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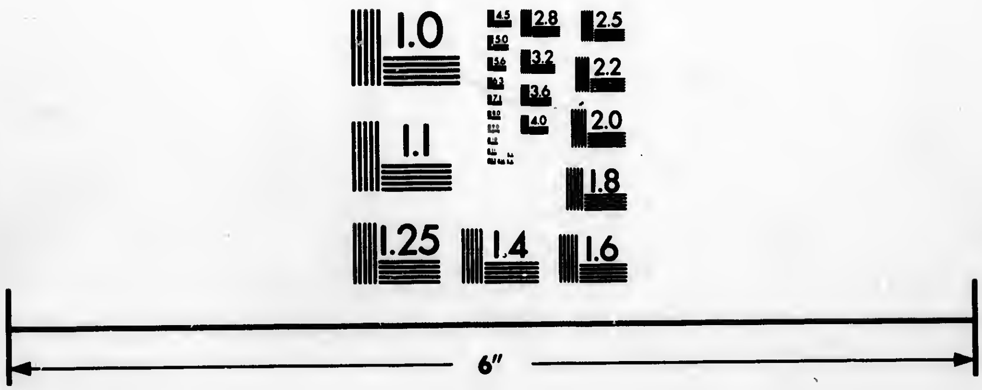

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INVARIANTS AND EQUATIONS

ASSOCIATED WITH THE

General Linear Differential Equation

THESIS PRESENTED FOR THE DEGREE OF PH. D.

BY

GEORGE F.rederic METZLER.

JOHNS HOPKINS UNIVERSITY,
BALTIMORE.
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Introduction.

The formation of functions, associated with differential equations and analogous to the invariants of algebraic quantics, has occupied the attention of several mathematicians for some years, because of their great value in leading to practical as well as theoretical solutions of such equations.

Starting with the work of M. Laguerre and of Professor Brioschi, M. Halphen, in two important memoirs,* indicated a method for the formation of invariants, but involving very difficult analysis. He derived the two simplest invariants for the cubic and quartic and such derivatives as may be deduced from them. For this purpose he, by means of the transformation $Y = ye^{-\int \frac{R_1}{R_0} dx}$, brings the equation to a form having zero for the coefficient of the second term.

Meanwhile Mr. Forsyth, starting with the letter of Professor Brioschi, prepared a very valuable memoir,† in which, by means of the following transformations, he obtains a canonical form in which the coefficients of both the second and third terms vanish. This may be stated as follows:

When the linear differential equation

$$\frac{d^n y}{dx^n} + \binom{n}{2} P_2 \frac{d^{n-2} y}{dx^{n-2}} + \binom{n}{3} P_3 \frac{d^{n-3} y}{dx^{n-3}} + \binom{n}{4} P_4 \frac{d^{n-4} y}{dx^{n-4}} + \dots + P_n = 0$$

---

* "Mémoire sur la réduction des équations differentielles linéaires aux formes integrables" (*Mémoires des Savants Étrangers*, Vol. 28, No. 1, 301 pp., 1880). Also, "Sur les invarients des équations differentielles linéaires du quatrième ordre" (*Acta Math.*, Vol. 3, 1883, pp. 325–380).

† "Invariants, Covariants and Quotient Derivatives associated with Linear Differential Equations."—*Philosophical Transactions of the Royal Society of London*, Vol. 179 (1888), A, pp. 377–489.

has its dependent variable y transformed to u by the equation $y = u\lambda$, λ being a function of x and its independent variable changed from x to z, where z and λ are determined by

$$\lambda = \varphi^{n-1}, \quad \frac{dz}{dx} = \varphi^{-2}, \tag{1}$$

$$\frac{d^2\varphi}{dx^2} + \frac{3}{n+1} P_2\varphi = 0, \tag{2}$$

the transformed in u is in the canonical form

$$\frac{d^n u}{dz^n} + \binom{n}{3} Q_3 \frac{d^{n-3}u}{dz^{n-3}} + \binom{n}{4} Q_4 \frac{d^{n-4}u}{dz^{n-4}} + \ldots + Q_n = 0,$$

$\binom{n}{r}$ being the binomial coefficient $\frac{n!}{r!\,\overline{n-r}!}$.

The coefficients P and Q of these equations are so connected that there exist $n-2$ algebraically independent functions $\theta\ (x)$ of the coefficients P and their derivatives which are such that, when the same function $\theta_\sigma(z)$ is formed of the coefficients Q and their derivatives, the equation

$$\theta_\sigma(x) = (z')^\sigma \theta_\sigma(z) \tag{3}$$

is identically satisfied. For this form of the differential equation

$$\theta_\sigma(z) \equiv Q_\sigma + \frac{\sigma}{2} \sum_{r=1}^{r=\sigma-3} (-1)^r a_{r,\sigma} \frac{d^r Q_{\sigma-r}}{dz^r},$$

where

$$a_{r,\sigma} = \frac{\overline{\sigma-1}!\,\overline{\sigma-2}!\,\overline{2\sigma-r-2}!}{r!\,\overline{2\sigma-3}!\,\overline{\sigma-r}!\,\overline{\sigma-r-1}!}.$$

Thus $\theta_\sigma(z)$ is independent of the order of the equation. In this z is completely determined by equations (1) and (2). But there may be difficulties in the way of solving (2), and thus it is desirable to form the invariants for the uncanonical form of the equation.

For this purpose Mr. Forsyth establishes relations between the coefficients P and Q for the case in which z, being arbitrary, is given the value $x + \epsilon\mu$, where ϵ is so small that the square and higher powers may be neglected, and μ is an arbitrary non-constant function of x. These relations are expressed thus:

$$\left.\begin{aligned} Q_s = P_s(1 - s\epsilon\mu') - \frac{\epsilon}{2} \sum_{\theta=0}^{\theta=s-1} \Bigg[\frac{s!}{\theta!\, \overline{s-\theta+1}!} \\ \{n(s-\theta-1)+s+\theta-1\}\, P_\theta \frac{d^{s-\theta+1}\mu}{dx^{s-\theta+1}} \Bigg] \end{aligned}\right\}. \quad (5)$$

These relations are fully developed in Mr. Forsyth's memoir; also in Dr. Craig's excellent work* they will be found, and such a general treatment of the whole subject of differential equations and differential quantics as makes the work an invaluable help and guide to any student of the subject.

