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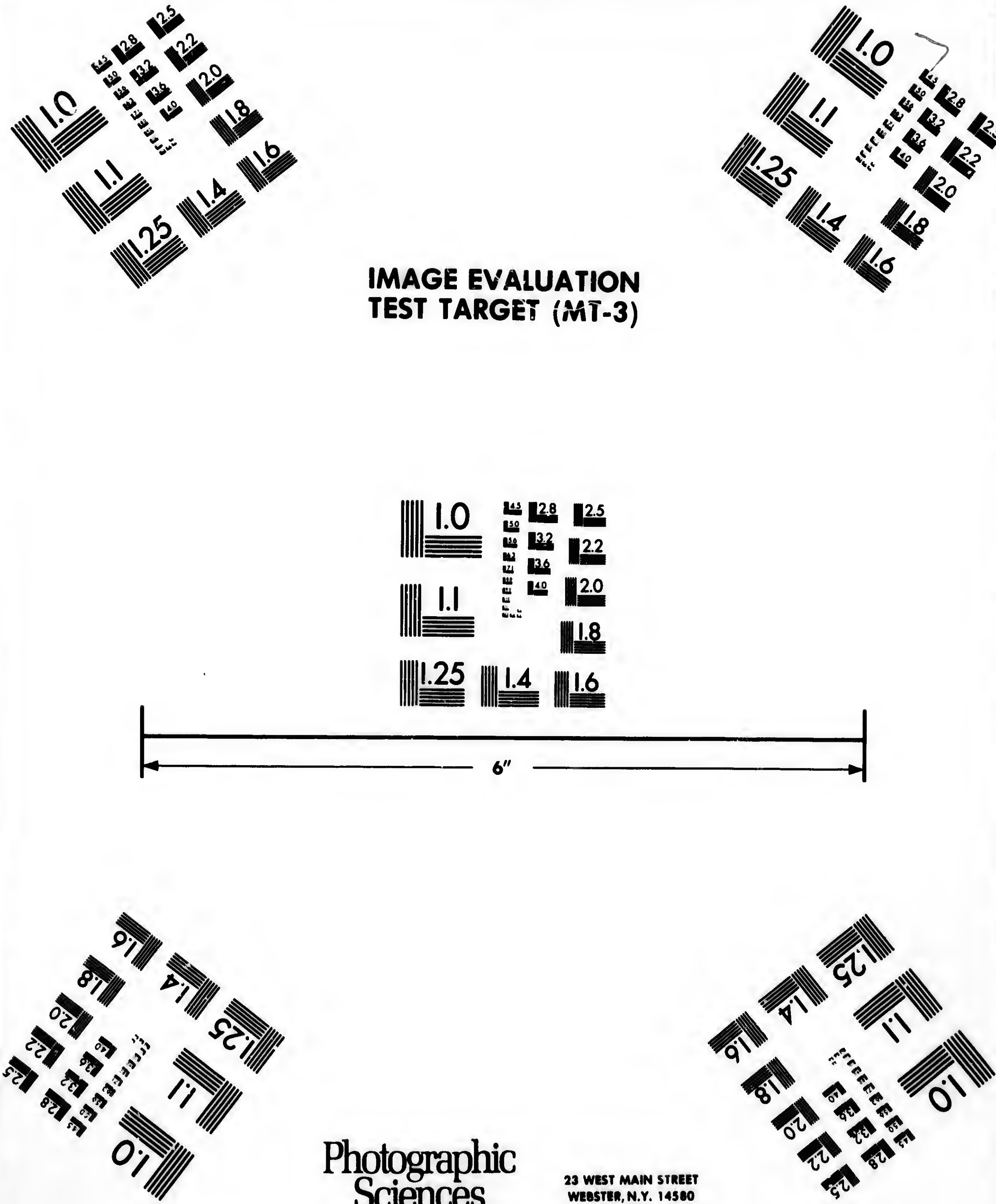

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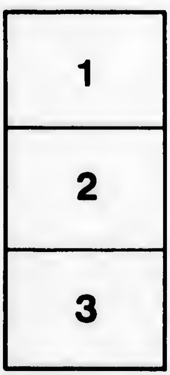
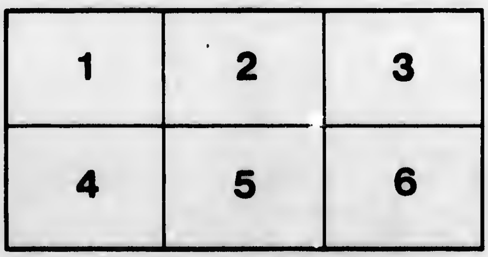




CENTRIFUGAL FORCE AND GRAVITATION

BY

(KÚKLOS)

JOHN HARRIS.

Parts.

1. The Attractive Force and Tangential Motion.
2. The Planetary axis of rotation, and plane of the Ecliptic.
3. The neighbouring Stellar System, and Aberration of Light.
4. The Cometary Orbit of Revolution.

5., 6., 7. } Parts I. & II. The Undulatory and other Theories of Light.
Part III. Light, and other manifestations of Force.

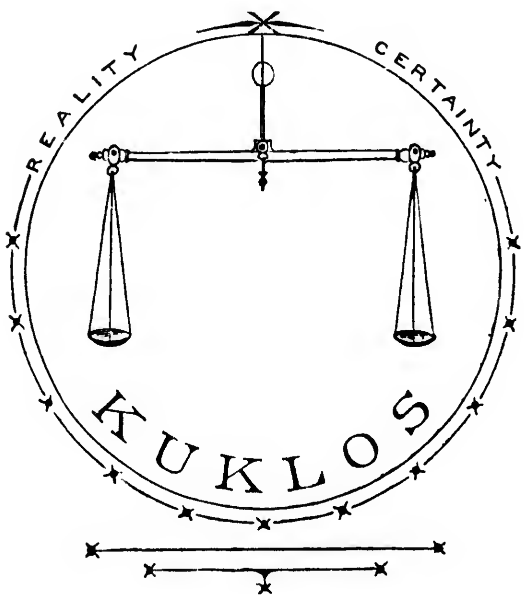
THE CIRCLE AND STRAIGHT LINE,

1. The Geometrical Relationship demonstrated.
2. The Construction of the Circle.
3. Conclusion.

THE CIRCLE

AND

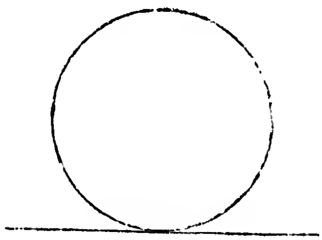
STRAIGHT LINE.



THE CIRCLE

AND

STRAIGHT LINE.



BY

JOHN HARRIS.

MONTREAL:

JOHN LOVELL, ST. NICHOLAS STREET.

JANUARY, 1874.

MONTREAL.—JOHN LOVELL, PRINTER.

PREFACE.

Deeming the subject of this work, viz: the solution of that geometrical problem which demonstrates the relation of the circle to the straight line, to be peculiarly of public importance, we will briefly state what has taken place in respect to it, and the present position of the matter.

The discovery ot the solution was communicated by letter, dated 29th December, 1870, accompanied with demonstration, &c., to the Astronomer Royal. It was afterwards found that the case, as contained in the papers thus presented in the first instance, was in a strictly geometrical sense imperfect and faulty; the solution and demonstration, however, were virtually contained in them, and no objection whatever was made to their form.

The papers were acknowledged, but, after some considerable time, a decisive intimation was received from Greenwich that the Astronomer Royal declined to examine the case.

In the month of January, 1873, the papers were presented to the president of the Royal Society with a formal request (claim) in writing to have the case judicially examined by that Society. The documents were returned with a note from the President, (G. B. Airy, Esq.,) decidedly refusing, as President, to receive the papers or to acknowledge the claim for investigation.

Before this refusal was received, considering that the subject if correctly understood would be found to have an especial interest for the public of a practical and immediate character, four letters containing a general explanation of the great importance of the subject with regard to the immediate interests of the public, were forwarded to the Editor of the "Times" (London) for publication in that Journal; and, when the letter of the President of the Royal Society, refusing to entertain the case, was received, a copy of that letter was also forwarded to the Editor of the "Times."

About the same time that the case was presented for investigation to the Royal Society, a communication thereof was made to McGill College, also formally requesting an investigation. Not long afterwards, however, the papers were returned without, as it would seem, having undergone examination.

Copies of the several documents referred to will be found at the end of the Appendix to this book.

INTRODUCTION.

The subject of this work is that geometrical problem which requires a straight line to be drawn equal in length to the given arc of a circle; or, a circle to be described equal in length to a given straight line—accompanied, in either case, with demonstration that the conditions of the requisition have been mathematically fulfilled.

