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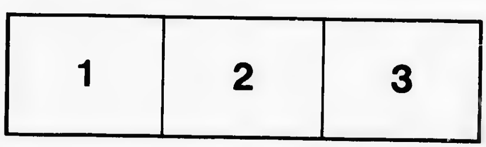

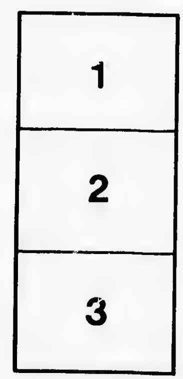

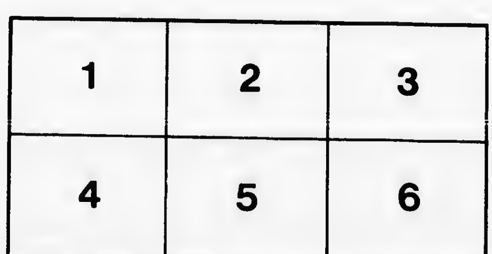

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EUCLID'S

Elements of Geometry:

BOOK I.,

With Explanatory Notes and Questions.

DESIGNED FOR THE USES OF JUNIOR CLASSES IN PUBLIC AND PRIVATE SCHOOLS.

BY

ROBERT POTTS, M. A.

FIVE HUNDREDTH THOUSAND.

TORONTO:
ADAM MILLER & CO.
1876.

EUCLID'S
ELEMENTS OF GEOMETRY.

BOOK I.

DEFINITIONS.

I.

A POINT is that which has no parts, or which has no magnitude.

II.

A line is length without breadth.

III.

The extremities of a line are points.

IV.

A straight line is that which lies evenly between its extreme points.

V.

A superficies is that which has only length and breadth.

VI.

The extremities of a superficies are lines.

VII.

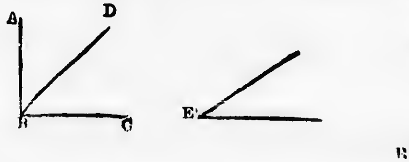
A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

VIII.

A plane angle is the inclination of two lines to each other in a plane, which meet together, but are not in the same direction.

IX.

A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.



N.B. If there be only one angle at a point, it may be expressed by a letter placed at that point, as the angle at *E*: but when several angles are at one point *B*, either of them is expressed by three letters, of which the letter that is at the vertex of the angle, that is, at the point in which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two is somewhere upon one of these straight lines, and the other upon the other line. Thus the angle which is contained by the straight lines *AB*, *CB*, is named the angle *ABC*, or *CBA*; that which is contained by *AB*, *DB*, is named the angle *ABD*, or *DBA*; and that which is contained by *DB*, *CB*, is called the angle *DBC*, or *CBD*.

X.

When a straight line standing on another straight line, makes the adjacent angles equal to one another, each of these angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.

XI.

An obtuse angle is that which is greater than a right angle.

XII.

An acute angle is that which is less than a right angle.

XIII.

A term or boundary is the extremity of any thing.

XIV.

A figure is that which is enclosed by one or more boundaries.

XV.

A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another.

XVI.

And this point is called the center of the circle.

XVII.

A diameter of a circle is a straight line drawn through the center, and terminated both ways by the circumference.

XVIII.

A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XIX.

The center of a semicircle is the same with that of the circle.

XX.

Rectilineal figures are those which are contained by straight lines.

XXI.

Trilateral figures, or triangles, by three straight lines.

XXII.

Quadrilateral, by four straight lines.

XXIII.

Multilateral figures, or polygons, by more than four straight lines, XXIV.

Of three-sided figures, an equilateral triangle is that which has three equal sides.

XXV.

An isosceles triangle is that which has two sides equal.

XXVI.

A scalene triangle is that which has three unequal sides.

XXVII.

A right-angled triangle is that which has a right angle.

XXVIII.

An obtuse-angled triangle is that which has an obtuse angle.

XXIX.

An acute-angled triangle is that which has three acute angles.

XXX.

Of quadrilateral or four-sided figures, a square has all its sides equal and all its angles right angles.

XXXI.

An oblong is that which has all its angles right angles, but has not all its sides equal.

XXXII.

A rhombus has all its sides equal, but its angles are not right angles.

XXXIII.

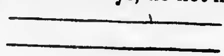
A rhomboid has its opposite sides equal to each other, but all its sides are not equal, nor its angles right angles.

XXXIV.

All other four-sided figures besides these, are called Trapeziums.

XXXV.

Parallel straight lines are such as are in the same plane, and which being produced ever so far both ways, do not meet.



A.

A parallelogram is a four-sided figure, of which the opposite sides are parallel: and the diameter, or the diagonal is the straight line joining two of its opposite angles.

POSTULATES.

I.

Let it be granted that a straight line may be drawn from any one point to any other point.

II.

That a terminated straight line may be produced to any length in a straight line.

III.

And that a circle may be described from any center, at any distance from that center.

AXIOMS.

I.

THINGS which are equal to the same thing are equal to one another.

II.

If equals be added to equals, the wholes are equal.

III.

If equals be taken from equals, the remainders are equal.

IV.

If equals be added to unequals, the wholes are unequal.

V.

If equals be taken from unequals, the remainders are unequal.

VI.

Things which are double of the same, are equal to one another.

VII.

Things which are halves of the same, are equal to one another.

VIII.

Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

IX.

The whole is greater than its part.

X.

Two straight lines cannot enclose a space.

XI.

All right angles are equal to one another.

XII.

If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles; these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.

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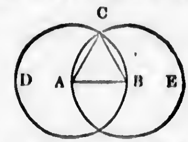
PROPOSITION I. PROBLEM.

To describe an equilateral triangle upon a given finite straight line.

Let *AB* be the given straight line.
It is required to describe an equilateral triangle upon *AB*.

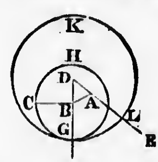


From the center *A*, at the distance *AB*, describe the circle *BCD*; (post. 3.)
from the center *B*, at the distance *BA*, describe the circle *ACE*;
and from *C*, one of the points in which the circles cut one another,
draw the straight lines *CA*, *CB* to the points *A*, *B*. (post. 1.)
Then *ABC* shall be an equilateral triangle.
Because the point *A* is the center of the circle *BCD*,
therefore *AC* is equal to *AB*; (def. 15.)
and because the point *B* is the center of the circle *ACE*,
therefore *BC* is equal to *AB*;
but it has been proved that *AC* is equal to *AB*;
therefore *AC*, *BC* are each of them equal to *AB*;
but things which are equal to the same thing are equal to one another;
therefore *AC* is equal to *BC*; (ax. 1.)
wherefore *AB*, *BC*, *CA* are equal to one another:
and the triangle *ABC* is therefore equilateral,
and it is described upon the given straight line *AB*.
Which was required to be done.

PROPOSITION II. PROBLEM.

From a given point, to draw a straight line equal to a given straight line

Let *A* be the given point, and *BC* the given straight line.
It is required to draw from the point *A*, a straight line equal to *BC*.



From the point *A* to *B* draw the straight line *AB*; (post. 1.)
upon *AB* describe the equilateral triangle *ABD*, (I. 1.)
and produce the straight lines *DA*, *DB* to *E* and *F*; (post. 2.)
from the center *B*, at the distance *BC*, describe the circle *CGH*. (post. 3.) cutting *DF* in the point *G*:
and from the center *D*, at the distance *DG*, describe the circle *GKL*. cutting *AE* in the point *L*.

Then the straight line AL shall be equal to BC.
Because the point B is the center of the circle CGH,
therefore BC is equal to BG; (def. 15.)
and because D is the center of the circle GKL,
therefore DL is equal to DG,
and DA, DB parts of them are equal; (I. 1.)
therefore the remainder AL is equal to the remainder BG; (ax. 3.)
but it has been shewn that BC is equal to BG,
wherefore AL and BC are each of them equal to BG;
and things that are equal to the same thing are equal to one another
therefore the straight line AL is equal to BC. (ax. 1.)

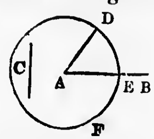
Wherefore from the given point A, a straight line AL has been drawn equal to the given straight line BC. Which was to be done.

PROPOSITION III. PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, of which AB is the greater.

It is required to cut off from AB the greater, a part equal to C, the less.



From the point A draw the straight line AD equal to C; (I. 2.) and from the center A, at the distance AD, describe the circle DEF (post. 3.) cutting AB in the point E.

Then AE shall be equal to C.
Because A is the center of the circle DEF,
therefore AE is equal to AD; (def. 15.)
but the straight line C is equal to AD; (constr.)
whence AE and C are each of them equal to AD;
wherefore the straight line AE is equal to C. (ax. 1.)

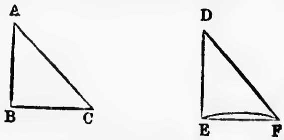
And therefore from AB the greater of two straight lines, a part AE has been cut off equal to C, the less. Which was to be done.

PROPOSITION IV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to each other; they shall likewise have their bases or third sides equal, and the two triangles shall be equal, and their other angles shall be equal, each to each, viz. those to which the equal sides are opposite.

Let ABC, DEF be two triangles, which have the two sides AB, AC equal to the two sides DE, DF, each to each. viz. AB to DE. and AC to DF. and the included angle BAC equal to the included angle EDF.

Then shall the base BC be equal to the base EF; and the triangle ABC to the triangle DEF; and the other angles to which the equal sides are opposite shall be equal, each to each, viz. the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

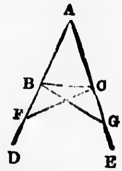


For, if the triangle ABC be applied to the triangle DEF,
so that the point A may be on D, and the straight line AB on DE;
then the point B shall coincide with the point E,
because AB is equal to DE;
and AB coinciding with DE,
the straight line AC shall fall on DF,
because the angle BAC is equal to the angle EDF;
therefore also the point C shall coincide with the point F,
because AC is equal to DF;
but the point B was shewn to coincide with the point E;
wherefore the base BC shall coincide with the base EF;
because the point B coinciding with E, and C with F,
if the base BC do not coincide with the base EF, the two straight lines BC and EF would enclose a space, which is impossible. (ax. 10.)
Therefore the base BC does coincide with EF, and is equal to it;
and the whole triangle ABC coincides with the whole triangle DEF, and is equal to it;
also the remaining angles of one triangle coincide with the remaining angles of the other, and are equal to them,
viz. the angle ABC to the angle DEF,
and the angle ACB to DFE.
Therefore, if two triangles have two sides of the one equal to two sides, &c. Which was to be demonstrated.

PROPOSITION V. THEOREM.

The angles at the base of an isosceles triangle are equal to each other; and if the equal sides be produced, the angles on the other side of the base shall be equal.

Let ABC be an isosceles triangle of which the side AB is equal to AC,
and let the equal sides AB, AC be produced to D and E.
Then the angle ABC shall be equal to the angle ACB,
and the angle DBC to the angle ECB.
In BD take any point F;
from AE the greater, cut off AG equal to AF the less, (I. 3.)
and join FC, GB.
Because AF is equal to AG, (constr.) and AB to AC; (hyp.)
the two sides FA, AC are equal to the two GA, AB, each to each;
and they contain the angle FAG common to the two triangles AFC, AGB;



therefore the base *FC* is equal to the base *GB*, (I. 4.)
and the triangle *AFC* is equal to the triangle *AGB*,
also the remaining angles of the one are equal to the remaining angles of the other, each to each, to which the equal sides are opposite;
viz. the angle *ACF* to the angle *ABG*,
and the angle *AFC* to the angle *AGB*.
And because the whole *AF* is equal to the whole *AG*,
of which the parts *AB*, *AC*, are equal;
therefore the remainder *BF* is equal to the remainder *CG*; (ax. 3.)
and *FC* has been proved to be equal to *GB*;
hence, because the two sides *BF*, *FC* are equal to the two *CG*, *GB*, each to each;
nd the angle *BFC* has been proved to be equal to the angle *CGB*,
also the base *BC* is common to the two triangles *BFC*, *CGB*;
wherefore these triangles are equal, (I. 4.)
and their remaining angles, each to each, to which the equal sides are opposite;
therefore the angle *FBC* is equal to the angle *GCB*,
and the angle *BCF* to the angle *CBG*.
And, since it has been demonstrated,
that the whole angle *ABG* is equal to the whole *ACF*,
the parts of which, the angles *CBG*, *BCF* are also equal;
therefore the remaining angle *ABC* is equal to the remaining angle *ACB*.
which are the angles at the base of the triangle *ABC*;
and it has also been proved,
that the angle *FBC* is equal to the angle *GCB*,
which are the angles upon the other side of the base.
Therefore the angles at the base, &c. Q.E.D.
Cor. Hence an equilateral triangle is also equiangular.

PROPOSITION VI. THEOREM.

If two angles of a triangle be equal to each other; the sides also which subtend, or are opposite to, *the equal angles, shall be equal to one another.*

Let *ABC* be a triangle having the angle *ABC* equal to the angle *ACB*
Then the side *AB* shall be equal to the side *AC*.

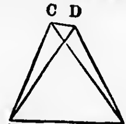
For, if AB be not equal to AC,
one of them is greater than the other.
If possible, let AB be greater than AC;
and from BA cut off BD equal to CA the less, (I. 3.) and join DC.
Then, in the triangles DBC, ABC,
because DB is equal to AC, and BC is common to both triangles,
the two sides DB, BC are equal to the two sides AC, CB, each to each;
and the angle DBC is equal to the angle ACB; (hyp.)
therefore the base DC is equal to the base AB, (I. 4.)
and the triangle DBC is equal to the triangle ABC,
the less equal to the greater, which is absurd. (ax. 9.)
Therefore AB is not unequal to AC, that is, AB is equal to AC.
Wherefore, if two angles, &c. Q.E.D.

COR. Hence an equiangular triangle is also equilateral.

PROPOSITION VII. THEOREM.

Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base, equal to one another, and likewise those which are terminated in the other extremity.

If it be possible, on the same base AB, and upon the same side of it, let there be two triangles ACB, ADB, which have their sides CA, DA, terminated in the extremity A of the base, equal to one another, and likewise their sides CB, DB, that are terminated in B.

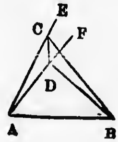


Join CD.

First. When the vertex of each of the triangles is without the other triangle.

Because AC is equal to AD in the triangle ACD,
therefore the angle ADC is equal to the angle ACD; (I. 5.)
but the angle ACD is greater than the angle BCD; (ax. 9.)
therefore also the angle ADC is greater than BCD;
much more therefore is the angle BDC greater than BCD.
Again, because the side BC is equal to BD in the triangle BCD, (hyp.)
therefore the angle BDC is equal to the angle BCD; (I. 5.)
but the angle BDC was proved greater than the angle BCD,
hence the angle BDC is both equal to, and greater than the angle BCD;
which is impossible.

Secondly. Let the vertex D of the triangle ADB fall within the triangle ACB.



Produce AC to E, and AD to F, and join CD.
Then because AC is equal to AD in the triangle ACD, therefore the angles ECD, FDC upon the other side of the base CD, are equal to one another; (I. 5.)
but the angle ECD is greater than the angle BCD; (ax. 9.)
therefore also the angle FDC is greater than the angle BCD; much more then is the angle BDC greater than the angle BCD.
Again, because BC is equal to BD in the triangle BCD, therefore the angle BDC is equal to the angle BCD, (I. 5.)
but the angle BDC has been proved greater than BCD, wherefore the angle BDC is both equal to, and greater than the angle BCD; which is impossible.

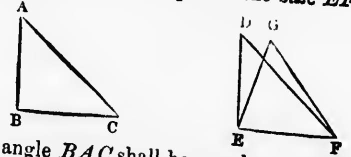
Thirdly. The case in which the vertex of one triangle is upon a side of the other, needs no demonstration.

Therefore, upon the same base and on the same side of it, &c. Q.E.D.

PROPOSITION VIII. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them, of the other.

Let ABC, DEF be two triangles, having the two sides AB, AC, equal to the two sides DE, DF, each to each, viz. AB to DE, and AC to DF, and also the base BC equal to the base EF.



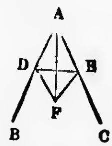
Then the angle BAC shall be equal to the angle EDF.
For, if the triangle ABC be applied to DEF, so that the point B be on E, and the straight line BC on EF;
then because BC is equal to EF, (hyp.)
therefore the point C shall coincide with the point F.
wherefore BC coinciding with EF,
BA and AC shall coincide with ED, DF;
for, if the base BC coincide with the base EF, but the sides BA, AC, do not coincide with the sides ED, DF, but have a different situation as EG, GF:
then, upon the same base, and upon the same side of it, there can be two triangles which have their sides which are terminated in one extremity of the base, equal to one another, and likewise those sides which are terminated in the other extremity; but this is impossible. (I. 7.)
Therefore, if the base BC coincide with the base EF, the sides BA, AC cannot but coincide with the sides ED, DF; wherefore likewise the angle BAC coincides with the angle EDF, and is equal to it. (ax. 8.)