Then we derive

$$\left.\begin{aligned} \frac{d^r Q_s}{dz^r} = \frac{d^r P_s}{dx^r}\{1-(r+s)\epsilon\mu'\} - s\epsilon P_s \frac{d^{r+1}\mu}{dx^{r+1}} \\ - \epsilon \sum_{m=1}^{m=r-1} \Bigg[\frac{r!}{m!\, \overline{r-m+1}!} \{s(r+1) \\ - m(s-1)\} \frac{d^m P}{dx^m} \frac{d^{r-m+1}\mu}{dx^{r-m+1}} \Bigg] \\ - \frac{\epsilon}{2} \sum_{\theta=0}^{\theta=s-1} \Bigg[\frac{s!}{\theta!\, \overline{s-\theta+1}!} \{n(s-\theta-1) \\ + s + \theta - 1\} \frac{d^r}{dx^r}\left(P_\theta \frac{d^{s-\theta+1}\mu}{dx^{s-\theta+1}}\right) \Bigg] \end{aligned}\right\}. \quad (6)$$

The only invariants that have been formed, so far as I know, are $\theta_3, \theta_4, \theta_5, \theta_6$ and θ_7, where θ_r is the invariant of the rth order of an equation of order n.

In Section I of this thesis the general invariant θ_s is considered, and it is there shown that in the non-linear part every term is of the form ABC. Where A is a number, B is a function of P_3 and its derivatives, and C is an invariant or the derivative of an invariant with suffix differing from s by an even number. When s is even C may be a number.

Section II deals with the coefficients of θ_s, giving some

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* Treatise on Linear Differential Equations. By Thomas Craig, Ph.D. Vol. I.

general expressions by which they may be calculated for any value of s.

Section III treats of associate variables and associate equations, showing which are identical and which may not be.

Dr. Craig having discovered that the condition for the self-adjointness of the sextic and octic was that their invariants with odd suffix all vanish, suggested to me the general theorem announced in his treatise, pp. 293–295. The proof given at that time only applied to equations in Mr. Forsyth's canonical form. By aid of what is established in Section I, it is shown to apply also to equations in any form.

A fuller history of the subject will be found in the works to which reference has been made.

This paper was not only suggested by Dr. Craig, but has had his valuable criticism.

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Section I.

THE FORM OF THE GENERAL LINEAR PRIME INVARIANT θ_s.

Since θ_s has only a linear part when P_3 vanishes, its form must be as follows:

$$
\begin{array}{llll}
 & [AP_s + BP'_{s-1} + CP''_{s-2} + \dots + WP_2^{(s-2)} & &] \\
+ & [P_3\{a_3\theta_{s-3} & + a_4\theta'_{s-4} + a_5\theta''_{s-5} + \dots + & a_{s-3}P_3^{(s-6)}\}] \\
+ & [P'_3\} & b_4\theta_{s-4} + b_5\theta'_{s-5} + \dots + & b_{s-3}P_3^{(s-6)}\}] \\
+ & [P''_3\} & c_5\theta_{s-5} + c_6\theta'_{s-6} + \dots + & c_{s-3}P_3^{(s-6)}\}] \\
+ & [P_3^2\} & d_6\theta_{s-6} + d_7\theta'_{s-7} + \dots + & d_{s-3}P_{s-3}^{(s-6)}\}] \\
+ & [P'''_3\} \; . \; . \; . \; . \; . \; . \; . & &] \\
+ & [\text{etc.} \; . \; . \; . \; . \; . \; . \; . & &] \\
+ & [\text{etc.} \; . \; . \; . \; . \; . \; . \; . & &] \\
 & \text{etc., etc.} \; . \; . \; . \; . \; . \; . \; . & &
\end{array}
$$

In this (r) is the differential index, so that

$$P_3^{(r)} \equiv \frac{d^r P_3}{dx^r}, \qquad \theta_{s-\kappa}^{(r)} \equiv \frac{d^r \theta_{s-\kappa}}{dx^r} \dots$$

The sum of the suffixes and differential indices, it will be noticed, equals s for every term; that is, θ_s possesses a kind of homogeneity.* s is called the index or dimension number of θ_s; the dimension number of $P_3^{(p)^r}\theta_{s-\kappa}^{\mu}$ being

$$(p + 2)r + \mu + s - \kappa.$$

Denoting the terms within the square parenthesis by L, α, β, γ, δ, etc., then $\theta_s \equiv L + \alpha + \beta + \gamma + \delta + \dots$

The notation used here will be nearly that used by Mr. Forsyth, but to simplify the work the μ's and their derivatives arising from $z = x + \epsilon\mu$ will be dropped, that is, they will be

* Philosophical Transactions, Vol. 179 (1888) A, pp. 391–92.

treated as unity, when the result will not be changed by doing so. Also $P_r^{(0)} \equiv \dfrac{d^0 P_r}{dx^0}$ will be considered $\equiv$ with P_r.

The general form of the terms in L is

$$(-1)^\tau \frac{s!\,\overline{s-2}!\,\overline{2s-\tau-2}!}{2.\tau!\,\overline{s-\tau}!\,\overline{s-\tau-1}!\,\overline{2s-3}!}\,P_{s-\tau}^{(\tau)},\ \tau = 0, 1, 2 \ldots s-2. \quad \text{(a)}$$

I shall now show that when κ is odd each of the numerical coefficients a_κ, b_κ, c_κ, d_κ, etc., of the non-linear part of θ_s equals zero.

From page 4 of the introduction we have

$$\Theta_s(x) = z'^s\,\Theta_s(z) = (1 - s\epsilon\mu')\,\Theta_s(z)$$

identically satisfied. If in the right member of this identity the Q's and their derivatives are replaced by their values in terms of the P's and their derivatives, as expressed by formulae (5) and (6) (page 5, introduction), then the terms of dimension 's' in each member cancel, those of dimension '$s-1$' furnish the numerical coefficients in the linear part L, and there remain terms of dimension equal to and less than $s-2$ with which we may determine the coefficients of the non-linear part.

Remembering the convention $P_r^{(0)} \equiv P_r$, formulae (5) and (6) are included in

$$\left.\begin{aligned}
\frac{d^r Q_s}{dz^r} = {} & P_s^{(r)}\{1 - (r+s)\epsilon\mu'\} \\
& - \epsilon \sum_{m=0}^{m=r-1} \left[\frac{r!}{m!\,\overline{r-m+1}!}\{s(r-m+1) \right. \\
& \qquad\qquad \left. + m\}\,P_s^{(m)}\mu^{(r-m+1)}\right] \\
& - \frac{\epsilon}{2} \sum_{\theta=0}^{\theta=s-1} \left[\frac{s!}{\theta!\,\overline{s-\theta+1}!}\{n(s-\theta-1) \right. \\
& \qquad\qquad \left. + s + \theta - 1\}(P_\theta\mu^{(s-\theta+1)})^{(r)}\right] \\
& \qquad\qquad r = 0, 1, 2, 3 \ldots s
\end{aligned}\right\} . \quad (7)$$

