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H. B. Bruce's

PLANE TRIGONOMETRY

AS FAR AS THE

SOLUTION OF TRIANGLES;

BY

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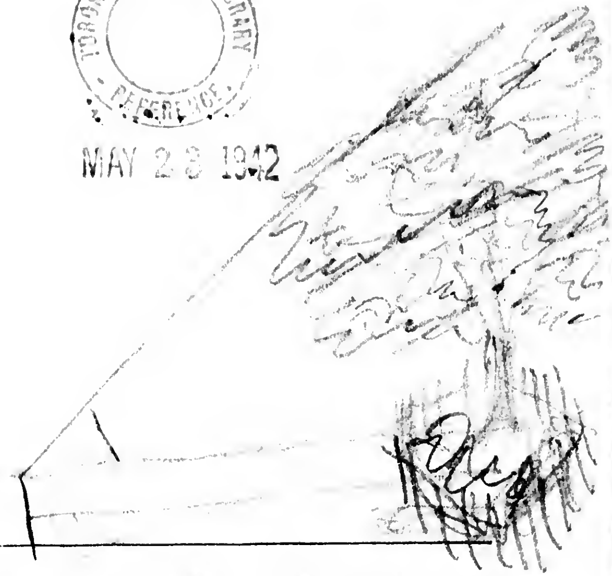
SECOND EDITION.

TORONTO:

COPP, CLARK & CO., PRINTERS, KING STREET EAST.

1869.



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Bruce's

LOGARITHMS.

1. The common logarithm of a number is the index of the power to which *ten* must be raised in order to produce that number; so that in the equation A logarithm defined.

$$10^x = a,$$

x is the logarithm of the number a, and this is written

$$x = \log a.$$

2. The logarithms of numbers which are integral powers of ten are immediately known; for example:

$$\begin{array}{llll} 10^3 = 1000, & \log 1000 = 3, \\ 10^2 = 100, & \log 100 = 2, \\ 10^1 = 10, & \log 10 = 1, \\ 10^0 = 1, & \log 1 = 0, \\ 10^{-1} = 0{\cdot}1, & \log 0{\cdot}1 = -1, \\ 10^{-2} = 0{\cdot}01, & \log 0{\cdot}01 = -2, \\ 10^{-3} = 0{\cdot}001, & \log 0{\cdot}001 = -3, \end{array}$$

For numbers greater than ten, the logarithms will be positive integers or mixed numbers; for numbers between 10 and 1, the logarithms will be positive decimals; for numbers less than 1, the logarithms will be negative quantities; the logarithm of zero is negative infinity, and negative numbers have no logarithms.

3. When the logarithm of a number is a negative quantity, it is convenient to express it so that the integral part alone is Characteristic and Mantissa.

negative, the decimal part remaining always positive, and the negative sign is written *over* the integral part to indicate this:

$$\begin{aligned} \text{Thus, } \log 0{\cdot}05 &= -(1{\cdot}30103) \\ &= -1 - 0{\cdot}30103 \\ &= -2 + (1 - 0{\cdot}30103) \\ &= -2 + 0{\cdot}69897 \\ \text{and this is written } &= \bar{2}{\cdot}69897. \end{aligned}$$

With this convention, the integral part of the logarithm is called the *characteristic*, and the decimal part the *mantissa*.

4. Since numbers which have $(n + 1)$ figures in their integral part commence with 10^n and run up to 10^{n+1}, their logarithms will commence with n and run up to $(n + 1)$, and the characteristic for all such numbers will therefore be n. Again, since pure decimals in which the first significant digit occurs in the n^{th} place from the decimal point commence with 10^{-n} and run up to $10^{-(n-1)}$, their logarithms will commence with $-n$ and run up to $-(n-1)$, that is, will be $-n$ increased by some decimal, and the characteristic for all such will therefore be $\bar{n}$. Hence we have the following rule for finding the characteristic of the logarithm for any number.

Rule for finding the characteristic.

If the number be an integer or a mixed number, the characteristic is positive and is less by unity than the number of figures in the integral part; if the number be a decimal the characteristic is the number of the place of the first significant digit, counting from the decimal point, and is negative.

Thus for the following numbers

$$12345, \quad 12{\cdot}345, \quad 1{\cdot}23, \quad 0{\cdot}54, \quad 0{\cdot}000543,$$

the characteristics are respectively

$$4, \quad 1, \quad 0, \quad \bar{1}, \quad \bar{4}.$$

Conversely, when a logarithm is given, the position of the decimal point in the corresponding number depends only on the characteristic, and we have the following rule for placing it.

Rule for placing the decimal point in the number.

If the characteristic be positive or zero, the number of figures in the integral part of the number will be greater by one than the characteristic; if the characteristic be negative, the number will be a pure decimal having its first significant digit in the place indicated by the number of the characteristic.

Investigation of the rules for using logarithms in arithmetical operations.

5. The following are the rules on which are founded the uses of logarithms in performing arithmetical operations.

(1)..................$\log (a\,b) = \log a + \log b.$

Let

$$x = \log a, \quad y = \log b$$

so that

$$10^x = a, 10^y = b.$$

Then,

$$a\,b = 10^x \times 10^y = 10^{x+y}$$

so that $x + y$ is the logarithm of $(a\,b)$,

or, $$\log (a\,b) = \log a + \log b.$$

(2)..................$\log \frac{a}{b} = \log a - \log b.$

Let $$x = \log a, \quad y = \log b,$$

so that $$10^x = a, 10^y = b.$$

Then,

$$\frac{a}{b} = \frac{10^x}{10^y} = 10^{x-y}$$

so that $x - y$ is the logarithm of $\frac{a}{b}$

or $$\log \left(\frac{a}{b}\right) = \log a - \log b.$$

(3)..................$\log (a^n) = n \log a$.

Let $\qquad x = \log a$, so that $10^x = a$.

Then,

$$a^n = (10^x)^n = 10^{nx}$$

so that $\qquad nx$ is the logarithm of a^n

or $\qquad \log (a^n) = n \log a$.

(4) ....$\log (\sqrt[n]{a}) = \frac{1}{n} \log a$.

Let $\qquad x = \log a$, so that $10^x = a$.

Then $\qquad \sqrt[n]{a} = a^{\frac{1}{n}} = (10^x)^{\frac{1}{n}} = 10^{\frac{x}{n}}$,

so that $\qquad \frac{x}{n}$ is the logarithm of $\sqrt[n]{a}$,

or $\qquad \log (\sqrt[n]{a}) = \frac{1}{n} \log a$.

6. Any of these operations may be combined: thus

$$\log (abcd) = \log a + \log b + \log c + \log d;$$

$$\log \left(\frac{a}{b\,c}\right) = \log a - \log b - \log c;$$

$$\log \frac{a}{c^2} \cdot \frac{\sqrt{b}}{\sqrt[3]{d}} = \log a + \tfrac{1}{2} \log b - 2 \log c - \tfrac{1}{3} \log d.$$

The mantissa independent of the place of the decimal point in the number.

7. *The mantissa of the logarithm is the same for all numbers which differ only in the position of the decimal point.*

For any change in the position of the decimal point in a number is effected by a continued multiplication or division by ten; and since $\log 10 = 1$, each such multiplication or division alters the characteristic of the logarithm only by the addition or subtraction of 1, thus leaving the mantissa unchanged.

8. In the tables of logarithms of numbers, the mantissas alone are given (exact to a certain number of decimals), and the characteristics must be supplied by the rule of § 4. The number of figures in the given mantissas determines the number of figures for which the logarithm is given with sufficient accuracy in these tables. Thus when six figures are given in the mantissas, the tables will be available only for numbers consisting of six figures or less, that is (disregarding the decimal point) for numbers ranging from 1 to 1000000. The mantissas however are not entered for *all* those numbers, but only for those terminating in the hundreds: for the intermediate numbers, the mantissas must be calculated by aid of the principle that the difference between the logarithms of two numbers is proportional to the difference between the numbers, when the numbers are taken sufficiently close. Thus the difference between two consecutive mantissas in the table corresponds to a difference of 100 between the numbers, and we obtain by a simple proportion the difference of mantissa corresponding to any less difference than 100 in the numbers.

Arrangement of tables of logarithms.

E.g. .. Required the mantissa for the logarithm of 675347.

From the tables,

Number,	675400;	mantissa,	829561
" ,	675300;	" ,	829497
Difference,	100;	difference,	64

Then, by the principle,

$$\text{required difference for } 47 = \frac{47}{100} \times 64 = 30{\cdot}08,$$

and therefore the mantissa for 675347 is 829497 + 30, or 829527.

In many tables, the trouble of performing the multiplication in the above is avoided by the insertion of tables of *proportional parts*, in which are set down the products of the difference for 100 by the respective units, so that these products can at sight be taken out and added to the mantissa.

Table of proportional parts.

Thus, in the previous example,

From the table, Number	675300;	Mantissa,	829497
From table of *p.p.*, for	40,	difference,	25·2
.. ..	7,	....	4.4
Therefore for Number	675347,	Mantissa,	829527·

According to the usual rule in decimals, in carrying out to a certain number only of places, the last figure must be increased by 1 when the first of the neglected figures is 5 or a higher digit.

To take out the logarithm of a number.

9. The following is then the rule for finding the logarithm of a number of six or less figures.

Disregarding the decimal point, look in the table for the first three figures of the number in the left-hand column, and for the fourth figure in the top line; at the intersection of the corresponding line and column will be found the mantissa; for the fifth figure, look in the table of proportional parts and take out the number for that column; and for the sixth figure, also from the table of proportional parts, take out the corresponding number, removing the decimal point one place to the left. Add these two latter numbers to the mantissa previously found, and then, by consideration of the position of the decimal point in the original number, prefix the proper characteristic.*

---

* In five-figure tables, the first three figures are to be looked for in the left-hand column, the fourth figure in the top line, and the fifth must be calculated for from the table of proportional parts. In seven-figure tables, the first four are given in the left-hand column, the fifth in the top line, and the sixth and seventh must be calculated for. As the arrangement of the tables varies according to the fancy of the compiler, the student must learn the peculiarities of the set he uses. The remarks in the text apply to Law's Mathematical Tables (abridged), Toronto: Chewett & Co. In practice, five figures will generally be found sufficient, and in the sequel five only will be used.

Example. Required the logarithm of 327·695.