This problem we have succeeded in solving according to the strict rules of geometry, and the demonstration that we have so done, is herein presented to the public.

We publish our solution with the distinct statement that it is essentially in strict accordance with that scientific system known as Euclid's. We claim to have our demonstration admitted or disproved, and we challenge objection or adverse argument on that system.

The book known as 'Euclid's Elements of Geometry,' although possessing a high degree of completeness compared with other scientific treatises, and including a considerable part of the subjects belonging to that division of science, is a human production and imperfect—it is neither absolutely complete, nor absolutely comprehensive.

However desirable it may be to retain the formal method in and by which Euclid taught his application of the system, there can be no good reason why the method should be restricted to any particular number of problems, or why it should not be extended to include cases of the same character as those treated in the 'Elements,' but which did not come under the consideration of Euclid.

Let us briefly examine the essential characteristics of the scientific system taught by Euclid,—and also the arrangement of Euclid's formal method.

The first essential of the system is that there shall be a simple (elementary) basis . . . of which the reality, actuality, or truth, is absolutely certain. This fundamental basis (or each such basis, because there may be an indefinite number) being elementary and manifestly real and true, requires merely to be defined or stated with precision ; and it is termed, accordingly,—*a definition.*

A geometrical definition may therefore be called the verbal equivalent of a fact. *

When the basis is not (elementary) quite simple, but requires only a brief explanation to satisfy the reasonable mind as to its unquestionable truth or reality . . it is called—*an axiom.*

(Strictly speaking an axiom is a proposition or theorem, of which the manifest truth becomes so readily apparent as to render formal demonstration unnecessary ; or in some instances, the definite statement of the axiom may include its demonstration.) † The postulates of Euclid's 'elements' are the rules of his systematic method,

* It may be said to assert the existence of the fact which it defines.

† Therefore it is in the same case with the definition, and it may be considered a compound definition ; or, the definition may be considered an elementary axiom. In either case there is the definition only of a reasonable recognition by the mind of an actual existence or reality. (Def.) *e. g.*, 'A straight line is that which lies evenly between its points,' the mind at once recognizes and *reasonably* accepts the *fact*, which is manifest whether the line be considered only a natural or only an ideal reality. (Axiom.) *e. g.*, 'The whole is greater than its part,' the comparison has already been made by the mind, and the inter-relation of the two things, thus defined, is at once recognized by the mind as actual, (*i. e.*, as an ideal fact.)

as distinguished from the laws or rules of the philosophy, which last are to be learnt from their application and illustration throughout the work. In stating this, we are not making a fanciful distinction, but indicating a difference which it is of great moment to correctly appreciate.

Euclid's 'Ideal Philosophy,' or 'Scientific System,' if perfected, would be a perfect system of human reasoning, that is, to say, a system under which, its rules and regulations being strictly observed, the guidance of reason could be obtained (in compounding knowledge) on all subjects to which human knowledge is permitted to extend, so as to insure the attainment of certainty and truth in all cases.

Euclid's systematic method, as taught in the work known as the 'Elements of Geometry,' is an endeavour to apply that system to one of the divisions of science,—namely, the science of Form and Magnitude; an endeavour which, with the exception of one grievous and calamitous mistake, must be considered as having been successful in an astonishing degree; for, as a complete, comprehensive and important work in itself, the (so-called) 'Elements of Geometry' may be justly regarded as the greatest product, and as the proudest monument, of the human intellect the world has yet to show.

The postulates of the 'Elements' are the arbitrary rules* framed to insure method and precision in the application of the system to the science of 'Form and Mag-

* Arbitrary, however, in a relative, not in an absolute sense; for the postulates of 'Euclid's Elements' are reasonable, *i. e.*, subordinate to the rules of reason. The postulates are arbitrary in a general sense, in the same relative manner that the laws of material Nature are arbitrary in a spiritual sense; in either case there is a limitation and restriction which may be termed artificial, but . . the one is a contrivance of human reason . . the other, an arrangement by Divine reason.

nitude.' * Formally, they give permission to conduct the necessary process of construction in a particular manner; thereby strictly prohibiting other methods of conducting the process. Now the importance of such strict rules in science generally, and in each of the divisions of science, is not to be lightly esteemed. For the purpose of preserving order and distinctness, and of enabling a number of persons to work harmoniously (in concert) on the same subject, some such rules may be considered practically indispensable; the indirect benefit of their influence, also, in training the mind to a due regard for law and system, in its endeavours to acquire increased knowledge by compounding the elements thereof, is unquestionably very great.

It would be, however, superstitious, slavish, and quite unreasonable, for men to allow themselves collectively or individually to be absolutely bound and fettered by a particular set of rules, framed long since by other men, viz., by Euclid and his predecessors, as the most advan-

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* The science of 'Form and Magnitude' may be correctly termed an abstract science, because the expression 'abstract' is so used in a definite and intelligible sense. To us as human beings, cognizing even our ideas and thoughts through a material medium, the abstract property... such as form, magnitude, number, or quantity, is not conceivable as an existence distinct from a material existence or thing, the same as a spiritual existence or force is conceivable as distinct. The abstract sciences belong, therefore, to natural science, (*i. e.*) to the science of Material Nature, which includes:—

The material sciences.

The sciences of spiritual force manifested on matter.

The science of the abstract properties belonging to matter.

By the aid of ideal philosophy, we are enabled to separate and to study separately these ... all and each of which form a part of compound matter (*i. e.*, of matter naturally recognized as such by us.)

tageous which could at that time be devised. On the contrary, if circumstances render an extension or modification of the rules desirable, it is quite reasonable, deliberately and circumspectly, to make and agree to such alteration as the advancement of science may call for.

The formal method adopted and taught by Euclid may be described as consisting of:— (1) The definite statement of the requisition. (2) The construction and arrangement of the case, (*i.e.*, of the circumstances and facts belonging to the case). (3) The answer to the requisition. (4) Demonstration that the answer is the true one. In this manner, the legitimate process of compounding knowledge is, by the aid of reason, systematically and methodically applied to the particular facts of the science; and which science in Euclid's treatise is that of Form and Magnitude.

The proposition or theorem is first stated in precise but general terms. A particular case, suitable for the illustration of the question, is then taken, and by aid of the geometrical figure is distinctly defined. Next comes the construction. The construction may be considered as equivalent to an experiment in physical or chemical science*; certain additional lines are drawn for the pur-

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* The construction in Euclid's method is the orderly mode of submitting the solution of the problem for the approval of reason, . . . it is in the first place a result, ... the mind has arranged the elements of the case and contemplates the conclusion resulting therefrom; but, before definitively accepting the conclusion, formally submits the arranged case to reason for approval. If demonstration follows, reason approves; if demonstration fails, reason disapproves. Therefore, 'Euclid's construction' is equivalent to the (so-called) experiment in physical or chemical science: the experiment is, also, in the first place, a result; . . . the conditions of the case are arranged by the mind which contemplates the conclusion, but which, before venturing to accept that conclusion, submits the arranged case (together with

pose of giving completeness to the figure, so as to enable the necessary combinations and comparisons to be made in an orderly and systematic manner. The geometrician is now in a position to obtain the answer to the requisition. This is at first briefly stated as a positive or decided conclusion. The demonstration then follows; in which the answer to the requisition as stated is justified by the facts upon which it is based, and shown thereby to be the only correct explanation of the result.

We have elsewhere stated that the primary and most important of the two-fold purpose which Euclid's work had in view was to teach and illustrate the philosophy—*i.e.*, the scientific system of reasoning, and that the application of the philosophy in his treatise on the science of 'Form and Magnitude,' does not justify the inference that the philosophy has a peculiar connection with that one division of science, and that it is not, with the requisite modifications, equally applicable to the other divisions of science.

* *Note* (*a*). In writing thus it is to be understood that we are idealizing Euclid and considering him as the representative author of a work which cannot reasonably be supposed the production of one individual only; it may be that a large part of the 'Elements,' as well as the arrangement of the parts and the coherent complete-

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the conclusion that it contemplates accepting) for the approval of reason. If reason approves, the experiment succeeds; if reason disapproves, the experiment fails. It may be an exhibition experiment: the experimenter has himself accepted the result (he has repeatedly performed the experiment and has knowledge that reason approves the arranged case), but he exhibits the experiment. For what purpose? In order to demonstrate to the spectators the approval of reason; or, more strictly speaking, to demonstrate the legitimacy of the arrangement and soundness of its conclusion, which conclusion is thus authorized and commended by reason for the mind's acceptance as knowledge.

ness of the exposition as a whole, is attributable to Euclid himself, but, to some extent, at least, the book must be considered as the arrangement of the work of his predecessors by Euclid.