Also, differentiating the invariants, we find

$$\frac{d^r\theta_s(x)}{dx^r} = \frac{d^r}{dz^r}\left[z'^s\,\theta_s(z)\right], \tag{8}$$

$$\left.\begin{aligned}\frac{d^r\theta_s(z)}{dz^r} = \theta_s^{(r)}(x)\{1-(s+r)\epsilon\mu'\} - \epsilon \sum_{m=0}^{m=r-1}&\left[\frac{r!}{m!\,\overline{r-m+1}!}\right. \\ &\left.\{s(r-m+1)+m\}\,\theta_s^{(m)}\mu^{(r+1-m)}\right]\end{aligned}\right\}. \tag{9}$$

Then $Q_2 = P_2(1-2\epsilon\mu') - \dfrac{n+1}{6}\,\epsilon\mu'''$,

$$Q_3 = P_3(1-3\epsilon\mu') - 3\epsilon\mu''P_2 - \frac{n+1}{4}\,\epsilon\mu^{(4)},$$

. . . .

. . . .

. . . .

$$\begin{aligned}Q_6 = P_6(1-6\epsilon\mu') &- 15\epsilon\mu''P_5 - \tfrac{5}{2}(n+9)\,\epsilon\mu'''P_4 \\ &- 5(n+4)\,\epsilon\mu^{(4)}P_3 - \tfrac{3}{2}(3n+7)\,\epsilon\mu^{(5)}P_2 - \frac{n+1}{14}\,5\epsilon\mu^{(7)}.\end{aligned}$$

$$\frac{dQ_2}{dz^2} = P_2'(1-3\epsilon\mu') - 2\epsilon\mu''P_2 - \frac{n+1}{6}\,\epsilon\mu^{\mathrm{IV}},$$

$$\frac{d^2Q_2}{dz^2} = P_2''(1-4\epsilon\mu') - 5\epsilon\mu''P_2' - 2\epsilon\mu'''P_2 - \frac{n+1}{6}\,\epsilon\mu^{\mathrm{V}},$$

. . . .

. . . .

$$\begin{aligned}\frac{d^2Q_4}{dz^2} = P_4^{(2)}(1-6\epsilon\mu') &- 9\epsilon\mu''P_4' - 4\epsilon\mu'''P_4 \\ &- \epsilon\frac{d^2}{dx^2}\{6\mu''P_3 + (n+5)\,\mu'''P_2\} - \tfrac{3}{10}(n+1)\,\epsilon\mu^{\mathrm{V}},\end{aligned}$$

etc. . . . . etc.

From these follow

$$Q_2^r = P_2^r\{1-2r\epsilon\mu'\} - rP_2^{r-1}\frac{n+1}{6}\,\epsilon\mu^{(3)},$$

$$\begin{aligned}\left(\frac{d^2Q_2}{dz^2}\right)^m = (P_2^{(2)})^m(1-m4\epsilon\mu') - m(P_2^{(2)})^{m-1}&\left\{5\epsilon\mu''P_2'\right. \\ &\left.+ 2\epsilon\mu'''P_2 + \frac{n+1}{6}\,\epsilon\mu^{\mathrm{V}}\right\}.\end{aligned}$$

If $P_2'''\theta_{s-\kappa}^{(\kappa-5)}$ be a term in θ_s, then will the term $\frac{d^2Q_2}{dz^2}\ \frac{d^{\kappa-5}\theta_{s-\kappa}(z)}{dz^{\kappa-5}}$ be multiplied by $(z')^s$ or $(1 - s\mu'\epsilon)$, and

$$
\begin{aligned}
P_2'''\theta_{s-\kappa}^{(\kappa-5)}(x) &= (1 + s\epsilon\mu')\frac{d^2Q_2}{dz^2}\ \frac{d^{\kappa-5}\theta_{s-\kappa}(z)}{dz^{\kappa-5}} \\
&= (1 + \epsilon s\mu')\Big\{P_2'''(1 - 5\epsilon\mu') - 9\epsilon\mu''P_2'' \\
&\qquad - 7\epsilon\mu'''P_2' - 2\epsilon\mu^{\mathrm{IV}}P_2 - \frac{n+1}{5}\epsilon\mu^{\mathrm{VI}}\Big\} \\
&\quad \times\Big(\theta_{s-\kappa}^{(\kappa-5)}(x)\{1-(s-5)\epsilon\mu'\} - \epsilon\sum_{m=0}^{m=\kappa-6}\Big[\frac{\varkappa-5!}{m!\,\varkappa-4-m!} \\
&\qquad \{(\varkappa-4-m)(s-\varkappa)+m\}\,\theta_{s-\kappa}^{(m)}\mu^{(\kappa-6-m)}\Big]\Big) \\
&= P_2'''\theta_{s-\kappa}^{(\kappa-5)} - \theta_{s-\kappa}^{(\kappa-5)}\Big(9\epsilon\mu''P_2'' + 7\epsilon\mu'''P_2' \\
&\qquad + 2\epsilon\mu^{\mathrm{IV}}P_2 + \frac{n+1}{6}\epsilon\mu^{\mathrm{VI}}\Big) \\
&\qquad - P_2'''\epsilon\sum_{m=0}^{m=\kappa-6}\Big[\frac{\varkappa-5!}{m!\,\varkappa-4-m!} \\
&\qquad \{(s-\varkappa)(\varkappa-4-m)+m\}\theta_{s-\kappa}^{(m)}\mu^{(\mu-6-m)}\Big].
\end{aligned}
$$

In this equation the terms of dimension 's' cancel and $-\epsilon$ is a factor of the remaining terms, so that when every term in θ_s is treated in this way, all terms of dimension 's' cancel each other and the remainder is divisible by $-\epsilon$. Denoting by RL the remainder of the linear part L, by R_a the remainder of the terms in a, etc., and by $\left(\frac{n}{r}\right)$ the binomial coefficient $\frac{n!}{r!\,n-r!}$, also omitting the μ's and dividing by $-\epsilon$, we get

$$
\begin{aligned}
RL \equiv A\Big[\frac{s.s-1}{2}P_{s-1} + \frac{s.s-1}{2.3!}(n+1+2(s-2)P_{s-3} + \text{etc.}\Big] \\
+ B[s-1\,P_{s-1} + \ldots] + \ldots \\
= A\Big|\Big\{\left(\frac{s}{\varkappa}\right)\frac{n+1}{2}\,\frac{\varkappa-1}{\varkappa+1} + \left(\frac{s}{\varkappa+1}\right)\Big\}P_{s-\kappa}.\quad \varkappa=1,2,3\ldots s\Big] \\
+ B\sum_{\kappa=1}^{\kappa=s}\Big[\left(\frac{s-1}{\varkappa-1}\right)\frac{n+1}{2}\,\frac{\varkappa-2}{\varkappa} + \left(\frac{s-1}{\varkappa}\right)(\mu^{(\kappa+1)}P_{s-\kappa})'\Big]
\end{aligned}
$$