Therefore if two triangles have two sides, &c. Q.E.D.

PROPOSITION IX. PROBLEM.

To bisect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle.
It is required to bisect it.

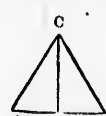


In AB take any point D;
from AC cut off AE equal to AD, (I. 3.) and join DE;
on the side of DE remote from A,
describe the equilateral triangle DEF (I. 1.), and join AF.
Then the straight line AF shall bisect the angle BAC.
Because AD is equal to AE, (constr.)
and AF is common to the two triangles DAF, EAF;
the two sides DA, AF, are equal to the two sides EA, AF, each to each;
and the base DF is equal to the base EF: (constr.)
therefore the angle DAF is equal to the angle EAF. (I. 8.)
Wherefore the angle BAC is bisected by the straight line AF. Q.E.F.

PROPOSITION X. PROBLEM.

To bisect a given finite straight line, that is, to divide it into two equal parts.

Let AB be the given straight line.
It is required to divide AB into two equal parts.
Upon AB describe the equilateral triangle ABC; (I. 1.)



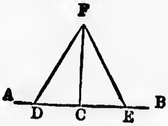
and bisect the angle ACB by the straight line CD meeting AB in the point D. (I. 9.)
Then AB shall be cut into two equal parts in the point D.
Because AC is equal to CB, (constr.)
and CD is common to the two triangles ACD, BCD;
the two sides AC, CD are equal to the two BC, CD, each to each;
and the angle ACD is equal to BCD; (constr.)
therefore the base AD is equal to the base BD. (I. 4.)
Wherefore the straight line AB is divided into two equal parts in the point D. Q.E.F.

PROPOSITION XI. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be the given straight line, and C a given point in it.

It is required to draw a straight line from the point C at right angles to AB



In AC take any point D, and make CE equal to CD; (I. 3.)
upon DE describe the equilateral triangle DEF (I. 1,) and join CF.

Then CF drawn from the point C, shall be at right angles to AB.

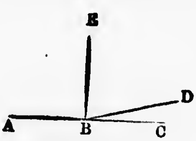
Because DC is equal to EC, and FC is common to the two triangles DCF, ECF;
the two sides DC, CF are equal to the two sides EC, CF, each to each;
and the base DF is equal to the base EF; (constr.)
therefore the angle DCF is equal to the angle ECF: (I. 8.)
and these two angles are adjacent angles.

But when the two adjacent angles which one straight line makes with another straight line, are equal to one another, each of them is called a right angle: (def. 10.)
therefore each of the angles DCF, ECF is a right angle.

Wherefore from the given point C, in the given straight line AB, FC has been drawn at right angles to AB. Q.E.F.

COR. By help of this problem, it may be demonstrated that two straight lines cannot have a common segment.

If it be possible, let the segment AB be common to the two straight lines ABC, ABD.



From the point B, draw BE at right angles to AB; (I. 11.)
then because ABC is a straight line,
therefore the angle ABE is equal to the angle EBC. (def. 10.)
Similarly, because ABD is a straight line,
therefore the angle ABE is equal to the angle EBD;
but the angle ABE is equal to the angle EBC,
wherefore the angle EBD is equal to the angle EBC, (ax. 1.)
the less equal to the greater angle, which is impossible.

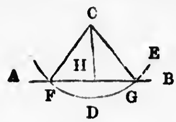
Therefore two straight lines cannot have a common segment.

PROPOSITION XII. PROBLEM.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced any length both ways, and let C be a point without it.

It is required to draw a straight line perpendicular to AB from the point C.



Upon the other side of AB take any point D,
and from the center C, at the distance CD, describe the circle EGF
meeting AB, produced if necessary, in F and G: (post. 3.)
bisect FG in H (I. 10.), and join CH.

Then the straight line CH drawn from the given point C, shall be perpendicular to the given straight line AB.

Join FC, and CG.
Because FH is equal to HG, (constr.)
and HC is common to the triangles FHC, GHC;
the two sides FH, HC, are equal to the two GH, HC, each to each;
and the base CF is equal to the base CG; (def. 15.)
therefore the angle FHC is equal to the angle GHC; (I. 8.)
and these are adjacent angles.

But when a straight line standing on another straight line, makes the adjacent angles equal to one another, each of them is a right angle, and the straight line which stands upon the other is called a perpendicular to it. (def. 10.)

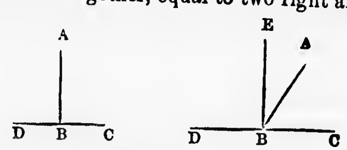
Therefore from the given point C, a perpendicular CH has been drawn to the given straight line AB. Q.E.F.

PROPOSITION XIII. THEOREM.

The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line AB make with CD, upon one side of it, the angles CBA, ABD.

Then these shall be either two right angles,
or, shall be together, equal to two right angles.



For if the angle CBA be equal to the angle ABD,
each of them is a right angle. (def. 10.)
But if the angle CBA be not equal to the angle ABD,
from the point B draw BE at right angles to CD. (I. 11.)
Then the angles CBE, EBD are two right angles. (def. 10.)

And because the angle CBE is equal to the angles CBA, ABE,
add the angle EBD to each of these equals;
therefore the angles CBE, EBD are equal to the three angles CBA, ABE, EBD. (ax. 2.)

Again, because the angle DBA is equal to the two angles DBE, EBA,
add to each of these equals the angle ABC;
therefore the angles DBA, ABC are equal to the three angles DBE, EBA, ABC.

But the angles CBE, EBD have been proved equal to the same three angles;
and things which are equal to the same thing are equal to one another;
therefore the angles CBE, EBD are equal to the angles DBA, ABC;
but the angles CBE, EBD are two right angles;
therefore the angles DBA, ABC are together equal to two right angles. (ax. 1.)

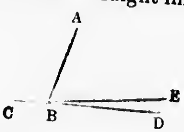
Wherefore, when a straight line, &c. Q.E.D

PROPOSITION XIV. THEOREM.

If at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles; then these two straight lines shall be in one and the same straight line.

At the point B in the straight line AB, let the two straight lines BC, BD upon the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles.

Then BD shall be in the same straight line with BC.



For, if BD be not in the same straight line with BC,
if possible, let BE be in the same straight line with it.

Then because AB meets the straight line CBE;
therefore the adjacent angles CBA, ABE are equal to two right angles; (I. 13.)
but the angles CBA, ABD are equal to two right angles; (hyp.)
therefore the angles CBA, ABE are equal to the angles CBA, ABD: (ax. 1.)
take away from these equals the common angle CBA,
therefore the remaining angle ABE is equal to the remaining angle ABD; (ax. 3.)
the less angle equal to the greater, which is impossible:
therefore BE is not in the same straight line with BC.

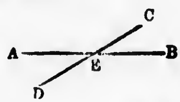
And in the same manner it may be demonstrated, that no other can be in the same straight line with it but BD, which therefore is in the same straight line with BC.

Wherefore, if at a point, &c. Q.E.D.

PROPOSITION XV. THEOREM.

If two straight lines cut one another, the vertical, or opposite angles shall be equal.

Let the two straight lines *AB*, *CD* cut one another in the point *E*. Then the angle *AEC* shall be equal to the angle *DEB*, and the angle *CEB* to the angle *AED*.



Because the straight line *AE* makes with *CD* at the point *E*, the adjacent angles *CEA*, *AED*;

these angles are together equal to two right angles. (I. 13.)

Again, because the straight line *DE* makes with *AB* at the point *E*, the adjacent angles *AED*, *DEB*;

these angles also are equal to two right angles;

but the angles *CEA*, *AED* have been shewn to be equal to two right angles;

wherefore the angles *CEA*, *AED* are equal to the angles *AED*, *DEB*:

take away from each the common angle *AED*,

and the remaining angle *CEA* is equal to the remaining angle *DEB*. (ax. 3.)

In the same manner it may be demonstrated, that the angle *CEB* is equal to the angle *AED*.

Therefore, if two straight lines cut one another, &c. Q. E. D.

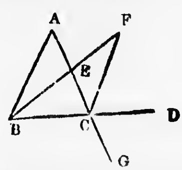
Cor. 1. From this it is manifest, that, if two straight lines cut each other, the angles which they make at the point where they cut, are together equal to four right angles.

Cor. 2. And consequently that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

PROPOSITION XVI. THEOREM.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Let *ABC* be a triangle, and let the side *BC* be produced to *D*. Then the exterior angle *ACD* shall be greater than either of the interior opposite angles *CBA* or *BAC*.



Bisect *AC* in *E*, (I. 10.) and join *BE*;

produce *BE* to *F*, making *EF* equal to *BE*, (I. 3.) and join *FC*.

Because AE is equal to EC, and BE to EF; (constr.) the two sides AE, EB are equal to the two CE, EF, each to each, in the triangles ABE, CFE;

and the angle AEB is equal to the angle CEF, because they are opposite vertical angles; (I. 15.) therefore the base AB is equal to the base CF, (I. 4.) and the triangle AEB to the triangle CEF, and the remaining angles of one triangle to the remaining angles of the other, each to each, to which the equal sides are opposite; wherefore the angle BAE is equal to the angle ECF; but the angle ECD or ACD is greater than the angle ECF; therefore the angle ACD is greater than the angle BAE or BAC.

In the same manner, if the side BC be bisected, and AC be produced to G; it may be demonstrated that the angle BCG, that is, the angle ACD, (I. 15.) is greater than the angle ABC.

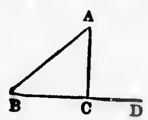
Therefore, if one side of a triangle, &c. Q. E. D.

PROPOSITION XVII. THEOREM.

Any two angles of a triangle are together less than two right angles.

Let ABC be any triangle.

Then any two of its angles together shall be less than two right angles.



Produce any side BC to D.

Then because ACD is the exterior angle of the triangle ABC; therefore the angle ACD is greater than the interior and opposite angle ABC; (I. 16.)

to each of these unequals add the angle ACB; therefore the angles ACD, ACB are greater than the angles ABC, ACB;

but the angles ACD, ACB are equal to two right angles; (I. 13.) therefore the angles ABC, ACB are less than two right angles.

In like manner it may be demonstrated, that the angles BAC, ACB are less than two right angles, as also the angles CAB, ABC.

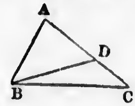
Therefore any two angles of a triangle, &c. Q. E. D.

PROPOSITION XVIII. THEOREM.

The greater side of every triangle is opposite to the greater angle.

Let ABC be a triangle, of which the side AC is greater than the side AB.

Then the angle ABC shall be greater than the angle ACB



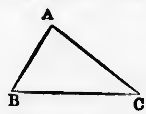
Since the side AC is greater than the side AB, (hyp.)
make AD equal to AB, (I. 3.) and join BD.
Then, because AD is equal to AB, in the triangle ABD,
therefore the angle ABD is equal to the angle ADB, (I. 5.)
but because the side CD of the triangle BDC is produced to A
therefore the exterior angle ADB is greater than the interior and opposite angle DCB; (I. 16.)
but the angle ADB has been proved equal to the angle ABD,
therefore the angle ABD is greater than the angle DCB;
wherefore much more is the angle ABC greater than the angle ACB
Therefore the greater side, &c. Q. E. D.

PROPOSITION XIX. THEOREM.

The greater angle of every triangle is subtended by the greater side, or, has the greater side opposite to it.

Let ABC be a triangle of which the angle ABC is greater than the angle BCA.

Then the side AC shall be greater than the side AB.



For, if AC be not greater than AB,
AC must either be equal to, or less than AB;
if AC were equal to AB,
then the angle ABC would be equal to the angle ACB; (I. 5.)
but it is not equal; (hyp.)
therefore the side AC is not equal to AB.
Again, if AC were less than AB,
then the angle ABC would be less than the angle ACB; (I. 18.)
but it is not less, (hyp.)
therefore the side AC is not less than AB;
and AC has been shewn to be not equal to AB;
therefore AC is greater than AB.
Wherefore the greater angle, &c. Q. E. D.

PROPOSITION XX. THEOREM.

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle.

Then any two sides of it together shall be greater than the third side,
viz. the sides BA, AC greater than the side BC;

AB, *BC* greater than *AC*;
and *BC*, *CA* greater than *AB*.

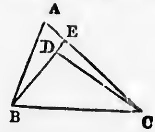
Produce the side *BA* to the point *D*,
make *AD* equal to *AC*, (I. 3.) and join *DC*.
Then because *AD* is equal to *AC*, (constr.)
therefore the angle *ACD* is equal to the angle *ADC*; (I. 5.)
but the angle *BCD* is greater than the angle *ACD*; (ax. 9.)
therefore also the angle *BCD* is greater than the angle *ADC*.
And because in the triangle *DBC*,
the angle *BCD* is greater than the angle *BDC*,
and that the greater angle is subtended by the greater side; (I. 19.)
therefore the side *DB* is greater than the side *BC*;
but *DB* is equal to *BA* and *AC*,
therefore the sides *BA* and *AC* are greater than *BC*.
In the same manner it may be demonstrated,
that the sides *AB*, *BC* are greater than *CA*;
also that *BC*, *CA* are greater than *AB*.
Therefore any two sides, &c. Q.E.D.

PROPOSITION XXI. THEOREM.

If from the ends of a side of a triangle, there be drawn two straight lines to a point within the triangle; these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let *ABC* be a triangle, and from the points *B*, *C*, the ends of the side *BC*, let the two straight lines *BD*, *CD* be drawn to a point *D* within the triangle.

Then *BD* and *DC* shall be less than *BA* and *AC* the other two sides of the triangle,
but shall contain an angle *BDC* greater than the angle *BAC*.



Produce *BD* to meet the side *AC* in *E*.
Because two sides of a triangle are greater than the third side, (I. 20.)
therefore the two sides *BA*, *AE* of the triangle *ABE* are greater than *BE*;

to each of these unequals add *EC*;
therefore the sides *BA*, *AC* are greater than *BE*, *EC*. (ax. 4.)
Again, because the two sides *CE*, *ED* of the triangle *CED* are greater than *DC*; (I. 20.)
add *DB* to each of these unequals;

therefore the sides *CE*, *EB* are greater than *CD*, *DB*. (ax. 4.)

But it has been shewn that *BA*, *AC* are greater than *BE*, *EC*;

much more then are *BA*, *AC* greater than *BD*, *DC*.

Again, because the exterior angle of a triangle is greater than the interior and opposite angle; (I. 16.)

therefore the exterior angle *BDC* of the triangle *CDE* is greater than the interior and opposite angle *CED*;

for the same reason, the exterior angle *CED* of the triangle *ABE* is greater than the interior and opposite angle *BAC*;

and it has been demonstrated,

that the angle *BDC* is greater than the angle *CEB*;

much more therefore is the angle *BDC* greater than the angle *BAC*.

Therefore, if from the ends of the side, &c. Q.E.D.

PROPOSITION XXII. PROBLEM.

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

Let *A*, *B*, *C* be the three given straight lines,

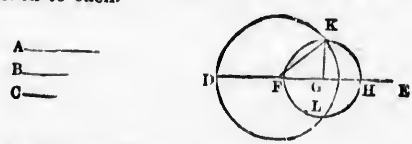
of which any two whatever are greater than the third, (I. 20.)

namely, *A* and *B* greater than *C*;

A and *C* greater than *B*;

and *B* and *C* greater than *A*.

It is required to make a triangle of which the sides shall be equal to *A*, *B*, *C*, each to each.



Take a straight line *DE* terminated at the point *D*, but unlimited towards *E*.

make *DF* equal to *A*, *FG* equal to *B*, and *GH* equal to *C*; (I. 3.)

from the center *F*, at the distance *FD*, describe the circle *DKL*; (post 3.)

from the center *G*, at the distance *GH*, describe the circle *HLK*;

from *K* where the circles cut each other, draw *KF*, *KG* to the points *F*, *G*;

Then the triangle *KFG* shall have its sides equal to the three straight lines *A*, *B*, *C*.

Because the point *F* is the center of the circle *DKL*,

therefore *FD* is equal to *FK*; (def. 15.)

but *FD* is equal to the straight line *A*;

therefore *FK* is equal to *A*.

Again, because *G* is the center of the circle *HKL*,

therefore *GH* is equal to *GK*, (def. 15.)

but *GH* is equal to *C*;

therefore also *GK* is equal to *C*; (ax. 1.)

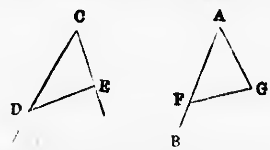
and *FG* is equal to *B*;

therefore the three straight lines *KF, FG, GK*, are respectively equal to the three, *A, B, C*:
and therefore the triangle *KFG* has its three sides *KF, FG, GK*, equal to the three given straight lines *A, B, C*. Q.E.F.

