00000032
Sum $= \dfrac{1.02}{.098} =$	1.04081632

which is much more expeditious than dividing 1 02 by .98.

It will be seen that every term we add makes the quotient accurate to one or two more decimals, so that there is no limit to the precision which may be attained by the use of the series.

If, however, x had been greater than unity, the series would give no result, because the terms $2x$, $2x^2$, $2x^3$, would have gone on increasing indefinitely, whereas the true value of the fraction $\frac{1+x}{1-x}$ would have been negative.

This example illustrates the following two cases of series:

I. *There may be a certain limit to which the sum of the series shall approach, as we increase the number of terms, but which it can never reach, how great soever the number of terms added.*

For example, the series we have just tried,

$$1 + \frac{2}{50} + \frac{2}{50^2} + \frac{2}{50^3} + \frac{2}{50^4} + \text{etc.},$$

approaches the limit $\frac{1.02}{0.98}$, but never absolutely reaches it.

II. *As we increase the number of terms, the sum may increase without limit, or may vibrate back and forth in consequence of some terms being positive and others negative.*

These two classes of series are distinguished as *convergent* and *divergent*.

Def. A **Convergent Series** is one of which the sum approaches a limit as the number of terms is increased.

Refer to § 213 for an example of infinite series in geometrical progressions which have limits.

Def. A **Divergent Series** is one of which the sum does not approach a limit.

EXAMPLES. The series $1+2+3+4+$etc., *ad infinitum*, is divergent, because there is no limit to the sum of its terms.

The series $1-1+1-1+1-$ etc., is divergent, because its sum continually fluctuates between $+1$ and 0.

REM. When we consider only a limited number of terms, the question of convergence or divergence is not important. But when the sum of the whole series to infinity is to be considered, only convergent series can be used.

Notation of Sums.

274. The sum of a series of terms represented by common symbols may be expressed by the symbol Σ, followed by one of the terms.

EXAMPLE. The expression

$$\Sigma a$$

means "the sum of several terms, each represented by a."

When it is necessary to distinguish the different terms, different accents or indices are affixed to them, and represented by some common symbol.

EXAMPLE. The expression

$$\Sigma a_i$$

means the sum of several terms represented by the symbol a with indices attached; that is, the sum of several of the quantities a_1, a_2, a_3, a_4, etc.

When the particular indices included in the summation are to be expressed, the greatest and least of them are written above and below the symbol Σ.

EXAMPLES. The expression

$$\sum_{i=5}^{i=15} a_i$$

means: "Sum of all the symbols a_i formed by giving i all integral values from $i = 5$ to $i = 15$." That is,

$$\sum_{i=5}^{i=15} a_i = a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15}.$$

$$\sum_{i=0}^{i=5} im \text{ means } 0 + m + 2m + 3m + 4m + 5m.$$

$$\sum_{i=1}^{i=4} (i, j) \text{ means } (1, j) + (2, j) + (3, j) + (4, j).$$

$$\sum_{j=2}^{j=6} (i, j) = (i, 2) + (i, 3) + (i, 4) + (i, 5) + (i, 6).$$

$$\sum_{n=1}^{n=4} n! = 1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33.$$

$$\sum_{i=7}^{i=11} i = 7 + 8 + 9 + 10 + 11 = 45.$$

$$\sum_{i=2}^{i=5} i^2 = 2^2 + 3^2 + 4^2 + 5^2 = 54.$$

EXERCISES.

Write out the following summations, and compute their values when they are purely numerical:

1. $\sum_{j=1}^{j=7} j^2$.
2. $\sum_{n=1}^{n=6} n(n-1)$.
3. $\sum_{n=1}^{n=6} n(n+1)$.
4. $\sum_{i=4}^{i=8} m_i$.
5. $\sum_{n=4}^{n=7} nk$.
6. $\sum_{n=0}^{n=6} (n+1)(j-1)$.
7. $\sum_{i=2}^{i=4} im_i$.
8. $\sum_{n=2}^{n=5} n^2m^2$.
9. $\sum_{n=0}^{n=5} \dfrac{n-1}{n+1}$.

Express the following sums by the sign Σ:

10. $h_0 + h_1 + h_2 + h_3 + h_4$.
11. $1^3 + 2^3 + 3^3 + 4^3$.
12. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5$.
13. $\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} + \dfrac{4}{5} + \dfrac{5}{6}$.

CHAPTER II.

DEVELOPMENT IN POWERS OF A VARIABLE.

279. Among the most common series employed in mathematics are those of which the terms are multiplied by the successive powers of some one quantity.

An example of such a series is

$$1 + 2z + 3z^2 + 4z^3 + 5z^4 + \text{etc.},$$

in which each coefficient is greater by unity than the power of z which it multiplies.

A geometrical progression, it will be remarked, is a seric of this kind, in which the terms contain the successive powers of the common ratio.

The general form of such a series is

$$a_0 + a_1z + a_2z^2 + a_3z^3 + \text{etc.},$$

in which the successive coefficients a_0, a_1, a_2, etc., are formed according to some law, but do not contain z.

Such a series as this is said to proceed according to the ascending powers of the variable z.

Rem. The sum of a series is often equal to some algebraic expression containing the variable. Conversely, we may find a series the sum of all the terms of which shall be equal to a given expression.

Def. A series equal to a given expression is called the **Development** of that expression.

To **Develop** an expression means to find a series the sum of all the terms of which are equal to the expression.

The most extensively used method of development is that of indeterminate coefficients.

Method of Indeterminate Coefficients.

280. The method of indeterminate coefficients is based upon the following principles:

Let us have two equal expressions, each containing a variable z, and one or both containing also certain *indeterminate quantities*, that is, quantities introduced hypothetically, and not given by the original problem, the values of which are to be subsequently assigned so as to fulfil a certain condition.

The condition to be fulfilled by the values of the indeterminate quantities is that the two expressions containing z and these quantities shall be made identically equal.

Then, because the equations are to be identically equal, we can assign any values we please to z, and thus form as many equations as we please between the indeterminate quantities.

If these equations can be all satisfied by one set of values of these quantities, then by assigning these values to them in the original equation, the latter will be an identical one, as required.

The student should trace the above general method in the following examples of its application.

281. THEOREM I. *If a series proceeding according to the ascending powers of a quantity is equal to zero for all values of that quantity, the coefficient of each separate term must be zero.*

Proof. Let the several coefficients be a_0, a_1, a_2, etc., and z the quantity, so that the series, put equal to zero, is

$$a_0 + a_1z + a_2z^2 + a_3z^3 + \text{etc.} = 0.$$

Because the equation is true for all values of z, it must be true when $z = 0$. Putting $z = 0$, it becomes

$$a_0 = 0.$$

Dropping a_0, the equation becomes

$$a_1z + a_2z^2 + a_3z^3 + \text{etc.} = 0.$$

Dividing by z, $\quad a_1 + a_2z + a_3z^2 + \text{etc.} = 0.$

From this we derive, by a repetition of the same reasoning,

$$a_1 = 0.$$

Continuing the process, we find

$$a_2 = 0, \quad a_3 = 0, \quad \text{etc., indefinitely.}$$

THEOREM II. *If two series proceeding by ascending powers of a quantity are equal for all values of that quantity, the coefficients of the equal powers must be equal.*

Proof. Let the two equal series be

$$a_0 + a_1 z + a_2 z^2 + \text{etc.} = b_0 + b_1 z + b_2 z^2 + \text{etc.} \qquad (a)$$

Transposing the second member to the left-hand side and collecting the equal powers of z, the equation becomes

$$a_0 - b_0 + (a_1 - b_1)\, z + (a_2 - b_2)\, z^2 + \text{etc.} = 0.$$

Since this equation is to be satisfied for all values of z, the coefficients of the separate powers of z must all be zero.

Hence,

$$a_0 - b_0 = 0, \qquad a_1 - b_1 = 0, \qquad a_2 - b_2 = 0, \qquad \text{etc.}$$

or

$$a_0 = b_0, \qquad a_1 = b_1, \qquad a_2 = b_2, \qquad \text{etc.}$$

EXERCISE. Let the student demonstrate these last equations independently from (a), by supposing $z = 0$, then subtracting from both sides of (a) the quantities found to be equal; then dividing by z; then supposing $z = 0$, etc.

REM. The hypothesis that (a) is satisfied for all values of z is equivalent to the supposition that it is an *identical equation.* In general, when we find different expressions for the same functions of a variable quantity, these expressions ought to be identically equal, because they are expected to be true for all values of the variable.

THEOREM III. *A function of a variable can only be developed in a single way in ascending powers of the variable.*

For if we should have

$$Fz = A_0 + A_1 z + A_2 z^2 + A_3 z^3 + \text{etc.},$$

and also

$$Fz = B_0 + B_1 z + B_2 z^2 + B_3 z^3 + \text{etc.},$$

these two series, being each identically equal to Fz, must be identically equal to each other. But, by Th. II, this cannot be the case unless we have

$$A_0 = B_0, \quad A_1 = B_1, \quad A_2 = B_2, \quad \text{etc.}$$

The coefficients being equal, the two series are really one and the same.

282. *Expansion by Indeterminate Coefficients.* The above principle is applied to the development of functions in powers of the variable. The method of doing this will be best seen by an example.

1. Develop $\dfrac{1}{1+x}$ in powers of x.

Let us call the coefficients of the powers of x a_0, a_1, etc. The series will be known as soon as these coefficients are known. Let us then suppose

$$\frac{1}{1+x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \text{etc.}$$

Here we remark that, so far as we have shown, this equation is purely hypothetical. We have not proved that any such equation is possible, and the question whether it is possible must remain open for the present. We must find whether we can assign such values to the indeterminate coefficients, a_0, a_1, a_2, etc., that the equation shall be identically true.

Assuming the equation to be true, we multiply both sides by $1 + x$. It then becomes

$$1 = a_0 + (a_0 + a_1)x + (a_1 + a_2)x^2 + \text{etc.};$$

or transposing 1,

$$0 = a_0 - 1 + (a_0 + a_1)x + (a_1 + a_2)x^2 + (a_2 + a_3)x^3 + \text{etc.}$$

By Theorem I, the coefficients must be identically zero. Hence,

$$\begin{array}{llll}
a_0 - 1 = 0, & \text{which gives} & a_0 = 1; \\
a_1 + a_0 = 0, & \text{"} \quad \text{"} & a_1 = -a_0 = -1; \\
a_2 + a_1 = 0, & \text{"} \quad \text{"} & a_2 = -a_1 = 1; \\
a_3 + a_2 = 0, & \text{"} \quad \text{"} & a_3 = -a_2 = -1; \\
\quad \text{etc.} & & \quad \text{etc.}
\end{array}$$

Substituting these values of the coefficients in the original equation, it becomes

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \text{etc.}$$

This same method can be applied to the development of any rational fraction of which the terms are entire functions of some one quantity. Let us, for instance, suppose

$$\frac{a+bx}{m+nx+px^2} = A_0 + A_1x + A_2x^2 + \ldots\ldots + A_nx^n.$$

Multiplying by the denominator of the fraction, this equation gives

$$a + bx = mA_0 + (nA_0 + mA_1)x + (pA_0 + nA_1 + mA_2)x^2 \\ + (pA_1 + nA_2 + mA_3)x^3 + \text{etc.}$$

We now see that when $i > 1$, the coefficient of x^i in this equation is $mA_i + nA_{i-1} + pA_{i-2}$.

Equating the coefficients of like powers of x,

$$\begin{array}{rlrl} mA_0 = a, & \text{whence} & A_0 = & \frac{a}{m}; \\ mA_1 + nA_0 = b, & \text{"} & A_1 = & \frac{b}{m} - \frac{n}{m}A_0; \\ mA_2 + nA_1 + pA_0 = 0, & \text{"} & A_2 = & -\frac{p}{m}A_0 - \frac{n}{m}A_1; \\ mA_3 + nA_2 + pA_1 = 0, & \text{"} & A_3 = & -\frac{p}{m}A_1 - \frac{n}{m}A_2. \end{array}$$

We have from the general coefficient above written, when $i > 1$,

$$A_i = -\frac{n}{m}A_{i-1} - \frac{p}{m}A_{i-2}.$$

That is, *each coefficient after the second is the same linear function of the two coefficients next preceding.*

Such a series is called a **Recurring Series.**

EXERCISES.

Develop by indeterminate coefficients:

1. $\frac{1}{1-x}$.
2. $\frac{1}{1-2x}$.

3. $\dfrac{1-x}{1+x}$. 4. $\dfrac{1+x}{1-x}$.

5. $\dfrac{1+x}{1+2x+3x^2}$. 6. $\dfrac{1-x}{1-2x+x^2}$.

7. $\dfrac{1-2x+3x^2}{1+2x+3x^2}$. 8. $\dfrac{1-x}{1+x-x^3}$.

283. The development of a rational fraction may also be effected by division, after the manner of §§ 96, 97, the operation being carried forward to any extent.

EXAMPLE. Develop $\dfrac{1+x}{1-x}$.

$$
\begin{array}{l|l}
1+x & 1-x \\
\underline{1-x} & \overline{1+2x+2x^2+2x^3+\text{etc.}} \\
\quad 2x & \\
\quad \underline{2x-2x^2} & \\
\qquad 2x^2+0 & \\
\qquad \underline{2x^2-2x^3} & \\
\qquad\qquad 2x^3, \text{ etc.} &
\end{array}
$$

EXERCISES.

Develop by division the expressions:

1. $\dfrac{1-2x}{1+x}$. 2. $\dfrac{1+x}{1-x+x^2}$.

284. *Elimination by Undetermined Multipliers.* There is an application of the method of undetermined coefficients to the problem of eliminating unknown quantities, which merits special attention on account of its instructiveness. Let any system of simultaneous equations between three unknown quantities be

$$ax + by + cz = h, \tag{1}$$
$$a'x + b'y + c'z = h', \tag{2}$$
$$a''x + b''y + c''z = h''. \tag{3}$$

Can we find two such factors that, if we multiply two of the equations by them, and add the results to the third, two of the three unknown quantities shall be eliminated?

This question is answered in the following way:

If there be such factors, let us call them m and n. If we multiply the first equation by m, the second by n, and add the product to the third equation, we shall have

$$\left.\begin{array}{r}(am + a'n + a'')\,x\\ +\,(bm + b'n + b'')\,y\\ +\,(cm + c'n + c'')\,z\end{array}\right\} = hm + h'n + h''. \qquad (b)$$

In order that the quantities y and z may disappear from this equation, we must have

$$bm + b'n + b'' = 0,$$
$$cm + c'n + c'' = 0.$$

Since we have these two equations between the quantities m and n, we can determine their values.

Solving the equations, we find:

$$m = \frac{b'c'' - b''c'}{bc' - b'c},$$

$$n = \frac{b''c - bc''}{bc' - b'c}.$$

These are the required values of the multipliers. Substituting them in the equation (b), we find that the coefficients of y and z vanish, and that the equation becomes

$$\left[\frac{a\,(b'c'' - b''c') + a'\,(b''c - bc'')}{bc' - b'c} + a''\right]x$$
$$= \frac{h\,(b'c'' - b''c') + h'\,(b''c - bc'')}{bc' - b'c} + h''.$$

Clearing of denominators and dividing by the coefficient of x, we find

$$x = \frac{h\,(b'c'' - b''c') + h'\,(b''c - bc'') + h''\,(bc' - b'c)}{a\,(b'c'' - b''c') + a'\,(b''c - bc'') + a''\,(bc' - b'c)}.$$

EXERCISES.

1. Find the values of y and z by the above process for finding x.

For this purpose we may begin with the equation (b) and find values of m and n such that the coefficients of x and z in (b) shall vanish. These values will be different from those given in (c). By substituting them in (b), x and z will be eliminated, and we shall obtain the value of y.

We then find a third set of values of m and n, such that the coefficients of x and y shall vanish, and thus obtain the value of z.

2. Solve by the method of indeterminate multipliers the exercise 3 of § 140.

Multiplication of Two Infinite Series.

284*a*. Problem. To express the product of the two series

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \text{etc.},$$

and

$$b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \text{etc.}$$

The method is similar to that by which the square of an entire function is formed (§ 173, 2).

We readily find the first two terms of the product to be

$$a_0 b_0 + (a_0 b_1 + a_1 b_0)\, x.$$

The combinations which produce terms in x^2 are

$$a_0 b_2 x^2 + a_1 b_1 x^2 + a_2 b_0 x^2.$$

Those which produce terms in x^3 are

$$a_0 b_3 x^3 + a_1 b_2 x^3 + a_2 b_1 x^3 + a_3 b_0 x^3.$$

In general, to find the terms in x^n we begin by multiplying a_0 into the term $b_n x^n$ of the lower series, and then multiplying each succeeding of the first series by each preceding term of the second, until we end with $a_n b_0 x^n$. Hence, if we suppose

$$\text{Product} = A_0 + A_1 x + A_2 x^2 + \dots + A_n x^n + \text{etc.},$$

we shall have, for all values of n,

$$A_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_n b_0.$$

By giving n all integral values, we shall form as many values as we choose of A_n, and so as many terms as we choose of the series.

EXERCISES.

1. Form the product of the two series:

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \text{etc.},$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \text{etc.}$$

2. Form the square of each of these series.

3. Can you, by adding the squares together, show that their sum is equal to unity, whatever be the value of x?

To effect this, multiply each coefficient of x^n in the sum of the squares by $n!$, substitute for each term its value C_r^n given in § 257, and apply § 262, Th. II.

285. *Series proceeding according to the Powers of Two Variables.* Such a series is of the form

$$a_0 + b_0 x + a_1 y + c_0 x^2 + b_1 xy + a_2 y^2 + \text{etc.},$$

in which the products of all powers of x and y are combined. By collecting the coefficients of each power of x, the series will become

$$\begin{aligned} &a_0 + a_1 y + a_2 y^2 + a_3 y^3 + \ldots . \\ &+ (b_0 + b_1 y + b_2 y^2 + b_3 y^3 + \ldots .)\, x \\ &+ (c_0 + c_1 y + c_2 y^2 + c_3 y^3 + \ldots .)\, x^2 \\ &+ \text{etc.,} \quad \text{etc.,} \quad \text{etc.,} \quad \text{etc.} \end{aligned}$$

Hence, the series is one proceeding according to the powers of one variable, in which the coefficients are themselves series, proceeding according to the ascending powers of another variable.

Let us have the identically equal series proceeding according to the ascending powers of the same variables,

$$\begin{aligned} &A_0 + A_1 y + A_2 y^2 + \ldots . \\ &+ (B_0 + B_1 y + B_2 y^2 + \ldots .)\, x \\ &+ (C_0 + C_1 y + C_2 y^2 + \ldots .)\, x^2 \\ &+ \text{etc.,} \quad \text{etc.,} \quad \text{etc.} \end{aligned}$$

Since these series are to be equal for all values of x, the coefficients of like powers of x must be equal. Hence,

$$a_0 + a_1 y + a_2 y^2 + \text{etc.} = A_0 + A_1 y + A_2 y^2 + \text{etc.}$$
$$b_0 + b_1 y + b_2 y^2 + \text{etc.} = B_0 + B_1 y + B_2 y^2 + \text{etc.}$$
$$\text{etc.} \qquad\qquad\qquad \text{etc.}$$

Again, since these series are to be equal for all values of y, we must have

$$a_0 = A_0, \quad a_1 = A_1, \quad a_2 = A_2, \quad \text{etc.}$$
$$b_0 = B_0, \quad b_1 = B_1, \quad b_2 = B_2, \quad \text{etc.}$$
$$\text{etc.} \qquad \text{etc.} \qquad \text{etc.}$$

Hence, *in order that two series proceeding according to the ascending powers of two variables may be identically equal, the coefficients of every like product of the powers must be equal.*

CHAPTER III.

SUMMATION OF SERIES.

Of Figurate Numbers.

286. The numbers in the following columns are formed according to these rules:

1. The first column is composed of the natural numbers, 1, 2, 3, etc.

2. In every succeeding column each number is the sum of all the numbers above it in the column next preceding.

Thus, in the second column, the successive numbers are:

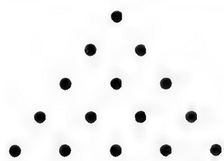
$$1, \quad 1+2=3, \quad 1+2+3=6, \quad 1+2+3+4=10, \text{ etc.}$$

In the third column we have

$$1, \quad 1+3=4, \quad 1+3+6=10, \quad \text{etc.}$$

1						
	1					
2		1				
	3		1			
3		4		1		
	6		5		1	
4		10		6		(A)
	10		15		7	
5		20		21		
	15		35			
6		35				
	21					
7	etc.		etc.		etc.	

It is evident from the mode of formation that each number is the difference of the two numbers next above and below it in the column next following.



$N = 1+2+3+4+5.$

The numbers 1, 3, 6, 10, etc., in the second column are called **triangular numbers**, because they represent numbers of points which can be regularly arranged over triangular surfaces.

The numbers 1, 4, 10, etc., in the third columns are called **pyramidal numbers,** because each one is composed of a sum of triangular numbers, which being arranged in layers over each other, will form a triangular pyramid.

All the numbers of the scheme are called **figurate numbers.**

The numbers in the i^{th} column are called figurate numbers of the i^{th} order.

287. If we suppose a column of 1's to the left of the first column, and take each line of numbers from left to right inclined upward, we shall have the successive lines 1, 1; 1, 2, 1; 1, 3, 3, 1, etc. These numbers are formed by addition in the same way as the binomial coefficients in § 171, 2. We may therefore conclude that all the numbers obtained by the preceding process are binomial coefficients, or combinatory expressions. This we shall now prove.

THEOREM. *The n^{th} number in the i^{th} column is equal to C_i^{n+i-1} or to*

$$\frac{n(n+1)(n+2)\ldots.(n+i-1)}{1\cdot 2\cdot 3\ldots.i}. \qquad (1)$$

Proof. Because the combinations of 1 in any number are equal to that number, we have, when $i = 1$,

$$n^{th} \text{ number in 1st column} = n = C_1^n,$$

which agrees with the theorem.

When $i = 2$, we have, by the law of formation of the numbers,

$$n^{th} \text{ number in 2d column} = C_1^1 + C_1^2 + C_1^3 + \ldots. + C_1^n,$$

which, by equation (a) (§ 260, 3), is equal to C_2^{n+1}.

Therefore the successive numbers in the second column, found by supposing $n = 1$, $n = 2$, etc., are

$$C_2^2,\ C_2^3,\ C_2^4,\ \ldots.\ C_2^{n+1}.$$

Since the n^{th} number in the third column is equal to the sum of all above it in the second, we have

$$n^{th} \text{ number in 3d column} = C_2^2 + C_2^3 + C_2^4 + C_2^{n+1} = C_3^{n+2},$$

which still corresponds to the theorem, because, when $i = 3$, $n + i - 1 = n + 2$.

To prove that the theorem is true as far as we choose to carry it, we must show that if it is true for any value of i, it is also true for a value 1 greater. Let us then suppose that, in the r^{th} column the first n numbers are

$$C_r^r,\ C_r^{r+1},\ C_r^{r+2},\ \ldots\ C_r^{r+n-1}.$$

Since the n^{th} number in the next column is the sum of these numbers, it will be equal to

$$C_{r+1}^{r+n},$$

which is the expression given by the theorem when we suppose $i = r + 1$.

Now we have proved the theorem true when $i = 3$; therefore (supposing $r = 3$) it is true for $i = 4$. Therefore (supposing $r = 4$) it is true for $i = 5$, and so on indefinitely.

If in the general expression (1) we put $i = 2$, we shall have the values of the triangular numbers; by putting $i = 3$, we shall have the pyramidal numbers, etc. Therefore,

$$\text{The } n^{th} \text{ triangular number} = \frac{n(n+1)}{1 \cdot 2}.$$

$$\text{The } n^{th} \text{ pyramidal number} = \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}.$$

By supposing $n = 1, 2, 3, 4$, etc., in succession, we find the succession of triangular numbers to be

$$\frac{1 \cdot 2}{1 \cdot 2},\quad \frac{2 \cdot 3}{1 \cdot 2},\quad \frac{4 \cdot 5}{1 \cdot 2},\quad \text{etc.};$$

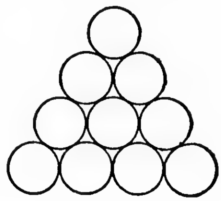
and the pyramidal numbers,

$$\frac{1 \cdot 2 \cdot 3}{1 \cdot 2 \cdot 3},\quad \frac{2 \cdot 3 \cdot 4}{1 \cdot 2 \cdot 3},\quad \frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3},\quad \text{etc.},$$

which we readily see correspond to the values in the scheme (A).

Enumeration of Triangular Piles of Shot.

288. An interesting application of the preceding theory is that of finding the number of cannon-shot in a pile. There are two cases in which a pile will contain a figurate number:



I. Elongated projectiles, in which each rests on two projectiles below it.

II. Spherical projectiles, each resting on three below it, and the whole forming a pyramid.

Case I. *Elongated Projectiles.* Here the vertex of a pile of one vertical layer will be formed of one shot, the next layer below of two, the third of three, etc. Hence the sum of n layers from the vertex down will be the n^{th} triangular number.

It is evident that the number of shot in the bottom row is equal to the number of rows. Hence, if n be this number, and N the entire number of shot in the pile, we shall have,

$$N = \frac{n(n+1)}{2}.$$

If the pile is incomplete, in consequence of all the layers above a certain one being absent, we first compute how many there would be if the pile were complete, and subtract the number in that part of the pile which is absent.

Example. The bottom layer has 25 shot, but there are only 11 layers in all. How many shot are there?

If the pile were complete, the number would be $\frac{25 \cdot 26}{2}$. There being 14 layers wanting from the top, the total number of shot wanting is $\frac{14 \cdot 15}{2}$. Hence the number in the pile is

$$N = \frac{25 \cdot 26 - 14 \cdot 15}{2} = \frac{(14+11)(15+11) - 14 \cdot 15}{2}$$

$$= \frac{11(14+15+11)}{2} = 220.$$

NOTE. This particular problem could have been solved more briefly by considering the number of shot in the several layers as an arithmetical progression, but we have preferred to apply a general method.

EXERCISES.

1. A pile of cylindrical shot has n in its bottom row, and r rows. How many shot are there?

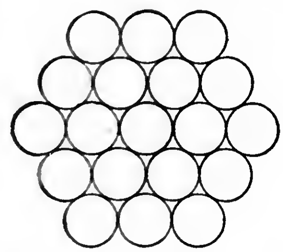
2. From a complete pile having h layers, s layers are removed. How many shot are left?

3. A pile has n shot in its bottom row, and m in its top row. How many rows and how many shot are there?

4. A pile has p rows and k shot in its top row. How many shot are there?

5. Explain the law of succession of even and odd numbers in the series of triangular numbers.

6. How many balls are necessary to fill a hexagon, having n balls in each side?



NOTE. In the adjoining figure, $n = 3$.

289. CASE II. *Pyramid of Balls.* If a course of balls be laid upon the ground so as to fill an equilateral triangle, having n balls on each side, a second course can be laid upon these having $n - 1$ balls on each side, and so on until we come to a single ball at the vertex.

Commencing at the top, the first course will consist of 1 ball, the next of 3, the third of 6, and so on through the triangular numbers. Because each pyramidal number is the sum of all the preceding triangular numbers, the whole number of balls in the n courses will be the n^{th} pyramidal number, or

$$N = \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}.$$

EXERCISES.

1. How many balls in a triangular pyramid having 9 balls on each side?

2. If from a triangular pyramid of n courses k courses be removed from the top, how many balls will be left?

3. How many balls in the frustum of a triangular pyramid having n balls on each side of the base and m on each side of the upper course?

Sum of the Similar Powers of an Arithmetical Progression.

290. Put a_1, the first term of the progression;
d, the common difference;
n, the number of terms;
m, the index of the power.

It is required to find an expression for the sum,

$$a_1^m + (a_1 + d)^m + (a_1 + 2d)^m + \dots + [a_1 + (n-1)d]^m,$$

which sum we call S_m.

Let us put, for brevity, $a_1, a_2, a_3, a_4, \dots a_n$ for the several terms of the progression. Then

$$\begin{aligned} a_2 &= a_1 + d, \\ a_3 &= a_1 + 2d \qquad = a_2 + d, \\ &\vdots \\ a_n &= a_1 + (n-1)d = a_{n-1} + d. \end{aligned}$$

Raising these equations to the $(m+1)^{th}$ power, and adding the equation $a_{n+1} = a_n + d$, we have

$$\begin{aligned} a_2^{m+1} &= a_1^{m+1} + (m+1)a_1^m d + \frac{(m+1)m}{1\cdot 2} a_1^{m-1}d^2 + \text{etc.} \\ a_3^{m+1} &= a_2^{m+1} + (m+1)a_2^m d + \frac{(m+1)m}{1\cdot 2} a_2^{m-1}d^2 + \text{etc.} \\ a_4^{m+1} &= a_3^{m+1} + (m+1)a_3^m d + \frac{(m+1)m}{1\cdot 2} a_3^{m-1}d^2 + \text{etc.} \\ &\vdots \\ a_{n+1}^{m+1} &= a_n^{m+1} + (m+1)a_n^m d + \frac{(m+1)m}{1\cdot 2} a_n^{m-1}d^2 + \text{etc.} \end{aligned}$$

If we add these equations together, and cancel the common terms, $a_2^{m+1} + a_3^{m+1} + \dots + a_n^{m+1}$, which appear in both members, we shall have

$$a_{n+1}^{m+1} = a_1^{m+1} + (m+1)\,dS_m + \frac{(m+1)m}{1\cdot 2}\,d^2S_{m-1}$$
$$+ \frac{(m+1)\,m\,(m-1)}{1\cdot 2\cdot 3}\,d^3S_{m-2}, \text{ etc.}$$

From this we obtain, by solving with respect to S_m,

$$S_m = \frac{a_{n+1}^{m+1} - a_1^{m+1}}{(m+1)\,d} - \frac{m}{2}\,dS_{m-1} - \frac{m\,(m-1)}{1\cdot 2\cdot 3}\,d^2S_{m-2} - \text{etc.}, \quad (2)$$

which will enable us to find S_m when we know $S_1, S_2, \ldots$ S_{m-1}, that is, to find the sum of the n^{th} powers when we know the sum of all the lower powers. It will be noted that S_1 means the sum of the arithmetical series itself, as found in Book VII, Chap. I; and that $S_0 = n$, because there are n terms and the zero power of each is 1.

By § 209, Prob. V,

$$S_1 = n\,\frac{a_n + a_1}{2}.$$

To find the sum of the squares, we put $m = 2$, which gives

$$S_2 = \frac{a_{n+1}^3 - a_1^3}{3d} - dS_1 - \frac{d^2}{3}\,S_0. \quad (3)$$

291. The simplest application of this expression is given by the problem:

To find the sum of the squares of the first n natural numbers, namely,

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2.$$

Here $d = 1$, $a_n = n$, etc., $S_1 = 1 + 2 \ldots + n = \dfrac{n\,(n+1)}{2}$, so that (3) gives

$$S_2 = \frac{(n+1)^3 - 1}{3} - \frac{n\,(n+1)}{2} - \frac{n}{3}.$$

Noting that $n + 1$ is a factor of the second member, we may reduce this equation to

$$S_2 = \frac{n\,(n+1)\,(2n+1)}{6}, \quad (4)$$

which is the required expression for the sum of the squares of the first n numbers.

292. To find the sum of the cubes of any progression, we put $m = 3$ in the equation (2), which then gives

$$S_3 = \frac{a_{n+1}^4 - a_1^4}{4d} - \frac{3}{2}dS_2 - d^2S_1 - \frac{1}{4}d^3S_0. \qquad (6)$$

Applying this as before to the case in which a_1, a_2, a_3, etc., are the natural numbers, 1, 2, 3, etc., we find

$$S_3 = \frac{(n+1)^4 - 1}{4} - \frac{3}{2}S_2 - S_1 - \frac{1}{4}S_0$$

$$= \frac{(n+1)^4 - 1}{4} - \frac{n(n+1)(2n+1)}{4} - \frac{n(n+1)}{2} - \frac{n}{4}.$$

Separating the factor $n + 1$ and then reducing, this equation becomes

$$S_3 = \left[\frac{n(n+1)}{2}\right]^2. \qquad (5)$$

But $\frac{n(n+1)}{2}$ is the sum of the natural numbers

$$1 + 2 + 3 + \text{etc.},$$

and S_3 being the sum of the cubes, we have the remarkable relation,

$$1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = (1 + 2 + 3 + \ldots\ldots + n)^2.$$

That is, *the sum of the cubes of the first n numbers is equal to the square of their sum.*

We may verify this relation to any extent, thus:

When $n = 2$, $1^3+2^3 = 1+8 = 9 = (1+2)^2$.

When $n = 3$, $1^3+2^3+3^3 = 1+8+27 = 36 = (1+2+3)^2$.

When $n = 4$, $1^3+2^3+3^3+4^3 = 1+8+27+64 = 100 = (1+2+3+4)^2$.

etc. etc. etc. etc.

293. *Enumeration of a Rectangular Pile of Balls.* The preceding theory may be applied to the enumeration of a pile of balls of which the base is rectangular and each ball rests on four balls below it. Let us put p, q, the number of balls in two adjacent sides of the base.

Then the second course will have $p-1$ and $q-1$ balls on its sides; the third $p-2$ and $q-2$, and so on to the top, which will consist of a single row of $p-q+1$ balls (supposing $p \geqq q$). The bottom course will contain pq balls, the next course $(p-1)(q-1)$, etc. The total number of balls in the pile will be

$$N = pq + (p-1)(q-1) + (p-2)(q-2) + \ldots + (p-q+1). \quad (6)$$

To find the sum of this series, let us first suppose $p = q$, and the base therefore a square. We shall then have

$$N' = q^2 + (q-1)^2 + (q-2)^2 + \ldots + 1,$$

which is the sum of the squares of the first q numbers.

Therefore, by § 291 (4),

$$N' = \frac{q(q+1)(2q+1)}{6}. \quad (7)$$

Next let us put r for the number by which p exceeds q in the general expression (6). This expression will then become

$$\begin{aligned} N &= q(q+r) + (q-1)(q-1+r) + (q-2)(q-2+r) + \ldots \\ &\qquad + (1+r) \\ &= q^2 + (q-1)^2 + (q-2)^2 + \ldots + 2^2 + 1 \\ &\qquad + [q + (q-1) + (q-2) + \ldots + 1]\, r \\ &= \frac{q(q+1)(2q+1)}{6} + \frac{q(q+1)}{2} r \qquad (\S\ 291,\ 4.) \\ &= \frac{q(q+1)(3r+2q+1)}{6}. \end{aligned}$$

EXERCISES.