$$+ C\left[(s-2)P_{s-2} + \ldots + \frac{2!}{1.2!}\left\{2(s-2)+1\right]P'_{s-2}\right.$$
$$+ \sum_{\kappa=3}^{\kappa=s}\left\{\binom{s-2}{x-2}\frac{n+1}{2}\cdot\frac{x-3}{x-1} + \binom{s-3}{x-1}\right\}(P_{s-\kappa}\mu^{(\kappa+1)})''$$

for the first three terms of θ_s. Replacing A, B, C, etc., by their values ((a) p. 8) $1, -\frac{s}{2}, \frac{s.s-1.s-2}{2.2.2s-3}$, etc., the $(r+1)$st term gives

$$\left.\begin{array}{l}(-1)^r\dfrac{s!\,s-2!\,2s-r-2!}{2.r!\,s-r!\,s-r-1!\,2s-3!} \\ \left[\begin{array}{l}\quad\left\{\dbinom{r}{m}(s-r)\right. \\ \qquad\left.+\dbinom{r}{m-1}\right\}P^{(m)}_{s-r},\ m=0,1,2,\ldots r-1 \\ +\left\{\dbinom{s-r}{x-r}\dfrac{n+1}{2}\dfrac{x-r-1}{x-r+1}\right. \\ \qquad\left.+\dbinom{s-r}{x-r+1}\right\}(\mu^{(\kappa+1)}P_{s-\kappa})^{(r)}, \\ \hfill x=r+1, r+2, \ldots s\end{array}\right]\end{array}\right\} \quad (10)$$

as a remainder. By giving r all values 0, 1, 2, 3, ... (10) expresses the whole of RL.

$$R\alpha = P_3\left[\{(s-3)a_3\theta_{s-3} + a_4\}s-4.\theta_{s-4} + (2s-7)\theta_{s-4}\} + \text{etc.}\right]$$
$$+ \frac{n+1}{6}\left[a_2\theta_{s-2} + a_3\theta_{s-3} + a_4\theta''_{s-4} + \ldots\right],$$

$$R\beta = \left(2P_3 + \frac{n+1}{6}\right)\left[b_3\theta_{s-3} + b_4\theta'_{s-4} + b_5\theta''_{s-5} + \ldots\right]$$
$$+ P'_3\sum_{\kappa=4}^{\kappa=s-3}\left[b_\kappa\sum_{m=0}^{m=\kappa-4}\left[\left\{\binom{r}{m}(s-x) + \binom{r}{m-1}\right\}\theta^{(m)}_{s-\kappa},\right.\right.$$
$$\left. r = x-2\right],$$

with similar expressions for the other parts, $R\gamma$, $R\delta$, etc. Suppose that the coefficient of $P^{(s)}_{s-\kappa}$ in RL is $A_1(n+1) + B_1 + C_1$. Then the general forms

$$(-1)^r\frac{1}{2}\binom{s}{r}\binom{s-2}{r-1}\left(\frac{2s-r-2}{2s-3}\right)\binom{s-r}{x-r}\frac{x-r-1}{x-r+1}(\mu^{(\kappa+1)}P_{s-\kappa})^{(r)},$$
$$r = v, v+1, \ldots x-1,$$

$$(-1)^r \frac{1}{2}\left(\frac{s}{r}\right)\left(\frac{s-2}{r-1}\right)\left(\frac{2s-r-2}{2s-3}\right)\left(\frac{s-r}{\kappa-r+1}\right)(\mu^{\kappa+1}P_{s-\kappa})^{(r)},$$
$$r = v, v+1, \ldots \kappa - 1,$$

and

$$(-1)^\kappa \left(\frac{s}{\kappa}\right)\left(\frac{s-2}{\kappa-1}\right)\left(\frac{2s-\kappa-2}{2s-3}\right)\left[\left(\frac{\kappa}{v}\right)(s-\kappa) + \left(\frac{\kappa}{v-1}\right)\right]$$

will, when expanded, give A_1, B_1 and C_1 respectively. In these $\left(\frac{2s-\kappa-2}{2s-3}\right)$ is the reciprocal of $\left(\frac{2s-3}{2s-\kappa-2}\right)$. Thus A_1 is found to be

$$A_1 = (-1)^v\left(\frac{s}{\kappa}\right)\left(\frac{s-2}{v-1}\right)\left(\frac{\kappa}{2s-3}\right)\frac{1}{4v}\left[\frac{2s-v-2 \ldots 2s-\kappa-2}{\kappa-v+1\,!}\,\kappa-v-1\right.$$
$$- \frac{2s-v-3 \ldots 2s-\kappa-2}{\kappa-v\,!}\,\kappa-v-2\,\frac{s-v-1}{1}$$
$$+ \frac{2s-v-4 \ldots 2s-\kappa-2}{\kappa-v-1\,!}\,\kappa-v-3\,\frac{s-v-1\,.\,s-v-2}{1\,.\,2}$$
$$-+\ldots(-1)^{\kappa-v}\,\frac{2s-\kappa\,.\,2s-\kappa-1 \ldots 2s-\kappa-2}{3\,!}$$
$$\left.\times\frac{s-v-1 \ldots s-\kappa+2}{\kappa-v-2\,!} \pm \text{etc.}\right]$$
$$= (-1)^v\left(\frac{s}{\kappa}\right)\left(\frac{\kappa}{2s-3}\right)\left(\frac{s-2}{v-1}\right)\frac{1}{4v}\left[2s-\kappa-2\left(\frac{s-\kappa+v \ldots s-1}{\kappa-v\,!}\right.\right.$$
$$\left.\mp\frac{s-v-1 \ldots s-\kappa}{\kappa-v\,!}\right) - 2\left(\frac{s-\kappa+v-1 \ldots s-1}{s-v+1\,!}\right.$$
$$\left.\left.\mp\frac{s-v-1 \ldots s-\kappa-1}{\kappa-v-1}\right)\right].$$

Use the upper or lower signs according as $\kappa - v$ is odd or even. To obtain this result expand

$$\left.\begin{aligned} x^2(1-x)^{s-v-1} &= x^2 - (s-v-1)x^3 \\ &+ \frac{s-v-1\,.\,s-v-2}{2}\,x^4 - \ldots - (-1)^{\kappa-v}\,\frac{s-v-1\,!}{\kappa-v-2\,!}\,x^{\kappa-v} \end{aligned}\right\}\text{(a)}$$

and

$$x^{-2}(1-x)^{-(2s-\kappa-2)} = \frac{1}{x^2} + (2s-\kappa-2)\frac{1}{x} + \frac{2s-\kappa-2\,.\,2s-\kappa-1}{2} + \binom{2s-\kappa}{3}x + \ldots + \frac{2s-v-2\,!}{\kappa-v+1\,!\,2s-\kappa-3\,!}\,x^{\kappa-v-1}.$$

