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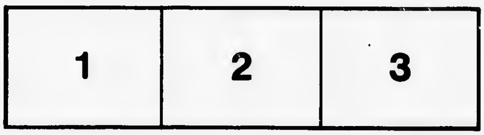

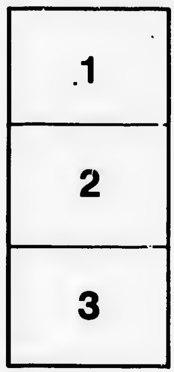

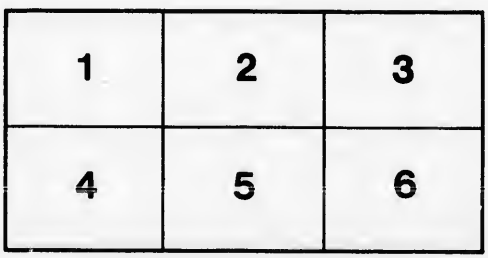

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To be read Thursday, March 13th.

COLUMNS.†

By Charles F. Findlay, M. Inst. C. E.

In the whole range of subjects with which the art of engineering is concerned, there are few in which theory has contributed less to the advance of intelligent practice than in this. The reason cannot be found in any want of importance in the subject itself, for, of the materials used in construction, a much greater part is used in compression than in tension, and a very considerable part is used in the form of columns or struts. Until recently too, the experiments on compressive strength were very few in number and narrow in range, compared with the multitude of tensile experiments that have been made. Even now, many of the laboratories with expensive and elaborate apparatus for tensile tests have no means of making compressive tests. Of late years, however, American experimenters have done a great deal to remedy this state of things, among whom Mr. Christie, Mr. Bouscaren, and Clarke, Reeves & Co. may be mentioned in particular.

What is now wanted even more than experiments, is a theory of the subject which will enable us to interpret their results in a general way, and also to indicate the kind of experiments desirable. We shall therefore attempt here to give a treatment of the subject somewhat more adequate than is usually found in text books.

A column differs from a beam or girder in the nature of the forces it is intended to resist, a beam being used to resist forces transverse to its length, a column to resist forces directed between its two ends. Of course, the same material may be used to resist forces of both kinds at once, but for the sake of clearness we shall neglect transverse forces altogether in treating of the column. The leading feature in the behaviour of a loaded column is that it bends, the deflection constantly increasing with the load. The stress which might at first be almost uniformly distributed in the column, becomes thus more and more unequal at the outer and inner edges as the load increases. With a tensile stress on the other hand, although a small flexure undoubtedly must result from the same causes that give rise to flexure of columns, the direction of the flexure is opposite, and therefore tends to produce a more uniform distribution of stress. The difference between the two cases may be illustrated by that between stable and unstable equilibrium.

The equation which forms the basis of all investigations into the flexure of beams and columns is

$$(1) \qquad M = EI\left(\frac{1}{R} - \frac{1}{R_0}\right)^*$$

where M moment of external forces at a cross section of the beam or column, whose moment of inertia about an axis through its centre of gravity at right angles to the plane of flexure is I;

R = radius of curvature of axis of beam or column at same cross section, and R_0 = initial value of R when M = 0

E = modulus of elasticity of material.

By the axis of a column we here and hereafter mean the axis of figure and not the neutral axis (which for columns generally lies at a considerable distance outside the column).

It will be well here to call attention to the assumptions on which (1) rests, because it has often been applied to circumstances in which it

† This investigation coincides in certain parts with papers by Professors Ayrton and Perry, in the *Engineer* (London), December 10th and 24th, 1886, and by Professor Krohn in the Proc. Am. Soc. C.E., 1887.