PROPOSITION XXIII. PROBLEM.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let *AB* be the given straight line, and *A* the given point in it, and *DCE* the given rectilineal angle.
It is required, at the given point *A* in the given straight line *AB*, to make an angle that shall be equal to the given rectilineal angle *DCE*.

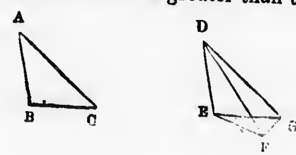


In *CD, CE*, take any points *D, E*, and join *DE*;
on *AB*, make the triangle *AFG*, the sides of which shall be equal to the three straight lines *CD, DE, EC*, so that *AF* be equal to *CD*, *AG* to *CE*, and *FG* to *DE*. (I. 22.)
Then the angle *FAG* shall be equal to the angle *DCE*.
Because *FA, AG* are equal to *DC, CE*, each to each,
and the base *FG* is equal to the base *DE*;
therefore the angle *FAG* is equal to the angle *DCE*. (I. 8.)
Wherefore, at the given point *A* in the given straight line *AB*, the angle *FAG* is made equal to the given rectilineal angle *DCE*. Q.E.F.

PROPOSITION XXIV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other; the base of that which has the greater angle, shall be greater than the base of the other.

Let *ABC, DEF* be two triangles, which have the two sides *AB, AC*, equal to the two *DE, DF*, each to each, namely, *AB* equal to *DE*, and *AC* to *DF*; but the angle *BAC* greater than the angle *EDF*.
Then the base *BC* shall be greater than the base *EF*.



Of the two sides *DE*, *DF*, let *DE* be not greater than *DF*, at the point *D*, in the line *DE*, and on the same side of it as *DF*, make the angle *EDG* equal to the angle *BAC*; (I. 23.) make *DG* equal to *DF* or *AC*, (I. 3.) and join *EG*, *GF*.

Then, because *DE* is equal to *AB*, and *DG* to *AC*, the two sides *DE*, *DG* are equal to the two *AB*, *AC*, each to each, and the angle *EDG* is equal to the angle *BAC*; therefore the base *EG* is equal to the base *BC*. (I. 4.)

And because *DG* is equal to *DF* in the triangle *DFG*, therefore the angle *DFG* is equal to the angle *DGF*; (I. 5.) but the angle *DGF* is greater than the angle *EGF*; (ax. 9.) therefore the angle *DFG* is also greater than the angle *EGF*; much more therefore is the angle *EFG* greater than the angle *EGF*.

And because in the triangle *EFG*, the angle *EFG* is greater than the angle *EGF*, and that the greater angle is subtended by the greater side; (I. 19.) therefore the side *EG* is greater than the side *EF*; but *EG* was proved equal to *BC*; therefore *BC* is greater than *EF*.

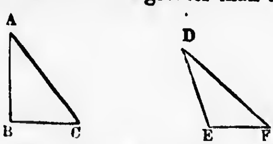
Wherefore, if two triangles, &c. Q.E.D.

PROPOSITION XXV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other; the angle contained by the sides of the one which has the greater base, shall be greater than the angle contained by the sides, equal to them, of the other.

Let *ABC*, *DEF* be two triangles which have the two sides *AB*, *AC*, equal to the two sides *DE*, *DF*, each to each, namely, *AB* equal to *DE*, and *AC* to *DF*; but the base *BC* greater than the base *EF*.

Then the angle *BAC* shall be greater than the angle *EDF*.



For, if the angle *BAC* be not greater than the angle *EDF*, it must either be equal to it, or less than it.

If the angle *BAC* were equal to the angle *EDF*, then the base *BC* would be equal to the base *EF*; (I. 4.) but it is not equal, (hyp.) therefore the angle *BAC* is not equal to the angle *EDF*.

Again, if the angle *BAC* were less than the angle *EDF*, then the base *BC* would be less than the base *EF*; (I. 24.) but it is not less, (hyp.) therefore the angle *BAC* is not less than the angle *EDF*; and it has been shewn, that the angle *BAC* is not equal to the angle *EDF*; therefore the angle *BAC* is greater than the angle *EDF*.

Wherefore, if two triangles, &c. Q.E.D.

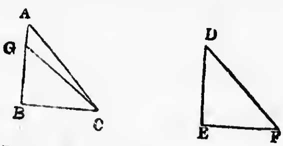
PROPOSITION XXVI. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, viz. either the sides adjacent to the equal angles in each, or the sides opposite to them; then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.

Let ABC, DEF be two triangles which have the angles ABC, BCA, equal to the angles DEF, EFD, each to each, namely, ABC to DEF, and BCA to EFD; also one side equal to one side.

First, let those sides be equal which are adjacent to the angles that are equal in the two triangles, namely, BC to EF.

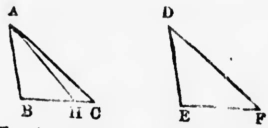
Then the other sides shall be equal, each to each, namely, AB to DE, and AC to DF, and the third angle BAC to the third angle EDF.



For, if AB be not equal to DE,
one of them must be greater than the other.
If possible, let AB be greater than DE,
make BG equal to ED, (I. 3) and join GC.
Then in the two triangles GBC, DEF,
because GB is equal to DE, and BC to EF, (hyp.)
the two sides, GB, BC are equal to the two DE, EF, each to each;
and the angle GBC is equal to the angle DEF;
therefore the base GC is equal to the base DF, (I. 4.)
and the triangle GBC to the triangle DEF,
and the other angles to the other angles, each to each, to which the equal sides are opposite;
therefore the angle GCB is equal to the angle DFE;
but the angle ACB is, by the hypothesis, equal to the angle DFE;
wherefore also the angle GCB is equal to the angle ACB; (ax. 1.)
the less angle equal to the greater, which is impossible;
therefore AB is not unequal to DE,
that is, AB is equal to DE.
Hence, in the triangles ABC, DEF;
because AB is equal to DE, and BC to EF, (hyp.)
and the angle ABC is equal to the angle DEF; (hyp.)
therefore the base AC is equal to the base DF, (I. 4.)
and the third angle BAC to the third angle EDF.

Secondly, let the sides which are opposite to one of the equal angles in each triangle be equal to one another, namely AB equal to DE.

Then in this case likewise the other sides shall be equal, AC to DF, and BC to EF and also the third angle BAC to the third angle EDF.

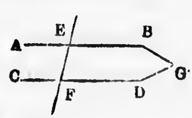


For if BC be not equal to EF,
one of them must be greater than the other.
If possible, let BC be greater than EF;
make BH equal to EF, (I. 3.) and join AH.
Then in the two triangles ABH, DEF,
because AB is equal to DE, and BH to EF,
and the angle ABH to the angle DEF; (hyp.)
therefore the base AH is equal to the base DF, (I. 4.)
and the triangle ABH to the triangle DEF.
and the other angles to the other angles, each to each, to which the equal sides are opposite;
therefore the angle BHA is equal to the angle EFD;
but the angle EFD is equal to the angle BCA; (hyp.)
therefore the angle BHA is equal to the angle BCA, (ax. 1.)
that is, the exterior angle BHA of the triangle AHC, is equal to its interior and opposite angle BCA;
which is impossible; (I. 16.)
wherefore BC is not unequal to EF,
that is, BC is equal to EF.
Hence, in the triangles ABC, DEF;
because AB is equal to DE, and BC to EF, (hyp.)
and the included angle ABC is equal to the included angle DEF; (hyp.)
therefore the base AC is equal to the base DF, (I. 4.)
and the third angle BAC to the third angle EDF.
Wherefore, if two triangles, &c. Q. E. D.

PROPOSITION XXVII. THEOREM.

If a straight line falling on two other straight lines, make the alternate angles equal to each other; these two straight lines shall be parallel.

Let the straight line EF, which falls upon the two straight lines AB, CD, make the alternate angles AEF, EFD, equal to one another.
Then AB shall be parallel to CD.



For, if AB be not parallel to CD,
then AB and CD being produced will meet, either towards A and C, or towards B and D.
Let AB, CD be produced and meet, if possible, towards B and D, in the point G,
then GEF is a triangle.

And because a side *GE* of the triangle *GEF* is produced to *A*,
therefore its exterior angle *AEF* is greater than the interior and opposite angle *EFG*; (I. 16.)
but the angle *AEF* is equal to the angle *EFG*; (hyp.)
therefore the angle *AEF* is greater than, and equal to, the angle *EFG*; which is impossible.

Therefore *AB*, *CD* being produced, do not meet towards *B*, *D*.
In like manner, it may be demonstrated, that they do not meet when produced towards *A*, *C*.

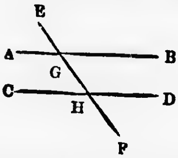
But those straight lines in the same plane, which meet neither way, though produced ever so far, are parallel to one another; (def. 35.)
therefore *AB* is parallel to *CD*.
Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION XXVIII. THEOREM.

If a straight line falling upon two other straight lines, make the exterior angle equal to the interior and opposite upon the same side of the line; or make the interior angles upon the same side together equal to two right angles; the two straight lines shall be parallel to one another.

Let the straight line *EF*, which falls upon the two straight lines *AB*, *CD*, make the exterior angle *EGB* equal to the interior and opposite angle *GHD*, upon the same side of the line *EF*; or make the two interior angles *BGH*, *GHD* on the same side together equal to two right angles.

Then *AB* shall be parallel to *CD*.



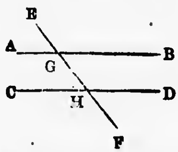
Because the angle *EGB* is equal to the angle *GHD*, (hyp.)
and the angle *EGB* is equal to the angle *AGH*, (I. 15.)
therefore the angle *AGH* is equal to the angle *GHD*; (ax. 1.)
and they are alternate angles,
therefore *AB* is parallel to *CD*. (I. 27.)

Again, because the angles *BGH*, *GHD* are together equal to two right angles, (hyp.)
and that the angles *AGH*, *BGH* are also together equal to two right angles; (I. 13.)
therefore the angles *AGH*, *BGH* are equal to the angles *BGH*, *GHD*; (ax. 1.)
take away from these equals, the common angle *BGH*;
therefore the remaining angle *AGH* is equal to the remaining angle *GHD*; (ax. 3.)
and they are alternate angles;
therefore *AB* is parallel to *CD*. (I. 27.)
Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION XXIX. THEOREM.

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.

Let the straight line *EF* fall upon the parallel straight lines *AB, CD*.
Then the alternate angles *AGH, GHD* shall be equal to one another;
the exterior angle *EGB* shall be equal to the interior and opposite angle *GHD* upon the same side of the line *EF*;
and the two interior angles *BGH, GHD* upon the same side of *EF* shall be together equal to two right angles.

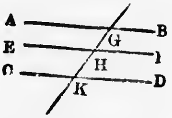


First. For, if the angle *AGH* be not equal to the alternate angle *GHD*, one of them must be greater than the other;
if possible, let *AGH* be greater than *GHD*,
then because the angle *AGH* is greater than the angle *GHD*,
add to each of these unequals the angle *BGH*;
therefore the angles *AGH, BGH* are greater than the angles *BGH GHD*; (ax. 4.)
but the angles *AGH, BGH* are equal to two right angles; (I. 13.)
therefore the angles *BGH, GHD* are less than two right angles;
but those straight lines, which with another straight line falling upon them, make the two interior angles on the same side less than two right angles, will meet together if continually produced; (ax. 12.)
therefore the straight lines *AB, CD*, if produced far enough, will meet towards *B, D*;
but they never meet, since they are parallel by the hypothesis;
therefore the angle *AGH* is not unequal to the angle *GHD*,
that is, the angle *AGH* is equal to the alternate angle *GHD*.
Secondly. Because the angle *AGH* is equal to the angle *EGB*, (I. 15.)
and the angle *AGH* is equal to the angle *GHD*,
therefore the exterior angle *EGB* is equal to the interior and opposite angle *GHD*, on the same side of the line.
Thirdly. Because the angle *EGB* is equal to the angle *GHD*,
add to each of them the angle *BGH*;
therefore the angles *EGB, BGH* are equal to the angles *BGH, GHD*: (ax. 2.)
but *EGB, BGH* are equal to two right angles: (I. 13.)
therefore also the two interior angles *BGH, GHD* on the same side of the line are equal to two right angles. (ax. 1.)
Wherefore, if a straight line, &c. Q.E.D.

PROPOSITION XXX. THEOREM.

Straight lines which are parallel to the same straight line are parallel to each other.

Let the straight lines *AB*, *CD*, be each of them parallel to *EF*.
Then shall *AB* be also parallel to *CD*.

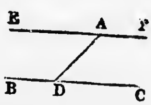


Let the straight line *GHK* cut *AB*, *EF*, *CD*.
Then because *GHK* cuts the parallel straight lines *AB*, *EF*, in *G*, *H*:
therefore the angle *AGH* is equal to the alternate angle *GHF*. (I. 29.)
Again, because *GHK* cuts the parallel straight lines *EF*, *CD*, in *H*, *K*;
therefore the exterior angle *GHF* is equal to the interior angle *HKD*;
and it was shewn that the angle *AGH* is equal to the angle *GHF*;
therefore the angle *AGH* is equal to the angle *GKD*;
and these are alternate angles;
therefore *AB* is parallel to *CD*. (I. 27.)
Wherefore, straight lines which are parallel, &c. Q. E. D.

PROPOSITION XXXI. PROBLEM

To draw a straight line through a given point parallel to a given straight line.

Let *A* be the given point, and *BC* the given straight line.
It is required to draw, through the point *A*, a straight line parallel to the straight line *BC*.



In the line *BC* take any point *D*, and join *AD*;
at the point *A* in the straight line *AD*,
make the angle *DAE* equal to the angle *ADC*, (I. 23.) on the opposite side of *AD*;
and produce the straight line *EA* to *F*.
Then *EF* shall be parallel to *BC*.
Because the straight line *AD* meets the two straight lines *EF*, *BC*, and makes the alternate angles *EAD*, *ADC*, equal to one another,
therefore *EF* is parallel to *BC*. (I. 27.)
Wherefore, through the given point *A*, has been drawn a straight line *EAF* parallel to the given straight line *BC*. Q.E.F.

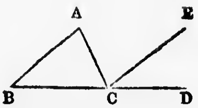
PROPOSITION XXXII. THEOREM.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

Let *ABC* be a triangle, and let one of its sides *BC* be produced to *D*.

Then the exterior angle *ACD* shall be equal to the two interior and opposite angles *CAB*, *ABC*:

and the three interior angles *ABC*, *BCA*, *CAB* shall be equal to two right angles.



Through the point *C* draw *CE* parallel to the side *BA*. (I. 31.)

Then because *CE* is parallel to *BA*, and *AC* meets them,

therefore the angle *ACE* is equal to the alternate angle *BAC*. (I. 29.)

Again, because *CE* is parallel to *AB*, and *BD* falls upon them,

therefore the exterior angle *ECD* is equal to the interior and opposite angle *ABC*; (I. 29.)

but the angle *ACE* was shewn to be equal to the angle *BAC*;

therefore the whole exterior angle *ACD* is equal to the two interior and opposite angles *CAB*, *ABC*. (ax. 2.)

Again, because the angle *ACD* is equal to the two angles *ABC*, *BAC*,

to each of these equals add the angle *ACB*,

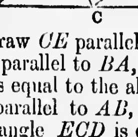
therefore the angles *ACD* and *ACB* are equal to the three angles *ABC*, *BAC*, and *ACB*. (ax. 2.)

but the angles *ACD*, *ACB* are equal to two right angles, (I. 13.)

therefore also the angles *ABC*, *BAC*, *ACB* are equal to two right angles. (ax. 1.)

Wherefore, if a side of any triangle be produced, &c. Q.E.D.

Cor. 1. All the interior angles of any rectilineal figure together with four right angles, are equal to twice as many right angles as the figure has sides.



For any rectilineal figure *ABCDE* can be divided into as many triangles as the figure has sides, by drawing straight lines from a point *F* within the figure to each of its angles.

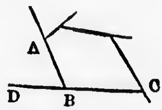
Then, because the three interior angles of a triangle are equal to two right angles, and there are as many triangles as the figure has sides,

therefore all the angles of these triangles are equal to twice as many right angles as the figure has sides;

but the same angles of these triangles are equal to the interior angles of the figure together with the angles at the point *F*:

and the angles at the point F, which is the common vertex of all the triangles, are equal to four right angles, (I. 15. Cor. 2.)
therefore the same angles of these triangles are equal to the angles of the figure together with four right angles;
but it has been proved that the angles of the triangles are equal to twice as many right angles as the figure has sides;
therefore all the angles of the figure together with four right angles, are equal to twice as many right angles as the figure has sides.