Differentiating the last equation,

$$\left.\begin{aligned} -2x^{-3}(1-x)^{-(2s-\kappa-2)} & \\ -(2s-\kappa-2)\,x^{-2}(1-x)^{-(2s-\kappa-1)} & \\ = \frac{-1}{x^2} - \frac{2s-\kappa-2}{x^2} + 0 + \frac{2s-\kappa\,!}{3\,!\,2s-\kappa-3\,!} & \\ + \ldots \kappa-v-1\,\frac{2s-v-2\,!}{\kappa-v+1\,!\,2s-\kappa-3\,!}\,x^{\kappa-v-2} & \\ + \ldots & \end{aligned}\right\}. \quad \text{(b)}$$

The coefficient of $x^{\kappa-v}$ in the product of the right members of (a) and (b) is the series of terms in square parenthesis in the expression of A_1 above, and the coefficient of $x^{\kappa-v}$ in the product of the left members is the quantity within square parenthesis in the final value given for A_1.

B_1 is found by putting $(1-x)^{s-v-1}$ and $(1-x)^{2s-\kappa-2}$ equal to their expansions and taking the coefficients of $x^{\kappa-v+1}$ from the product of the left members and also from the product of the right members. Then

$$B_1 = (-1)^v\binom{s}{\kappa}\binom{s-2}{v-1}\binom{\kappa}{2s-3}\frac{s-\kappa}{2v}\left[\frac{s-1\ldots s-\kappa+v-1}{\kappa-v+1} \mp \binom{s-v-1}{\kappa-v+1} \pm \binom{s-v-1}{\kappa-v}\,2s-\kappa-2\right].$$

If in these expressions for A_1, B_1 and C_1, v is made equal to zero, then for all odd values of κ

$$A_1 = 0 = B_1 + C_1, \tag{11}$$

while for even values of κ

$$\left.\begin{aligned} A_1(n+1) + B_1 + C_1 = \binom{s}{\kappa}\binom{s-2}{\kappa-1}\binom{\kappa}{2s-3} & \\ \left[\frac{2s-\kappa-1}{2\,.\,\kappa+1}(n+1) + \frac{s-\kappa\,.\,s-\kappa-1}{\kappa\,.\,\kappa+1}\right] & \end{aligned}\right\} \tag{12}$$

For $v = 1$

and x increased by unity, $A_1(n+1)+B_1+C_1$ becomes the same as in (12) multiplied by $-\frac{s-x}{2}$. Then in RL, if W be the coefficient of P_{s-x} when x is even,

$$-W.\frac{s-x}{2} \text{ is the coefficient of } P'_{s-x-1}. \qquad (13)$$

When $v=x-2$, let $A_1(n+1)+B_1+C_1$ be denoted by a_{1x}. The following are the values of A_1, B_1 and C_1 when $v=x-2$:

$$A_1=(-1)^x\left(\frac{s}{x}\right)\left(\frac{s-2}{x-1}\right)\left(\frac{x}{2s-3}\right)\frac{x-1\,.\,2s-x-2\,.\,2s-x-1\,.\,2s-x}{s-x+1\,.\,s-x\,.\,4},$$

$$B_1=(-1)^x\left(\frac{s}{x}\right)\left(\frac{s-2}{x-1}\right)\left(\frac{x}{2s-3}\right)\frac{2s-x-1\,.\,2s-x-2\,.\,2s-2x+3\,.\,x-1}{2\,.\,6}$$

$$C_1=(-1)^{x+1}\left(\frac{s}{x}\right)\left(\frac{s-2}{x-1}\right)\left(\frac{x}{2s-3}\right)\frac{2s-x-2\,.\,3s-2x-2\,.\,x-1}{2\,.\,6}.$$

Now, when the whole remainder is considered, the coefficient of each of the $(P^{(e)}_{s-\lambda})$'s must be zero. Let us now consider those terms of dimension $s-2$. They will be found only in RL and Ra. The coefficient of P_{s-2} is $-a_{12}+\frac{n+1}{6}a_2$. This equals zero, and when $v=0$ and $x=2$

$$a_{12}=\frac{1}{6}\left(\frac{s}{2}\right)\left(\frac{s-2}{1}\right)\left(\frac{2}{2s-3}\right)\{(2s-3)(n+1)+s^2-5s+6\},$$

therefore

$$a_2=-\frac{1}{n+1}\left(\frac{s}{2}\right)\left(\frac{s-2}{1}\right)\left(\frac{2}{s-3}\right)\{(2s-3)(n+1)+(s-2).(s-3)\}.$$

The coefficient of P'_{s-3} is, by (13),

$$\frac{n+1}{6}a_3+\frac{n+1}{6}a_2\,\frac{s-2}{2}-\frac{s-2}{2}\,a_{12}=\frac{n+1}{6}a_3;$$

then

$$a_3=0.$$

The coefficient of P''_{s-4} is

$$\frac{n+1}{6}a_4+\frac{n+1}{6}a_2\left(\frac{s-2}{2}\right)\frac{1}{4\,.\,2s-3}-a_{14}.$$

Substituting for a_3 and a_{14} their values,

$$a_4 = \frac{-4}{n+1}\left(\frac{s}{4}\right)\left(\frac{s-2}{2}\right)\frac{2s-8\,!}{2s-3\,!}\{2.(n+1)(2s-5)+s-4.s-5\}.$$

Calling the three terms whose sum gave the coefficient of P''_{s-4} λ, μ, ν, then the coefficient of $P^{(3)}_{s-5}$ is

$$\frac{n+1}{6}a_5 + \frac{s-4}{2}\lambda + \frac{s-4.s-5}{3.2s-8}\mu - a_{15} \equiv \sigma_1 + \lambda_1 + \mu_1 + \nu_1,$$

say. The last three terms reduce to zero; therefore

$$a_5 = 0.$$

The coefficient of $P^{(4)}_{s-6} = \sigma_2 + \lambda_2 + \mu_2 + a_{16}$, say

$$= \frac{n+1}{6}a_6 + \frac{s-5.s-6}{2.2s-11}\lambda_1 - \frac{s-5.s-6}{4.2s-9}\mu_1 - a_{16} = 0.$$