* This equation is usually given as $M = \frac{EI}{R}$, thus assuming the initial form of the axis to be a straight line. For a beam, although the assumption cannot be true in fact, it does not affect the distribution of stress in the beam. With a column, however, it is of vital importance, as we shall see.

is not true, and most misleading conclusions have been drawn in consequence. Equation (1) supposes:

First, that every plane section of the beam or column normal to the axis in the unstrained state remains plane and normal to the axis throughout. This is probably approximately true for homogeneous beams or columns formed out of one mass of uniform material, as, for instance, a log of timber, a cast iron column, or a rolled joist, but it is very doubtful how far it can be relied on for a framework built up of a number of separate pieces.

Secondly, that the value of E is constant throughout the cross section. This is approximately true in most cases, and it can be shown that the variations of E occurring in practice cannot largely affect the strength of iron columns, though some writers have attached an exaggerated importance to this point.

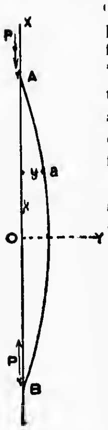
Thirdly, that the fibres of the beam or column are all strained in accordance with Hooke's Law. As soon as the most strained fibres pass the limit of elasticity, equation (1) ceases to express the condition of equilibrium, and all conclusions based on it cease to have an application. When therefore, as is sometimes done, a comparison is made between the stress in the extreme fibres of a beam as deduced from equation (1), when M is the moment causing fracture, and the direct tensile stress the same fibres will bear in a testing machine, it should cause no surprise that there is an enormous discrepancy. Equation (1) is based upon a supposed distribution of stress throughout the various parts of the cross section which is utterly wide of the truth when the moment of flexure is such as to cause rupture.*

Equation (1) being the equation of moments at any section of a column, we have also an equation between the sum of the applied forces parallel to the axis and the sum of the internal stresses at the same cross section which will be found to be:

$$(2) \qquad p = Ey_1\left(\frac{1}{R}-\frac{1}{R_0}\right) \text{ whence by (1) } y_1 = p\frac{1}{M}$$

where p is the total applied force per unit of area of cross section, and y_1 is the distance of the neutral axis from the axis of figure at the same section.

In order to determine M we must know the form of the bent column under a given load, and in order to ascertain that, equation (1) must be solved. Now the essential difference between a beam and a column as regards this solution is, that for a beam M is independent of the deflection, while for a column it depends directly on the deflection. In other words, if x and y be rectangular co-ordinates defining the position of any point on the axis of the column, x being measured parallel to the length of the column, M is for beams a function of x only, while for columns it is a function of y. This entirely alters the character of the differential equation to be solved, and makes it quite inadmissible to assume, as is frequently done, that the deflections of a column will be governed by the same law as are deduced from equation (1) for beams.



Let AQB be the axis of a column bending under the action of a load P. Take AB as axis of x, its middle point O as origin, and let AB $= l$. Since equilibrium of the column as a whole requires that the forces acting on its two ends should be reducible to two equal and opposite forces P, let the line of action of these forces be at a distance a from O and we shall suppose it parallel to AB, because it is found that the principal effect in producing flexure is due to the mean value of the eccentricities of the loads at the two ends. Of course a is small since the intention is supposed to be to load the column centrally so far as possible. Let a be the cross section of the column at Q, and let $pa = $P. Let y_0 be the initial value of y for the unstrained column. We shall suppose $x_0 = x$, thereby neglecting the elastic shortening of the column. Let $\rho =$ radius of gyration at Q, $= \sqrt{\frac{I}{a}}$; then since $\frac{1}{R} = \frac{d^2y}{dx^2}$ as long as R is very large and similarly $\frac{1}{R_0} = \frac{d^2y_0}{dx^2}$ and since M $=$ $pa\,(y+a)$, (1) becomes

$$-\,p\,(y+a) = E\rho^2\left(\frac{d^2y}{dx^2} - \frac{d^2y_0}{dx^2}\right)$$

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*For a full discussion of the question of the relation of transverse to direct strength of iron, v. Etudes sur l'Emploi du fer et de l'acier, by Mr. Considère, Paris, 1886.

write for brevity K^2 for $\frac{p}{E\rho^2}$ and suppose K to be constant throughout the length of the column, *i.e.*, suppose the column to be uniform in section and in material.