From the table,	3276..,	Mantissa	515344
From *p.p.*,	9 ,	diff.	119 7
....	5,	"	6 6
	327695,	Mant.	515470

and the characteristic is 2; therefore the logarithm of 327·695 is 2·515470.

To take out the number corresponding to a given logarithm.

10. The reverse process of finding the number corresponding to a given logarithm is performed on the same principle. Disregarding the characteristic, look out in the tables for the mantissa next below the given mantissa. In the corresponding line and column will be found the first three and the fourth figures of the number. Then taking the difference between the mantissa thus found and the given one, and also that between the former and the next higher in the tables (which will be the difference for 100 in the number), by a simple proportion the tens and units in the required number are found. The decimal point must then be inserted by consideration of the characteristic of the given logarithm.

Example. Find the number corresponding to the given logarithm, $\bar{2}$·767198. The mantissa next below is 767156, and the corresponding number is 585000. The difference between the two mantissas is 42.

Again in the tables,

Mantissa corresponding to	585100	is	767230
....	585000	"	767156
Difference of mantissa for	100	is	74

Then, by the proportion, the required difference in the number for a difference of 42 in the mantissa is

$$100 \times \frac{42}{74} = 56·7,$$

and the number for this mantissa is 585000 + 57, or 585057. The characteristic in the given logarithm being $\bar{2}$, the number required will be 0 0585057.

Table of proportional parts.

As in the previous case, the trouble of performing the division in the above is avoided by the tables of proportional parts in which the quotients corresponding to the division are set down. Thus, having taken the difference between the given mantissa and the one next below it in the tables, look out in the corresponding table of proportional parts for the number next below this difference, and the column in which this is found gives the fifth figure: again take the difference between the previous difference and the number found in the table of proportional parts, and removing the decimal point in it one place to the right, look out again in the table of proportional parts for the number nearest to it, and the column in which this is found gives the sixth figure.

The previous example would be thus worked:

Given mantissa,	767198;			
Mantissa next below,	767156,	corresponding number, 5850 ..		
Difference	42,			
In table of p. p., diff. next below is	37·0,	"	"	5
Residual difference	5.0	"	"	7

This gives for the six figures, 585057, and the number required is therefore 0·0585057.

Use of logarithms in multiplication.

We shall now exemplify the rules for performing arithmetical operations by aid of logarithms, demonstrated in § 5, using five-figure logarithms only.

11. To multiply numbers together.

Rule. *Add the logarithms of the numbers, and take from the tables the number corresponding to this sum as a logarithm.*

Ex. (1). Multiply 379·45 into 2·4672.

Number,	379·45;	log,	2·57915
"	2·4672;	",	0·39220
Product,	936·16;	log,	2·97135

Ex. (2). Multiply 997 into 0·0325

Number,	997;	log,	2·99870
"	0·0325;	",	$\bar{2}$·51188
Product,	32,403;	log,	1·51058

Observe that the addition is + 2 + (— 2) + 0·9 .. + 0·5 ..

Ex. (3). Multiply 7240000 into 93201

Number,	7240000;	log,	6.85974
"	93201;	",	4·96942
Product,	674780000000;	log,	11·82916

Here the product has 12 figures in its integral part, of which only five are determined; the remaining 7 being unknown are replaced by cyphers.

Ex. (4). Multiply 0·076905 into 0·000094397

Number,	0·076905;	log,	$\bar{2}$·88595
"	0·000094397;	",	$\bar{5}$·97496
Product,	0·0000072596;	log,	$\bar{6}$·86091

Here the addition is — 2 — 5 + 0·8 .. + 0·9 ..

12. To divide one number by another. **Division.**

Rule. *Subtract the logarithm of the divisor from that of the dividend, and take from the tables the number corresponding to this difference as a logarithm.*

Ex. (1). Divide 32·495 by 7·6993.

Dividend,	32·495;	log,	1·51182
Divisor,	7.6993;	",	0·88645
Quotient,	4·2206;	log,	0·62537

Ex. (2). Divide 2·7045 by 312·79.

Dividend,	2·7045;	log,	0·43209
Divisor,	312·79;	log,	2·49525
Quotient,	0·0086465;	log,	$\bar{3}$·93684

Here the subtraction is 1·43.... — 0·49.... — 2 — 1.

Ex. (3). Divide 465·94 by 0·793.

Dividend,	465·94;	log,	2·66833
Divisor,	0·793;	",	$\bar{1}$·89927
Quotient,	587·57;	log,	2·76906

Here the subtraction is 2·6.... — 0·8.... — (— 1).

Ex. (4). Divide 0·0037095 by 0·00001605.

Dividend,	0·0037095;	log,	$\bar{3}$·56932
Divisor,	0·00001605;	",	$\bar{5}$·20548
Quotient,	231·12;	log,	2·36384

Here the subtraction is 0·5.... — 0·2.... + (— 3) —(— 5).

In cases of this kind, it may be easier to make both characteristics positive by adding the same number to each: for example, add 10 in the above, and the process is

7·5.... — 5·2.... = 2·3....

Use of arithmetical complements.

13. It is convenient to convert the process of subtraction into one of addition by the use of what is called the *arithmetical complement.* Thus if b is to be subtracted from a, instead of subtracting b, add 10 — b, and subtract 10 from the result; for

$$a - b = a + (10 - b) - 10.$$

This quantity $(10 - b)$ is called the arithmetical complement of b, and is found by subtracting the first significant digit, beginning from the right hand, from 10, and each following digit from 9, including, in the case of a logarithm, the characteristic with its proper sign.

For example,

Number, 239·31; log, 2·37896; co-log, 7.62104;
" 0·0025177; log, $\bar{3}$·40100; co-log, 12.59900.

The working of the previous examples would then stand thus,

Ex. (1).

Dividend,	32·495;	log,	1·51182
Divisor,	7·6993;	co-log,	9·11355
			0·62537

Ex. (2).

Dividend,	2·7045;	log,	0·43209
Divisor,	312·79;	co-log,	7·50475
			$\bar{3}$·93684

Ex. (3).

Dividend,	465·94;	log,	2·66833
Divisor	0·793;	co-log,	10·10073
			2·76906

Ex. (4).

Dividend,	0·0037095;	log,	$\bar{3}$·56932
Divisor,	0·00001605;	co-log,	14·79452
			2·36384

14. To raise a number to any power. Involution.

Rule. *Multiply the logarithm of the number by the power, and take from the tables the number corresponding to this product as a logarithm.*

Ex. (1). Find the sixth power of 23·91.

$$\begin{array}{rr} \text{Number, } 23{\cdot}91; \quad \text{log,} & 1{\cdot}37858 \\ & 6 \\ \hline \text{Required power, } 186840000; \quad \text{log,} & 8{\cdot}27148 \end{array}$$

Here the power has 9 figures in its integral part, of which only 5 are determined, the remaining 4 being unknown are replaced by cyphers.

Ex. (2). Find $(0{\cdot}032507)^{10}$.

$$\begin{array}{rr} \text{Number, } 0{\cdot}032507; \quad \text{log,} & \bar{2}{\cdot}51198 \\ & 10 \\ \hline \text{Power} = \underline{0{\cdot}0000000000000013177}; \quad \text{log,} & \overline{15}{\cdot}11980 \end{array}$$

Here the multiplication is $10(-2) + 10(\cdot 5\,.\,.)$.

Evolution.

15. To extract any root of a number.

Rule. *Divide the logarithm of the number by the root, and take from the tables the number corresponding to this quotient as a logarithm.*

Ex. (1). Required the fifth root of 2.

$$\begin{array}{rr|l} \text{Number, } 2; & \text{log,} & 0{\cdot}30103 \\ & 5 & \\ \hline \text{Required root, } \underline{1{\cdot}1487}; & \text{log,} & \underline{0{\cdot}06021} \end{array}$$

Ex. (2). Required the 8th root of 0·79635.

$$\begin{array}{rr|l} \text{Number, } 0{\cdot}79635; & \text{log,} & \bar{1}{\cdot}90110 \\ & 8 & \\ \hline \text{Root, } \underline{0{\cdot}97194}; & \text{log,} & \underline{\bar{1}{\cdot}98764} \end{array}$$

Here the characteristic being negative and not exactly divisible by the root, we add to it a sufficient number (negative) to make it exactly divisible, and therefore the same number (positive) to the mantissa. Thus,

$-8+7\cdot9$.., which on division gives $-1 + 0\cdot9$ .. or $\bar{1}\cdot9$ ..

16. As before remarked, any of these operations may be combined, but when more than one arithmetical complement is used, a ten must be subtracted from the result for each complement. *Combined operations.*

Ex. (1). Find the value of $\dfrac{(12\cdot345)^5}{670\cdot59 \times 50\cdot323}$.

Number, 12·345;	log,	1·09149			
		5			
		5·45745	......		5·45745
" 670·59;	log,	2·82646	co-log,		7·17354
" 50·323;	log,	1·70177	" ,		8·29823
Required value,		8·4961	log,		0·92922

Ex. (2). Find $\sqrt[3]{\frac{5}{6}}$.

Number, 5;	log,	0·69897
" 6;	co-log,	9·22185
		3)$\bar{1}\cdot92082$
Required value, 0·94105;	log,	$\bar{1}\cdot97361$

The operation here is this:

$$\log \sqrt[3]{\tfrac{5}{6}} = \tfrac{1}{3}\log\tfrac{5}{6} = \tfrac{1}{3}(\log 5 - \log 6).$$
$$= \tfrac{1}{3}(\log 5 + \text{co-log } 6 - 10);$$

THE TRIGONOMETRICAL RATIOS.

17. It is proved by Euclid that in a right-angled triangle, when one of the other angles is given, the ratios of the sides are also given. To these ratios, six in number, distinctive *The trigonometrical ratios of an angle defined.* Fig. 1.

names are attached, and they are called the *trigonometrical ratios* of the given angle. Thus in the triangle ABC, (fig. 1) having the angle C right, with reference to the angle A, calling the side opposite to A the perpendicular, the other side the base, that opposite to C being the hypothenuse, the ratio of perpendicular to hypothenuse is called the *sine* of the angle A; the ratio of perpendicular to base the *tangent*; and the ratio of hypothenuse to base, the *secant*; or, as they are written,

when the angle increases fr: 0° to 90° $\dfrac{BC}{AB} = \sin A,$ increases from 0 to 1

" " $\dfrac{BC}{AC} = \tan A,$ " " 0 " ∞

" " $\dfrac{AB}{AC} = \sec A.$ " " 1 " ∞

The other three ratios—namely:

$$\frac{AC}{AB}, \qquad \frac{AC}{BC}, \qquad \frac{AB}{BC},$$

are evidently the sine, tangent, and secant with reference to the angle B, and this angle being the complement of A, the term "sine of the complement of A" is abbreviated into the *cosine* of A; and similarly the names, *cotangent*, *cosecant* are formed for the other two. These are written,

" " $\dfrac{AC}{AB} = \cos A,$ " 1 to 0

" " $\dfrac{AC}{BC} = \cot A,$ diminishes fr: ∞ to 0

" " $\dfrac{AB}{BC} = \operatorname{cosec} A.$ " ∞ to 1

Their nature.