We do not assert that Euclid himself had, in arranging the work, a clear and distinct appreciation of the philosophy as a system of reasoning not peculiarly connected with that one division of science to which it is applied in the 'Elements.' It is almost certain that he had not, and it is, humanly speaking, almost impossible that he could have had such a distinct appreciation of the fact. In the first place, the choice of the subject of the application was not originally his, the compound of the philosophy with that particular division of science was known as Geometry before his time, and was already regarded, not as a compound, but as a peculiar (mystical) division of knowledge; if Euclid had clearly perceived the fallacy of thus regarding it, some decisive indication that he did so would appear in the work. And, secondly, it must be remembered that most of the divisions of science, as we now recognize them, were at that time quite unknown, and, it may be said that, scientific knowledge was in a great measure comprised in that one division which alone had been scientifically arranged. It was, therefore, evidently almost impossible for Euclid to clearly appreciate the general relationship of philosophy to all the divisions of science, because he was quite unacquainted with even the existence of the major part of those divisions, and was consequently not in a position to appreciate the inter-relation of the divisions to each other as parts of one general science. *(Note b.)*—There is, however, some apparent evidence that Euclid did in a measure regard the exposition of the particular science *primarily* as a means of illustrating and teaching the general philosophy; instances of propositions treated in a very indirect and elaborate manner, at least, suggest such an explanation. Notable instances of this kind are the 1st prop. of Book III, and the 2nd prop. of Book XII.)*

The rules of philosophy are not in the same case with the postulates of Euclid's formal method. The laws of reason—that is, the authoritative rules by which knowledge is to be compounded, and according to which compound knowledge is to be accepted as sound or rejected as unsound, cannot be altered, and may not be tampered with in any degree, nor is it permissible to disregard them. An unintentional neglect of these rules through ignorance or carelessness entails in all cases some degree of retributive punishment on the offender, according to the circumstances and the greater or lesser importance of the subject on which the mind thus neglects the laws of its intellectual existence. A wilful disregard, contempt, or defiance, of these rules is an intellectual crime, and, it may be, if the subject be of great importance, and, especially, if the mind be of great capacity and the acknowledged representative of many others, a crime of frightful magnitude, of which the consequences fall, not only on the individual offender, but on an indefinite number (a vast number, perhaps) of other persons.

It is our purpose to bring, almost immediately, this subject, namely, the 'law of intellectual existence' and the 'responsibilities belonging to knowledge,' particularly before the public.

We will now proceed to the demonstration of the geometrical problem, in its several forms—that is to say, the several forms of the general relationship in form and magnitude between the straight-line and the perimeter of the circle.

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* We will in the Appendix make some examination of these props. relatively to the above suggestion.

THE CIRCLE AND STRAIGHT LINE.

1. *Definition.*—If a circle be applied upon a straight line, and the circle be then moved upon the straight line in such wise that each point in the circumference, successive to the point in the circumference which is first in contact with the straight line, be brought successively into contact with the straight line—to wit, with each similar successive point in the straight line, commencing from the point in the straight line which is first in contact with the circle, and if the circle be in such wise continually moved until a given point in the circumference, at

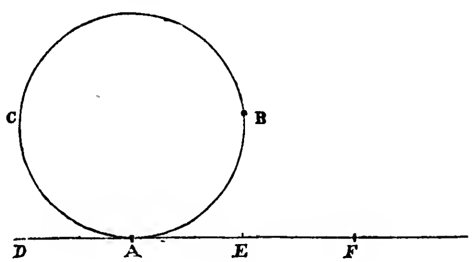

Fig. 1.

a definite distance from the first point of contact, become in contact with the straight line—the circle is said to be rolled upon the straight line from the point in the circumference first in contact with the straight line to the given point in the circumference.

(Fig. 1.) Let *A. B. C.* be the circle applied upon the straight line *D. E. F.*, and in contact with the straight line at *A.*, and let *B.* be the given point in the circumference. If the circle be moved upon the straight line in the direction *D. E. F.*, and be so moved that each successive point in the circumference between *A.* and *B.* become successively in contact with each similar point successive in the straight line from the point *A.*, and be so moved continually until the given point *B.* in the circumference of the circle be in contact with the straight line—the circle *A. B. C.* is said to be rolled upon the straight line *D. E. F.*, from *A.* to *B.* on the circle. (*a.*)

But if the given point be upon the straight line, at a definite distance from the first point of contact between the circle and the straight line, and the circle be moved, in such wise as before, until some point in the circumference become in contact with the given point on the straight line—the circle is said to be rolled upon the straight line from the first point of contact to the given point on the straight line.

(Fig. 1.) Let *F.* be the given point on the straight line, and let the circle *A. B. C.* be continually moved from the point *A.*, in the direction *D. E. F.*, in the manner directed, until some point in the circumference become in contact with the point *F.* on the line—the circle, *A. B. C.* is said to be rolled upon the straight line from *A.* to *F.* on the line. (*b.*)

2. *Definition.*—If a straight line be made to deviate from its evenness between the two extreme points thereof in such wise that the line be made to contain a circle or to be any part of the circumference of a circle—the straight line is said to be bent or curved into the arc of a circle.

Similarily if a curved line be extended until it lies evenly between its extreme points—the line is said to be unbent into a straight line.

Postulates.—Let it be granted that:

1. A circle, or that any arc of a circle may be rolled upon a straight line.
2. A straight line may be bent into a curved line.
3. The arc of a circle, or other curved line may be unbent (extended) into a straight line. *

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* Should any one be disposed to object that the operations for the performance of which leave is here taken are of a mechanical nature, a little consideration may convince them

Axioms.—1. The arc of a circle formed by bending a straight line into a curved line is equal in length to that straight line.

2. The straight line formed by unbending the arc of a circle into a straight line is equal in length to that arc.

3. If a circle be rolled upon a straight line from the point of contact in that straight line until a given point in the circumference become in contact with the straight line . . . the distance between the first point of contact and the last point of contact on the straight line, is equal to the distance between the first point of contact and the given point on the circumference of the circle.

CONSTRUCTION, PLATE 1, FIG. 2.

With the centre *A.* and radius *A. B.* describe the quadrant *B. F.* Bisect *B. F.* at *M.* Draw *M. N.*, the sine of the arc *B. M.*, at right angles to *A. B.*, intercepting *A. B.* at *N.* From *B.* at right angles to *A. B.*, draw *B. E.* of indefinite length, tangential to the arc *B. M.* Divide the arc *M. F.* into ten equal parts at the points of equal division *a. b. c. d. e. f. g. h. i.* From *M.* draw *M. D.* perpendicular to *B. E.* and intercepting *B. E.* at *D.*

Scholium.—If the curved line, forming the arc *B. M.* cut off from the quadrant at *M.*, is supposed to be straightened upon the line *B. E.*, the line *B. M.* will then throughout its entire length be applied upon the line *B. E.* (*i.e.*, will coincide with a part of the straight line *B. E.*) and the point *M.* at the extremity of *B. M.* will

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that describing a circle with a centre and radius is quite as much so. But in fact the circle is to be geometrically not mechanically rolled, and the reasoning is quite independent of any mechanical operation in both cases. It may be thus explained:—The circle is merely supposed to be rolled. The reasoning investigates and determines the alteration which would be occasioned in the relative positions of certain points if the circle were to be rolled.

manifestly be in contact with (*i.e.* coincide with) a point in the line *B. E.*, at some place more distant than the point *D.* from *B.* Now the distance of the point of contact of *M.* from *B.* can be approximately determined, and is approximately known. Let *O.* indicate the unknown locality of the actual point of contact of *M.* when the curved line *B. M.* is straightened upon the line *B. E.*

Again, if the arc *B. M.* in contact with the line *B. E.* at *B.* is supposed to be rolled from *B.* upon the line *B. E.*, until the point *M.* at the opposite extremity of the arc, become in contact with the line *B. E.*, then, since each and every constituent (or component) part of the arc *B.M.* is successively brought into contact with each similar constituent (or component) part of the line *B. E.*, the point of contact must necessarily be the same point *O.* touched by *M.* when the curved line forming the arc *B. M.* is straightened upon *B. E.*, and the straight line *B. O.* (*i.e.* the straight line contained between the point *B.* and the unknown point which is indicated by *O.*) must necessarily contain the same quantity of length contained in the arc *B. M.*

Construction continued.—Suppose the arc *B. M.* to be straightened upon the line *B. E.*; and let *O.* indicate the point on the line *B. E.* which coincides with the point *M.* at the extremity of *B. M.* when so straightened. From *O.* perpendicular to *B. E.* draw *O. C.* equal in length to *A. B.* With centre *C.* and radius *C. O.* describe the quadrant *O. P.* Bisect *O. P.* at *S.* Through *C. C.* join *S. M.* intersecting *C. O.* at *R.*, and through *S.* draw *C. Q.* intercepting *B. E.* at *Q.* Join *C. P.*, *P. Q.*