Then $\frac{d^2y}{dx^2} - \frac{d^2y_0}{dx^2} = -K^2(y+a)$

To solve this equation we must know what $\frac{d^2y_0}{dx^2}$ is in terms of x, or, in other words, what the unstrained form of the axis is. It being supposed that the column is initially as straight as the ordinary conditions of manufacture permit, the actual form of the axis will be an irregular curve departing very little from the line AB. We shall assume that the curve is the curve of sines $y_0 = \delta_0 \cos \frac{\pi x}{l}$. This assumption can be justified by the fact that the curve, whatever its form, can be expressed by equating y_0 to an infinite series proceeding by sines and cosines of multiples of $\frac{\pi x}{l}$ of which $\delta_0 \cos \frac{\pi x}{l}$ may be considered the first term. If this be done, it will be found that in ordinary cases the effect of the successive terms of the series on the resulting solution rapidly decreases.

We have then $\frac{d^2y_0}{dx^2} = -\delta_0 \frac{\pi^2}{l^2} \cos \frac{\pi x}{l}$

and $\frac{d^2y}{dx^2} = -K^2(y+a) - \delta_0 \frac{\pi^2}{l^2} \cos \frac{\pi x}{l}$

The solution of this equation with the conditions that $y = o$ when $x = \frac{l}{2}$ or $-\frac{l}{2}$ is:

$$(3) \quad y + a = a \frac{\cos kx}{\cos \frac{kl}{2}} + \frac{\delta_0}{1 - \frac{k^2 l^2}{\pi^2}} \cos \frac{\pi x}{l}$$

Before proceeding to obtain from this equation the stresses in the column, it will be well briefly to point out the relation it bears to the theory of flexure given in most text books, and known by the name of Euler's theory from the name of its author, who published it in the middle of last century. In that theory δ_0 and a are neglected, so that the differential equation is $\frac{d^2y}{dx^2} = -K^2y$. The direct solution of this is $y = A \cos kx + B \sin kx$, and the terminal conditions require $kl = \pi$ and $B = o$, while A is indeterminate. In other words $p = \pi^2 E \left(\frac{\rho}{l}\right)^2$, and p having that value y may be anything. The same result may be deduced from (3), for if a and δ_0 vanish, y must always be zero except when $Kl = \pi$, and in that case $\frac{a}{\cos \frac{kl}{2}}$ is an indeterminate fraction for which an arbitrary constant A may be substituted.

This theory is not only inconsistent with facts, which might be expected from its assuming ideal conditions that cannot be realized in practice, but in the form given above, which is that usually published, it evidently contains some other serious error; for while it is quite conceivable that under the ideal conditions supposed, no flexure can occur until p reaches a certain limit, it is not conceivable that the flexure, when it does occur, should be independent of the load, so that the same load which first causes flexure will bend the column double. The error lies in neglecting the elastic shortening of the column. In the theory here given this is permissible, but in Euler's theory, a and δ_0 being neglected, a different order of small quantities is dealt with and the contraction of the column becomes important. If this be taken into account, and p_1 is the load for which flexure first becomes possible, or $\pi^2 E \left(\frac{\rho}{l}\right)^2$, then the deflection δ, for a given load p per sq. in. greater than p_1, is given by

$$\delta^2 = \frac{4E}{p} \rho^2 \left\{ \sqrt{\frac{p}{p_1}} - 1 \right\}$$

It must also be observed that although by Euler's theory flexure cannot take place until p reaches the value p_1, it does not follow that it will then occur. For every value of p greater than p_1, there will be two positions of equilibrium for the column, one straight and one curved. The former will be unstable, and the chord of the arc of the bent column will be shorter than the straight column under the same load by an amount $\frac{\pi^2 \delta^2}{4l}$.

