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THE VALUE OF MATHEMATICS AS AN INSTRUMENT OF EDUCATION.

AN ADDRESS

DELIVERED BEFORE

THE ONTARIO TEACHERS' ASSOCIATION,

AUGUST 12th, 1879,

BY THE PRESIDENT, J. A. McLELLAN, M.A., LL.D.

In discussing Education, one of the most important questions that has arisen is—*What branches of study possess the greatest educational value as tending at once to secure the highest development of mental power, and contribute most largely to human progress?* From the variety of the intellectual powers, it may be fairly inferred that different subjects of study produce different results, and that it is absurd to expect to accomplish the great end of education by the *exclusive* study of any single department of human knowledge. But there are some subjects which educate the powers of the mind to a pre-eminent degree of activity and energy; and as I believe that the Mathematical Sciences can claim a high place among these, it is my present purpose to point out, in part, their beneficial influence as a discipline of the mind, and as an instrument of progress. I shall consider the subject under the following heads:—

I. THE CHARACTER OF MATHEMATICAL KNOWLEDGE, AND ITS GENERAL INFLUENCE ON THE MIND.

II. THE VALUE OF MATHEMATICS AS A MEANS OF INVIGORATING THE INTELLECTUAL POWERS.

III. THEIR VALUE AS A LOGICAL EXERCISE OF MIND.

IV. THEIR VALUE AS AN INSTRUMENT OF MATERIAL PROGRESS.

I.

THE CHARACTER OF MATHEMATICAL KNOWLEDGE AND ITS GENERAL INFLUENCE ON THE MIND.

1 Compared with Mathematics, no other department of knowledge so clearly illustrates the processes of the human mind in the establishment of true science. In none, has investigation led to the discovery of so great a number of important and recondite truths. This is no doubt due to their *method* and the nature of their principles. The value of any *system* of knowledge and its influence in education, depend upon the nature of its first principles and the mode of its development. If these principles be founded on imperfect observation or hasty generalization, the superstructure reared upon them, though possessing an external aspect of solidity and beauty, and dignified by its builders with the name of Science, will partake of the instability of its basis, and crumble into ruins before the invincible progress of true philosophy. But, if they bear the impress of indubitable truths, and be developed by a strictly scientific method, there will result a knowledge that can be shaken neither by the attacks of scepticism, nor the cavils of ignorance—a knowledge which is indeed worthy of the name of Science. Now, the principles of pure Mathematics are necessary truths, and consequently the knowledge founded upon them is a science of pure reason. It has indeed been asserted that the Mathematician has no right to enquire into the nature of the first principles of the science, and that a clear apprehension of their character as universal truths cannot produce any modification of their influence in education. The former assumption is partly wrong, the latter entirely so. For, though the Mathematician has no right to enquire *how* pure synthetic propositions are possible as the offspring of the understanding—this being properly the province of the metaphysician—he has a perfect right, since his science is *demonstrative*, to enquire into the validity of the principles which underlie his demonstrations, or form his links of method. And again, if the first principles are considered as empirical, the entire science is viewed as empirical; the necessary is sacrificed to the contingent, the mental to the material, a standing doubt as to the possibility of knowledge is likely to be generated in the mind, and a foundation laid for an unimproving [illegible] with all its consequences. Hence, a clear apprehension of the nature of the first principles of the science, is necessary to the recognition of its real nature as a production of the understanding, and the science itself is thus enabled to exert a beneficial influence on the mind, by familiarizing it with a body of truths not derived from experience.

There are some philosophers who maintain that the first principles of Mathematics are merely generalizations from experience—a view which reduces the science to empiricism, and virtually implies the impossibility of any knowledge independent of sensuous impressions. If the meaning be, that these principles are first made manifest through the agency of something out of the mind, it may be readily admitted, since there is no knowledge developed in the mind in point of time absolutely *antecedent* to experience.

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But though experience may be the *occasion* of our knowledge, as furnishing the materials of thought, and calling into activity the latent energies of the mind, does it follow that all our knowledge is derived from experience, uninfluenced by the creative laws of the "INTELLECTUS IPSE?"

Nay, the possibility of experience itself depends on the validity of certain cognitions which must be the property of the mind alone. For from what source, independent of the universal laws of the intellect, can these principles spring, which recognize the validity of experience? They cannot be derived from the fortuitous teaching of that experience whose trustworthiness they affirm, and they must consequently be the genuine offspring of the understanding. The principles of pure Mathematics are characterized by *necessity* and *universality*—properties which cannot possibly be given in the limited and fortuitous teachings of the sensible world. These can indeed inform us that a thing *is*, but not that it *has been, is,* and *will be,* in consequence of the universal necessity that it MUST be, which are the clearly conceived characteristics of Mathematical truths. The laws of the sensible world may change, and phenomena occur in violation of all past experience; there can be imagined a time when the sun shall no longer rise in the eastern and set in the western heavens; but there can never come a time when two parallel lines can meet, or two straight lines enclose a space. Pure Mathematics, therefore, furnish striking instances of intuitive truths as eternal as the intellectual principle itself, and present a brilliant example of the almost limitless extent to which the mind can develop clear and abiding knowledge, independent of the laws of the external world.

2. The principles of *Applied Mathematics* are generalizations from experience, which, by the application of pure Mathematics, are elevated to the rank of universal truths that constitute the basis of sciences commonly classified with those of pure reason. The principles of pure Mathematics are *pure, a prori* truths; those of the physical sciences founded on Mathematics, contain elements of experience. The truths of pure Mathematics are absolutely universal and necessary; while the fundamental principles of Applied Mathematics are universal only on the assumption of permanence in the operation of nature's laws. Of the former we are assured that they must be, of the latter we know only that they MUST BE *under the present condition of things.* To this extent these principles are universal and necessary, and by the aid of those of pure Mathematics, have become the elements of sciences which, as I have already said, are classed as rational. Now, *Applied Mathematics* affords a striking illustration of the application of pure mathematics to the varied phenomena of the Universe, and exhibit in a remarkable degree the harmony of deduction and induction as methods of investigation, as well as the power of the former to aid the latter in bringing the contingent and variable within the compass of universal laws. These laws are sometimes *clearly* indicated by a few well observed facts and do not seem to require the aid of pure science to demonstrate their universality. But even in these instances, Mathematical investigations frequently indicate the existence of phenomena overlooked by observation, and show the assumed law to be more general than mere experience had indicated, by demonstrating that certain facts considered as consequences of other and unknown agencies, are the legitimate results of the already discovered law. But in many instances the law is only faintly pointed out by the observed facts, or suggested by a fortunate anticipation of genius, and Mathematics are needed to verify its claims to be ranked as a GENERAL LAW, by submitting the deductions logically derived from it, to the test of coincidence with observed phenomena. Thus the logic of induction as applied to external nature, is eminently aided by the logic of pure Mathematics; the former often indicating laws beyond the scope of the latter, while, on the other hand, the latter gives them their character as general laws, and guides experience to results it would otherwise never reach. Hence, while pure Mathematics constitute a world of ideas independent of material forms, applied Mathematics grasp the phenomena of the material world as the tangible forms of the pure ideal, and show the harmony between the world without, and the inner world of thought. If the Mathematics of the world of mind, exhibit the "harmony of thought with thought," their application to external phenomena demonstrates the harmony of thought with existence, and secures to intelligence its conquest over matter. We can thus understand the importance of the knowledge given to the mind by the application of Mathematics to the investigation of external phenomena—a knowledge which alone can dispel the mists that hide the treasures of nature from our intellectual vision—which wrests the universe from the grasp of an inexorable fate, and with reverence places it before the THRONE OF GOD.