From *S.* perpendicular to *B. E.* draw *S. T.* Produce *M. S.* through *S.* intercepting *P. Q.*; and, from the point *X.* on the line so produced, taken at the distance *S. X.* equal to the distance *M. R.* on the same line, draw the perpendicular *X. Y.* intercepting *B. E.* at *Y.* Divide the line *R. S.* into nine equal parts at the points of equal division *b. c. d. e. f. g. h. i.*

Examination by Hypothesis. (Euclid. 2. xii.) On the assumption that M. *a.* on the line M. S. equals M. a. on the arc M. F. *

Scholium.—Since, if the quadrant *B. F.* be rolled upon the line *B. E.* until the point *M.* bisecting the quadrant arrive at *O.* (which indicates the unknown actual point of contact), the point *F.* at the extremity of the quadrant, must necessarily arrive at *S.*, (*O. S.* represents what will then be the place of *M. F.*, because *M.* arrives at the point indicated by *O.*, and *F.* then, necessarily, arrives at *S.*), the point *e.* bisecting the arc *M F.* must also, necessarily, (when the arc *B. M.* has rolled through half its length) arrive at *e.* on the line *M. S.*, because the line *B. O.* is manifestly equal to the line *M. S.* which is consequently equal to the arc *M. F.*; it, therefore, follows that each of the divisional points *a. b. c. d. e. f. g. h. i.* on the arc *M. F.* must arrive successively at each of the divisional points *a. b. c. d. e. f. g. h. i.* on the line *M. S.* Again, if the quadrant be further rolled upon the line *B. E.* until the point *F.*, at the extremity of the quadrant, be in contact with *B. E.*, the point of contact must necessarily be the point *Y.* (because the arc *O. S.* (or *M. F.*) is similar and equal to the arc *B. M.*, and the movement of the one arc is similar and equal to the movement of the other) and *T. Y.* is equal to *D. O.*, therefore, since *B.* arrives at the point indicated by *O.*, and since *F.* (or *S.*) must then necessarily (if the rolling be continued) arrive at *Y.*, the line *O. Y.* is equal to the line *M. S.* which is equal to the line *B. O.*, which by the construction is the same length as the arc *B. M.*

Verbal definition.—When a magnitude is said to contain a certain number of lesser magnitudes, it is defined that the greater magnitude is wholly compounded of that certain number of those lesser magnitudes; similarly, when a line (or a definite quantity of longitudinal space) is stated to contain a certain number of (equal) parts, the line (or definite quantity of space), is defined by the statement to be wholly compounded of that certain number of those parts.

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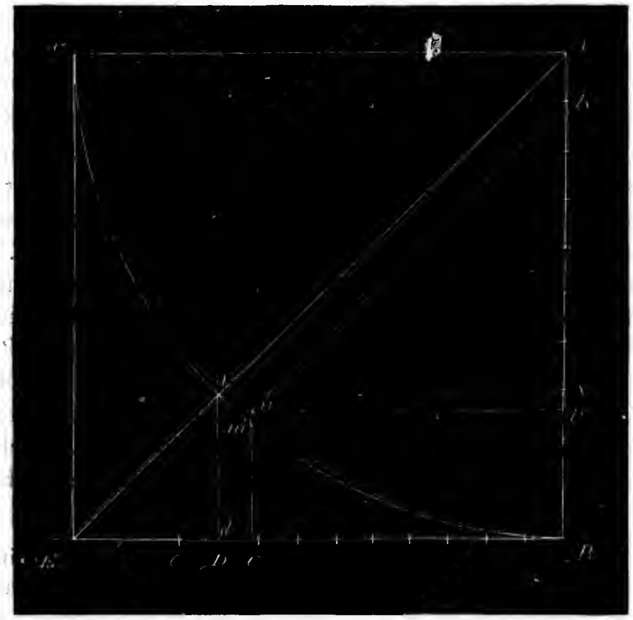
* See Demonst. (*a*) Part Second.

Prop. A.—Theorem.

If an arc containing one-eighth of a circle, be applied upon a straight line, and, from the terminal extremity of the arc, a perpendicular be drawn intercepting the straight line, and if from the arc one-tenth thereof be cut off, then, if the remaining arc (to wit, the arc containing nine-tenths of the whole arc,) be rolled upon the straight line, the point of contact shall be the same point on the straight line intercepted by the perpendicular drawn from the terminal extremity of the whole arc.

Fig. 1. Let the arc *B. M.*, containing one-eighth of a circle and described with the radius *A. B.*, be the arc applied upon the stiaight line *B. E.*, and let *M. D.* be the perpendicular drawn from *M.* intercepting the straight



line at *D.* And let the arc *B. m.*, nine-tenths the length of *B. M.*, be the arc cut off from *B. M.* If the arc *B. m.* be rolled upon the straight line until *m.* arrive at the straight line, the point of contact shall be the point *D.* intercepted by the perpendicular *M. D.*

From the radius A. B. take K. B. nine-tenths the length of A. B., and with centre K. and radius K. B., describe the arc B. n. similar to the arc B. M., and equal in length to the arc B. m.* Draw M. N. the sine of the arc B. M., and n. p. the sine of the arc B. n. From n. at the extremity of the arc B. n. draw the perpendicular n. C. intercepting the straight line at C.

Let the arc B. M. be rolled upon the straight line until M. become in contact, and let O. indicate the point of contact. Let also the arc B. n. be rolled until n. become in contact on the straight line, and let d. indicate the point of contact.

Because the radius K. B. is nine-tenths of the radius A. B., the sine n. p. is nine-tenths of the sine M. N., and the arc length of B. n., which is indicated by B. d., is nine-tenths the arc length of B. M. which is indicated by B. O., and C. d., the difference of the sine and arc length of B. n., is equal to nine-tenths of D. O. the difference of the sine and arc length of the arc B. M.

Now if the arc B. M. be rolled upon the straight line (back again) from the point of contact (indicated by O.) until the opposite extremity of the arc be in contact with the opposite extremity of the straight line at B., the point M. must again arrive at M. the extremity of the perpendicular M. D., as in the first position of the arc, and since the difference of the arc length and sine of B. n. has the same proportion to the difference of the arc length and sine of B. M. which the arc B. n. has to the arc B. M., therefore, if the arc B. n. be rolled upon the straight line (back again) from the point of contact indicated by d., until the opposite extremity of the arc be in contact with B. at the opposite extremity of the straight

* Circles are proportional to each other directly as their radii; therefore, since the radius K. B. is nine-tenths of A. B. the arc B. n. is nine-tenths the arc B. M. But the arc B. m. is nine-tenths of B. M., therefore, the arc B. n. is equal to the arc B. m.

line, *n.* the point at the extremity of the perpendicular *n. C.*, at which the extremity *n.* of the arc arrives, must be arrived at by the point *n.* from a horizontal distance *C. d.* having the same proportion to *D. O.*, which the sine *n. p.* has to the sine *M. N.* Now the arc *B. n.* is equal to the arc *B. M.* diminished by one-tenth,* and, *C. O.* is the difference (*C. d.*) of the sine and arc-length of *B. n.*, and the difference (*D. O.*) of the sine and arc-length of *B. M.* taken together . . . that is, the difference of the sine of the lesser arc and the arc-length of the greater arc. But, because the ratio of *B. D.* to *B. C.* is the same as the ratio of *B. O.* to *B. d.* and the same as the ratio of *d. O.* to *c. d.*, the part *C. D.* of the whole distance *C. O.* has necessarily the same proportion to the part *D. O.* of the whole distance which the sine *n. p.* has to the sine *M. N.*, and, therefore, the distance *C. d.* is the same as the distance *C. D.*, and the point indicated by *d.* is the same point *D.* at the extremity of the perpendicular *M. D.* *Wherefore it is demonstrated* that if an arc containing one-eighth of a circle, &c., &c. *Q. E. D.*

Corollary.—Hence it follows,—because the point *n.*, when the arc *B. n.* is rolled, becomes in contact on the straight line at the point intercepted by the perpendicular *M. D.*, and, because the distance *B. D.* is (consequently) nine-tenths of *B. O.*,—that the sine of an arc containing one-eighth of a circle is nine-tenths the arc length. † (*i. e.*, the ratio of the length of the sine to the arc-length is the ratio of nine to ten.