The difference in results between Euler's theory and equation (3) is strikingly shown by observing that if either a or δ_0 has any finite value, however small, equation (3) shows that y is infinite for the value $Kl = \pi$, where Euler's theory first allows flexure to become possible. It should also be observed that for ordinary proportions of columns p_1 approaches the limit of elasticity of the column, so that at ordinary working loads no flexure would ever occur if Euler's theory applied. For instance, with wrought iron having an elastic limit of 12 tons, the length of a column would have to be more than 100 times its radius of gyration in order that any flexure should occur with a load of 12 tons per sq. in., and 100 times to bend with a load of 6 tons per sq. in. We have given so much space to Euler's theory only because it has been so widely spread in treatises on engineering subjects.

We will now return to equation (3), and, before going further, it becomes necessary to distinguish between the various ways in which the ends of the column may be constrained since the value of a depends upon the nature of the constraint.

i. The ends may rest in hinges in which the friction is insufficient to prevent rotation. In this a is independent of p.

ii. The ends may be rigidly attached to fixed objects, in which case a is a determinate function of p. (Flat-ended columns are another variety which corresponds mechanically neither with i or ii, and presents great elements of uncertainty. As it does not represent any column used in practice we shall not consider it here. The same remarks apply to round-ended columns).

iii. The ends may be rigidly attached to objects which themselves change in position as the load varies. This is a very common case and is that of the struts of a rivetted bridge, but we shall not deal with it, as it would require a very lengthy investigation. We only call attention to the fact that the conditions vary from those of case ii, and that a rivetted bridge strut has in consequence less strength than a similar strut whose ends are really fixed. Not only does it endure secondary stresses arising from the bending moments applied to its ends by the booms of the bridge, but its axis is thereby thrown out of line, by which its strength is impaired for bearing the direct load. We can refer students who wish to pursue the subject (which is one of great complexity) to Dr. Winkler's treatise, "Theorie der brucken," Vienna, 1886.

Considering case i, we must first remark that a hinged strut differs in no way from a fixed-end strut so long as the friction of the hinges is below its limiting value, or if the circumstances are such that the flexure takes place in some other plane than that normal to the axes of the hinges. Presuming that rotation of the hinges actually occurs, let h be the radius of the hinge, a_0 the distance of the centre of the hinge from the axis of the column, ϕ the angle of friction;

then $a = a_0 \pm h \sin\phi$, where the sign of $h \sin\phi$ depends on the values of a_0 and δ_0, the direction of friction being always opposite to the tendency to rotation at the end. For iron on iron without lubrication $\sin\phi =$ about $\frac{1}{10}$, and $h \sin\phi$ will usually be much greater than a_0.

In equation (3) it is permissible to expand in powers of kl and neglect the higher powers because $kl = \frac{l}{\rho}\sqrt{\frac{p}{E}}$ and must always be small for such values of p as would constitute a safe working load. This may be verified by calculating the values of y direct from equation (3) for certain cases, and comparing them with the values given by the approximation. If this expansion be made, we obtain

$$(4)\quad y + a = a\left(1 + \frac{k^2 l^2}{8}\right)\cos kx + \delta_0\left(1 + \frac{k^2 l^2}{\pi^2}\right)\cos\frac{\pi x}{l}$$

$$\text{when } x = 0,\ y = \delta_0 + k^2 l^2\left(\frac{a}{8} + \frac{\delta_0}{\pi^2}\right)$$

so that the central deflection is $\frac{p}{E}\left(\frac{l}{\rho}\right)^2\left(\frac{a}{8} + \frac{\delta_0}{\pi^2}\right)$

Let r be the distance of the innermost or most strained fibres of the column from the axis and f the stress per sq. in. in them, then, so long as Hooke's Law holds, the moment of resistance of the column may be expressed by $\frac{a\rho^2}{r}(f - p)$ which must therefore be equal to $ap\,(y + a)$.