3. *Mathematical Principles and Propositions are Synthetic—leading to New Truths.* It has sometimes been said that Mathematical propositions are merely analytic—that they only resolve conceptions into their contained elements, and consequently furnish the mind with no new truths. It may be admitted that there are certain axioms and definitions in Mathematics which are identical propositions: but most of these are only minute links in the chain of method, and are not absolutely necessary to the development of the Science. Every principle really necessary in Mathematical investigation is a synthetic proposition: the predicate is not affirmed of the subject through the principle of identity, as something necessarily given in our conception of it; but on the contrary, the predicate is added as a new attribute to the contents already cogitated in our conception of the subject. For it appears evident that Mathematics could not possibly have been developed, as they undoubtedly have been, into the most perfect of the rational sciences, by combining merely identical propositions. Even the fundamental propositions of Arithmetic are not identical propositions. For, to use an illustration from Kant, in the addition of the simplest numbers, does the simple cogitation of their *union,* evolve the conception of their *sum?* The conception of such a union can never give the required predicate, and the synthesis must be effected by intuitions supplied by external objects. It is the same with the primary truths of pure Geometry. The proposition, three straight lines may enclose a space, is not an assertion of identity. For by no *analysis* of the subject—one conception of which (straight) is merely *qualitative,* the other (line) a quantity of but *one* dimension—is it possible, to evolve the conception of a quantity of *two* dimensions, and the proposition is therefore synthetic. And thus we might show that every proposition essential to the establishment of the pure Mathematical sciences, is a synthetical or augmentative proposition. If such, therefore, is the real nature of Mathematic[illegible]sitions, the science itself must be augmentative, and hence appea[illegible]fairness of the assertion that as wholly *given* in its princip[illegible]re is merely *explicative,*—the sim[illegible] evolution of a potential into [illegible]ctual knowledge—and that therefore it can be of but little worth as an exercise of the higher faculties of mind. The worth of this philosophy can be easily illustrated. Take the definition of a circle; this definition includes all its properties; one property is that the circumference is to the diameter in the approximate ratio of 3.1416 to 1: is this recondite property given in the definition, so that it is conceived by a mere "negation of thought?" On the contrary it was discovered only by a train of complicated reasoning, and it is a mere perversion of language to say that it formed part of the concept *circle,* before it had been discovered by mathematical investigation. Every science worthy of the name must proceed from certain principles, and may in one sense be said to be given in its principles; but are its highest developments therefore attained with a minimum of thought? The pure Mathematical sciences are indeed founded on principles which are immutable and given intuitively; but *their law is progress.* They proceed from their primary conceptions to the most comprehensive generalizations—from the simplest abstractions to the highest within the compass of the human intellect—and yet we are asked to believe that they do nothing more than evolve the truths *contained in* their principles! If the objection has any weight when urged against Mathematics, it is equally valid against all the rational sciences, and especially against Metaphysics. For the very possibility of mental science seems to depend on the existence of propositions similar to the *a priori* synthetic principles of pure Mathematics. It is the province of Metaphysics not to analyze conceptions of things, but like Mathematics, to proceed from the synthetic judgments of the reason, to enlarge their boundaries by combination with others, till organized knowledge takes the place of isolated principles. It is true that Metaphysics either from error in method, or from the tendency of reason to attempt the solution of the insoluble, have never attained that certainty which distinguishes mathematics. But surely the latter are not of less worth because instead of groping in obscurity on a sea of doubt, they are constantly making unerring progress on the great ocean of truth.

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4. *But the Method of Mathematics is as rigorous as its principles are sure.* Proceeding from intuitive truths by logical reasoning, we discover new truths which are in turn to become the *data* for higher and more complicated results. Not only the premises but every succeeding step in the reasoning is cautiously examined. In all mathematical investigations, "even those of the poorest and most meagre form" the mind is *habituated* to resolve every train of reasoning into steps, and to make sure each step before proceeding to another. If practice in such reasoning "gives us nothing else, it give us wariness of mind;" it accustoms us to demand a sure footing: and though it leaves us no better judges of ultimate premises than it found us (which is no more than may be said of almost all metaphysics) at least it does not suffer us to let in, at any of the joints of the reasoning, any assumption which we have not previously faced in the shape of an axiom, postulate, or definition." Nor can any error be *incorporated into* the science; for though a mistake may be made in effecting the synthesis between the known and the unknown terms of a proposition, the results, either directly or indirectly, indicate a fallacy in the reasoning, and the error must be corrected before further progress can be made. Such then, being the nature of Mathematics, and such the character of *their method*, we are prepared to understand their perfection as a science. It is a science whose foundations cannot be shaken by any possible revolutions of experience, and whose symmetry and beauty can never be obscured by the gloom of scepticism: it stands forth, a world of pure ideas, supreme over the fluctuations of the world of sense—the genuine creation of the understanding

Having thus glanced at the nature and method of Mathematical Science, let me refer briefly to the *general* influence which they may be supposed to exert on the mind.

When uncertainty and contradiction in so many other departments of investigation, incline us to despair of the attainment of truth, Mathematics in their wonderful certainty, in the almost measureless extent of the truths they reveal, in the marvellous power over the secrets of the universe which they give, furnish undoubted evidence of the possibility of knowledge, and a standing refutation of philosophic scepticism. Dealing with necessary truths, in the study of Mathematics, the mind becomes familiar with their nature as independent of sensuous impressions, and acquires a knowledge that has all the marks of a true science. Instead of looking in bewilderment at the ever-varying phenomena of the external world, we are enabled to discover, by the application of Mathematics, the general laws which underlie all changes, and clear and abiding knowledge takes the place of isolated, perplexing facts. Looking towards the arena of Metaphysics and beholding the unending warfare of systems which is there exhibited—a warfare in which no victory has ever yet secured permanent possession—we may well doubt the possibility of a true science of mind. But turning towards the brilliant success of Mathematical science, which like Metaphysics, is a science of *a priori* truths, we confidently look for the time when a true method shall be found to guide us to the solution of the great problems which have so long baffled human reason and which still knock for answer at the human heart.

To this result I have no doubt, the judicious study of mathematics will contribute. Elevated beyond mere sensuous impressions, the mind is constantly contemplating those pure forms which are its own creation, and therefore independent of matter and its phenomena. Does not this contemplation of the pure ideal, qualify the mind for the examination of its own attributes? To grasp and analyze the phenomena of mind, we must rise above the physical and concentrate attention on the mental—the world of thought; and to accomplish this the highest degree of the power of abstraction is required—abstraction is the only method, the only guide to success. And thus the processes of mathematics which, as already said, are constantly familiar with a world of supersensuous ideas, and which develop to its highest range the abstractive powers of the intellect, must eminently prepare the mind for the investigation of its own activities, and contribute materially to the devolopment of a true science of mind.

It has been said that, "in consequence of their disqualifying us for the examination of moral liberty in the soul, and familiarizing us with the phenomena of a mechanical necessity in nature," mathematics induce scepticism relative to the spiritual or moral world. Now, there is nothing in the habits of thought engendered by the study of either pure or applied mathematics that would predispose the mind to such scepticism. On the contrary, I hold that, the very reverse is the case because the study of mathematics, pure and applied, tends to the *exaltation of mind* and the *subordination of matter*—to magnify the spiritual and subordinate the material. The voluntary energies of mind are necessarily brought into prominence by the study of pure mathematics. Independently of the material world, the mind by its own creative energies, constructs a supersensible world—does the process tend to the "negation of a hyperphysical and immaterial principle of thought?" In fact mathematical idealism is opposed to metaphysical materialism. After contemplating the eternal truths of the pure mathematics — exploring a boundless world of the pure ideal called into being by its own free and independent energies—the mind becomes conscious of its creative energy—of its personalty—and feels its worth as an intelligence elevated even to infinitude.

With reference to the phenomena of the material world, it might seem that their exclusive study by giving undue prominence to the physical, may tend to degrade the immaterial. But these phenomena can be profoundly investgated only by the application of pure mathematics, and therefore it may be replied that mathematical idealism will correct the tendency, if such exist, towards materialism induced by the earnest investigation of external phenomena: While, therefore, the exclusive study of these phenomena may induce, in the mind of the non-mathematical observer, the belief that everything is the result of a "mechanical necessity in nature,"—the decree of an inexorable fate—a different effect is produced in the mind of the mathematical observer.

But I hold that in the *applications* of mathematics to investigate nature, the dignity and superiority of MIND are still proclaimed. For the triumphs which the human race has won are still the triumphs of *mind* It is *mind* that passes in review nature's grand domain and with authority demands the surrender of her secrets and her treasures. It is *mind* that has penetrated the star-depths, where suns and systems pursue in harmony their everlasting march. It is *mind* under whose forming power the world wears the aspect of a new creation, in which the impress of the finite *human mind* blends with that of the Infinite and Divine.