18. These ratios, when the angle is given, are independent of the magnitude of the triangle, and are in effect determinate positive numbers. Since the perpendicular and base are always less than the hypothenuse, it is plain that the sines and cosines are proper fractions, while the secants and cosecants are whole numbers or improper fractions, but the tangents and cotangents may have any positive values.

19. As the angle A increases, retaining the same hypothenuse, the perpendicular increases and the base diminishes continually, and therefore the sine, tangent, and secant increase, while the cosine, cotangent, and cosecant diminish, and when A approaches near to 90°, the perpendicular approaches to coincidence with the hypothenuse, while the base vanishes, and we have therefore for 90°, *Their changes of value.*

Particular values for 90°, 0°, 45°, 30°, 60°.

$$\sin 90^\circ = 1,\ \tan 90^\circ = \infty,\ \sec 90^\circ = \infty,\ \cos 90^\circ = 0,$$
$$\cot 90^\circ = 0,\ \operatorname{cosec} 90^\circ = 1.$$

Also since 0° is the complement of 90°, these values give $\cos 0^\circ = 1$, $\cot 0 = \infty$, $\operatorname{cosec} 0 = \infty$, $\sin 0 = 0$, $\tan 0 = 0$, $\sec 0 = 1$.

20. The following intermediate values may be noticed.

Take a right-angled isosceles triangle (fig. 2), in which the perpendicular and base are each $= 1$, and the hypothenuse therefore $= \sqrt{2}$. Fig. 2.

Then either angle being 45°, it is seen by inspection that $\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}$; $\tan 45^\circ = \cot 45^\circ = 1$; $\sec 45^\circ = \operatorname{cosec} 45^\circ = \sqrt{2}$.

Hence also the tangent of an angle less than 45° is less than 1, and of an angle greater than 45° is greater than 1, while the reverse is the case for the cotangent.

Again, take an equilateral triangle (fig. 3) each of whose sides $= 2$, and from one of the vertexes drop a perpendicular on the opposite side; this perpendicular bisects both the side and the angle, giving two right-angled triangles with the angles 30°, 60°, and the length of this perpendicular is $\sqrt{3}$. Hence by inspection Fig. 3.

$$\sin 30^\circ \text{ or } \cos 60^\circ = \frac{1}{2};\quad \cos 30^\circ \text{ or } \sin 60^\circ = \frac{\sqrt{3}}{2};$$

$$\tan 30^\circ \text{ or } \cot 60^\circ = \frac{1}{\sqrt{3}};\quad \cot 30^\circ \text{ or } \tan 60^\circ = \sqrt{3};$$

$$\sec 30^\circ \text{ or } \operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}};\quad \operatorname{cosec} 30^\circ \text{ or } \sec 60^\circ = 2.$$

Five independent relations connect them.

21 It is also proved by Euclid that when the ratio of two sides in a right-angled triangle is given, the angles are also given. Consequently when any one of the six trigonometrical ratios of an angle is given, the angle itself is determinate, and the other five ratios can be found. Hence there must be five independent relations connecting the six ratios of an angle. By inspection it is seen that the sine and cosecant, the tangent and cotangent, the cosine and secant are reciprocals, so that

$$\sin A = \frac{1}{\operatorname{cosec} A}, \tan A = \frac{1}{\cot A}, \cos A = \frac{1}{\sec A}.$$

Again,

$$\frac{\sin A}{\cos A} = \frac{BC}{AB} \div \frac{AC}{AB} = \frac{BC}{AC} = \tan A.$$

These are four of the relations; a fifth, connecting sine and cosine is given by Euclid, B. I. Prop. 47*; for

$$AB^2 = BC^2 + AC^2,$$

and therefore

$$1 = \left(\frac{BC}{AB}\right)^2 + \left(\frac{AC}{AB}\right)^2$$

$$= (\sin A)^2 + (\cos A)^2,$$

or, as it is usually written

$$\sin^2 A + \cos^2 A = 1.$$

Numerous other relations exist between these ratios, but they are all deducible from the five above given, which enable us by a simple algebraic process to express any one ratio in terms of any other.

Tables of their values.

22. The values of all these ratios are calculated for all angles between 0 and 90°, and are entered in tables called *natural sines*, &c.; but these values are not so useful as the logarithms of them which form the tables called *logarithmic sines*, &c. Since, however, the sines and cosines are proper fractions, and so also are some of the tangents and cotangents,

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* Another proof, not depending on this proposition, will be subsequently given.

their logarithms will have negative characteristics, and to avoid the inconvenience of printing these, every logarithm of a trigonometrical ratio is increased by 10 before being entered in the table. To distinguish therefore the real logarithm from that given in the tables, the latter will always be written with an italic capital L, and it must always be borne in mind that 10 is to be taken from each such logarithm when used instead of the real logarithm, the operation being either expressed or understood.

The tabular logarithm as distinguished from the real logarithm.

For instance

$$\sin 30^\circ = \tfrac{1}{2} = 0 \cdot 5$$
$$\log \sin 30^\circ = \log (0 \cdot 5) = \bar{1} \cdot 69897$$
$$L \sin 30^\circ = 9 \cdot 69897.$$

Also,

$$\tan 45^\circ = 1,$$
$$\log \tan 45^\circ = 0,$$
$$L \tan 45^\circ = 10 \cdot 00000.$$

23. Again, since

$$\sin A \times \operatorname{cosec} A = 1, \text{ we have}$$
$$\log \sin A + \log \operatorname{cosec} A = 0,$$
$$L \sin A - 10 + L \operatorname{cosec} A - 10 = 0,$$

or,

$$L \sin A + L \operatorname{cosec} A = 20.$$

And similarly,

$$L \tan A + L \cot A = 20,$$
$$L \cos A + L \sec A = 20.$$

Also,

$$\tan A = \frac{\sin A}{\cos A}$$
$$\log \tan A = \log \sin A - \log \cos A$$
$$L \tan A - 10 = L \sin A - 10 - (L \cos A - 10)$$
$$L \tan A = L \sin A + 10 - L \cos A.$$

By aid of these formulas, if $L \sin A$ and $L \cos A$ be tabulated from 0 to 45°, the values of the other logarithmic functions from 0 to 90° can be formed.

Arrangement of the tables of logarithmic sines, &c.

24. In the ordinary tables, these logarithmic sines, cosines, &c., are given for all angles from 0° to 90° at intervals of one minute, and it will be sufficient for most purposes to take out any required angle to the nearest minute, but if greater accuracy be needed, recourse must be had to the principle of proportional parts already explained in discussing the logarithms of numbers.

The usual arrangement is that the angles from 0 to 45° are placed at intervals of one degree at the head of the page, the minutes running down the left-hand column, while the angles from 45° to 90° are placed at the foot of the page, and the minutes run up the right-hand column. By this arrangement the same column is used for the sine of an angle and for the cosine of its complement; and in the same way for the tangent and cotangent, and for secant and cosecant.

25. Since sines and cosines are proper fractions, the tabular logarithms of them will always be less than 10; and since secants and cosecants are integers or improper fractions, their tabular logarithms will always be greater than 10. The logarithmic tangents will be less than 10 up to 45°, and after this will be greater than 10, and the reverse will be the case for the cotangents. The following table exhibits the changes as the angle passes from 0 to 90°:

sine increases	from 0 to 1;	L sin increases	from $-\infty$ to 10
cosine decreases	" 1 " 0;	L cos decreases	" 10 " $-\infty$
tangent increases	" 0 " ∞;	L tan increases	" $-\infty$ " $+\infty$
cotangent decreases	" ∞ " 0;	L cot decreases	" $+\infty$ " $-\infty$
secant increases	" 1 " ∞;	L sec increases	" 10 " $+\infty$
cosecant decreases	" ∞ " 1;	L cosec decreases	" $+\infty$ " 10.

L tan and L cot are each 10 at 45°.

SOLUTION OF RIGHT-ANGLED TRIANGLES.

Direct relations connecting the sides and the trigonometrical ratios of one of the angles in a right-angled triangle.

26. Taking the triangle ABC, where C is 90°, and denoting the lengths of the sides opposite to each angle by the

small letter corresponding, the definitions of the trigonometrical ratios give the following relations:

Fig. 4.

$$\sin A = \frac{a}{c}, \quad \text{or} \quad a = c \sin A;$$

$$\tan A = \frac{a}{b} \quad \ldots\ldots\ldots\ldots \quad a = b \tan A;$$

$$\sec A = \frac{c}{b} \quad \ldots\ldots\ldots\ldots \quad c = b \sec A;$$

$$\cos A = \frac{b}{c} \quad \ldots\ldots\ldots\ldots \quad b = c \cos A;$$

$$\cot A = \frac{b}{a} \quad \ldots\ldots\ldots\ldots \quad b = a \cot A;$$

$$\operatorname{cosec} A = \frac{c}{a} \quad \ldots\ldots\ldots\ldots \quad c = a \operatorname{cosec} A;$$

Two parts being given (one at least being a line), the triangle can be solved.

27. From these relations, any two of the four quantities a, b, c, A being given, the other two could be found by aid of the tables of *natural* sines, cosines, &c.; and the remaining angle B, which is the complement of A, being thus found also, the triangle would be completely determined. Such a mode of solution would however be inconvenient, as involving long processes of multiplication, and we shall proceed to discuss the different cases of the solution of right-angled triangles by means of the logarithmic tables.

Four cases of solution.

28. Four distinct cases will arise, (1), an angle and a side; (2), an angle and the hypothenuse; (3), the two sides; (4), a side and the hypothenuse. In cases (1) and (2), it is indifferent which angle be given, as the other is at once known. The solution will be effected in each case by picking out from among the foregoing relations one which connects the quantity sought for with two quantities which have been given or found, and it will be noticed that in each case there will be *two* of these relations which would serve this purpose. If one involves a process of addition, and the other a process of subtraction, we shall always take the former.