Therefore $\frac{\rho^2}{r}(f - p) = p\,(y + a)$.

f is greatest where y is greatest, *i.e.* at the middle of the column. Substitute from (4) for $y + a$ and write b for $\frac{a + \delta_0}{r}$ and c for $\frac{1}{r}\left(\frac{a}{8} + \frac{\delta_0}{\pi^2}\right)$. We have then $\frac{\rho^2}{r}(f - p) = p\left\{br + cr\frac{p}{E}\left(\frac{l}{\rho}\right)^2\right\}$

The reason for introducing r into the values of b and c is that a and δ_0 are likely to vary roughly with the size of the column. We have seen that the principal part of a generally will arise from friction and be proportional to the radius of the hinge, and δ_0 is likely to increase with the length of the column with respect to which the diameter is generally fixed. At any rate $\frac{\delta_0}{r}$ will more nearly be independent of the dimensions of any particular column than δ_0 simply.

Rearranging the terms, $f = p\left\{1 + \left(\frac{r}{\rho}\right)^2\left(b + \frac{cp}{E}\left(\frac{l}{\rho}\right)^2\right)\right\}$

This is a quadratic to obtain p in terms of f, but b and c being small, we may, as a first approximation, write f for p in the last term inside the bracket, the error so committed being in the direction of safety, and we thus obtain

$$(5)\quad p = \frac{f}{1 + \left(\frac{r}{\rho}\right)^2\left\{b + \frac{cf}{E}\left(\frac{l}{\rho}\right)^2\right\}}$$

Case ii. Fixed ends.—Here the terminal conditions are that $\frac{dy}{dx}$ when $x = \frac{l}{2}$ or $-\frac{l}{2}$, must be constant, whatever the value of p may be. If the same assumption as before with regard to the initial form of the axis be made, these conditions give

$$a = -\delta_0 \frac{\frac{kl}{\pi}}{1 - \frac{k^2 l^2}{\pi^2}} \cot \frac{kl}{2}$$

or expanding in powers of kl as before $a = -\frac{2\delta_0}{\pi} - \delta_0 k^2 l^2 \frac{12 - \pi^2}{6\pi^3}$

The maximum bending moment is in this case at the end of the column and equal to $p\,a$. Proceeding as before to equate $p\,a$ to $\frac{\rho^2}{r}(f - p)$ we obtain a result of exactly the same form as (5), but in this case $b = \frac{2}{\pi}\frac{\delta_0}{r}$ or nearly $\frac{2}{3}\frac{\delta_0}{r}$ and $c = \frac{12 - \pi^2}{6\pi^3}\frac{\delta_0}{r}$ = nearly $\frac{1}{87}\frac{\delta_0}{r}$. The central deflection will be found to be $\frac{12 - \pi^2}{12\pi^2}\delta_0 k^2 l^2$ or nearly

$$\frac{3}{40}\delta_0 \frac{p}{E}\left(\frac{l}{\rho}\right)^2$$

It might, however, perhaps be a fairer assumption for a fixed-end column to suppose the initial form of the axis to be the reversed curve $y_0 = \frac{\delta_0}{2}\left(1 + \cos\frac{2\pi x}{l}\right)$, having the same central ordinate δ_0 but tangent to the line AB at both ends. On this assumption a will be found to be equal to $\frac{\delta_0}{2}\left(1 + \frac{k^2 l^2}{4\pi^2}\right)$

and the equation to the curve of the axis will be

$$y = \frac{\delta_0}{2}\left(1 + \frac{k^2 l^2}{4\pi^2}\right)\left(1 + \cos\frac{2\pi x}{l}\right)$$

giving a central deflection due to the load of $\delta_0 \frac{k^2 l^2}{4\pi^2}$ or nearly $\frac{\delta_0}{40}\frac{p}{E}\left(\frac{l}{c}\right)$. The relation between f and p is of the form of (5) as before, only that now $b = \frac{1}{2}\frac{\delta_0}{r}$ and $c = \frac{1}{8\pi^2}\frac{\delta_0}{r}$ = nearly $\frac{1}{80}\frac{\delta_0}{r}$

The following conclusions may be drawn from the investigation which has resulted in this formula.