Since, then, the study of mathematics tends to give preeminence to the mind—to demonstrate the transcendent power of the thinking principle whatever that may be—it cannot fairly be maintained that they predispose to scepticism as to the spiritual or moral world, and to a denial of the existence of the ETERNAL MIND, whose thoughts are embodied in the universe, and of which the investigating mind is but the finite reflex—the dim and shadowy reflection. Nor do I believe that applied science, by revealing the mysteries of nature to the common gaze, will so "reave the heaven of its divinities and disenchant the universe," that admiration and reverence shall find place no more in the human heart. On the contrary, seen under the light of science, the manifold wonders of the world around us, and the majesty of the heaven above us, kindle a sublimer admiration, and become objects of a still profounder "adoration to an infant world." Guided by the light of science, the mind surveys the phenomena of nature, and beholds in the impress of intelligence everywhere apparent—not the mere reflex of organization—but the wisdom of the Great First Cause. If the wondrous mechanisms displayed in material organisms suggest the operation of a personal intelligence to the ordinary observer, probability gives place to demonstration under the clear, steady light of mathematical science.

If, for example, the human eye alone, as has been said, be a cure for atheism, how sublime its teachings when all its perfection, as exhibited in the adaptability of means to end, is revealed by the magic hand of science! It is capable of demonstration, that in all the varied organizations of the natural world, wherever velocity is to be secured, or power generated, or adaptation to natural laws effected, the most perfect means are adopted—means whose conception indicates omniscience, their execution, omnipotence Thus the application of mathematics to the familiar phenomena daily within the scope of our observation, brings into a pre-eminence attainable by no other means, the wondrous *design* universally exhibited and leads the mind from the design to the all-wise Designer—from the law to the ETERNAL LEGISLATOR. Passing beyond the phenomena of our earth

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which after all, is but an atom of the universe, directed by the pure light of mathematical science, we behold the gorgeous majesty of the Heavens, and the conviction of *God's* existence, strikes us with the certainty of demonstration. By this science alone does the universe give a clear reflection of its Creator's power. To the unmathematical observer, the mechanism of the heavens is shrouded in obscurity, or stamped with imperfections.

A superficial view of its architecture, may produce a doubt of its perfection as the work of an infinite intelligence, since it appeais to be marked with variations, which threaten its stability. But seen under the light of mathematical science, every change becomes constancy; every diversity, uniformity. The supposed irregularities which characterize the motions of the heavenly bodies, are shown to be the necessary consequences of the great commanding law, established for the government of the universe, and, instead of indicating instability in the system of worlds, are absolutely necessary to preserve it from dissolution.—While, therefore, pure mathematics elevate the mind above the sensible world, into a world of its own creation, in which the spiritual is pre-eminently exalted in conceiving pure forms and contemplating the beauty of everlasting truths, applied mathematics disclose the glories of the universe as the embodied thoughts of the infinite mind in whom all things "live and move and have their being,"—they unveil the *terrestrial*, and it points to a *God*—they pour a flood of light upon the celestial, and "the Heavens declare His glory and the firmament showeth His handiwork."

II.

Let us now proceed to notice more particularly, THE VALUE OF MATHEMATICS AS AN INVIGORATOR OF THE INTELLECTUAL POWERS.

(1.) *And first as to Memory.* That the faculty of memory is highly cultivated by the study of mathematics, appears evident from the method of their processes and the unity and extent of the knowledge they embrace. In consequence of the necessary connection of principles which gives the science its peculiar unity, the memory must be constantly ready with previously acquired knowledge, to assist the reason in its efforts for the discovery of new truths. For if any of the previous truths be forgotten, the numerous results founded upon them, cannot be attained, and the mind is forced to return to the forgotten truths, till they are made thoroughly its own. Hence appears the error of the assertion, that the mind is required to retain only its last results as a foundation for further investigation, and that thus the memory becomes "stupified" from want of exercise!—To believe the assertion, originated in a speculative unbelief of the utility of the science, that mathematics tend to stupify the memory, requires a credulity born of dense ignorance of the subject. The synthetic processes of pure Geometry demand the constant excercises of this faculty; for how can there be *progress* unless memory supplies the necessary principles from those already attained and classified by the understanding? From the comparatively few first principles of the science, we proceed to the more elementary propositions, and thence to higher truths, till a perfect web of pure science is produced by the necessary laws of thought. And, since every succeeding proposition depends upon many preceding ones which are absolutely necessary to its demonstration, how can it be said that the memory is stultified by progress in the science? But if the memory is exercised in acquiring a knowledge of already demonstrated truths, how much more, in searching out new methods of demonstration, and establishing new truths,—exercises which form a leading part in every mathematical training deserving the name. Is it said that we are required to remember only the results deduced and not the trains of reasoning involved? Then we cannot claim to have received a truly mathematical training, and the science cannot justly be held responsible for results which it never had an opportunity of accomplishing. Still, even under this assumption, the memory is exercised, though certainly not to so high a degree as a rational training in the subject would ensure. For so numerous are the results deduced—the rules and formulas and principles—and so frequently and necessarily are they employed, that the constant exercise of the faculty of memory is imperatively demanded. But is the charge under consideration of any greater weight, when urged against arithmetic and the analytic methods? I reply in the negative. From the primary operations of arithmetic—from the multiplication table—to the highest applications of the Calculus, a necessary condition of progress is the distinct recollection of the results already acquired. Is it said, that in arithmetical investigations "the second mark being discovered, we no longer think of the first," and therefore, but little exertion of memory required? I ask how do we arrive at the discovery of the second mark, but through the active exercise of memory? Every process of the kind, is but the evolution of new truths from principles, which having been once discovered, are rendered effective for higher investigations, by a trustworthy memory. As to the algebraic analysis, even admitting, that as soon as the second mark is discoveed we ignore the first, does it follow that the facu ty in question is not exercised, or if at all, only in the lowest possible degree? Is the mind driven onward, by an irresistible agency, through a series of processes and results, in which it is a "mere spectator?" On the contrary, all algebraic investigations, like those of arithmetic, but still more imperatively, are founded on results previously deduced, and must call into continuous exercise the retentive powers. But it is of no more true of mathematical reasoning than of any other, that the second mark being discovered we no longer think of the first. In any process of reasoning, whatever be the nature of the truths involved, is it necessary, or even possible, to recollect at *every instant*, all the results previously obtained? Or do we not rather, withdrawing our attention from steps already taken, concentrate our energies on those about to be taken and recall previous reasonings when necessary, by a special exertion of mental power.

In the transformation of equations—a process which at first sight may seem to demand but a slight effort of this faculty—it must furnish the materials for ingenuity to work upon, and bring up from its depths the principles necessary to effect the synthesis of the known with the unknown. And, in the investigation of general principles it is exercised in a still higher degree. For, though its individual principles are analytic, the science as a whole is synthetic—it proceeds from the simple to the complex—and therefore every investigation depends on a multitude of preceding truths which memory must be constantly ready to supply, or the desired results, with the consequences that flow from them, are completely unattainable Thus it may be shown that each of the mathematical sciences demands as an indispensable condition of its attainment, the vigorous exercise of the faculty in question.

But as each of the mathematical sciences possesses a logical unity which absolutely compels the exercise of memory, so the same unity binds them all into an harmonious whole, and hence the farther we progress in the boundless realms of knowledge they spread before us, the more completely is this faculty cultivated and its tenacity and power increased. Each branch of the science has certain principles peculiar to itself, but in addition to these, it demands the truths already demonstrated by subordinate departments, and therefore an abiding knowledge of these truths is absolutely necessary. From these and like considerations which, if time permitted might be abundantly illustrated, it seems clear that mathematics *demand* the constant exercise of memory, and stand pre-eminent as an invigorator of its powers. Yet it has been asserted that instead of exercising the memory they actually dwarf its powers. I have no doubt that every man who has anything more than a superficial knowledge of even the elementary branches will acknowledge the groundlessness of the assertion, and consider it but another instance of the fallibility of metaphysical speculators: especially when, like the sophists of old, they lay claim to universal wisdom, and dogmatize on subjects of which they are either totally ignorant, or view only through the distorted vision of the bigot.