Note.—Having regard to the great importance of the subject, and as this theorem is fundamental, it will be now repeated in a somewhat different form :—

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* That is, *B. M.* is the arc *B. n.* increased by one-ninth of *B. n.*

† Independent demonstration will be given of this proposition, but the corollary becomes apparent so soon as the point of contact, at which the extremity of the lesser arc arrives on the straight line, is determined.

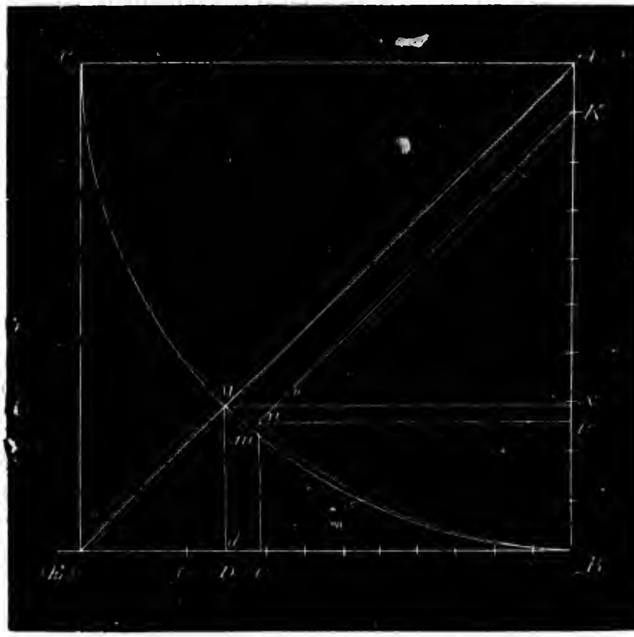
Scholium.—In considering the following case, since if several similar arcs of different magnitudes be rolled on a straight line, the motion of the terminal point at the extremity of each arc is compounded of the vertical and oblique motion of each and all the component parts of that arc, it is requisite only in comparing the results of the motions of two or more of the similar arcs with each other, to take the straight horizontal longitudinal advance—that is, the horizontal motion—into consideration; because the result of the horizontal advance includes the result of the vertical motion of which it is in part compounded, and which must be necessarily proportional to the horizontal advance and, also, to the magnitude of the arc.

Theorem A. (repeated.) Fig. 1.

When the arc *B. M.* is rolled, *M.* advances to a point (*i. e.*, some point) on the straight line, indicated by *O.* When the arc *B. n.*, which is similar to the arc *B. M.*, and contains nine-tenths of the length contained by *B. M.*, is rolled upon the straight line, *n.* advances to a point (some point) indicated by *d.* Now the distances *O.* and *d.* from *B.* are so related to each other by the construction that *B. d.* contains nine-tenths the length contained by *B. O.* Also the sine *n. p.* of the arc *B. n.* is nine-tenths the sine *M. N.* of the arc *B. M.*

(2.) The distance *D. O.*, which is the horizontal advance of the point *M.*, contains the difference between the sine and arc length of *B. M.* Now, if *B. M.* be reduced by one-tenth part thereof, and the remaining arc, by diminishing the radius in the same proportion, is converted into a similar arc, containing one-tenth less length, the sine of the lesser arc *B. n.* thus described will have the same proportion to the sine of the greater arc *B. M.*,—to wit, the sine *n. p.* of the lesser arc will be nine-tenths the length of the sine *M. N.* The difference *(D. O.)* between the arc length and sine of *B. M.* is to the difference *(C. d.)* between the arc length and sine of

B. n. in the proportion of ten to nine. But the one-tenth by which *D. O.* exceeds *C. d.* is manifestly the increase belonging to the one-tenth of the arc *B. M.* by which that arc exceeds the arc *B. n.* Therefore, when



the arc *B. M.* is diminished by one-tenth thereof, if the lesser arc, thence resulting, is rolled until the terminal point become in contact on the straight line, the advance of the terminal point must be less than *D. O.* by one-tenth of *D. O.* But the sine of the arc *B. n.* is one-tenth less than the sine of the similar arc *B. M.*, therefore the distance between *C.* and *d.* must be less than the distance *C. O.* by one-tenth of *D. O.* and one-tenth of *B. D.* taken together; that is by the distance *D. O.*; because *B. D.* and *D. O.* taken together equal *B. O.*, and *D. O.* is one-tenth of *B. O.*; therefore *B. d.* is equal to *B. O.* diminished by *D. O.*; but the distance *B. O.* diminished by the distance *D. O.* is *B. D.*, and therefore the distance *B. d.*, is the same as the distance *B. D.* Wherefore it is demonstrated that the point of contact indicated by *d.* on the straight line *B. E.*, at which the terminal point of the arc *B. n.* arrives, is the point intercepted by the perpendicular *M. D.* drawn from the terminal point *M.* of the greater arc *B. M.*

(3.) Because :—

$$\left.\begin{array}{r} B.\,C.\,:\,B.\,D.\,::\,B.\,d.\,:\,B.\,O. \\ \text{and } B.\,C.\,:\,B.\,d.\,::\,B.\,d.\,:\,B.\,O. \end{array}\right\} \begin{array}{l} \text{therefore} \\ B.\,d. = B.\,D. \end{array}$$

That is :—

The sine of *B. n.* is to the sine of *B. M.* as the arc-length of *B. n.* is to the arc-length of *B. M.*
The sine of *B. n.* is to the arc-length of *B. n.* as the arc-length of *B. n.* is to the arc-length of *B. n.* increased by one-ninth.

Wherefore the arc-length of *B. n.* equals the sine of *B. M.*

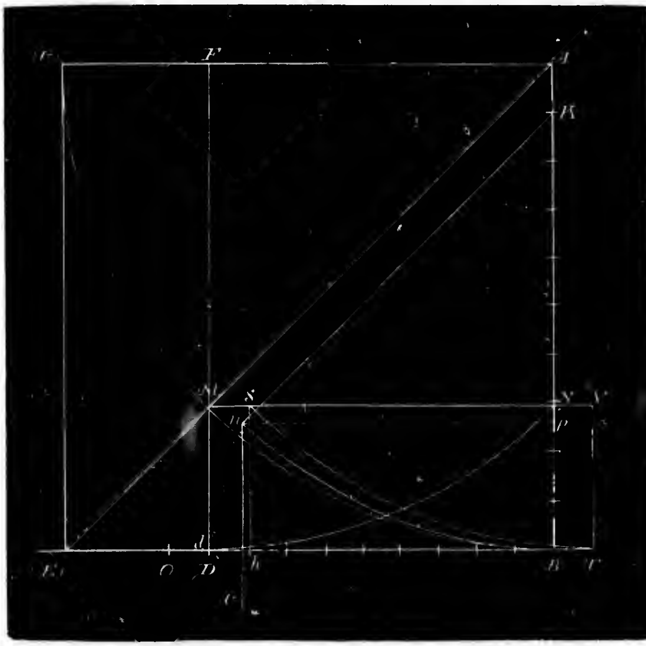
It immediately follows — because *d. O.* is one-tenth of *B. O.*, that *D. O.* (which is the same) is also the one-tenth of *B. O.*

(4.) Again; let it be assumed to be possible that *n.* advances to some point, *d.*, less distant than *D.* from *B.*, then must the ratio of the distance *B. O.*, to the distance *B. d.*, be greater than the ratio of *M. N.*, to *n. p.*, because the point *D.* is, by the assumption, in advance of the point *d.*, and the greater arc must advance from (its starting point) the point *D.* proportionally to the advance of the lesser arc (from its starting point) the point *C.* Now the distance *B. O.* contains the distance *B. C.*, together with the advance of the two arcs taken together—to wit, the advance of *M.* from *D.*, together with the advance of *n.* from *C.*, and together with the difference, if there be any, between *d.*, the advanced place to which *n.* advances, and *D.* the place from which *M.* commences to advance—that is, if *d.* be less distant than *D.* from *C.*, *C. O.* includes the diff. *D. O.* together with the diff. *C. d.* and together with the diff. *d. D.*; (thus . . . *C. O.* includes the advance of *n.* through *C. d.*, of *M.* through $C.\ d. + \frac{C.\ d.}{9}$ and the distance *d. D.*) But *B. C.* and *C. d.* taken together equal *B. d.* which is equal to *n. p.* And *M. N.* is to *n. p.* as ten to nine. Therefore $B.\ d. + C.\ d. + \frac{C.\ d.}{9}$ is to *B. C.* + *C. d*, as ten to nine.