First, the actual strength of any column depends partly on known facts with regard to its dimensions, material, etc., and partly on accidental circumstances which can neither be predicted nor observed. This is true in some measure of all constructions, but with columns these accidental circumstances do not merely require corrections in the calculations of strength that would otherwise be true, but are governing factors in the calculations themselves. It follows then that any formula such as (5), supposing the constants determined, should express, not the strength which may be expected from a particular column of certain dimensions and material, but the *minimum* strength which experiments show may fairly be expected with ordinary care and skill in manufacture. The strength of a number of columns apparently alike and loaded alike may be expected to vary widely without apparent cause, and a formula for practical use should express the minimum and not the average of the results given by a very wide and numerous range of experiments.

Secondly, experiments on the crippling or destruction of columns cannot be expected to give coherent results when applied to the determination of the constants in any formula such as (5). The reasons for this have been fully indicated already. The constants can only properly be determined by experiments within the limits for which the column is perfectly elastic, and unfortunately very few experiments have been made of that kind. One method of determining the constants would be to measure the variations of deflection for given increments of load. If δ be the increase of deflection for an increase of load p, we have for a hinged column $\delta = cr \frac{p}{E} \left(\frac{l}{\rho}\right)^2$ and b = nearly $9c$.

For a fixed-end column (on the second hypothesis, which is probably the truer) $\delta = 2cr \frac{p}{E} \left(\frac{l}{\rho}\right)^2$ and $b = 40c$.

Another method of determining b and c is to observe the limit of elasticity of the column as a whole and assume that this corresponds with the stress in the extreme fibres, f being the limit of elasticity of the material under direct stress. In Mr. Christie's experiments (Proc. Am. Soc. C. E., 1884) there are some observations which enable the first method to be applied. In Mr. Bouscaren's experiments (Proc. Am. Soc. C. E., 1880) are some which enable the second method to be applied. Unfortunately the majority of experiments are made on flat-ended columns, and no deduction can be made for our present purpose from them.

According to the experiments mentioned above, it would seem safe to place c for hinged columns at .05, and b at .45. If then we take E to be 13,500 tons and f (limit of elasticity in direct stress) 15 tons, which figures would apply to mild steel, and if, further, we allow the working load to be half the elastic load, we shall have as the working in lbs. per sq. in. of a mild steel hinged column

$$(6)\quad p = \frac{16{,}800}{1 + \left(\frac{r}{\rho}\right)^2 \left\{ .45 + \frac{1}{16{,}000}\left(\frac{l}{\rho}\right)^2 \right\}}$$

For fixed-end columns it would seem that c might be put at .0075 and b at .3, and the safe load on a fixed-end column in lbs. per sq. in. would then be

$$(7)\quad p = \frac{16{,}800}{1 + \left(\frac{r}{\rho}\right)^2 \left\{ .3 + \frac{1}{120{,}000}\left(\frac{l}{\rho}\right)^2 \right\}}$$

These determinations cannot, however, be recommended with any confidence on such a narrow basis of experiment. They would give higher working loads for long columns relatively to short ones than the formulæ and tables now in use. The fact that they disagree with present practice we do not regard as a condemnation, because present practice judges of the strength of a column for ordinary working loads solely by its ultimate strength, which we regard as a false principle, but at the same time so radical a departure as is now advocated could not prudently be made until a sufficient experimental basis is found for it as well as full theoretical justification.

The method chiefly used by engineers now is to make the working load a definite fraction of the load of rupture as determined by experiments for a similar section, and this is the best plan available; but it must not be supposed that it secures anything like a definite limit of working stress in the column. The term "crippling load" means different things with different experimenters. "Crippling" occurs in various ways with different columns, and the ratio of the crippling load to the elastic load varies very greatly with different proportions of columns. Further, the stress in the innermost fibres of a column at its point of failure is not the ultimate stress of the fibres, or any constant amount whatever.