1. Find the sum of the first 20 numbers, $1+2+3+\ldots+20$, then the sum of their squares, and the sum of their cubes, by successive substitutions in the general equation (2).

2. Express the sum and the sum of the squares of the first r odd numbers, namely,

$$1 + 3 + 5 + \ldots + (2r-1),$$

and

$$1^2 + 3^2 + 5^2 + \ldots + (2r-1)^2.$$

3. Express the sum of the first r even numbers and the sum of their squares, namely,

$$2 + 4 + 6 + \ldots + 2r,$$

and

$$2^2 + 4^2 + 6^2 + \ldots + (2r)^2.$$

4. A rectangular pile of balls is started with a base of p balls on one side and q on the other. How many balls will there be in the pile after 3 courses have been laid? How many after s courses?

5. Find the value of the expression

$$\sum_{x=1}^{x=5}(a+bx+cx^2).$$

6. Find the value of

$$\sum_{x=1}^{x=b}(a+bx+cx^2).$$

294. *To find the sum of n terms of the series*

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots\ldots+\frac{1}{n(n+1)}.$$

Each term of this series may be divided into two parts, thus:

$$\frac{1}{1\cdot 2}=\frac{1}{1}-\frac{1}{2},\qquad \frac{1}{2\cdot 3}=\frac{1}{2}-\frac{1}{3},$$

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}.$$

Therefore the sum of the series is

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right),$$

in which the second part of every term except the last is cancelled by the first part of the term next following. Therefore the sum of the n terms is

$$1-\frac{1}{n+1}=\frac{n}{n+1}.$$

If we suppose the number of terms n to increase without limit, the fraction $\dfrac{1}{n+1}$ will reduce to zero, and we shall have

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\text{etc., } \textit{ad infinitum}=1.$$

This is the same as the sum of the geometrical progression, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$

+ etc., *ad infinitum*. It will be interesting to compare the first few terms of the two series. They are

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}.$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}.$$

We see that the first term is the same in both, while the next three are larger in the geometrical progression. After the fourth term, the terms of the progression become the smaller, and continue so.

295. *Generalization of the Preceding Result.* Let us take the series of which the n^{th} term is

$$\frac{p}{(i+n-1)(j+n-1)}.$$

The series to n terms will then be

$$\frac{p}{ij}+\frac{p}{(i+1)(j+1)}+\frac{p}{(i+2)(j+2)}+\ldots\ldots$$
$$+\frac{p}{(i+n-1)(j+n-1)}.$$

If we suppose $j > i$, and put, for brevity,

$$k = j - i,$$

the terms may be put into the form

$$\frac{p}{ij}=\frac{p}{k}\left(\frac{1}{i}-\frac{1}{j}\right),$$

$$\frac{p}{(i+1)(j+1)}=\frac{p}{k}\left(\frac{1}{i+1}-\frac{1}{j+1}\right),$$

etc. etc.

$$\frac{p}{(i+n-1)(j+n+1)}=\frac{p}{k}\left(\frac{1}{i+n-1}-\frac{1}{j+n-1}\right).$$

When we add these quantities, the second part of each term will be cancelled by the first part of the k^{th} term next following, leaving only the first part of the first k terms and the second part of the last k terms. Hence the sum will be

$$\frac{p}{k}\left(\frac{1}{i}+\frac{1}{i+1}+\ldots\ldots+\frac{1}{j+1}-\frac{1}{i+n}-\frac{1}{i+n-1}\ldots-\frac{1}{j+n-1}\right).$$

EXAMPLE. To find the sum of n terms of the series

$$\frac{1}{2\cdot5}+\frac{1}{3\cdot6}+\frac{1}{4\cdot7}+\frac{1}{5\cdot8}+\ldots\ldots+\frac{1}{(n+1)(n+4)}.$$

Each term may be expressed in the form

$$\frac{1}{2\cdot5}=\frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right),$$

$$\frac{1}{3\cdot6}=\frac{1}{3}\left(\frac{1}{3}-\frac{1}{6}\right),$$

$$\frac{1}{4\cdot7}=\frac{1}{3}\left(\frac{1}{4}-\frac{1}{7}\right),$$

$$\frac{1}{n(n+3)}=\frac{1}{3}\left(\frac{1}{n}-\frac{1}{n+3}\right),$$

$$\frac{1}{(n+1)(n+4)}=\frac{1}{3}\left(\frac{1}{n+1}-\frac{1}{n+4}\right).$$

Therefore, separating the positive and negative terms, we find the sum of the series to be

$$\frac{1}{3}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\ldots+\frac{1}{n}+\frac{1}{n+1}\right.$$
$$\left.-\frac{1}{5}-\frac{1}{6}-\ldots\ldots-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}-\frac{1}{n+4}\right),$$

or, omitting the terms which cancel each other,

$$\frac{1}{3}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{n+2}-\frac{1}{n+3}-\frac{1}{n+4}\right).$$

When n is infinite, the sum becomes

$$\frac{1}{3}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=\frac{1}{3}\cdot\frac{13}{12}=\frac{13}{36}.$$

EXERCISES.

What is the sum of n terms of the series:

1. $\frac{1}{3\cdot4}+\frac{1}{4\cdot5}+\frac{1}{5\cdot6}+$ etc.

2. $\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\frac{1}{7\cdot9}+\ldots\ldots+\frac{1}{(2n+1)(2n+3)}.$

3. $\frac{2}{2 \cdot 5} + \frac{2}{3 \cdot 6} + \frac{2}{4 \cdot 7} + \ldots . + \frac{2}{(n+1)(n+4)}$.

4. $\frac{3}{1 \cdot 3} + \frac{3}{2 \cdot 4} + \frac{3}{3 \cdot 5} + \ldots . + \frac{3}{n(n+2)}$.

5. Sum the series

$$\frac{1}{a(a+1)} + \frac{1}{(a+1)(a+2)} + \frac{1}{(a+2)(a+3)} + \text{etc., } ad\ inf.$$

296. To sum the series

$$S = 1 + 2r + 3r^2 + 4r^3 + \text{etc.}$$

Let us first find the sum of n terms, which we shall call S_n. Then

$$S_n = 1 + 2r + 3r^2 + 4r^3 + \ldots . nr^{n-1}.$$

Multiplying by r, we have

$$rS_n = r + 2r^2 + 3r^3 + 4r^4 + \ldots . + nr^n.$$

By subtraction,

$$(1-r)S_n = 1 + r + r^2 + r^3 \ldots . + r^{n-1} - nr^n$$
$$= \frac{1-r^n}{1-r} - nr^n \text{ (§ 212, Prob. V).}$$

Therefore, $\quad S_n = \frac{1-r^n}{(1-r)^2} - \frac{nr^n}{1-r}$.

Now suppose n to increase without limit. If $r \geqq 1$, the sum of the series will evidently increase without limit.

If $r < 1$, both r^n and nr^n will converge toward zero as n increases (as we shall show hereafter), and we shall have

$$S = \frac{1}{(1-r)^2}.$$

EXERCISES.

Find in the above way the sum of the following series to n terms and to infinity, supposing $r < 1$:

1. $a + 3ar + 5ar^2 + 7ar^3 \ldots . + (2n-1)ar^{n-1}$.
2. $2a + 4ar + 6ar^2 + 8ar^3 \ldots . + 2nar^{n-1}$.
3. $(a+b)r + (a+2b)r^2 + \ldots . + (a+nb)r^n$.

297. Sum the series

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\text{etc.,} \qquad (a)$$

of which the general term is $\dfrac{1}{n(n+1)(n+2)}$.

Let us find whether we can express this series as the sum of two series. Assume

$$\frac{1}{n(n+1)(n+2)}=\frac{A}{n(n+1)}+\frac{B}{(n+1)(n+2)},$$

where, if possible, the values of the indeterminate coefficients A and B are to be so chosen that this equation shall be true identically.

Reducing the second member to a common denominator, we have

$$\frac{1}{n(n+1)(n+2)}=\frac{(A+B)n+2A}{n(n+1)(n+2)}.$$

In order that these fractions may be identically equal, we must have

$$(A+B)n+2A=1, \textit{ identically,}$$

which requires that we have (§ 281),

$$A+B=0, \qquad 2A=1.$$

This gives $\qquad A=\dfrac{1}{2}, \qquad B=-\dfrac{1}{2}.$

Therefore,

$$\frac{1}{n(n+1)(n+2)}=\frac{1}{2}\,\frac{1}{n(n+1)}-\frac{1}{2}\,\frac{1}{(n+1)(n+2)},$$

so that each term of the series (a) may be divided into two terms. The whole series will then be

$$\frac{1}{2}\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\text{etc.}\right)-\frac{1}{2}\left(\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\text{etc.}\right).$$

We see on sight, that by cancelling equal terms, the sum of n terms is

$$S_n=\frac{1}{4}-\frac{1}{2(n+1)(n+2)},$$

and the sum to infinity is $\dfrac{1}{4}$.

298. Consider the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.},$$

of which the n^{th} term is $\frac{1}{n}\cdot$ This series is divergent, because we may divide it into an unlimited number of parts, each equal to or greater than $\frac{1}{2}$, as follows:

$$\begin{aligned} \text{1st term} = 1, &> \frac{1}{2}; \\ \text{2d term} &= \frac{1}{2}; \\ \text{3d and 4th terms} &> \frac{1}{2}; \\ \text{etc.} \quad & \quad \text{etc.} \end{aligned}$$

In general, if we consider the n consecutive terms,

$$\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}, \qquad (a)$$

the smallest will be $\frac{1}{2n}$, and therefore their sum will be greater than $\frac{1}{2n} \times n$, that is, greater than $\frac{1}{2}$.

Now if in (a) we suppose n to take the successive values, 1, 2, 4, 8, 16, etc., we shall divide the series into an unlimited number of parts of the form (a), each greater than $\frac{1}{2}\cdot$ Therefore, the sum has no limit and so is divergent.

Of Differences.

299. When we have a series of quantities proceeding according to any law, we may take the difference of every two consecutive quantities, and thus form a series of differences. The terms of this series are called **First Differences.**

Taking the difference of every two consecutive differences, we shall have another series, the terms of which are called **Second Differences.**

The process may be continued so long as there are any differences to write.

Example. In the second column of the following table are given the seven values of the expression

$$x^4 - 10x^3 + 30x^2 - 40x + 25 = \phi x,$$

for $x = 0, 1, 2, 3, 4, 5, 6$.

In the third column Δ' are given the differences,

$$6 - 25 = -19, \quad 1 - 6 = -5, \quad -14 - 1 = -15, \quad \text{etc.}$$

In column Δ'' are given the differences of these differences, namely,

$$-5 - (-19) = +14, \quad -15 - (-5) = -10, \quad \text{etc.}$$

x	ϕx	Δ'	Δ''	Δ'''	Δ^{iv}	Δ^{v}
0	+ 25					
		− 19				
1	+ 6		+ 14			
		− 5		− 24		
2	+ 1		− 10		+ 24	
		− 15		0		0
3	− 14		− 10		+ 24	
		− 25		+ 24		0
4	− 39		+ 14		+ 24	
		− 11		+ 48		
5	− 50		+ 62			
		+ 51				
6	+ 1					

The process is continued to the fourth order of differences, which are all equal, whence those of the fifth and following orders are all zero.

It will be noted that the sign of each difference is taken so that it shall express each quantity *minus* the quantity next preceding. We have therefore the following definitions:

300. *Def.* The **First Difference** of a function of any variable is the increment of the function caused by an increment of unity in the variable.

The **Second Difference** is the difference between two consecutive first differences.

In general, the n^{th} **Difference** is the difference between two consecutive $(n-1)^{st}$ differences.

To investigate the relation among the differences, let us represent the successive numbers in each column by the indices 1, 2, 3, etc., and let us put Δ_1, Δ_2, Δ_3, etc., for the values of ϕx. We shall then have the following scheme of differences, in which

$$\Delta'_0 = \Delta_1 - \Delta_0, \quad \Delta'_1 = \Delta_2 - \Delta_1, \quad \Delta'_2 = \Delta_3 - \Delta_2;$$

$$\Delta''_0 = \Delta'_1 - \Delta'_0, \quad \Delta''_1 = \Delta'_2 - \Delta'_1, \quad \Delta''_2 = \Delta'_3 - \Delta'_2;$$

$$\Delta'''_0 = \Delta'_1 - \Delta''_0, \quad \Delta'''_1 = \Delta''_2 - \Delta''_1, \quad \Delta'''_2 = \Delta''_3 - \Delta''_2;$$

etc. etc. etc.

the n^{th} order of differences being represented by the symbol Δ with n accents.

$$\begin{array}{ccccc}
\Delta_0 & & & & \\
 & \Delta'_0 & & & \\
\Delta_1 & & \Delta''_0 & & \\
 & \Delta'_1 & & \Delta'''_0 & \\
\Delta_2 & & \Delta''_1 & & \Delta'''_0 \\
 & \Delta'_2 & & \Delta'''_1 & . \\
\Delta_3 & . & \Delta''_2 & & \\
. & . & & & \\
. & . & & & \\
. & \Delta'_{n-1} & & & \\
\Delta_n & & & &
\end{array}$$

Let us now consider the following problem:

To express Δ_i *in terms of* Δ_0, Δ'_0, Δ''_0, etc.

We have, by the mode of forming the differences,

$$\Delta_1 = \Delta_0 + \Delta'_0, \quad \Delta'_1 = \Delta'_0 + \Delta''_0, \quad \Delta''_1 = \Delta''_0 + \Delta'''_0, \text{ etc.} \qquad (a)$$

$$\Delta_2 = \Delta_1 + \Delta'_1, \quad \Delta'_2 = \Delta'_1 + \Delta''_1, \quad \Delta''_2 = \Delta''_1 + \Delta''' \text{ etc.}$$

If in this last system of equations, we substitute the values of Δ_1, Δ'_1, etc., from the system (a), we have

$$\Delta_2 = \Delta_0 + 2\Delta'_0 + \Delta''_0, \quad \Delta'_2 = \Delta'_0 + 2\Delta''_0 + \Delta'''_0, \text{ etc.} \qquad (b)$$

Again,

$$\Delta_3 = \Delta_2 + \Delta'_2, \quad \Delta'_3 = \Delta'_2 + \Delta''_2, \quad \Delta''_3 = \Delta''_2 + \Delta'''_2, \text{ etc.}$$

Substituting the values of Δ_2, Δ'_2, etc., from (b), we have

$$\begin{array}{rl} \Delta_3 = \Delta_0 + 2\Delta'_0 + & \Delta''_0 \\ + \ \Delta'_0 + & 2\Delta''_0 + \Delta'''_0 \\ \hline \end{array}$$

or
$$\Delta_3 = \Delta_0 + 3\Delta'_0 + 3\Delta''_0 + \Delta'''_0 \qquad (c)$$

$$\begin{array}{rl} \Delta'_3 = \Delta'_0 + 2\Delta''_0 + & \Delta'''_0 \\ + \ \Delta''_0 + & 2\Delta'''_0 + \Delta^{\text{iv}}_0 \\ \hline \end{array}$$

$$\Delta'_3 = \Delta'_0 + 3\Delta''_0 + 3\Delta'''_0 + \Delta^{\text{iv}}_0$$

Forming $\Delta_4 = \Delta_3 + \Delta'_3$, etc., we see that the coefficients of Δ_0, Δ'_0, etc., which we add, are the same as the coefficients of the successive powers of x in raising $1 + x$ to the n^{th} power by successive multiplication, as in § 171. That is, to form Δ_4, Δ'_4, etc., the coefficients to be added are

$$\begin{array}{ccccc} 1 & 3 & 3 & 1 & \\ & 1 & 3 & 3 & 1 \\ \hline 1 & 4 & 6 & 4 & 1 \end{array}$$

and these are to be added in the same way to form Δ_5, and so on indefinitely. Hence we conclude that if i be any index, the law will be the same as in the binomial theorem, namely,

$$\left.\begin{array}{l} \Delta_i = \Delta_0 + i\Delta'_0 + \binom{i}{2}\Delta''_0 + \binom{i}{3}\Delta'''_0 + \text{etc.} \\ \Delta'_i = \Delta'_0 + i\Delta''_0 + \binom{i}{2}\Delta'''_0 + \binom{i}{3}\Delta^{\text{iv}}_0 + \text{etc.} \end{array}\right\} \qquad (d)$$

To show rigorously that this result is true for all values of i, we have to prove that if true for any one value, it must be true for a value one greater. Now we have, by definition, whatever be i,

$$\Delta_{i+1} = \Delta_i + \Delta'_i, \qquad \Delta'_{i+1} = \Delta'_i + \Delta''_i, \qquad \text{etc.}$$

Hence, substituting the above value of Δ_i and Δ'_i,

$$\Delta_{i+1} = \Delta_0 + (i + 1)\Delta'_0 + \left[\binom{i}{2} + i\right]\Delta''_0$$
$$+ \left[\binom{i}{3} + \binom{i}{2}\right]\Delta'''_0 + \text{etc.} \qquad (e)$$

We readily prove that

$$\binom{i}{2} + i = \binom{i+1}{2},$$

$$\binom{i}{3} + \binom{i}{2} = \binom{i+1}{3},$$

etc. etc.

Substituting these values in (e), the result is the same given by the equation (d) when we put $i + 1$ for i.

The form (c) shows the formula to be true for $i = 3$.

Therefore it is true for $i = 4$.

Therefore it is true for $i = 5$, etc., indefinitely.

EXAMPLES AND EXERCISES.

1. Having given $\Delta_0 = 7$, $\Delta'_0 = 5$, $\Delta''_0 = -2$, and Δ''', Δ^{iv}, etc. $= 0$, it is required to find the values of Δ_1, Δ_2, Δ_3, etc., indefinitely, both by direct computation and by the formula (d).

We start the work thus:

The numbers in column Δ'' are all equal to -2, because $\Delta''' = 0$.

Each number in column Δ' after the first is found by adding Δ'' or -2 to the one next above it.

Each value of Δ_i is then obtained from the one next above it by adding the appropriate value of Δ'_i.

This process of addition can be carried to any extent. Continuing it to $i = 10$, we shall find $\Delta_{10} = -33$.

i	Δ_i	Δ'_i	Δ''_i
0	7		
		$+5$	
1	$+12$		-2
		$+3$	
2	$+15$		-2
		$+1$	
3	etc.		-2
		-1	
4			-2
		etc.	
etc.			etc.

Next, the general formula (d) gives, by putting $\Delta_0 = 7$, $\Delta'_0 = 5$, $\Delta''_0 = -2$, and all following values $= 0$,

$$\Delta_i = 7 + 5i - 2\frac{i(i-1)}{2},$$

and the student is now to show that by putting $i = 1$, $i = 2$, etc., in this expression, we obtain the same values of Δ_1, Δ_2, Δ_3, ... Δ_{10}, that we get by addition in the above scheme.

It is moreover to be remarked that we can reduce the last equation to an entire function of i, thus:

$$\Delta_i = 7 + 6i - i^2.$$

2. Having given $\Delta_0 = 5$, $\Delta'_0 = -20$, $\Delta''_0 = -30$, $\Delta'''_0 = +9$, it is required to find in the same way the values of Δ_1 to Δ_5, and to express Δ_i as an entire function of i by formula (d).

3. On March 1, 1881, at Greenwich noon, the sun's longitude was $341° 5' 10''.9$; on March 2 it was greater by $1° 0' 9''.6$, but this daily increase was diminishing by $2''$ each day. It is required to compute the longitude for the first seven days of the month, and to find an expression for its value on the n^{th} day of March.

4. A family had a reservoir containing, on the morning of May 5, 495 gallons of water, to which the city added regularly 50 gallons per day. The family used 35 gallons on May 5, and 5 gallons more each subsequent day than it did on the day preceding. Find a general expression for the quantity of water on the n^{th} day of May; and by equating this expression to zero, find at what time the water will all be gone. Also explain the two answers given by the equation.

Theorems of Differences.

301. To investigate the general properties of differences, we use a notation slightly different from that just employed.

If u be any function of x, which we may call ϕx, so that we put

$$u = \phi x,$$

then

$$\Delta u = \phi(x+1) - \phi x. \qquad (a)$$

Here the symbol Δ does not represent a multiplier, but merely the words *difference of.*

The second difference of u being the difference of the difference, may be represented by $\Delta\Delta u$.

For brevity, we put

$$\Delta^2 u \text{ for } \Delta\Delta u,$$

where the index 2 is not an exponent, but a symbol indicating a second difference.

Continuing the same notation, the n^{th} difference will be represented by Δ^n.

EXAMPLE.

To find the successive differences of the function

$$u = ax^3 + bx^2.$$

By the formula (a), we have

$$\Delta u = a(x+1)^3 + b(x+1)^2 - ax^3 - bx^2;$$

and, by developing,

$$\Delta u = 3ax^2 + (3a + 2b)x + a + b.$$

Taking the difference of this last equation,

$$\begin{aligned}\Delta^2 u &= 3a(x+1)^2 + (3a+2b)(x+1) + a + b \\ &\qquad - 3ax^2 - (3a+2b)x - a - b \\ &= 6ax + 6a + 2b.\end{aligned}$$

Again taking the difference, we have

$$\Delta^3 u = 6a(x+1) - 6ax = 6a.$$

This expression not containing x, $\Delta^4 u$, $\Delta^5 u$, etc., all vanish.

EXERCISES.

Compute the differences of the functions:

1. $x^3 + mx^2 + nx + p$.   2. $2x^4 + 3x^2 + 5$.

3. $5x^3 + 10x^2 + 15$.

4. In the case of the last expression, prove the agreement of results by computing the values of Δu, $\Delta^2 u$, etc., for $x = 0$, $x = 1$, and $x = 3$, and comparing them with those obtained by the method of § 299. The latter are shown in the following table:

$$u = 5x^3 + 10x^2 + 15.$$

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$
0				
	15			
1		15		
	30		50	
2		65		30
	95		80	
3		145		30
	240		110	
4		255		
	495			
5				

5. Do the same thing for exercise 2, and for the function tabulated in § 299.

302. It will be seen by the preceding examples and exercises, that for each difference of an entire function of x which we form, the degree of the function is diminished by unity. This result is generalized in the following theorem:

The n^{th} differences of the function x^n are constant and equal to $n!$

Proof. If $u = x^n$, we have, by the definition of the symbol Δ,

$$\Delta u = (x + 1)^n - x^n,$$

or

$$\Delta u = nx^{n-1} + \binom{n}{2}x^{n-2} + \text{etc.}$$

That is, *in taking the difference, the highest power of x is multiplied by its exponent and the latter is diminished by unity.*

Continuing the process, we shall find the n^{th} difference to be

$$n(n-1)(n-2) \ldots . 1 = n!$$

Cor. If we have an entire function of x of the degree n,

$$ax^n + bx^{n-1} + cx^{n-2} + \text{etc.},$$

the $(n-1)^{st}$ difference of bx^{n-1}, the $(n-2)^d$ difference of cx^{n-2}, etc., will all be constant, and therefore the n^{th} difference of these terms will all vanish. Therefore, the n^{th} difference of the entire function will be the same as the n^{th} difference of ax^n; that is, we have

$$\Delta^n (ax^n + bx^{n-1} + \text{etc.}) = an!$$

Hence, *the n^{th} difference of a function of the n^{th} degree is constant, and equal to $n!$ multiplied by the coefficient of the highest power of the variable.*

CHAPTER IV.

THE DOCTRINE OF LIMITS.

303. The doctrine of limits embraces a set of principles applicable to cases in which the usual methods of calculation fail, in consequence of some of the quantities to be used vanishing or increasing without limit.

We have already made extensive use of some of the principles of this doctrine, and thus familiarized the student with their application, but our further advance requires that they should be rigorously developed.

Axiom I. Any quantity, however small, may be multiplied so often as to exceed any other fixed quantity, however great.

Ax. II. *Conversely*, any quantity, however great, may be divided into so many parts that each part shall be less than any other fixed quantity, however small.

Def. An **Independent Variable** is a quantity to which we may assign any value we please, however small or great.

Theorem I. *If a fraction have any finite numerator, and an independent variable for its denominator, we may assign to this denominator a value so great that the fraction shall be less than any quantity, however small, which we may assign.*

Proof. Let a be the numerator of the fraction, x its denominator, and α any quantity, however small, which we may choose to assign.

Let n be the number of times we must multiply α to make it greater than a. (Axiom I.) We shall then have

$$a < n\alpha.$$

Consequently, $$\frac{a}{n} < \alpha.$$

Hence, by taking x greater than n, we shall have

$$\frac{a}{x} < \alpha.$$

Example. Let $a = 10$. Then if we take for α in succession, $\frac{1}{100}$, $\frac{1}{10{,}000}$, $\frac{1}{1{,}000{,}000}$, etc., we have only to take

$$x > 1{,}000, \quad x > 100{,}000, \quad x > 10{,}000{,}000, \quad \text{etc.},$$

to make $\frac{10}{x}$ less than α.

In the language of limits, the above theorem is expressed thus:

The limit of $\frac{a}{x}$, when x is indefinitely increased, is zero.

Theorem II. *If a fraction have any finite numerator, and an independent variable for its denominator, we may assign to this denominator a value so small that the fraction shall exceed any quantity, however great, which we may assign.*

Proof. Put as before $\frac{a}{x}$ for the fraction, and let A be any number however great, which we choose to assign.

Let n be a number greater than A. Divide a into n parts, and let α be one of these parts; then

$$a = n\alpha.$$

Consequently, $$\frac{a}{\alpha} = n.$$

Therefore, if we take for x a quantity less than α, we shall have

$$\frac{a}{x} > n > A,$$

or $$\frac{a}{x} > A.$$

Rem. If we have two independent variables, x and y: We may make x any number of times greater than y.

Then we may make y any number of times greater than this value of x.

Then we may make x any number of times greater than this value of y.

And we can thus continue, making each variable outstrip the other to any extent in a race toward infinity, without either ever reaching the goal.

THEOREM III. *If k be any fixed quantity, however great, and a a quantity which we may make as small as we please, we may make the product ka less than any assignable quantity.*

Proof. If there is any smallest value of ka, let it be s. Because we may make a as small as we please, let us put

$$a < \frac{s}{k}.$$

Multiplying by k, we find

$$ka < s.$$

So that ka may be made less than s, and s cannot be the smallest value.

Def. The **Limit** of a variable quantity is a value which it can never reach, but to which it may approach so nearly that the difference shall be less than any assignable quantity.

REM. In order that a variable X may have a limit, it must be a function of some other variable, and there must be certain values of this other variable for which the value of X cannot be directly computed.

EXAMPLES.

1. The value of the expression

$$X = \frac{x^3 - a^3}{x - a}$$

can be computed directly for any pair of numerical values of x and a, except those values which are equal. If we suppose $x = a$, the expression becomes

$$\frac{a^3 - a^3}{a - a} = \frac{0}{0},$$

which, considered by itself, has no meaning.

2. The sum of any finite number of terms of a geometrical progression may be computed by adding them. But if the number of terms is infinite, an infinite time would be required for the direct calculation, which is therefore impossible.

3. The area of a polygon of any number of sides, and having a given apothegm, may be computed. But if the number of sides becomes infinite, and the polygon is thus changed into a circle, the direct computation is not practicable.

EXERCISE.

If we have the fraction, $X = \frac{7x - 8}{3x - 1}$, show that we may make x so great that X shall differ from $\frac{7}{3}$ by less than $\frac{1}{100}$, less than $\frac{1}{100,000}$, lesss than $\frac{1}{1,000,000}$, and so on indefinitely.

Notation of the Method of Limits.

304. Put X, the quantity of which the value is to be found;

x, the independent variable on which X depends, so that X is a function of x;

a, the particular value of x for which we cannot compute X;

L, the limit of X, or the value to which it approaches as x approaches to a.

Then the limit L must be a quantity fulfilling these two conditions:

1st. Supposing x to approach as near as we please to a, we must always be able to find a value of x so near to a that the difference $L - X$ shall become less than any assignable quantity.

2d. X must not become absolutely equal to L, however near x may be to a.

REM. The quantity a, toward which x approaches, may be either zero, infinity, or some finite quantity.

EXAMPLE 1. Suppose

$$X = \frac{x^3 - a^3}{x - a}.$$

By § 93, this expression is equal to

$$x^2 + ax + a^2, \tag{a}$$

except when $x = a$. But suppose δ to be the difference between x and a, so that

$$x = a + \delta.$$

Substituting this value in the expression (a), the equation becomes

$$\frac{x^3 - a^3}{x - a} = 3a^2 + 3a\delta + \delta^2.$$

Now we may suppose δ so small that $3a\delta + \delta^2$ shall be less than any quantity we choose to assign. Hence we may choose a value of x so near to a that the value of $\frac{x^3 - a^3}{x - a}$ shall differ from $3a^2$ by less than any assignable quantity. Hence, if

$$X = \frac{x^3 - a^3}{x - a},$$

then $$L = 3a^2,$$

or $3a^2$ is the limit of the expression $\frac{x^3 - a^3}{x - a}$ as x approaches a.

EX. 2. The limit of $\frac{x}{x+1}$, when x becomes indefinitely great, is unity.

For, subtracting this expression from unity, we find the difference to be

$$\frac{1}{x+1}.$$

By taking x sufficiently great, we may make this expression less than any assignable quantity. (§ 303, Th. I.) Therefore, $\frac{x}{x+1}$ approaches to unity as x increases, whence unity is its limit.

Notation. The statement that L is the limit of X as x approaches a is expressed in the form

$$\text{Lim. } X_{(x=a)} = L.$$

The conclusions of the last two examples may be expressed thus:

$$\text{Lim. } \frac{x^3 - a^3}{x - a}{}_{(x=a)} = 3a^2. \qquad \text{Lim. } \frac{x}{x+1}{}_{(x=\infty)} = 1.$$

REM. This form of notation is often used for the following purpose. Having a function of x which we may call X, the form $X_{(x=a)}$ means, "the value of X when $x = a$."

EXAMPLES.

$$(x^2 + a)_{(x=a)} = a^2 + a. \qquad (x^2 - a^2)_{(x=a)} = 0.$$

$$(u^2 + 2ub)_{(u=-b)} = -b^2.$$

If we require the limit of a fraction when both terms become zero or infinite, *divide both terms by some common factor which becomes zero or infinity.*

REM. If the beginner has any difficulty in understanding the preceding exposition, it will be sufficient for him to think of the limit as simply the value of the expression when the quantity on which it depends becomes zero or infinity.

For instance, $\quad \text{Lim. } \dfrac{x}{x+1} \quad (x = \infty)$,

the value of which we have found to be unity, may be regarded as simply the value of the expression, $\quad \dfrac{\infty}{\infty + 1}$.

Although this way of thinking is convenient, and generally leads to correct results, it is not mathematically rigorous, because neither zero nor infinity are, properly speaking, mathematical quantities, and people are often led into paradoxes by treating them as such.

EXERCISES.

Find the limit of

1. $\dfrac{x - a}{x}$ when x approaches infinity.

   Divide both terms by x.

2. $\dfrac{ax + b}{bx + a}$ when x approaches infinity.

3. $\dfrac{mx^2}{px^2 - ax}$ when x approaches infinity.

4. $\dfrac{1-x}{1-ax}$ when x approaches infinity.

5. $\dfrac{x^2-a^2}{x-a}$ when x approaches a.

6. $\dfrac{a+x}{a-x}$ when x approaches infinity.

Properties of Limits.

305. THEOREM I. *If two functions are equal, they must have the same limit.*

Proof. If possible, let L and L' be two different limits for the respective functions. Put

$$z = \frac{1}{2}(L - L'),$$

so that L and L' differ by $2z$.

Because L is the limit of the one function, the latter may approach this limit so nearly as to differ from it by less than z. In the same way, the other function may differ from L' by less than z. Then, because L and L' differ by $2z$, the functions would differ, which is contrary to the hypothesis.

THEOREM II. *The limit of the sum of several functions is equal to the sum of their separate limits.*

Proof. Let the functions be X, X', X'', etc.
Let their limits be L, L', L'', etc.

Let their differences from their limits be α, α', α'', etc.

Then
$$\begin{aligned} X &= L - \alpha, \\ X' &= L' - \alpha', \\ X'' &= L'' - \alpha'', \\ \text{etc.} & \quad \text{etc.} \end{aligned}$$

Adding, we have

$$X + X' + X'' + \text{etc.} = L + L' + L'' + \text{etc.} - (\alpha + \alpha' + \alpha'' + \text{etc.})$$

The theorem asserts that we may take the functions so near their limits that the sums of the differences $\alpha + \alpha' + \alpha'' + \text{etc.}$ shall be less than any quantity we can assign.

Let k be this quantity, which may be ever so small;
n, the number of the quantities α, α', α'', etc.;
α, the largest of them.

Because we can bring the functions as near their limits as we please, we may bring them so near as to make

$$\alpha < \frac{k}{n}, \qquad \text{or} \qquad n\alpha < k.$$

Then $\alpha + \alpha' + \alpha'' + \text{etc.} < n\alpha$ (because α is the largest);

whence, $\alpha + \alpha' + \alpha'' + \text{etc.} < k$.

Therefore the sum $X + X' + X'' + \text{etc.}$ will approach to the sum $L + L' + L'' + \text{etc.}$, so as to differ from it by less than k. Because this quantity k may be as small as we please, $L + L' + L'' + \text{etc.}$ is the limit of $X + X' + X'' + \text{etc.}$

THEOREM III. *The limit of the product of two functions is equal to the product of their limits.*

Proof. Adopting the same notation as in Th. II, we shall have

$$XX' = LL' - \alpha L' - \alpha' L + \alpha\alpha'.$$

Because L and L' are finite quantities, we may take α and α' so small that $\alpha L' + \alpha' L - \alpha\alpha'$ shall be less than any quantity we can assign. Hence XX' may approach as near as we please to LL', whence the latter is its limit.