Case 1. A side and an angle given.

Case (1). Given a, A; to find B, b, c.

$B = 90° - A$. .... B found.

$b = a \cot A$.

Taking the logarithms of both sides,

$$\log b = \log a + \log \cot A$$

or

$$\log b = \log a + L \cot A - 10 \text{ .... } b \text{ found.}$$

$$c = a \operatorname{cosec} A$$

or

$$\log c = \log a + L \operatorname{cosec} A - 10 \text{........} c \text{ found.}$$

Case 2.

The hypothenuse and an angle given.

Case (II). Given c, A; to find B, a, b.

$$B = 90° - A. \text{ .... } B \text{ found.}$$

$$a = c \sin A,$$
$$\log a = \log c + L \sin A - 10 \text{ .... } a \text{ found.}$$

$$b = c \cos A,$$
$$\log b = \log c + L \cos A - 10 \text{ .... } b \text{ found.}$$

Case 3.

The two sides given.

Case (III). Given a, b; to find A, B, c.

$$\tan A = \frac{a}{b},$$
$$\log \tan A = \log a - \log b,$$
$$L \tan A - 10 = \log a + \operatorname{colog} b - 10$$

and therefore

$$L \tan A = \log a + \operatorname{colog} b \text{.... ....} A \text{ found.}$$

$$B = 90° - A. \text{ .... } B \text{ found.}$$

$$c = a \operatorname{cosec} A,$$
$$\log c = \log a + L \operatorname{cosec} A - 10 \text{ ...... } c \text{ found.}$$

In this case it is indifferent whether we determine A from the formula $\tan A = \frac{a}{b}$, or from $\cot A = \frac{b}{a}$. Also there is not among our relations one connecting c with the given quantities a, b, and although we know from Euclid that $c^2 = a^2 + b^2$, this formula is not convenient for logarithmic computation, and we therefore determine c by means of A, which though not *given* has been already found. We might also have determined c by means of $c = b \sec A$.

Case 4.

A side and the hypothenuse given.

Case (IV). Given a, c; to find A, B, b.

$$\sin A = \frac{a}{c},$$

$$\log \sin A = \log a - \log c$$
$$L \sin A - 10 = \log a + \operatorname{colog} c - 10,$$

and therefore

$L \sin A = \log a + \operatorname{colog} c$ .... A found.

$B = 90^\circ - A$ .... B found.

$b = a \cot A$

$\log b = \log a + L \cot A - 10$ .... b found.

In this case it is indifferent whether we determine A from the formula $\sin A = \frac{a}{c}$, or from $\operatorname{cosec} A = \frac{c}{a}$. Also, there being none of the relations which connects b directly with the given quantities a, c, it is determined by means of A which has previously been found; it might also have been found from the formula $b = c \cos A$. It is known from Euclid that $b^2 = c^2 - a^2$, and b might have thus been found directly, but the formula is not convenient for logarithms.

29. The solution of an isosceles triangle can be effected by aid of the preceding; for such a triangle can be divided by a perpendicular dropped from the vertex on the base into two right-angled triangles, equal in all respects, and by solving these, the parts of the isosceles triangle also are determined. Isosceles triangles solved.

30. Examples of right-angled triangles. Examples.

Case (1). Given $a = 129{\cdot}5$, $A = 37^\circ\ 07'$.

$B = 90^\circ - A$.

$$\begin{array}{rl} & 90^\circ\ 00' \\ A = & 37^\circ\ 07' \\ \hline B = & 52^\circ\ 53' \end{array} \qquad (B \text{ found.})$$

$\log b = \log a + L \cot A - 10$.

$$\begin{array}{lll} a = 129{\cdot}5; & \log a, & 2{\cdot}11227 \\ A = 37^\circ\ 07'; & L \cot A, & 10\ 12105 \\ \hline b = 171{\cdot}13; & \log b, & 2{\cdot}23332 \end{array} \qquad (b \text{ found.})$$

$\log c = \log a + L \operatorname{cosec} A - 10$.

$$\begin{array}{lll} & \log a, & 2{\cdot}11227 \\ A = 37^\circ\ 07'; & L \operatorname{cosec} A, & 10{\cdot}21937 \\ \hline c = 214{\cdot}61; & \log c, & 2{\cdot}33164 \end{array} \qquad (c \text{ found.})$$

Case (II). Given $c = 31459$, $A = 46° 32'$.

$B = 90° - A.$

$$\begin{array}{rr} & 90° \ 00' \\ A = & 46 \ \ \ 32 \\ \hline B = & 43° \ 28' \end{array}$$

(*B* found.)

$\log a = \log c + L \sin A - 10.$

$c = 31459$;	$\log c$,	4·49774
$A = 46°32$;	$L \sin A$,	9·86080
$a = 22832$;	$\log a$,	4·35854

(*a* found.)

$\log b = \log c + L \cos A - 10.$

	$\log c$,	4·49774
$A = 46° \ 32'$;	$L \cos A$,	9·83755
$b = 21642$;	$\log b$,	4·33529

(*b* found.)

Case (III). Given $a = 2{\cdot}7039$, $b = 3{\cdot}4505$.

$L \tan A = \log a + \text{colog } b.$

$a = 2{\cdot}7039$;	$\log a$,	0·43199
$b = 3{\cdot}4505$;	colog b,	9·46212
$A = 38° \ 05'$;	$L \tan A$,	9·89411

(*A* found.)

$B = 90° - A.$

$$\begin{array}{rr} & 90° \ 00' \\ A = & 38 \ \ \ 05' \\ \hline B = & 51° \ 55' \end{array}$$

(*B* found.)

$\log c = \log a + L \text{ cosec } A - 10.$

	$\log a$,	0·43199
$A = 38° \ 05'$;	L cosec A,	10·20985
$c = 4{\cdot}3837$;	$\log c$,	0·64184

(*c* found.)

Case (IV). Given $a = 21$, $c = 21{\cdot}981$.

$$L \sin A = \log a + \text{colog } c.$$

$$\begin{array}{ll} a = 21 \quad ; & \log a,\ 1{\cdot}32222 \\ c = 21{\cdot}981; & \text{colog } c,\ 8{\cdot}65795 \\ \hline A = 72^\circ\ 49'; & L \sin A,\ 9{\cdot}98017 \end{array}$$

(*A* found.)

$B = 90^\circ - A.$

$$\begin{array}{r} 90^\circ\ 00' \\ A = 72^\circ\ 49 \\ \hline B = 17^\circ\ 11' \end{array}$$

(*B* found.)

$$\log b = \log a + L \cot A - 10.$$

$$\begin{array}{ll} & \log a,\ 1{\cdot}32222 \\ A = 72^\circ\ 49'; & L \cot A,\ 9{\cdot}49029 \\ \hline b = \ 6{\cdot}4940\ ; & \log b,\ 0{\cdot}81251 \end{array}$$

(*b* found.)

TRIGONOMETRICAL FORMULAS.

Extension of the definition of the trigonometrical ratios to the case of an angle greater than 90°.

31. It is necessary now to extend our definitions to the case of an angle greater than one, but less than two, right angles. Let CAB be such an angle and be denoted by A. Produce CA through A and drop BC' perpendicularly upon it. The angle BAC' is called the *supplement* of A, and $= 180^\circ - A$. We now define the trigonometrical ratios of the angle A to be the corresponding ratios for the angle BAC' in the triangle $BC'A$, with the convention that AC' is to be considered a negative magnitude. Let p, b, h be the numerical values of the lengths of the perpendicular, base, and hypothenuse in the triangle: then

Fig. 5.

Relations between the ratios of an angle and its supplement.

$$\sin A = \frac{BC'}{AB} = \frac{p}{h} = \sin BAC' = \sin(180^\circ - A);$$

$$\tan A = \frac{BC'}{AC'} = \frac{p}{-b} = -\frac{p}{b} = -\tan BAC' = -\tan(180^\circ - A);$$

$$\sec A = \frac{AB}{AC'} = \frac{h}{-b} = -\frac{h}{b} = -\sec BAC' = -\sec(180^\circ - A);$$

$$\cos A = \frac{AC'}{AB} = \frac{-b}{h} = -\frac{b}{h} = -\cos BAC' = -\cos(180^\circ - A);$$

$$\cot A = \frac{AC'}{BC'} = \frac{-b}{p} = -\frac{b}{p} = -\cot BAC' = -\cot(180^\circ - A);$$

$$\operatorname{cosec} A = \frac{AB}{BC'} = \frac{h}{p} = \operatorname{cosec} BAC' = \operatorname{cosec}(180^\circ - A).$$

32. It will be seen on inspection that the ratios according to this extended definition still satisfy the same five fundamental relations as before; and although the complement of an angle (A) which is greater than 90°, being 90° − A, is a negative quantity, and ceases at present to have any signification, we shall still say that the cosine, cotangent, cosecant of such an angle are the sine, tangent, and secant of its complement, and hereafter, if necessary, give a consistent interpretation to the quantity.

The ratios for angles greater than 90° found from those of angles less than 90°.

33. From the above it is seen that the trigonometrical ratio of an angle is the same in numerical value as the corresponding ratio of its supplement, but bears a different sign except in the cases of sine and cosecant which bear the same sign. It is therefore unnecessary to construct additional tables for angles greater than 90°, as the ratios for such angles can be found from those of their supplements, which are less than 90°. Further, for such angles, the tangents, secants, cosines, and cotangents being negative quantities, have no logarithms, and it is only for the sines and cosecants that the logarithms have real values, being the same as those given in the tables for the supplements of these angles.

The object of the convention in § 31 being to distinguish as far as possible between the ratios of an angle and its supplement, it will be noticed that we have succeeded in distinguishing between four only of the six; now the signs of *all* the lines of the triangle BAC' being at our disposal, if we were to make them all negative, we should have the same values for all the ratios; if we were to make two of them negative, we should have four of the ratios with changed signs, and the other two the same, which is the same result as obtained by making one only of the lines negative. Hence the latter method as being more simple is adopted, and of the three lines the base is selected as the one to be changed, because (as will be seen in the next article) the relations between the sides of a triangle and the ratios for its angles can thus be expressed by the same formulas, whatever be the nature of the triangle.

34. We can now proceed to the discussion of triangles in general, to the angles of which, whether acute or obtuse, our definitions of the ratios will now apply.