2. *But Mathematics also cultivate in a high degree the powers of abstraction and generalization.* Although quantity, in its general sense, is the object of mathematical investigation, the conceptions involved are not connected with material substance nor limited by its finite nature. They are the product of the reason itself, and possess an immutability and a universality that cannot originate from material forms, though they may comprehend them. Sensible objects may give us our first ideas of numbers, but the mind soon passes to the abstract conceptions, and the particular is comprehended in the universal. So intuitions may be supplied, in the first instance, by imperfect geometrical figures, to give the mind conceptions of fundamental definitions, but the perception of the particular figures fades

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away, to be replaced by universal conceptions. In the first stages of the algebraic methods, material objects may be used to aid the mind in gaining clear notions of the things considered; but no sooner does the mind obtain these notions than it loses sight of the particular and grasps the general. The language employed in analytical investigations is eminently suited to the pure abstractions involved—presenting ideas entirely unconnected with material objects, it is yet capable of representing such objects—universal in power, it is equally applicable to the particular. Every principle in the most elementary of the mathematical sciences is founded on abstraction; every successive stage is reached by a still higher effort of abstraction, while the fundamental principles and ultimate results of the calculus and its applications can be attained only by its highest possible development. This power of mathematics to cultivate the faculty of abstraction establishes one of its most important claims to a high position as a means of intellectual discipline For the faculty of abstraction is undoubtedly connected with the loftiest efforts of the human mind, whether directed to the attainment of moral or intellectual truths. It is the foundation of intellectual and moral philosophy, since the phenomena of the mind, varied, complex and transient as they are, can be carefully observed and truly investigated only by a high degree of abstractive power.

But, in the power of generalization, as well as abstraction, cultivated by the study of mathematics, or is there no generalizations in the sciences, as some assert, because their universal truths are not derived *a posteriori* from experience?

In the opinion of some philosophers, abstraction necessarily implies generalization. Without adopting the view that there can be no abstraction without generalization, since it seems evident that the mind can contemplate certain abstracted qualities of any object, without necessarily establishing a class whose essential marks are given in these qualities, it must be admitted that abstraction is the foundation and necessary condition of all generalization, —Abstraction gives the elements of the concept; generalization moulds them into convenient forms as materials of thought. Hence as mathematics pre-eminently cultivate the power of abstraction, they must qualify the mind for generalization. Admitting that, in obtaining our first conceptions of geometrical truths, "the general is viewed in the particular," the power of abstraction is necessary to give the mind the pure notion which enables it to dispense with sensible objects, and lay the foundation of a pure science. If it be said that the object is still presented to the mind, as a concrete form, by the imagination, I reply that abstraction is necessary to enable the mind to grasp the general as an *a priori* intuition, before the imagination can present the concrete as the representative of the *universal*. And further, in recalling *any* conception to the mind, do we necessarily view all the marks given by abstraction and generalization in the formation of the conception? Do we cognize the general as it is, or grasp it in the particular? It is believed that, though the mind can, by a special exertion of its powers, view the general in its comprehended marks, a particular object is usually recalled as a representative of the class, though with the consciousness that the *individual* possesses many attributes not given in the conception of the *class*. In all the higher geometrical investigations, we are constantly within the confines of the universal—is the universal reached without the generalizing power? In the fundamental propositions and principles there is a classification, and from these the science is unceasingly discovering properties peculiar to distinct classes of conceptions —does not this process of development involve the principle of classification and the power of generalization?

Generalization is also a characteristic feature in Analytical Geometry. "Every process,"—to use the language of J. S. Mill—in Universal Geometry "is a practical exercise in the management of wide generalizations, and abstraction of the points of agreement from those of difference among objects of great and confusing diversity, to which the most purely inductive science cannot furnish many superior." If we pursue the synthetic method of investigation, we shall find that every result, though so far general that it includes a multitude of particulars, is relatively particular, and can be shown to be comprehended in results still more general; and hence every step of our progress demands the exercise of the power of generalizing. Investigating, for instance, the equation of any of the conic sections, we obtain a general expression comprehending a great number of truths—proceeding with the investigation of a second, another result is found equally comprehensive and equally general, and thus, for each figure of the entire class; but the results, though exhibiting each the special property of the conic to which it refers, have nevertheless common characteristics which facilitate their combination into a general expression embracing all the results separately deduced from the independent equations; and if we follow the analytic method, a high degree of abstraction is necessary to enable us to clearly comprehend reasonings founded on conceptions so comprehensive.

But is it true that the analytic method employed in Algebra and the higher mathematics, do not cultivate the power in question, because they substitute a sign for a notion and thus relieve the mind from all intellectual effort? I think not. For though it may not be always so necessary in analytical as in geometrical investigation, to keep in view for the purposes of comparison, the results deduced, a high degree of mental effort, aided by accurate discrimination, is required to enable us to select from the many preceding generalizations, and skilfully apply, the principles necessary to effect the desired synthesis. It is true that the analytical methods, from their precise notation, and higher power of generalization, simplify many geometrical investigations—or rather attain, in a comparitively simple manner, results which geometry can give only by long and cumbrous processes—but the utility they thus lose as an invigorator of mind, is more than restored by their wondrous powers of bringing within its grasp, truths which otherwise would be completely unattainable. But in *any* process of abstract reasoning, do we constantly cogitate the general conception in its essential marks? Or do we not rather use "a sign for a notion," by elevating words to the rank of thoughts? Unless we did so, how complicated would be our mental processes, how unsatisfactory their results, since the difficulty of reasoning increases with the abstruseness of the abstractions involved. So it is with the language of the higher analysis. The reasonings are upon abstract conceptions so comprehensive, that the relation between their successive steps cannot be understood without a vigorous intellectual effort. And though arbitrary symbols are used in analysis, the student must have so clear a conception of the things signified, and their complicated relations, that he is constantly prepared to translate into ordinary language, or interpret by geometry, the results deduced, or he certainly cannot be said to *know* the subject of investigation. Does the difficulty of any process of reasoning increase with the degree of abstraction and generalization of the terms employed? Then analytical investigations must demand a very high degree of mental activity, since they employ the most comprehensive generalization, and are capable of representing in a single view processes and results which would require pages of ordinary language for their elucidation.

As before shown, every first principle of arithmetic and ordinary algebra must proceed from abstraction, and every succeeding principle is a generalized result—from the contemplation of particular examples we attain the general, the universal truth. Every student of the science, has at the outset of his course, experienced the difficulty of rising from the particular illustrations to the universal principles, in consequence of the generalizations involved requring a higher effort of abstraction than his comparatively undeveloped powers can easily attain; but from the cultivation this faculty receives by thorough progress in the science, he ultimately comprehends truths involving a higher degree of abstraction, with greater ease than he had mastered its elementary principles. But the generalizations of the higher analysis and geometric methods, demand a pre-eminent degree of mental energy.—The fundamental principles of those sciences are the result of generalization, or reached only by a high degree of abstraction, and as every demonstration is a generalizing of abstract conceptions, or the analysis of the universal into its comprehended elements, thorough progress cannot be made without a constant exercise of the higher faculties of the mind. By methods of investigation essentially geometric, though aided by analysis, Newton effected the solution of the Lunar inequalities—a problem which had mystified the philosophers of all preceding ages—is there no generalization in the results which comprehend these complicated movements? By a more extensive application of analysis, the dynamics of the planetary worlds may be represented in a single view,—does the evolution of results so comprehensive involve no generalization? But it seems to me quite unnecessary to enter into a lengthened defence of the utility of mathematical science as an invigorator of the powers

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in question, when the origin of its principles, the method of procedure, and its highest developments, are comprehended in two words *abstraction* and *generalization*. It may be said that mathematical abstraction and generalization, are different from those of common observation and experience, and therefore they do not cultivate the higher intellectual faculties. We admit that mathematical processes evolve truths characterized by an immutability that can never distinguish the generalizations of the world of probabilites. But the *mental powers that abstract and generalize are the same*, whether directed to the investigation of *necessary* or *contingent* matter, and they surely cannot be less efficiently cultivated in the one case in consequence of their attaining clear and certain results, than in the other, because they are limited by probability.

3. But the study of Mathematics is of great value as correcting the vice of mental distraction, and forming the opposite habit of continuity of thought.

The importance of early forming the habit of deliberate concentration, can hardly be too highly estimated. In the commencement of all intellectual effort there is difficulty, and the success attending such effort must be in proportion to the power of continuous thought we are capable of exercising.