Reducing this,

$$a_6 = \frac{-6}{n+1}\left(\frac{s\,!\,s-2\,!\,2s-12\,!}{s-6\,!\,s-6\,!\,2s-3\,!\,3\,!\,2}\right)\{3(n+1)(2s-7)$$
$$+s-6.s-7\}.$$

Similarly a_7 may be shown equal to zero and

$$a_8 = \frac{-6}{n+1}\,\frac{s\,!\,s-2\,!\,2s-16\,!}{2.3\,!\,s-8\,!\,s-8\,!\,2s-3\,!}\{4(n+1)(2s-9)$$
$$+s-8.s-9\}.$$

Had the terms in the coefficient of $P^{(5)}_{s-7}$ been denoted by $\frac{n+1}{6}a_7$, σ_3, λ_3, μ_3 and a_{17}, then those giving a_8 would be

$$\frac{n+1}{6}a_8 + \frac{s-7.s-8}{2.2s-15}\sigma_3 + \frac{s-7.s-8}{4.2s-13}\lambda_3 + \frac{s-7.s-8}{6.2s-11}\mu_3 - a_{18}.$$

It thus appears that λ_3, μ_3, σ_3 have a relation between them similar to λ_2, μ_2, σ_2 and λ_1, μ_1, σ_1, etc., and if we follow the same law the coefficient of $P^{\kappa-2}_{s-\kappa}$ becomes

$$\frac{n+1}{6}a_\kappa - a_{1\kappa} + (-1)^{\kappa+1}\left[\frac{s\,!\,s-2\,!\,2s-\kappa-2p-2\,!}{4\,!\,\kappa-2p\,!\,s-\kappa\,!\,s-\kappa+1\,!\,2s-3\,!}\right.$$
$$\left.2s-2\kappa.2s-2\kappa-2.\beta\theta_p\right],$$

where $\beta = 2s - 4p - 1$ and

$$\theta_p = \{p(n+1)(2s-2p-1) + (s-2p)(s-2p-1)\},$$
$$2p = 2, 4, 6 \ldots \kappa - 1 \text{ or } \kappa.$$

Also a_κ would equal zero when κ is odd, and when κ is even

$$a_\kappa = \frac{-6}{n+1} \frac{s!\, s-2!\, 2s-2\kappa!}{2.3!\, s-\kappa!\, s-\kappa!\, 2s-3!} \left(\frac{\kappa}{2}(n+1)(2s-\kappa-1) \right.$$
$$\left. + (s-\kappa)(s-\kappa-1)\right).$$

To prove that this law holds, consider the series

$$\Gamma \equiv (-1)^\kappa \left[(n+1) \frac{s!\, s-2!\, 2s-\kappa!}{4!\, s-\kappa!\, s-\kappa-1!\, \kappa-2!\, 2s-3!} \right.$$
$$- \frac{s!\, s-2!\, 2s-\kappa-1!\, (2s-2\kappa)(s-2\kappa+3)}{4!\, s-\kappa!\, s-\kappa-1!\, \kappa-2!\, 2s-3!}$$
$$+ \frac{s!\, s-2!\, 2s-\kappa-2!\, (3s-2\kappa-2)(2s-2\kappa)(s-\kappa+1)}{4!\, s-\kappa!\, s-\kappa+1!\, \kappa-2!\, 2s-3!}$$
$$- \Sigma \frac{s!\, s-2!\, 2s-\kappa-2p-2!}{4!\, \kappa-2p!\, s-\kappa!\, s-\kappa+1!\, 2s-3!}\, 2s-2\kappa$$
$$\times\, 2s-2\kappa+2 \,.\, 2s-4p-1 \,.\, \theta p,$$

$$2p = 2, 4, 6 \ldots \kappa - 1 \text{ or } \kappa.$$

The first three terms are what A_1, B_1 and C_1 become when v is made equal to $\kappa - 2$. As the series is to be shown to be equal to zero, the common factor $(-1)^\kappa \dfrac{s!\, s-2!\, 2s-2\kappa-2!}{4!\, s-\kappa!\, s-\kappa+1!\, 2s-3!}$ may be omitted. Then

$$\binom{2s-\kappa-g}{\kappa-g+2} = \frac{2s-\kappa-g \,.\, 2s-\kappa-g-1 \ldots 2s-2\kappa-1}{\kappa-g+2!}$$
$$\equiv \chi(g), \text{ say,}$$

and

$$\frac{2s-\kappa-g \,.\, 2s-\kappa-g-1}{\kappa-g+2 \,.\, \kappa-g+1}\, \chi(g+2) = \chi(g). \qquad (14)$$

The series to be considered now becomes

$$\chi(4) . 2s - x . 2s - x - 1 . 2s - x - 2 . 2s - x - 3 . (n + 1)$$
$$- \chi(4) . 2s - x - 1 . 2s - x - 2 . 2s - x - 3 . 2s - 2x . s - 2x + 3$$
$$+ \chi(4) . 2s - x - 2 . 2s - x - 3 . 2s - 2x . s - x + 1 . 3s - 2x - 2$$
$$- \chi(4) . 2s - 2x . 2s - 2x + 2 . 2s - 5 \left(n(2s - 3) + s^2 - 3s + 3\right)$$
$$- \chi(6) . 2s - 2x . 2s - 2x + 2 . 2s - 9 \left(2n . (2s - 5) + s^2 - 5s + 10\right)$$
$$- \chi(8) . 2s - 2x . 2s - 2x + 2 . 2s - 13 \left(3n . (2s - 7) + s^2 - 7s + 21\right)$$

. . . .

. . . .

. . . .

$$- \chi(2p + 2) . 2s - 2x . 2s - 2x + 2 . 2s - 4p - 1 \{pn(2s - 2p - 1)$$
$$+ s^2 - (2p + 1)s + p(2p + 1)\},$$

$$2p = x - 1 \quad \text{or} \quad x.$$

Consider the coefficient of n,

$$\chi(4)[2s - x . 2s - x - 1 . 2s - x - 2 . 2s - x - 3$$
$$- 2s - 2x . 2s - 2x + 2 . 2s - 5 . 2s - 3]$$
$$= \chi_4 [8s^2 - 4s(2x + 3) + x(x + 11)] x - 2 . x - 3$$
$$= x - 2 . x - 3 . \chi(4) \varDelta_1, \text{ say,}$$
$$= 2s - x - 4 . 2s - x - 5 \varDelta_1 \chi(6) \text{ by (14).}$$

Take from this

$$\chi(6) . 2s - 2x . 2s - 2x + 2 . 2s - 9 . 2s - 5 . 2;$$

and the second remainder is

$$\chi(6) x - 4 . x - 5 . [12s^2 - 6s(2x + 5) + x(x + 29)]$$
$$\equiv x - 4 . x - 5 . \chi(6) \varDelta_2, \text{ say.}$$