Three independent relations connect the six parts of an oblique triangle.

The triangle being ABC, the lengths of the sides opposite to the respective angles will be denoted by the small letters corresponding. The triangle then is said to have six parts:—namely, the three angles, A, B, C, and the three sides, a, b, c. It is proved by Euclid that when three of these parts are given (one of them being a side), the other parts can be found. There must therefore be three independent relations connecting these six quantities. One such relation is already established by Euclid, namely:

One relation.

$$A + B + C = 180^\circ \text{..................} (1)$$

Two others we proceed to investigate.

From C drop the perpendicular CD on AB (fig. 6) or on BA produced (fig. 7).

Then in the right-angled triangle CBD,

$$CD = BC \sin CBD = a \sin B.$$

And in the right-angled triangle CAD,

$$CD = AC \sin CAD = b \sin A, \text{ in fig. 6,}$$
$$= b \sin (180^\circ - A) = b \sin A, \text{in fig.7.}$$

Hence

$$a \sin B = b \sin A.$$

Similarly, by dropping a perpendicular from A, we should obtain

$$b \sin C = c \sin B,$$

The other two.

And hence

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Another relation found independently, but actually deducible from the above.

35. From these three relations (1), (2), all others can be deduced, but for such as we require at present, it will sometimes be easier to give proofs which do not directly depend on these.

Resuming the figures and construction of the previous proposition,

$$\begin{aligned} AB &= DB + AD, \text{ in fig. 6.} \\ &= BC \cos CBD + AC \cos CAD \\ &= a \cos B + b \cos A. \end{aligned}$$

Also,

$$\begin{aligned} AB &= DB - AD, \text{ in fig. 7.} \\ &= BC \cos CBD - AC \cos CAD \\ &= a \cos B - b \cos (180° - A) \\ &= a \cos B + b \cos A. \end{aligned}$$

* Hence, universally,

$$c = a \cos B + b \cos A. \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad (3)$$

Deduction of certain general formulas.

36. Multiplying the respective terms of this equation by the equal quantities $\frac{\sin C}{c}$, $\frac{\sin A}{a}$, $\frac{\sin B}{b}$, we obtain

$$\sin C = \sin A \cos B + \cos A \sin B,$$

---

* In this formula, writing it

$$1 = \frac{a}{c} \cos B + \frac{b}{c} \cos A,$$

suppose that C is a right angle. Then $\cos B = \sin A$,

$\frac{a}{c} = \sin A, \frac{b}{c} = \cos A$, and, making these substitutions, it becomes

$$1 = (\sin A)^2 + (\cos A)^2$$

This is the proof alluded to in page 18, as not depending on Euclid, B. I. Prop. 47, but in fact being also a proof of that proposition.

but C is the supplement of $(A + B)$; therefore

$$\sin (A + B) = \sin A \cos B + \cos A \sin B. \quad \ldots\ldots (4)$$ (sin $A + B$).

37. We can express the sine of the difference of two angles in a similar way, for, by § 31,

$$\sin(A-B) = \sin \left\{ 180^\circ - (A-B) \right\} = \sin \left\{ (180^\circ - A) + B \right\}$$
$$= \sin (180^\circ - A) \cos B + \cos (180^\circ - A) \sin B,$$
$$= \sin A \cos B - \cos A \sin B \ldots\ldots \ldots\ldots\ldots (5)$$ sin $(A - B)$.

Also we can thus express the cosine of the sum of two angles; for

$$\cos (A+B) = \sin \left\{ 90^\circ - (A+B) \right\} = \sin \left\{ (90^\circ - A) - B \right\}$$
$$= \sin (90^\circ - A) \cos B - \cos (90^\circ - A) \sin B$$
$$= \cos A \cos B - \sin A \sin B \quad \ldots\ldots\ldots\ldots (6)$$ cos $(A + B)$

The above proof of the last three formulas restricts the angles A and B to have their sum less than 180°. The formulas however are universal, but it is not necessary to extend them beyond this case, as it is the only case in which their use is at present required.

38. In (4), and (6), putting $B = A$, we obtain

$$\sin 2A = \sin A \cos A + \cos A \sin A$$
$$= 2 \sin A \cos A$$
$$\cos 2A = \cos A \cos A - \sin A \sin A$$
$$= \cos^2 A - \sin^2 A$$

and therefore, (since $\cos^2 A + \sin^2 A = 1$,)

$$= 2 \cos^2 A - 1$$
or
$$= 1 - 2 \sin^2 A.$$

Writing $\frac{1}{2}A$ instead of A, these become

$$\sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A, \ldots\ldots\ldots\ldots (7)$$
$$\cos A = 2 \cos^2 \tfrac{1}{2} A - 1 = 1 - 2 \sin^2 \tfrac{1}{2} A \ldots (8)$$

sin A and cos A in terms of sin $\frac{1}{2}A$, cos $\frac{1}{2}A$.

39. Adding (4) and (5), we obtain

$$\sin (A + B) + \sin (A - B) = 2 \sin A \cos B,$$

And subtracting (5) from (4),

$$\sin(A+B) - \sin(A-B) = 2\cos A \sin B.$$

Dividing the terms of these two equalities, we obtain

$$\frac{\sin(A+B)+\sin(A-B)}{\sin(A+B)-\sin(A-B)} = \frac{2\sin A\cos B}{2\cos A\sin B}$$
$$= \frac{\sin A}{\cos A} \div \frac{\sin B}{\cos B}$$
$$= \frac{\tan A}{\tan B}$$

In this formula, instead of $(A+B)$ write A, and instead of $(A-B)$ write B, and therefore also instead of A write $\frac{1}{2}(A+B)$, and instead of B write $\frac{1}{2}(A-B)$, and we obtain

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} \quad \ldots\ldots\ldots (9)$$

40. To express the cosine of an angle of a triangle in terms of the sides.

Resuming (3),

cos A in terms of a, b, c.

$$c = a\cos B + b\cos A.$$

From the analogy we see that

$$a = b\cos C + c\cos B$$
$$b = c\cos A + a\cos C$$

If from these three equations we eliminate cos B and cos C, the required result will be obtained. Multiplying the first by c, and the third by b, and then adding; we have

$$c^2 + b^2 = a\,c\cos B + a\,b\cos C + 2\,b\,c\cos A$$
$$= a(c\cos B + b\cos C) + 2\,b\,c\cos A$$
$$= a^2 + 2\,b\,c\cos A, \text{ (from the second),*}$$

* Written in the form,

$$a^2 = b^2 + c^2 - 2\,b\,c\cos A$$

this is identical with Euclid p.p 12, 13, B II; for in fig. (6) $AD = b\cos A$, and in fig. (7), $AD = -b\cos A$, and therefore

$$BC^2 = AC^2 + AB^2 \mp 2\,AB.\,AD,$$

— or + according as A is acute or obtuse.

or,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,b\,c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Analogous expressions can now be written down for cos B and cos C. These expressions are not adapted to logarithmic calculation, and we therefore proceed to modify them.

41. From (8),

The previous expression modified for logarithmic use.

$$2 \sin^2 \tfrac{1}{2} A = 1 - \cos A$$

$$= 1 - \frac{b^2 + c^2 - a^2}{2\,b\,c} \quad \ldots\ldots\ldots \text{from } (10)$$

$$= \frac{a^2 - (b^2 - 2\,b\,c + c^2)}{2\,b\,c}$$

(bracket in wrong place)

$$= \frac{a^2 - (b - c)^2}{2\,b\,c}$$

$$= \frac{(a + b - c)\,(a - b + c)}{2\,b\,c}$$

Again from (8),

$$2 \cos^2 \tfrac{1}{2} A = 1 + \cos A$$

$$= 1 + \frac{b^2 + c^2 - a^2}{2\,b\,c}$$

$$= \frac{(b^2 + 2\,b\,c + c^2) - a^2}{2\,b\,c}$$

(no index over a where there ought to be)

$$= \frac{(b + c)^2 - a^2}{2\,b\,c}$$

$$= \frac{(b + c + a)\,(b + c - a)}{2\,b\,c}$$

Now putting

$$a + b + c = 2\,s$$

s the semi-perimeter.

and therefore

$$a + b - c = 2\,(s - c)$$
$$b + c - a = 2\,(s - a)$$
$$c + a - b = 2\,(s - b),$$

these become

$$\left.\begin{aligned} \sin^2 \tfrac{1}{2} A &= \frac{(s-b)\,(s-c)}{b\,c} \\ \cos^2 \tfrac{1}{2} A &= \frac{s\,(s-a)}{b\,c} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

And dividing the former by the latter,

$$\tan^2 \tfrac{1}{2} A = \frac{(s-b)\,(s-c)}{s\,(s-a)},$$

or

tan $\frac{1}{2}A$ in terms of s and the sides.

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)\,(s-c)}{s\,(s-a)}} \ldots\ldots\ \ldots\ldots\ldots\ldots (12)$$

SOLUTION OF OBLIQUE TRIANGLES.

Solution of oblique triangles.

42. Four distinct cases occur in the solution of oblique triangles, according to the way in which three parts out of the six which compose the triangle are selected, one at least of the given parts being a side.

These are,

Four cases

(1), two angles and a side. (Euclid, B. I. Prop. 26.)
(2), the three sides. (..... Prop. 8)
(3), two sides and the included angle. (...... Prop. 4)
(4), two sides and an angle not included. (.........The omitted case.)*

* *If two triangles have two sides of the one equal to two sides of the other, each to each, and have also one angle in each equal, being opposite to equal sides; then if each of the angles opposite to the other equal sides be greater than a right angle, or be less than a right angle, or if one of them be a right angle, the triangles will be equal in every respect.*

S = Semiperimeter

43. Case I. Given A, B, a; to find C, b, c. **Case I.** *Two angles and a side given.*

To find C,

$$C = 180^\circ - A - B \ldots\ldots\ldots\ldots\ldots\ldots C \text{ found.}$$

To find b,

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

or

$$b = \frac{a \sin B}{\sin A} = a \sin B \operatorname{cosec} A,$$

and taking logarithms

$$\log b = \log a + L \sin B - 10 + L \operatorname{cosec} A - 10$$
$$= \log a + L \sin B + L \operatorname{cosec} A - 20$$

from which there is $\qquad\qquad b$ found.