COR. 1. *The limit of the product of any number of functions is equal to the product of their limits.*

COR. 2. *The limit of any power of a function is equal to the power of its limit.*

THEOREM IV. *The limit of the quotient of two functions is equal to the quotient of their limits.*

Proof. Using the same notation as before, we have for the quotient of the functions,

$$\frac{X'}{X} = \frac{L' - \alpha}{L - \alpha},$$

while the quotient of their limits is $\frac{L'}{L}$.

The difference between the two quotients is

$$\frac{L'}{L} - \frac{L' - a'}{L - a} = \frac{La' - L'a}{L(L - a)}.$$

If L is different from zero, we may make the quantities a and a' so small that this expression shall be less than any quantity we choose to assign. Therefore, $\frac{L'}{L}$ is the limit of $\frac{L' - a'}{L - a}$, that is, of $\frac{X'}{X}$.

306. Problem. *To find the limit of $\frac{x^n - a^n}{x - a}$ as x approaches a.*

Case I. *When n is a positive whole number.*

We have from § 93, when x is different from a,

$$\frac{x^n - a^n}{x - a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1}.$$

Now suppose x to approach the limit a. Then x^{n-1} will approach the limit a^{n-1}, x^{n-2} the limit a^{n-2}, etc. Multiplying by a, a^2, etc., we see that each term of the second member approaches the limit a^{n-1}. Because there are n such terms, we have

$$\text{Lim.}\ \frac{x^n - a^n}{x - a}{}_{(x=a)} = na^{n-1}.$$

Case II. *When n is a positive fraction.*

Suppose $n = \frac{p}{q}$, p and q being whole numbers. Then

$$\frac{x^n - a^n}{x - a} = \frac{x^{\frac{p}{q}} - a^{\frac{p}{q}}}{x - a}.$$

Let us put, for convenience in writing,

$$x^{\frac{1}{q}} = y, \qquad a^{\frac{1}{q}} = b;$$

then

$$x = y^q, \qquad a = b^q;$$

and

$$\frac{x^n - a^n}{x - a} = \frac{y^p - b^p}{y^q - b^q} = \frac{\dfrac{y^p - b^p}{y - b}}{\dfrac{y^q - b^q}{y - b}}.$$

As x approaches indefinitely near to a, and consequently y to b, the numerator of this fraction (Case I) approaches to pb^{p-1} as its limit and the denominator to qb^{q-1}. Hence, the fraction itself approaches to

$$\frac{pb^{p-1}}{qb^{q-1}} = \frac{p}{q}b^{p-q}.$$

Substituting for b its value $a^{\frac{1}{q}}$, we have

$$\text{Lim. } \frac{x^n - a^n}{x - a}\,_{(x=a)} = \frac{p}{q}b^{p-q} = \frac{p}{q}a^{\frac{p-q}{q}} = \frac{p}{q}a^{\frac{p}{q}-1}$$
$$= na^{n-1}.$$

Hence the same formulæ holds when n is a positive fraction.

CASE III. *When n is negative.*

Suppose $n = -p$, p itself (without the minus sign) being supposed positive. Then

$$\frac{x^n - a^n}{x - a} = \frac{x^{-p} - a^{-p}}{x - a} = x^{-p}a^{-p}\left(\frac{a^p - x^p}{x - a}\right)$$
$$= -x^{-p}a^{-p}\frac{x^p - a^p}{x - a}$$

When x approaches a, then x^{-p} approaches a^{-p}, and $\dfrac{x^p - a^p}{x - a}$ approaches pa^{p-1}. Substituting these limiting values, we have

$$\text{Lim. } \frac{x^n - a^n}{x - a}\,_{(x=a)} = -a^{-2p}pa^{p-1} = -pa^{-p-1}.$$

Substituting for $-p$ its value n, we have

$$\text{Lim. } \frac{x^n - a^n}{x - a}\,_{(x=a)} = na^{n-1}.$$

Hence,

THEOREM. *The formulæ*

$$\text{Lim. } \frac{x^n - a^n}{x - a}\,_{(x=a)} = na^{n-1}$$

is true for all values of n, whether entire or fractional, positive or negative.

CHAPTER V.

THE BINOMIAL AND EXPONENTIAL THEOREMS.

The Binomial Theorem for all Values of the Exponent.

307. We have shown in §§ 171, 264, how to develop $(1+x)^n$ when n is a positive whole number. We have now to find the development when n is negative or fractional. Assume

$$(1 + x)^n = B_0 + B_1x + B_2x^2 + B_3x^3 + \text{etc.}, \qquad (a)$$

B_0, B_1, etc., being indeterminate coefficients. Because this equation is by hypothesis true for all values of x, it will remain true when we put another quantity a in place of x. Hence,

$$(1 + a)^n = B_0 + B_1a + B_2a^2 + B_3a^3 + \text{etc.} \qquad (b)$$

Subtracting (b) from (a), and putting for convenience

$$X = 1 + x, \qquad A = 1 + a,$$

the difference of the two equations (a) and (b) will be

$$X^n - A^n = B_1(x - a) + B_2(x^2 - a^2) + B_3(x^3 - a^3) + \text{etc.}$$

The values we have assumed for X and A give

$$X - A = x - a.$$

Dividing the left-hand member by $X - A$, and the right-hand member by the equal quantity $x - a$, we have

$$\frac{X^n - A^n}{X - A} = B_1 + B_2\frac{x^2 - a^2}{x - a} + B_3\frac{x^3 - a^3}{x - a} + \text{etc.}$$

Now suppose x to approach a. The limit of the left-hand member will be nA^{n-1}. Taking the sum of the corresponding limits of the right-hand member, we shall have

$$nA^{n-1} = B_1 + 2B_2a + 3B_3a^2 + 4B_4a^3 + \text{etc.}$$

Replace A by its value, $1 + a$, and multiply by $1 + a$. We then have

$$n(1+a)^n = B_1(1+a) + 2B_2a(1+a) + 3B_3a^2(1+a) + 4B_4a^3(1+a) + \text{etc.}$$
$$= B_1 + (B_1 + 2B_2)a + (2B_2 + 3B_3)a^2 + (3B_3 + 4B_4)a^3 + \text{etc.}$$

Multiplying the equation (b) by n, we have

$$n(1+a)^n = nB_0 + nB_1a + nB_2a^2 + nB_3a^3.$$

Equating the coefficients of the like powers of a in these equations (§ 281), we have, first,

$$B_1 = nB_0.$$

By putting $a = 0$ in equation (b), we find $B_0 = 1$, whence

$$B_1 = n = \binom{n}{1}.$$

Then we find successively,

$$2B_2 = (n-1)B_1, \text{ whence } B_2 = \frac{n-1}{2}B_1 = \frac{n(n-1)}{1\cdot 2}.$$

$$3B_3 = (n-2)B_2, \quad \text{"} \quad B_3 = \frac{n-2}{3}B_2 = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}.$$

Substituting these values of B_0, B_1, B_2, etc., in the equation (a) and using the abbreviated notation, we obtain the equation

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \text{etc.}, \qquad (c)$$

which equation is true for all values of n.

308. There is an important relation between the form of this development when n is a positive integer, as in §§ 171 and 264 and when it is negative or fractional. In the former case, when we form the successive factors $n-1$, $n-2$, $n-3$, etc., the n^{th} factor will vanish, and therefore all the coefficients after that of x^n will vanish.

But if n is negative or fractional, none of the factors $n-1$, $n-2$, etc., can become zero, and, in consequence, the series will go on to infinity. It therefore becomes necessary, in this case, to investigate the convergence of the development.

If $x > 1$, the successive powers of x will go on increasing indefinitely, while the coefficients $\binom{n}{1}$, $\binom{n}{2}$, etc., will not go on diminishing indefinitely in the same ratio. For, let us consider two successive terms of the development, the i^{th} and the $(i+1)^{st}$, namely,

$$\binom{n}{i} x^i \quad \text{and} \quad \binom{n}{i+1} x^{i+1}.$$

The quotient of the second by the first is

$$\binom{n}{i+1} x \div \binom{n}{i} = \frac{n-i}{i+1} x.$$

As i increases indefinitely, this coefficient of x will approach the limit -1 (§ 304), while x is by hypothesis as great as 1. Therefore, by continuing the series, a point will be reached from which the terms will no longer diminish. Therefore,

The development of $(1+x)^n$ *in powers of* x *is not convergent unless* $x < 1$.

In consequence, if we develop $(a+b)^n$ when n is negative or fractional, we must do so in ascending powers of the lesser of the two quantities, a or b.

EXAMPLES.

1. Develop $(1+x)^{\frac{1}{2}}$, or the square root of $1+x$.

Putting $n = \frac{1}{2}$, we have

$$\binom{n}{1} = \frac{1}{2}.$$

$$\binom{n}{2} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\cdot 2} = -\frac{1\cdot 1}{2\cdot 4}.$$

$$\binom{n}{3} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1\cdot 2\cdot 3} = \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}.$$

$$\binom{n}{4} = \frac{\frac{1}{2}-3}{4}\binom{n}{3} = -\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}.$$

etc. etc. etc.

Whence,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1\cdot 1}{2\cdot 4}x^2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 + \text{etc.}$$

If x is a small fraction, the terms in x^2, x^3, etc., will be much smaller than $\frac{1}{2}x$ itself, and the first two terms of the series will give a result very near the truth. We therefore conclude:

The square root of 1 *plus a small fraction is approximately equal to* 1 *plus half that fraction.*

2. To develop $\sqrt{10}$.

We see at once that $\sqrt{10}$ is between 3 and 4. We put 10 in the form

$$3^2 + 1 = 3^2\left(1 + \frac{1}{9}\right),$$

when

$$\sqrt{10} = 3\left(1 + \frac{1}{9}\right)^{\frac{1}{2}}.$$

Then, by the development just performed,

$$\left(1 + \frac{1}{9}\right)^{\frac{1}{2}} = 1 + \frac{1}{2\cdot 9} - \frac{1}{8\cdot 9^2} + \frac{1}{16\cdot 9^3} - \frac{5}{128\cdot 9^4} + \text{etc.}$$

We now sum the terms:

1st term,			1.0000000
2d "	$=$ 1st $\div 18$,	$+$	.0555556
3d "	$=$ 2d $\div -36$,	$-$	.0015432
4th "	$=$ 3d $\div -18$,	$+$	.0000857
5th "	$=$ 4th $\times -5 \div 72$,	$-$	.0000060
6th "	$=$ 5th $\times -7 \div 90$,	$+$	.0000005

$$\text{Sum} = \left(1 + \frac{1}{9}\right)^{\frac{1}{2}} = 1.0540926$$

Whence, $$\sqrt{10} = 3 \times \text{sum} = 3.1622778$$

which may be in error by a few units in the last place, owing to the omission of the decimals past the seventh.

3. To develop $\sqrt{8}$.

We see that 3 is the nearest whole number of the root. So we put

$$\sqrt{8} = \sqrt{(3^2 - 1)} = \sqrt{3^2\left(1 - \frac{1}{9}\right)} = 3\left(1 - \frac{1}{9}\right)^{\frac{1}{2}},$$

from which the development may be effected as before.

EXERCISES.

1. Compute the square root of 8 to 6 decimals, and from it find the square root of 2 by § 183.

2. Develop $(1 - x)^{\frac{1}{2}}$.

3. Develop $(1 - x)^{-\frac{1}{2}}$, and express the term in x^i.

$$\textit{Ans.} \quad 1 + \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \text{etc.}$$

$$\text{Term in } x^i = \frac{1 \cdot 3 \cdot 5 \ldots . 2i - 1}{2 \cdot 4 \cdot 6 \ldots . 2i}x^i.$$

4. Develop $\dfrac{1}{(1 + x)^{\frac{1}{2}}}$ and express the general term.

5. Develop $\left(1 + \dfrac{1}{x}\right)^m$ and express the general term.

6. Develop $(1 - x)^{\frac{1}{m}}$, and express the general term.

7. Develop the m^{th} root of $1 + m$.

8. Develop $(a - b)^{-3}$, when $a < b$.

9. Develop $(1 - x)^{-m}$, when $x > 1$.

Because the development will not be convergent in ascending powers of x when $x > 1$, we transform thus:

$$1 - x = -x\left(1 - \frac{1}{x}\right),$$

and so put $\quad (1 - x)^{-m} = (-x)^{-m}\left(1 - \dfrac{1}{x}\right)^{-m}.$

10. Develop the m^{th} power of $1 + \dfrac{1}{m}$.

11. Compute the cube root of 1610 to six decimals.

12. Develop $(\sqrt{a} + \sqrt{b})^n$.

13. Using the functional notation,

$$\phi(m) = 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \binom{m}{3}x^3 + \text{etc.},$$

multiply the two series, $\phi(m)$ and $\phi(n)$, and show by the formulæ of § 261 that the product is equal to $\phi(m + n)$.

The Exponential Theorem.

309. Let it be required, if possible, to develop a^x in powers of x, a being any quantity whatever. Assume

$$a^x = C_0 + C_1x + C_2x^2 + C_3x^3 + \text{etc.} \qquad (1)$$

to be true for all values of x. Putting any other quantity y in place of x, we shall have

$$a^y = C_0 + C_1y + C_2y^2 + C_3y^3 + \text{etc.} \qquad (2)$$

By the law of exponents we must always have

$$a^x \times a^y = a^{x+y}.$$

Now the value of a^{x+y} is found by writing $x + y$ for x in (1), which gives

$$a^{x+y} = C_0 + C_1(x+y) + C_2(x+y)^2 + C_3(x+y)^3 + \text{etc.} \qquad (3)$$

On the other hand, by multiplying equations (1) and (2), we find

$$\begin{array}{rrrrrr} a^xa^y = & C_0{}^2 + & C_0C_1y + & C_0C_2y^2 + & C_0C_3y^3 & + \text{etc.} \\ & & + C_0C_1x + & C_1{}^2xy + & C_1C_2xy^2 & + \text{etc.} \\ & & & + C_0C_2x^2 + & C_1C_2x^2y & + \text{etc.} \\ & & & & + C_0C_3x^3 & + \text{etc.} \end{array} \qquad (4)$$

By § 285, the coefficients of all the products of like powers of x and y must be equal. By equating them, we shall have more equations than there are quantities to be determined, and, unless these equations are all consistent, the development is impossible. To facilitate the process of comparison, we have in equation (4) arranged all terms which are homogeneous in x and y under each other.

By putting $x = 0$ in (1), we find

$$a^0 = C_0, \qquad \text{whence} \qquad C_0 = 1. \quad (\S\ 103.)$$

Comparing the terms of the first degree in x and y in (3) and (4), we find

$$\begin{aligned} &\text{Coefficient of } x, & C_1 &= C_0 C_1\,; \\ &\text{“} \qquad \text{“} \quad y, & C_1 &= C_0 C_1. \end{aligned}$$

These two equations are the same, and agree with $C_0 = 1$; but neither of them gives a value for C_1, which must therefore remain undetermined.

Comparing the terms of the second degree, we find, by developing $(x + y)^2$,

$$C_2\,(x^2 + 2xy + y^2) = C_2 x^2 + C_1{}^2 xy + C_2 y^2,$$

which gives $$2C_2 = C_1{}^2,$$

whence $$C_2 = \frac{1}{1 \cdot 2}\, C_1{}^2.$$

Comparing the terms of the third order in the same way, we have

$$C_3\,(x^3 + 3x^2y + 3xy^2 + y^3) = C_3 x^3 + C_1 C_2 x^2 y + C_1 C_2 xy^2 + C_3 y^3,$$

which gives $$3C_3 = C_1 C_2 = \frac{1}{2}\, C_1{}^3;$$

whence $$C_3 = \frac{1}{1 \cdot 2 \cdot 3}\, C_1{}^3.$$

If the successive values of C follow the same law, we shall have

$$C_4 = \frac{1}{4!}\, C_1{}^4;$$

and in general, $$C_n = \frac{1}{n!}\, C_1{}^n. \qquad (5)$$

Let us now investigate whether these values of C render the equations (3) and (4) identically equal.

Let us consider the corresponding terms of the n^{th} degree, n being any positive integer. In (3) this term will be

$$C_n\,(x + y)^n.$$

Expanding, it will be

$$C_n\left[x^n + nx^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \text{etc.}\right] \quad (6)$$

In (4) the sum of the corresponding terms will be, putting $C_0 = 1$,

$$C_n x^n + C_1 C_{n-1} x^{n-1} y + C_2 C_{n-2} x^{n-2} y^2 + C_3 C_{n-3} x^{n-3} y^3 + \text{etc.} \quad (7)$$

The first terms in the two expressions are identical.

The comparison of the second terms gives

$$nC_n = C_1 C_{n-1}, \qquad \text{whence} \qquad C_n = \frac{C_1}{n} C_{n-1}.$$

This corresponds with (5), because (5) gives

$$C_{n-1} = \frac{1}{(n-1)!} C_1^{n-1},$$

and if we substitute this value of C_{n-1} in the preceding expression for C_n, it will become

$$C_n = \frac{C_1^n}{n(n-1)!} = \frac{C_1^n}{n!},$$

which agrees with (5).

The third terms of (6) and (7) being equated give

$$\binom{n}{2} C_n = C_2 C_{n-2}.$$

Substituting the values of C_n, C_2, and C_{n-2} assumed in the general form (5), we have

$$\binom{n}{2}\frac{1}{n!}C_1^n = \frac{1}{2!}\cdot\frac{1}{(n-2)!}C_1^n,$$

and we wish to know if this equation is true.

Multiplying both sides by $n!$ and dropping the common factor C_1^n, it becomes

$$\binom{n}{2} = \frac{n!}{2!\,(n-2)!},$$

which is an identical equation.

In the same way, the comparison of the following terms in (6) and (7) give

$$\binom{n}{3} = \frac{n!}{3!\,(n-3)!}, \qquad \binom{n}{4} = \frac{n!}{4!\,(n-4)!}, \qquad \text{etc.,}$$

all of which are identical equations. Hence the conditions of the development, namely, that (6) and (7), and therefore (3) and (4), shall be identically equal, are all satisfied by the values of the coefficients C in (5). Substituting those values in (1), the development becomes

$$a^x = 1 + C_1x + \frac{1}{1\cdot 2}C_1^2x^2 + \frac{1}{1\cdot 2\cdot 3}C_1^3x^3 + \text{etc.} \qquad (8)$$

This development is called the **Exponential Theorem,** as the development of $(a+b)^n$ is called the binomial theorem.

310. The value of C_1 is still to be determined. To do this, assign to x the particular value $\frac{1}{C_1}$. Then the equation (8) becomes

$$a^{\frac{1}{C_1}} = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \text{etc., } ad\ inf. \qquad (9)$$

The second member of this equation is a pure number, without any algebraic symbol. We can readily compute its approximate value, since dividing the third term by 3 gives the fourth term, dividing this by 4 gives the fifth, etc. Then

$1 + 1 =$	2.000000
$1 \div 1\cdot 2 =$	.500000
$1 \div 1\cdot 2\cdot 3 =$	.166667
$1 \div 1\cdot 2\cdot 3\cdot 4 =$	.041667
$1 \div 1\cdot 2\cdot 3\cdot 4\cdot 5 =$	.008333
$1 \div 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 =$	.001389
$1 \div 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7 =$	.000198
$1 \div 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8 =$	.000025
$1 \div 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9 =$	.000003
Sum of the series to 6 decimals,	2.718282

This constant number is extensively used in the higher mathematics and is called the *Naperian base.** It is represented for shortness by the symbol e, so that $e = 2.718282....$

The last equation is therefore written in the form

$$a^{\frac{1}{C_1}} = e.$$

* After Baron Napier, the inventor of logarithms.

Raising to the $C_1{}^{th}$ power, we have $a = e^{C_1}$. Hence:

The quantity C_1 is the exponent of the power to which we must raise the constant e to produce the number a.

We may assign one value to a, namely, e itself, which will lead to an interesting result. Putting $a = e$, we have $C_1 = 1$, and the exponential series gives

$$e^x = 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.} \qquad (10)$$

If we put $x = 1$, we have the series for e itself, and if we put $x = -1$, we have

$$e^{-1} = \frac{1}{e} = 1 - 1 + \frac{1}{1 \cdot 2} - \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} - \text{etc.}$$

We thus have the curious result that this series and (9) are the reciprocals of each other.

EXERCISES.

1. Substitute in the first four or five terms of the expressions (6) and (7) the values of C_2, C_3, C_{n-2}, etc., given by (5), and show that (6) and (7) are thus rendered identically equal.

NOTE. This is merely a slight modification of the process we have actually followed in comparing the coefficients of like powers of x and y in (6) and (7).

2. Compute arithmetically the values of 2.7183^2, 2.7183^{-1}, and 2.7183^{-2}, and show that they are the same numbers, to three places of decimals, that we obtain by putting $x = 2$, $x = -1$, and $x = -2$ in (10), and computing the first eight or ten terms of the series.

3. Since $e^{1+x} = ee^x$, the equation (10) gives, by substituting the developments of e^{1+x} and e^x,

$$1 + 1 + x + \frac{(1+x)^2}{2!} + \frac{(1+x)^3}{3!} + \frac{(1+x)^4}{4!} + \text{etc.}$$
$$= e\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \text{etc.}\right).$$

It is required to prove the identity of these developments, by showing that the coefficients of like powers of x are equal.

CHAPTER VI.

LOGARITHMS.

311. To form the logarithm of a number, a constant number is assumed at pleasure and called the *base*.

Def. The **Logarithm** of a number is the exponent of the power to which the base must be raised to produce the number.

The logarithm of x is written $\log x$.

Let us put a, the base ;
x, the number ;
l, the logarithm of x.

Then $$a^l = x.$$

Rem. For every positive value we assign to x there will be one and only one value of l, so long as the base a remains unchanged.

Def. A **System of Logarithms** means the logarithms of all positive numbers to a given base. The base is then called the base of the system.

Properties of Logarithms.

312. Consider the equations,

$$\left.\begin{aligned} a^0 &= 1; \\ a^1 &= a; \\ a^2 &= a^2; \end{aligned}\right\} \text{ whence by definition, } \left\{\begin{aligned} \log 1 &= 0; \\ \log a &= 1; \\ \log a^2 &= 2. \end{aligned}\right.$$

Hence,

I. *The logarithm of* 1 *is zero, whatever be the base.*

II. *The logarithm of the base is* 1.

III. *The logarithm of any number between* 1 *and the base is a positive fraction.*

IV. *The logarithms of powers of the base are integers, but no other logarithms are.*

Again we have

$$\left.\begin{aligned} a^{-1} &= \frac{1}{a}; \\ a^{-2} &= \frac{1}{a^2}; \\ a^{-n} &= \frac{1}{a^n}; \end{aligned}\right\} \text{ whence by definition, } \left\{\begin{aligned} \log\frac{1}{a} &= -1; \\ \log\frac{1}{a^2} &= -2; \\ \log\frac{1}{a^n} &= -n. \end{aligned}\right.$$

Hence,

V. *The logarithm of a number between* 0 *and* 1 *is negative.*

Again, as we increase n, the value of a^n increases without limit, and that of $\frac{1}{a^n}$ approaches zero as its limit. Hence,

VI. *The logarithm of* 0 *is negative infinity.*

VII. THEOREM. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

Proof. Let p and q be two factors, and suppose

$$h = \log p, \qquad k = \log q.$$

Then $\quad a^h = p, \qquad a^k = q.$

Multiplying, $\quad a^h a^k = a^{h+k} = pq.$

Whence, by definition,

$$h + k = \log(pq),$$

or

$$\log p + \log q = \log(pq).$$

The proof may be extended to any number of factors.

VIII. THEOREM. *The logarithm of a quotient is found by subtracting the logarithm of the divisor from that of the dividend.*

Proof. Dividing instead of multiplying the equations in the last theorem, we have

$$\frac{a^h}{a^k} = a^{h-k} = \frac{p}{q}.$$

Hence, by definition, $$h - k = \log \frac{p}{q},$$

or $$\log p - \log q = \log \frac{p}{q}.$$

IX. THEOREM. *The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

Proof. Let $h = \log p$, and let n be the exponent.

Then $$a^h = p.$$

Raising both sides to the n^{th} power,

$$a^{nh} = p^n.$$

Whence $$nh = \log p^n,$$

or $$n \log p = \log p^n.$$

X. THEOREM. *The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root.*

Proof. Let s be the number, and let p be its n^{th} root, so that

$$p = \sqrt[n]{s} \quad \text{and} \quad s = p^n.$$

Hence, $$\log s = \log p^n = n \log p. \quad \text{(IX.)}$$

Therefore, $$\log p = \frac{\log s}{n},$$

or $$\log \sqrt[n]{s} = \frac{\log s}{n}.$$

NOTE. We may also apply Th. IX, since $p = s^{\frac{1}{n}}$. Considering $\frac{1}{n}$ as a power, the theorem gives

$$\log p = \frac{1}{n} \log s.$$

EXERCISES.

Express the following logarithms in terms of $\log p$, $\log q$, $\log x$, and $\log y$, a being the base of the system:

1. Log p^2q. *Ans.* $2 \log p + \log q$.
2. Log pq^3.
3. Log p^2q^5.
4. Log $pq^2x^3y^4$.
5. Log $\dfrac{x}{p} = \log xp^{-1}$, and explain the identity.
6. Log $\dfrac{xy}{pq} = \log xyp^{-1}q^{-1}$.
   Ans. $\text{Log } x + \log y - \log p - \log q$.
7. Log $\dfrac{xy^2}{pq^2}$.
8. Log $\dfrac{x^ny^3}{p^mq^3}$.
9. Log $\sqrt{x}$ (Note, § 123).
10. Log $\sqrt[3]{x}\sqrt{y}$.
11. Log $\sqrt{\dfrac{p}{q}}$.
12. Log $\sqrt{a}$.
13. Log ax.
14. Log $\dfrac{x}{a}$.
15. Log $\dfrac{x}{a^n}$.
16. Log $\dfrac{a^np^m}{x^2y^3}$.
17. Log $\sqrt{a^2 - x^2}$. *Ans.* $\dfrac{\text{Log}(a+x) + \log(a-x)}{2}$.
18. Log $\sqrt{1 - x^2}$.
19. Log $(a^2 - x^2)$.

Logarithmic Series.

313. Rem. The logarithm of a number cannot be developed in powers of the number. For, if possible, suppose

$$\log x = C_0 + C_1x + C_2x^2 + \text{etc.}$$

Supposing $x = 0$, we have

$$C_0 = \log 0,$$

which we have found to be negative infinity (§ 312, VI). Hence the development is impossible.

But we can develop $\log(1 + y)$ in powers of y. For this purpose, we develop $(1 + y)^x$ by both the binomial and exponential theorems, and compare the coefficients of the first power of x. First, the binomial theorem gives

$$(1 + y)^x = 1 + xy + \frac{x(x-1)}{1 \cdot 2}y^2 + \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3}y^3 + \text{etc.}$$

If we develop the coefficients of y^2, y^3, etc., by performing the multiplications, we have

$$\text{Coef. of } y^2 = \frac{x^2 - x}{1 \cdot 2}; \qquad \text{part in } x = -\frac{x}{2}.$$

$$\text{“} \quad \text{“} \quad y^3 = \frac{x(x^2 - 3x + 2)}{2 \cdot 3}; \qquad \text{“} \quad \text{“} \quad x = +\frac{x}{3}.$$

In general, in the coefficient of y^n, or

$$x(x-1)(x-2) \dots (x-n+1),$$

the term containing the first power of x will be

$$\frac{\pm 1 \cdot 2 \cdot 3 \dots (n-1)x}{1 \cdot 2 \cdot 3 \dots n} = \pm \frac{x}{n}.$$

Hence,

$$(1+y)^x = 1 + x\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \text{etc.}\right) + \text{terms in } x^2, x^3, \text{etc.}$$

On the other hand, the exponential development, § 309, (8), gives, by putting $1 + y$ for a.

$$(1+y)^x = 1 + C_1 x + \text{terms in } x^2, x^3, \text{etc.}$$

Equating the coefficients of x in these two identical series we have

$$C_1 = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \text{etc.} \tag{1}$$

The value of C_1 is given by the theorem of § 310, putting $1 + y$ for a; that is, C_1 is here defined by the equation

$$e^{C_1} = 1 + y.$$

Hence, if we take the number e (§ 310) as the base of a system of logarithms, we shall have

$$C_1 = \log(1+y).$$

Comparing with (1), we reach the conclusion:

THEOREM. *Assuming the Naperian base e as a base, we have*

$$\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \text{etc., } ad\ inf. \tag{2}$$

Def. Logarithms to the base e are called **Naperian Logarithms**, or **Natural Logarithms.**

The appellation "natural" is used, because this is the simplest system of logarithms.

Rem. The series (2) is not convergent when $y > 1$, and therefore must be transformed for use.

Putting $-y$ for y in (2), we have

$$\log(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \text{etc.}$$

Subtracting this from (2), and noticing that

$$\log(1+y) - \log(1-y) = \log\frac{1+y}{1-y} \text{ (Th. VIII)},$$

we have
$$\log\frac{1+y}{1-y} = 2y + \frac{2y^3}{3} + \frac{2y^5}{5} + \text{etc.} \qquad (3)$$

Now n being any number of which we wish to investigate the logarithm, let us suppose $y = \frac{1}{2n+1}$. This will give

$$\frac{1+y}{1-y} = \frac{n+1}{n},$$

whence
$$\log\frac{1+y}{1-y} = \log\frac{n+1}{n} = \log(n+1) - \log n.$$

Substituting these values in (3), we have

$$\log(n+1) - \log n = \frac{2}{2n+1} + \frac{2}{3(2n+1)^3} + \frac{2}{5(2n+1)^5} + \text{etc.} \qquad (4)$$

This series enables us to find $\log(n+1)$ when we know $\log n$. To find $\log 2$, we put $n = 1$, which, because $\log 1 = 0$, gives

$$\log 2 = 2\left(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \text{etc.}\right)$$

Summing five terms of this series, we find

$$\log 2 = 0.693147 \ldots.$$

Putting $n = 2$ in (4), we have

$$\log 3 = \log 2 + 2\left(\frac{1}{5} + \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} + \frac{1}{7\cdot 5^7} + \text{etc.}\right),$$

which gives $\qquad \log 3 = 1.098612.$

Because $9 = 3^2$, $\quad \log 9 = 2 \log$ [illegible] .197224.

Putting $n = 9$ in (4), we have

$$\log 10 = \log 9 + 2\left(\frac{1}{19} + \frac{1}{3\cdot 19^3} + \frac{1}{5\cdot 19^5} + \text{etc.}\right),$$

whence $\qquad \log 10 = 2.302585.$

In this way the Naperian logarithms of all numbers may be computed. It is only necessary to compute the logarithms of the prime numbers from the series, because those of the composite numbers can be formed by adding the logarithms of their prime factors. (§ 312, VII.)

314. *Definitive Form of the Exponential Series.* We are now prepared to give the exponential series (§ 309, 8) its definite form. Since the coefficient C_1 is defined by the equation

$$e^{C_1} = a,$$

the quantity C is the Naperian logarithm of a. Hence, the exponential series is

$$a^x = 1 + \frac{x \log a}{1} + \frac{(x \log a)^2}{2\,!} + \frac{(x \log a)^3}{3\,!} + \text{etc.},$$

which is a fundamental development in Algebra.

By putting $a = e$, we have $\log a = 1$, and the series becomes that for e^x already found.

By putting $x = 1$, we have an expression for any number in terms of its natural logarithm, namely,

$$a = 1 + \frac{\log a}{1} + \frac{(\log a)^2}{2\,!} + \frac{(\log a)^3}{3\,!} + \frac{(\log a)^4}{4\,!} + \text{etc.}$$

Comparison of Two Systems of Logarithms.

315. Put $\quad e$, the base of one system ;
a, the base of another;
n, a number of which we take the logarithm in both systems.

Putting l and l' for the logarithms in the two systems, we have

$$e^{l} = n, \qquad a^{l'} = n,$$

and therefore $$e^{l} = a^{l'}. \tag{1}$$

Now put k for the logarithm of a to the base e. Then

$$e^{k} = a,$$

and raising both members to the l'^{th} power,

$$e^{kl'} = a^{l'}.$$

Comparing with (1), $$l = kl',$$

or $$l' = l \times \frac{1}{k}. \tag{2}$$

This equation is entirely independent of n, and is therefore the same for all values of n. Hence,

THEOREM. *If we multiply the logarithm of any number to the base a by the logarithm of a to the base e, we shall have the logarithm of the number to the base e.*

316. Although there may be any number of systems of logarithms, only two are used in practice, namely:

1. The natural or Naperian system, base $= e =$ 2.718282 . . . .
2. The common system, base $= 10$.

The natural system is used for purely algebraic purposes.

The common system is used to facilitate numerical calculations.

Assigning these values to e and a in the preceding section, the constant k is the natural logarithm of 10, which we have found to be 2.302585.

Therefore, by (2), for any number,

$$\text{nat. log} = \text{common log} \times 2.302585,$$

and $$\text{common log} = \frac{\text{nat. log.}}{2.302585\ldots.}$$
$$= \text{nat. log} \times 0.4342944\ldots.$$

Hence,

THEOREM. *The common logarithm of any number may be found by multiplying its natural logarithm by* 0.4342944 .... *or by the reciprocal of the Naperian logarithm of* 10.

Def. The number 0.4342944 is called the **Modulus** of the common system of logarithms.

EXERCISES.

1. Show that if a and b be any two bases, the logarithm of a to the base b and the logarithm of b to the base a are the reciprocals of each other.

2. What does this theorem express in the case of the natural and common systems of logarithms?

Common Logarithms.

317. Because

$$\left.\begin{array}{rl} 10^2 &= 100, \\ 10^1 &= 10, \\ 10^0 &= 1, \\ 10^{-1} &= \frac{1}{10}, \\ 10^{-2} &= \frac{1}{100}, \end{array}\right\} \text{ we have to base 10, } \left\{\begin{array}{rl} \log 100 &= 2, \\ \log 10 &= 1, \\ \log 1 &= 0, \\ \log \frac{1}{10} &= -1, \\ \log \frac{1}{100} &= -2, \end{array}\right.$$

etc. etc.