For, when we first direct our attention to subjects of investigation, our minds are distracted by numerous extraneous thoughts arising from the manifold circumstances that surround us, and we are thus prevented from bringing our intellectual energies fully to bear upon the subject. Nor is the difficulty overcome by a simple exertion of will. Even when we feel the necessity of concentrating power for the accomplishment of our object, and determine to exclude all distracting elements, a thousand obtrusive notions will spring up to confuse our thoughts and dissipate our energies; and for a long time even after the subject of investigation begins to allure us by presenting interesting thoughts for our contemplotion, the glimmerings and associations of preceding reflections will obtrude themselves and prevent the full and free exercise of our intellectual powers. When opposed by these obtrusive elements, is it possible for the mind to energize freely and successfully prosecute difficult investigations? Can it reach its highest attainable success when its operations are impeded by distracting thoughts, and its energies dissipated by efforts to concentrate its powers upon the object in view? On the contrary, the *habit* of concentration must be acquired before the mind can energize with the probability of attaining the highest success within its reach. But this can be accomplished only by time and frequent practice in the effort The first effort at concentration is attended with great difficulty; but power is gained by repeated effort; every succeeding effort will become less difficult than the preceding; the elements of distraction gradually diminish in frequency and power, and at last the mind can almost involuntarily concentrate its energies upon its object, and attain its highest pleasure and most certain results, in the unimpeded operation of its powers. So important is this power of continuity of thought, that some have considered it identical with genius, while all admit that it is an invariable concomitant of genius, and a necessary condition of its greatest achievements. Sir Isaac Newton, with his usual modesty, attributed his success, not to the force of superior genius, but simply to his power of patient attention. "Genius," says a celebrated philosopher, "is only a protracted patience." Granting that there may be great powers of mind without a corresponding power of concentration, they cannot accomplish great results; but on the other hand, mediocrity accompanied by a high degree of this power may be elevated almost to the rank of genius.

Now, in Mathematics, we are accustomed to emancipate ourselves from the disturbing influences that surround us, We *must* concentrate attention on the truths bearing on the investigations, or there is no possibility of reaching a successful issue. The mind is required to keep constantly before it a vast number of already established premises, and to exercise its powers in the selection and application of those specially adapted to the end in view. And as the connection between the successive steps is, in general, difficult to grasp, in consequence of the abstract nature of the conceptions involved, the continuity of thought necessarily required must be both intensive and extensive,—intensive, as requiring a high degree of mental power to grasp the conceptions and their relations; extensive, as involving the contemplation and analysis of a large number of distinct, though related truths. But the fact that a high degree of abstractive power is required—as already shown—in both elementary and advanced Mathematics, proves conclusively their utility in cultivating the power of continuous thought. I may remark, in preparing the mind for the careful contemplation of its own transient phenomena. The habit of concentration, to which I have referred, can be formed only by acts of "patient attention" which gradually increase in difficulty. This is accomplished by the study of Mathematics. The elementary propositions *demand* a certain amount of concentrative power, and every successive step demands a somewhat higher degree in consequence of the greater difficulty and greater number of the terms involved, and from this constant repetition and exercise, commencing with the simpler acts and rising gradually to the more profound, the power of continuous thought is increased to a higher degree than can be reached by any other course of discipline. If it be said that Mathematics, from the very fact that they *demand* a degree of continuity of thought in their most elementary propositions, are thus unfitted to remedy the vice of distraction, we reply that if there be any possible corrective discipline, it must be such as begins with the simpler efforts of the power of continuity, and gradually proceeds to the more difficult and prolonged; and the mind that *cannot* exert concentration sufficient to master the elementary propositions of Mathematics, though it may sometimes give evidence of latent power, will never be likely to attain by any other means this invaluable habit of mind.

4. The study of Mathematics developes the *power of observation*, and cultivates the *imagination*, whether considered as a representative or a creative power. By observation we mean the power of fixing attention on material or mental objects so as to note their distinctive properties, and their points of resemblance and of difference. From the most elementary Mathematical notions to the highest range of Mathematical investigation, this power is exercised. Even the first notions of number and of form which underlie Arithmetic and pure Geometry, are acquired by observation. Nor is it alone in securing the fundamental notions of number and form that observation plays so important a part. "The very genius of common Geometry is that it is but a series of observations. The figure being before the eye in actual representation or before the mind in conception is so closely scrutinized that all its distinctive features are perceived, auxiliary lines are drawn (the imagination leading in this) and a new series of inspections is made; and thus aided by direct simple observations, the investigation proceeds. So necessary is observation in Geometry that Comte, the ablest writer on the philosophy of Mathematics, is disposed to class Geometry—in view of its methods—with the natural sciences, as being based on observation. When we consider applied Mathematics, we have only to notice that the exercise of this faculty is so essential that the basis of investigation, the very materials with which we build, have received the name *observations*.

Further, the *representative* power of *imagination*, as constantly exercised in presenting to the mind intuitions of space, and the complicated relations of external things, must be greatly strengthened and developed; and the beauty, order, and harmony disclosed in terrestrial phenomena, and in the starry regions where worlds on worlds arise, must permit abundant materials for the exercise of its creative powers. And this creative faculty has constant exercise in all original Mathematical investigation from the solution of the simplest problem to the discovery of the most recondite truths; for it is not by intuitive, consecutive steps that we advance from the known to the unknown, the imagination, rather that the logical faculty, leads in this advance. In fact, practical observation is often in advance of logical exposition. In the discovery of truth, the imagination habitually presents hypothesis, and observation supplies facts which it may take ages to connect logically with the known. That the imagination, and not the logical faculty, *leads* in all original investigations will be admitted by any student who has ever succeeded in producing an original demonstration of even one of the simple propositions in Geometry.

III.

Let us now proceed to notice the VALUE OF MATHEMATICS AS A LOGICAL EXERCISE OF MIND.

1. *It habituates the mind to the use of correct forms of reasoning.*

However valuable pure logic may be as the science of the laws of thought, its highest utility is realized only in the practical application of its principles. The bare study of the formal laws of thought cannot *exercise* the mind in the forms which it must follow in all sound reasoning. Logic may unfold the characteristics of these laws, and thence deduce the necessary conditions of cogency in reasoning. But the mind can acquire the

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habit of observing these conditions only by practice in their application, and as affording such practice, mathematics stand pre-eminent. They constitute the most perfect application of pure logic, and especially of the deductive method of ascertaining truth. This method of deduction—of which Mathematics give the most scientific form—is a most powerful instrument in the discovery of truth. The laws of extension and number—that is, the laws of Mathematics—underlie all the other laws of the material universe; and in the great inquiries of the moral and social sciences, Mathematics afford the only sufficiently perfect type. "Up to this time," says Mill, "I venture to say that no one ever knew what deduction is as a means of investigating the laws of nature, who had not learned it from Mathematics, nor can any one hope to understand it thoroughly who has not, at some time of his life, known enough of Mathematics to be familiar with the instrument at work." The logical definition of terms—the explicit statement of premises—the clear and well-defined steps in the trains of reasoning—the exclusion of intermediate propositions the truth of which is not clearly seen—the precise and constant meanings of the terms employed—these are characteristics of the deductive method, which finds its highest type in Mathematics.

But it has been said that the Logic of Mathematics, dealing only with necessary matter, and concerned only with demonstrative evidence, does not prepare the mind for researches in contingent matter, that is, to correctly estimate probable evidence. It may be remarked in passing, that this objection is equally valid against Metaphysics and indeed all the rational sciences, which demand not probable but certain evidence.

But the objection has no weight when urged against Mathematical logic. For, to say nothing of the fact that the Mathematical theory of probabilities is a most valuable contribution to logic, and lays the foundation for a sound knowledge of the rules of probable evidence, what is to be our guide in estimating probable evidence, but those logical forms of reasoning deduced from the laws of mind, and practically exhibited in their highest perfection in Mathematics? Is pure logic of great value as pointing out the conditions of cogency to which the *probable* argument must conform in order to secure the acceptance of its conclusion as an article of belief? How much greater the value of Mathematics, which demand a continued application of these conditions, and hence educate the mind to a sagacity in detecting error, that the mere study of formal logic cannot impart? When we consider the multiplicity of circumstances likely to invalidate our investigations in the "field of probabilities," we can scarcely think too highly of those methods of discipline which develope so acute a perception of the form and essence of sound reasoning, that the mind is enabled instinctively, as it were, to detect the presence of fallacies. Mathematics proceed from data which have the certainty of necessary truths by a demonstrative process in which the connection between the successive steps of the reasoning is clearly comprehended. No obscure terms, nor imperfectly understood propositions, are either admitted as data or mark their processes. Should not a similar rigor be observed in reasoning on contingent matter? They assume no principle as the basis of an argument, or as a means of effecting a synthesis, whose truth has not already been established, and they submit to the severest test everything having the slightest element of uncertainty. Should not the same method distinguish inquiries in which there is a balancing of probabilities? The highest ingenuity and skill in analysis and combination, are required in Mathematical research—surely the same qualities are essential to correct reasoning in matters of observation and experience.