COR. 2. All the exterior angles of any rectilineal figure, made by producing the sides successively in the same direction, are together equal to four right angles.

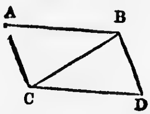


Since every interior angle ABC with its adjacent exterior angle ABD, is equal to two right angles, (I. 13.)
therefore all the interior angles, together with all the exterior angles, are equal to twice as many right angles as the figure has sides;
but it has been proved by the foregoing corollary, that all the interior angles together with four right angles are equal to twice as many right angles as the figure has sides;
therefore all the interior angles together with all the exterior angles, are equal to all the interior angles and four right angles, (ax. 1.)
take from these equals all the interior angles,
therefore all the exterior angles of the figure are equal to four right angles. (ax. 3.)

PROPOSITION XXXIII. THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

Let AB, CD be equal and parallel straight lines,
and joined towards the same parts by the straight lines AC, BD.
Then AC, BD shall be equal and parallel.



Join BC.

Then because AB is parallel to CD, and BC meets them,
therefore the angle ABC is equal to the alternate angle BCD; (I. 29.)
and because AB is equal to CD, and BC common to the two triangles ABC, DCB; the two sides AB, BC, are equal to the two DC, CB, each to each, and the angle ABC was proved to be equal to the angle BCD;
therefore the base AC is equal to the base BD, (I. 4.)
and the triangle ABC to the triangle BCD.

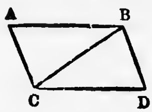
and the other angles to the other angles, each to each, to which the equal sides are opposite;
therefore the angle ACB is equal to the angle CBD.
And because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another;
therefore AC is parallel to BD; (I. 27.)
and AC was shewn to be equal to BD.
Therefore, straight lines which, &c. Q.E.D

PROPOSITION XXXIV. THEOREM.

The opposite sides and angles of a parallelogram are equal to one another, and the diameter bisects it, that is, divides it into two equal parts.

Let $ACDB$ be a parallelogram, of which BC is a diameter.

Then the opposite sides and angles of the figure shall be equal to one another; and the diameter BC shall bisect it.



Because AB is parallel to CD, and BC meets them,
therefore the angle ABC is equal to the alternate angle BCD. (I. 29.)
And because AC is parallel to BD, and BC meets them,
therefore the angle ACB is equal to the alternate angle CBD. (I. 29.)
Hence in the two triangles ABC, CBD,
because the two angles ABC, BCA in the one, are equal to the two angles BCD, CBD in the other, each to each;
and one side BC, which is adjacent to their equal angles, common to the two triangles;
therefore their other sides are equal, each to each, and the third angle of the one to the third angle of the other, (I. 26.)
namely, the side AB to the side CD, and AC to BD, and the angle BAC to the angle BDC.
And because the angle ABC is equal to the angle BCD,
and the angle CBD to the angle ACB,
therefore the whole angle ABD is equal to the whole angle ACD; (ax. 2.)
and the angle BAC has been shewn to be equal to BDC;
therefore the opposite sides and angles of a parallelogram are equal to one another.

Also the diameter BC bisects it.

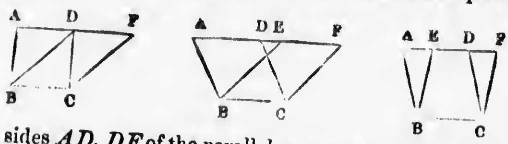
For since AB is equal to CD, and BC common, the two sides AB, BC, are equal to the two DC, CB, each to each,
and the angle ABC has been proved to be equal to the angle BCD;
therefore the triangle ABC is equal to the triangle BCD; (I. 4.) and the diameter BC divides the parallelogram $ACDB$ into two equal parts.
Q.E.D.

PROPOSITION XXXV. THEOREM.

Parallelograms upon the same base, and between the same parallels, are equal to one another.

Let the parallelograms *ABCD*, *EBCF* be upon the same base *BC*, and between the same parallels *AF*, *BC*.

Then the parallelogram *ABCD* shall be equal to the parallelogram *EBCF*.



If the sides *AD*, *DF* of the parallelograms *ABCD*, *DBCF*, opposite to the base *BC*, be terminated in the same point *D*;

then it is plain that each of the parallelograms is double of the triangle *BDC*; (I. 34.)

and therefore the parallelogram *ABCD* is equal to the parallelogram *DBCF*. (ax. 6.)

But if the sides *AD*, *EF*, opposite to the base *BC*, be not terminated in the same point;

Then, because *ABCD* is a parallelogram,
therefore *AD* is equal to *BC*; (I. 34.)
and for a similar reason, *EF* is equal to *BC*;
wherefore *AD* is equal to *EF*; (ax. 1.)
and *DE* is common;
therefore the whole, or the remainder *AE*, is equal to the whole, or the remainder *DF*; (ax. 2 or 3.)
and *AB* is equal to *DC*; (I. 34.)
hence in the triangles *EAB*, *FDC*,
because *FD* is equal to *EA*, and *DC* to *AB*,
and the exterior angle *FDC* is equal to the interior and opposite angle *EAB*; (I. 29.)
therefore the base *FC* is equal to the base *EB*, (I. 4.)
and the triangle *FDC* is equal to the triangle *EAB*.
From the trapezium *ABCF* take the triangle *FDC*,
and from the same trapezium take the triangle *EAB*,
and the remainders are equal, (ax. 3.)
therefore the parallelogram *ABCD* is equal to the parallelogram *EBCF*.

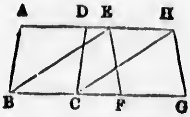
Therefore, parallelograms upon the same, &c. Q.E.D.

PROPOSITION XXXVI. THEOREM.

Parallelograms upon equal bases and between the same parallels, are equal to one another.

Let *ABCD*, *EFGH* be parallelograms upon equal bases *BC*, *FG*, and between the same parallels *AH*, *BG*.

Then the parallelogram *ABCD* shall be equal to the parallelogram *EFGH*.



Join BE, CH.

Then because BC is equal to FG, (hyp.) and FG to EH, (I. 34.)
therefore BC is equal to EH; (ax. 1.)
and these lines are parallels, and joined towards the same parts by the straight lines BE, CH;
but straight lines which join the extremities of equal and parallel straight lines towards the same parts, are themselves equal and parallel; (I. 33.)
therefore BE, CH are both equal and parallel;
wherefore $EBCH$ is a parallelogram. (def. A.)

And because the parallelograms $ABCD$, $EBCH$, are upon the same base BC, and between the same parallels BC, AH;
therefore the parallelogram $ABCD$ is equal to the parallelogram $EBCH$. (I. 35.)

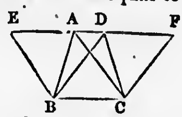
For the same reason, the parallelogram $EFGH$ is equal to the parallelogram $EBCH$;
therefore the parallelogram $ABCD$ is equal to the parallelogram $EFGH$. (ax. 1.)

Therefore, parallelograms upon equal, &c. Q.E.D.

PROPOSITION XXXVII. THEOREM.

Triangles upon the same base and between the same parallels, are equal to one another.

Let the triangles ABC, DBC be upon the same base BC, and between the same parallels AD, BC.
Then the triangle ABC shall be equal to the triangle DBC.



Produce AD both ways to the points E, F;
through B draw BE parallel to CA, (I. 31.)
and through C draw CF parallel to BD.

Then each of the figures $EBCA$, $DBCF$ is a parallelogram;
and $EBCA$ is equal to $DBCF$, (I. 35.) because they are upon the same base BC, and between the same parallels BC, EF.

And because the diameter AB bisects the parallelogram $EBCA$,
therefore the triangle ABC is half of the parallelogram $EBCA$; (I. 34.)
also because the diameter DC bisects the parallelogram $DBCF$,
therefore the triangle DBC is half of the parallelogram $DBCF$,
but the halves of equal things are equal; (ax. 7.)
therefore the triangle ABC is equal to the triangle DBC.

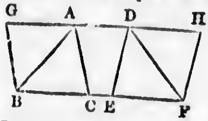
Wherefore, triangles, &c. Q.E.D.

PROPOSITION XXXVIII. THEOREM.

Triangles upon equal bases and between the same parallels, are equal to one another.

Let the triangles *ABC*, *DEF* be upon equal bases *BC*, *EF*, and between the same parallels *BF*, *AD*.

Then the triangle *ABC* shall be equal to the triangle *DEF*.



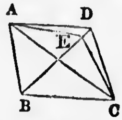
Produce *AD* both ways to the points *G*, *H*;
through *B* draw *BG* parallel to *CA*, (I. 31.)
and through *F* draw *FH* parallel to *ED*.
Then each of the figures *GBCA*, *DEFH* is a parallelogram;
and they are equal to one another, (I. 36.)
because they are upon equal bases *BC*, *EF*,
and between the same parallels *BF*, *GH*.
And because the diameter *AB* bisects the parallelogram *GBCA*, therefore the triangle *ABC* is the half of the parallelogram *GBCA*; (I. 34.)
also, because the diameter *DF* bisects the parallelogram *DEFH*, therefore the triangle *DEF* is the half of the parallelogram *DEFH*:
but the halves of equal things are equal; (ax. 7.)
therefore the triangle *ABC* is equal to the triangle *DEF*.
Wherefore, triangles upon equal bases, &c. Q.E.D.

PROPOSITION XXXIX. THEOREM.

Equal triangles upon the same base and upon the same side of it, are between the same parallels.

Let the equal triangles *ABC*, *DBC* be upon the same base *BC* and upon the same side of it.

Then the triangles *ABC*, *DBC* shall be between the same parallels.



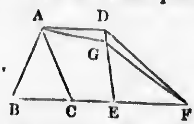
Join *AD*; then *AD* shall be parallel to *BC*.
For if *AD* be not parallel to *BC*,
if possible, through the point *A*, draw *AE* parallel to *BC*, (I. 31.) meeting *BD*, or *BD* produced, in *E*, and join *EC*.
Then the triangle *ABC* is equal to the triangle *EBC*, (I. 37.)
because they are upon the same base *BC*,
and between the same parallels *BC*, *AE*:
but the triangle *ABC* is equal to the triangle *DBC*; (hyp.)
therefore the triangle *DBC* is equal to the triangle *EBC*,

the greater triangle equal to the less, which is impossible:
therefore *AE* is not parallel to *BC*.
In the same manner it can be demonstrated,
that no other line drawn from *A* but *AD* is parallel to *BC*;
AD is therefore parallel to *BC*.
Wherefore, equal triangles upon, &c. Q. E. D.

PROPOSITION XL. THEOREM.

Equal triangles upon equal bases in the same straight line, and towards the same parts, are between the same parallels.

Let the equal triangles *ABC*, *DEF* be upon equal bases *BC*, *EF*, in the same straight line *BF*, and towards the same parts.
Then they shall be between the same parallels.

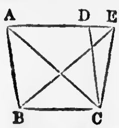


Join *AD*; then *AD* shall be parallel to *BF*.
For if *AD* be not parallel to *BF*,
if possible, through *A* draw *AG* parallel to *BF*, (I. 31.)
meeting *ED*, or *ED* produced in *G*, and join *GF*.
Then the triangle *ABC* is equal to the triangle *GEF*, (I. 38.)
because they are upon equal bases *BC*, *EF*,
and between the same parallels *BF*, *AG*;
but the triangle *ABC* is equal to the triangle *DEF*; (hyp.)
therefore the triangle *DEF* is equal to the triangle *GEF*, (ax. 1.)
the greater triangle equal to the less, which is impossible:
therefore *AG* is not parallel to *BF*.
And in the same manner it can be demonstrated,
that there is no other line drawn from *A* parallel to it but *AD*;
AD is therefore parallel to *BF*.
Wherefore, equal triangles upon, &c. Q. E. D.

PROPOSITION XLI. THEOREM.

If a parallelogram and a triangle be upon the same base, and between the same parallels; the parallelogram shall be double of the triangle.

Let the parallelogram *ABCD*, and the triangle *EBC* be upon the same base *BC*, and between the same parallels *BC*, *AE*.
Then the parallelogram *ABCD* shall be double of the triangle *EBC*



Join *AC*.
Then the triangle *ABC* is equal to the triangle *EBC*, (I. 37.)

because they are upon the same base BC, and between the same parallels BC, AE.

But the parallelogram $ABCD$ is double of the triangle ABC, because the diameter AC bisects it; (I. 34.)

wherefore $ABCD$ is also double of the triangle EBC.

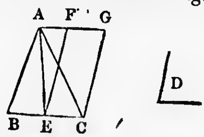
Therefore, if a parallelogram and a triangle, &c. Q.E.D.

PROPOSITION XLII. PROBLEM.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle.

It is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.



Bisect BC in E, (I. 10.) and join AE;

at the point E in the straight line EC, make the angle CEF equal to the angle D; (I. 23.)

through C draw CG parallel to EF, and through A draw AFG parallel to BC, (I. 31.) meeting EF in F, and CG in G.

Then the figure $CEFG$ is a parallelogram. (def. A.)

And because the triangles ABE, AEC are on the equal bases BE, EC, and between the same parallels BC, AG;

they are therefore equal to one another; (I. 38.)

and the triangle ABC is double of the triangle AEC;

but the parallelogram $FECG$ is double of the triangle AEC, (I. 41.) because they are upon the same base EC, and between the same parallels EC, AG;

therefore the parallelogram $FECG$ is equal to the triangle ABC, (ax. 6.) and it has one of its angles CEF equal to the given angle D.

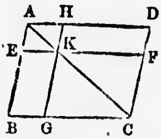
Wherefore, a parallelogram $FECG$ has been described equal to the given triangle ABC, and having one of its angles CEF equal to the given angle D. Q.E.F.

PROPOSITION XLIII. THEOREM.

The complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.

Let $ABCD$ be a parallelogram, of which the diameter is AC: and EH, GF the parallelograms about AC, *that is, through which AC passes:* also BK, KD the other parallelograms which make up the whole figure $ABCD$, which are therefore called the complements.

Then the complement BK shall be equal to the complement KD.

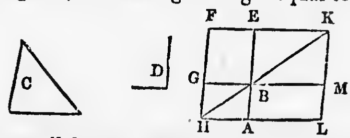


Because $ABCD$ is a parallelogram, and AC its diameter,
therefore the triangle ABC is equal to the triangle ADC. (I. 34.)
Again, because $EKHA$ is a parallelogram, and AK its diameter,
therefore the triangle AEK is equal to the triangle AHK; (I. 34.)
and for the same reason, the triangle KGC is equal to the triangle KFC.
Wherefore the two triangles AEK, KGC are equal to the two triangles AHK, KFC, (ax. 2.)
but the whole triangle ABC is equal to the whole triangle ADC;
therefore the remaining complement BK is equal to the remaining complement KD. (ax. 3.)
Wherefore the complements, &c. Q.E.D.

PROPOSITION XLIV. PROBLEM.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, and C the given triangle, and D the given rectilineal angle.
It is required to apply to the straight line AB, a parallelogram equal to the triangle C, and having an angle equal to the angle D.



Make the parallelogram $BEFG$ equal to the triangle C,
and having the angle EBG equal to the angle D, (I. 42.)
so that BE be in the same straight line with AB;
produce FG to H,
through A draw AH parallel to BG or EF, (I. 31.) and join HB.
Then because the straight line HF falls upon the parallels AH, EF,
therefore the angles AHF, HFE are together equal to two right angles; (I. 29.)
wherefore the angles BHF, HFE are less than two right angles:
but straight lines which with another straight line, make the two interior angles upon the same side less than two right angles, do meet if produced far enough: (ax. 12.)
therefore HB, FE shall meet if produced;
let them be produced and meet in K,
through K draw KL parallel to EA or FH,
and produce HA, GB to meet KL in the points L, M.
Then $HLKF$ is a parallelogram, of which the diameter is HK;

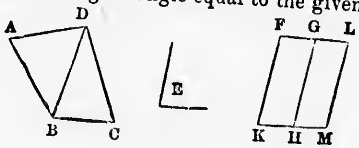
and *AG*, *ME*, are the parallelograms about *HK*;
also *LB*, *BF* are the complements;
therefore the complement *LB* is equal to the complement *BF*; (I. 43.)
but the complement *BF* is equal to the triangle *C*; (constr.)
wherefore *LB* is equal to the triangle *C*.
And because the angle *GBE* is equal to the angle *ABM*, (I. 15.)
and likewise to the angle *D*; (constr.)
therefore the angle *ABM* is equal to the angle *D*. (ax. 1.)
Therefore to the given straight line *AB*, the parallelogram *LB* has been applied, equal to the triangle *C*, and having the angle *ABM* equal to the given angle *D*. Q.E.F.

PROPOSITION XLV. PROBLEM.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let *ABCD* be the given rectilineal figure, and *E* the given rectilineal angle.