By the assumption *B. O.* contains (*B. C* + *C. O.*) $B.\ d. + C.\ d. + \frac{C.\ d.}{9} + d.\ D.$, and therefore the ratio of *B. O.* to *B. d.* is greater than the ratio of *M. N.* to *n. p.* by the distance *d. D.* Now, it is manifestly impossible that *B. M.* can increase by advancing in a greater ratio of proportion to the increase of *B. n.* than the proportion of the sine *M. N.* to the sine *n. p.*, because *B. M.* and *B. n.* are similar arcs; therefore the distance *C. d.* cannot be less than the distance *C. D.* By similar reasoning it may be shown that the distance *C. d.* cannot be greater than *C.D.*, because then the advance of the arc *B. M.* would be proportional to the advance of the arc *B. n.* in a ratio of proportion less than the ratio of the sine *M. N.*, to the sine *n. p.*, to suppose which would be absurd. Wherefore it is demonstrated that the point of contact of the lesser arc *B. n.* on the straight line *B. E.*, indicated assumptively by *d.*, is the same point *D.* at the extremity of the perpendicular *M. D.*

Q. E. D.

(5.) Again—Fig. 3 (repeats the construction of Fig. 1.) Produce *D. M.* through *M.*, and make *F. D.* equal to *A. B.* With centre *F.* and radius *F. D.* describe the arc *D. N.* equal and similar to the arc *B. M.*, produce the straight line *E. B.* through *B.* indefinitely, and upon the line so produced roll the arc *D. N.* from *D.* in the direction *D. B.* until the extremity *N.* of the arc becomes in contact upon the line at *T.* Now the distance *B. T.* is manifestly equal to the distance *D. O.*, because if the arc be rolled upon the line to the extremity of the arc opposite to the point of contact, the advance in that direction must necessarily be equal to the advance in the opposite direction, if the arc be similarly rolled in the opposite direction; but the other extremity of the arc—to wit, the extremity which is in contact with the line at *D.*, must likewise advance (to *S.*) in the direction *D. B.*, a horizontal distance *D. h.* equal to the distance *B. T.*; manifestly the arc *T. S.*



must be equal to the arc *D. N.* (because both are the same arc in an altered position.) Also *D. h.*, which is equal to *D. O.*, has the same ratio to *C. d.* which *M. N.* has to *n. p.*—to wit, the ratio of ten to nine. And the arc *S. T.* has likewise the same ratio to the arc *B. n.*, and the line *B. T.* the same ratio to the line *C. d.*

Therefore *d. B.* + *B. T.* (= *d. T.*) = *B. d.* + *d. O.*

(= *B. O.*) Now *B. O.* contains *B. C.* + *C. D.* + *D. O.*, and *d. T.* contains *B. T.* + *B. C.* + *C. d.*, and since *B. T.* equals *D. O.*, *C. d.* must equal *C. D.* Wherefore it is demonstrated, &c.

Note.—This last method is given as furnishing an independent means of testing and verifying the preceding demonstrations.

Prop. B.—Theorem.

That if a straight line be drawn from the radius of a circle at right angles to the radius, and the line be drawn to the circumference of the circle, such that one-eighth part of the circle be contained between the point on the circumference intercepted by the straight line, and the point intercepted by the radius — then if the one eighth-part of the circle be divided into ten equal parts, the straight line so drawn shall contain nine equal parts, each of them equal to each of the ten equal parts contained in the one-eighth part of the circle.

Fig. 4. Let *M. N.* be the straight line drawn from the radius *A. B.* of the circle, of which circle the quadrant *B. M. F.* is bisected by the straight line in the point *M.* The straight line *M. N.* shall contain nine equal parts, each of them equal to each of ten equal parts contained in the half-quadrant *B. M.*

From *B.* at right angles to *A. B.* draw the straight line of indefinite length, *B. E.*; and, through *M.*, produce the straight line *N. M.* indefinitely. From *M.* draw the perpendicular *B. D.*, intercepting the line *B. E.* at *D.* Roll the arc *B. M.* upon the straight line from *B.* in the direction *B. E.*, and from the point *O.*,* where the terminal extremity, *M.*, of the arc becomes in contact, draw the perpendicular *O. C.* equal to *A. B.*, intersecting the production of *N. M.* in the point *R.* With centre *C.*,

* Theorem A.

and radius *C. O.*, describe the quadrant *O. P.* bisected by the production of *N. M.* in the point *S.* Divide the line *R. S.*, cut off from the production of *N. M.* by the arc, into nine equal parts at the points of equal division *b. c. d. e. f. g. h. i.*

Because the lines *N. B.*, *M D.*, *R. O.* are all of them perpendicular to the straight line *B. E.*, the line *M. N.* is equal to the line *B. D.*, and *R. M.* is equal to *O. D.* Now it has been demonstrated (Prop. A.) that if the arc *B. M.* be rolled upon the straight line *B. E.* until the terminal extremity *M.* of the arc becomes in contact on the line, the magnitude *B. O.*, cut off from the line *B. E.* by the point of contact *O.* is greater than the magnitude *B. D.*, cut off by the perpendicular *M. D.*, in the ratio of ten to nine. But the arc-length *M. F.* is equal to the arc-length *B. M.*, and *R. S.* is equal to *B. D.*, and *M. R.* is equal to *D. O.*, therefore *M. S.* is equal to the arc-length of *M. F.* And since *M. R.* is one-tenth of *M. S.* and *R. S.* is divided into nine equal parts, *M. R.* is equal to one of those equal parts. But the line *M. S.* is equal to the arc-length of *M. F.*, therefore, *M. F.* contains ten equal parts, each of them equal to each of the equal parts contained in the line *R. S.* Now the line *R. S.* is equal to the line *M. N.*, and the arc *M. F.* is equal to the arc *B. M.* *Wherefore it is demonstrated* that if a straight line drawn from the radius of a circle, &c. Q. E. D.

Corollary.—*Hence*—since if the line, drawn from the radius, intercepting the circle on the one side, be drawn from the circle through the radius until it intercepts the circle on the other side of the radius, the part of the line on the one side of the radius is necessarily equal to the part of the line on the other side, and the fraction of the circle, contained between the point of interception of the line on the one side and the extremity of the radius, is equal to the similar fraction of the circle cut off between the extremity of the radius and the point of interception of the line on the other side—*it follows that*, if a straight line be drawn from a circle through the radius, at right angles to the radius, until the line intercept the circle at the opposite side of the radius, and the line be so drawn that the part of the circle next the extremity of the radius be the one-fourth part of the circle, then—if the straight line so drawn be divided into nine equal parts, the fourth part of the circle shall contain ten equal parts each of them equal to each of the nine equal parts contained by the line, and the whole circle shall contain forty such equal parts.

Prop. C.—Problem.

Requisition.—It is required upon a given straight line to describe an arc, such that the arc shall be equal in length to the given straight line, and shall contain a definite fraction of a circle of definite magnitude.

Definition.—Fig. 5. Let *B. O.* be the given straight line: it is required upon *B. O.*, to describe a definite arc, equal in length to *B. O.*

Construction.—Divide the line *B. O.* into ten equal parts at the points of equal division *a. b. c. d. e. f. g. h. i.* From the point *B.*, at the extremity of the straight line, draw the perpendicular of indefinite length *B. ë.*, and from the point *i.*, on the straight line *B. O.*, draw the perpendicular of indefinite length *i. H.* On the line *B. ë.* take *B. d'.*, equal to *B. i.*, and join *i. d'.* From *B.* at right angles to *i. d'.*, draw the line of indefinite length *B. E.*, and on the line *B. E.* take *B. C.* equal to *B. i.* Produce *B. O.* indefinitely through *O.* and, through the point *C.*, draw a line at right angles to *B. E.*, intercepting *B. ë.* at *A.* and intercepting the production of *B. O.* at *D.* With centre *A.* and radius *A. B.* describe the quadrant *B. M. F.* bisected at *M.* by the perpendicular *i. H.* From the point *M.*, and at right angles to *A. B.*, draw *M. N.*, intercepting *A. B.* at *N.*

Result.—The arc *B. M.* described with the definite radius *A. B.* shall be the required arc, equal in length to the given straight line *B. O.*

Demonstration.—Because the arc *B. M. F.* is a quadrant, and the point *M.* bisects the quadrant, therefore the arc *B. M.* contains the eighth part of a circle.

Now *M. N.* is the sine of the arc *B. M.* and *M. N.* is manifestly equal to *B. i.* on the line *B. O.;* but *B. i.* contains nine of the ten equal divisional parts of the line *B. O.*, and it has been demonstrated (Prop. B.) that, if the sine of an arc, containing the eighth part of a circle, be divided into nine equal parts, the arc contains ten equal parts, each of them equal to each of the nine equal parts contained by the sine. Therefore *B. M* is equal to *B. O.*

Wherefore the arc *B. M.* containing the eighth part of a circle has been described with the definite radius *A. B.* upon the given straight line *B. O.*, and it has been shown to be equal in length to the given straight line, as was required to be done.