It has been also said that the *matter* and the *method* of Mathematics preclude the possibility of error, and that therefore the science does not, like probable reasoning, educate to sagacity in its detection. But, as already suggested, it is impossible to discover the fallacies of probable reasoning without practical skill in the methods of sound reasoning; and this, the study of Mathematics imparts by rigorous adherence to the forms of strict logical inference. "Let us be assured," says the great thinker already quoted, "that for the formation of a well-trained intellect, it is no slight recommendation of a study, that it is the means by which the mind is earliest and most easily brought to maintain within itself a standard of complete proof." It is true that Mathematics have continued to make unerring progress, while contradiction and abberration have distinguished most other sciences and retarded their development. This progress, however, does not prove the impossibility of mental sophistries in Mathematical investigation, but rather that the *matter* and the *method* of the science lead quickly to the detection of fallacies, and prevent the introduction of permanent error. Such fallacies are probably due to the abstract and comprehensive nature of the conceptions involved in the demonstrations; and that their discovery and elimination often require great skill, as well as acuteness and soundness of judgment is well known to every student of the science. If a vigorous exercise of intellectual power is necessary to grasp such conceptions, a still higher degree of mental energy is required to comprehend their *relations*; and thus the mind is led sometimes to confound abstractions which are really distinct; at others, to assume an analogy where none exists. Hence, if fallacies creep, into Mathematical demonstrations *in spite of the logical rigor of their method*, they must be such as are most likely to deceive the mind, and their frequent occurrence—with their discovery and correction—must habituate the student to a discriminating caution which is of great value in the probable reasonings of experience. A mind thoroughly trained in Mathematical reasoning may indeed commit the error of expecting in all proof too close an adherence to the type with which it is familiar; "but he who has never acquired this type has no just sense of the difference between what is proved and what is not proved; the first foundation of the scientific habit of mind has not been laid."

2. But further. *The study of Mathematics requires the exercise of ingenuity, acuteness in discrimination, and caution in the admission and combination of data*, and consequently affords a still more effective preparation for conquering the difficulties and avoiding the dangers in the reasonings of experience. Though Mathematical science is demonstrative—occupied with the deduction of conclusions—we are as often required to establish certain truths to serve as premises for the deduction of a proposed truth, as to deduce the necessary consequences from given premises. These premises are to be selected from the numerous truths already acquired by the understanding, and combined to effect the required proof: and this can be accomplished only by a careful analysis of the proposition to be proved, and an accurate discrimination of the results previously known. A careful examination of the given proposition is needed to guide the mind to the necessary data; accurate discrimination and ingenuity, to select, from the many principles bearing on the question, those necessary and sufficient for the demonstration.

In the solution of Mathematical problems, how is the synthesis between the known and the unknown to be effected, without skilful analysis, acute comparison, and judicious application? The relative bearings of principles previously determined, and their connection with those to be established, must be carefully examined and clearly comprehended as a preliminary to the required solution—does this require no acuteness in comparison and discrimination? The intermediate terms employed in the investigation must be sought among general truths which from their complex relations, are the more difficult to distinguish—is there no ingenuity required in the selection and application of those which will lead most directly to the desired result? The very fact of mathematics being a demonstrative yet a *progressive* science, proves at once the necessary *connection* yet *distinctiveness* of its propositions, and implies as a condition of progress the constant exercise of the powers in question. Hence as sagacity and skill are required in common reasoning to obtain the needed premises, and ingenuity in analysis and comparison to free them from everything irrelevant to the argument, it seems evident that mathematics must prepare us for overcoming the difficulties by which such reasoning is characterized, and for moulding the isolated facts furnished by observation and experience into the symmetry and stability of science. This seems to receive corroboration from the great success which mathematicians have achieved in the application of the science to external phenomena. For most of the physical sciences are founded on observation and experiment necessarily carried on by mathematicians, and eminently exhibiting subtlety in discrimination and analysis, and skill in comparison and generalization. It is a fair inference, too that mathematics qualify the mind for observation and experiment, since these sciences owe their origin to mathematical skill in observing and generalizing physical facts, as well as their development to the power of mathematical analysis.

But Mathematics induce a cautiousness in the admission and combination of data which still further fortifies the mind against the fallacies that occur in reasoning on practical affairs. However opposed to the progress of truth violation of the forms of true reasoning may be, a source of error equally fertile is to be found in the rash assumption of false premises, and

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the introduction of intermediate propositions essentially inadmissible. To be beguiled by the fallacy of *rigorous* reasoning from *erroneous* data, and illogical reasoning from true data, seems to be one of the most common of our intellectual failings. On the one hand, the force of the premises blinds us to the fallacy of the reasoning; on the other, the soundness of the reasoning leads us to lose sight of the falsity of the premises. Now, the primary principles of Mathematics are clear and certain, and stamp the conviction of their realty as soon as comprehended by the mind; and no proposition is admitted unless its certainty is clearly recognized as a first principle, or as a clearly established truth. The mind is trained to the examination of premises. Even in the elementary branches, a vigorous exercise of mental power is needed to fully apprehend the data; while the first principles of the calculus and higher branches of pure Mathematics, are products of a very high abstraction, and, the ultimate propositions and the trains of reasoning involved, demand the exercise of great intellectual power. Are the fundamental propositions of Applied Mathematics, which are rendered more complex by the union of conceptions from physical laws with the difficult abstractions of pure Mathematics—*passively* received? Dynamics, Optics, Acoustics, Astronomy, Electricity and other sciences in which analysis reaches its highest applications--do their principles condemn to mere "mental inertion"—their highest development to an absolute "minimum" of thought? Hence it appears that mathematics exact the critical examination of data as a necessary condition of conquering their difficulties; we *must* concentrate our attention on first principles till these are fully comprehended and become genuine elements of knowledge; thus trained we acquire—not a "blind credulity" but a *habit of caution* in the admission of premises.

But if their utility is great in guarding us against errors in data, it is still greater in fortifying us against fallacies in *reasoning*. On this point little need be added to what has already been advanced. There appears to be in the human mind a natural tendency to perceive resemblances where none exist, and to be led astray by false analogies. Hence the necessity of caution in admitting the connection between the successive steps in any argument. Now, granting for a moment that mathematics preclude the possibility of sophistry in thought—tolerate no false analogy from deceptive resemblances—the successive steps in their processes must be immediately comprehended as necessary. Hence the mind becomes habituated to the evident connection between them and hesitates to admit their validity when it does not clearly perceive their relation. Is there not thus formed a habit of caution which is of the highest importance in the reasonings of experience? If we refuse to sanction any step in the reasoning till we clearly comprehend its logical connection with the preceding one, do we not adopt the surest possible safeguard against a fruitful source of error?

But, as before stated, I do not believe that mental sophistries are excluded from mathematical reasoning. Owing to the abstruseness of the conceptions employed, there is danger of including something irrelevant, excluding something comprehended, and supposing an analogy where none exists. And since these fallacies occur in spite of a *rigorous method*, they must be such as arise from the admission of false premises or propositions; and the frequency of their occurrence, and of their discovery and elimination, must develope a *habit* of caution in the examination of connecting propositions till their relevancy is plainly seen.

I think, then, I am justified in maintaining the value of mathematics as imparting habits of caution in the admission of premises and intermediate principles. And yet it has been asserted that their tendency is to develope a blind credulity and an uncompromising scepticism! If any Mathematitian has exhibited a blind credulity in the admission of erroneous data and the deduction of extravagant conclusions, it must have been *in spite* of his mathematical training, and not in consequence of it. The *post hoc, ergo propter hoc* style of argument has been a common weapon with the speculative opponents of mathematical discipline. Mathematicians have sometimes proved unfortunate in the management of their business affairs, and forth with mathematical discipline is charged with the failure, and pronounced to disquality for the affairs of life and for common reasoning. But, I suspect we know of many failures which cannot possibly be traced to the influence of mathematics. Despite the caution and sagacity constantly required in their own science, they have sometimes been too prone to manifest a 'facile credence' in the reception of principles and theories which rested mainly on the authority of their originators and supporters; but it would not be difficult to find illustrations of "facile credence" that can hardly be traced to the influence of mathematics. Mathematical metaphysicians have occasionally been guilty of absurd theories in metaphysics; but why should mathematics rather than metaphysics, be held responsible for the absurdities? Would it not be well to consider the legions of *non*—mathematical metaphysicians who have been guilty of equal or still greater absurdities? The history of metaphysics thus far is a history of mental aberration; are mathematics responsible for the ceaseless recurrence of erroneous systems? The great modern champion of the paramount importance of metaphysical research admits that the "past history of philosophy has, in a great measure, been only a history of variation and error"—have mathematics been the cause of this endless uncertainty?