It is required to describe a parallelogram that shall be equal to the figure *ABCD*, and having an angle equal to the given angle *E*.



Join *DB*.
Describe the parallelogram *FH* equal to the triangle *ADB*, and having the angle *FKH* equal to the angle *E*; (I. 42.)
to the straight line *GH*, apply the parallelogram *GM* equal to the triangle *DBC*, having the angle *GHM* equal to the angle *E*. (I. 44.)
Then the figure *FKML* shall be the parallelogram required.
Because each of the angles *FKH*, *GHM*, is equal to the angle *E*,
therefore the angle *FKH* is equal to the angle *GHM*;
add to each of these equals the angle *KHG*;
therefore the angles *FKH*, *KHG* are equal to the angles *KHG*, *GHM*;
but *FKH*, *KHG* are equal to two right angles; (I. 29.)
therefore also *KHG*, *GHM* are equal to two right angles;
and because at the point *H*, in the straight line *GH*, the two straight lines *KH*, *HM*, upon the opposite sides of it, make the adjacent angles *KHG*, *GHM* equal to two right angles,
therefore *HK* is in the same straight line with *HM*. (I. 14.)
And because the line *HG* meets the parallels *KM*, *FG*,
therefore the angle *MHG* is equal to the alternate angle *HGF*; (I. 29.)
add to each of these equals the angle *HGL*;
therefore the angles *MHG*, *HGL* are equal to the angles *HGF*, *HGL*;
but the angles *MHG*, *HGL* are equal to two right angles; (I. 29.)
therefore also the angles *HGF*, *HGL* are equal to two right angles,
and therefore *FG* is in the same straight line with *GL*. (I. 14.)

And because KF is parallel to HG, and HG to ML,
therefore KF is parallel to ML; (I. 30.)
and FL has been proved parallel to KM,
wherefore the figure $FKML$ is a parallelogram;
and since the parallelogram HF is equal to the triangle ABD,
and the parallelogram GM to the triangle BDC;
therefore the whole parallelogram $KFLM$ is equal to the whole rectilineal figure $ABCD$.

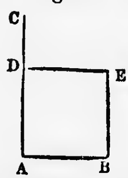
Therefore the parallelogram $KFLM$ has been described equal to the given rectilineal figure $ABCD$, having the angle FKM equal to the given angle E. Q.E.F.

COR. From this it is manifest how, to a given straight line, to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure; viz. by applying to the given straight line a parallelogram equal to the first triangle ABD, (I. 44.) and having an angle equal to the given angle.

PROPOSITION XLVI. PROBLEM.

To describe a square upon a given straight line.

Let AB be the given straight line.



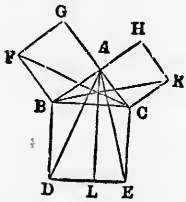
It is required to describe a square upon AB.
From the point A draw AC at right angles to AB; (I. 11.)
make AD equal to AB; (I. 3.)
through the point D draw DE parallel to AB; (I. 31.)
and through B, draw BE parallel to AD, meeting DE in E;
therefore $ABED$ is a parallelogram;
whence AB is equal to DE, and AD to BE; (I. 34.)
but AD is equal to AB,
therefore the four lines AB, BE, ED, DA are equal to one another,
and the parallelogram $ABED$ is equilateral.
It has likewise all its angles right angles;
since AD meets the parallels AB, DE,
therefore the angles BAD, ADE are equal to two right angles; (I. 29.)
but BAD is a right angle; (constr.)
therefore also ADE is a right angle.
But the opposite angles of parallelograms are equal; (I. 34.)
therefore each of the opposite angles ABE, BED is a right angle;
wherefore the figure $ABED$ is rectangular,
and it has been proved to be equilateral;
therefore the figure $ABED$ is a square, (def. 30.)
and it is described upon the given straight line AB. Q.E.F.

Cor. Hence, every parallelogram that has one of its angles a right angle, has all its angles right angles.

PROPOSITION XLVII. THEOREM.

In any right-angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.

Let *ABC* be a right-angled triangle, having the right angle *BAC*. Then the square described upon the side *BC*, shall be equal to the squares described upon *BA*, *AC*.



On *BC* describe the square *BDEC*, (I. 46.)
and on *BA*, *AC* the squares *GB* *HC*;
through *A* draw *AL* parallel to *BD* or *CE*; (I. 31.)
and join *AD*, *FC*.
Then because the angle *BAC* is a right angle, (hyp.)
and that the angle *BAG* is a right angle, (def. 30.)
the two straight lines *AC*, *AG* upon the opposite sides of *AB*, make with it at the point *A*, the adjacent angles equal to two right angles;
therefore *CA* is in the same straight line with *AG*. (I. 14.)
For the same reason, *BA* and *AH* are in the same straight line.
And because the angle *DBC* is equal to the angle *FBA*,
each of them being a right angle,
add to each of these equals the angle *ABC*,
therefore the whole angle *ABD* is equal to the whole angle *FBC*. (ax. 2.)
And because the two sides *AB*, *BD*, are equal to the two sides *FB*, *BC*, each to each, and the included angle *ABD* is equal to the included angle *FBC*,
therefore the base *AD* is equal to the base *FC*, (I. 4.)
and the triangle *ABD* to the triangle *FBC*.
Now the parallelogram *BL* is double of the triangle *ABD*, (I. 41.)
because they are upon the same base *BD*, and between the same parallels *BD*, *AL*;
also the square *GB* is double of the triangle *FBC*,
because these also are upon the same base *FB*, and between the same parallels *FB*, *GC*.
But the doubles of equals are equal to one another; (ax. 6.)
therefore the parallelogram *BL* is equal to the square *GB*.
Similarly, by joining *AE*, *BK*, it can be proved,
that the parallelogram *CL* is equal to the square *HC*.

Therefore the whole square $BDEC$ is equal to the two squares GB, HC; (ax. 2.)
and the square $BDEC$ is described upon the straight line BC,
and the squares GB, HC, upon AB, AC:
therefore the square upon the side BC, is equal to the squares upon the sides AB, AC.
Therefore, in any right-angled triangle, &c. Q.E.D.

PROPOSITION XLVIII. THEOREM.

If the square described upon one of the sides of a triangle, be equal to the squares described upon the other two sides of it; the angle contained by these two sides is a right angle.

Let the square described upon BC, one of the sides of the triangle ABC, be equal to the squares upon the other two sides, AB, AC.
Then the angle BAC shall be a right angle.

From the point A draw AD at right angles to AC, (I. 11.)
make AD equal to AB, and join DC.
Then, because AD is equal to AB,
the square on AD is equal to the square on AB;
to each of these equals add the square on AC;
therefore the squares on AD, AC are equal to the squares on AB, AC:
but the squares on AD, AC are equal to the square on DC, (I. 47.)
because the angle DAC is a right angle;
and the square on BC, by hypothesis, is equal to the squares on BA, AC;
therefore the square on DC is equal to the square on BC;
and therefore the side DC is equal to the side BC.
And because the side AD is equal to the side AB,
and AC is common to the two triangles DAC, BAC;
the two sides DA, AC, are equal to the two BA, AC, each to each:
and the base DC has been proved to be equal to the base BC;
therefore the angle DAC is equal to the angle BAC; (I. 8.)
but DAC is a right angle;
therefore also BAC is a right angle.
Therefore, if the square described upon, &c. Q.E.D.

NOTES TO BOOK I.

ON THE DEFINITIONS.

Geometry is one of the most perfect of the deductive Sciences, and seems to rest on the simplest inductions from experience and observation.

The first principles of Geometry are therefore in this view consistent hypotheses founded on facts cognizable by the senses, and it is a subject of primary importance to draw a distinction between the conception of things and the things themselves. These hypotheses do not involve any property contrary to the real nature of the things, and consequently cannot be regarded as arbitrary, but in certain respects, agree with the conceptions which the things themselves suggest to the mind through the medium of the senses. The essential definitions of Geometry therefore being inductions from observation and experience, rest ultimately on the evidence of the senses..

It is by experience we become acquainted with the existence of individual forms of magnitudes; but by the mental process of abstraction, which begins with a particular instance, and proceeds to the general idea of all objects of the same kind, we attain to the general conception of those forms which come under the same general idea.

The essential definitions of Geometry express generalized conceptions of real existences in their most perfect ideal forms: the laws and appearances of nature, and the operations of the human intellect being supposed uniform and consistent.

But in cases where the subject falls under the class of simple ideas, the terms of the definitions so called, are no more than merely equivalent expressions. The simple idea described by a proper term or terms, does not in fact admit of definition properly so called. The definitions in Euclid's Elements may be divided into two classes, those which merely explain the meaning of the terms employed, and those, which, besides explaining the meaning of the terms, suppose the existence of the things described in the definitions.

Definitions in Geometry cannot be of such a form as to explain the nature and properties of the figures defined: it is sufficient that they give marks whereby the thing defined may be distinguished from every other of the same kind. It will at once be obvious, that the definitions of Geometry, one of the pure sciences, being abstractions of space, are not like the definitions in any one of the physical sciences. The discovery of any new physical facts may render necessary some alteration or modification in the definitions of the latter.

Def. I. Simson has adopted Theon's definition of a point. Euclid's definition is, *σημεῖον ἐστιν οὗ μέρος οὐδέν*, "A point is that, of which there is no part," or which cannot be parted or divided, as it is explained by Proclus. The Greek term *σημεῖον*, literally means, a visible *sign* or *mark* on a surface, in other words, *a physical point*. The English term *point*, means the sharp end of any thing, or a mark made by it. The word *point* comes from the Latin *punctum*, through the French word *point*. Neither of these terms, in its literal sense, appears to give a very exact notion of what is to be understood by a point in Geometry. Euclid's definition of a point merely expresses a negative property, which excludes the proper and literal meaning of the Greek term, as applied to denote a physical point, or a mark which is visible to the senses.

Pythagoras defined a point to be *μονὰς θέσιν ἔχουσα*, "a monad having position." By uniting the positive idea of position, with the negative idea of defect of magnitude, the conception of a point in Geometry may be rendered perhaps more intelligible. A point is defined to be that which has no magnitude, but position only.

Def. II. Every visible line has both length and breadth, and it is impossible to draw any line whatever which shall have no breadth. The definition requires the conception of the length only of the line to be considered, abstracted from, and independently of, all idea of its breadth.

Def. III. This definition renders more intelligible the exact meaning of the definition of a point: and we may add, that, in the Elements, Euclid supposes that the intersection of two lines is a point, and that two lines can intersect each other in one point only.

Def. IV. The straight line or right line is a term so clear and intelligible as to be incapable of becoming more so by formal definition. Euclid's definition is Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται, wherein he states it to lie *evenly*, or *equally*, or *upon an equality* (ἐξ ἴσου) between its extremities, and which Proclus explains as being stretched between its extremities, ἡ ἐπ' ἄκρων τεταμένη.

If the line be conceived to be drawn on a plane surface, the words ἐξ ἴσου may mean, that no part of the line which is called a straight line deviates either from one side or the other of the direction which is fixed by the extremities of the line; and thus it may be distinguished from a curved line, which does not lie, in this sense, evenly between its extreme points. If the line be conceived to be drawn in space, the words ἐξ ἴσου, must be understood to apply to every direction on every side of the line between its extremities.

Every straight line situated in a plane, is considered to have two sides; and when the direction of a line is known, the line is said to be given in position; also, when the length is known or can be found, it is said to be given in magnitude.

From the definition of a straight line, it follows, that two points fix a straight line in position, which is the foundation of the first and second postulates. Hence straight lines which are proved to coincide in two or more points, are called, "one and the same straight line," Prop. 14, Book I, or, which is the same thing, that "two straight lines cannot have a common segment," as Simson shews in his Corollary to Prop. 11, Book I.

The following definition of straight lines has also been proposed. "Straight lines are those which, if they coincide in any two points, coincide as far as they are produced." But this is rather a criterion of straight lines, and analogous to the eleventh axiom, which states that, "all right angles are equal to one another," and suggests that all straight lines may be made to coincide wholly, if the lines be equal; or partially, if the lines be of unequal lengths. A definition should properly be restricted to the description of the thing defined, as it exists, independently of any comparison of its properties or of tacitly assuming the existence of axioms.

Def. VII. Euclid's definition of a plane surface is 'Επίπεδος ἐπιφάνειά ἐστιν ἥτις ἐξ ἴσου ταῖς ἐφ' ἑαυτῆς εὐθείαις κεῖται, "A plane surface is that which lies evenly or equally with the straight lines in it;" instead of which Simson has given the definition which was originally proposed by Hero the Elder. A plane superficies may be supposed to be situated in any position, and to be continued in every direction to any extent.

Def. VIII. Simson remarks that this definition seems to include the angles formed by two curved lines, or a curve and a straight line, as well as that formed by two straight lines.

Angles made by straight lines only, are treated of in Elementary Geometry.

Def. IX. It is of the highest importance to attain a clear conception of an angle, and of the sum and difference of two angles. The literal meaning of the term *angulus* suggests the Geometrical conception of an angle, which may be regarded as formed by the divergence of two straight lines from a point. In the definition of an angle, the magnitude of the angle is independent of the lengths of the two lines by which it is included; their mutual divergence from the point at which they meet, is the criterion of the magnitude of an angle, as it is pointed out in the succeeding definitions. The point at which the two lines meet is called the angular point or the vertex of the angle, and must not be confounded with the magnitude of the angle itself. The right angle is fixed in magnitude, and, on this account, it is made the standard with which all other angles are compared.

Two straight lines which actually intersect one another, or which when produced would intersect, are said to be inclined to one another, and the inclination of the two lines is determined by the angle which they make with one another.

Def. X. It may be here observed that in the Elements, Euclid always assumes that when one line is perpendicular to another line, the latter is also perpendicular to the former; and always calls *a right angle*, ὀρθὴ γωνία; but *a straight line*, εὐθεῖα γραμμή.

Def. XIX. This has been restored from Proclus, as it seems to have a meaning in the construction of Prop. 14, Book II; the first case of Prop. 33, Book III, and Prop. 13, Book VI. The definition of the segment of a circle is not once alluded to in Book I, and is not required before the discussion of the properties of the circle in Book III. Proclus remarks on this definition: "Hence you may collect that the center has three places: for it is either within the figure, as in the circle; or in its perimeter, as in the semicircle; or without the figure, as in certain conic lines."

Def. XXIV-XXIX. Triangles are divided into three classes, by reference to the relations of their sides; and into three other classes, by reference to their angles. A further classification may be made by considering both the relation of the sides and angles in each triangle.

In Simson's definition of the isosceles triangle, the word *only* must be omitted, as in the Cor. Prop. 5, Book I, an isosceles triangle may be equilateral, and an equilateral triangle is considered isosceles in Prop. 15, Book IV. Objection has been made to the definition of an acute-angled triangle. It is said that it cannot be admitted as a definition, that all the three angles of a triangle are acute, which is supposed in Def. 29. It may be replied, that the definitions of the three kinds of angles point out and seem to supply a foundation for a similar distinction of triangles.

Def. XXX-XXXIV. The definitions of quadrilateral figures are liable to objection. All of them, except the trapezium, fall under the general idea of a parallelogram; but as Euclid defined parallel straight lines after he had defined four-sided figures, no other arrangement could be adopted than the one he has followed; and for which there appeared to him, without doubt, some probable reasons. Sir Henry Savile, in his Seventh Lecture, remarks on some of the definitions of Euclid, "Nec dissimulandum aliquot harum in manibus exiguum esse usum in Geometriâ." A few verbal emendations have been made in some of them.

A square is a four-sided plane figure having all its sides equal, and one angle a right angle: because it is proved in Prop. 46, Book I, that if a parallelogram have one angle a right angle, all its angles are right angles.

An oblong, in the same manner, may be defined as a plane figure of four sides, having only its opposite sides equal, and one of its angles a right angle.

A rhomboid is a four-sided plane figure having only its opposite sides equal to one another and its angles not right angles.

Sometimes an irregular four-sided figure which has two sides parallel, is called a trapezoid.

Def. xxxv. It is possible for two right lines never to meet when produced, and not be parallel.

Def. A. The term parallelogram literally implies a figure formed by parallel straight lines, and may consist of four, six, eight, or any even number of sides, where every two of the opposite sides are parallel to one another. In the Elements, however, the term is restricted to four-sided figures, and includes the four species of figures named in the Definitions xxx—xxxiii.

The synthetic method is followed by Euclid not only in the demonstrations of the propositions, but also in laying down the definitions. He commences with the simplest abstractions, defining a point, a line, an angle, a superficies, and their different varieties. This mode of proceeding involves the difficulty, almost insurmountable, of defining satisfactorily the elementary abstractions of Geometry. It has been observed, that it is necessary to consider a solid, that is, a magnitude which has length, breadth, and thickness, in order to understand aright the definitions of a point, a line, and a superficies. A solid or volume considered apart from its physical properties, suggests the idea of the surfaces by which it is bounded: a surface, the idea of the line or lines which form its boundaries: and a finite line, the points which form its extremities. A solid is therefore bounded by surfaces; a surface is bounded by lines; and a line is terminated by two points. A point marks position only: a line has one dimension, length only, and defines distance: a superficies has two dimensions, length and breadth, and defines extension: and a solid has three dimensions, length, breadth, and thickness, and defines some portion of space.