This equals

$$2s - x - 6 . 2s - x - 7 . \chi(8) \varDelta_2 \text{ by (14).}$$

Take from this the next term of the series,

$$\chi(8) . 2s - 2x . 2s - 2x + 2 . 2s - 13 . 2s - 7 . 3;$$

the remainder is

$$x - 6 . x - 7 \chi(8) [16s^2 - 8s(2x + 7) + x(x + 55)]$$
$$= 2s - x - 8 . 2s - x - 9 . \chi(10) \varDelta_3, \text{ say.}$$

Supposing this law to hold for all differences till the $(m - 1)$th, it can be shown to hold for the mth. The $(m - 1)$th is

$$x + 2 - 2m \,.\, x + 1 - 2m \,.\, \chi(2m)[4ms^2 - 2ms(2x + 2m - 1)$$
$$+ x(x + 4m^2 + 2m - 1)] = \chi(2m + 2)\, \Delta_{m-1}$$
$$2s - x - 2m \,.\, 2s - x - 2m - 1.$$

Taking from this

$$m\chi(2m + 2)\, 2s - 2x \,.\, 2s - 2x + 2 \,.\, 2s - 2m - 1 \,.\, 2s - 4m - 1,$$

there remains

$$x - 2m \,.\, x - 2m - 1 \,.\, \chi(2m + 2)[4(m + 1)s^2$$
$$- 2(m + 1)(2x + 2m + 1) + x(x + 4m^2 + 6m + 1)]$$
$$= x - 2m \,.\, x - 2m - 1\chi(2m + 2)\, \Delta_m;$$

that is, the mth difference is the same function of m as the $(m - 1)$th is of $m - 1$.

When $2m = 2p = x - 1$ or x the subtrahend is the last term of the series and the difference vanishes. Thus we see the coefficient of n in the series vanishes.

The algebraic sum of the first four terms independent of n is

$$x - 2 \,.\, x - 3\chi(4)[2s^3 - s^2(2x + 8) + 10s(x + 1)$$
$$- x(x + 11)] = x - 2 \,.\, x - 3\chi_4 \Delta_1, \text{ say,}$$

then by (14) it equals

$$\Delta_1 2s - x - 4 \,.\, 2s - x - 5 \,.\, \chi(6).$$

Taking from this

$$\chi(6)\, 2s - 2x \,.\, 2s - 2x + 2 \,.\, 2s - 9 \,.\, s^2 - 5s + 10$$

there remains

$$\chi(6)\, x - 4 \,.\, x - 5\, [2s^3 - (2x + 12)s^2 + (14x + 25)s - x(x + 29)]$$
$$= \chi(6)\, x - 4 \,.\, x - 5\, \Delta_2, \text{ say.}$$

If the $(m - 1)$th difference be

$$x - 2m + 2 \,.\, x - 2m + 1 \,.\, \chi(2m)[2s^3 - (2x + 4m)s^2 + \{2x(2m + 1)$$
$$+ 2(m - 1)(2m + 1)\}s - x(x + 4m^2 - 2m - 1)],$$

which we will denote by $\psi(m - 1)$; then the mth is

$$\psi(m - 1) - \chi(2m + 2)[2s - 2x \,.\, 2s - 2x + 2 \,.\, 2s - 4m - 1$$
$$\times \{mn(2s - 2m - 1) + s^2 - (2m + 1)s + m(2m + 1)\}]$$
$$= \chi(2m + 2) \,.\, x - 2m \,.\, x - 2m - 1\, [2s^3 - s^2(2x + 4(m + 1))$$
$$+ \{2x(2m + 3) + 2m(2m + 3)\}s - x(x + 4m^2 + 6m + 1)]$$
$$= \psi(m).$$

This vanishes when $2m = x$ or $x - 1$, and also completes the series.

Thus the whole series has been shown to vanish whatever be the value of x. (15)

Assuming that $a_x = 0$ when x is odd, and

$$= \frac{-6}{n+1} \frac{s!\,(s-2)!\,(2s-2x)!}{2.3!\,(s-x)!\,(s-x)!\,(2s-3)!} \left\{ \frac{x}{2} (n+1)(2s-x-1) + (s-x)(s-x-1) \right\}$$

for all even values of x less than $2w+1$, then it may be shown to be true when $x = 2w+1$ and $2w+2$. The coefficient of $P^{(2w-1)}_{s-2w-1}$ in RL is $a_{1.2w+1}$, and if M^s_x represent the value of a_x when x is even, and N^τ_s represent the expression

$$(-1)^\tau \frac{s!\,(s-2)!\,(2s-\tau-2)!}{2!\,\tau!\,(s-\tau)!\,(s-\tau-1)!\,(2s-3)!},$$

i. e. the coefficient of $P^{(\tau)}_{s-\tau}$ in L, then the whole coefficient of $P^{(2w-1)}_{s-2w-1}$ is

$$\frac{n+1}{6} a_{2w+1} + a_{1.2w+1} - [M^s_2 N^{2w-1}_{s-2} + M^s_4 N^{2w-3}_{s-4} + M^s_6 N^{2w-5}_{s-6} + \ldots + M^s_{2w} N^1_{s-2w}] \frac{n+1}{6}.$$

Now $a_{1.2w+1}$ is the sum of the first three terms of Γ, and the following terms are those of Γ also; for taking any one of them, as

$$M^s_{2z} N^{2w-2z+1}_{s-2z} \qquad z \leqq w,$$

it becomes, when written in full,

$$\frac{6}{n+1} \frac{n+1}{6} \frac{s!\,(s-2)!\,(2s-4z)!}{2.3!\,(s-2z)!\,(s-2z)!\,(2s-3)!} \{z(n+1)(2s-2z-1) + (s-2z)(s-2z-1)\}$$

$$\times \frac{(s-2z)!\,(s-2z-2)!\,(2s-2w-2z-3)}{2!\,(2w-2z+1)!\,(s-2w-1)!\,(s-2w-2)!\,(2s-4z-3)!}$$

$$= \frac{s!\,(s-2)!\,(2s-2w-2z-3)!\,.\,\theta,}{4!\,(2w-2z+1)!\,s\,.\,(2w-1)!\,(s-2w)!\,(2s-3)!}\,(2s-4w) \times (2s-4w+2)\,.\,(2s-4z-1),$$

which coincides with the last terms of Γ when $x = 2w + 1$ and $s = p$. Thus the coefficient of $P^{(2w-1)}_{s-2w-1}$ consists of $\frac{n+1}{6}\,a_{2w+1}$ plus a series of terms which vanish by (15); then

$$a_{2w+1} = 0. \tag{16}$$

The coefficient of $P^{(2w)}_{s-2w-2}$ is

$$\frac{n+1}{6}\,a_{2w+2} + a_{1.2w+2} + \frac{n+1}{6}\left[M^s_2 N^{2w}_{s-2} + M^s_4 N^{2w-2}_{s-4} + \ldots + M^s_{2w} N^2_{s-2w}\right] = 0.$$