As to scepticism, I suppose that there is some ground for the long-standing complaint against mathematicians, that they are hard to convince "But it is a far greater disqualification both for philosophy and for the affairs of life to be too easily convinced; to have too low a standard of proof. The only sound intellects are those which in the first instance set their standard of proof high. Practice in concrete affairs soon teaches them to make the necessary abatement; but they retain the consciousness without which there is no sound, practical reasoning, that in accepting inferior evidence because there is none better to be had, they do not, by that acceptance raise it to completeness."

3. But not only do mathematics educate to the use of correct forms of reasoning, and sagacity in the discovery and correction of fallacies, they educe a general vigor and comprehension of thought which still further prepare the mind for every kind of logical investigation. In support of this propositi on but little more need be advanced as I have already shown heir beneficial influence in expanding and strengthening the several mental powers. The first principles of mathematics—especially of the higher branches—though universal and necessary truths are not *passively* received but exact a conscious activity of mind for their clear apprehension; while the constant exercise in discerning the relations of truths so abstract and comprehensive, tends to the highest development of the intellectual powers. And the application of mathematics to physical laws, necessitates a grasp of mind still more comprehensive; for with the difficult abstractions of the pure mathematics are combined new conceptions from physical laws which increase the complexity of the data, the abstruseness of the connecting propositions and the consequent laboriousness of the trains of reasoning. Yet it has been said that mathematics call forth but a minimum of thought because the principles are self-evident, and every step in their reasonings are equally self evident, though the discovery of new truths may indicate a philosophic genius! Such an assertion could never have been uttered by any one possessing a knowledge of the subject beyond its most elementary principles. If by self-evident principles be meant such as are *passively* received by the mind, then mathematical principles, even in the mere elements of the science, are not *self-evident*; and still less the propositions employed in the demonstrations. The fundamental principles of abstract mathematics strike the mind with the conviction of their certainty *as soon as they are understood*; and the successive steps of a mathematical demonstration are equally self-evident *as soon as their relation is clearly comprehended*. But, as already shown, a vigorous exercise of intellect is required, especially in the higher mathematics, to understand the necessary data, and to comprehend the logical relation of the several propositions, before their *self-evident* nature is viewed in their *necessity* and *universality*.

Is there no energy of thought required to comprehend the successive steps of the demonstrations in the sublime geometry of Newton? The eleventh section of his Principia has been pronounced by a great philosopher to be characterized by "a spirit of far-reaching thought which distinguishes it beyond any other production of the human intellect"—does it require only a minimun af thought to understand his reasonings and to grasp, in all its comprehensiveness the fruitfulness of the results? By the application of analysis the complicated dynamics of the solar system are brought within reach of the human intellect—do the investigations determine thought to its "feeblest development?"

Nor is it true that though original discoveries and inventions require a

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philosophic genius, they may, like a fact in chemistry, when once discovered, be reproduced and applied by the dullest intellect. They could be thus easily attained if mathematical reasoning were a series of mere mechanical steps "passively" taken by the mind, But from what has been already advanced, we are justified in declaring this to be a groundless assumption—the utterance of an uncandid critic, or of a novice in the science. The discovery of new truths, or an original application of the old may have been a work of comparative ease to the man of superior genius. But the clear comprehension of the modes of investigation, and the complete appropriation of the discoveries, compel from the ordinary intellect the highest exertion of its powers. This will be corroborated by every student who has made himself master of any important branch of the science. The assertion that the works of the immortal Newton can be mastered by the exertion of a minimum of mental power, is too astounding a paradox to merit serious consideration. In the unquestioned judgment of mankind they stamp him as the philosopher *qui genus humanum ingenio superavit.* They stand conspicuous as the grandest monument of intellectual power that the world has ever seen, and shed a lustre on his age, before which the glories of all preceding times grow dim. They have established a greatness that does not vanish in the mists of years, but is carried onward down the stream of time with a splendor ever gathering from the triumphs of a distinction that can never die. They constitute not the transient and visionary philosophy of an epoch, but the creed of all time, and their author has become not the forgotten representative of a metaphysical sect, but the educator of the human race.

I may add further that in every mathematical training worthy of the name, the inventive powers which in their highest degree constitute genius, are called into exercise and fostered in teaching mathematics to the merest tyro. For in every rational training the solution of problems forms an important part—problems which are not mere repetitions of the type—questions given in illustration of principles--but so constructed as to test both the knowledge and the inventive powers of the pupil. This is true even in the simpler branches of the science. In elementary algebra, for example, a great variety of problems can be constructed to illustrate even the simple formulas in multiplication, which require for their solution no small degree of ingenuity. Of course, if problems are merely ceaseless repetitions of a certain type, their solution soon becomes as mechanical a process as repeating the multiplication table. But no mathematical teacher worthy of the name is ever guilty of such palpable cramming; and no mathematical examiner worthy of his trust will, by setting questions of this purely mechanical type, commit the serious error of encouraging a system of mathematical teaching which condemns the pupil to a minimum of thought. The thorough teacher and the competent examiner will so direct and control mathematical training as to expand and invigorate the same faculties of the mind, which are of closest kin to those of the greatest philosopher, and which in their highest degrees have produced the greatest discoveries in mathematical science.

4. Mathematics are of scarcely less importance in educating to an *accurate use of language and consequent skill in detecting the fallacies arising from its ambiguous use.*

Though words should be the passive subjects of the understanding, they sometimes, as it were, revolt from its authority, and create universal anarchy in the empire of thought. It is generally admitted that to inadequacy and ambiguity of words may be attributed a large portion of the errors which ensnare the understanding and impede its progress in the discovery of truth. Among "the four species of idols" which Lord Bacon has distinguished as "besetting the human mind," he ranks the *idola fori*- those which arise from the imperfection of words,—as the "most troublesome of all." He observes, "words still manifestly force the understanding, throw everything into confusion and lead mankind into vain and innumerable controversies and fallacies, hence the great and solemn disputes of learned men often terminate in controversies about words and names, in regard to which it would be better, imitating the caution of mathematicians, to proceed more advisedly in the first instance, and to bring such disputes to a regular issue by definitions." And Locke uses still stronger language in reference to the same subject, attributing to the incompleteness of words, almost all the errors that have obscured genuine knowledge and characterized the disputes of mankind. Though the latter may have stated the case somewhat too strongly, since it seems hardly possible that the solemn responsibilities of life should have been so generally sacrificed in a mere contest about words, it is nevertheless true as stated above that the incompleteness and ambiquity of words have proved a fruitful source of error, and a serious hindrance to the progress of knowledge. Hence the importance of being *accustomed* to the accurate use of words, and skilled in detecting the illusions lurking in their ambiguity. To this, we think, the study of mathematics eminently conduces.—Their language is precise and adequate, in consequence of the clear and distinct conceptions which they involve. No word is defective from inadequately representing the conception for which it stands: nor ambiguous from admitting anything extraneous; all are complete representatives of the things signified, and preclude the possibility of vitiating demonstration either in the admission of any foreign element, or the exclusion of any part of the case under consideration. Does not this constant and necessary accuracy in the use of words, habituate the mind to a corresponding accuracy of language in other departments of knowledge, and educate to skill in the detection of its fallacies? Or, as has been asserted, is mathematical science in consequence of this unerring exactness in its terms, utterly incapable of fortifying the mind against illusions from which it is itself exempt? Must we plunge at once into the reeling tempest of conflicting meanings to become accustomed to accuracy in the use of words? It seems evident that the necessary use of accurate forms of expression must tend to the formation of a *habit* of accuracy. Are we familiarized with the characteristics of the perfect by being first accustomed to imperfection? Are habits of certainty and precision in the use of language, best formed by our being first familiarized with its variable and ambiguous meanings? On the contrary, admitting the possibility of the formation of such habits by this mode of procedure, they could only be acquired from repeated experience in the illusions of language, and would consequently require the unnecessary expenditure of mental energy.