It may also be remarked that two points are sufficient to determine the position of a straight line, and three points not in the same straight line, are necessary to fix the position of a plane.

ON THE POSTULATES.

The definitions assume the possible existence of straight lines and circles, and the postulates predicate the possibility of drawing and of producing straight lines, and of describing circles. The postulates form the principles of construction assumed in the Elements; and are, in fact, problems, the possibility of which is admitted to be self-evident, and to require no proof.

It must, however, be carefully remarked, that the third postulate only admits that when any line is given in position and magnitude, a circle may be described from either extremity of the line as a center, and with a radius equal to the length of the line, as in Euc. i, 1. It does not admit the description of a circle with any other point as a center than one of the extremities of the given line.

Euc. i. 2, shews how, from any given point, to draw a straight line equal to another straight line which is given in magnitude and position.

ON THE AXIOMS.

Axioms are usually defined to be self-evident truths, which cannot be rendered more evident by demonstration; in other words, the axioms of Geometry are theorems, the truth of which is admitted without proof. It is by experience we first become acquainted with the different forms of geometrical magnitudes, and the axioms, or the fundamental ideas of their equality or inequality appear to rest on the same basis. The conception of the truth of the axioms does not appear to be more removed from experience than the conception of the definitions.

These axioms, or first principles of demonstration, are such theorems as cannot be resolved into simpler theorems, and no theorem ought to be admitted as a first principle of reasoning which is capable of being demonstrated. An axiom, and (when it is convertible) its converse, should both be of such a nature as that neither of them should require a formal demonstration.

The first and most simple idea, derived from experience is, that every magnitude fills a certain space, and that several magnitudes may successively fill the same space.

All the knowledge we have of magnitude is purely relative, and the most simple relations are those of equality and inequality. In the comparison of magnitudes, some are considered as given or known, and the unknown are compared with the known, and conclusions are synthetically deduced with respect to the equality or inequality of the magnitudes under consideration. In this manner we form our idea of equality, which is thus formally stated in the eighth axiom: "Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another."

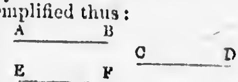
Every specific definition is referred to this universal principle. With regard to a few more general definitions which do not furnish an equality, it will be found that some hypothesis is always made reducing them to that principle, before any theory is built upon them. As for example, the definition of a straight line is to be referred to the tenth axiom; the definition of a right angle to the eleventh axiom; and the definition of parallel straight lines to the twelfth axiom.

The eighth axiom is called the principle of superposition, or, the mental process by which one Geometrical magnitude may be conceived to be placed on another, so as exactly to coincide with it, in the parts which are made the subject of comparison. Thus, if one straight line be conceived to be placed upon another, so that their extremities are coincident, the two straight lines are equal. If the directions of two lines which include one angle, coincide with the directions of the two lines which contain another angle, where the points, from which the angles diverge, coincide, then the two angles are equal: the lengths of the lines not affecting in any way the magnitudes of the angles. When one plane figure is conceived to be placed upon another, so that the boundaries of one exactly coincide with the boundaries of the other, then the two plane figures are equal. It may also be remarked, that the converse of this proposition is not universally true, namely, that when two magnitudes are equal, they coincide with one another: since two magnitudes may be equal in area, as two parallelograms or two triangles, Euc. I. 35, 37; but their boundaries may not be equal: and, consequently, by superposition, the figures could not exactly coincide: all such figures, however, having equal areas, by a different arrangement of their parts, may be made to coincide exactly.

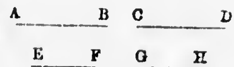
This axiom is the criterion of Geometrical equality, and is essentially different from the criterion of Arithmetical equality. Two geometrical magnitudes are equal, when they coincide or may be made to coincide: two abstract numbers are equal, when they contain the same aggregate of units; and two concrete numbers are equal, when they contain the same number of units of the same kind of magnitude. It is at once obvious, that Arithmetical representations of Geometrical magnitudes are not admissible in Euclid's criterion of Geometrical Equality, as he has not fixed the unit of magnitude of either the straight line, the angle, or the superficies. Perhaps Euclid intended that the first seven axioms should be applicable to numbers as well as to Geometrical magnitudes, and this is in accordance with the words of Proclus, who calls the axioms, *common notions*, not peculiar to the subject of Geometry.

Several of the axioms may be generally exemplified thus:

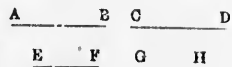
Axiom I. If the straight line AB be equal to the straight line CD; and if the straight line EF be also equal to the straight line CD; then the straight line AB is equal to the straight line EF.



Axiom II. If the line AB be equal to the line CD; and if the line EF be also equal to the line GH: then the sum of the lines AB and EF is equal to the sum of the lines CD and GH.

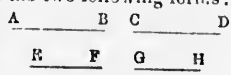


Axiom III. If the line AB be equal to the line CD; and if the line EF be also equal to the line GH; then the difference of AB and EF, is equal to the difference of CD and GH.

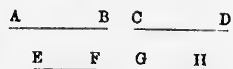


Axiom IV. admits of being exemplified under the two following forms:

1. If the line AB be equal to the line CD; and if the line EF *be greater than* the line GH; then the sum of the lines AB and EF *is greater than* the sum of the lines CD and GH.

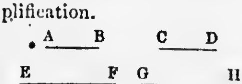


2. If the line AB be equal to the line CD; and if the line EF *be less than* the line GH; then the sum of the lines AB and EF *is less than* the sum of the lines CD and GH.

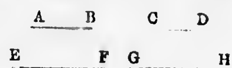


Axiom V. also admits of two forms of exemplification.

1. If the line AB be equal to the line CD; and if the line EF *be greater than* the line GH; then the difference of the lines AB and EF is *greater than* the difference of CD and GH.



2. If the line AB be equal to the line CD; and if the line EF *be less than* the line GH; then the difference of the lines AB and EF is *less than* the difference of the lines CD and GH.



The axiom, "If unequals be taken from equals, the remainders are unequal," may be exemplified in the same manner.

Axiom VI. If the line AB be double of the line CD; and if the line EF be also double of the line CD;
then the line AB is equal to the line EF.

Axiom VII. If the line AB be the half of the line CD; and if the line EF be also the half of the line CD;
then the line AB is equal to the line EF.

It may be observed that when equal magnitudes are taken from unequal magnitudes, the greater remainder exceeds the less remainder by as much as the greater of the unequal magnitudes exceeds the less.

If unequals be taken from unequals, the remainders are not always unequal; they may be equal: also if unequals be added to unequals the wholes are not always unequal, they may also be equal.

Axiom IX. The whole is greater than its part, and conversely, the part is less than the whole. This axiom appears to assert the contrary of the eighth axiom, namely, that two magnitudes, of which one is greater than the other, cannot be made to coincide with one another.

Axiom X. The property of straight lines expressed by the tenth axiom, namely, "that two straight lines cannot enclose a space," is obviously implied in the definition of straight lines; for if they enclosed a space, they could not coincide between their extreme points, when the two lines are equal.

Axiom XI. This axiom has been asserted to be a demonstrable theorem. As an angle is a species of magnitude, this axiom is only a particular application of the eighth axiom to right angles.

Axiom XII. See the notes on Prop. XXIX. Book I.

ON THE PROPOSITIONS.

WHENEVER a judgment is formally expressed, there must be something respecting which the judgment is expressed, and something else which constitutes the judgment. The former is called the *subject* of the proposition, and the latter, the *predicate*, which may be anything which can be affirmed or denied respecting the *subject*.

The propositions in Euclid's Elements of Geometry may be divided into two classes, *problems* and *theorems*. A proposition, as the term imports, is something proposed; it is a *problem*, when some Geometrical *construction* is required to be *effected*: and it is a *theorem* when some Geometrical *property* is to be *demonstrated*. Every proposition is naturally divided into two parts; a problem consists of the *data*, or *things given;* and the *quæsita*, or *things required:* a theorem, consists of the *subject* or *hypothesis*, and the *conclusion*, or *predicate*. Hence the distinction between a problem and a theorem is this, that a problem consists of the data and the quæsita, and requires solution: and a theorem consists of the hypothesis and the predicate, and requires demonstration.

All propositions are *affirmative* or *negative;* that is, they either asser some property, as Euc. I. 4, or deny the existence of some property, as Euc. I. 7; and every proposition which is affirmatively stated has a contradictory corresponding proposition. If the affirmative be proved to be true, the contradictory is false.

All propositions may be viewed as (1) *universally affirmative*, or *universally negative*; (2) as *particularly affirmative*, or *particularly negative*.

The connected course of reasoning by which any Geometrical truth is established is called a demonstration It is called a *direct* demonstration when the predicate of the proposition is inferred directly from the premisses, as the conclusion of a series of successive deductions. The demonstration is called *indirect*, when the conclusion shows that the introduction of any other supposition contrary to the hypothesis stated in the proposition, necessarily leads to an absurdity.

It has been remarked by Pascal, that "Geometry is almost the only subject as to which we find truths wherein all men agree; and one cause of this is, that Geometers alone regard the true laws of demonstration."

These are enumerated by him as eight in number. "1. To define nothing which cannot be expressed in clearer terms than those in which it is already expressed. 2. To leave no obscure or equivocal terms undefined. 3. To employ in the definition no terms not already known. 4. To omit nothing in the principles from which we argue, unless we are sure it is granted. 5. To lay down no axiom which is not perfectly evident. 6. To demonstrate nothing which is as clear already as we can make it. 7. To prove every thing in the least doubtful by means of self-evident axioms, or of propositions already demonstrated. 8. To substitute mentally the definition instead of the thing defined." Of these rules, he says, "the first, fourth and sixth are not absolutely necessary to avoid error, but the other five are indispensable; and though they may be found in books of logic, none but the Geometers have paid any regard to them.'

The course pursued in the demonstrations of the propositions in Euclid's Elements of Geometry, is always to refer directly to some expressed principle, to leave nothing to be inferred from vague expressions, and to make every step of the demonstrations the object of the understanding.

It has been maintained by some philosophers, that a genuine definition contains some property or properties which can form a basis for demonstration, and that the science of Geometry is deduced from the definitions, and that on them alone the demonstrations depend. Others have maintained that a definition explains only the meaning of a term, and does not embrace the nature and properties of the thing defined.

If the propositions usually called postulates and axioms are either tacitly assumed or expressly stated in the definitions; in this view, demonstrations may be said to be legitimately founded on definitions. If, on the other hand, a definition is simply an explanation of the meaning of a term, whether abstract or concrete, by such marks as may prevent a misconception of the thing defined; it will be at once obvious that some constructive and theoretic principles must be assumed, besides the definitions to form the ground of legitimate demonstration. These principles we conceive to be the postulates and axioms. The postulates describe constructions which may be admitted as possible by direct appeal to our experience; and the axioms assert general theoretic truths so simple and self-evident as to require no proof, but to be admitted as the assumed first principles of demonstration. Under this view all Geometrical reasonings proceed upon the admission of the hypotheses assumed in the definitions, and the unquestioned possibility of the postulates, and the truth of the axioms.

Deductive reasoning is generally delivered in the form of an enthymeme, or an argument wherein one enunciation is not expressed, but is readily supplied by the reader: and it may be observed, that although this is the ordinary mode of speaking and writing, it is not in the strictly syllogistic form; as either the *major* or the *minor* premiss only is formally stated before the conclusion: Thus in Euc. I. 1.

Because the point *A* is the center of the circle *BCD*;
therefore the straight line *AB* is equal to the straight line *AC*.

The premiss here omitted, is: all straight lines drawn from the center of a circle to the circumference are equal.

In a similar way may be supplied the reserved premiss in every enthymeme. The conclusion of two enthymemes may form the major and minor premiss of a third syllogism, and so on, and thus any process of reasoning is reduced to the strictly syllogistic form. And in this way it is shown that the general theorems of Geometry are demonstrated by means of syllogisms founded on the axioms and definitions.

Every syllogism consists of three propositions, of which, two are called the premisses, and the third, the conclusion. These propositions contain three terms, the subject and predicate of the conclusion, and the middle term which connects the predicate and the conclusion together. The subject of the conclusion is called *the minor*, and the predicate of the conclusion is called *the major* term, of the syllogism. The major term appears in one premiss, and the minor term in the other, with the middle term, which is in both premisses. That premiss which contains the middle term and the major term, is called the *major premiss;* and that which contains the middle term and the minor term, is called the *minor premiss* of the syllogism. As an example, we may take the syllogism in the demonstration of Prop. 1, Book 1, wherein it will be seen that the middle term is the subject of the major premiss and the predicate of the minor.

Major premiss: because the straight line *AB* is equal to the straight line *AC*;
Minor premiss: and, because the straight line *BC* is equal to the straight line *AB*;
Conclusion: therefore the straight line *BC* is equal to the straight line *AC*.

Here, *BC* is the subject, and *AC* the predicate of the conclusion.
BC is the subject, and *AB* the predicate of the minor premiss.
AB is the subject, and *AC* the predicate of the major premiss.

Also, *AC* is the major term, *BC* the minor term, and *AB* the middle term of the syllogism.

In this syllogism, it may be remarked that the definition of a straight line is assumed, and the definition of the Geometrical equality of two straight lines; also that a general theoretic truth, or axiom, forms the ground of the conclusion. And further, though it be impossible to make any point, mark or sign (σημεῖον) which has not both length and breadth, and any line which has not both length and breadth; the demonstrations in Geometry do not on this account become invalid. For they are pursued on the hypothesis that the point has no parts, but position only: and the line has length only, but no breadth or thickness: also that the surface has length and breadth only, but no thickness: and all the conclusions at which we arrive are independent of every other consideration.

The truth of the conclusion in the syllogism depends upon the truth of the premisses. If the premisses, or only one of them be not true, the conclusion is false. The conclusion is said *to follow from* the premisses; whereas, in truth, it is *contained in* the premisses. The expression must be understood of the mind apprehending in succession, the truth of the premisses, and subsequent to that, the truth of the conclusion; so that the conclusion *follows from* the premisses in order of time as far as reference is made to the mind's apprehension of the whole argument.

Every proposition, when complete, may be divided into six parts, as Proclus has pointed out in his commentary.

1. *The proposition*, or *general enunciation*, which states in general terms the conditions of the problem or theorem.

2. *The exposition*, or *particular enunciation*, which exhibits the *subject* of the proposition in particular terms as a fact, and refers it to some diagram described.

3. *The determination* contains the *predicate* in particular terms, as it is pointed out in the diagram, and directs attention to the demonstration, by pronouncing the thing sought,

4. *The construction* applies the postulates to prepare the diagram for the demonstration.

5. *The demonstration* is the connexion of syllogisms, which prove the truth or falsehood of the theorem, the possibility or impossibility of the problem, in that particular case exhibited in the diagram.

6. *The conclusion* is merely the repetition of the general enunciation, wherein the predicate is asserted as a demonstrated truth.

Prop. I. In the first two Books, the circle is employed as a mechanical instrument, in the same manner as the straight line, and the use made of it rests entirely on the third postulate. No properties of the circle are discussed in these books beyond the definition and the third postulate. When two circles are described, one of which has its center in the circumference of the other, the two circles being each of them partly within and partly without the other, their circumferences must intersect each other in two points; and it is obvious from the two circles cutting each other, in two points, one on each side of the given line, that two equilateral triangles may be formed on the given line.

Prop. II. When the given point is neither in the line, nor in the line produced, this problem admits of eight different lines being drawn from the given point in different directions, every one of which is a solution of the problem. For, 1. The given line has two extremities, to each of which a line may be drawn from the given point. 2. The equilateral triangle may be described on either side of this line. 3. And the side *BD* of the equilateral triangle *ABD* may be produced either way.

But when the given point lies either in the line or in the line produced, the distinction which arises from joining the two ends of the line with the given point, no longer exists, and there are only four cases of the problem.

The construction of this problem assumes a neater form, by first describing the circle *CGH* with center *B* and radius *BC*, and producing *DB* the side of the equilateral triangle *DBA* to meet the circumference in *G*: next, with center *D* and radius *DG*, describing the circle *GKL*, and then producing *DA* to meet the circumference in *L*.

By a similar construction the less of two given straight lines may be produced, so that the less together with the part produced may be equal to the greater.

Prop. III. This problem admits of two solutions, and it is left undetermined from which end of the greater line the part is to be cut off.

By means of this problem, a straight line may be found equal to the sum or the difference of two given lines.