Corollary 1. — *To describe upon the given straight line a quadrant equal in length to the given straight line.*

(Fig. 5 R.) Bisect the radius *A. B.* at the point *a.*, and from *a.* at right angles to *A. B.* draw the line of indefinite length *a. g.* With centre *a.* and radius *a. B.* describe the quadrant *B. c.*

Now because *a. B.* is the one half of *A. B.*, the quadrant *B. c.* is the fourth part of a circle of half the magnitude of the circle of which the arc *B. M.* (Fig. 5.) is the one eighth part; consequently the quadrant *B. c.* is equal in length to the arc *B. M.*; and it has been demonstrated that the arc *B. M.* is equal in length to the line *B. O.*, which is the given straight line. Wherefore the quadrant *B. c.* is also equal to the given straight line.

Corollary 2. — To describe upon the given straight line an arc containing the sixteenth part of a circle such that the arc shall be equal in length to the given straight line.

(Fig. $5^{R.}$) Produce *A. B.* through *A.*, and from the production take *R. B.* twice the length of *A. B.* From *R.* through *F.* draw *R. F. G.* of indefinite length. With centre *R.* and radius *R. B.* describe the arc *B. P.*, intercepting *R. F. G.* at *P.* Bisect the arc *B. P.* at *Q.* Now, since the arc *B. Q.* is the one half of the arc *B. P.*, which contains the eighth part of the circle; and, because the radius *R. B.* is twice the length of the radius *A. B.*; the arc *B. Q.* contains the sixteenth part of a circle twice the magnitude of the circle of which the arc *B. M.* contains the eighth part. Therefore the arc *B. Q.* is equal in length to the arc *B. M.*, and it has been shown that the arc *B. M.* is equal in length to the given straight line *B. O.* Wherefore the arc *B. Q.*, which contains the sixteenth part of a circle, is also equal to the given straight line.

(*Note.*)—It would be more strictly methodical to give these two corollaries as separate propositions, formally supported by demonstration that the magnitudes of circles are in the same ratio one to another as the ratios one to another of their respective radii. But the theorem to which such demonstration belongs is one of the fundamental facts upon which the science of trigonometry is indirectly based. It seems, therefore, preferable to avoid complicating the immediate subject, and to give these secondary demonstrations in the less formal but more concise manner in which they are here presented.

This same Note has reference, also, to the corollary of Prop. A. which in strictness requires formal definition of the *sine* of an arc; but the approved trigonometrical nomenclature of the lines belonging to the circle is quite definite, and we may, for the present purpose, assume the definition to be contained in the expression.

PROP. D.—PROBLEM.

Requisition.—Through a given point in the circumference of a given circle it is required to draw a straight line tangential to the circle, which shall be equal in length to the given circle.

Definition.—Fig. 6.—Let *B. E. G. H.* be the given circle, and let *B.* be the given point,—it is required to draw a straight line through the point *B.* which shall be equal in length to the circle *B. E. G. H.*

Construction.—Through the point *B.* draw the line *d. B. a.* of indefinite length, and from *B.* perpendicular to *d. B. a.* draw the diameter *B. G.;* bisect *B. G.* at *C,* the centre of the circle; and, through *C.* at right angles to *B. G.,* draw the diameter *E. H.* Bisect the quadrant *B. E.* at *M.;* and, at right angles to *B. C.,* draw the line *M. O.,* intersecting *B. C.* at *N.,* and intercepting the circle and bisecting the quadrant *B. H.* at *O.*

Divide the chord *M. O.* of the quadrant *M. B. O.* into eighteen equal parts. On the line *d. B. a.* take, from *B.* in the direction *B. d., B. f.* containing ten equal parts, each of them equal to each of the eighteen equal parts contained in the chord *M. O.* (Theor. B.,) and on the same line *d. B. a.,* from *B.* in the opposite direction, take *B.* c. containing likewise ten equal parts, each of them equal to each of the eighteen equal parts contained in the chord *M. O.* Increase *B. f.* through *f.* on the line *B. d.,* and take *B. D.* equal to four times the length of *B. f.,* and on the same line in the opposite direction increase *B. e.* through *e.,* and take *B. A.* equal to *B. D.*

Result.—The straight line *D. B. A.* contained between the points *D.* and *A.* shall be the required straight line equal in length to the circle *B. E. G. H.*

Demonstration.—Because the point *M.* bisects the quadrant *B. M. E.,* and because the sine *M. N.* of the arc *B. M.* is divided into nine equal parts, therefore *B. f.*, part of the line *d. B. a.*, containing ten equal parts each of them equal to each of the nine equal parts contained in the sine *M. N.*, is equal in length to the arc *B. M.* (Prop. B.) But the part *B. e.* of the same line *d. B. a.* is equal to the part *B. f.*, and is similarly related to the arc *B. O.*, which is similar and equal to the arc *B. M.*; therefore *f. B. e.*, the two parts of the line taken together, is equal to the quadrant *M. B. O.* containing the two arcs taken together. Now the line *D. B. A.* is (by the construction) equal to the part thereof *f. B. e.*, taken four times together, and the circle *B. E. G. H.* is equal to the quadrant *M. B. O.* taken four times together; therefore the line *D. B. A.* is equal to the circle *B. E. G. H.*

Wherefore the line *D. B. A.*, drawn through the given point *B.* in the circumference of the circle, and tangential to the circle, is equal in length to the circle *B. E. G. H.*

Q. E. D.

Corollary.—Hence it follows that, if the chord of a quadrant be divided into (9) eighteen equal parts, a straight line containing (10) twenty equal parts, each of them equal to each of the (9) eighteen equal parts contained in the chord, is equal in length to the quadrant. Wherefore if a square be inscribed in a circle the ratio of the inscribed square to the circle is the ratio of nine to ten.

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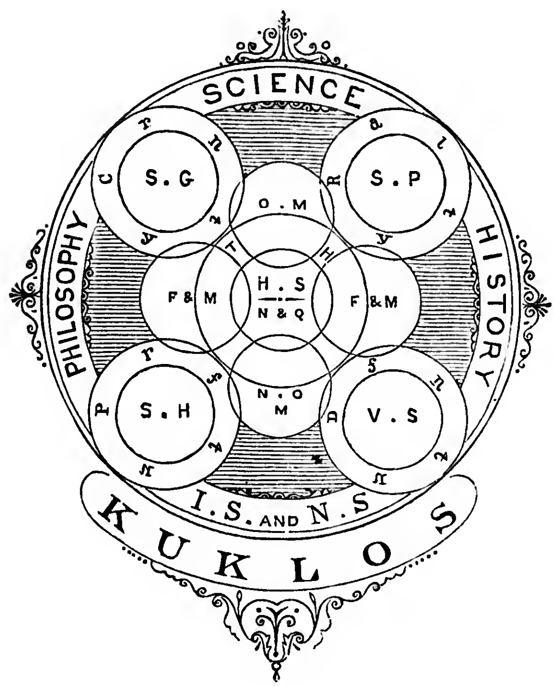

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THE APPENDIX. (a)

The question to be here briefly noticed is, whether 'Euclid's Elements,' having regard to the apparent intention of the author (or authors) of that work, should be considered a treatise on one peculiar division of knowledge (one science) only—that is to say, a treatise on the inter-relation of the subjects of the science and of the laws which govern and belong to that peculiar (so-called) science of geometry; or whether the work should be considered, in regard to its purpose, a practical treatise on applied reasoning—teaching, by illustration, the correct mode of building-up (compounding) science from its elements.

We will briefly examine two important propositions of those belonging to the 'Elements' as to any evidence they may afford in this respect.

The first prop. of the third book: 'To find the centre of a circle.' It is at once evident that this prop., because of the use made by Euclid of the circle, must be considered, if regarded as one of *the Elements* of a peculiar science (geometry), as a prop. of great relative importance. The plan of the book, assuming the purpose of that plan to be a treatise on the science only, would call for a solution of this prop. in such a form as to constitute it a primary or fundamental prop., upon which secondary propositions and corollaries could be based and shown to be directly dependant. One or more definitions or axioms might obviously be provided to furnish any necessary support for a concise and comprehensive solution. At least, in regard to such supposed purpose, a direct character for the solution would suggest itself as almost imperative. As it stands, the character of the solution is indirect, and may be termed negative; it gets constructively at the answer to the requisition by a very simple operation, but, as a reasonable and subjective proceeding, this operation is only supported by the negative results of an exhaustive process. The question naturally suggests itself whether the construction has been, in the first place, merely the result of a fortunate guess. If it is based upon and has been compounded from elements previously verified, why is it not shown to be so? As it stands, we cannot take the result, and by inverting the order of the reasoning, get at the construction as subjective. No reason for performing the (given) constructive operation is apparent.