It is surely far better to enter upon any subject of investigation in which errors are likely to arise through the imperfection of language—not depending upon the successive corrections of erroneous results for ultimately creating habits of precision in expression—but already possessing such habits from the constant use of words characterized by distinct, invariable, and adequate meanings. In mathematics, each term is used in the same, invariable sense—does not this secure us against the fallacies of *fluctuating meanings?* No term is employed which is not a full and clear representative of the thing signified—are we not thus guarded against the illusions of obscurity? No expression involving a plurality of meanings is ever admitted—do we not thus become prepared to detect instinctively the sophistries of ambiguity? It hence appears that mathematics, in exacting an absolute strictness in their language, must conduce to an accuracy in the use of words and a skill in detecting their fallacies, which enable us instead of groping in obscurity in the field of probability to advance steadily amidst the obstructions that surround us, with the greatest assurance of a rapid and enlightened progress.

IV.

Their Value as an Instrument of Material Progress.

The time at my disposal will permit little more than a reference to the objective utility of mathematics as shown in their necessary connection with other sciences, and with the progress of mankind. It may be said that it is illogical to attempt to enhance the value of mathematics as a means of education by an appeal to their value as essential to human progress. But, in determining "what knowledge is of most worth" in education, it is not only proper but necessary to take into account its influence on material progress. For what is this progress, but the conquest of human liberty and intelligence over matter and material phenomena? It is certainly a part of the destiny of man to achieve such a conquest. In the earlier stages of civilization, but few of the secrets of nature are given up to man, and material forms contribute but little to his happiness—he does not yet appear as the master of his habitation. But soon the world begins to change its aspect before the operations of intelligence—it surrenders its secrets and its treasues and acknowledges its subjection to its appointed Master. Material progress is therefore but the reflex of intellectual development. Now, it must be admitted that the most effectual triumphs of mind over matter, have been won through mathematics. Take away from what has been secured to civilization through the long

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struggle of ages, all that is due to mathematics, and we shall be centuries nearer the primitive barbarism of the race. Permit me to give in this connection Herbert Spencer's rapid review of the facts as to the worth of mathematics. For all the higher arts of construction, some acquaintance with mathematics is indispensable. The village carpenter, who, lacking rational instruction, lays out his work by Empirical rules learnt in his apprenticeship, equally with the builder of a Britannia Bridge, makes hourly reference to the laws of quantitative relations. The surveyor on whose survey the land is purchased; the architect in designing a mansion to be built on it, the builder in preparing his estimates; his foreman in laying out the foundations; the masons in cutting the stones; and the various artisans who put up the fittings; are all guided by geometrical truths. Railway making is regulated from beginning to end by mathematics; alike in the preparation of plans and sections; in staking out the line, in the measurement of cuttings and embankments; in the designing, estimating, and building bridges, culverts, viaducts, tunnels, stations. And similarly with the harbors, docks, piers, and various engineering and architectural works that fringe the coasts and overspread the face of the country; as well as the mines that run underneath it. Out of geometry too, as applied to astronomy, the art of navigation has grown, and so, by this science, has been made possible that enormous foreign commerce which supports a large part of our population, and supplies us with many necessaries and most of our luxuries. And now-a-days even the farmer, for the correct laying out of his drains, has recourse to the level—that is to geometrical principles. When from those divisions of mathematics which deal with *space* and *number*, some small smattering, of which is given in schools, we turn to that other division which deals with *force*, of which even a smattering is scarcely ever given, we meet with another large class of activities *which this science presides over*. On the application of rational mechanics depends the success of nearly all modern manufacture. The properties of the lever, the wheel and axle, &c., are involved in every machine—every machine is a solidified mechanical theorem; and to machinery in these times we owe nearly all production. Add to which that for the means of distribution over both land and sea, we are similarly indebted. And then let it be remembered that according as the principles of mechanics are well or ill used to these ends, comes success or failure—individual and national. The engineer who misapplies his formulæ for the strength of materials, builds a bridge that breaks down. The manufacturer whose apparatus is badly devised, cannot compete with another whose apparatus wastes less in friction and inertia. The ship-builder adhering to the old model, is out sailed by one who builds on the mechanically justified wave-line principle. And as the ability of a nation to hold its own against other nations, depends on the skilled activity of its units, we see that on such knowledge may turn the national fate. Judge, then the worth of mathematics." Having determined beyond question what knowledge is of most worth—indeed actually essential to the progress of mankind—we have thereby determined what is of most value as a means of intellectual discipline; for "it would be utterly contrary to the beautiful economy of *nature* if one kind of culture were needed for the gaining of information, and another for the development of intellect. Everywhere throughout creation we find faculties developed through the performance of those functions which it is their office to perform; not through the performance of artificial exercises devised to fit them for these functions.

I shall be charged, perhaps, in consequence of these views with believing in the *absolute* sufficiency of Mathematics as a means of education, and with arrogance in rejecting the testimony of certain eminent men against the utility of Mathematical discipline. But I neither hold the all-sufficiency of Mathematics nor possess credulity enough to render a passive belief in any utterance, simply because it is sanctioned by illustrious names. There is a great deal of contradiction among the authorities cited against the value of Mathematics in education; and those whose evidence is not nullified by mutual contradiction, are opposed by a greater number of more *credible*, because more *competent* witnesses. In fact the extravagance of the opposing testimony demonstrates its falsity. It is asserted that Mathematics are difficult only because they are too easy, that they determine thought to its feeblest developement—that they actually dwarf the mental powers—that they contribute no advantage as a passport to Psychology or other sciences—that they lead to credulity and scepticism—and that a great genius cannot be a great mathematician. Descartes, Leibnitz, Newton, Euler, La Place, and a host of others—were they not men of genius? Or has the admiration of successive generations been only the tribute of a "blind credulity?" Then humanity has produced nothing great, and subjects for veneration must be sought among the forgotten—and soon to be forgotten—assailants of Mathematics. To show that I do not stand alone in my estimate of the value of Mathematics, it may be well to quote the evidence of a few of the witnesses who are *really competent* to give an opinion on the subject. *Kant*, after stating that the sure path of Metaphysical science has not yet been found, says "It seems to me that the examples of Mathematics and Natural Philosophy are sufficiently remarkable to fix our attention on the essential circumstances which have proved so beneficial to them, and to induce us to imitate them so far as the analogy which, as rational sciences, they bear to Metaphysics may permit." *Cousin*, "the greatest philosopher of France," asserts the influence of Mathematics in the philosophy of Kant, and speaking of the Mathematic and idealistic character of the Pythagorean philosophy, says, "For Mathematics are founded on abstraction, and there is an intimate alliance between Mathematics and idealism; thence the Mathematical idealism that penetrates all parts of the Pythagorean system." And in reference to the Platonic philosophy, he observes.—"Abstraction is, therefore, the process, the instrument, of all good philosophy; this is also the process which characterizes the genius of Plato; hence, all that is true and sublime in the philosophy of Plato; hence his morality, his politics, and his *decided taste for Mathematics*; you perceive, in fact, that the Mathematical habit of considering, in quantities and dimensions, only their essential properties, was a happy preparation to Platonic abstraction." This is clear evidence of the power of Mathematics to develop abstraction, and prepare the mind for the accurate investigation of its own phenomena. As to credulity and scepticism, we have the testimony of the celebrated *Dr. Barrow*; "Mathematics deliver us from credulous simplicity, most strongly fortify us against the vanity of scepticism, effectually restrain us from rash presumption, and most easily incline us to due assent." *Lord Bacon* says:—"In the Mathematics I can report no deficiency, except it be that men do not sufficiently understand the excellent use of the pure Mathematics (he could have added *now* and the *still more excellent use of the applied Mathematics*,) in that they do remedy and cure many defects in the wit and intellectual faculties; for if the wit be too dull they sharpen it; if too wandering they fix it; if too inherent in the sense they abstract it." And *John Stuart Mill* says: "The value of Mathematical instruction as a preparation for those more difficult investigations, society, government, etc., consists in the applicability, not of its doctrines, but of its method; the applications of Mathematics to the simpler branches of physics furnish the only school in which philosophers can effectually learn the most difficult and important portion of their art—the employment of the laws of simpler phenomena for explaining and predicting those of the more complex; these grounds are quite sufficient for deeming Mathematical training an indispensable basis of real scientific education, and one of the most essential qualifications for the higher branches of philosophy."

I think, then, that notwithstanding the dogmatic utterances of certain Metaphysicians who were almost totally ignorant of Mathematics, and the careless admissions of a few Mathematical Metaphysicians who sacrificed the certainty and stability of Mathematics for the aberrations of Psychology, enough has been adduced to establish the proposition that Mathematics are entitled to a high position as an Instrument of Education.