Prop. IV. This forms the first case of equal triangles, two other cases are proved in Prop. VIII. and Prop. XXVI.

The term *base* is obviously taken from the idea of a building, and the same may be said of the term *altitude*. In Geometry, however, these terms are not restricted to one particular position of a figure, as in the case of a building, but may be in any position whatever.

Prop. V. Proclus has given, in his commentary, a proof for the equality of the angles at the base, without producing the equal sides. The construction follows the same order, taking in *AB* one side of the isosceles triangle *ABC*, a point *D* and cutting off from *AC* a part *AE* equal to *AD*, and then joining *CD* and *BE*.

A corollary is a theorem which results from the demonstration of a proposition.

Prop. VI. is the converse of one part of Prop. V. One proposition is defined to be the *converse* of another when the hypothesis of the former becomes the predicate of the latter; and vice versa.

There is besides this, another kind of conversion, when a theorem has several hypotheses and one predicate; by assuming the predicate and one, or more than one of the hypotheses, some one of the hypotheses may be inferred as the predicate of the converse. In this manner, Prop. VIII. is the converse of Prop. IV. It may here be observed, that converse theorems are not universally true: as for instance, the following direct proposition is universally true; "If two triangles have their three sides respectively equal, the three angles of each shall be respectively equal." But the converse is not universally true; namely, "If two triangles have the three angles in each respectively equal, the three sides are respectively equal." Converse theorems require, in some instances, the consideration of other conditions than those which enter into the proof of the direct theorem. *Converse* and *contrary* propositions are by no means to be confounded; the *contrary* proposition denies what is asserted, or asserts what is denied, in the *direct* proposition, but the subject and predicate in each are the same. A *contrary proposition* is a *completely contradictory proposition*, and the distinction consists in this—that *two contrary propositions* may both be false, but of *two contradictory propositions*, one of them must be true, and the other false. It may here be remarked, that one of the most common intellectual mistakes of learners, is to imagine that the denial of a proposition is a legitimate ground for affirming the contrary as true: whereas the rules of sound reasoning allow that the affirmation of a proposition as true, only affords a ground for the denial of the contrary as false.

Prop. VI. is the first instance of indirect demonstrations, and they are more suited for the proof of converse propositions. All those propositions which are demonstrated ex absurdo, are properly analytical demonstrations, according to the Greek notion of analysis, which first supposed the thing required, to be done, or to be true, and then shewed the consistency or inconsistency of this construction or hypothesis with truths admitted or already demonstrated.

In indirect demonstrations, where hypotheses are made which are not true and contrary to the truth stated in the proposition, it seems desirable that a form of expression should be employed different from that in which the hypotheses are true. In all cases therefore, whether noted by Euclid or not, the words *if possible* have been introduced, or some such qualifying expression, as in Euc. I. 6, so as not to leave upon the mind of the learner, the impression that the hypothesis which contradicts the proposition, is really true.

Prop. VIII. When the three sides of one triangle are shewn to coincide with the three sides of any other, the equality of the triangles is at once obvious. This, however, is not stated at the conclusion of Prop. VIII. or of Prop. XXVI. For the equality of the areas of two coincident triangles, reference is always made by Euclid to Prop. IV.

A direct demonstration may be given of this proposition, and Prop. VII. may be dispensed with altogether.

Let the triangles ABC, DEF be so placed that the base BC may coincide with the base EF, and the vertices A, D may be on opposite sides of EF. Join AD. Then because EAD is an isosceles triangle, the angle EAD is equal to the angle EDA; and because CDA is an isosceles triangle, the angle CAD is equal to the angle CDA. Hence the angle *EAF* is equal to the angle *EDF*, (ax. 2 or 3): or the angle *BDC* is equal to the angle *EDF*.

Prop. IX. If *BA*, *AC* be in the same straight line. This problem then becomes the same as Prob. XI, which may be regarded as drawing a line which bisects an angle equal to two right angles.

If *FA* be produced in the fig. Prop. 9, it bisects the angle which is the defect of the angle *BAC from four right angles.*

By means of this problem, any angle may be divided into four, eight, sixteen, &c. equal angles.

Prop. X. A finite straight line may, by this problem, be divided into four, eight, sixteen, &c. equal parts.

Prop. XI. When the point is at the extremity of the line; by the second postulate the line may be produced, and then the construction applies. See note on Euc. III. 31.

The distance between two points is the straight line which joins the points; but the distance between a point and a straight line, is *the shortest line* which can be drawn from the point to the line.

From this Prop. it follows that only one perpendicular can be drawn from a given point to a given line; and this perpendicular may be shewn to be less than any other line which can be drawn from the given point to the given line: and of the rest, the line which is nearer to the perpendicular is less than one more remote from it: also only two equal straight lines can be drawn from the same point to the line, one on each side of the perpendicular or the least. This property is analogous to Euc. III. 7, 8.

The corollary to this proposition is not in the Greek text, but was added by Simson, who states that it "is necessary to Prop. 1, Book XI., and otherwise."

Prop. XII. The third postulate requires that the line *CD* should be drawn before the circle can be described with the center *C*, and radius *CD*.

Prop. XIV. is the converse of Prop. XIII. "Upon the opposite sides of it." If these words were omitted, it is possible for two lines to make with a third, two angles, which together are equal to two right angles, in such a manner that the two lines shall not be in the same straight line.

The line *BE* may be supposed to fall above, as in Euclid's figure, or below the line *BD*, and the demonstration is the same in form.

Prop. XV. is the development of the definition of an angle. If the lines at the angular point be produced, the produced lines have the same inclination to one another as the original lines, but in a different position.

The converse of this Proposition is not proved by Euclid, namely:—If the vertical angles made by four straight lines at the same point be respectively equal to each other, each pair of opposite lines shall be in the same straight line.

Prop. XVII. appears to be only a corollary to the preceding proposition, and it seems to be introduced to explain Axiom XII, of which it is the converse. The exact truth respecting the angles of a triangle is proved in Prop. XXXII.

Prop. XVIII. It may here be remarked, for the purpose of guarding the student against a very common mistake, that in this proposition and in the converse of it, the *hypothesis* is stated before the *predicate*.

Prop. XIX. is the converse of Prop. XVIII. It may be remarked, that Prop. XIX. bears the same relation to Prop. XVIII., as Prop. VI does to Prop. V.

Prop. xx. The following corollary arises from this proposition:— A straight line is the shortest distance between two points. For the straight line *BC* is always less than *BA* and *AC*, however near the point *A* may be to the line *BC*.

It may be easily shewn from this proposition, that the difference of any two sides of a triangle is less than the third side.

Prop. xxii. When the sum of two of the lines is equal to, and when it is less than, the third line; let the diagrams be described, and they will exhibit the impossibility implied by the restriction laid down in the Proposition.

The same remark may be made here, as was made under the first Proposition, namely:—if one circle lies partly within and partly without another circle, the circumferences of the circles intersect each other in two points.

Prop. xxiii. *CD* might be taken equal to *CE*, and the construction effected by means of an isosceles triangle. It would, however, be less general than Euclid's, but is more convenient in practice.

Prop. xxiv. Simson makes the angle *EDG* at *D* in the line *ED*, the side which is not the greater of the two *ED*, *DF*; otherwise, three different cases would arise, as may be seen by forming the different figures. The point *G* might fall below or upon the base *EF* produced as well as above it. Prop. xxiv. and Prop. xxv. bear to each other the same relation as Prop. iv. and Prop. viii.

Prop. xxvi. This forms the third case of the equality of two triangles. Every triangle has three sides and three angles, and when any three of one triangle are given equal to any three of another, the triangles may be proved to be equal to one another, whenever the three magnitudes given in the hypothesis are independent of one another. Prop. iv. contains the first case, when the hypothesis consists of two sides and the included angle of each triangle. Prop. viii. contains the second, when the hypothesis consists of the three sides of each triangle. Prop. xxvi. contains the third, when the hypothesis consists of two angles, and one side, either adjacent to the equal angles, or opposite to one of the equal angles in each triangle. There is another case, not proved by Euclid, when the hypothesis consists of two sides and one angle in each triangle, but these not the angles included by the two given sides in each triangle. This case however is only true under a certain restriction, thus:

If two triangles have two sides of one of them equal to two sides of the other, each to each, and have also the angles opposite to one of the equal sides in each triangle, equal to one another, and if the angles opposite to the other equal sides be both acute, or both obtuse angles; then shall the third sides be equal in each triangle, as also the remaining angles of the one to the remaining angles of the other.

Let *ABC*, *DEF* be two triangles which have the sides *AB*, *AC* equal to the two sides *DE*, *DF*, each to each, and the angle *ABC* equal to the angle *DEF*: then, if the angles *ACB*, *DEF*, be *both acute*, or *both obtuse angles*, the third side *BC* shall be equal to the third side *EF*, and also the angle *BCA* to the angle *EFD*, and the angle *BAC* to the angle *EDF*.

First. Let the angles *ACB*, *DFE* opposite to the equal sides *AB*, *DE*, be *both acute angles*.

If *BC* be not equal to *EF*, let *BC* be the greater, and from *BC*, cut off *BG* equal to *EF*, and join *AG*.

Then in the triangles *ABG*, *DEF*, Euc. i. 4. *AG* is equal to *DF*, and the angle AGB to DFE. But since AC is equal to DF, AG is equal to AC: and therefore the angle ACG is equal to the angle AGC, which is also an acute angle. But because AGC, AGB are together equal to two right angles, and that AGC is an acute angle, AGB must be an obtuse angle; which is absurd. Wherefore, BC is not unequal to EF, that is, BC is equal to EF, and also the remaining angles of one triangle to the remaining angles of the other.

Secondly. Let the angles ACB, DFE, be both *obtuse angles*. By proceeding in a similar way, it may be shewn that BC cannot be otherwise than equal to EF.

If ACB, DFE be *both right angles:* the case falls under Euc. I. 26.

Prop. XXVII. Alternate angles are defined to be the two angles which two straight lines make with another at its extremities, but upon opposite sides of it.

When a straight line intersects two other straight lines, two pairs of alternate angles are formed by the lines at their intersections, as in the figure, BEF, EFC are alternate angles as well as the angles AEF, EFD.

Prop. XXVIII. One angle is called "the exterior angle," and another "the interior and opposite angle," when they are formed on the same side of a straight line which falls upon or intersects two other straight lines. It is also obvious that on each side of the line, there will be two exterior and two interior and opposite angles. The exterior angle EGB has the angle GHD for its corresponding interior and opposite angle: also the exterior angle FHD has the angle HGB for its interior and opposite angle.

Prop. XXIX is the converse of Prop. XXVII and Prop. XXVIII.

As the definition of parallel straight lines simply describes them by a statement of the negative property, that they never meet; it is necessary that some positive property of parallel lines should be assumed as an axiom, on which reasonings on such lines may be founded.

Euclid has assumed the statement in the twelfth axiom, which has been objected to, as not being self-evident. A stronger objection appears to be, that the converse of it forms Euc. I. 17; for both the assumed axiom and its converse, should be so obvious as not to require formal demonstration.

Simson has attempted to overcome the objection, not by any improved definition and axiom respecting parallel lines; but, by considering Euclid's twelfth axiom to be a theorem, and for its proof, assuming two definitions and one axiom, and then demonstrating five subsidiary Propositions.

Instead of Euclid's twelfth axiom, the following has been proposed as a more simple property for the foundation of reasonings on parallel lines; namely, "If a straight line fall on two parallel straight lines, the alternate angles are equal to one another." In whatever this may exceed Euclid's definition in simplicity, it is liable to a similar objection, being the converse of Euc. I. 27.

Professor Playfair has adopted in his Elements of Geometry, that "Two straight lines which intersect one another cannot be both parallel to the same straight line." This apparently more simple axiom follows as a direct inference from Euc. I. 30.

But one of the least objectionable of all the definitions which have been proposed on this subject, appears to be that which simply expresses the conception of equidistance. It may be formally stated thus: "Parallel lines are such as lie in the same plane, and which neither recede from, nor approach to, each other." This includes the conception stated by Euclid, that parallel lines never meet. Dr. Wallis observes on this subject, "Parallelismus et æquidistantia vel idem sunt, vel certe se mutuo comitantur."

As an additional reason for this definition being preferred, it may be remarked that the meaning of the terms γραμμαὶ παράλληλοι, suggests the exact idea of such lines.

An account of thirty methods which have been proposed at different times for avoiding the difficulty in the twelfth axiom, will be found in the appendix to Colonel Thompson's "Geometry without Axioms."

Prop. xxx. In the diagram, the two lines *AB* and *CD* are placed one on each side of the line *EF*: the proposition may also be proved when both *AB* and *CD* are on the same side of *EF*.

Prop. xxxii. From this proposition, it is obvious that if one angle of a triangle be equal to the sum of the other two angles, that angle is a right angle, as is shewn in Euc. iii. 31, and that each of the angles of an equilateral triangle, is equal to two thirds of a right angle, as it is shewn in Euc. iv. 15. Also, if one angle of an isosceles triangle be a right angle, then each of the equal angles is half a right angle, as in Euc. ii. 9.

The three angles of a triangle may be shewn to be equal to two right angles without producing a side of the triangle, by drawing through any angle of the triangle a line parallel to the opposite side, as Proclus has remarked in his Commentary on this proposition. It is manifest from this proposition, that the third angle of a triangle is not independent of the sum of the other two; but is known if the sum of any two is known. Cor. 1 may be also proved by drawing lines from any one of the angles of the figure to the other angles. If any of the sides of the figure bend inwards and form what are called re-entering angles, the enunciation of these two corollaries will require some modification. As Euclid gives no definition of re-entering angles, it may fairly be concluded, he did not intend to enter into the proofs of the properties of figures which contain such angles.

Prop. xxxiii. The words "towards the same parts" are a necessary restriction: for if they were omitted, it would be doubtful whether the extremities *A*, *C*, and *B*, *D* were to be joined by the lines *AC* and *BD*; or the extremities *A*, *D*, and *B*, *C*, by the lines *AD* and *BC*.

Prop. xxxiv. If the other diameter be drawn, it may be shewn that the diameters of a parallelogram bisect each other, as well as bisect the area of the parallelogram. If the parallelogram be right angled, the diagonals are equal; if the parallelogram be a square or a rhombus, the diagonals bisect each other at right angles. The converse of this Prop., namely, "If the opposite sides or opposite angles of a quadrilateral figure be equal, the opposite sides shall also be parallel; that is, the figure shall be a parallelogram," is not proved by Euclid.

Prop. xxxv. The latter part of the demonstration is not expressed very intelligibly. Simson, who altered the demonstration, seems in fact to consider two trapeziums of the same form and magnitude, and from one of them, to take the triangle *ABE*; and from the other, the triangle *DCF*; and then the remainders are equal by the third axiom: that is, the parallelogram *ABCD* is equal to the parallelogram *EBCF*. Otherwise, the triangle, whose base is *DE*, (fig. 2.) is taken twice from the trapezium, which would appear to be impossible, if the sense in which Euclid applies the third axiom, is to be retained here.

It may be observed, that the two parallelograms exhibited in fig. 2 partially lie on one another, and that the triangle whose base is *BC* is a common part of them, but that the triangle whose base is *DE* is entirely without both the parallelograms. After having proved the triangle *ABE* equal to the triangle *DCF*, if we take from these equals (fig. 2.) the triangle whose base is *DE*, and to each of the remainders add the triangle whose base is *BC*, then the parallelogram *ABCD* is equal to the parallelogram *EBCF*. In fig. 3, the equality of the parallelograms *ABCD*, *EBCF*, is shewn by adding the figure *EBCD* to each of the triangles *ABE*, *DCF*.

In this proposition, the word *equal* assumes a new meaning, and is no longer restricted to mean coincidence in all the parts of two figures.

Prop. XXXVIII. In this proposition, it is to be understood that the bases of the two triangles are in the same straight line. If in the diagram the point *E* coincide with *C*, and *D* with *A*, then the angle of one triangle is supplemental to the other. Hence the following property:—If two triangles have two sides of the one respectively equal to two sides of the other, and the contained angles supplemental, the two triangles are equal.

A distinction ought to be made between *equal* triangles and *equivalent* triangles, the former including those whose sides and angles mutually coincide, the latter those whose areas only are equivalent.

Prop. XXXIX. If the vertices of all the equal triangles which can be described upon the same base, or upon the equal bases as in Prop. 40, be joined, the line thus formed will be a straight line, and is called the locus of the vertices of equal triangles upon the same base, or upon equal bases.

A locus in plane Geometry is a straight line or a plane curve, every point of which and none else satisfies a certain condition. With the exception of the straight line and the circle, the two most simple loci; all other loci, perhaps including also the Conic Sections, may be more readily and effectually investigated algebraically by means of their rectangular or polar equations.