The following conclusions respecting common logarithms will be evident from an inspection of the above examples:

I. *The logarithm of any number between* 1 *and* 10 *is a fraction between* 0 *and* 1.

II. *The logarithm of a number with two digits is* 1 *plus some fraction.*

III. *In general, the logarithm of a number of i digits is $i - 1$, plus some fraction.*

IV. *The logarithm of a fraction less than unity is negative.*

V. *The logarithms of two numbers, the reciprocal of each other, are equal and of opposite signs.*

VI. *If one number is 10 times another, its logarithm will be greater by unity.*

Proof. If $$10^l = n,$$
then $$10^{1+l} = 10 \times 10^l = 10n.$$

Hence, if $$l = \log n,$$
then $$l + 1 = \log 10n.$$

318. To give an idea of the progression of logarithms, the following table of logarithms of the first 11 numbers should be studied.

The logarithms are not exact, because all logarithms, except those of powers of 10, are irrational numbers, and therefore when expressed as decimals extend out indefinitely. We give only the first two decimals.

$\log 1 = 0.00,$	$\log 7 = 0.85,$
$\log 2 = 0.30,$	$\log 8 = 0.90,$
$\log 3 = 0.48,$	$\log 9 = 0.95,$
$\log 4 = 0.60,$	$\log 10 = 1.00,$
$\log 5 = 0.70,$	$\log 11 = 1.04.$
$\log 6 = 0.78,$	

It will be noticed that the difference between two consecutive logarithms continually diminishes as the numbers increase. For instance, the difference between log 20 and log 10 must by § 312, VIII, be the same as between log 1 and log 2.

319. *Computation of Logarithms.* Since the logarithms of all composite numbers may be found by adding the logarithms of their factors, it is only necessary to show how the logarithms of prime numbers are computed. We have already shown (§ 313) how the natural logarithms may be computed, and (§ 316) how the common ones may be derived from them by multiplying by the modulus 0.4342944.... It is not however necessary to multiply the whole logarithm by this factor, but we may proceed thus:

We have, putting M for the modulus,

$$\text{com. } \log n = M \text{ nat. } \log n,$$
$$\text{com. } \log (n + 1) = M \text{ nat. } \log (n + 1);$$

whence, by subtraction,

$$\text{com.}\log(n+1) - \text{com.}\log n = M[\text{nat.}\log(n+1) - \text{nat.}\log n];$$

and, by substituting for nat. log $(n+1)$ — nat. log n its value, § 313,

$$\text{com.}\log(n+1) = \text{com.}\log n + 2M\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \text{etc.}\right]$$

By means of this series, the computations of the successive logarithms may be carried to any extent.

Tables of the logarithms of numbers up 100,000, to seven places of decimals, are in common use for astronomical and mathematical calculations. One table to ten decimals was published about the beginning of the present century. The most extended tables ever undertaken were constructed under the auspices of the French government about 1795, and have been known under the name of *Les Grandes Tables du Cadastre.* Many of the logarithms were carried to nineteen places of decimals. They were never published, but are preserved in manuscript.

320. It may interest the student who is fond of computation to show how the common logarithms of small numbers may be computed by a method based immediately on first principles.

Let n be a number, and let $\frac{p}{q}$ be an approximate value of its logarithm. We shall then have,

$$n = 10^{\frac{p}{q}},$$

or, raising to the q^{th} power,

$$n^q = 10^p.$$

Hence, could we find a power of the number which is also a power of 10, the ratio of the exponents would at once give the logarithm. This can never be exactly done with whole numbers, but, if the condition be approximately fulfilled, we shall have an approximate value of the logarithm.

Let us seek log 2 in this way. Forming the successive powers of 2, we find

$$2^{10} = 1024 = 10^3\,(1.024). \qquad (1)$$

Hence, $3 : 10 = 0.3$ is an approximation to log 2. To find a second approximation, we form the powers of 1.024 until we reach a number nearly equal to 2 or 10, or the quotient of any power of 2 by a power of 10. Suppose, for instance, that we find

$$1.024^x = 2.$$

Because $1.024 = 2^{10} \div 10^3$, this equation will give

$$\left(\frac{2^{10}}{10^3}\right)^x = 2, \quad \text{or} \quad 2^{10x} = 2.10^{3x}, \quad \text{or} \quad 2^{10x-1} = 10^{3x},$$

which will give
$$\log 2 = \frac{3x}{10x - 1}.$$

If we form the powers of 1.024 by the binomial theorem, or in any other way, we shall find that x is between 29 and 30, from which we conclude that $\log 2 = 0.301$ nearly.

To obtain a yet more exact value, we form the 30th power of 1.024 to six or seven decimals, and put it in the form

$$1.024^{30} = 2(1 + \alpha),$$

where α will be a small fraction.

Then we find what power of $1 + \alpha$ will make 2. Let y be this power. Raising the last equation to the yth power, we have

$$1.024^{30y} = 2^y(1 + \alpha)^y = 2^{y+1}.$$

Putting for 1.024 its value, $2^{10} \div 10^3$, this equation becomes

$$\frac{2^{300y}}{10^{90y}} = 2^{y+1}, \qquad \text{or} \qquad 2^{299y-1} = 10^{90y},$$

whence,
$$\log 2 = \frac{90y}{299y - 1}.$$

By a little care, the value of y can be obtained so accurately that the value of log 2 shall be correct to 8, 9, or 10 places of decimals.

The power to which we must raise $1 + \alpha$ to produce 2 will be approximately $\dfrac{\text{Nap. log } 2}{\alpha}$, when α is very small.

EXERCISES.

1. In the common system ($a = 10$) we have

$$\log 2 = 0.30103, \qquad \log 3 = 0.47712.$$

Hence find the logarithms of 4, 5, 6, 8, 9, 12, $12\frac{1}{2}$, 15, 16, $16\frac{2}{3}$, 18, 20, 250, 6250.

Note that $5 = \frac{10}{2}$, $12\frac{1}{2} = \frac{100}{8}$, $16\frac{2}{3} = \frac{100}{6}$, and apply Th. VIII.

2. How many digits are there in the hundredth power of 2?

3. Given $\log 49 = 1.690196$; find $\log 7$.

4. Given $\log 1331 = 3.124178$; find $\log 11$.

5. Find the logarithm of 105 and 1.05 from the above data?

6. Find the logarithm of 1.05^{10}.

7. If $1 is put out at 5 per cent. per annum compound interest for 1000 years, how many digits will be required to express the amount? (Compare § 216.)

8. Prove the equation

$$\log x = \frac{1}{2}\log(x+1) + \frac{1}{2}\log(x-1)$$
$$+ M\left[\frac{1}{2x^2-1} + \frac{1}{3(2x^2-1)^3} + \frac{1}{5(2x^2-1)^5} + \text{etc.}\right]$$

9. If $y = \log n$, of what numbers will $y+1$, $y+2$, $y-1$, and $y-2$ be the logarithms?

10. Find x from the equation $c^x = h$.

Solution. Taking the logarithms of both members, we have

$$x \log c = \log h;$$

whence,
$$x = \frac{\log h}{\log c}.$$

11. $c^{ax} = n$.

12. $c^{bx} = \dfrac{1}{m}$.

13. $b^x = \dfrac{1}{p}$.

14. $b^{-x} = p$.

Show that the answers to (13) and (14) are and ought to be identical.

15. $a^{cx} = m$.

16. $bc^x = k$.

17. Find x and y from the equations

$$a^x b^y = p, \qquad a^y b^x = q.$$

BOOK XII.

IMAGINARY QUANTITIES.

CHAPTER I.

OPERATIONS WITH THE IMAGINARY UNIT.*

321. Since the square of either a negative or a positive quantity is always positive, it follows that if we have to extract the square root of a negative quantity, no answer is possible, in ordinary positive or negative numbers (§§ 170, 200).

In order to deal with such cases, mathematicians have been led to *suppose* or *imagine* a kind of numbers of which the squares shall be negative. These numbers are called **Imaginary Quantities,** and their units are called **Imaginary Units,** to distinguish them from the ordinary positive and negative quantities, which are called *real.*

322. *The Imaginary Unit.* Let us have to extract the square root of -9. It cannot be equal to $+3$ nor to -3, because the square of each of these quantities is $+9$. We may therefore call the root $\sqrt{-9}$, just as we put the sign $\sqrt{\ }$ before any other quantity of which the root cannot be extracted. But the root may be transformed in this way:

Since $\qquad -9 = +9 \times -1,$

it follows from § 183 that

$$\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3\sqrt{-1}.$$

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* It is not to be expected that a beginner will fully understand this subject at once. But he should be drilled in the mechanical process of operating with imaginaries, even though he does not at first understand their significance, until the subject becomes clear through familiarity.

Def. The surd $\sqrt{-1}$ is the **Imaginary Unit.** The imaginary unit is commonly expressed by the symbol i.

This symbol is used because it is easier to write i than $\sqrt{-1}$.

The unit i is a supposed quantity such that, when squared, the result is -1.

That is, i is defined by the equation

$$i^2 = -1.$$

THEOREM. *The square root of any negative quantity may be expressed as a number of imaginary units.*

For let $-n$ be the number of which the root is required.

Then $\sqrt{-n} = \sqrt{+n}\sqrt{-1} = \sqrt{n}i.$

Hence,

To extract the square root of a negative quantity, extract the root as if the quantity were positive, and affix the symbol i to it.

323. *Complex Quantities.* In ordinary algebra, any number may be supposed to mean so many units. 7 or a, for example, is made up of 7 units or a units, and might be written $7\cdot 1$ or $a1$.

When we introduce imaginary quantities, we consider them as made up of a certain number of imaginary units, each represented by the sign i, just as the real unit is represented by the sign 1. A number b of imaginary units is therefore written bi.

A sum of a real units and b imaginary units is written

$$a + bi,$$

and is called a complex quantity. Hence,

Def. A **Complex Quantity** consists of the sum of a certain number of real units plus a certain number of imaginary units.

Def. When any expression containing the symbol of the imaginary unit is reduced to the form of a complex quantity, it is said to be expressed in its **Normal Form.**

Addition of Complex Expressions.

324. The algebraic operations of addition and subtraction are performed on imaginary quantities according to nearly the same rules which govern the case of surds (§ 181), the surd being replaced by i. Thus,

$$a\sqrt{-1} + b\sqrt{-1} = ai + bi = (a + b)\,i.$$

Hence the following rule for the addition and subtraction of imaginary quantities:

Add or subtract all the real terms, as in ordinary algebra. Then add the coefficients of the imaginary unit, and affix the symbol i to their sum.

Example. Add $a + bi$, $6 + 7i$, $5 - 10i$, and subtract $3a - 2bi + z$ from the sum.

We may arrange the work as follows:

$$\begin{array}{rl}
 & a + bi \\
 & 6 + 7i \\
 & 5 - 10i \\
 & \underline{-\,z - 3a + 2bi} \quad \text{(sign changed).} \\
\text{Sum,} & -\,z - 2a + 11 + (3b - 3)\,i.
\end{array}$$

EXERCISES.

1. Add $3x + 4yi + m$, $2m + 5xi$, $6m - 6yi$.
2. Add $4ai$, $17i$, $3a + 6bi$, $x + yi$.
3. From the sum $a + bi + m - ni - p + qi$ subtract the sum $+\,yi - z - ui$.

Reduce to the normal form:

4. $a + bi - (m - ni) - (x + yi)$.
5. $m\,(a - bi) - n\,(x - yi)$.

Multiplication of Complex Quantities.

325. Theorem. *All the even powers of the imaginary unit are real units, and all its odd powers are imaginary units, positive or negative.*

Proof. The imaginary unit is by definition such a symbol as when squared will make -1. Hence,

$$i^2 = -1.$$

Now multiply both sides of this equation by i a number of times in succession, and substitute for each power of i its value given by the preceding equation. We then have

$$\begin{aligned}
i^3 &= -i, \\
i^4 &= -i^2 = +1 \text{ (because } i^2 = -1), \\
i^5 &= -i^3 = +i, \\
i^6 &= -i^4 = +i^2 = -1, \\
i^7 &= -i^5 = +i^3 = -i, \\
\text{etc.} & \quad \text{etc.} \qquad\qquad \text{etc.}
\end{aligned}$$

It is evident that the successive powers of i will always have one of the four values, i, -1, $-i$, or $+1$.

$$\left\{\begin{array}{llllll}
i, & i^5, & i^9, & \text{etc., will be equal to} & & i; \\
i^2, & i^6, & i^{10}, & \text{etc.,} \quad `` \quad `` & & -1; \\
i^3, & i^7, & i^{11}, & \text{etc.,} \quad `` \quad `` & & -i; \\
i^4, & i^8, & i^{12}, & \text{etc.,} \quad `` \quad `` & & +1.
\end{array}\right.$$

We may express this result thus:

If n is any integer, then:

$$i^{4n} = 1, \quad i^{4n+1} = i, \quad i^{4n+2} = -1, \quad i^{4n+3} = -i.$$

To multiply or divide imaginary quantities, we proceed as if they were real and substitute for each power of i its value as a real or imaginary, positive or negative unit.

Ex. 1. Multiply ai by xi.

By the ordinary method, we should have the product, axi^2. But $i^2 = -1$. The product is therefore $-ax$.

That is, $$ai \times xi = -ax.$$

Ex. 2. Multiply $a + bi$ by $m + ni$.

$$\begin{aligned}
ni\,(a + bi) &= ani - bn \text{ (because } ni \times bi = -bn) \\
m\,(a + bi) &= \underline{bmi + am} \\
(m + ni)\,(a + bi) &= am - bn + (an + bm)\,i,
\end{aligned}$$

which is the product required.

EXERCISES.

Multiply

1. $x + yi$ by $a - b$.
2. $m + ni$ by ai.
3. $m - ni$ by bi.
4. $1 + i$ by $1 - i$.
5. $x - yi$ by $a + bi$.
6. $x - yi$ by $x + yi$.
7. $a - ai - bi$ by $a + ai + bi$.

Develop

8. $(a + bi)^2$.
9. $(m + ni)^3$.
10. $(1 + i)^2$.
11. $(1 - i)^2$.

326. *Imaginary Factors.* The introduction of imaginary units enables us to factor expressions which are prime when only real factors are admitted. The following are the principal forms:

$$a^2 + b^2 = (a + bi)(a - bi),$$

$$a^2 - b^2 \pm 2abi = (a \pm bi)^2.$$

The first form shows that the sum of two squares can always be expressed as a product of two complex factors.

For example, $17 = 4^2 + 1^2 = (4 + i)(4 - i)$.

EXERCISES.

Factor the expressions:

1. $x^2 + 4$.
2. $x^2 + 2$.
3. $x^2 - 2x + 5 = (x - 1)^2 + 4$.
4. $x^2 - 4x + 13$
5. $a + b$.
6. $a^2 + 2an + 5n^2$.
7. $x^2 + 2xy + 2y^2$.

327. FUNDAMENTAL PRINCIPLE. *A complex quantity $A + Bi$ cannot be equal to zero unless we have both*

$$A = 0 \qquad \text{and} \qquad B = 0.$$

Proof. If A and B were not zero, the equation $A + Bi = 0$ would give

$$i = -\frac{A}{B},$$

that is, the imaginary unit equal to a real fraction, which is impossible.

Cor. If both members of an equation containing imaginary units are reduced to the normal form, so that the equation shall be in the form

$$A + Bi = M + Ni,$$

we must have the two equations,

$$A = M, \qquad B = N.$$

For, by transposition, we obtain

$$A - M + (B - N)\,i = 0,$$

whence the theorem gives $A - M = 0$, $B - N = 0$. Hence,

Every equation between complex quantities involves two equations between real quantities, formed by equating the numbers of real and imaginary units.

Reduction of Functions of i to the Normal Form.

328. 1. If we have an entire function of i,

$$a + bi + ci^2 + di^3 + ei^4 + f\,i^5 + \text{etc.},$$

we reduce it by putting

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad \text{etc.,} \quad \text{etc.,}$$

and the expression will become

$$(a - c + e - \text{etc.}) + (b - d + f - \text{etc.})\,i;$$

which, when we put

$$x = a - c + e - \text{etc.}, \qquad y = b - d + f - \text{etc.},$$

becomes $x + yi$, as required.

2. To reduce a rational fraction of i to the normal form, we reduce both numerator and denominator. The fraction will then take the form

$$\frac{a + bi}{m + ni}.$$

Since this is to be reduced to the form $x + yi$, let us put

$$\frac{a + bi}{m + ni} = x + yi,$$

x and y being indeterminate coefficients.

Clearing of fractions,

$$a + bi = mx - ny + (my + nx)\,i.$$

Comparing the number of real and imaginary units on each side of the equation, we have the two equations

$$mx - ny = a, \qquad nx + my = b.$$

Solving them, we find

$$x = \frac{ma + nb}{m^2 + n^2}, \qquad y = \frac{mb - na}{m^2 + n^2}.$$

Therefore, $\dfrac{a + bi}{m + ni} = \dfrac{ma + nb}{m^2 + n^2} + \dfrac{mb - na}{m^2 + n^2}i,$

which is the normal form.

EXERCISES.

Reduce to the normal form :

1. $7 - 3i - 6i^2 + 2i^3 + i^4 - i^5$.

2. $1 + i - i^2 + i^3 - i^4 - i^5 + i^6$.

3. $\dfrac{2}{i - 1}$.

4. $\dfrac{6 + 5i}{6 - 5i}$.

5. $\dfrac{1 + i}{1 - i}$.

6. $\dfrac{mi(x - ai)}{x + ai}$.

7. $\dfrac{1 - i}{2 + 4i}$.

8. $\dfrac{a + bi}{a - bi}$.

9. $\dfrac{(a + bi)(a - bi)}{(x + bi)^2}$.

10. What is the value of the exponential series which gives the development of e^i? We put $x = i$ in § 310, Eq. 10.

11. Develop $(1 + xi)^n$ by the binomial theorem.

12. What are the developed values of

$$(1 + bi)^n + (1 - bi)^n$$

and

$$(1 + bi)^n - (1 - bi)^n?$$

13. Write eight terms of the geometrical progression of which the first term is a and the common ratio i.

14. Find the limit of the sum of the geometrical progression of which the first term is a and the common ratio $\dfrac{i}{2}$.

329. To reduce the square root of an imaginary expression to the normal form.

Let the square root be $\sqrt{a + bi}$.

We put $\qquad x + yi = \sqrt{a + bi}.$

Squaring, $\quad x^2 - y^2 + 2xyi = a + bi.$

Comparing units, $x^2 - y^2 = a,$

$$2xy = b.$$

Solving this pair of quadratic equations, we find

$$x = \frac{\sqrt{(\sqrt{a^2+b^2}+a)}}{\sqrt{2}},$$

$$y = \frac{\sqrt{(\sqrt{a^2+b^2}-a)}}{\sqrt{2}}.$$

Therefore,

$$\sqrt{a+bi} = \sqrt{\left(\frac{\sqrt{a^2+b^2}+a}{2}\right)} + \sqrt{\left(\frac{\sqrt{a^2+b^2}-a}{2}\right)}\, i.$$

EXERCISES.

Reduce the square roots of the following expressions to the normal form:

1. $3 + 4i$. 2. $4 + 3i$. 3. $12 + 5i$.

4. Find the square roots of the imaginary unit i, and of $-i$, and prove the results by squaring them.

Note that this comes under the preceding form when $a = 0$ and $b = \pm 1$.

5. Find the fourth roots of the same quantities by extracting the square roots of these roots.

330. *Quadratic Equations with Imaginary Roots.* The combination of the preceding operations will enable us to solve any quadratic equation, whether it does or does not contain imaginary quantities.

EXAMPLE 1. Find x from the equation

$$x^2 + 4x + 13 = 0.$$

Completing the square and proceeding as usual, we find

$$x^2 + 4x + 4 = -9,$$

whence $x + 2 = \sqrt{-9} = \pm 3i,$

and $x = -2 \pm 3i.$

Ex. 2. $x^2 + bxi - c = 0.$

Completing the square,

$$x^2 + bxi - \frac{b^2}{4} = c - \frac{b^2}{4}.$$

Extracting the root,

$$x + \frac{bi}{2} = \frac{\sqrt{4c - b^2}}{2},$$

whence $$x = \pm \frac{1}{2}\sqrt{(4c - b^2)} - \frac{bi}{2}.$$

EXERCISES.

Solve the quadratic equations:

1. $x^2 + x + 1 = 0.$ 2. $x^2 - x + 1 = 0.$

3. $x^2 + 3x + 10 = 0.$ 4. $x^2 + 10x + 34 = 0.$

Form quadratic equations (§ 199) of which the roots shall be

5. $a + bi$ and $a - bi.$ 6. $ai + b$ and $ai - b.$

331. *Exponential Functions.* When in the exponential function a^z we suppose z to represent an imaginary expression $x + yi$, it becomes

$$a^{x+yi}.$$

This expression could have no meaning in any of our previous definitions of an exponent, because we have not shown what an imaginary exponent could mean. But if we suppose the effect of the exponent to be defined by the exponential theorem (§§ 309, 314), we can develop the above expression. First we have, by the fundamental law of exponents,

$$a^{x+yi} = a^x a^{yi}.$$

Next, if we put $c =$ Nap. log a, we have

$$a = e^c;$$

whence, $$a^{yi} = e^{cyi}.$$

If we put, for brevity, $cy = u$, we shall now have

$$a^{x+yi} = a^x e^{ui}.$$

The value of a^x being already perfectly understood, we may leave it out of consideration for the present, and investigate the development of e^{ui}. By the exponential theorem (§ 310, 10),

$$e^{ui} = 1 + ui + \frac{u^2i^2}{2!} + \frac{u^3i^3}{3!} + \frac{u^4i^4}{4!} + \frac{u^5i^5}{5!} + \text{etc.}$$

Substituting for the powers of i their values (§ 325),

$$e^{ui} = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \text{etc.} + \left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \text{etc.}\right) i.$$

These two series are each functions of u, to which special names have been given, namely:

Def. The series $1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \frac{u^8}{8!} - \text{etc.}$, is called the **cosine of u,** and is written $\cos u$.

Def. The series $u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \frac{u^9}{9!} - \text{etc.}$, is called the **sine of u,** and is written $\sin u$.

Using this notation, the above development becomes,

$$e^{ui} = \cos u + i \sin u, \qquad (a)$$

which is a fundamental equation of Algebra, and should be memorized.

REMARKS. These functions, $\cos u$ and $\sin u$, have an extensive use in both Trigonometry and Algebra. To familiarize himself with them, it will be well for the student to compute their values from the above series for $i = 0.25$, $i = 0.50$, $i = 1$, $i = 2$, to three or four places of decimals. This can be done by a process similar to that employed in computing e in § 310. If the work is done correctly, he will find:

For	$u = \frac{1}{4}$,	$\cos \frac{1}{4} = 0.969$,	$\sin \frac{1}{4} = 0.247$.
"	$u = \frac{1}{2}$,	$\cos \frac{1}{2} = 0.878$,	$\sin \frac{1}{2} = 0.479$.
"	$u = 1$,	$\cos 1 = 0.540$,	$\sin 1 = 0.841$.
"	$u = 2$,	$\cos 2 = -0.416$,	$\sin 2 = 0.909$.

332. Let us now investigate the properties of the functions $\cos u$ and $\sin u$, which are defined by the equations,

$$\left.\begin{aligned} \cos u &= 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \text{etc.} \\ \sin u &= u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \text{etc.} \end{aligned}\right\} \qquad (b)$$

Since $\cos u$ includes only even powers of u, its value will remain unchanged when we change the sign of u from $+$ to $-$, or *vice versa*. Hence,

$$\cos(-u) = \cos u. \tag{1}$$

Since $\sin u$ contains only odd powers of u, its sign will change with that of u. Hence,

$$\sin(-u) = -\sin u. \tag{2}$$

If in the equation (a) we change the sign of u, we have, by (1) and (2),

$$e^{-ui} = \cos(-u) + i\sin(-u),$$

or

$$e^{-ui} = \cos u - i\sin u.$$

Now multiply this equation by (a). Since

$$e^{ui} \times e^{-ui} = e^{ui} \times \frac{1}{e^{ui}} = 1,$$

we have

$$1 = (\cos u)^2 - i^2(\sin u)^2,$$

or

$$1 = (\cos u)^2 + (\sin u)^2.$$

It is customary to write $\cos^2 u$ and $\sin^2 u$ instead of $(\cos u)^2$ and $(\sin u)^2$, to express the square of the cosine and of the sine of u. The last equation will then be written

$$\cos^2 u + \sin^2 u = 1. \tag{c}$$

Although we have deduced this equation with entire rigor, it will be interesting to test it by squaring the equations (b). First squaring $\cos u$, we find (§ 284),

$$\cos^2 u = 1 - u^2 + u^4\left(\frac{1}{4!} + \frac{1}{2!\,2!} + \frac{1}{4!}\right) - \text{etc.}$$

The coefficient of u^n is found to be

$$\frac{1}{n!} + \frac{1}{2!\,(n-2)!} + \frac{1}{4!\,(n-4)!} + \dots + \frac{1}{n!}$$

when n is double an even number, and to the negative of this expression when n is double an odd number.

Again, taking the square of $\sin u$, we find

$$\sin^2 u = u^2 + u^4\left(-\frac{1}{1!\,3!} - \frac{1}{1!\,3!}\right) + \text{etc.}$$

the coefficient of u^n being

$$-\frac{1}{1!\,(n-1)!}-\frac{1}{3!\,(n-3)!}-\frac{1}{5!\,(n-5)!}-\dots-\frac{1}{(n-1)!\,1!},$$

or the negative of this expression, according as $\frac{1}{2}n$ is even or odd.

Adding $\sin^2 u$ and $\cos^2 u$, we see that the terms u^2 cancel each other, and that the sum of the coefficients of u^4 can be arranged in the form

$$\frac{1}{4!}-\frac{1}{1!\,3!}+\frac{1}{2!\,2!}-\frac{1}{3!\,1!}+\frac{1}{4!}.$$

Let us call this sum A. If we multiply all the terms by $4!$, and note that by the general form of the binomial coefficients,

$$\frac{n!}{s!\,(n-s)!}=\binom{n}{s},$$

we find $$4!\,A=1-\binom{4}{1}+\binom{4}{2}-\binom{4}{3}+\binom{4}{4},$$

which sum is zero, by § 262, Th. II. Therefore the coefficients of u^n cancel each other.

Taking the sum of the coefficients of u^n, we arrange them in the form

$$\frac{1}{n!}-\frac{1}{1!\,(n-1)!}+\frac{1}{2!\,(n-2)!}-\frac{1}{3!\,(n-3)!}+\text{etc.},$$

which call A. Then multiplying by $n!$, we have

$$n!\,A=1-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\dots+\binom{n}{n},$$

which sum is zero. Therefore all the coefficients of u^n cancel each other in the sum $\sin^2 u+\cos^2 u$, leaving only the first term 1 in $\cos^2 u$, thus proving the equation (c) independently.

This example illustrates the consistency which pervades all branches of mathematics when the reasoning is correct. The conclusion (c) was reached by a very long process, resting on many of the fundamental principles of Algebra; and on reaching a simple conclusion of this kind in such a way, the mathematician always likes to test its correctness by a direct process, when possible.

Let us now resume the fundamental equation (a). Since u may here be any quantity whatever, let us put nu for u. The equation then becomes,

$$e^{nui} = \cos nu + i \sin nu.$$

But by raising the equation (a) to the n^{th} power, we have

$$e^{nui} = (\cos u + i \sin u)^n.$$

Hence we have the remarkable relation,

$$(\cos u + i \sin u)^n = \cos nu + i \sin nu.$$

Supposing $n = 2$, and developing the first member, we have

$$\cos^2 u - \sin^2 u + 2i \sin u \cos u = \cos 2u + i \sin 2u.$$

Equating the real and imaginary parts (§ 327, *Cor.*), we have

$$\cos^2 u - \sin^2 u = \cos 2u,$$

$$2 \sin u \cos u = \sin 2u,$$

relations which can be verified from the series representing $\cos u$ and $\sin u$, in a way similar to that by which we verified $\sin^2 u + \cos^2 u = 1$.

EXERCISES.

1. Find the values of $\cos^3 u$, $\sin^3 u$, $\cos^4 u$, and $\sin^4 u$ by the preceding process.

2. Write the three equations which we obtain by putting $u = a$, $u = b$, and $u = a + b$ in equation (a). Then equate the product of the first two to the third, and show that

$$\cos (a + b) = \cos a \cos b - \sin a \sin b,$$
$$\sin (a + b) = \sin a \cos b + \cos a \sin b.$$

3. Reduce to the normal form,

$$(x - i)(x - 2i)(x - 3i)(x - 4i).$$

4. Develop $(a + bi)^{\frac{1}{2}}$ by the binomial theorem, and reduce the result to the normal form.

CHAPTER II.

THE GEOMETRIC REPRESENTATION OF IMAGINARY QUANTITIES.

333. In Algebra and allied branches of the higher mathematics, the fundamental operations of Arithmetic are extended and generalized. In Elementary Algebra we have already had several instances of this extension, and as we are now to have a much wider extension of the operations of addition and multiplication, attention should be directed to the principles involved.

In the beginning of Algebra, we have seen the operation of addition, which in Arithmetic necessarily implies *increase,* so used as to produce *diminution.*

The reason of this is that Arithmetic does not recognize negative quantities as Algebra does, and therefore in employing the latter we have to extend the meaning of addition, so as to apply it to negative quantities. When thus applied, we have seen that it should mean to subtract the quantity which is negative.

In its primitive sense, as used in the third operation of Arithmetic, the word *multiply* means to add a quantity to itself a certain number of times. In this sense, there would be no meaning to the words "multiply by a fraction." But we extend the meaning of the word multiply to this case by defining it to mean taking a fraction of the quantity to be multiplied. We then find that the rules of multiplication will all apply to this extended operation.

This extension of multiplication to fractions does not take account of negative multipliers. In the latter case we can extend the meaning of the operation by providing that the algebraic sign of the quantity shall be changed when the multiplier is negative. We thus have a result for multiplication by every positive or negative algebraic number.

Now that we have to use imaginary quantities as multipliers, a still further extension is necessary. Hitherto our operations with imaginary units have been purely symbolic; that is, we have used our symbols and performed our operations without assigning any definite meaning to them. We shall now assign a geometric signification to operations with imaginary units, subject to these three necessary conditions:

1. The operations must be subject to the same rules as those of real quantities.

2. The result of operating with an imaginary quantity must be totally different from that of operating with a real one, and the imaginary quantity must signify something which a real quantity does not take account of.

3. If the imaginary quantity changes into a real one, the operation must change into the corresponding one with real quantities.

334. *Geometric Representation of Imaginary Units.* Certain propositions respecting the geometric representation of multiplication have been fully elucidated in Part I, and are now repeated, to introduce the corresponding representations of complex quantities.

I. All real numbers, positive and negative, may be arranged along a line, the positive numbers increasing in one direction, the negative ones in the opposite direction from a fixed zero point. Any number may then be represented in magnitude by a line extending from 0 to the place it occupies.

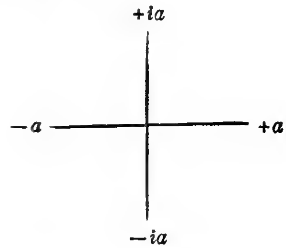
We call this line a **Vector.**

II. If a number a be multiplied by a positive multiplier (for simplicity, suppose $+1$), the direction of its vector will remain unaltered. If it be multiplied by a negative multiplier (suppose -1), its vector will be turned in the opposite direction (from $0-a$ to $0+a$, or *vice versa*). Compare § 72, where the coarse lines are the vectors of the several quantities.

III. If the number be multiplied twice by -1, that is, by $(-1)^2$, its vector will be restored to its first position, being twice turned, and if it be multiplied twice by $+1$, that is, by $(+1)^2$, its vector will not be changed at all. Its vector will therefore be found in its first position, whether we multiply it by the square of a positive or of a negative unit; in other words, both squares are positive.

IV. To multiply the line $+a$ twice by the imaginary unit i, is the same as multiplying it by i^2 or -1. Hence,

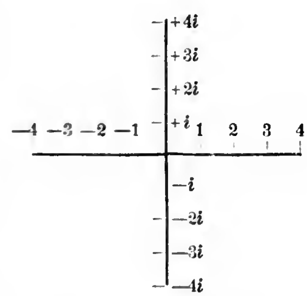
Multiplying by the imaginary unit i *must give the vector such a motion as, if repeated, will change it from* $+a$ *to* $-a$.



Such a motion is given by turning the vector through a right angle, into the position $+ia$. A second motion brings it to the position $-a$, the opposite of $+a$. A third motion brings it to $-ia$, a position the opposite of $+ia$. A fourth motion restores it to the original position $+a$.

If we call each of these motions *multiplying by* i, we have, from the diagram, $a = a$, $ia = ia$, $i^2a = -a$, $i^3a = -ia$, $i^4a = a$, which corresponds exactly to the law governing the powers of i (§ 325). Hence:

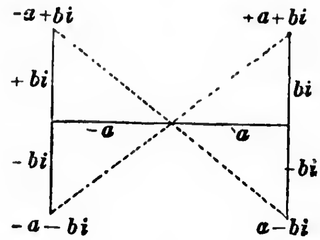
If a quantity is represented by a vector extending from a zero point, the multiplication of this quantity by the imaginary unit may be represented by turning the vector through 90°.



V. In order that multiplier and multiplicand may in this operation be interchanged without affecting the product, we must suppose that the vertical line which we have called ia is the same as ai, that is, that this line represents a imaginary units.

We have therefore to count the imaginary units along a vertical line on the same system that we count the real units on a horizontal line.

335. *Geometric Representation of a Complex Quantity.* We have shown (§ 15) that algebraic addition may be represented by putting lines end to end, the zero point of each line added being at the end of the line next preceding. The distance of the end of the last line from the zero point is the algebraic sum.



On the same system, to represent the algebraic sum of the real and imaginary quantities $a+bi$, we lay off a units on the real (horizontal) line, and then b units from the end of this line in a vertical direction. The end of the vertical line will then be the position corresponding to $a+bi$.

It is evident that we should reach the same point if we first laid off b units from 0 on the imaginary line, and then a units horizontally. Hence this system gives

$$bi + a = a + bi,$$

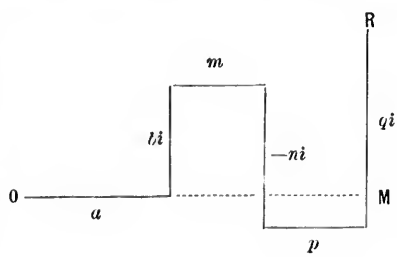
as it ought to, to represent addition.