Again, the corollary is stated to be manifest; but, it is certainly not manifest as a corollary to the negative solution of the prop. If the corollary is manifest, it is manifest independently (in itself), and the solution of the prop. might be very well based upon the corollary. By exhibiting the fact stated in the corollary, in the form of a demonstrated theorem, the problem (Prop. I.) might be directly and positively solved in a few lines, and Prop. III. (Theorem) might become unnecessary.

Now, if we suppose the primary purpose of the work was to illustrate the applied method, and that the particular science (chosen for the purpose of illustration) was considered quite subordinate to that purpose, it is much easier to understand how such a form of solution might suggest itself as desirable in this and in some other similar instances. The negative exhaustive process illustrated in this prop. is undoubtedly in some cases of much value, and it may be considered, in some sense, a distinct kind of reasoning; hence, having regard to the primary purpose, it might be thought desirable to exhibit it as illustrating a proposition which from its fundamental character and importance would be more likely to attract attention and interest the mind.

We cannot consider this as furnishing an altogether satisfactory explanation of the form here given to the solution, but it seems to be much more intelligible when so considered, viz., as primarily intended to illustrate the applied method of reasoning.

The second proposition of the 12th book is a similar and still more striking instance, which appears to strengthen and confirm the explanation just suggested in the case of Prop. I. Book III. This proposition (Prop. II. Book XII.) 'Circles are to one another as the squares of their diameters,' is also evidently a proposition of a primary and important character because defining the nature of a fundamental relation between circles, and therefore here again we should expect a direct, positive, and simple treatment of the case, and, on the contrary, we again find the indirect negative treatment exhibited in a long solution of considerable complexity; the only characteristic difference between the treatment in this case and that of Prop. I. Book III., is that in this, the construction, like the solution to which it belongs, is complex and only indirectly related to the proposition, whereas, in the former instance, the construction is simple and directly related to its proposition. But the particular mode of the reasoning, for the illustration of which this proposition appears to have been selected, is in itself of a very refined and instructive character. In the kindred science of 'Number and Quantity,' calculations of great importance are derived from and based upon the quantitive equivalent of this proposition: for example —'Legendre's Numerical and Trigonometrical Propositions' concerning the quantitive relations of polygons and the perimeters of circles.* It may be also useful to

* We may remark here the confusion liable to arise from using the same expression 'geometrical' which is applied to the propositions of the science of Form and Magnitude, for those of the science of Number and Quantity. To these last some of Legendre's propositions belong exclusively, whilst others are of a hybrid character. It is true these sciences have a common boundary where they approach each other very nearly, but it is a stumbling-block in the way of the student to find them thus mixed together without indication as to their distinct and different characters.

point out that this is one of the propositions in which Euclid defines, by illustration, the use of the hypothesis. In this instance, the hypothesis is assumed to be true, and is then made to temporarily occupy the place of, and serve as a fact, so that combination can be carried forward and the results tested by comparison. The terms in which this proposition is stated require particular notice, in regard to the sense in which the expression 'square of the diameter' is used; for, the very same expression is used in the kindred science of 'Number and Quantity,' and is used therein in an essentially different sense. Since, therefore, we are here, almost on the border land which connects these two sciences, there is much danger of the expression used in the sense belonging to the one science being mistaken for the same expression used in the sense belonging to the other science. The proposition under consideration is thus stated in the Elements of Euclid, 'Circles are to each other as the squares of their diameters.' The expression 'square of the diameter' here means (as used by Euclid) a square of which the diameter of the circle is one of the four equal sides. The statement would therefore have the same (equivalent) meaning if the word 'square' be left out and we read simply 'circles are to each other as their diameters.' To demonstrate the one statement is to demonstrate the other; because if two magnitudes be proportional one to the other, equimultiples of those magnitudes are proportional in the same ratio; and, if equimultiples of those magnititudes be taken from them respectively, the remainders are likewise proportional in the same ratio. But in the applied science of 'Number and Quantity,' the same expression, 'square of the diameter,' has a different meaning.

To correctly appreciate the nature of the difference it is necessary to observe that all 'Number' or 'Quantity' is relative. It has, always, reference to a standard of comparison * (an apparent exception to this is the number one, or unity, but here the number is itself equivalent to its standard of comparison). In trigonometry 'square of the diameter' means 'a quantity of magnitude (diameter) taken as many times as there are units contained in that quantity of magnitude (diameter).' An example will at once make the character of this difference quite obvious. Let the two diameters be proportional one to the other in the ratio of '*four*' to '*two*'—then the square of their diameters in the sense intended by Euclid will be—*four* taken *four* times, and *two* taken *four* times; that is, *sixteen* to *eight*, magnitudes proportional in the same ratio as before. But in the numerical or quantitive sense we get—*four* multiplied by *four* gives sixteen ($4 \times 4 = 16$); and—*two* multiplied by *two* gives *four* ($2 \times 2 = 4$). The numerical proportion becomes therefore 16 : 4, instead of 16 : 8;—the ratio of the proportion is no longer the same as before.

We have recently made public a notice of very grave errors in astronomical science, certain of which appear to have arisen from a fundamental misapprehension (non-appreciation) as to the relation of the semi-diameter (radial-distance) to the circumference of the circle—*e. g.*, the doctrine of the (supposed) law of equable areas. The question may suggest itself whether the use of the expression 'square of the diameter' with the two different meanings undistinguished, may not have assisted to prepare the deceptive foundation for that superstructure of unsound knowledge which has been since built upon it.

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* Which is commonly termed a 'unit.'

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COPIES OF THE DOCUMENTS REFERRED TO IN THE PREFACE.

MONTREAL, 20th December, 1872.

To G. B. AIRY, ESQ.,

President of the Royal Society, London.

SIR,—I have the honor herewith to send you papers containing the solution to that geometrical problem (commonly or vulgarly known as 'Squaring the Circle'), which requires 'that a straight line shall be drawn equal in length to a given arc of a circle of definite magnitude,' or 'that a definite arc belonging to a circle of definite magnitude shall be described equal in length to a given straight line,' and which requires that the demonstration (proving that every condition of the requisition has been fulfilled,) shall be in strict accordance with the system of reasoning laid down in that work generally known, and recognized by mathematicians, as the 'Elements of Euclid,' or 'Euclid's Geometry.'

The papers are submitted to you for the purpose of their being examined by those whose mathematical knowledge entitles them to express an authoritative judgment as to the merits of the case thus laid before the Royal Society; and, as the subject is one of great public importance, I have to request that as soon as the necessary examination has been made, (*i.e.*, within reasonable time and without unnecessary delay,) the correctness of my demonstration and the result of that demonstration shall be publicly acknowledged and admitted as established fact; or, otherwise, if disputed—then I require, which I submit that I have an unquestionable right to do, that the objection or objections shall be made strictly in accordance with the same system of reasoning—namely, that laid down and illustrated by Euclid,—in order that I may have the opportunity to meet such objections and, if I am able, to disprove their validity, and so to put myself in a position to insist upon my demonstration being publicly acknowledged as sound and true.

Yours respectfully,

JOHN HARRIS.

(NOTE.—The above copy is taken from a rough draft of the letter, and may be possibly not a strictly accurate copy of the letter forwarded to the President of the R. S.)

The letter accompanying the papers sent to McGill College, was of the same tenor, and in form substantially the same, as the foregoing.

ROYAL OBSERVATORY, GREENWICH,
LONDON, S.E.,

February 4, 1873.

SIR,

I have to acknowledge your letter of December 20, addressed to me as President of the Royal Society, (which has reached me through the hands of John Marshall, Esq.,) inclosing papers which are supposed to contain an investigation on trustworthy principles of the problem of "Squaring the Circle."

You are perhaps aware that the Foreign Academies, in general, by express statutes, refuse to receive communications on this subject. I do not know that the Royal Society of London is prevented by statute from receiving them, but I know that its practice is unvarying; and, so far as my private judgment is concerned, I approve of that practice. I must, therefore, decline to present the papers to the Royal Society.

I cannot myself give any time to their examination; nor should I think it right to force them on the attention of others. I have therefore thought it best at once to return them entire to Mr. Marshall.

I am, Sir,

Your obedient servant,

(Signed,) G. B. AIRY.

JNO. HARRIS, ESQ.

MONTREAL, 21st December, 1872.

TO MR. JOHN HARRIS,

SIR,

I herewith return your papers relating to the problem known as Squaring the Circle. I have not received from any of the professors in McGill College any observations on the subject.

Your obedient servant,

(Signed,) CHAS. D. DAY.

ENWICH,

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