To find c,

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

or

$$c = a \sin C \operatorname{cosec} A$$
$$\log c = \log a + L \sin C + L \operatorname{cosec} A - 20,$$

from which there is $\qquad\qquad c$ found.

In this case it is indifferent which of the sides is given, as all three angles are at once known.

44. Case II. Given a, b, c; to find A, B, C. **Case II.** *The three sides given.*

To find A, we have, (where $s = \frac{1}{2}(a+b+c)$),

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)\,(s-c)}{s\,(s-a)}} \quad \ldots\ldots\ldots\ldots\ldots \text{from (12)}$$

and taking logarithms,

$$L \tan \tfrac{1}{2} A - 10 = \tfrac{1}{2} \log \frac{(s-b)\,(s-c)}{s\,(s-a)}$$

$$= \tfrac{1}{2}\left\{\log(s-b) + \log(s-c) - \log s - \log(s-a)\right\}$$

$$= \tfrac{1}{2}\left\{\log(s-b) + \log(s-c) + \operatorname{colog} s + \operatorname{colog}(s-a) - 20\right\}$$

and therefore

$L \tan \frac{1}{2} A = \frac{1}{2} \{ \log(s-b) + \log(s-c) + \text{colog}\,(s-a) + \text{colog}\, s \}$,
from which there is $\frac{1}{2} A$ and therefore A found.

By the analogous formula, B can be found and then C which is $180° - A - B$. It is however better in practice to find C also by its analogous formula, and the sum of the three angles amounting to $180°$ will serve as verification.

We might also have used either of the formulas (11) for $\sin \frac{1}{2} A$, $\cos \frac{1}{2} A$, but that for the tangent is practically preferable. If the sum of two of the quantities a, b, c, be not greater than the third, one of the quantities $s-a$, $s-b$, $s-c$, will be negative, and its logarithm imaginary.

Case 3.

Two sides and the included angle given.

45. Case III. Given a, b, C; to find A, B, c. $(a > b)$.

To find A, B.

$$\frac{\sin A}{a} = \frac{\sin B}{b},$$

or

$$\frac{a}{b} = \frac{\sin A}{\sin B},$$

and therefore

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}$$

$$= \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}, \text{ ... .from (9)}$$

or

$$\tan \frac{1}{2}(A-B) = \frac{a-b}{a+b} \tan \frac{1}{2}(A+B).$$

Now

$\frac{1}{2}(A+B) = \frac{1}{2}(180° - C) = 90° - \frac{1}{2} C$, and is known;
also $\tan \frac{1}{2}(A+B) = \cot \frac{1}{2} C$,

and therefore

$$\tan \frac{1}{2}(A-B) = \frac{a-b}{a+b} \cot \frac{1}{2} C,$$

and taking logarithms,

$$L \tan \frac{1}{2}(A-B) - 10 = \log(a-b) + \text{colog}\,(a+b) - 10 + L \cot \frac{1}{2} C - 10,$$

or

$$L \tan \tfrac{1}{2}(A-B) = \log(a-b) + \text{colog}(a+b) + L \cot \tfrac{1}{2} C - 10,$$
from which $\frac{1}{2}(A-B)$ is found; also $\frac{1}{2}(A+B)$ being known, we have by addition and subtraction............A and B found.

A having thus been found, we obtain c from the formula

$$\frac{\sin C}{c} = \frac{\sin A}{a},$$

$$c = a \sin C \operatorname{cosec} A$$

$$\log c = \log a + L \sin C + L \operatorname{cosec} A - 20,$$

(c found,)

in which formula b, B might also be used in place of a, A.

In this case, c is known directly in terms of the given parts from

$$c^2 = a^2 + b^2 - 2\, a\, b \cos C,$$

but this formula is not adapted to logarithmic calculation, and it is better to find c by aid of one of the angles which have been previously found.

46. Case IV. Given A, a, b; to find B, C, c. Case IV.

Two sides and an angle not included by them given. The ambiguity discussed.

In this case there are sometimes two triangles which have the given parts. For let A be acute, and (fig. 8) drop the perpendicular CD, which is equal to $b \sin A$; then there can be drawn two lines, each $= a$, one on each side of CD, and if both these fall (as CB_1, CB_2) on the right of b, the two triangles ACB_1, ACB_2 will have the same three given parts. This requires a to be less than b and greater than CD; if however $a = CD$, the two triangles coincide in a right-angled triangle, and if a be less than CD, no triangle exists having the given quantities for parts. Also if $a = b$, the triangle ACB_2 vanishes, and only one is left, and if a be greater than b, the line CB_2 falls to the left of b, and the triangle so formed would not have the angle A, and in this case there is only one triangle. Fig. 8. Fig. 9.

Again if A be obtuse (fig. 10), in order that a triangle may exist, a must be greater than b, and the other line equal to a will fall to the left of b, so that only one triangle exists. Fig. 10.

Collecting these results, we see that, when A is acute, if $a < b \sin A$, there is no triangle; if $a = b \sin A$, there is one only; if $a > b \sin A$ and $< b$, there are two; if $a =$ or $> b$, there is only one, and when A is obtuse, if $a <$ or $= b$, there is no triangle; and if $a > b$, there is one only. If A be a right angle, then a must be $> b$, and the two triangles on opposite sides of b are equal in every respect, and therefore only give the *same* triangle in different positions.

The analytical solution which follows will of itself show which of these varieties occurs in any particular case.

Solution. To find B;

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

or

$$\sin B = \frac{b \sin A}{a}$$

and taking logarithms

$$L \sin B - 10 = \log b + \operatorname{colog} a - 10 + L \sin A - 10,$$

whence

$$L \sin B = \log b + \operatorname{colog} a + L \sin A - 10.$$

This gives $L \sin B$, but as the L sin of an angle is the same as the L sin of the supplement of that angle, there are two angles which have this value of L sin, and both must be taken. Let B_1, B_2 be these two angles, the former being less than 90° and taken directly from the tables, the latter being its supplement. Let C_1, C_2 be the corresponding values for C, so that

$$C_1 = 180° - A - B_1,$$
$$C_2 = 180° - A - B_2.$$

If both these values are positive, two triangles exist.

Let c_1, c_2 be the corresponding values of c. To find them;

$$\frac{\sin C_1}{c_1} = \frac{\sin A}{a}$$

$$c_1 = a \sin C_1 \operatorname{cosec} A,$$

$$\log c_1 = \log a + L \sin C_1 + L \operatorname{cosec} A - 20.$$

Similarly

$$\log c_2 = \log a + L \sin C_2 + L \operatorname{cosec} A - 20.$$

If the second value of C be 0 or negative, the second solution has no existence; and if both values of C are nega-

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tive, no solution exists. Also if the value of $L \sin B$ be greater than 10, there is no solution.

47. **Examples.** — Examples.

Case I.

Given $A = 120° \, 08'$, $B = 24° \, 40'$, $a = 981{\cdot}23$. — A, B, a given to find C

$$C = 180° - A - B.$$

$$\begin{array}{rrr} A = & 120° & 08' \\ B = & 24 & 40 \\ \hline & 144 & 48 \\ & 180 & \\ \hline C = & 35° & 12' \end{array} \qquad (C \text{ found.})$$

$$\log b = \log a + L \sin B + L \operatorname{cosec} A - 20.$$ — b.

$$\begin{array}{lll} a = 981{\cdot}23\,; & \log a & 2{\cdot}99177 \\ B = 24° \, 40'\,; & L \sin B, & 9{\cdot}62049 \\ A = 120° \, 08'\,; & L \operatorname{cosec} A, & 10{\cdot}06305 \\ \hline b = 473{\cdot}49\,; & \log b, & 2{\cdot}67531 \end{array} \quad (b \text{ found.})$$

$$\log c = \log a + L \sin C + L \operatorname{cosec} A - 20.$$

$$\begin{array}{lll} & \log a, & 2{\cdot}99177 \\ C = 35° \, 12'\,; & L \sin C, & 9{\cdot}76075 \\ & L \operatorname{cosec} A, & 10{\cdot}06305 \\ \hline c = 653{\cdot}99\,; & \log c, & 2{\cdot}81557 \end{array} \quad (c \text{ found.})$$

Case II.

Given $a = 753{\cdot}09$, $b = 333{\cdot}33$, $c = 666{\cdot}66$. — a, b, c given to find

$$\begin{array}{rr} a = & 753{\cdot}09 \\ b = & 333{\cdot}33 \\ c = & 666{\cdot}66 \\ \hline 2s = & 1753{\cdot}08 \end{array}$$

	log	colog.
$s = 876{\cdot}54$	2·94277	7·05723
$s - a = 123{\cdot}45$	2·09149	7·90850
$s - b = 543{\cdot}21$	2·73497	7·26503
$s - c = 209{\cdot}88$	2·32197	7·67803

$L \tan \frac{1}{2} A = \frac{1}{2} \{\log (s-b) + \log (s-c) + \operatorname{colog}(s-a) + \operatorname{colog} s\}$

A.

$$\begin{array}{lr} \log (s - b), & 2{\cdot}73497, \\ \log (s - c), & 2{\cdot}32197 \\ \operatorname{colog} (s - a), & 7{\cdot}90850 \\ \operatorname{colog}\ s, & 7{\cdot}05723 \\ \hline & 2)\underline{20{\cdot}02267} \\ \frac{1}{2} A = 45° \ 45'; \quad L \tan \frac{1}{2} A, & 10{\cdot}01133 \end{array}$$

$A = 91° \ 30'$ (*A* found.)

$L \tan \frac{1}{2} B = \{\log (s-c) + \log (s-a) + \operatorname{colog} (s-b) + \operatorname{colog} s\}.$

B.

$$\begin{array}{lr} \log (s - c), & 2{\cdot}32197 \\ \log (s - a), & 2{\cdot}09149 \\ \operatorname{colog} (s - b), & 7{\cdot}26503 \\ \operatorname{colog}\ s, & 7{\cdot}05723 \\ \hline & 2)\underline{18{\cdot}73572} \\ \frac{1}{2} B = 13° \ 08'; \quad L \tan \frac{1}{2} B, & 9{\cdot}36786 \end{array}$$

$B = 26° \ 16'$ (*B* found.)