Prop. XLI. The converse of this proposition is not proved by Euclid; viz. If a parallelogram is double of a triangle, and they have the same base, or equal bases upon the same straight line, and towards the same parts, they shall be between the same parallels. Also, it may easily be shewn that if two equal triangles are between the same parallels; they are either upon the same base, or upon equal bases.

Prop. XLIV. A parallelogram described on a straight line is said to be *applied* to that line.

Prop. XLV. The problem is solved only for a rectilineal figure of four sides. If the given rectilineal figure have more than four sides, it may be divided into triangles by drawing straight lines from any angle of the figure to the opposite angles, and then a parallelogram equal to the third triangle can be applied to *LM*, and having an angle equal to *E*: and so on for all the triangles of which the rectilineal figure is composed.

Prop. XLVI. The square being considered as an equilateral rectangle, its area or surface may be expressed numerically if the number of lineal units in a side of the square be given, as is shewn in the note on Prop. I., Book II.

The student will not fail to remark the analogy which exists between the *area of a square* and *the product of two equal numbers*; and between *the side of a square* and *the square root of a number*. There is, however, this distinction to be observed; it is always possible to find the product of two equal numbers, (or *to find the square of a number*, as it is usually called,) and to describe a square on a given line; but conversely, though the side of a given square is known from the figure itself, the exact number of units in the side of a square of given area, can only be found exactly, in such cases where the given number is a square number. For example, if the area of a square contain 9 square units, then the square root of 9 or 3, indicates the number of lineal units in the side of that square. Again, if the area of a square contain 12 square units, the side of the square is greater than 3, but less than 4 lineal units, and there is no number which will exactly express the side of that square: an approximation to the true length, however, may be obtained to any assigned degree of accuracy.

Prop. XLVII. In a right-angled triangle, the side opposite to the right angle is called the hypotenuse, and the other two sides, the base and perpendicular, according to their position.

In the diagram the three squares are described on the *outer* sides of the triangle *ABC*. The Proposition may also be demonstrated (1) when the three squares are described upon the *inner* sides of the triangle: (2) when one square is described on the outer side and the other two squares on the inner sides of the triangle: (3) when one square is described on the inner side and the other two squares on the outer sides of the triangle.

As one instance of the third case. If the square *BE* on the hypotenuse be described on the inner side of *BC* and the squares *BG*, *HC* on the outer sides of *AB*, *AC*; the point *D* falls on the side *FG* (Euclid's fig.) of the square *BG*, and *KH* produced meets *CE* in *E*. Let *LA* meet *BC* in *M*. Join *DA*; then the square *GB* and the oblong *LB* are each double of the triangle *DAB*, (Euc. I. 41.); and similarly by joining *EA*, the square *HC* and oblong *LC* are each double of the triangle *EAC*. Whence it follows that the squares on the sides *AB*, *AC* are together equal to the square on the hypotenuse *BC*.

By this proposition may be found a square equal to the sum of any given squares, or equal to any multiple of a given square: or equal to the difference of two given squares.

The truth of this proposition may be exhibited to the eye in some particular instances. As in the case of that right-angled triangle whose three sides are 3, 4, and 5 units respectively. If through the points of division of two contiguous sides of each of the squares upon the sides, lines be drawn parallel to the sides (see the notes on Book II.), it will be obvious, that the squares will be divided into 9, 16 and 25 small squares, each of the same magnitude; and that the number of the small squares into which the squares on the perpendicular and base are divided is equal to the number into which the square on the hypotenuse is divided.

Prop. XLVIII is the converse of Prop. XLVII. In this Prop. is assumed the Corollary that "the squares described upon two equal lines are equal," and the converse, which properly ought to have been appended to Prop. XLVI.

The First Book of Euclid's Elements, it has been seen, is conversant with the construction and properties of rectilineal figures. It first lays down the definitions which limit the subjects of discussion in the First Book, next the three postulates, which restrict the instruments by which the constructions in Plane Geometry are effected; and thirdly, the twelve axioms, which express the principles by which a comparison is made between the ideas of the things defined.

This Book may be divided into three parts. The first part treats of the origin and properties of triangles, both with respect to their sides and angles; and the comparison of these mutually, both with regard to equality and inequality. The second part treats of the properties of parallel lines and of parallelograms. The third part exhibits the connection of the properties of triangles and parallelograms, and the equality of the squares on the base and perpendicular of a right-angled triangle to the square on the hypotenuse.

When the propositions of the First Book have been read with the notes, the student is recommended to use different letters in the diagrams, and where it is possible, diagrams of a form somewhat different from those exhibited in the text, for the purpose of testing the accuracy of his knowledge of the demonstrations. And further, when he has become sufficiently familiar with the method of geometrical reasoning, he may dispense with the aid of letters altogether, and acquire the power of expressing in general terms the process of reasoning in the demonstration of any proposition. Also, he is advised to answer the following questions before he attempts to apply the principles of the First Book to the solution of Problems and the demonstration of Theorems.

QUESTIONS ON BOOK I.

1. What is the name of the Science of which Euclid gives the Elements? What is meant by *Solid Geometry!* Is there any distinction between *Plane Geometry*, and the *Geometry of Planes?*

2. Define the term *magnitude*, and specify the different kinds of magnitude considered in Geometry. What dimensions of space belong to figures treated of in the first six Books of Euclid?

3. Give Euclid's definition of a "straight line." What does he really use as his test of rectilinearity, and where does he first employ it? What objections have been made to it, and what substitute has been proposed as an available definition? How many points are necessary to fix the position of a straight line in a plane? When is one straight line said to *cut*, and when to *meet* another?

4. What positive property has a Geometrical point? From the definition of a straight line, shew that the intersection of two lines is a point.

5. Give Euclid's definition of a plane rectilineal angle. What are the limits of the angles considered in Geometry? Does Euclid consider angles greater than two right angles?

6. When is a straight line said to be drawn at *right angles*, and when *perpendicular*, to a given straight line?

7. Define a *triangle*; shew how many kinds of triangles there are according to the variation both of the *angles*, and of the *sides*.

8. What is Euclid's definition of a circle? Point out the assumption involved in your definition. Is any axiom implied in it? Shew that in this as in all other definitions, some geometrical fact is assumed as somehow previously known.

9. Define the quadrilateral figures mentioned by Euclid.

10. Describe briefly the use and foundation of definitions, axioms; and postulates: give illustrations by an instance of each.

11. What objection may be made to the method and order in which Euclid has laid down the elementary abstractions of the Science of Geometry? What other method has been suggested?

12. What distinctions may be made between definitions in the Science of Geometry and in the Physical Sciences?

13. What is necessary to constitute an exact definition? Are definitions propositions? Are they arbitrary? Are they convertible? Does a Mathematical definition admit of proof on the principles of the Science to which it relates?

14. Enumerate the principles of construction assumed by Euclid.

15. Of what instruments may the use be considered to meet approximately the demands of Euclid's postulates? Why only *approximately?*

16. "A circle may be described from any center, with any straight line as radius." How does this postulate differ from Euclid's, and which of his problems is assumed in it?

17. What principles in the Physical Sciences correspond to axioms in Geometry?

18. Enumerate Euclid's twelve axioms and point out those which have special reference to Geometry. State the converse of those which admit of being so expressed.

19. What two tests of equality are assumed by Euclid? Is the assumption of the principle of superposition (ax. 8.), essential to all Geometrical reasoning? Is it correct to say, that it is "an appeal, though of the most familiar sort, to external observation"?

20. Could any, and if any, which of the axioms of Euclid be turned into definitions; and with what advantages or disadvantages?

21. Define the terms, Problem, Postulate, Axiom and Theorem. Are any of Euclid's axioms improperly so called?

22. Of what two parts does the enunciation of a Problem, and of a Theorem consist? Distinguish them in Euc. I. 4, 5, 18, 19.

23. When is a problem said to be indeterminate? Give an example.

24. When is one proposition said to be the converse or reciprocal of another? Give examples. Are converse propositions universally true? If not, under what circumstances are they necessarily true? Why is it necessary to demonstrate converse propositions? How are they proved?

25. Explain the meaning of the word *proposition*. Distinguish between *converse* and *contrary* propositions, and give examples.

26. State the grounds as to whether Geometrical reasonings depend for their conclusiveness upon axioms or definitions.

27. Explain the meaning of *enthymeme* and *syllogism*. How is the enthymeme made to assume the form of the syllogism? Give examples.

28. What constitutes a demonstration? State the laws of demonstration.

29. What are the principal parts, in the entire process of establishing a proposition?

30. Distinguish between a *direct* and *indirect* demonstration.

31. What is meant by the term *synthesis*, and what, by the term *analysis?* Which of these modes of reasoning does Euclid adopt in his Elements of Geometry?

32. In what sense is it true that the conclusions of Geometry are necessary truths?

33. Enunciate those Geometrical definitions which are used in the proof of the propositions of the First Book.

34. If in Euclid I. 1, an equal triangle be described on the other side of the given line, what figure will the two triangles form?

35. In the diagram, Euclid I. 2, if DB a side of the equilateral triangle DAB be produced both ways and cut the circle whose center is B and radius BC in two points G and H; shew that either of the dis-

tances DG, DH may be taken as the radius of the second circle; and give the proof in each case.

36. Explain how the propositions Euc. I. 2, 3, are rendered necessary by the restriction imposed by the third postulate. Is it necessary for the proof, that the triangle described in Euc. I. 2, should be equilateral? Could we, at this stage of the subject, describe an isosceles triangle on a given base?

37. State how Euc. I. 2, may be extended to the following problem: "From a given point to draw a straight line *in a given direction* equal to a given straight line."

38. How would you cut off from a straight line unlimited in both directions, a length equal to a given straight line?

39. In the proof of Euclid I. 4, how much depends upon Definition, how much upon Axiom?

40. Draw the figure for the third case of Euc. I. 7, and state why it *needs no demonstration.*

41. In the construction Euclid I. 9, is it indifferent in all cases on which side of the joining line the equilateral triangle is described?

42. Shew how a given straight line may be bisected by Euc. I. 1.

43. In what cases do the lines which bisect the interior angles of plane triangles, also bisect one, or more than one of the corresponding opposite sides of the triangles?

44. "Two straight lines cannot have a common segment." Has this corollary been tacitly assumed in any preceding proposition?

45. In Euc. I. 12, must the given line necessarily be "of unlimited length"?

46. Shew that (fig. Euc. I. 11) every point without the perpendicular drawn from the middle point of every straight line DE, is at unequal distances from the extremities D, E of that line.

47. From what proposition may it be inferred that a straight line is the shortest distance between two points?

48. Enunciate the propositions you employ in the proof of Euc. I. 16.

49. Is it essential to the truth of Euc. I. 21, that the two straight lines be drawn from the extremities of the base?

50. In the diagram, Euc. I. 21, by how much does the greater angle BDC exceed the less BAC?

51. To form a triangle with three straight lines, any two of them must be greater than the third: is a similar limitation necessary with respect to the three angles?

52. Is it possible to form a triangle with three lines whose lengths are 1, 2, 3 units: or one with three lines whose lengths are 1, $\sqrt{2}$, $\sqrt{3}$?

53. Is it possible to construct a triangle whose angles shall be as the numbers 1, 2, 3? Prove or disprove your answer.

54. What is the reason of the limitation in the construction of Euc. I. 24. viz. "that DE is that side which is not greater than the other?"

55. Quote the first proposition in which the equality of two areas which cannot be superposed on each other is considered.

56. Is the following proposition universally true? "If two plane triangles have three elements of the one respectively equal to three elements of the other, the triangles are equal in every respect." Enumerate all the cases in which this equality is proved in the First Book. What case is omitted?

57. What parts of a triangle must be given in order that the triangle may be described?

58. State the converse of the second case of Euc. I. 26? Under what limitations is it true? Prove the proposition so limited?

59. Shew that the angle contained between the perpendiculars drawn to two given straight lines which meet each other, is equal to the angle contained by the lines themselves.

60. Are two triangles necessarily equal in all respects, where a side and two angles of the one are equal to a side and two angles of the other, each to each?

61. Illustrate fully the difference between analytical and synthetical proofs. What propositions in Euclid are demonstrated analytically?

62. Can it be properly predicated of any two straight lines that they never meet if indefinitely produced either way, antecedently to our knowledge of some other property of such lines, which makes the property first predicated of them a necessary conclusion from it?

63. Enunciate Euclid's definition and axiom relating to parallel straight lines; and state in what Props. of Book I. they are used.

64. What proposition is the converse to the twelfth axiom of the First Book? What other two propositions are complementary to these?

65. If lines being produced ever so far do not meet; can they be otherwise than parallel? If so, under what circumstances?

66. Define *adjacent angles*, *opposite angles*, *vertical angles*, and *alternate angles*; and give examples from the First Book of Euclid.

67. Can you suggest anything to justify the assumption in the twelfth axiom upon which the proof of Euc. I. 29, depends?

68. What objections have been urged against the definition and the doctrine of parallel straight lines as laid down by Euclid? Where does the difficulty originate? What other assumptions have been suggested and for what reasons?

69. Assuming as an axiom that two straight lines which cut one another cannot both be parallel to the same straight line; deduce Euclid's twelfth axiom as a corollary of Euc. I. 29.

70. From Euc. I. 27, shew that the distance between two parallel straight lines is constant?

71. If two straight lines be not parallel, shew that all straight lines falling on them, make alternate angles, which differ by the same angle.

72. Taking as the definition of parallel straight lines that they are equally inclined to the same straight line towards the same parts; prove that "being produced ever so far both ways they do not meet?" Prove also Euclid's axiom 12, by means of the same definition.

73. What is meant by *exterior* and *interior* angles? Point out examples.

74. Can the three angles of a triangle be proved equal to two right angles without producing a side of the triangle?

75. Shew how the corners of a triangular piece of paper may be turned down, so as to exhibit to the eye that the three angles of a triangle are equal to two right angles.

76. Explain the meaning of the term *corollary*. Enunciate the two corollaries appended to Euc. I. 32, and give another proof of the first. What other corollaries may be deduced from this proposition?

77. Shew that the two lines which bisect the exterior and interior angles of a triangle, as well as those which bisect any two interior angles of a parallelogram, contain a right angle.

78. The opposite sides and angles of a parallelogram are equal to one another, and the diameters bisect it. State and prove the converse of this proposition. Also shew that a quadrilateral figure, is a paral-

lelogram, when its diagonals bisect each other: and when its diagonals divide it into four triangles, which are equal, two and two, viz. those which have the same vertical angles.

79. If two straight lines join the extremities of two parallel straight lines, but *not* towards the same parts, when are the joining lines equal, and when are they unequal?

80. If either diameter of a four-sided figure divide it into two equal triangles, is the figure necessarily a parallelogram? Prove your answer.

81. Shew how to divide one of the parallelograms in Euc. I. 35, by straight lines so that the parts when properly arranged shall make up the other parallelogram.

82. Distinguish between *equal* triangles and *equivalent* triangles, and give examples from the First Book of Euclid.

83. What is meant by the locus of a point? Adduce instances of loci from the first Book of Euclid.

84. How is it shewn that equal triangles upon the same base or equal bases have equal altitudes, whether they are situated on the same or opposite sides of the same straight line?

85. In Euc. I. 37, 38, if the triangles are not towards the same parts, shew that the straight line joining the vertices of the triangles is bisected by the line containing the bases.

86. If the complements (fig. Euc. I. 43) be squares, determine their relation to the whole parallelogram.

87. What is meant by a parallelogram being applied to a straight line?

88. Is the proof of Euc. I. 45, perfectly general?

89. Define a square without including superfluous conditions, and explain the mode of constructing a square upon a given straight line in conformity with such a definition.

90. The sum of the angles of a square is equal to four right angles. Is the converse true? If not, why?

91. Conceiving a square to be a figure bounded by four equal straight lines not necessarily in the same plane, what condition respecting the angles is necessary to complete the definition?

92. In Euclid I. 47, why is it necessary to prove that one side of each square described upon each of the sides containing the right angle, should be in the same straight line with the other side of the triangle?

93. On what assumption is an analogy shewn to exist between the product of two equal numbers and the surface of a square?

94. Is the triangle whose sides are 3, 4, 5 right-angled, or not?

95. Can the side and diagonal of a square be represented simultaneously by any finite numbers?

96. By means of Euc. I. 47, the square roots of the natural numbers, 1, 2, 3, 4, &c. may be represented by straight lines.

97. If the square on the hypotenuse in the fig. Euc. I. 47, be described on the other side of it: shew from the diagram how the squares on the two sides of the triangle may be made to cover exactly the square on the hypotenuse.

98. If Euclid II. 2, be assumed, enunciate the form in which Euc. I. 47 may be expressed.

99. Classify all the properties of *triangles* and *parallelograms*, proved in the First Book of Euclid.

100. Mention any propositions in Book I. which are included in more general ones which follow.