If a or b is negative, it is to be laid off in the opposite direction from the positive one. We then have the points corresponding to $-a+bi$, $-a-bi$, and $a-bi$, shown in the diagram, which should be carefully studied by the pupil.

The result we have reached is the following:

Every complex quantity $a+bi$ is considered as belonging to a certain point on the plane, namely, that point which is reached by laying off from the zero point a units in the horizontal direction and b units in the vertical direction.

336. *Addition of Complex Quantities.* If we have several complex terms to add, as $a+bi$, $m-ni$, $p+qi$, we may lay them off separately in their appropriate magnitude and di-



rection, as in the figure, the last line terminating in a point R.

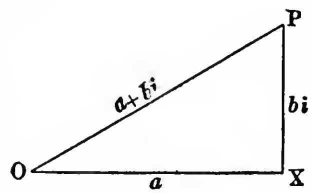
If we first add the quantities $a + bi$, etc., algebraically (§ 324), the result will be

$$a + m + p + (b - n + q)\,i.$$

We may lay off this sum in one operation. The sum $a + m + p$ will carry us from 0 to M, and the sum $(b - n + q)\,i$ from M to R, because $MR = b - n + q$. Therefore we shall reach the same point R whether we lay the quantities off separately, or take their sum and lay off its real and imaginary parts separately.

337. *Vectors of Complex Quantities.* The question now arises by what straight line or vector shall we represent a sum of complex quantities? The answer is:

The vector of a sum of several vectors is the straight line from the beginning of the first to the end of the last vector added.



For example, the sum of the quantities $OX = a$ and $XP = bi$ is the vector OP.

It might seem to the student that the length of the vector representing the sum should be equal to the combined lengths of all the separate vectors. This difficulty is of the same kind as that encountered by the beginner in finding the sum of a positive and negative quantity less than either of them. The solution of the difficulty is simply that by addition we now mean something different from both arithmetical and algebraic addition. But the operation reduces to arithmetical addition when the quantities are all real and positive, because the vectors are then all placed end to end in the same straight line. Therefore there is no inconsistency between the two operations.

Two imaginary quantities are not equal, unless both their real and imaginary parts are equal, so that their sum shall terminate at the same point P. Their vectors will then coincide with each other. Hence:

Two vectors are not considered equal unless they agree in direction as well as length.

In other words, *in order to determine a vector completely, we must know its direction as well as its length.*

This result embodies the theorem of the preceding chapter (§ 327), that two complex quantities are not equal unless both their real and imaginary parts are equal. It is only in case of this double equality that the two complex quantities will belong to the same point on the plane.

Because OXP is a right angle, we have by the Pythagorean theorem of Geometry,

$$(\text{length of vector})^2 = a^2 + b^2,$$

or

$$\text{length of vector} = \sqrt{a^2 + b^2}.$$

We are careful to say *length* of vector, and not merely vector, because the vector has *direction* as well as length, and the direction is as important an element as length.

To avoid repeating the words "length of," we shall put a dash over the letters representing a vector when we consider only its length. Then $\overline{\text{OX}}$ will mean *length* of the line OX.

Def. The length of the vector, or the expression $\sqrt{a^2 + b^2}$, is called the **Modulus** of the complex expression $a + bi$.

The modulus is the absolute value of the expression, considered without respect to its being positive or negative, real or imaginary. Thus the different expressions,

$$-5, \quad +5, \quad 3 + 4i, \quad 4 - 3i, \quad 5i,$$

all have the modulus 5 (because $\sqrt{3^2 + 4^2} = 5$). The points which represent them are all 5 units distant from the zero point, and so lie on a circle, and their vectors are all 5 units in length.

The German mathematicians therefore call the modulus the *absolute value* of the complex quantity, and this is really a better term than the English expression *modulus*.

Def. The **Angle** of the vector is the angle which it makes with the line along which the real units are measured.

If OA is this line, and OB the vector, the angle is AOB.

EXERCISES.

Lay off the following complex quantities, draw the vectors corresponding to them, and find the modulus both by measurement and calculation :

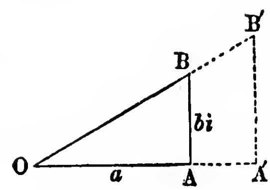
1.	$4 + 3i$.	2.	$4 - 3i$.	3.	$-4 + 3i$.
4.	$-4 - 3i$.	5.	$3 + 4i$.	6.	$3 - 4i$.
7.	$-3 + 4i$.	8.	$-3 - 4i$.	9.	$5 + 7i$.
10.	$5 + 6i$.	11.	$5 + 5i$.	12.	$5 + 4i$.
13.	$3 + 2i$.	14.	$3 + i$.	15.	$3 - i$.
16.	$3 - 2i$.				

17. Draw a horizontal and vertical line; mark several points on the plane of these lines, and find by measurement the complex expressions for each point. Also, draw the several vectors and measure their length. Continue this exercise until the relation between the complex expressions and their points is well apprehended.

NOTE. The student may adopt any scale he pleases, but a scale of millimeters will be found convenient.

338. *Geometric Multiplication.* The question next arises whether the results we obtain for multiplication of complex quantities follow, in all respects, the usual laws of multiplication, especially the commutative and distributive laws.

I. *To multiply a vector by a real factor.*



Let the vector be $a + bi$ and the factor m. The product will be

$$ma + mbi.$$

In the geometric construction, let $OA = a$ and $AB = bi$. We shall then have, by the rule of addition,

$$\text{Vector } OB = a + bi.$$

When we multiply a by m, let OA' be the product ma, and $A'B'$ the product mbi. Because the lines OA and AB are both multiplied by the same real factor m to form OA' and $A'B'$, we shall have

$$OA : AB : OB = OA' : A'B' : OB'.$$

Therefore the triangles OAB and OA′B′ are similar and equiangular, so that

$$\text{angle } A'OB' = \text{angle } AOB.$$

This shows that the lines OB and OB′ coincide, so that BB′ is the continuation of OB in the same straight line. Moreover, the above proportion gives

$$OB' = mOB,$$

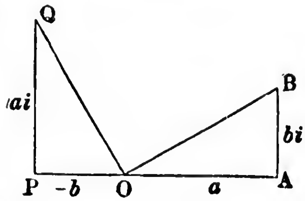
or, from (1), $\quad$ vector $OB' = m$ vector OB.

Therefore, *multiplying a vector by a real factor changes its length without altering its direction.*

II. *To multiply a vector by the imaginary unit.*

Multiplying $a + bi$ by i, the result is

$$-b + ai.$$



The construction of the two vectors being made as in the figure, we have

$$OB = a + bi,$$
$$OQ = -b + ai.$$

Because the triangles OPQ and OAB are right-angled at P and B, and have the sides containing the right angle equal in length, they are identically equal, and

$$\text{angle } POQ = \text{angle } OBA = 90° - \text{angle } BOA.$$

Hence the sum of the angles POQ and BOA is a right angle, and because POA is a straight line, therefore,

$$\text{angle } BOQ = 90°.$$

Therefore, *the result of multiplying the vector* OB *by the imaginary unit is to turn it* 90° *without changing its length.*

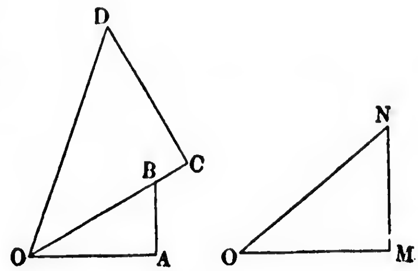
We have assumed this to be the case when the vector represents a real quantity, or lies along the line OB; we now see that the same thing holds true when the vector represents a complex quantity.

If instead of the multiplier being simply the imaginary unit, it is of the form ni, then, by (I), in addition to turning the vector through 90°, we multiply it by n.

III. *To multiply a vector by a complex quantity,*

$$m + ni.$$

This will consist in multiplying separately by m and ni, and adding the two products. Put $\text{OB} = a + bi$, the vector to be multiplied; $\text{ON} = m + ni$, the multiplier.



To multiply OB by m, we take a length OC, determined by the proportion,

$$\text{OC} : \text{OB} = m : 1, \quad \text{(I)}$$

whence by (I),

$$\begin{aligned}\text{OC} &= m \cdot \text{OB} \\ &= m\,(a + bi).\end{aligned}$$

To multiply OB by ni, we take a length CD determined by the condition,

$$\text{length CD} = n \text{ length OB},$$

or
$$\overline{\text{CD}} : \overline{\text{OB}} = n : 1;$$

and to multiply by i, we place it perpendicular to OB. (II)
We then have,

$$\text{CD} = \text{OB} \times ni.$$

In order to add it to OC, the other product, we place it as in the diagram, and thus find a point D which corresponds to the sum

$$\text{OC} + \text{CD} = \text{OB} \times m + \text{OB} \times ni;$$

that is, to the product

$$(m + ni)\,(a + bi).$$

Now because $\overline{\text{OC}} = \overline{\text{OB}} \times m$ and $\overline{\text{CD}} = \overline{\text{OB}} \times n$, we have

$$\overline{\text{OC}} : \overline{\text{CD}} = m : n = \overline{\text{OM}} : \overline{\text{MN}},$$

and because the angles at M and C are right angles, the triangles OCD and OMN are similar. Therefore,

$$\text{angle COD} = \text{angle MON}.$$

Hence the angle AOD of the product-vector is equal to the sum of the angles of the multiplier and multiplicand.

For the length OD of the product-vector we have,

$$\text{length } \overline{OD}^2 = \overline{OC}^2 + \overline{CD}^2$$
$$= m^2\overline{OB}^2 + n^2\overline{OB}^2$$
$$= (m^2 + n^2)\,\overline{OB}^2.$$

Extracting the square root,

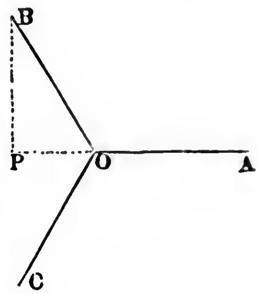
$$\text{length } \overline{OD} = \sqrt{m^2 + n^2} \cdot \overline{OB}$$
$$= \sqrt{m^2 + n^2} \cdot \sqrt{a^2 + b^2}.$$

Therefore the length of the product-vector is equal to the products of the lengths of the vectors of the factors.

Combining these two results, we reach the conclusion:

The modulus of the product of two complex factors is equal to the product of their moduli.

The angle of the product is equal to the sum of the angles of the factors.



339. *The Roots of Unity.* We have the following curious problem:

Given, a vector OA, which call a; it is required to find a complex factor x, such that when we multiply a n times by x, the last product shall be a itself. That is, we must have

$$x^n a = a.$$

The required factor must be one which will turn the vector round without changing its length. Let us begin with the case of $n = 3$.

Since three equal motions must restore OA to its original position, the condition will be satisfied by letting x indicate a motion through 120°, so that OA shall take the position OB when angle AOB = 120°. Then, P being the foot of the perpendicular from B upon AO produced, we shall have angle POB = 60°, and angle PBO = 30°. Therefore,

$$\overline{PO} = \frac{1}{2}a, \qquad \overline{PB} = \frac{\sqrt{3}}{2}a,$$

and $$\text{vector OB} = xa = -\frac{1}{2}a + \frac{\sqrt{3}}{2}ai.$$

Because the factor x has not changed the length of the line, the modulus of x is unity, and because it has turned the line through 120°, its angle is 120°. Therefore its value is

$$-\text{OP} + \text{PB}i$$

on a scale of numbers in which $\text{OB} = 1$; that is,

$$x = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

Reasoning in the same way with respect to the product x^2a, which produces the vector OC, we find

$$x^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i,$$

an equation which we readily prove by squaring the preceding value of x and reducing.

Multiplying these values of x and x^2, we find

$$x^3 = 1,$$

which ought to be the case, because $x^3a = a$. Hence,

The complex quantity $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ *is a cube root of unity.*

But the vector OC, of which the angle is 240°, also represents a cube root of unity, if we suppose $\text{OC} = 1$, because three motions of 240° each turn a vector through 720°, or two revolutions, and thus restore it to its original position. This also agrees with the algebraic process, because, by squaring the above value of x^2, we have

$$\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^2 = \frac{1}{4} - \frac{3}{4} + \frac{\sqrt{3}}{2}i = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = x,$$

and by repeating the process we find

$$\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1.$$

Since 1 itself is a cube root of unity, because $1^3 = 1$, we conclude :

There are three cube roots of unity.

We readily find, by the process of § 334, IV, that

$$i, \quad -1, \quad -i, \quad \text{and} \quad 1,$$

are all fourth roots of unity.

By a course of reasoning similar to the above for any value of n, we conclude:

The n^{th} roots of unity are n in number.

EXERCISES.

1. Form the first eight powers of the expression

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i;$$

show that the eighth power is 1, and lay off the vector corresponding to each power.

2. Form the first twelve powers of

$$\frac{\sqrt{3}}{2} + \frac{1}{2}i,$$

and show that the twelfth power is $+1$.

3. Find the fifth and sixth roots of unity by dividing the circle into five and six parts, and either computing or measuring the lengths of the lines which determine the expression.

Note. The student will remark the similarity of the general problem of the n^{th} roots of unity to that of dividing the circle into n equal parts (Geom., Book VI).

BOOK XIII.

THE GENERAL THEORY OF EQUATIONS.

Every Equation has a Root.

340. In Book III, equations containing one unknown quantity were reduced to the normal form

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \dots + F = 0.$$

If we divide all the terms of this equation by the coefficient A, and put, for brevity,

$$p_1 = \frac{B}{A},$$

$$p_2 = \frac{C}{A},$$

$$\text{etc.} \quad \text{etc.}$$

$$p_n = \frac{F}{A},$$

the equation will become

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0. \qquad (a)$$

This equation is called the **General Equation of the n^{th} Degree,** because it is the form to which every algebraic equation can be reduced by assigning the proper values to n, and to p_1, p_2, p_3, etc.

The n quantities $p_1, p_2, \dots p_n$ are called the **Coefficients** of the equation.

We may consider p_n as the coefficient of $x^0 = 1$.

341. THEOREM I. *Every algebraic equation has a root, real or imaginary.*

That is, whatever numbers we may put in place of p_1, p_2, p_3, ... p_n, there is always some expression, real or imaginary, which, being substituted for x in the equation, will satisfy it.

REM. The theorem that every equation has a root is demonstrated in special treatises on the theory of equations, but the demonstration is too long to be inserted here.

If we suppose the values of the coefficients p_1, p_2, etc., to vary, the roots will vary also. Hence,

THEOREM II. *The roots of an algebraic equation are functions of its coefficients.*

EXAMPLE. In Chapter VI we have shown that the roots of a quadratic equation are functions of the coefficients, because if the equation is

$$x^2 + px + q = 0,$$

the root is $$x = \frac{-p \pm \sqrt{p^2 - q}}{2},$$

which is a function of p and q.

342. *Equations which can be solved.* If the degree of the equation is not higher than the *fourth*, it is always possible to express the root algebraically as a function of the coefficients.

But if the equation is of the fifth or any higher degree, it is not possible to express the value of the root of the general equation by any algebraic formulæ whatever.

This important theorem was first demonstrated by Abel in 1825. Previous to that time, mathematicians frequently attempted to solve the general equation of the fifth degree, but of course never succeeded.

This restriction applies only to the *general* equation, in which the coefficients p_1, p_2, p_3, etc., are all represented by separate algebraic symbols. Such special values may be assigned to these coefficients that equations of any degree shall be soluble.

343. The problem of finding a root of an equation of the higher degrees is generally a very complex one. If, however, the equation has the roots -1, 0, or $+1$, they can easily be discovered by the following rules:

I. *If the algebraic sum of the coefficients in the equation vanishes, then* $+1$ *is a root.*

II. *If the sum of the coefficients of the even powers of x is equal to that of the coefficients of the odd powers, then* -1 *is a root.*

III. *If the absolute term p_n is wanting, then* 0 *is a root.*

These rules are readily proved by putting $x = +1$, then $x = -1$, then $x = 0$ in the general equation (a) and noticing what it then reduces to. The demonstration of II will be a good exercise for the student.

Number of Roots of General Equation.

344. In the equation (a), the left-hand number is an entire function of x, which is equal to zero when the equation is satisfied. Instead of supposing an equation, let us suppose x to be a variable quantity, which may have any value whatever, and let us study the function of x,

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_{n-1} x + p_n,$$

which for brevity we may call Fx.

Whatever value we assign to x, there will be a corresponding value of Fx.

Example. Consider the expression

$$Fx = x^3 - 7x^2 + 36.$$

Let us suppose x to have in succession the values -4, -3, -2, -1, 0, 1, 2, etc., and let us compute the corresponding values of Fx. We thus find,

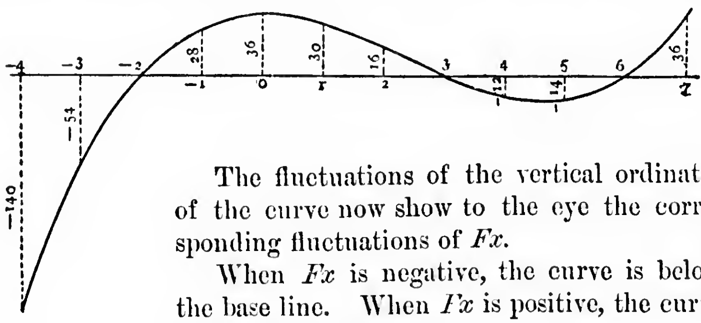
$$\begin{array}{rrrrrr} x = & -4, & -3, & -2, & -1, & 0, \\ Fx = & -140, & -54, & 0, & +28, & +36, \end{array}$$

$$\begin{array}{rrrrrrrrr} x = & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8. \\ Fx = & +30, & +16, & 0, & -12, & -14, & 0, & +36, & +100. \end{array}$$

We see that while x varies from -4 to $+8$, the value of Fx fluctuates, being first negative, then changing to positive, then back to negative again, and finally becoming positive once more.

We also see that there are three special values of x, namely, -2, $+3$, and $+6$, which satisfy the equation $Fx = 0$, and which are therefore roots of this equation.

345. *Representation of Fx by a Curve.* In Book VIII it was shown how a function of a variable of the first degree might be represented to the eye by a straight line. The relation between a variable and any function of it may be represented to the eye in the same way by a curve, as shown in Geometry, Book VII. We take a base line, mark a zero point upon it, and lay off any number of equidistant values of x. At each point we erect a perpendicular proportional to the corresponding value of Fx at that point, and draw a curve through the ends.



The fluctuations of the vertical ordinates of the curve now show to the eye the corresponding fluctuations of Fx.

When Fx is negative, the curve is below the base line. When Fx is positive, the curve is above the base line.

The roots of the equation $Fx = 0$ are shown by the points at which the curve crosses the base line. In the present case these points are -2, $+3$, $+6$.

In order to distinguish the roots from the variable quantity x, we may call them α, β, γ, δ, etc., or x_1, x_2, x_3, etc., or a_1, a_2, a_3, etc., the symbol x being reserved for the variable.

The distinction between x and the roots will then be this:

x is an independent variable, which may have any value whatever.

Fx is a function of x of which the value is fixed by that of x.

α, β, γ, etc., or x_1, x_2, x_3, etc., are special values of x which, being substituted for x, satisfy the equation

$$Fx = 0.$$

THEOREM. *An equation with real coefficients, of which the degree is an odd number, must have at least one real root.*

Proof. 1. When n is odd, x^n will have the same sign (+ or —) as x.

2. So large a value, positive or negative, may be assigned to x that the term x^n shall be greater in absolute magnitude than all the other terms of the expression Fx. For, let us put the expression Fx in the form

$$Fx = x^n\left(1 + \frac{p_1}{x} + \frac{p_2}{x^2} + \ldots + \frac{p_n}{x^n}\right). \qquad 1)$$

If we suppose x to increase indefinitely either in the positive or negative direction, the terms $\frac{p_1}{x}$, $\frac{p_2}{x^2}$, etc., will all approach 0 as their limit (§ 303, Th. I). Therefore the expression $1 + \frac{p_1}{x} + \frac{p_2}{x^2} +$ etc. will approach unity as its limit, and will therefore be positive for large values of x, both positive and negative. The whole expression will then have the same sign as the factor x^n, and, n being odd, will have the same sign as x.

3. Therefore, between the value of x for which Fx is negative and that for which it is positive there must be some value of x for which $Fx = 0$, that is, some root of the equation $Fx = 0$.

For illustration, take the preceding cubic equation.

Cor. *An equation of odd degree has an odd number of real roots.*

For, as Fx changes from negative to positive infinity, it must cross zero an odd number of times.

346. Theorem I. *If we divide the expression Fx by $x - a$, the remainder will be Fa, or*

$$\text{Remainder} = a^n + p_1a^{n-1} + p_2a^{n-2} + \ldots + p_n.$$

Special Illustration. Let the student divide

$$x^3 + 5x^2 + 3x + 1$$

by $x - a$, according to the method of § 96. He will find the remainder to come out

$$a^3 + 5a^2 + 3a + 1.$$

General Proof. When we divide Fx by $x - a$, let us put

Q, the quotient;
R, the remainder.

Then, because the dividend is equal to the product, Divisor $\times$ Quotient + Remainder,

$$(x - a)\,Q + R = Fx.$$

Two things are here supposed:

1. That this equation is an identical one, true for all values of x. This must be true, because we have made no supposition respecting the value of x.

2. That we have carried the division so far that the remainder R does not contain x.

Because it is true for all values of x, it will remain true when we put $x = a$ on both sides. It thus reduces to

$$R = F(a),$$

which is the theorem enunciated.

The value of x being still unrestricted, let us in dividing take for a a root α of the general equation $Fx = 0$. Then, by supposing $x = \alpha$, the equation (a) will be satisfied, or

$$F\alpha = 0.$$

Therefore if we divide the general expression Fx by $x - \alpha$, the remainder $F\alpha$ will be zero. Hence.

Theorem II. *If we denote by α a root of the equation $Fx = 0$, the expression Fx will be exactly divisible by $x - \alpha$.*

Illustration. One root of the equation

$$x^3 - x^2 - 11x + 15 = 0$$

is 3. If we divide the expression

$$x^3 - x^2 - 11x + 15$$

by $x - 3$, we shall find the remainder to be zero.

347. When we divide Fx by $x - \alpha$, the highest power of x in the quotient will be x^{n-1}. Therefore the quotient will be an entire function of x of the degree $n - 1$.

Illustration. The quotient from the last division was

$$x^2 + 2x - 5,$$

which is of the second degree, while the original expression was of the third degree.

If we call this quotient F_1x, we shall have, by multiplying divisor and quotient,

$$Fx = (x - \alpha)\, F_1x.$$

Now suppose β a root of the equation

$$F_1x = 0;$$

then F_1x will, by the preceding theorem, be exactly divisible by $x - \beta$.

The quotient from this division will be an entire function of x of the degree $n - 2$. This function may again be divided by $x - \gamma$, representing by γ the root of the equation obtained by putting the function equal to zero, and so on.

The results of these successive divisions may therefore be expressed in the form

$$\left.\begin{aligned} Fx &= (x - \alpha)\, F_1x \ldots . \text{ (Degree } n - 1), \\ F_1x &= (x - \beta)\, F_2x \ldots . \text{ (Degree } n - 2), \\ F_2x &= (x - \gamma)\, F_3x \ldots . \text{ (Degree } n - 3), \end{aligned}\right\} \qquad (1)$$
$$\text{etc.} \qquad \text{etc.} \qquad \text{etc.}$$

Since the degree is diminished by unity with every division, we shall at length have a quotient of the first degree in x, of the form

$$x - \varepsilon,$$

ε being a constant.

Then, by substituting in the equations (1) for each function of x its value in the equation next below, we shall have

$$Fx = (x - \alpha)(x - \beta)(x - \gamma) \ldots . (x - \varepsilon),$$

the number of factors being equal to the degree of the original equation. Hence,

THEOREM I. *Every entire function of x of the nth degree may be divided into n factors, each of the first degree in x.*

Since a product of several factors becomes zero whenever any of the factors is zero, it follows that the equation

$$Fx = 0$$

will be satisfied by putting x equal to any one of the quantities $\alpha, \beta, \gamma, \ldots . \varepsilon$, because in either case the product

$$(x - \alpha)(x - \beta)(x - \gamma) \ldots . (x - \varepsilon)$$

will vanish. Therefore the quantities

$$\alpha, \beta, \gamma, \ldots . \varepsilon,$$

are all roots of the original equation $Fx = 0$. Hence,

THEOREM II. *An algebraic equation of the n^{th} degree has n roots.*

We have seen (§ 195) that a quadratic equation has two roots. In the same way, a cubic equation has three roots, one of the fourth degree four roots, etc.

Moreover, a product cannot vanish unless one of the factors vanishes. Hence the product

$$Fx \quad \text{or} \quad (x - \alpha)(x - \beta)(x - \gamma) \ldots . (x - \varepsilon)$$

cannot vanish unless x is equal to some one of the quantities, $\alpha, \beta, \gamma, \ldots . \varepsilon$. Hence,

An equation of the n^{th} degree can have no more than n roots.

348. We may form an equation of which the roots shall be any given quantities, a, b, c, etc., by forming the product,

$$(x - a)(x - b)(x - c), \text{ etc.}$$

EXAMPLE. Form an equation of which the roots shall be

$$-1, \quad +1, \quad 1 + 2i, \quad 1 - 2i.$$

Solution. We form the product

$$(x + 1)(x - 1)(x - 1 - 2i)(x - 1 + 2i),$$

which we find to be

$$x^4 - 2x^3 + 4x^2 + 2x - 5.$$

Therefore the required equation is

$$x^4 - 2x^3 + 4x^2 + 2x - 5 = 0.$$

EXERCISES.

Form equations with the roots:

1. $2+\sqrt{3},\quad 2-\sqrt{3},\quad -2,\quad +1.$
2. $3+\sqrt{5},\quad 3-\sqrt{5},\quad -3.$
3. $2,\quad -2,\quad 4+\sqrt{7},\quad 4-\sqrt{7}.$
4. $1+\sqrt{3},\quad 1-\sqrt{3},\quad 1+\sqrt{5},\quad 1-\sqrt{5}.$

349. When we can find one root of an equation, then, by dividing the equation by x *minus* that root, we shall have an equation of lower degree, the roots of which will be the remaining roots of the given equation.

EXAMPLE. One root of the equation

$$x^3 - x^2 - 11x + 15 = 0$$

is 3. Find the other two roots.

Dividing the given equation by $x-3$, the quotient is

$$x^2 + 2x - 5.$$

Equating this to zero, we have a quadratic equation of which the roots are

$$-1+\sqrt{6} \quad \text{and} \quad -1-\sqrt{6}.$$

Hence the three roots of the original equation are

$$3, \quad -1+\sqrt{6}, \quad -1-\sqrt{6}.$$

EXERCISES.

1. One root of the equation

$$x^3 - 3x^2 - 14x + 12 = 0$$

is -3. Find the other two roots.

2. Find the five roots of the equation

$$x^5 - 4x^4 + 12x^3 + 4x^2 - 13x = 0.$$

(Compare § 343.)

350. *Equal Roots.* Sometimes, in solving an equation, several of the roots may be identical.

For example, the equation

$$x^3 - 6x^2 + 12x - 8 = 0$$

has no root except 2. If we divide it by $x-2$, and solve the resulting quadratic, its roots will also be 2. Hence, when we factor it the result is

$$(x-2)(x-2)(x-2)=0.$$

In this case the equation is said to have three equal roots. Hence, in general,

The n roots of an equation of the n^{th} degree are not all necessarily different from each other, but two or more of them may be equal.

Relations between Coefficients and Roots.

351. Let us suppose the roots of the general equation of the n^{th} degree

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n=0$$

to be $\alpha, \beta, \gamma, \ldots\ldots \varepsilon$.

We have shown (§ 341) that these roots are functions of the coefficients $p_1, p_2, \ldots\ldots p_n$. To find these functions is to solve the equation, which is generally a very difficult problem.

But the coefficients can also be expressed as functions of the roots, and this is a very simple process which we have already performed in some special cases by forming equations having given roots (§ 348).

If we form an equation with the two roots, α and β, the result will be

$$0=(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta.$$

Comparing this with the general form,

$$x^2+p_1x+p_2=0,$$

we see that

$$p_1=-(\alpha+\beta),$$

$$p_2=\alpha\beta,$$

a result already reached (§§ 198, 199).

Next form an equation with the three roots, α, β, γ.

Multiplying $(x-\alpha)(x-\beta)$ by $x-\gamma$, we find the equation to be

$$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma=0.$$

So in this case,
$$p_1 = -(\alpha + \beta + \gamma),$$
$$p_2 = \alpha\beta + \beta\gamma + \gamma\alpha,$$
$$p_3 = -\alpha\beta\gamma.$$

Adding another root δ, we find the result to be

$$\begin{aligned} p_1 &= -(\alpha + \beta + \gamma + \delta), \\ p_2 &= \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta, \\ p_3 &= -\alpha\beta\gamma - \alpha\beta\delta - \alpha\gamma\delta - \beta\gamma\delta, \\ p_4 &= \alpha\beta\gamma\delta. \end{aligned} \qquad (2)$$

Generalizing this process, we reach the following conclusions:

The coefficient p_1 of the second term of the general equation is equal to the sum of the roots taken negatively.

The coefficient p_2 of the third term is equal to the sum of the products of every combination of two roots.

The coefficient p_3 of the fourth term is equal to the sum of the products of every combination of three roots taken negatively.

The last term is equal to the continued product of the negatives of the roots.

352. *Symmetric Functions.* It will be remarked that the preceding expressions for the coefficients p_1, p_2, etc., are all *symmetric functions* of the roots α, β, γ, etc. (§ 256.)

The following more extended theorem is true:

THEOREM. *Every rational symmetric function of the roots of an equation may be expressed as a rational function of the coefficients.*

EXAMPLE. From the equations (2) we find

$$p_1^2 - 2p_2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2,$$
$$3p_1p_2 - p_1^3 - 3p_3 = \alpha^3 + \beta^3 + \gamma^3 + \delta^3.$$

We thus reach the curious conclusion that although we may not be able to find any individual root of an equation, yet there is no difficulty in finding the continued product of the roots, their sum, the sum of their squares, of their cubes, etc.

The general demonstration of this theorem, and the methods by which any rational symmetrical function of the roots may be determined, are found in more advanced treatises.

Derived Functions.

353. *Def.* If in the expression

$$Fx = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n,$$

we substitute $x+h$ for x, and then develop in powers of h, the coefficient of the first power of h is called the **First Derived Function of** x.

To find the First Derived Function. Putting $x + h$ for x, the result is

$$F(x+h) = (x+h)^n + p_1(x+h)^{n-1} + \ldots + p_{n-1}(x+h) + p_n. \quad (a)$$

Developing the several terms of the second member by the binomial theorem, we have

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \text{etc.},$$
$$(x+h)^{n-1} = x^{n-1} + (n-1)x^{n-2}h + \text{etc.},$$
$$(x+h)^{n-2} = x^{n-2} + (n-2)x^{n-3}h + \text{etc.},$$
$$\text{etc.} \qquad \text{etc.} \qquad \text{etc.}$$

Substituting these expressions in the equation (a) and leaving out the terms in h^2, h^3, etc. (because we do not want them), we have

$$\begin{aligned} F(x+h) &= x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1}x + p_n \\ &+ [nx^{n-1} + (n-1)p_1 x^{n-2} + (n-2)p_2 x^{n-3} + \ldots + p_{n-1}]h \\ &+ \textit{omitted terms multiplied by } h^2, h^3, \textit{etc.} \qquad (b) \end{aligned}$$

We see that the first line is here the original Fx, while the coefficient of h in the second line is by definition the derived function. So, if we put

$F'x$, the derived function of Fx,

we have $F(x+h) = Fx + hF'x +$ terms $\times\, h^2, h^3$, etc.

Let the student, as an exercise, now find the derived function of

$$x^4 + 3x^3 - 5x^2 + 7x - 9$$

by the process just followed, commencing with equation (a).

Examining the coefficient of h in (b), we see that the derived function is formed by the following rule:

Multiply each term by the exponent of the variable in that term, and diminish the exponent by unity.

The last or constant term disappears entirely from the expression.

EXERCISES.

Form the derived function of the following expressions :

1. $x^5 + 5x^4 + 8x^3 - 2x^2 - x + 1$.

   Ans. $5x^4 + 20x^3 + 24x^2 - 4x - 1$.

2. $x^7 - 2x^5 - 2x^3 - 2x$.
3. $x^6 + 12x^5 - 24x^3 + x^2 + 7$.
4. $x^4 - 2ax^3 + 3b^2x^2 + a^2bx$.
5. $x^5 - 5mx^4 + 10mx^3 - 15mx^2$.

REM. The student should obtain the result by substituting $x+h$ for h in each equation and developing, until he is master of the process.

354. *Second Form of the Derived Function.* If, as before, we put α, β, γ, δ, etc., for the roots of the equation $Fx = 0$, we shall have

$$Fx = (x - \alpha)(x - \beta)(x - \gamma) \ldots . (x - \varepsilon). \qquad (c)$$

Let us form the derived function from this expression.

Putting $x + h$ for x, it will become

$$(h + x - \alpha)(h + x - \beta)(h + x - \gamma) \ldots . (h + x - \varepsilon).$$

Studying this expression, and forming the products which contain h when three or four factors only are included, we see that the coefficient of the h in the first factor is $(x-\beta)(x-\gamma) \ldots .$, in the second factor $(x-\alpha)(x-\gamma) \ldots .$, etc. That is, the total coefficient of h will be

$$\begin{array}{ll} (x - \beta)(x - \gamma) \ldots . (x - \varepsilon), & \text{omitting first term;} \\ + (x - \alpha)(x - \gamma) \ldots . (x - \varepsilon), & \text{omitting second term;} \\ \quad \text{etc.} \quad \text{etc.} \quad \text{etc.} & \\ + (x - \alpha)(x - \beta)(x - \gamma) \ldots . & \text{omitting last term.} \end{array}$$

But comparing with (c), we see that the first of these products is $\dfrac{Fx}{x - \alpha}$, the second is $\dfrac{Fx}{x - \beta}$, etc., to the last, which is $\dfrac{Fx}{x - \varepsilon}$. Hence,

$$F'x = \frac{Fx}{x-\alpha} + \frac{Fx}{x-\beta} + \frac{Fx}{x-\gamma} + \ldots + \frac{Fx}{x-\varepsilon}. \qquad (d)$$

Illustration. Let us take once more the expression of § 344,

$$Fx = x^3 - 7x^2 + 36,$$

of which the three roots are -2, 3, and 6. Its derived function, by method (1), is

$$3x^2 - 14x.$$

Expressing Fx as a product of factors, it is

$$Fx = (x+2)(x-3)(x-6).$$

By (d) the derived function is

$$(x-3)(x-6) + (x+2)(x-6) + (x+2)(x-3),$$

which reduces to $\qquad 3x^2 - 14x,$

the same value as by the first method.