$L \tan \frac{1}{2} C = \frac{1}{2} \{\log(s-a) + \log(s-b) + \operatorname{colog}(s-b) + \operatorname{colog} s\}.$

C.

$$\begin{array}{lr} \log (s - a), & 2{\cdot}09149 \\ \log (s - b), & 2{\cdot}73497 \\ \operatorname{colog} (s - c), & 7{\cdot}67803 \\ \operatorname{colog}\ s, & 7{\cdot}05723 \\ \hline & 2)\underline{19{\cdot}56172} \\ \frac{1}{2} C = 31° \ 07'; \quad L \tan \frac{1}{2} C, & 9{\cdot}78086 \end{array}$$

$C = 62° \ 14'$ (*C* found.)

Verification.

Verification.

$$\begin{aligned} A &= 91^\circ\ 30' \\ B &= 26\ \ \ 16 \\ C &= 62\ \ \ 14 \\ \hline A + B + C &= 180^\circ \end{aligned}$$

Case III.

Given $a = 209{\cdot}88$, $b = 333{\cdot}33$, $C = 122^\circ\ 26'$. *a, b, C given to find*

Here, b being greater than a, we must interchange a, A, with b, B in the formulas of solution.

$$C = 122^\circ\ 26';\quad \tfrac{1}{2}\,C = \begin{array}{r} 90^\circ\ 00 \\ 61\ \ \ 13 \\ \hline 28^\circ\ 47' \end{array}$$

$$\tfrac{1}{2}\,(B + A) = 90^\circ - \tfrac{1}{2}\,C = 28^\circ\ 47'$$

$$L \tan \tfrac{1}{2}\,(B - A) = \log\,(b - a) + \operatorname{colog}\,(b + a) + L \cot \tfrac{1}{2}\,C - 10.$$ *A and B.*

$$\begin{aligned} b &= 333{\cdot}33 \\ a &= 209{\cdot}88;\quad \log,\ 2{\cdot}32197 \\ b - a &= 123{\cdot}45;\quad \log,\ 2{\cdot}09149 \\ b + a &= 543{\cdot}21;\quad \log,\ 2{\cdot}73497;\quad \operatorname{colog},\ 7{\cdot}26503 \end{aligned}$$

$$\begin{aligned} \log\,(b - a),&\ 2{\cdot}09149 \\ \operatorname{colog}\,(b + a),&\ 7{\cdot}26503 \\ \tfrac{1}{2}\,C = 61^\circ\ 13';\quad L \cot \tfrac{1}{2}\,C,&\ 9{\cdot}73987 \\ \hline \tfrac{1}{2}\,(B - A) = 7^\circ\ 07';\quad L \tan \tfrac{1}{2}\,(B - A),&\ 9{\cdot}09639 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}\,(B - A) &= 7^\circ\ 07' \\ \tfrac{1}{2}\,(B + A) &= 28\ \ \ 47 \\ \hline B &= 35^\circ\ 54' \\ A &= 21^\circ\ 40' \end{aligned}$$

(*B* and *A* found.)

$$\log c = \log a + L \sin C + L \operatorname{cosec} A - 20.$$ *c.*

$$\begin{aligned} \log a,&\ 2{\cdot}32197 \\ C = 122^\circ\ 26';\quad L \sin C,&\ 9{\cdot}92635 \\ A = 21^\circ\ 40';\quad L \operatorname{cosec} A,&\ 10{\cdot}43273 \\ \hline c = 479{\cdot}79;\quad \log c,&\ 2{\cdot}68105 \end{aligned}$$

(*c* found.)

Case IV.

A, a, b, given, to find

Ex. (1). Given $A = 57° 34'$, $a = 47{\cdot}979$, $b = 54{\cdot}321$.

B.

$$L \sin B = \log b + \operatorname{colog} a + L \sin A - 10.$$

$$\begin{array}{rlr}
b = 54{\cdot}321\,; & \log b, & 1{\cdot}73497 \\
a = 47{\cdot}979\,; & \operatorname{colog} a, & 8{\cdot}31895 \\
A = 57° 34'; & L \sin A, & 9{\cdot}92635 \\
\hline
\left\{\begin{array}{l} B_1 = 72° 52'; \\ B_2 = 107° 08'; \end{array}\right. & L \sin B, & 9{\cdot}98027
\end{array}$$

(B_1 and B_2 found.)

C.

$$\begin{array}{ll}
C_1 = 180 - (A + B_1) & C_2 = 180 - (A + B_2) \\
\quad A = 57° \; 34' & \quad A = 57° \; 34' \\
\quad B_1 = 72 \quad 52 & \quad B_2 = 107 \quad 08 \\
\hline
C_1 = 180° - 130° \; 26' & C_2 = 180° - 164 \quad 42 \\
\quad = 49° \; 34' & \quad = 15° \; 18'.
\end{array}$$

Solutions.

Hence there are two solutions.

c_1.

$$\log c_1 = \log a + L \sin C_1 + L \operatorname{cosec} A - 20.$$

$$\begin{array}{rlr}
a = 47{\cdot}979\,; & \log a, & 1{\cdot}68105 \\
C_1 = 49° 34'; & L \sin C_1, & 9{\cdot}88148 \\
A = 57° 34'; & L \operatorname{cosec} A, & 10{\cdot}07365 \\
\hline
c_1 = 43{\cdot}269\,; & \log c_1, & 1{\cdot}63618
\end{array}$$

(c_1 found.)

c_2.

$$\log c_2 = \log a + L \sin C_2 + L \operatorname{cosec} A - 20.$$

$$\begin{array}{rlr}
 & \log a, & 1{\cdot}68105 \\
C_2 = 15° \; 18'; & L \sin C_2, & 9{\cdot}42139 \\
 & L \operatorname{cosec} A, & 10{\cdot}07365 \\
\hline
c_2 = 15\,; & \log c_2, & 1{\cdot}17609
\end{array}$$

(c_2 found.)

A, a, b given, to find B.

Ex. (2). Given $A = 49° 41'$, $a = 323{\cdot}1$, $b = 21{\cdot}808$.

$$L \sin B = \log b + \operatorname{colog} a + L \sin A - 10.$$

$$\begin{array}{ll} b = 21{\cdot}808\ ; & \log b,\ 1{\cdot}33862 \\ a = 323{\cdot}1\ ; & \text{colog } a,\ 7{\cdot}49066 \\ A = 49^\circ\ 41';& L \sin A,\ 9{\cdot}88223 \\ \hline \left\{\begin{array}{l} B_1 = 2^\circ\ 57'; \\ B_2 = 177^\circ\ 03' \end{array}\right. & L \sin B,\ 8{\cdot}71151 \end{array}$$

(B_1 and B_2 found.)

$$\begin{array}{ll} C_1 = 180 - (A + B_1) & C_2 = 180 - (A + B_2) \\ A = 49^\circ\ 41' & A = 49^\circ\ 41' \\ B_1 = 2^\circ\ 57' & B_2 = 177^\circ\ 03' \\ C_1 = 180^\circ - 52^\circ\ 38' & C_2 = 180^\circ - 226^\circ\ 44' \\ = 127^\circ\ 22' & = - \end{array}$$

C.

One solution.

The second solution does not exist. The value of c_1 can be found as in the previous example.

c_1.

Ex. (3). Given $A = 30^\circ$, $a = 18{\cdot}4$, $b = 38{\cdot}9$.

A, a, b given to find B.

$$L \sin B = \log b + \text{colog } a + L \sin A - 10.$$

$$\begin{array}{ll} b = 38{\cdot}9; & \log b,\ 1{\cdot}58995 \\ a = 18{\cdot}4; & \text{colog } a,\ 8{\cdot}73518 \\ A = 30^\circ\ ; & L \sin A,\ 9{\cdot}69897 \\ \hline & L \sin B,\ 10{\cdot}02410 \end{array}$$

No solution.

No solution exists.

Ex. (4). Given $A = 128^\circ\ 57'$, $a = 21700$, $b = 19342$.

A, a, b given to find B.

$$L \sin B = \log b + \text{colog } a + L \sin A - 10.$$

$$\begin{array}{ll} b = 19342\ ; & \log b,\ 4{\cdot}28650 \\ a = 21700\ ; & \text{colog } a,\ 5{\cdot}66354 \\ A = 128^\circ\ 57; & L \sin A,\ 9{\cdot}89081 \\ \hline \left\{\begin{array}{l} B_1 = 43^\circ\ 53'; \\ B_2 = 136^\circ 07'. \end{array}\right. & L \sin B,\ 9{\cdot}84085 \end{array}$$

C.

$$\begin{array}{ll} C_1 = 180 - (A + B_1) & C_2 = 180 - (A + B_2) \\ A = 128^\circ\ 57' & A = 128^\circ\ 57' \\ B_1 = 43\ \ 53 & B_2 = 136\ \ 07 \\ C_1 = 180^\circ - 172^\circ\ 50' & C_2 = 180^\circ - 265\ \ 04 \\ = 7^\circ\ 10' & = - \end{array}$$

One solution.

The second solution does not exist.

A, a, b given to find *B*.

Ex. (5). Given $A = 163° 24'$, $a = 42$, $b = 53·004$.

$$L \sin B = \log b + \operatorname{colog} a + L \sin A - 10.$$

$b = 53·004$;	$\log b$, 1·72431
$a = 42$;	$\operatorname{colog} a$, 8·37675
$A = 163°24'$;	$L \sin A$, 9·45589
$B_1 = 21° 08'$;	$L \sin B$, 9·55695
$B_2 = 158° 52'$;	

C.

$$C_1 = 180° - (A + B_1) \qquad C_2 = 180° - (A + B_2)$$

$$A = 163° 24' \qquad A = 163° 24'$$

$$B_1 = 21° 08' \qquad B_2 = 158° 52'$$

No solution.

$$C_1 = 180° - 184° 32' \qquad C_2 = 180° - 322 \ 16$$

$$= - \qquad = -$$

No solution exists.

42. Expressions for the area of a triangle.

The area of a triangle.

It is proved by Euclid (*B* I. prop. 41) that the area of a triangle is half that of a rectangle having the same base and height. Now the number of square units in the area of a rectangle is equal to the product of the numbers of linear units in the base and height respectively, which is briefly expressed by saying that the area of a rectangle is the product of the base and height. Hence the area of a triangle is half the product of its base and height.

Fig. 6, 7.