Thirdly, since f (or more strictly p) is a coefficient of $\left(\frac{l}{\rho}\right)^2$ in the denominator of (5), the question is raised as to whether f should be made the elastic limit of the material in direct compression, and the working load made a definite fraction of the value of p so derived from (5), or whether f should be made the permissible working stress in the extreme fibres, and the working load p derived directly by (5) from it. The two methods would give different working loads which would agree for short columns and diverge more and more as $\left(\frac{l}{\rho}\right)$ increased, the first method giving the higher load. The first seems, however, the more logical plan, and has been followed in the tentative formulæ (6) and (7).

By it the working load is made a definite fraction of the load at which the column, as a whole, ceases to be perfectly elastic. As we have before said, the margin between this elastic limit of the column and its failing point may vary enormously for different proportions of columns.

Fourthly, comparing hinged and fixed-end columns, we see that the strength of hinged columns may be expected to be much more variable even than that of those with fixed ends, because it depends on two variable elements a and δ_0, while the fixed end gets rid of a. For a hinged column accurately centered $a_0 = 0$ and $a = \pm h \sin \alpha$, being opposite in sign to δ_0, unless the hinge were very small indeed, b and c would then be negative. The meaning of this is that the friction would never rise to its limiting value, and the column would behave exactly as if its ends were fixed. The moment at the end of a fixed column under a load p is nearly $p\frac{\delta_0}{2}\left\{1+\frac{p}{40\,E}\left(\frac{l}{\rho}\right)^2\right\}$, and while p is within the elastic load, the moment of friction in a hinge of ordinary size will be greater than this. When p reaches such a magnitude that the friction is not sufficient to resist rotation, we may expect the column suddenly to spring into a new position of equilibrium.

We see, then, that according to formula (5) an accurately centered pinned column will safely bear as great a load as if its ends were fixed, and would be much weaker if it rested on knife edges. It is possible, however, that in structures the vibration which usually accompanies increase of load may destroy the friction to a great extent, and, if so, it would largely diminish the strength we might otherwise allow for in pinned columns well centered, and besides that a_0 may in practice have a very appreciable value.

The above conclusions, which are strictly deduced from our investigation, agree in general with the observations made by Mr. Christie on his experiments referred to above.

At first sight it might seem an objection to formula (5) that when $\frac{l}{\rho}$ becomes very small p does not approach the value f as a limit, but a definite quantity less than f. A little consideration, however, will show that any theory which takes into account the variations from ideal conditions on which the flexure of columns really depends ought to give that result. It must be remembered that f is not the safe load per sq in. on a short column uniformly loaded, but the stress per sq. in. in the extreme fibres, and this will always be greater than the average stress, except under the ideal conditions of a and δ_0 being zero, which lead to Euler's theory.

The formulæ hitherto used to express the strength of columns have been empirical, that is, have been framed so as to agree as nearly as possible with the results of experiments. One of them has practically displaced all others, and is known as Gordon's formula, having been first made widely known by Professor Lewis Gordon, although it was put forward originally by Tredgold. Using the same symbols as hitherto it is as follows:

$$p = \frac{f}{1+c\left(\frac{l}{\rho}\right)^2}$$

[and is sometimes given in a less correct form as:

$$p = \frac{f}{1+c\left(\frac{l}{r}\right)^2}$$]

We see that for columns of similar section in which $\frac{r}{\rho}$ is the same, this is practically the same formula as (5), for the second term in the denominator of (5) could then be got rid of by using a different value for f, viz.: $\frac{f}{1+b\left(\frac{r}{\rho}\right)^2}$, the safe load per square inch on a short column of the section in question. It may be regarded as a confirmation of the truth of the principles on which we have proceeded that we have arrived at a similar formula in the main to that which is usually accepted as the nearest possible expression of experimental results. The agreement of Gordon's formula with experiments on the destruction of columns is, however, only comparative, and, as we have pointed out, could only be expected to be so. It would appear *a priori* that the introduction of $\left(\frac{r}{\rho}\right)^2$ into Gordon's formula, or some similar correction, ought to be an improvement, because, of two sections having an equal radius of gyration, that one with a smaller external diameter would have a higher elastic limit per square inch. The results of experiments also have shown that Gordon's formula requires a different value of c for different sections of column.