355. THEOREM I. *When the derived function is positive, the original function increases with x; when it is negative, the function decreases as x increases.*

Proof. When we increase x by the quantity h, Fx is changed to $F(x+h)$, and is increased by the difference

$$F(x+h) - Fx.$$

But, by (b) and (b'), we have

$$\begin{aligned} F(x+h) - Fx &= h\,F'x + h^2 \times \textit{other terms} \\ &= h\,(F'x + h \times \textit{other terms}). \qquad (e) \end{aligned}$$

Now we may take the increment h so small that $h \times$ *other terms* shall be less than $F'x$, and then $F'x + h \times$ *other terms* will have the same sign ($+$ or $-$) as $F'x$.

Then, supposing h positive, the increment

$$F(x+h) - Fx$$

will be positive when $F'x$ is positive, and negative when it is negative.

THEOREM II. *If an equation has equal roots, such root will also be a root of the derived function.*

Proof. Let β be the root which $Fx = 0$ has in duplicate. Then when Fx is factored, it will be of the form

$$Fx = (x - \alpha)(x - \beta)(x - \beta)(x - \gamma) \ldots . (x - \varepsilon).$$

Now when we form $F'x$ by method (2), the factor $(x - \beta)$ will be left in all the terms. Therefore $x - \beta$ will be a factor of $F'x$. Therefore, when $x = \beta$, then $F'x = 0$, so that β is a root of the equation $F'x = 0$.

356. If the equation $Fx = 0$ contains no equal roots, and if we suppose $x = \alpha$ in equation (d), all the terms except the first will vanish, because the common numerators Fx contain $x - \alpha$ as a factor.

In the case of the first term, both numerator and denominator vanish when $x = \alpha$; therefore we must find the limit of $\dfrac{Fx}{x - \alpha}$ when x approaches α. This is easy, because

$$\frac{Fx}{x - \alpha} = (x - \beta)(x - \gamma) \ldots . (x - \varepsilon).$$

Therefore, by supposing x to approach α, we shall have

$$\text{Lim.}\ \frac{Fx}{x - \alpha}{}_{(x=a)} = (\alpha - \beta)(\alpha - \gamma) \ldots . (\alpha - \varepsilon).$$

Therefore, by changing x into α in (d), we find

$$F'\alpha = (\alpha - \beta)(\alpha - \gamma) \ldots . (\alpha - \varepsilon).$$

Hence

The derived function of a root which has no other root equal to it is the continued product of its difference from all the other roots.

Significance of the Derived Function.

357. THEOREM. *The derived function expresses the rate of increase of the function as compared with that of the variable.*

Proof. The equation (e) may be expressed in the form

$$F(x + h) = Fx + h(F'x + Bh),$$

where Bh^2 is the sum of the remaining terms of the development in powers of h.

We then have

$$\text{Increment of } x = h.$$

$$\begin{aligned}\text{Corresponding increment of } Fx &= F(x+h) - Fx\\ &= h\,(F'x + Bh).\end{aligned}$$

$$\text{Ratio of these increments, } \frac{h\,(F'x + Bh)}{h} = F'x + Bh.$$

If we suppose the increment h to approach zero as its limit, the product Bh will also approach zero, and the ratio will approach $F'x$ as its limit.

But this ratio of the increments may be considered as the ratio of the average rate of increase of the function F to that of the variable x.

Hence, when we plot the values of Fx by a curve, as in § 345, the derived function shows the slope of the curve at each point.

When the derived function is positive, the curve is running upward in the positive direction, as from $x = -3$ to $x = 0$, and from $x = +5$ to $x = +\infty$.

When the derived function is negative, the curve slopes downward, as from $x = 0$ to $x = +4$.

When the derived function is zero, the curve at the corresponding point runs parallel to the base line, as at 0 and $+4\frac{2}{3}$. If this point corresponds to a root of the equation, the curve will coincide with the base line at this point, and will therefore be tangent to it. Hence, from § 356, Th. II,

A pair of equal roots of an equation are indicated by the curve touching the base line without intersecting it.

Forms of the Roots of Equation.

358. THEOREM I. *Imaginary roots enter an equation with real coefficients in pairs.*

That is, if $a + bi$ be a root of such an equation, then $a - bi$ will also be a root.

Proof. Let

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0 \quad (1)$$

be the equation with real coefficients, and let us suppose that $a + bi$ is a root of this equation. If we substitute $a + bi$ for x, we shall have

$$x^n = a^n + na^{n-1}bi - \frac{n(n-1)}{2}a^{n-2}b^2 - \binom{n}{3}a^{n-3}b^3i + \text{etc.}$$

$$p_1x^{n-1} = p_1a^{n-1} + p_1a^{n-2}bi - \text{etc.}$$

If we substitute all the terms thus formed in equation (1), and collect the real and imaginary terms separately, we shall have a result

$$A + Bi = 0$$

(§ 324), A signifying the sum of all the real terms,

$$a^n, \quad -\frac{n(n-1)}{2}a^{n-2}b^2, \quad p_1a^{n-1}, \quad \text{etc.,}$$

and Bi the sum of all the imaginary ones.

In order that this equation may be satisfied, we must have identically

$$A = 0, \quad B = 0 \quad (\S\ 327).$$

Next let us substitute $a - bi$ for x. Since the even powers of bi are all real, and the odd powers all imaginary, this change of sign will leave all the real terms in (1) unchanged, but will change the signs of all the imaginary terms. Hence the result of the substitution will be

$$A - Bi.$$

But if $a + bi$ is a root, then, as already shown, $A = 0$ and $B = 0$; whence

$$A - Bi = 0$$

also, and therefore $a - bi$ is also a root.

Def. A pair of imaginary roots which differ only in the sign of the coefficients of the imaginary unit are called a pair of **Conjugate Imaginary Roots.**

THEOREM II. *In the expression Fx every pair of conjugate imaginary factors form a real product of the second degree in x.*

Proof. If in the expression

$$Fx = (x - \alpha)(x - \beta)(x - \gamma) \dots (x - \varepsilon),$$

we suppose α and β to be a pair of conjugate imaginary roots, which we may represent in the form

$$\alpha = a + bi, \qquad \beta = a - bi,$$

then the product of the terms $(x - a)(x - b)$ or of

$$(x - a - bi)(x - a + bi),$$

will be
$$(x - a)^2 + b^2,$$
or
$$x^2 - 2ax + a^2 + b^2,$$
a real expression of the second degree in x.

Cor. Since Fx can always be separated into factors of the first degree, either real or imaginary (§ 347, Th. I), and since all the imaginary factors enter in pairs of which the product is real, we conclude:

Every entire function of x with real coefficients may be divided into real factors of the first or second degree.

Decomposition of Rational Fractions.

359. *Def.* **A Rational Fraction** is one which may be reduced to the form

$$\frac{ax^m + bx^{m-1} + cx^{m-2} + \dots + l}{x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n}. \qquad (a)$$

If the exponent m of the numerator is equal to or greater than the exponent n of the denominator, we may divide the numerator by the denominator, obtaining a quotient, and a remainder of which the highest exponent will not exceed $n - 1$. If we put

fx, the numerator of the above fraction;
Fx, its denominator;
Q, the quotient;
ϕx, the remainder:

we shall have, Rational fraction $= \dfrac{fx}{Fx} = Q + \dfrac{\phi x}{Fx}$. (§ 96.)

Q will be an entire function of x, with which we need not now further concern ourselves.

The problem now is, if possible, to reduce the fraction $\frac{\phi x}{Fx}$ to the sum of a series of fractions of the form

$$\frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma} + \dots + \frac{E}{x-\varepsilon},$$

A, B, C, etc., being constants to be determined, and α, β, γ, etc., being the roots of the equation $Fx = 0$. Let us then suppose

$$\frac{\phi x}{Fx} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma} + \dots + \frac{E}{x-\varepsilon}. \qquad (b)$$

Multiplying both sides by Fx, we have

$$\phi x = \frac{AFx}{x-\alpha} + \frac{BFx}{x-\beta} + \frac{CFx}{x-\gamma} + \dots + \frac{EFx}{x-\varepsilon}. \qquad (b')$$

We require that this equation shall be an identical one, true for all values of x. Let us then suppose $x = \alpha$. Then because by hypothesis α is a root of the equation $Fx = 0$, we have $F\alpha = 0$, and the terms in the second member will all vanish except the first. If there is only one root α, we have (§ 357),

$$\text{Lim. } \frac{Fx}{x-\alpha}{}_{(x=\alpha)} = F'\alpha.$$

Therefore, changing x to α, we have

$$\phi\alpha = AF'\alpha,$$

which gives
$$A = \frac{\phi\alpha}{F'\alpha}.$$

In the same way we may find

$$B = \frac{\phi\beta}{F'\beta}, \qquad (c)$$

$$C = \frac{\phi\gamma}{F'\gamma},$$

etc. etc.

Substituting these values of A, B, etc., in the equation (b), it becomes

$$\frac{\phi x}{Fx} = \frac{\phi\alpha}{(x-\alpha)F'\alpha} + \frac{\phi\beta}{(x-\beta)F'\beta} + \frac{\phi\gamma}{(x-\gamma)F'\gamma} + \text{etc.}$$

NOTE. The critical student should remark that in the preceding analysis we have not proved that the expression of the rational fraction in the form (b) is always possible, but have only proved that *if* it be possible, *then* the coefficients A, B, C must have the values (c). To prove that the form is possible, the second member of (b) may be reduced to a common denominator, which common denominator will be Fx, and the sum of the numerators equated to ϕx. By equating the coefficients of the separate powers of x, we shall have n equations to determine the n unknown quantities A, B, C, etc. Since n quantities can, in general, be made to satisfy n equations, values of A, B, C, etc., will in general be possible.

It will be instructive to solve the following exercises, both directly and by the common denominator.

EXAMPLES.

1. Decompose $\dfrac{2x^2 - 3x + 5}{x^3 - 7x^2 + 36}$.

We have already found the roots of the denominator to be -2, 3, and 6. Using the formulæ (c), we find

$$\begin{aligned}
\phi x &= 2x^2 - 3x + 5,\\
Fx &= x^3 - 7x^2 + 36 = (x+2)(x-3)(x-6),\\
F'x &= 3x^2 - 14x;
\end{aligned}$$

$$\begin{array}{lll}
\alpha = -2, & \beta = 3, & \gamma = 6;\\
\phi\alpha = 19, & \phi\beta = 14, & \phi\gamma = 59;\\
F'\alpha = 40, & F'\beta = -15, & F'\gamma = 24.
\end{array}$$

$$\frac{2x^2 - 3x + 5}{x^3 - 7x^2 + 36} = \frac{19}{40(x+2)} - \frac{14}{15(x-3)} + \frac{59}{24(x-6)}.$$

2. Decompose $\dfrac{2x^2 - 7x + 3}{x^3 - 2x^2 - x + 2} = \dfrac{2x^2 - 7x + 3}{(x+1)(x-1)(x-2)}$.

Here the roots of the denominator are -1, 1, and 2. Let us effect the decomposition by the following method. Assume

$$\frac{2x^2 - 7x + 3}{(x + 1)(x - 1)(x - 2)} = \frac{A}{x + 1} + \frac{B}{x - 1} + \frac{C}{x - 2}. \quad (d)$$

Reducing the second member to a common denominator, it becomes

$$\frac{A(x^2 - 3x + 2) + B(x^2 - x - 2) + C(x^2 - 1)}{(x + 1)(x - 1)(x - 2)}.$$

Since both members now have the same denominator, their numerators must also be equal. Equating them, after arranging the last one according to powers of x, we have

$$(A + B + C)x^2 - (3A + B)x + 2A - 2B - C = 2x^2 - 7x + 3.$$

Since this must be true for all values of x, we equate the coefficients of x in each member, giving

$$A + B + C = 2,$$
$$3A + B = 7,$$
$$2A - 2B - C = 3.$$

These equations being solved give

$$A = 2, \qquad B = 1, \qquad C = -1.$$

Substituting in (d),

$$\frac{2x^2 - 7x + 3}{(x + 1)(x - 1)(x - 2)} = \frac{2}{x + 1} + \frac{1}{x - 1} - \frac{1}{x - 2}.$$

EXERCISES.

Decompose:

1. $\dfrac{x + 10}{x^2 - 4}$.

2. $\dfrac{x^2 + 8x + 4}{x^3 + x^2 - 4x - 4}$.

3. $\dfrac{2x^3 - 12x^2 - 8x + 12}{x^4 - 5x^2 + 4}$.

4. $\dfrac{x}{x^2 - a^2}$.

5. $\dfrac{2a}{x^2 - a^2}$.

6. $\dfrac{a^2b^2}{(x^2 - a^2)(x^2 - b^2)}$.

360. When the equation $Fx = 0$ has two or more equal roots, the preceding form fails, because all the terms of the second member of (b') will then vanish when we suppose x equal to one of the multiple roots. In this case we must proceed as follows:

If $$Fx = (x-\alpha)^m (x-\beta)^n (x-\gamma)^p,$$
we suppose

$$\frac{\phi x}{Fx} = \frac{A}{(x-\alpha)^m} + \frac{A_1}{(x-\alpha)^{m-1}} + \frac{A_2}{(x-\alpha)^{m-2}} + \cdots + \frac{A_{m-1}}{x-\alpha}$$
$$+ \frac{B}{(x-\beta)^n} + \frac{B_1}{(x-\beta)^{n-1}} + \cdots + \frac{B_{n-1}}{x-\beta}$$
$$+ \frac{C}{(x-\gamma)^p} + \frac{C_1}{(x-\gamma)^{p-1}} + \cdots + \frac{C_{p-1}}{x-\gamma}.$$
$$\text{etc.} \qquad \text{etc.} \qquad \text{etc.}$$

In the case of m, n, or $p = 1$, this form will be the same as (b), as it should.

By reducing the second member to a common denominator, and equating the sum of the numerators to ϕx, we shall have, as before, a number of equations the same as the degree of x in Fx.

EXAMPLE.

Decompose $$\frac{8x^3 - 9x^2 - 2x - 1}{x^5 - 2x^4 - 2x^3 + 4x^2 + x - 2},$$
of which the roots of the denominator are $-1, -1, 1, 1, 2$.

Solution. Because of the roots just given, the expression to which the fraction is to be equal is

$$\frac{A}{(x-1)^2} + \frac{A_1}{x-1} + \frac{B}{(x+1)^2} + \frac{B_1}{x+1} + \frac{C}{x-2}.$$

Reducing to a common denominator, and equating the coefficients of the powers of x to the coefficients of the corresponding powers in the numerator $8x^3 - 4x^2 - 2x - 1$, we have

$$\begin{aligned} A_1 + B_1 + C &= 0, \\ -A_1 + A - 3B_1 + B &= 8, \\ -3A_1 + B_1 - 4B - 2C &= -9, \\ A_1 - 3A + 7B_1 + 5B &= -2, \\ 2A_1 - 2A + 2B_1 + 2B + C &= -1. \end{aligned}$$

Solving these equations, we find,

$$A = 1, \qquad B = 2, \qquad C = 3.$$
$$A_1 = -2, \qquad B_1 = -1,$$

The given fraction is therefore equal to

$$\frac{1}{(x-1)^2} - \frac{2}{x-1} + \frac{2}{(x+1)^2} - \frac{1}{x+1} + \frac{3}{x-2}.$$

EXERCISES.

1. Decompose $\dfrac{x+1}{x^2-2x+1}$. *Ans.* $\dfrac{1}{x-1} + \dfrac{2}{(x-1)^2}$.

2. $\dfrac{x-1}{(x+1)^2}$.

3. $\dfrac{x^2-2}{x^3-x^2+x+1}$.

4. $\dfrac{x+2}{x^3+x^2-x-1}$.

Greatest Common Divisor of Two Functions.

361. When we have two equations, some values of the unknown quantity may satisfy them both. They are then said to have one or more common roots. Such equations, when factored as in § 347, will have a common factor or divisor for each common root. Hence,

THEOREM. *The common roots of two equations may be found from their greatest common divisor.*

PROBLEM. *To find the greatest common divisor of two equations.*

This problem is solved by dividing the two polynomials by the methods of §§ 96, 97, and 232.

EXAMPLE 1. To find the greatest common divisor of the two polynomials,

$$x^5 - 4x^4 + 12x^3 + 4x^2 - 13x$$

and $$x^4 - 2x^3 + 4x^2 + 2x - 5.$$

FIRST DIVISION.

$$\begin{array}{rrrrrr|l}
x^5 - 4x^4 & + 12x^3 & + 4x^2 & - 13x & & & x^4 - 2x^3 + 4x^2 + 2x - 5 \\ \cline{7-7}
\underline{x^5 - 2x^4} & \underline{+ 4x^3} & \underline{+ 2x^2} & \underline{- 5x} & & & x - 2 \\
- 2x^4 & + 8x^3 & + 2x^2 & - 8x & & & \\
\underline{- 2x^4} & \underline{+ 4x^3} & \underline{- 8x^2} & \underline{- 4x} & \underline{+ 10} & & \\
& 4x^3 & + 10x^2 & - 4x & - 10 & = \text{first remainder.} &
\end{array}$$

SECOND DIVISION.

$$
\begin{array}{rrrrr|l}
x^4 - 2x^3 & + \;\; 4x^2 & + 2x & - 5 & & 4x^3 + 10x^2 - 4x - 10 \\
\underline{x^4 + \frac{5}{2}x^3} & \underline{- \;\; x^2} & \underline{- \frac{5}{2}x} & & & \frac{1}{4}x - \frac{5}{8} \\
- \frac{9}{2}x^3 & + \;\; 5x^2 & + \frac{9}{2}x & - 5 & & \\
\underline{- \frac{9}{2}x^3} & \underline{- \frac{45}{4}x^2} & \underline{+ \frac{9}{2}x} & \underline{+ \frac{45}{4}} & & \\
& \frac{65}{4}x^2 & & - \frac{65}{4} & = \text{second remainder;} &
\end{array}
$$

or, $\quad \frac{65}{4}(x^2 - 1) = \text{second remainder.}$

In the next division, we may omit the fractional factor $\frac{65}{4}$, because every value of x which satisfies the equation $x^2 - 1 = 0$ will also make $\frac{65}{4}(x^2 - 1) = 0$, so that these two equations have the same roots. In this process we may always multiply or divide the terms of each remainder by any factor which will make their coefficients entire.

THIRD DIVISION.

$$
\begin{array}{rr|l}
4x^3 + 10x^2 & - 4x - 10 & x^2 - 1 \\
\underline{4x^3 \qquad\quad} & \underline{- 4x \qquad} & 4x + 10 \\
10x^2 & - 10 & \\
\underline{10x^2} & \underline{- 10} & \\
0 & 0 &
\end{array}
$$

Hence, the G.C.D. of the two functions is $x^2 - 1$, and their common roots are $+1$ and -1.

This result may also be reached by factoring the given equations, and multiplying the common factors, thus:

$$
\begin{aligned}
x^5 &- 4x^4 + 12x^3 + 4x^2 - 13x \\
&= x(x-1)(x+1)(x-2-3i)(x-2+3i),
\end{aligned}
$$

$$
\begin{aligned}
x^4 &- 2x^3 + 4x^2 + 2x - 5 \\
&= (x-1)(x+1)(x-1-2i)(x-1+2i).
\end{aligned}
$$

We see that the common factors are

$$(x-1)(x+1) = x^2 - 1.$$

The rules for throwing out factors from divisor or dividend are as follows:

I. *If both given polynomials contain the same factor in all their terms, remove this factor, and after the G. C. D. of the remaining factors of the two polynomials is found, multiply it by this factor.*

Proof. If a be such a factor, and X and Y the quotients after this factor is removed from the two polynomials, the latter, as given, will be

$$aX \quad \text{and} \quad aY.$$

Since a is now a common divisor of both given polynomials, if we call D the G.C.D. of X and Y, it is evident that aD will be the G.C.D. of aX and aY.

II. *Any factor common to all the terms of any divisor, and not contained in the dividend, may be thrown out.*

Proof. If this factor were any part of the G.C.D. sought, it would, by § 232, be a factor of each dividend. Since the only factors we require are those of the G.C.D, factors in a divisor only may be rejected.

EXERCISES.

Find the G.C.D. of the following polynomials:

1. $x^4 - 1$ and $x^6 - 1$.
2. $x^3 - 1$ and $x^4 - 1$.
3. $a^5 - 2a^4 - a^3 + 3a^2 - 2a - 15$ and $a^4 - a^3 - 4a^2 - a + 5$.
4. $25x^4 + 5x^3 - x - 1$ and $20x^4 + x^2 - 1$.
5. $a^4 + 2a^2 + 9$ and $a^4 + 2a^3 - 6a - 9$.
6. $m^3 + 3m^2 + 3m + 1$ and $m^2 - 1$.
7. $x^4 - 8x^3 + 21x^2 - 20x + 4$ and $2x^3 - 12x^2 + 21x - 10$.
8. $a^5 + a^4 - a - 1$ and $a^7 + a^6 - a - 1$.

362. The given polynomials may be functions of two or more symbols, as in § 97. We then arrange them according to the powers of one of the symbols, and perform the divisions by the precepts of § 97.

Ex. Find the greatest common divisor of

$$x^3 - ax^2 + a(b+c)x - abc - bx^2 - cx^2 + bcx$$

and $$x^3 - ax^2 - a(b+c)x - abc + bx^2 + cx^2 + bcx.$$

The quotient of the first division will be unity, so we write the two functions under each other, thus:

$$\begin{array}{l}
x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc \\
\underline{x^3 + (-a+b+c)x^2 - (ab-bc+ca)x - abc} \\
\qquad\quad -2(b+c)x^2 + \quad 2(ab+ac)x = \text{1st rem.}
\end{array}$$

Dividing this remainder by $-2(b+c)$, we have the next divisor. We then perform the next division as follows:

$$\begin{array}{l|l}
x^3 + (-a+b+c)x^2 - (ab-bc+ca)x - abc & x^2 - ax \\ \cline{2-2}
\underline{x^3 - ax^2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad} & x + (b+c) \\
\qquad (b+c)x^2 - (ab-bc+ca)x - abc & \\
\qquad \underline{(b+c)x^2 - (ab \qquad +ca)x \qquad\quad} & \\
\qquad\qquad\qquad\qquad bcx - abc = \text{2d rem.} &
\end{array}$$

Dividing this by the factor bc, which is contained in all its terms, we have $x - a$ for the next divisor, which we find to divide the last divisor, and therefore to be the G.C.D. required.

EXERCISES.

Find the G.C.D. of

1. $x^3 + 3bcx + b^3 - c^3$ and $x^3 + (c-b)x^2 + (b^2+bc+c^2)x$.
2. $x^3 + 3ax + a^3 - 1$ and $x^3 - (a^2 - 2a)x + a - 1$.
3. $(a+b+c)(ab+bc+ca) - abc$ and $a^2 + ab - ac - bc$.
4. $x^4 + 4a^4$ and $x^3 - 2a^2x + 4a^3$.
5. $x^3 - ax^2 - b^2x + ab^2$ and $x^2 - a^2$.
6. $x^3 + a^3 + b^3 - 3abx$ and $x^2 + 2ax + a^2 - b^2$.
7. $x^4 - 2x^2 + 2 - \dfrac{2}{x^2} + \dfrac{1}{x^4}$ and $x^4 - 2x^2 + \dfrac{2}{x^2} - \dfrac{1}{x^4}$.
8. $x^4 - x^3y + xy^3 - y^4$ and $x^4 + x^2y^2 + y^4$.

Transformation of Equations.

363. *Def.* An equation is said to be **Transformed** when a second equation is found whose roots bear a known relation to those of the given equation.

REM. Sometimes we may be able to find a root of the transformed equation, and thence the corresponding root of the original equation, more easily than by a direct solution.

PROBLEM I. *To change the signs of all the roots of an equation.*

Solution. By changing x into $-x$ in a given equation, the signs of the terms containing odd powers of x will be changed, while those of the even powers will be unchanged. Hence, if a be any root of the original equation, $-a$ will be a root of the equation after the signs of the alternate terms are changed. Hence the rule:

Change the signs of the alternate terms, of odd and even degree, in the equation.

PROBLEM II. *To diminish all the roots of an equation by the same quantity h.*

Solution. If the given equation is

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0,$$

and if y is the unknown quantity of the required equation, we must have

$$y = x - h.$$

Therefore, $$x = y + h.$$

Substituting this value of x in the equation, it will become

$$y^n + (p_1 + nh)\,y^{n-1} + \left[p_2 + (n-1)\,p_1h + \binom{n}{2}h^2\right]y^{n-2} + \text{etc.} \quad (a)$$

When h, n, and the p's are all given quantities, the coefficients of y become known quantities.

EXERCISES.

1. Transform the equation $x^2 - 3x - 4 = 0$ into one in which the roots shall be less by 1.

2. Transform $x^3 - 3x^2 + 57x - 7 = 0$ into one in which the roots shall be greater by 5.

364. *Removing Terms from Equations.* The quantity h may be so chosen that any required term after the first in the transformed equation shall vanish. For, if we wish the second term of the equation (a) to vanish, we have to suppose

$$p_1 + nh = 0,$$

which gives
$$h = -\frac{p_1}{n}.$$

We then substitute this value of h in the equation (a), which gives an equation in which the second term is wanting.

If we wish the third term to vanish, we must determine h by the condition

$$\binom{n}{2} h^2 + (n-1)p_1 h + p_2 = 0,$$

which requires the solution of a quadratic equation. Each consecutive term is one degree higher in the unknown quantity h, and the last term is of the same degree as the original equation.

This method is principally applied to make the second term disappear, which requires that we put

$$h = -\frac{p_1}{n}.$$

Example. Make the second term disappear from the following equation,

$$x^2 + px + q = 0.$$

Solution. Hence, $n = 2$ and $p_1 = p$, so that

$$h = -\frac{p}{2}.$$

$$x = y - \frac{p}{2}.$$

Making this substitution, the equation becomes

$$y^2 - \frac{p^2}{4} + q = 0,$$

which is the required equation.

REM. This process affords an additional elegant method of solving the quadratic equation.

The last equation gives

$$y = \sqrt{\frac{p^2}{4} - q} = \frac{1}{2}\sqrt{p^2 - 4q}.$$

The value of x, being equal to $y + h$, then becomes

$$x = -\frac{p}{2} + \frac{1}{2}\sqrt{p^2 - 4q},$$

which is the correct solution.

EXERCISES.

Remove the second term from the following equations:

1. $x^3 - 6x^2 + 6x - 1 = 0.$
2. $x^4 - 4x^3 + 3x^2 - 8 = 0.$
3. $x^5 - 5x^4 + 2x^3 + 2x^2 - 3x = 0.$
4. $x^6 - 12x^5 + 2x^3 - x = 0.$

REM. The theory of the above process will be readily comprehended by recalling that the coefficients of the second term is equal to the sum of the roots taken negatively, or if α, β, γ, etc., be the roots,

$$\alpha + \beta + \gamma + \ldots + \epsilon = -p_1.$$

It is evident that if we subtract the arithmetical mean of all the roots, that is, $-\frac{p_1}{n}$, from each of them, their sum will vanish, because

$$\alpha + \frac{p_1}{n} + \beta + \frac{p_1}{n} + \gamma + \frac{p_1}{n} + \text{etc.} = -p_1 + n\frac{p_1}{n} = 0.$$

Hence, when we put $y - \frac{p_1}{n}$ for x in the equation, the sum of the roots, and therefore the second term, vanish.

365. PROBLEM. *To transform an equation so that the roots shall be multiplied by a given factor m.*

Solution. Since the roots are to be multiplied by m, the new unknown quantity must be equal to mx. So if we call this quantity y, we have

$$y = mx,$$

which gives

$$x = \frac{y}{m}.$$

Substituting this in the general equation, it becomes

$$\frac{y^n}{m^n} + p_1\frac{y^{n-1}}{m^{n-1}} + p_2\frac{y^{n-2}}{m^{n-2}} + \ldots + p_n = 0.$$

Multiplying all the terms by m^n, the equation becomes

$$y^n + mp_1y^{n-1} + m^2p_2y^{n-2} + \ldots + m^np_n = 0.$$

Hence the rule,

Multiply the coefficient of the second term by m, that of the third by m^2, and so on to the last term, which will be multiplied by m^n.

If the roots are to be divided, we divide the terms in the same order.

EXERCISES.

1. Make the roots of $x^2 - 2x + 3 = 0$ four times as great.
2. Divide the same roots by 2.

366. PROBLEM. *To transform an equation so that its roots shall be squared.*

Solution. Let the given equation be

$$x^4 + p_1x^3 + p_2x^3 + p_3x + p_4 = 0.$$

If y be the unknown quantity of the new equation, we must have

$$y = x^2,$$

which gives

$$x = \pm y^{\frac{1}{2}}.$$

If we substitute $x = y^{\frac{1}{2}}$ in the given equation, it may be reduced to the form

$$y^2 + p_2y + p_4 + (p_1y + p_3)y^{\frac{1}{2}} = 0.$$

If we substitute $x = -y^{\frac{1}{2}}$, the result will be

$$y^2 + p_2 y + p_4 - (p_1 y + p_3)\,y^{\frac{1}{2}} = 0.$$

Since the value of y must satisfy one or the other of these equations, it must reduce their product to zero; we therefore multiply them together. Considering them as the sum and difference of a pair of expressions, the product will be

$$(y^2 + p_2 y + p_4)^2 - (p_1 y + p_3)^2 y = 0,$$

or

$$y^4 + (2p_2 - p_1^2)y^3 + (p_2^2 + 2p_4 - 2p_1 p_3)y^2 + (2p_2 p_4 - p_3^2)y + p_4^2 = 0.$$

EXERCISES.

1. Transform the quadratic,

$$x^2 - 5x + 6,$$

of which the roots are 2 and 3, into an equation in which the roots shall be the squares of 2 and 3, using the above process.

2. Transform in the same way

$$x^3 + 12x^2 + 44x + 48 = 0.$$

3. Transform

$$x^5 - 4x^4 - 10x^3 + 40x^2 + 9x - 36 = 0.$$

Generalization of the Preceding Problems.

367. PROBLEM. Given, *an equation of any degree in an unknown quantity x;*

Required, *to transform this equation into another of which the root shall be a given function of x.*

Solution. Let y be a root of the required equation, and fx the given function. We must then have

$$fx = y.$$

Solve this equation so as to obtain x as a function of y. Substitute this value of x in the original equation, and form as many equations as there are values of y.

The product of these equations will be the required equation in y.

EXERCISES.

1. Transform
$$x^2 - 7x + 10 = 0$$
so that the roots of the new equation shall be $3x^2$.

2. Transform $\qquad x^3 - 3x^2 + 2x = 0$

so that the roots shall be $ax + b$.

3. Transform $\qquad x^2 - 9x + 18 = 0$

so that the roots shall be $\frac{1}{3}x^2 - 3$.

Resolution of Numerical Equations.

368. *Convenient method of computing the numerical value of an entire function of x for an assumed value of x.*

If we have the entire function of x,
$$Fx = ax^4 + bx^3 + cx^2 + dx + e,$$
we may put it in the form
$$Fx = \{[(ax + b)x + c]x + d\}x + e.$$

Therefore, if we put
$$ax + b = b', \qquad b'x + c = c',$$
$$c'x + d = d', \qquad d'x + e = e',$$
we shall have $\qquad Fx = e'.$

Numerical Example. Compute the values of
$$Fx = 2x^5 - 3x^4 - 6x^3 + 8x - 9$$
for $x = 3$ and $x = -2$.

We arrange the work thus:

Coefficients,	2	-3	-6	0	$+8$	-9
Prod. by ($x=3$),		$+6$	$+9$	$+9$	$+27$	$+105$
		$+3$	$+3$	$+9$	$+35$	$+96$

Hence, $\qquad F3 = 96.$

	2	-3	-6	0	$+8$	-9
For $x = -2$,		-4	$+14$	-16	$+32$	-80
		-7	$+8$	-16	$+40$	-89

Hence, $\qquad F(-2) = -89.$

This, it will be noticed, is a more convenient process than that of forming the powers of x and multiplying and adding.

369. *Having an entire function of x, and putting $x = r + h$, it is required to develop the function in powers of h.*

It will be remarked that this problem is substantially identical with that of § 362, and the solution of this will be the solution of the former. But in the former case h was supposed to be a given quantity, whereas it is now the unknown quantity corresponding to y in the former problem.

EXAMPLE OF THE PROBLEM. If we have the expression

$$Fx = 2x^3 + 3x^2 + 4,$$

and put $x = 2 + h$, it will become, by developing the separate terms,

$$F(2 + h) = 2h^3 + 15h^2 + 36h + 32.$$

GENERAL RULE FOR THE PROCESS. *First compute the value of Fr by the process employed in § 366.*

Then repeat the process, using the successive sums obtained in the first process instead of the corresponding coefficients, and stopping one term before the last. The result will be the coefficient of h.

Repeat the process with the new sums, stopping yet one term sooner. The result will be the coefficient of h^2.

Continue the repetition until we have the first term only to operate upon, which will itself be the coefficient of the highest power of h.

Ex. 1. The example above given is performed as follows:

Coefficients,	+2	+3	0	+4
Product by r,		4	14	28
First sums,		7	14	32
Second products,		4	22	
Second sums,		11	36	
Third product,		4		
		15		

Result, $F(2+h) = 2h^3 + 15h^2 + 36h + 32.$

Ex. 2. In the function,

$$Fx = 2x^5 - 7x^4 + 5x^3 - 2x^2 + 6x - 8,$$

let us put $x = 3 + h$, and express the result in powers of h.