In fig. 6, 7, area of triangle $A B C$

$$= \tfrac{1}{2} A B . \ C D,$$
$$= \tfrac{1}{2} c b \sin A$$
$$= \tfrac{1}{2} b c \sin A.$$

Again

$\sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A$ ....from (7)

$$= 2\sqrt{\frac{(s-b)\,(s-c)}{b\,c}}\sqrt{\frac{s\,(s-a)}{b\,c}} \ldots \text{from (11)}$$

$$= \frac{2}{b\,c}\sqrt{\{s\,(s-a)\,(s-b)\,(s-c)\}}\,.$$

Therefore the area

$$= \sqrt{\{s\,(s-a)\,(s-b)\,(s-c)\}}\,.$$

TRIANGLES FOR VERIFICATION.

$a = 16{\cdot}39,\quad b = 23{\cdot}962,\quad c = 37{\cdot}024$
$A = 19°\ 08',\quad B = 28°\ 38',\quad C = 132°\ 14'$

$a = 325{\cdot}74,\quad b = 403{\cdot}58,\quad c = 250{\cdot}10$
$A = 53°\ 46',\quad B = 87°\ 58',\quad C = 38°\ 16'$

$a = 209{\cdot}88,\quad b = 333{\cdot}33,\quad c = 479{\cdot}79$
$A = 21°\ 40',\quad B = 35°\ 54',\quad C = 122°\ 26'$

$a = 7{\cdot}5316,\quad b = 3{\cdot}3342,\quad c = 6{\cdot}6666$
$A = 91°\ 30',\quad B = 26°\ 16',\quad C = 62°\ 14'$

$a = 4797{\cdot}9,\quad b = 5432{\cdot}3,\quad c = 1500$
$A = 57°\ 34',\quad B = 107°08',\quad C = 15°\ 18'$

$a = 12345,\quad b = 34516,\quad c = 45324$
$A = \ \ 8°\ 39',\quad B = 24°\ 52',\quad C = 146°\ 29'$

$a = 33{\cdot}45,\quad b = 69{\cdot}54,\quad c = 62{\cdot}09$
$A = 28°\ 44',\quad B = 88°\ 06',\quad C = 63°\ 10'$

$a = 200,\quad b = 235,\quad c = 48{\cdot}942$
$A = \ \ 40°,\quad B = 130°\ 57',\quad C = \ \ 9°\ 03'$

$a = 315,\quad b = 227,\quad c = 154$
$A - 110°\ 04',\quad B = 42°\ 36',\quad C = 27°\ 20'$

$a = 10{\cdot}1032,\quad b = 15{\cdot}2003,\quad c = 14.6884$
$A = 39°\ 28',\quad B = 73°,\quad C = 67°\ 32'$

APPENDIX.

FORMULAS, &c.

$\log 10 = 1$, $\log 1 = 0$, $\log 0 = -\infty$.

$\log (ab) = \log a + \log b$.

$\log \frac{a}{b} = \log a - \log b$.

$\log a^n = n \log a$.

$\log \sqrt[n]{a} = \frac{1}{n} \log a$.

Any trigonometrical ratio of an angle is the co-ratio of the complement.

$\sin A = \frac{1}{\operatorname{cosec} A}$, $\tan A = \frac{1}{\cot A}$, $\cos A = \frac{1}{\sec A}$, $\tan A = \frac{\sin A}{\cos A}$

$$\sin^2 A + \cos^2 A = 1.$$

$$\sin 30° = \tfrac{1}{2},\ \cos 30° = \frac{\sqrt{3}}{2},\ \tan 45° = 1.$$

As the angle changes from 0 to 90°

sin increases from.....	0 to 1 ;	L sin increases from	$-\infty$ to 10
tan	0 ... ∞ ;	L tan	$-\infty$ .. $+\infty$
sec	1 ... ∞ ;	L sec	10 .. $+\infty$
cos decreases	1 ... 0 ;	L cos decreases.....	10 .. $-\infty$
cot	∞ ... 0 ;	L cot	$+\infty$ .. $-\infty$
cosec	∞ ... 1 ;	L cosec	$+\infty$ .. 10

$$L \tan 45° = 10 = L \cot 45°.$$

In a right-angled triangle, C the right angle,

$a = c \sin A$; $a = b \tan A$; $b = c \cos A$; $b = a \cot A$; $c = b \sec A$; $c = a \operatorname{cosec} A$.

$\sin A = \sin(180^\circ - A)$, $\quad$ $\operatorname{cosec} A = \operatorname{cosec}(180^\circ - A)$;
$L \sin A = L \sin(180^\circ - A)$, $\quad$ $L \operatorname{cosec} A = L \operatorname{cosec}(180^\circ - A$;
$\cos A = -\cos(180^\circ - A)$, $\quad$ $\sec A = -\sec(180^\circ - A)$;
$\tan A = -\tan(180^\circ - A)$, $\quad$ $\cot A = -\cot(180^\circ - A)$.

General formulas,

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \quad \ldots\ldots\ldots \quad (4)$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B \quad \ldots\ldots\ldots \quad (5)$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \quad \ldots\ldots\ldots \quad (6)$$

$$\sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A \quad \ldots\ldots\ldots \quad (7)$$

$$\cos A = 2 \cos^2 \tfrac{1}{2} A - 1 = 1 - 2 \sin^2 \tfrac{1}{2} A \quad \ldots\ldots\ldots \quad (8)$$

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} \quad \ldots\ldots\ldots \quad (9)$$

In any triangle ABC,

$$A + B + C = 180^\circ \quad \ldots\ldots\ldots \quad (1)$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \ldots\ldots\ldots \quad (2)$$

$$c = a \cos B + b \cos A \quad \ldots\ldots\ldots \quad (3)$$

$$\left.\begin{array}{l} a^2 = b^2 + c^2 - 2\,b\,c \cos A \\ \cos A = \dfrac{b^2 + c^2 - a^2}{2\,bc} \end{array}\right\} \quad \ldots\ldots\ldots \quad (10)$$

$$\left.\begin{array}{l} \sin \frac{1}{2} A = \sqrt{\dfrac{(s-b)(s-c)}{b\,c}} \\ \cos \frac{1}{2} A = \sqrt{\dfrac{s(s-a)}{b\,c}} \end{array}\right\} \quad \ldots\ldots\ldots \quad (11)$$

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \quad \ldots\ldots\ldots \quad (12)$$

$$\tan \tfrac{1}{2}(A-B) = \frac{a-b}{a+b} \cot \tfrac{1}{2} C.$$

Area of triangle $= \frac{1}{2}$ (*base* $\times$ *height*)
$= \frac{1}{2}\, b\, c \sin A$
$= \sqrt{\{ s(s-a)(s-b)\, s-c) \}}$.

Circumference of a circle of radius $r = 2\pi r$.

Area .... $= \pi r^2$.

π is an incommensurable quantity of which an approximate value is $\frac{22}{7}$; and a still more approximate, $3{\cdot}14159$.

Length of arc in a circle (radius r) which subtends an angle of A degrees at the centre $= \frac{A}{180}\pi r$.

Angle subtended at the centre of a circle (radius r) by an arc of length $l = \frac{180^\circ}{\pi} \times \frac{l}{r}$.

Length of arc in a circle (radius r) subtending at the centre an angle of $1^\circ = \frac{\pi}{180} r = ({\cdot}01745) \times r$.

Angle subtended at the centre of a circle by an arc whose length is equal to the radius $= \frac{180^\circ}{\pi} = 57^\circ\,{\cdot}29578$.

Sin $1' = 0{\cdot}000291 = \tan 1'$.

Sin $1'' = 0{\cdot}000004848 = \tan 1''$.

Surface of a sphere (radius r) $= 4\pi r^2$.

Volume .... $= \frac{4}{3}\pi r^3$.

Volume of a pyramid or cone $= \frac{1}{3}$ (*base* $\times$ *height*).

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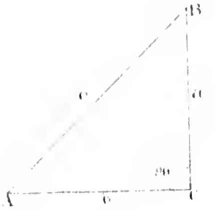
Fig. 3

Fig. 4



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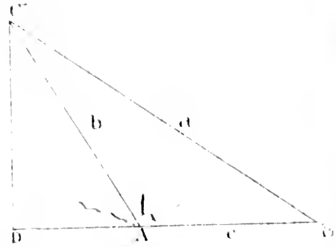

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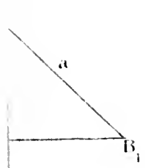

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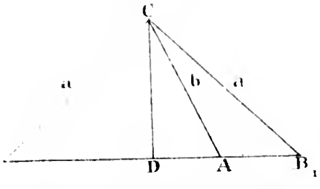

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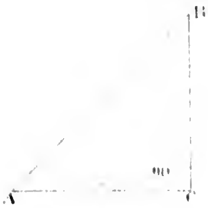
Fig. 1.



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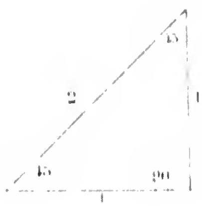
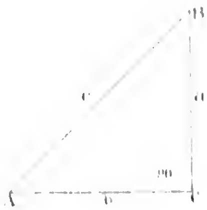


Fig. 3.

Fig. 4.



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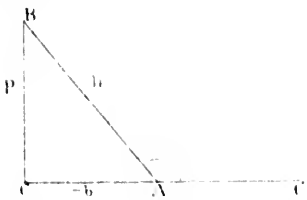

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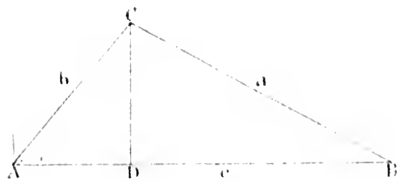

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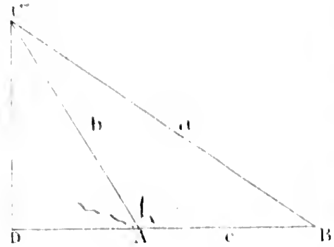

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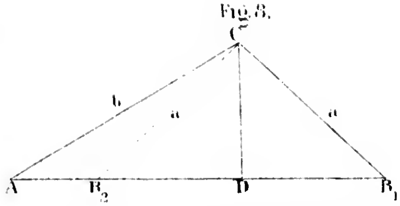

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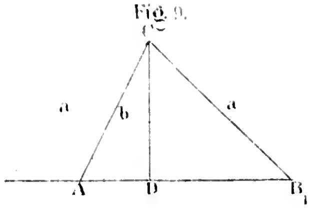

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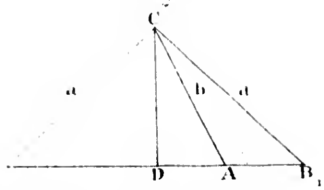

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