In designing a column it is clear that, according to formula (5), a given cross-section A will give the greatest strength: first, when $\frac{A}{\rho^2}$ is as small as possible; secondly, when $\left(\frac{r}{\rho}\right)^2$ is as small as possible. We therefore give a table of the values of $\frac{A}{\rho^2}$ and $\left(\frac{r}{\rho}\right)^2$ for some simple forms of cross-section.

Form of Cross-section.	Plane of flexure	$\frac{A}{\rho^2}$	$\left(\frac{r}{\rho}\right)^2$
1. Solid circular.........	Immaterial......	4π or 12·5	4
2. Solid square..........	Bisecting sides...	12	3
3. do do	Through diagonal	12	6
4. Hollow circular (thin shell of radius r and thickness d..........	Immaterial......	$\frac{8\pi^2 a^2}{A}\left(\text{or } 79\frac{d^2}{A}\right)$ or $\frac{2A}{r^2}$	2
5. Hollow square (thin shell of side $2a$ and thickness d)........	Bisecting sides...	$\frac{96d^2}{A}$ or $\frac{3A}{2a^2}$	$\frac{3}{2}$
6. do do	Through diagonal	$\frac{96d^2}{A}$ or $\frac{3A}{2a^2}$	3
7. Square, side $2a$, in which material is concentrated in the four angular points......	Bisecting sides...	$\frac{A}{a^2}$	1
8. do do	Through diagonal	$\frac{A}{a^2}$	2
9. Square, side $2a$, in which material is concentrated along two opposite sides of thickness d...............	Bisecting closed sides..........	$\frac{16d^2}{A}$ or $\frac{A}{a^2}$	1
10. do do	Bisecting open sides.........	$\frac{48d^2}{A}$ or $\frac{3A}{a^2}$	3
11. do do	Through diagonal	$\frac{24d^2}{A}$ or $\frac{3A}{2a^2}$	3
12. Angle bar, side a, small thickness d..	Through root of angle.........	$\frac{24A}{a^2}$	3

Nos. 7 and 9 are the ultimate forms of built up square columns with four sides latticed and with two plated and two latticed respectively. It appears doubtful, as was said before, however, how far the flexure of built up columns can be expected to follow the same laws as those of solid columns.

The table contradicts common impressions in one or two respects. It is commonly supposed that a square column is weakest for flexure in a plane bisecting its sides, but since the moment of inertia of the cross section is the same for all axes, the moment of resistance for a given curvature will be the same for all planes of flexure, and accidental circumstances will determine the plane in which flexure actually occurs. For a plane through a diagonal of the square, however, $\left(\frac{r}{\rho}\right)^2$ is twice as great as for a plane bisecting the sides, and therefore the diagonal plane is the one in which the column would soonest reach its limit of elasticity. For this plane a solid square column is therefore weaker than a solid circular column of equal length and weight, while, if the plane of flexure bisecting the sides of the square were alone regarded, the square column would be the stronger, ρ being almost the same for both. If hollow square and circular shells be compared, similar remarks apply, only in this case the circle has the advantage also of a larger radius of gyration nearly in the ratio of ten-ninths. If square columns of equal length and weight be compared, in which the material is in the first case distributed in a uniform shell, in the second along two sides of the square only, and in the third concentrated in the four angular points only, and taking for each case the most unfavorable plane of flexure $\frac{A}{\rho^2}$ varies in the three cases as $3 : 6 : 2$ and $\left(\frac{r}{\rho}\right)^2$ as $3 : 3 : 2$, showing that the last form has a considerable advantage in both respects over the others, and that the fully closed square is more economical than that latticed on two faces.