Coefficients,	2	-7	$+5$	-2	$+6$	-8
Products by 3,		6	-3	$+6$	$+12$	$+54$
First sums,		-1	$+2$	$+4$	$+18$	$+46$
Second products,		$+6$	$+15$	$+51$	$+165$	
Second sums,		$+5$	$+17$	$+55$	183	
Third products,		6	33	150		
Third sums,		11	50	205		
		6	51			
		17	101			
		6				
		23				

Result, $\quad F(3+h) = 2h^5 + 23h^4 + 101h^3 + 205h^2 + 183h + 46.$

EXERCISES.

1. Compute $2h^5 + 23h^4 + 101h^3 + 205h^2 + 183h + 46$, when $h = x - 3$.

2. Compute $x^3 - 7x + 7$ for $x = -4 + h, -3 + h$, etc., to $+3 + h$.

Proof of the Preceding Process. If we develop the expression

$$a(h+r)^n + b(h+r)^{n-1} + c(h+r)^{n-2} + d(h+r)^{n-3} + \text{etc.},$$

and collect the coefficients of like powers of h, we shall find

$$\begin{aligned}
\text{Coef. of } h^n &= a, \\
h^{n-1} &= nar + b, \\
h^{n-2} &= \binom{n}{2} ar^2 + (n-1) br + c, \qquad\qquad (A) \\
h^{n-3} &= \binom{n}{3} ar^3 + \binom{n-1}{2} br^2 + (n-2) cr + d, \\
&\vdots \\
h^{n-s} &= \binom{n}{s} ar^s + \binom{n-1}{s-1} br^{s-1} + \binom{n-2}{s-2} cr^{s-2} + \text{etc.}
\end{aligned}$$

Now examining Ex. 2 preceding, it will be seen that we can make the computation by columns, first computing the whole left-hand column and thus obtaining the coefficient of h^{n-1}, then computing the next column, thus obtaining the coefficient of h^{n-2}, and so on. Commencing in this way, and using the literal coefficients, a, b, c, etc., and the literal factor r, we shall have the results:

$$\begin{array}{lll}
a & b & c \\
 & \underline{ar} & \underline{ar^2 + br} \\
 & ar + b & ar^2 + br + c \\
 & \underline{ar} & \underline{2ar^2 + br} \\
 & 2ar + b & 3ar^2 + 2br + c \\
 & \underline{ar} & \underline{3ar^2 + br} \\
 & 3ar + b & 6ar^2 + 3br + c \\
 & \vdots & \vdots \\
 & nar + b & \binom{n}{2} ar^2 + (n-1)\,br + c.
\end{array}$$

If n is the degree of the equation, then, by the preceding process, we shall add the product ar to b n times, the n separate sums being

$$ar+b, \quad 2ar+b, \quad 3ar+b, \; .\; .\; .\; . \; nar+b.$$

To form the second column, we multiply each of these sums except the last by r, and add them to the coefficient c. The terms in ar added being ar^2, $2ar^2$, $3ar^2$, etc., the sum will be $(1+2+3+\ldots.+n-1)\,ar^2$. The coefficient is a figurate number equal to $\frac{n(n-1)}{2}$ (§§ 286, 287). The sum of the coefficients of br is $n-1$, because there are $n-1$ of them used, each equal to unity. Therefore the final result is

$$\binom{n}{2} ar^2 + (n-1)\,br + c,$$

which we have found to be the coefficient of h^{n-2}.

In this second column the partial sums or coefficients of ar^2 are

$1, \; 1+2=3, \; 1+2+3=6$, etc., to $1+2+3+\ldots.+(n-2)$.

Therefore the numbers successively added to form the coefficients of ar^3 in the third column are 1, $1+3=4$, $1+3+6=10$, etc. The coefficients of br^2 will be the same as these of ar^2 in the column next preceding.

Continuing the process, we see that the coefficients are formed by successive addition, as in the following table, where each number is the sum of the one above it plus the one on its

	r^0	r	r^2	r^3	r^4	r^5	r^6	etc.
h^0	1	1	1	1	1	1	1	etc.
h	1	2	3	4	5	6	etc.	
h^2	1	3	6	10	15	etc.		
h^3	1	4	10	20	etc.			
h^4	1	5	15	etc.				
h^5	1	6	etc.					
h^6	1	etc.						
etc.	etc.							

left. We have carried the table as far as $n = 6$, and the expressions at the bottom of each column will, when $n = 6$, be formed from the numbers in this table, taken in reverse order, thus:

Column under b, $6ar + b$;
" " c, $15ar^2 + 5br + c$;
" " d, $20ar^3 + 10br^2 + 4cr + d$;
" " e, $15ar^4 + 10br^3 + 6cr^2 + 3dr + e$;
" " f, $6ar^5 + 5br^4 + 4cr^3 + 3dr^2 + 2er + f$;
" " g, $ar^6 + br^5 + cr^4 + dr^3 + er^2 + fr + g$.

Now the numbers of the above scheme are the figurate numbers treated in § 287, where it is shown that the n^{th} number in the i^{th} column after the column of units is

$$\frac{n(n+1)(n+2)\ldots\ldots(n+i-1)}{1\cdot2\cdot3\ldots\ldots i} = \binom{n+i-1}{i}.$$

Comparing with the coefficients in the equations (A), we see that the two are identical, which proves the correctness of the method.

370. *Application of the Preceding Operation to the Extraction of the Roots of Numerical Equations.* Let the equation whose root is to be found be

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots + g = 0.$$

We find, by trial or otherwise, the greatest whole number in the root x. Let r be this number. We substitute $r+h$ for

x in the above expression, and, by the preceding process, get an equation in h, which we may put in the form

$$ah^n + b'h^{n-1} + c'h^{n-2} + d'h^{n-3} + \ldots + g' = 0.$$

Let r' be the first decimal of h. We put $r'+h'$ for h in this equation, and, by repeating the process, get an equation to determine h', which will be less than 0.1. If r'' be the greatest number of hundredths in h', we put $h' = r''+h''$, and thus get an equation for the thousandths, etc.

371. The first operation is to find the number and approximate values of the real roots. There are several ways of doing this, among which *Sturm's Theorem* is the most celebrated, but all are so laborious in application that in ordinary cases it will be found easiest to proceed by trial, substituting all entire numbers for x in the equation, until we find two consecutive numbers between which one or more roots must lie, and in difficult cases plotting the results by § 345.

It is, however, necessary to be able to set some limits between which the roots must be found, and this may be done by the following rules:

I. *An equation in which all the coefficients, including the absolute term, are positive, can have no positive real root.*

For no sum of positive quantities can be zero.

II. *If in computing the value of Fx for any assumed positive value of x, by the process of § 366, we find all the sums positive, there can be no root so great as that assumed.*

For the substitution of any greater number will make all the sums still greater, and so will carry the last sum, or Fx, still further from zero.

III. *If the sums are alternately positive and negative, the value of x we employ is less than any root.*

IV. *If two values of x give different signs to Fx, there must be one or some odd number of roots between these values* (compare § 345).

V. *Two values of x which lead to the same sign of Fx include either no roots or an even number of roots between them.*

Let us take as a first example the equation

$$x^3 - 7x + 7 = 0.$$

Let us first assume $x = 4$. We compute as follows:

Coefficients,	1	0	-7	$+7$
Products,		4	16	36
Sums,		$+4$	$+9$	$+43$

So $F(4) = +43$, and as all the coefficients are positive, there can be no root as great as 4.

Putting $x = -4$, the sums, including the first coefficient 1, are 1, -4, $+9$, -29. These being alternately positive and negative, there is no root so small as -4.

Substituting all integers between -4 and $+4$, we find

$$\begin{aligned} F(-4) &= -29, & F(0) &= +\ 7, \\ F(-3) &= +\ 1, & F(1) &= +\ 1, \\ F(-2) &= +13, & F(2) &= +\ 1, \\ F(-1) &= +13, & F(3) &= +13. \end{aligned}$$

If we draw the curve corresponding to these values (§ 345), we shall find one root between -3 and -4, and very near -3.05, and the curve will dip below the base line between $+1$ and $+2$, showing that there are two roots between these numbers; that is, there are two roots of the form $1+h$, h being a positive fraction. Transforming the equation to one in h, by putting $1 + h$ for x, we find the equation in h to be

$$h^3 + 3h^2 - 4h + 1 = 0. \qquad (1)$$

Substituting $h = 0.2$, 0.4, 0.6, 0.8, we find that there is one root between 0.3 and 0.4, and one between 0.6 and 0.7. Let us begin with the latter.

If in the last equation we put $h = 0.6 + h'$, we find the transformed equation in h' to be

$$Fh' = h'^3 + 4.8h'^2 + 0.68h' - 0.104 = 0. \qquad (2)$$

If we substitute different values of h' in this equation, we shall find that it must exceed .09, and as it must be less than 0.1, we conclude that 9 is the figure sought, and put

$$h' = .09 + h''.$$

Transforming the equation (2), we find the equation in h'' to be

$$h''^3 + 5.07h''^2 + 1.5683h'' - 0.003191 = 0. \qquad (3)$$

Since h'' is necessarily less than 0.01, its first digit, which is all we want, is easily found, because the two first terms of the equation are very small compared with the third. So we simply divide .003191 by 1.5683, and find that .002 is the required digit of h''. We now put

$$h'' = .002 + h''',$$

and transform again. The resulting equation for h''' is

$$h'''^3 + 5.076h'''^2 + 1.588592h''' - 0.000034112 = 0. \qquad (4)$$

The digits of x, h, h', and h'' which we have found show the true value of x to be

$$x = 1.692 + h'''.$$

By continuing this process, as many figures as we please may be found. But, after a certain point, the operation may be abbreviated by cutting off the last figures in the coefficients of the powers of h.

The work, so far as we have performed it, may be arranged in the following form (see next page).

The numbers under the double lines are the coefficients of the powers of h, h', h'', etc. It will be seen that for each digit we add to the root, we add one digit to the coefficient of h^2, two to that of h, and three to the absolute term. We have thus extended the latter to nine places of decimals, which, in most cases, will give nine figures of the root correctly. If this is all we need, we add no more decimals, but cut off one from the coefficient of h, two from that of h^2, and so on for each decimal we add to the root.

We shall find the next figure after 1.692 to be zero; so we cut off the figures without making any change in the coefficients. The next following is 2, so we cut off again for it, and multiply as shown in the following continuation of the process:

1	0	−7	+7 | 1.692
	+1	+1	−6
	+1	−6	+1.000
	+1	+2	−1.104
	+2	−4.00	− .104000
	+1	+2.16	+ .100809
	+3.0	−1.84	− .003191000
	+ .6	+2.52	+ .003156888
	+3.6	+0.6800	−34112
	.6	+0.4401	
	4.2	+1.1201	
	6	+ .4482	
	+4.80	+1.568300	
	9	10144	
	4.89	+1.578444	
	9	10148	
	4.98	+1.588592	
	9		
	+5.070		
	2		
	5.072		
	2		
	5.074		
	2		
	+5.076		

CONTINUATION OF PROCESS.

1	+|5.076	+1.5885|9|2	−34112 | 021471
		1	31774
		1.5887	−2338
		1	1589
		1.5|8|8|8	− 749
			636
			−113
			111
			−2

It will be seen that from this point we make no use of the coefficient 1 of h^3, and only with the second decimal do we use the coefficient of h^2. After that, the remaining four figures are obtained by pure division.

There is one thing, however, which a computer should always attend to in multiplying a number from which he has cut off figures in this way, namely:

Always carry to the product the number which would have been carried if the figures had not been cut off, and


1.0
2.8
2.5
3.2
2.2
3.6
4.0
2.0
1.8
1.6
1.1
1.0

increase it by 1 *if the figure following the one carried would have been* 5 *or greater.*

For instance, we had to multiply by 7 the number 15|888. If we entirely omit the figures cut off, the result would be 105. But the correct result is 111|216; we therefore take 111 instead of 105.

Again, in the operation preceding, we had to multiply 158|88 by 4. The true product is 635|52. But, instead of using the figures 635, we use 636, because the former is too small by |52, and the latter too great by |48, and therefore the nearer the truth. For the same reason, in multiplying 1.588|8 by 1, we called the result 1589.

Joining all the figures computed, we find the root sought to be 1.692021471.

Let us now find the negative root, which we have found to lie between —3 and —4. Owing to the inconvenience of using negative digits, and thus having to change the sign of every number we multiply, we transform the equation into one having an equal positive root by changing the signs of the alternate terms. The equation then is $x^3 - 7x - 7 = 0$.

The work, so far as it is necessary to carry it, is now arranged as follows:

<table>
<tr><td>1</td><td>0</td><td>−7</td><td>−7</td><td>3.0489173395</td></tr>
<tr><td></td><td>3</td><td>9</td><td>6</td><td></td></tr>
<tr><td></td><td>3</td><td>2</td><td>−1.000000</td><td></td></tr>
<tr><td></td><td>3</td><td>18</td><td>814464</td><td></td></tr>
<tr><td></td><td>6</td><td>20.0000</td><td>−0.185536000</td><td></td></tr>
<tr><td></td><td>3</td><td>.3616</td><td>.166382592</td><td></td></tr>
<tr><td></td><td>9.00</td><td>20.3616</td><td>− .19153408</td><td></td></tr>
<tr><td></td><td>4</td><td>.3632</td><td>18791228</td><td></td></tr>
<tr><td></td><td>9.04</td><td>20.724800</td><td>−362180</td><td></td></tr>
<tr><td></td><td>4</td><td>73024</td><td>208875</td><td></td></tr>
<tr><td></td><td>9.08</td><td>20.797824</td><td>−153305</td><td></td></tr>
<tr><td></td><td>4</td><td>73088</td><td>146213</td><td></td></tr>
<tr><td></td><td></td><td></td><td>−7092</td><td></td></tr>
<tr><td></td><td>9.120</td><td>20.87091|2</td><td>6266</td><td></td></tr>
<tr><td></td><td>8</td><td>823|0</td><td>−826</td><td></td></tr>
<tr><td></td><td>9.128</td><td>20.87914|2</td><td>627</td><td></td></tr>
<tr><td></td><td>8</td><td>823</td><td>−199</td><td></td></tr>
<tr><td></td><td>9.136</td><td>20.8873|7</td><td>188</td><td></td></tr>
<tr><td></td><td>8</td><td>9</td><td>−11</td><td></td></tr>
<tr><td></td><td>|9.1|44</td><td>20|.8|8|7|5</td><td></td><td></td></tr>
</table>

The negative root of the equation is therefore

$$-3.0489173395.$$

EXERCISES.

Find the roots of the following equations:

1. $x^3 - 3x^2 + 1 = 0$ (3 real roots).
2. $x^3 - 3x + 1 = 0$ (3 real roots).
3. $x^4 - 4x^2 + 2 = 0$ (2 positive roots).
4. $x^2 + x - 1 = 0$.
5. Prove that when we change the algebraic signs of the alternate coefficients of an equation, the sign of the root will be changed.

372. The preceding method may be applied without change to the solution of numerical quadratic equations, and to the extraction of square and cube roots. In fact, the square root of a number n is a root of the equation $x^2 - n = 0$, or $x^2 + 0x - n = 0$, and the cube root is a root of the equation $x^3 + 0x^2 + 0x - n = 0$.

Ex. 1. To compute $\sqrt{2}$.

1	0	−2	1.41421356
	1	1	
	1	−1.00	
	1	.96	
	2.0	−.0400	
	0.4	281	
	2.4	−11900	
	4	11296	
	2.80	−60400	
	1	56564	
	2.81	−3836	
	1	2828	
	2.820	−1008	
	4	849	
	2.824	−159	
	4	141	
	2.8280	−18	
	2	17	
	2.8282	1	
	2		
	2.8284		

Ex. 2. To compute the cube root of 9842036.

1	0	0	−9842036	214.30303242
	2	4	8	
	2	4	−1842	
	2	8	1261	
	4	1200	−581036	
	2	61	539344	
	60	1261	41692000	
	1	62	41274207	
	61	132300	−417793	
	1	2536	413326	
	62	134836	4467	
	1	2552	4133	
	630	137388.00	334	
	4	192.69	276	
	634	137580.69	58	
	4	192.78	55	
	638	137773.47	3	
	4	1.93		
	642.0	13|7|7|75|4		
	.3			
	642.3			
	3			
	642.6			
	3			
	642.9			

ANSWERS.

In the following list, answers to questions which do not require calculation or written work, or which it is supposed teachers would prefer to have in a separate Key, are omitted. The Key, published for the use of teachers, contains the complete solutions.

26. 1. -9. 2. -17. 3. $+9$. 4. -26. 5. $+10$. 6. -15. 7. -56. 9. $+840$ 10. -1056. 11. $+1$. 12. -306. 13. 0. 14. -1008.

28. 1. 1. 2. -2. 3. -5. 4. -14. 5. $+24$. 6. $\frac{24}{9}=\frac{8}{3}$. 7. $-\frac{26}{3}$. 8. $\frac{1}{9}$.

40. 1. 0. 2. 0. 3. 11. 4. 17. 5. -37. 6. -90. 7. 324. 8. 0. 9. -60. 10. -180. 11. 945. 12. 5040. 13. -41. 14. -1. 15. -17. 16. 26. 17. 99. 18. 675. 19. 74. 20. -468. 21. -218. 22. -529. 23. -9007. 24. -6800. 25. -420. 26. -840. 27. $\frac{47}{23}$. 28. $\frac{45}{-47}$. 29. 2. 30. 8. 31. When $x=2$, Exp. $=6$; $x=5$, Exp. $=18$; $x=7$, Exp. $=36$. 32. When $x=-5$, Exp. $=-\frac{23}{7}$; $x=2$, Exp. $=\frac{2}{14}$; $x=5$, Exp. $=-\frac{7}{23}$.

43. 1. When $x=-3$, Exp. $=0$; $x=-1$, Exp. $=0$; $x=1$, Exp. $=1$; $x=3$, Exp. $=15$. 2. When $x=-3$, Exp. $=\frac{144}{54}$; $x=-1$, Exp. $=\frac{16}{38}$; $x=1$, Exp. $=\frac{16}{22}$; $x=3$, Exp. $=24$. 3. When $x=-3$, Exp. $=46875$; $x=-1$, Exp. $=-\frac{3}{2}$; $x=1$, Exp. $=-88434$; $x=3$, Exp. $=-\frac{1}{4}(365)^3$. 4. When $x=-1$, Exp. $=(\sqrt{14}-\sqrt{2})4$; $x=1$, Exp. $=(\sqrt{8}-\sqrt{2})4$; $x=3$, Exp. $=(\sqrt{48}-\sqrt{42})4$.

48. 1. $a + bx - (x - y)$. 2. $x - y - (a + bx)$.

3. $a + bx - \frac{a - bx}{m}$. 4. $\frac{a - bx}{m} - mpq$. 5. $\sqrt{a + bx}$.

6. $\sqrt{(a + bx) + (x - y)}$. 7. $\sqrt{(a + bx) - (x - y)}$.

8. $(a + bx)^2 (x - y)^2$. 9. $(mpq)^3$. 10. $(x - y)^3 (mpq)^3$.

11. $$\frac{mpq\,(a + bx) - \frac{(a - bx)(x - y)}{m}}{\left(\frac{a - bx}{m}\right)^2 - (x - y)^2}.$$

etc., etc., etc.

54. 1. $5a + 4b - 8c - e$. 2. $-a + (x + y)$. 3. 6.

4. $9x - 13y$. 5. $22\,(a + b)^2 - x - y - z$. 6. $5\,(ab)$.

7. 0. 8. $7(m+n)^2 - x - 2y$. 9. $4(p+q)^2 + a + b + c - 6$.

10. $14a\,(x - y)$. 11. $15\,(m+n)\,x + 2\,(m - n)\,x - 17$.

12. $7\frac{x}{a} + 3\frac{y}{b} - 1$. 13. $10\frac{x}{y} - 10\frac{m}{n}$. 14. $16\frac{x + y}{m + n}$.

15. $5x - 7y$. 16. $8x$. 17. $4x - 30$.

55. 1. $(a+m)\,x + (b+n)\,y$. 2. $(mn + pq)\,x + (2b - 4b)\,y$.

3. $(3 + 6b + 7a)\,x + (-2 - 4)\,y + m + n$.

4. $(8a + 8b + 7 + 1)\,x + (b - 5 - 5)\,y$.

5. $(a - m)\,x + (b - n)\,y + (c - p)\,z$.

6. $(2d - 2f)\,x + (3e - 3d)\,y + (4f + 4e)\,z$.

7. $\left(\frac{3}{3}a + \frac{3}{4}b\right)y + (6a - 2)\,x$.

8. $(2a - 3b)\,x + (-b - 4d)\,y$.

9. $\left(\frac{1}{2}a - \frac{1}{6}m\right)x + \left(\frac{2}{3}b + \frac{3}{4}n\right)y$.

10. $\left(\frac{10}{3}m - 3a - 6c + \frac{1}{2}d\right)x + (2 + a)\,y$.

11. $(5ab - ab - d)\,x + (4cd - 3mn)\,y$.

12. $\left(-b - \frac{1}{4}d\right)x + 5ay$. 13. $-8x + \left(3 - \frac{1}{4}a\right)y$.

14. $(3m + 1 + a - a)\,x + \left(-1 - \frac{1}{2}a\right)y$.

15. $3abx + (2c + 1)\sqrt{x} + (-m - a)\,y$.

16. $-6x + (5m + 5)\sqrt{y} - y - 3\sqrt{x}$.

17. $cx + 5\sqrt{x} - 6y + (-3a - 1)\sqrt{y}$.

56. 3. $-11a + 16b - 4c + 7d - 7x + (4 + 3c)\,y$.

4. $117z + 283z^2 + 72y - 57ax - 20$. 5. $2a - 6b$.
6. $2a - 2b + 2c - 2d$. 7. $4a + 4b + 4c + 2d$.
8. $-3x^2 - 2x - 4$. 9. $3x^4 - x^3 + 14x + 18$.
10. $x^2 - ax + 2a^2$. 11. $2a^3 - 6a^2b + 3ab^2 - b^3$.
12. $3x^3 + 4x + 16$. 13. $-4(x - y) + 4(z - x)$.
14. $5(a - b) + 2(a + b) + 7a - 2b$.
15. $12\frac{x}{y} - 17\frac{y}{z} - 8\frac{z}{x} - 8\frac{a}{b}$.

58. 1. $2x$. 2. $2y$. 3. $4ab-4mp-3x$. 4. $mx-pz$. 5. $5\frac{a}{b}$.

59. 1. $-3ab - m - 2ax$. 2. $3x - 2a$. 3. $2b - 4c$.
4. $10x-7y+5z$. 5. $-9ax-2by$. 6. 0. 7. 0. 8. $3m$.

61. 1. $m-p+q+a-b+c+d$. 2. $m+a-b+p+q-n+k$.
3. $15ax - 4by$. 4. 0. 5. $p + b + s + t + m + n$.
6. $11ax$. 7. $-2ax-6by-cz$. 8. $-2x+2y$. 9. $-4bz$.
10. $2x - 6y - my + 4ab - 5$. 11. $ax + 2cx$.
12. $3ax - 3bx + 3ay + 3az - 3by - 3bz$.
13. $13ax - 3xy - 2d - 7ad$. 14. $m+3x+4y-ay-p$.
15. $2a\sqrt{y} + \sqrt{y} - 3m + 6n - b\sqrt{x}$.

69. 2. $6a^2bx^3$. 3. $15m^4xy$. 4. $42a^2m^2y$. 5. $4a^2m^2$.
6. $5x^9y^6z^2$. 7. $9x^2y^2z^2$. 8. $4a^2b^2m^2$. 9. $9a^4b^4x^4$.
10. $144mp^2q^2r^2s$. 11. $144ax^2y^2z$. 12. $m^{11}x^5y^4$.
13. $3mn^2k^2$. 14. $14abcd^2efg$.

70. 15. m^3xyz. 16. $abcdx^4$. 17. $12a^2b^2m^2n^2$. 18. $14a^2b^2c^2$.
19. $135m^3n^3p^3$. 20. $6a^4bcdmy^2z^2$. 21. $a^5m^3n^2x^5y^2z$.
22. $a^{10}x^7y^7$. 23. $48a^4m^4n^2x^2$.

72. 1. a^4bcdm. 2. $-abcdx^4$. 3. $-a^3b^2cx^4$. 4. $30a^6b^3mx^2$.
5. $105a^3m^2xy^3$. 6. $10n^7x^{m+n}yz^2$. 7. $4abmn$.
8. $168abm^2kx^2$. 9. $6bmngy^3$. 10. $4ax^4y^6$.
11. $-30agx^2y^3z^3$. 12. $15a^2b^2nx^3yz$. 13. $-4abgxyz^4$.
14. $4bc^2gnx^2z^5$. 15. $-3ab^2c^3x^2y$. 16. $4abcxy$.
17. $-24a^4x^2y^3$. 18. $a^4x^{\frac{7}{2}}y^3$. 19. $-3a^4x^3y^3$.
20. $-m^7n^4x^3$. 21. $a^5bx^4y^2$. 22. $-apqx^4y^3$.
23. $3a^3bcd^2x^3$. 24. $9acm^2y^2x^3$. 25. $-\frac{2}{5}acm^3n^2x^2$.
26. $3a^3bcxy^2$. 27. $-a^8bdx^5$. 28. $-30a^2m^4n^4y$.
29. $m^2n^2x^3y$. 30. $-\frac{1}{5}m^2pqx^3y^2$.

73. 2. $9x^3 - 3x^2y + 3xy^2$. 3. $3x^3 + 3x^2y + 3xy^2$.
4. $a^2x^2yz + abxy^2z + acxyz^2$.
5. $27a^2bx^4 - 45a^2bxy^2 - 63abx$. 6. $-12m^2pq + 18mnq^2$.
7. $40a^3by^3 - 56a^4by^2 - 56a^5by$.

74. 1. $ap + mp - p^2 + bq - cq - br - cr$.
2. $mx - anx - my - any + anz - mz$.
3. $acx - acy - bdx + bdy + fcdx + fcdy$.
4. $amx - a^2bm + a^2cm - abnx - b^2nc - b^3nd$.
5. $-apm - apn + bpm - bpn - bqm + bqn + aqm + aqn$.
6. $6qx - 3ncx + 10xy - 6cy - 2zm - 7zn$.
7. $a^2m^2c - an^2bc - 6amhk + 12amhd + 4amn$.
8. $6apq - 10bpq - 12cpq - 4mp^2q + 6np^2q^2$.
9. $-7abn - 7ab^2n + 7b^2cn - 3bn + abn + b^2n$. 10. 0.

76. 1. $(x^2+2x)y^3 + (3x^3-2x^2-1+5x)y^2 - 4x^3y + x^2 - 7x - 6$.
2. $x^2y^4 + xy^3 + (1-x^2)\,y^2 - xy - 1$.
3. $x^3y^5 + x^2y^4 + (x-2x^3)\,y^3 + (1-2x^2)\,y^2 - 2xy - 2$.
4. $x^4y^5 + x^3y^4 + (3x^3+x^2)\,y^3 + (3x^4+3)\,y^2 + 2x^2y + 3x$.

78. 1. $2a^2 - abn^2 - 2abn^3 + 2ab - b^2n^2 - 2b^2n^3$.
2. $3am + 2an - 5a^2bmn - 3bm - 2bn + 5ab^2mn$.
3. $2m^3n + pm^3 + qm^2n - 2mn^2 - pmn^3 + qn^3$.
4. $p^3q + p^2qr + p^3r + pq^3 + q^3r + pq^2r + pqr^2 + qr^2 + pr^3$.
5. $4a^2 - 2ab - 6b^2$. 6. $m^2x^2 - n^2y^2$.

79. 1. $6a^4 + a^3 + 11a^2 - a + 28$. 2. $a^3 - b^3$.
3. $a^4 + a^3 + a^2x^2 - a^3x - a^2x - x^4$.
4. $a^5 - 2a^4 + 3a^3 - 3a^2 + 2a - 1$. 5. $x^5 - a^5$.
6. $am + bnz + cmz^2 + dmz^3$. 7. $6a^4 + 19a^3 + 17a^2 + a - 28$.
8. $a^3 + b^3$. 9. $a^4 - x^4$. 10. $a^5 - a^3 + a^2 - 2a + 1$.
11. $x^5 + 2ax^4 + 2a^2x^3 + 2a^3x^2 + 2a^4x + a^5$.
12. $am + (an + bm)\,z + (bn + cm - ap)\,z^2$
$+ (dm + cn - bp)\,z^3 + (dn - cp)\,z^4 - dpz^5$.
13. $am + (an + bm)\,x + bnx^2$.
14. $am + (an+bm)x + (ap+bn+cm)x^2 + (bp+cn)x^3 + cpx^4$.
15. $y^5 - 5y^3 + 2y^2 + 6y - 4$. 16. $y^5 + 2y^4 + 3y^3 + y^2 + 1$.
17. $y^6 + 2y^4 - 7y^2 - 16$.
18. $(3a^{3m} - 3a^{2m+n})x + (-3a^{m+2} + 3a^{n+2})\,y + 2a^{m+2n} - 2a^{3n}$.
19. $a^3 + \frac{17}{3}a^2b + \frac{1}{3}ab - 2ab^2 - \frac{1}{9}b^3$. 20. $4ab$.
21. $a^4 + 2a^3 + a^2 - b^4 - 2b^3 - b^2$. 22. $a^2 + 2ac + c^2 - b^2$.

22. $a^2 + 2ac + c^2 - b^2$. 23. $-8a^2b$.
24. $-x^2 + (3b - a)x + y^2 + (b - 3a)y + 2a^2 - 2b^2$.
25. $a^2x^{m+2} + abx^{n+2} - a^2bx^3 + abx^{m+3} + b^2x^{m+3} - ab^2x^4$.
26. $a^{2m} - b^{2n}$.

96. 1. Q., $x - 3 + \dfrac{2}{x+1}$. 2. $x^2 + 3x + 1$.
3. Q., $x - 2 + \dfrac{-1}{x^2 - x}$. 4. Q., $2x^2 + 3 + \dfrac{2x - 2}{x^2 - x - 1}$.
7. $a^2 + a - 1$. 8. Q., $x - 1 + \dfrac{2}{x+1}$.
9. $4a^2 - 10a + 25$. 10. $a^4 - a^3 + a^2 - a + 1$.
11. $a^2 - 2a + 3$. 12. $a^3 - 2a^2 + 2a - 1$. 13. $x^4 - 10x^2 + 16$.
14. Q., $x^4 + 2x^3 - 15x + 56 - \dfrac{220}{x+4}$.
15. $1 + 2x + x^2$. 16. $1 - 3x + x^2$. 17. $3 - 2a + a^2$.
18. $1 - 2y + 2y^2 - y^3$. 19. $-16 + 8x - 4x^2 + 2x^3 - x^4$.
20. Q., $16 + 16x + 8x^2 + 4x^3 + 2x^4 + \dfrac{4x^5 - x^6}{4 - 4x + x^2}$.

97. 1. $x^2 - (a + c)x + ac$. 2. $x^2 - (a + b)x + ab$.
3. $a^2 + ac + c^2 - ab + b^2 + bc$. 4. $a^2 + a - ab + b^2 + b + 1$.
5. $ab + bx - ax$. 6. $a^4 - 4a^2bc + 7b^2c^2$.
7. $ab + ac + c^2 + bc$. 8. $c + b - a$.
9. $a^2 - ab + b^2 - ac - bc + c^2$. 10. $x^2 + 2ax + 2a^2$.
11. $ab + ax - bx$. 12. $x - b$. 13. $6a^2x^6 - 4a^3x^3 + a^4$.

104. 7. $\dfrac{m-n}{a-b} - \dfrac{m+n}{a+b}$.

105. 1. $\dfrac{1 - b + bc}{bc}$. 2. $\dfrac{1}{a-b}$. 3. $\dfrac{2}{a^2}$. 4. $\dfrac{a - b - c + d}{a - b}$.

106. 1. $\dfrac{x}{x-1}$. 2. $\dfrac{x}{x+1}$. 3. $\dfrac{2x}{1-x^2}$. 4. $\dfrac{2}{1-x^2}$. 5. 0.
6. $\dfrac{a^2+b^2}{a^2-b^2}$. 7. $\dfrac{a+x}{ax}$. 8. $\dfrac{3}{x(4x^2-1)}$. 9. 0.
10. $\dfrac{ab + bc + ca - (a^2 + b^2 + c^2)}{(a-b)(b-c)(c-a)}$. 11. $\dfrac{2ax}{x^2 - y^2}$.
12. $\dfrac{4ab}{a^2-b^2}$. 13. $\dfrac{2a}{a+b}$. 14. $\dfrac{1}{x^2(x^2-1)}$.
15. $\dfrac{2b}{a-b}$. 16. $\dfrac{2(nx + my)}{(m-n)(x+y)}$. 17. $\dfrac{-y^2 - m^2}{m^2(m-y)}$.

18. $\frac{x(a+2x)}{x^2-a^2}$. 19. 0. 20. $\frac{a+b}{b}$.

21. $\frac{2(ax-my+xy)}{x^2-y^2}$. 22. $\frac{a^2+b^2+c^2}{abc}$.

23. 0. 24. $\frac{4x^3}{x^2-1}$. 25. 0. 26. $\frac{-2ax}{x^2-a^2}$.

27. $\frac{2xy}{x^2+y^2}$. 28. $\frac{-(a-y)^2+x^2}{2ay}$. 29. $\frac{3a^2+b^2}{(a^2-b^2)^2}$.

30. $\frac{(a+b)^2}{4ab}$.

107. 1. $y\left(\frac{1}{c}-\frac{1}{a}-\frac{1}{b}\right)$. 2. $u+\frac{pu}{n}+\frac{pu}{m}$.

3. $(q+r)\left(\frac{1}{a}+\frac{1}{c}\right)$. 4. $\frac{x-4y}{2m}-\frac{b(y+3x)}{2am}$.

108. 1. $ab+y$. 12. $\frac{m^4}{m^2-n^2}$. 13. $ab+x^2-x\left(\frac{b^2}{a}+\frac{a^2}{b}\right)$.

14. $b\left(\frac{a}{x}-1\right)$. 16. $\frac{a}{a^2-b^2}$. 19. $\frac{2a+3m}{a^{2n}-b^{2n}}$.

110. 1. $\frac{y+x}{y-x}$. 2. $\frac{ax+b}{ax-b}$. 3. $\frac{(a-x)^2}{(a+x)^2}$. 4. $\frac{ak}{dn}$. 5. n.

6. $\frac{1+x^2}{2x}$. 7. $\frac{n(am^2+b)}{m(an^2-b)}$. 8. $\frac{2xy-3}{y(a+b-x)}$.

9. $\frac{(a+b)^2}{2(a^2+2ab-b^2)}$. 10. 1. 11. $\frac{a^2(a^2+2)+1}{a(1+a^2)}$.

12. $\frac{a^3+b^3}{b^2(a^2-a+b)}$. 13. 1.

14. $\frac{(x^2+y^2)(x-y)^2+(x+y)(x^2+y^2)}{(x+y)^2(x^2+y^2)-(x^2+y^2)}$.

111. 1. $\frac{b^2}{a-b}$. 4. $\frac{a^2+b^2}{a}$. 5. $x+1$. 6. $\frac{am(an+bm)}{bn(bm-an)}$.

123. 1. $x-27=0$. 2. $7x-5x=2450$. 3. $6x+4x-3x=60$.
4. $x^2+ax=ab$. 5. $abx+ab^2y+7a^2b=c$.
6. $5(4a+3b)=12x$. 7. $x^2-a^2=2ax$.
8. $x+b=2x-2a$. 9. $x+a=x^2+2ax$.
10. $x^2+3x-10=x^2-3x-10$. 11. $bx^2-by^2=axy$.
12. $x^3-5a^2x=0$. 13. $x-y=az-bz$.
14. $2x^2-ax-bx=a^2-ab$.

124. 1. $6y^2-3y+49=0$. 2. $3ax+a^2=0$. 3. $31x+23=0$.
4. $8x^4+9ax^3-6a^2x^2+a^4=0$.
5. $6a^2y^3+3(a^3-1)y^2-7a^4y+3a^2=0$.
6. $z^3+(a+b)z^2+(a^2+3ab+b^2)z+a^3+b^3+ab^2+a^2b=0$.
7. $2z^3+az^2+a^2z=0$. 8. $7y^3+6y^2+5y+4=0$.
9. $x^4-ax^3-2a^2x^2-a^3x-4a^4=0$.
10. $z^3+(b+c)z^2+c^2z+b^3+bc^2+c^3=0$.
11. $ax^2-a^2x-b^2x+b=0$. 12. $(1-n)x^2+n+1=0$.
13. $2a^2x^5+ax^4-a^5x^2+a^3=0$.
14. $10z^5-13z^4+6z^3+21z^2-6z-3=0$.
15. $(a-b)x^3+(a+b)x^2+(a^2-a^3+a^2b-ab)x=0$.
16. $a^2x^2+(-a^3+a^2b-ab-b^2)x+a^2b=0$.

129. 1. $\frac{33}{25}$. 2. $-a$. 3. 12. 4. 4. 5. $\frac{abc}{bc+ac-ab}$.
6. 25. 7. $36\frac{12}{43}$. 8. $\frac{a-b}{a+b}$. 9. 1. 10. $-\frac{3}{5}$.
11. $c+a$. 12. $4\frac{1}{2}$.
13. $b-a$. 14. 5. 15. $\frac{8a}{25}$. 16. $\frac{a^2(b-a)}{b(a+b)}$.
17. $\frac{a(1-b^2)}{b(a^2-1)}$. 18. $\frac{a(ac+b^2-1)+bc^2-b-c}{a(b+c)+bc-1}$.
19. $\frac{bn-am}{m-n}$. 20. $\frac{a^3+c^3+b^3-3abc}{3(a^2+b^2+c^2-ab-ac-bc)}$.
21. $a=\frac{d(c-b)}{b-d}$, $b=\frac{d(c+a)}{a+d}$,
$c=\frac{a(b-d)+db}{d}$, $d=\frac{ab}{a-b+c}$.
22. $a=-\frac{cd}{b}$, $b=-\frac{cd}{a}$, $c=-\frac{ab}{d}$, $d=-\frac{ab}{c}$.

130. 1. 20. 2. 72. 3. I, \$67; II, \$217. 4. 210. 5. 50.
6. 180. 7. 65. 8. A, \$130; B, \$110; C, \$260.
9. \$1000, \$1500, \$2000, \$2500, \$3000.
10. Man, 36; wife, 30. 11. I, $18\frac{1}{2}$; II, $26\frac{1}{2}$; III, 45.
12. 6 ft. 13. \$$2353\frac{11}{13}$. 14. 81 m. 15. $143\frac{19}{32}$ m.
16. A, \$600; B, \$1200. 17. $8\frac{1}{3}$ m. per h.
18. $\frac{h}{2(m-h)}$ h. 19. 15 and 24. 20. 15, 10.
21. Man, 40; wife, 35. 22. $19\frac{1}{5}$. 23. $6\frac{2}{3}$ days.

24. 30 m. 25. I, 6 ; II, 3 ; III, 2. 26. 3000. 27. 100.
28. 4. 29. \$8000. 30. \$142.50. 31. I, \$6 ; II, \$4.
32. 3 m. an h. 33. \$3600. 34. \$24800.
35. 3 h. $21\frac{9}{11}$ m.

36. I, $\dfrac{m}{1+2a+a^2}-a$; II, $\dfrac{m}{1+2a+a^2}+a$

III, $\dfrac{m}{a(1+2a+a^2)}$; IV, $\dfrac{ma}{1+2a+a^2}$.

38. I, $\dfrac{\$a-10n}{5}$; II, $\dfrac{\$a-5n}{5}$; III, $\dfrac{\$a}{5}$; IV, $\dfrac{\$a+5n}{5}$;

V, $\dfrac{\$a+10n}{5}$. 39. 16 h.; 160 m.

131. 40. 30 m.; 2 points.
41. 20 m.; 3 points. 42. $3\frac{1}{2}$ m.
43. $\dfrac{T'T}{T-T'}$.

138. 1. $y=2\frac{2}{5}$, $x=12\frac{3}{5}$. 2. $y=7$, $x=16$.

3. $x=a-b$, $y=\dfrac{7b}{6}-a$. 4. $y=\dfrac{m-n}{6}$, $x=\dfrac{m+n}{4}$.

5. $y=\dfrac{p-q}{2b}$, $x=\dfrac{q+p}{2a}$. 6. $x=84$, $y=84$.

7. $x=32$, $y=50$. 8. $x=a+b$, $y=\dfrac{3}{2}(a-b)$.

9. $x=9$, $y=3$. 10. $x=7$, $y=5$.
11. $y=6$, $x=4$. 13. $y=9$, $x=8$.
14. $y=8$, $x=6$. 15. $y=6$, $x=15$.
16. $y=7$, $x=14$. 17. $y=12$, $x=6$.

18. $y=\dfrac{2b}{c-d}$, $x=\dfrac{2a}{c+d}$. 19. $y=2$, $x=6$.

20. $y=a^2-2ab+b^2$, $x=a^2+2ab+b^2$.

140. 2. $x_1=27$, $x_2=22$, $x_3=8$, $x_4=7$.
3. $x=2$, $y=3$, $z=-2$.
4. $x=6$, $y=-1$, $z=3$, $w=2$.

5. $x=\dfrac{a-2b+c+d}{3}$, $y=\dfrac{a+b-2c+d}{3}$,

$z=\dfrac{a+b+c-2d}{3}$, $u=\dfrac{-2a+b+c+d}{3}$.

6. $x=\dfrac{2}{p+m+n}$, $y=\dfrac{2}{p+n-m}$, $z=\dfrac{2}{p-n-m}$.

1. A, \$225; B, \$150. 3. 54. 4. 42. 5. 67. 6. 81.
7. $\frac{28}{45}$. 8. A, 86; B, 72. 9. 16 good, 26 poor.
10. $\frac{14}{15}$. 11. $\frac{8}{15}$. 12. 25, 8. 13. 96, 37. 14. 24, 18.
15. A in 9, and B in 18 d. 16. 28, 23. 17. 35, 28.
18. I, 40; II, 30.
19. Bought, 72¢ and 24¢; sold, 90¢ and 32¢,
20. Coffee, $\frac{mp - ap}{mb - an}$; tea, $\frac{np - bp}{an - bm}$.
21. I, $\frac{1}{2}$; II, $4\frac{1}{2}$. 22. $\frac{a(b-c)}{b-a}$.
23. A, \$3000 @ 4%; B, \$4000 @ 5%; C, \$4500 @ 6%.
24. I, 120; II, 114; III, 110.

164. 3. 12, 24, 66. 4. $66\frac{2}{3}$, $133\frac{1}{3}$, 200, $266\frac{2}{3}$, $333\frac{1}{3}$.
5. $\frac{a}{a+b}$ and $\frac{b}{a+b}$. 6. 42. 18. 7. $\frac{m}{m-n}$ and $\frac{n}{m-n}$.
8. $x = \frac{a}{a-b}$, $y = \frac{b}{a+b}$. 10. $\frac{a+2}{a-2}$. 11. 2.
14. \$7536. 15. I, \$7700; II, \$12600. 16. 8.
17. 448 and 1008. 18. $\frac{b}{a+b}$, $\frac{a}{a+b}$.
19. 7 p. gold, 5 p. silver. 20. 5 p. gold, 3 p. silver.
21. $\frac{2am + an + bm}{(m+n)(a+b)}$, water; $\frac{bm + 2bn + an}{(a+b)(m+n)}$, alcohol.
22. $3am + 2an + bm : 3bn + 2bm + an$.
23. $(p+q)\,am + pan + qbm : (p+q)\,bn + pbm + qan$.
24. I, 5 : 3; II, 1 : 3.

173. 1. $1 + 4x + 10x^2 + 12x^3 + 9x^4$.
2. $1 + 4x + 10x^2 + 20x^3 + 25x^4 + 24x^5 + 16x^6$
3. $1 + 4x + 10x^2 + 20x^3 + 25x^4 + 34x^5 + 36x^6 + 30x^7 + 40x^8 + 25x^{10}$. 4. $1 + 4x + 10x^2 + 20x^3 + 25x^4 + 34x^5 + 48x^6 + 54x^7 + 76x^8 + 48x^9 + 25x^{10} + 60x^{11} + 36x^{12}$.
5. $1 - 4x + 10x^2 - 20x^3 + 25x^4 - 24x^5 + 16x^6$.

177. 1. $(a+b)^{\frac{3}{2}}$, $(a+b)$, $(a+b)^{\frac{3}{n}}$.
4. $(x+y)^{\frac{3}{4}}$, $(x+y)^{\frac{1}{2}}$, $(x+y)^{\frac{3}{2n}}$.

178. 17. $a^{\frac{1}{n}}(b-c)^{\frac{m}{n}}$.

184. 1. $10 + 3(5\sqrt{5} - 2\sqrt{2} - 3\sqrt{10})$. 2. $37\sqrt{2} - 17$.
4. $a + b + c + d + 2(\sqrt{ab} + \sqrt{ac} + \sqrt{ad} + \sqrt{bc} + \sqrt{bd} + \sqrt{cd})$.
8. $a^2 - 4a + 6 - \dfrac{4}{a} + \dfrac{1}{a^2}$. 9. $a^2 - b^2(x + y)$. 11. 1.
12. 1. 13. $\sqrt{2}(\sqrt{2} + 1)$. 17. $(x - y)^{\frac{1}{2}}[(x - y)^{\frac{1}{2}} - 1]$.
19. $\dfrac{1}{\sqrt{a + b}}$. 20. $\dfrac{ax + b}{ax - b}$. 21. $\dfrac{(a - x)^{\frac{1}{2}} + 1}{(a + x)^{\frac{1}{2}} - 1}$.

185. 1. $\dfrac{(a^2 - 36)^{\frac{1}{2}}}{a - 6}$. 2. $\dfrac{\sqrt{xy}}{y}$. 3. $\dfrac{\sqrt{1 - x^2}}{1 - x}$. 4. $\dfrac{7\sqrt{15}}{45}$.
5. $\dfrac{2\sqrt{3}}{3}$. 6. $5\sqrt{3}$. 7. $\dfrac{(a + \sqrt{b})^2}{a^2 - b}$. 8. $\dfrac{(a - \sqrt{x})^2}{a^2 - x}$.
9. $\dfrac{(\sqrt{x} + \sqrt{y})^2}{x - y}$. 10. $\dfrac{a^2 + a(x + y)^{\frac{1}{2}} - 2(x + y)}{a - x - y}$.
11. $\dfrac{9\sqrt{15} + 41}{2}$. 12. $\dfrac{(\sqrt{x} - \sqrt{x + y})^2}{-y}$.
13. $\dfrac{x + (x^2 - a^2)^{\frac{1}{2}}}{a^2}$. 14. $(a + 1)^{\frac{1}{2}} - a^{\frac{1}{2}}$.
15. $\dfrac{x + \sqrt{x^2 - a^2}}{a}$.

187. 1. $x^2 + 2xy = (x - y)^2 - y^2$.
2. $x^2 + 4xy = (x + 2y)^2 - 4y^2$.
3. $x^2 + 6ax = (x + 3a)^2 - 9a^2$.
4. $4x^2 + 4xy = (2x + y)^2 - y^2$.

190. 1. $\dfrac{p^2}{q^2}$. 2. $\dfrac{(a + b)^3}{c^3}$. 3. $(a + b)^3$. 4. 6. 5. $\sqrt{ab}$.
6. $a^{\frac{1}{4}}b^{\frac{1}{4}}$ 7. $(a^2 - b^2)^{\frac{nq}{mq + np}}$. 9. $(b^4 - 2a^2b^2 + 2a^4)^{\frac{1}{2}}$.
10. $b^2 + a$. 11. $\dfrac{a}{(1 - m^2)^{\frac{1}{2}}}$. 12. $\dfrac{b}{(1 - n^2)^2}$.

191. 1. 6, 12, 4. 2. 15, 12. 3. 47, 35. 4. 16. 5. $a + 1$.
6. 8, 16. 7. 64, 512. 8. 16, 48.
9. 10, 15. 10. $\dfrac{m^2 + mn}{(m - n)^2}, \dfrac{mn + n^2}{(m - n)^2}$. 11. 28, 36.

12. $y = \sqrt{\dfrac{ac' - a'c}{ab' - a'b}}$, $x = \sqrt{\dfrac{b'c - bc'}{ab' - a'b}}$. 13. 10, 24.

14. $\dfrac{c}{(m^2 + n^2)^{\frac{1}{2}}}$. 15. $\dfrac{1}{4}\sqrt{h}$.

195. 1. $x = 10, -\dfrac{2}{5}$. 2. $y = \pm 8$. 4. $y = a \pm b$.

5. $x = -a$ or $-b$. 7. $x = a(1 \pm \sqrt{2})$.

8. $y = \dfrac{1}{8}(-1 \pm \sqrt{129})$. 9. $y = \dfrac{1}{2}$.

10. $x = \pm a\sqrt{\dfrac{3}{5}}$. 1. ± 21, ± 27. 2. 4 and 10.

3. ± 17. 4. 36, 24. 5. 10, 15, 30.
6. 6, 10, 14, 18, or -18, -14, -10, -6.
7. 35. 8. 21 turkeys, 25 chickens. 9. 12. 10. 10.
11. 250. 12. 3. 13. Length, 45; breadth, 35.

14. $\dfrac{1}{2}(\sqrt{4m^2 + a^2} + a)$ and $\dfrac{1}{2}(\sqrt{4m^2 + a^2} - a)$.

15. 72 or 108.

196. 1. 81. 2. 121. 3. 225. 4. 289. 5. 256. 6. $\sqrt{3 \pm 3\sqrt{6}}$.

7. $\sqrt{\dfrac{1}{6}(7 \pm \sqrt{349})}$. 8. $\left(\dfrac{13}{5}\right)^4$, 1. 9. $\sqrt{\left(a^{\frac{1}{2}} + \dfrac{1}{a^{\frac{1}{2}}}\right)^{\frac{4n}{m}} - a^2}$.

202. 1. $x = \dfrac{a}{4}$. 2. $x = -a$. 3. $x = 13$. 4. $x = 50$.

5. $x = 1$. 6. $x = \dfrac{-1 \pm \sqrt{16a^2 + 1}}{8}$. 7. $x = 16$.

8. $x = 5$ or $\dfrac{9}{5}$. 9. $x = a^2 - b^2 \pm b\sqrt{b^2 - a^2}$.

10. $x = 4$. 11. $x = a$ or $\dfrac{1}{a}$.

203. 1. $x = \dfrac{37}{7}$ or 5, $y = \dfrac{10}{7}$ or 2.

2. $x = -4$ or $+13$, $y = 0$ or -17.

3. $x = -\dfrac{1}{3} \pm \dfrac{1}{3}\sqrt{-863}$; $y = \dfrac{1}{6}(1 \mp \sqrt{-863})$.

4. $x = 11$ or $-7\frac{54}{73}$, $y = 15$ or $-17\frac{58}{73}$.
5. $x = -21$ or 4; $y = 28$ or 3.

204. 1. $x = 1.37\ldots$ or $-0.156\ldots$; $y = -4.46\ldots$ or $-6.096\ldots$

2. $y = \dfrac{7}{2}$ or -4; $x = \dfrac{79}{12}$ or $-\dfrac{14}{3}$.

3. $x = 2$ or 5; $y = 6$ or 3.

205. 1. $y = \pm 1$ or $\pm\sqrt{\frac{1}{7}}$: $x = 2y$ or $-4y$.

2. $y = \pm\frac{1}{2}$ or $\pm\frac{4}{\sqrt{19}}$; $x = \mp\frac{3}{2}$ or $\pm\frac{3}{\sqrt{19}}$.

207. 1. $x = \pm 5$; $y = \pm 2$. 2. $x = \pm 8$; $y = \pm 3$.

3. $x = \frac{1}{2}(5 \pm \sqrt{5})$; $y = \frac{1}{2}(5 \mp \sqrt{5})$.

4. $y = 7$ or 2; $x = 2$ or 7. 5. $x = 5$ or 7; $y = 7$ or 5.

6. $x = \pm 9$; $y = \mp 2$. 7. $x = \pm 25$; $y = \pm 9$.

8. $x = \pm\frac{b}{\sqrt{a+b}}$; $y = \pm\frac{a}{\sqrt{a+b}}$. 9. $x=2$; $y=1$.

10. $x = a(a \pm b)$; $y = b(a \pm b)$. 11. $x = 4$; $y = 5$.

12. $x = \frac{3}{5}$; $y = \frac{1}{5}$. 13. $x = \frac{4}{15}$; $y = \frac{1}{15}$.

14. $x = 5$; $y = 1$ or 2; $z = 2$ or 1. 15. $x = 5$; $y = 3$.

16. Time, 6 or 7; rate, 7 or 6. 17. Dist., 30 or $46\frac{2}{3}$.

18. $x = \frac{1}{2}\left(\sqrt{a^2 + 4b^2} + \sqrt{a^2 - 4b^2}\right)$;

$y = \frac{1}{2}\left(\sqrt{a^2 + 4b^2} - \sqrt{a^2 - 4b^2}\right)$.

19. $\frac{1}{2}(1 \pm \sqrt{5})$ and $\frac{1}{2}(3 \pm \sqrt{5})$.

20. 24 and 9, or -12 and -18. 21. 49 and 25.

22. 64 and 8.

23. $m + n \mp \sqrt{m^2 + n^2}$ and $m + n \pm \sqrt{m^2 + n^2}$.

24. 12 men working 12 h. 25. 8; 10.

26. $x = \pm 6$; $y = \pm 4$. 27. 11; 3.

210. 7. 14075. 8. 5050. 10. n^2. 11. $n^2 + n$.

12. Lowest, $140 - 6m$; all, $137m - 3m^2$,

16. 0, 2, 4, 6, 8. 17. 951. 18. 4, 10, 16. 19. 11 or 8.

21. 10 or 16 d. 22. 9 days. 23. 2, 5, 8, 11, 14.

25. 2, 6, 10, 14, 18, 22, 26, 30, 34, 38. 27. 3, 5, ... 29.

28. a, $a + \frac{l-a}{i+1}$, $a + \frac{2(l-a)}{i+1}$, etc.

212. 6. Last nail, \$21474836.48; all, \$42949672.95. 7. 246.

12. 5 or $\frac{1}{5}$.

214. 1. $\frac{1}{2}$. 2. 2. 3. $\frac{1}{10}$. 4. $\frac{4}{5}$. 5. $\frac{1}{b}$. 6. $\frac{a}{b}$. 7. $\frac{m^2-2m}{m^2-1}$.

8. $12 - 6 + 3 - \frac{3}{2} + \text{etc.}, \textit{ad inf.}$ 9. $\frac{1}{3}$ from A to B.

215. 1. $\frac{1}{9}$. 2. $\frac{2}{9}$. 3. 1. 4. $\frac{1}{10}$. 5. $\frac{5}{11}$. 6. $\frac{27}{110}$. 7. $\frac{108}{999}$.
8. $\frac{797}{1100}$.

216. 1. \$216.74. 3. $2.72325a$.

4. $$a\frac{\left(1+\frac{c}{100}\right)^n - 1}{\left(1+\frac{c}{100}\right)^{n+1} - \left(1+\frac{c}{100}\right)^n}$$ and

$$a\frac{\left(1+\frac{c}{100}\right)^n - 1}{\left(1+\frac{c}{100}\right)^{n} - \left(1+\frac{c}{100}\right)^{n-1}}.$$

226 (*a*). 1. 440. 3. 74. 4. 148. 5. 0. 6. 0.
16. 1. 17. 4. 18. 10. 19. 20. 20. 35. 21. 56. 22. 0.
23. $-\frac{1}{4}$. 24. -1. 3. 74. 4. 148. 5. 0. 6. 0.

227. 1. $A_2 = A_1 - A_0$; $A_3 = -A_0$; $A_4 = -A_1$; etc.; $A_{10} = -A_1$.
2. $A_5 = 16A_1 - 15A_0$. 3. $A_5 = 43A_1 + 30A_0$.
4. $\frac{n+1}{2}(2A_0 + nh)$. 5. $\frac{r(r^n - 1)}{r - 1}A_0$.
6. $A_2 = kA_1 + A_0; \dots . A_6 = (120k^5 + 96k^3 + 9k)A_1 + (120k^4 + 36k^2 + 1)A_0$.

228. 1. $n - 4$ terms are omitted. 5. $s - 2$.
1. 120. 2. 720. 3. 40320. 4. 35. 5. 56.
1. 247. 2. 70.

230. 1. $2^3 \cdot 3$. 2. $2^3 \cdot 3^2$. 3. $2^2 \cdot 5 \cdot 13$. 4. 13^2. 5. $3^2 \cdot 5^2$.
6. 2^8.

232. 2. 7. 3. 1. 4. 1. 5. 4.

233. 6. First wheel = 7, second = 5, third = 3 turns.

245. 1. $\frac{113}{355}$. 2. $\frac{17}{58}$. 3. $\frac{19}{72}$. 4. $\frac{5x+1}{3(5x+1)+x}$.
5. $\frac{bc+1}{a(bc+1)+c}$.

251. 2. $\frac{1}{6}$. 4. 720. 5. 24.

6. (*a*) 2160 even, 2880 odd; (*b*) 144; (*c*) 720; (*d*) 576.
7. 720. 8. 120. 9. 120. 10. 120. 11. 12. 12. 72.
13. 144. 14. 720.

252. 3. 360. 5. 60. 6. 90 7. 24.

253. 1. 720. 2. 120. 3. 48. 4. 4. 5. 2880. 6. 14400.

254. 4. 140. 5. 6486480.

255. 1. 5. 2. 5. 3. 8. 4. 6. 5. 16. 6. 17.

257. 3. 3, 21, 36. 5. 10. 7. 3. 8. 3.
9. (*a*) 1; (*b*) 3; (*c*) 6 ways. 10. 3003.

258. 2. 35, $\left(\dfrac{2n-1}{n}\right)$. 3. $\left(\dfrac{2n-1}{n+1}\right)$.

261. 3. 15 and 20. 5. $\left(\dfrac{n-3}{s-3}\right)$.

263. 1. 120. 2. 240. 3. 2^n. 4. 81 routes.

267. 1. $\frac{1}{2}$. 2. $\frac{2}{3}, \frac{1}{3}$. 3. $\frac{1}{36}, \frac{5}{18}$. 4. $\frac{5}{36}$. 5. $\frac{1}{6}$. 6. $\frac{1}{3}$.
7. $\frac{3}{5}$. 8. $\frac{1}{3}$. 9. $\frac{5}{14}$. 10. $\frac{3}{10}$. 11. $\frac{1}{5}$. 13. $\frac{3}{7}$. 14. $\frac{7}{27}$.
15. $\dfrac{2mn}{(m+n)(m+n-1)}$. 16. $\frac{3}{10}$. 17. $\frac{n}{2^n}$.

269. 1. $\frac{2}{5}$ and $\frac{2}{15}$. 2. $\alpha, \frac{63}{80}$; $\beta, \frac{1}{80}$; $\gamma, \frac{7}{80}$; $\delta, \frac{9}{80}$.
3. 2 to 1 in favor. 4. $\frac{3}{14}$.
5. $\dfrac{3mn(m-1)}{(m+n)(m+n-1)(m+n-2)}$.
6. $A=\frac{2}{7}$, $B=\frac{5}{21}$, $C=\frac{4}{21}$, $D=\frac{1}{7}$, $E=\frac{2}{21}$, $F=\frac{1}{21}$.
7. $A=\frac{1}{2}$, $B=\frac{1}{4}$, $X=\frac{1}{4}$. 8. A, $\frac{2}{3}$; B, $\frac{1}{3}$.
9. $\frac{1}{2}\cdot\frac{2^n}{2^n-1}$; $\frac{1}{2^2}\cdot\frac{2^n}{2^n-1}$; .... $\frac{1}{2^n}\cdot\frac{2^n}{2^n-1}=\frac{1}{2^n-1}$.
10. $\frac{36}{91}, \frac{30}{91}, \frac{25}{91}$.
12. The chances are 41 to 25 in favor of the first purse.

271. 1. $\frac{80}{243}, \frac{80}{243}, \frac{40}{243}, \frac{10}{243}, \frac{1}{243}$. 2. $\frac{347}{2048}$. 3. $\frac{16}{27}$.

274. 1. (*a*) 0.429; (*b*) 0.159; (*c*) 0.813; (*d*) 0.655;
(*e*) 0.371; (*f*) 0.110; (*g*) 0.151; (*h*) 0.025.

2. 69. 4. \$296.30. 5. 0.4533.
6. \$1000; \$1666.67; \$2111.11. 7. $a\,[1-(1-p)^s]$.
8. 0.1123. 9. \$1894. 10. \$1224.

278. 1. 140. 2. 70. 3. 112. 5. $22k$. 6. $28\,(j-1)$.
8. $54m^2$. 9. $\frac{11}{10}$.

282. 1. $1+x+x^2+x^3+\text{etc.}$ 2. $1+2x+2^2x^2+2^3x^3+\text{etc.}$
3. $1-2x+2x^2-2x^3+\text{etc.}$ 4. $1+2x+2x^2+2x^3+\text{etc.}$
5. $1-x-x^2+5x^3-7x^4-\text{etc.}$ 6. $1+x+x^2+x^3+x^4+\text{etc.}$
7. $1-4x+8x^2-4x^3-16x^4+\text{etc.}$
8. $1-2x+2x^2-x^3-x^4+\text{etc.}$

283. 1. $1-3x+3x^2-3x^3+\text{etc.}$ 2. $1+2x+x^2-x^3-2x^4-\text{etc.}$

288. 1. $\frac{r\,(2n-r+1)}{2}$. 2. $\frac{h\,(h+1)-s\,(s+1)}{2}$.
3. $\frac{n\,(n+1)-m\,(m-1)}{2}$. 4. $\frac{p\,(p+2k-1)}{2}$.
6. $3n^2-3n+1$.

289. 1. 165. 2. $\frac{n\,(n+1)\,(n+2)-k\,(k+1)\,(k+2)}{1\cdot 2\cdot 3}$.
3. $\frac{n\,(n+1)\,(n+2)-(m-1)\,m\,(m+1)}{1\cdot 2\cdot 3}$.

293. 1. $S_1=210$; $S_2=2870$; $S_3=42665$.
2. $S_1=r^2$; $S_2=\frac{r\,(4r^2-1)}{3}$.
3. $S_1=r\,(r+1)$; $S_2=\frac{2r\,(r+1)\,(2r+1)}{3}$.
4. $N_3=3pq-3\,(p+q)+5$;
$N_p=spq-\frac{s\,(s-1)}{6}\,(3p+3q-2s+1)$.
5. $5a+15b+55c$. 6. $b\left[a+\frac{b+1}{2}\left(b+\frac{2b+1}{3}c\right)\right]$

295. 1. $\frac{1}{3}-\frac{1}{n+3}$. 2. $\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2n+3}\right)$.
3. $\frac{2}{3}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{n+2}-\frac{1}{n+3}-\frac{1}{n+4}\right)$.
4. $\frac{3}{2}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)$. 5. $\frac{1}{a}$.

296. 1. $\dfrac{2ar(1-r^{n-1})}{(1-r)^2} + \dfrac{a[1-(2n-1)r^n]}{1-r}$ and $\dfrac{a(1+r)}{(1-r)^2}$.

2. $\dfrac{2a(1-r^n)}{(1-r)^2} - \dfrac{2anr^n}{1-r}$ and $\dfrac{2a}{(1-r)^2}$.

3. $\dfrac{br(1-r^n)}{(1-r)^2} + \dfrac{ar-(a+nb)r^{n+1}}{1-r}$ and $\dfrac{(b+a)r-ar^2}{(1-r)^2}$.

300. 2. $\Delta_5 = -305$; $\Delta_i = \dfrac{3}{2}i^3 - \dfrac{39}{2}i^2 - 2i + 5$.

3. $341^\circ\, 5'\, 10''.9 + (n-1)(1^\circ\, 0'\, 9''.6) - (n-1)(n-2)''$.

4. $495 + 15(n-5) - 5\dfrac{(n-5)(n-6)}{2}$;

Morning of May 23 or Apr. 24.

304. 1. 1. 2. $\dfrac{a}{b}$. 3. $\dfrac{m}{p}$. 4. $\dfrac{1}{a}$. 5. $2a$. 6. -1.

308. 1. $\sqrt{8} = 2.828427$; $\sqrt{2} = 1.414214$.

2. $1 - \dfrac{1}{2}x - \dfrac{1\cdot 1}{2\cdot 4}x^2 - \dfrac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 - \dfrac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 - \text{etc.}$

3. General term $= -\dfrac{1\cdot 1\cdot 3\cdot 5\cdot 7\ldots\ldots 2i-3}{2\cdot 4\cdot 6\cdot 8\ldots\ldots 2i}x^i$.

4. $(-1)^i\dfrac{1\cdot 3\cdot 5\ldots\ldots(2i-1)}{2\cdot 4\cdot 6\ldots\ldots 2i}x^i$. 5. $\left(\dfrac{m}{i}\right)\dfrac{1}{x^i}$.

6. $\dfrac{(-1)(m-1)(2m-1)\ldots\ldots[(i-1)m-1]}{i!\;m^i}$.

7. $$1 + 1 + \frac{1-m}{2!} + \frac{(1-m)(1-2m)}{3!} + \frac{(1-m)(1-2m)(1-3m)}{4!} + \text{etc.}$$

8. $-\left(\dfrac{1}{b^3} + \dfrac{3}{1}\dfrac{a}{b^4} + \dfrac{3\cdot 4}{1\cdot 2}\dfrac{a^2}{b^5} + \ldots\ldots + \dfrac{3\cdot 4\cdot 5\ldots\ldots i+2}{1\cdot 2\cdot 3\ldots\ldots i}\dfrac{a^i}{b^{i+3}}\right)$

9. $$(-1)^m\left(\frac{1}{x^m} + \frac{m}{x^{m+1}} + \frac{m(m+1)}{1\cdot 2x^{m+2}} + \frac{m(m+1)(m+2)}{1\cdot 2\cdot 3x^{m+3}} + \ldots\ldots\right).$$

HINTS ON A COURSE IN ADVANCED ALGEBRA.

For the benefit of students who may contemplate a course of reading in the various branches of Advanced Algebra, the following list of subjects and books has been prepared. As a general rule, the most extended and thorough treatises are in the German Language, while the French works are noted for elegance and simplicity in treatment.

To pursue any of these subjects to advantage, the student should be familiar with the Differential Calculus.

I. THE GENERAL THEORY OF EQUATIONS.—In English, TODHUNTER'S is the work most read.

SERRET, *Algèbre Supérieure*, 2 vols., 8vo, is the standard French work, covering all the collateral subjects.

JORDAN, *Théorie des Substitutions et des Équations Algébriques*, 1 vol., 4to. is the largest and most exhaustive treatise, but is too abstruse for any but experts.

II. DETERMINANTS.—BALTZER, *Theorie der Determinanten*, is the standard treatise. There is a French but no English translation. A recent English work is ROBERT F. SCOTT, *The Theory of Determinants and their Applications in Analysis and Geometry.*

III. THE MODERN HIGHER ALGEBRA, resting on the theories of Invariants and Covariants.

SALMON, *Lessons introductory to the Modern Higher Algebra*, is the standard English work, and is especially adapted for instruction.

CLEBSCH, *Theorie der binären Algebraischen Formen*, is more exhaustive in its special branch and requires more familiarity with advanced systems of notation.

IV. THE THEORY OF NUMBERS. There is no recent treatise in English. GAUSS, *Disquisitiones Arithmeticæ*, and LEGENDRE, *Théorie des Nombres*, are the old standards, but the latter is rare and costly. LEJEUNE DIRICHLET, *Vorlesungen über Zahlentheorie*, is a good German Work. There is also a chapter on the subject in SERRET, *Algèbre Supérieure.*

V. SERIES.—This subject belongs for the most part to the Calculus, but CATALAN, *Traité elémentaire des Séries*, is a very convenient little French work on those Series which can be treated by Elementary Algebra.

VI. QUATERNIONS.—TAIT, *Elementary Treatise on Quaternions*, is prepared especially for students, and contains many exercises. The original works of HAMILTON, *Lectures on Quaternions* and *Elements of Quaternions*, are more extended, and the latter will be found valuable for both reading and reference.