Γ gives all the terms in this expression when $x = 2w + 2$, excepting the first or $\frac{n+1}{6}\,a_{2w+2}$. But the last term $M^s_{2w} N^2_{s-2w}$ is the second last in Γ when $x = 2w + 2$, $2p = 2, 4 \ldots 2w + 2$. Taking Γ from the above coefficient, $\frac{n+1}{6}\,a_{2w+2} - \frac{n+1}{6}\,M^s_{2w+2}$ is the coefficient, since $\Gamma = 0$ always. And as this must vanish,

$$a_{2w+2} = M^s_{2w+2}. \tag{17}$$

Thus (16) shows that if for any odd value of x and all lower odd values $a_x = 0$, then $a_{x+2} = 0$, and (17) shows that if for any even value and all lower even values $a_x = M^s_x$, then

$$a_{x+2} = M^s_{x+2}.$$

On pages 14 and 15 it is shown that $a_x = 0$ for $x = 3, 5, 7$ and $= M^s_x$ for $x = 2, 4, 6, 8$. Therefore it follows that (16) and (17) are true for all values of w.

It follows, then, that in θ, the row of terms designated a, of which P_s is a factor, contains no invariant or derivative of the form

$$\theta^{(2w-1)}_{s-2w-1}. \tag{18}$$

This is also the case for the terms entering in the row designated β and of which P'_4 is a factor, for the term $P_4\theta'_{s-4}$ is found only in Ra and $R\beta$. Its coefficient is

$$2b_4 + (2s - 7)\,a_4;$$

then

$$b_4 = -\frac{2s-7}{2}\,a_4.$$

Any term as $P_2\theta^{(\kappa-2)}_{s-\kappa}$, κ being odd, could appear only in $R\alpha$ and $R\beta$, and as it does not appear in $R\alpha$ it cannot in $R\beta$.

The coefficient of $P_2\theta^{(2u-2)}_{s-2u}$ is

$$2b_{2u} + \left\{ \binom{2u-2}{2u-3}(s-2u) + \binom{2u-2}{2u-4} \right\} a_{2u} = 0,$$

or

$$2b_{2u} + (u-1)(2s-2u-3)\,a_{2u} = 0. \qquad (19)$$

The terms of dimension $s-1$ and of form $P_2'\theta^{\kappa-4}_{s-\kappa}$ can appear only in $R\beta$ and $R\gamma$, and when κ is odd no such term appears in $R\beta$; therefore it does not enter into $R\gamma$.

When κ is even, the coefficient of $P_2'\theta^{(2u-4)}_{s-2u}$ is

$$5c_{2u} + \left\{ \binom{2u-3}{2u-4}(s-2u) + \binom{2u-3}{2u-5} \right\} b_{2u} = 0,$$

or

$$5c_{2u} + (2u-3)(s-u-2)\,b_{2u} = 0. \qquad (20)$$

In this way it is easy to see, by taking one row after another, that the non-linear part of θ_s contains no term having $\theta^{(0)}_{s-\kappa}$ as a factor when κ is odd. (21)

From this it follows that if all the invariants of a differential equation with even suffix vanish, the linear part of each vanishes. The same is true for those with odd suffix. (22)

Section II.

The Coefficients of θ_s.

θ_s has, as we have seen, a linear part expressed by

$$\sum_{\tau=0}^{\tau=s-2} N_s^\tau P_{s-\tau}^{(\tau)},$$

or

$$\sum_{\tau=0}^{\tau=s-2} (-1)^\tau \frac{s!\,s-2!\,2s-\tau-2!}{2.\tau!\,s-\tau!\,s-\tau-1!\,2s-3!} P_{s-\tau}^{(\tau)}. \quad (23)$$

Then follow a series of terms

$$P_2\{a_2\theta_{s-2} + a_4\theta''_{s-4} + a_6\theta^{\mathrm{IV}}_{s-6} + a_8\theta^{(6)}_{s-8} + \ldots\}$$

expressed generally by

$$P_2 \sum_{\kappa=1}^{\kappa=\left[\frac{s-2}{2}\right]} M^s_{2\kappa}\theta^{(2\kappa-2)}_{s-2\kappa},$$

or

$$\left.\begin{array}{l} -\dfrac{6}{n+1} P_2 \displaystyle\sum_{\kappa=1}^{\kappa=\left[\frac{s-2}{2}\right]} \dfrac{s!\,s-2!\,2s-4\kappa!}{2.3!\,s-2\kappa!\,s-2\kappa!\,2s-3!} \\ \qquad \{\kappa(n+1)(2s-2\kappa-1)+(s-2\kappa)(s-2\kappa-1)\}\,\theta^{(2\kappa-2)}_{s-2\kappa} \end{array}\right\}, \quad (24)$$

$\left[\frac{s-\lambda}{2}\right]$ meaning the greatest integer in $\frac{s-\lambda}{2}$. Then follow

$$\begin{array}{l} P'_2\{b_4\theta'_{s-4} + b_6\theta'''_{s-6} + b_8\theta^{\mathrm{V}}_{s-8} + \ldots\} \\ + P''_2\{c_4\theta_{s-4} + c_6\theta''_{s-6} + c_8\theta^{\mathrm{IV}}_{s-8} + \ldots\} \\ + P'''_2\{e_6\theta'_{s-6} + e_8\theta'''_{s-8} + e_{10}\theta^{\mathrm{V}}_{s-10} + \ldots\} \\ + P^{\mathrm{IV}}_2\{g_6\theta_{s-6} + g_8\theta''_{s-8} + e_{10}\theta^{\mathrm{IV}}_{s-10} + \ldots\} \\ + P^{(5)}_2\{l_8\theta'_{s-8} + l_{10}\theta'''_{s-10} + \ldots\} \\ + \ldots\ldots\ldots\ldots\ldots \end{array}$$

These are expressed generally by

$$P_2^{(\nu)} \sum_{\kappa=\frac{\nu+2}{2}}^{\kappa=\left[\frac{s-2}{2}\right]} n_{2\kappa}\theta^{(2\kappa-\nu-2)}_{s-2\kappa} + P_2^{(\nu+1)} \sum_{\kappa=\frac{\nu+4}{2}}^{\kappa=\left[\frac{s-2}{2}\right]} q_{2\kappa}\theta^{(2\kappa-\nu-2)}_{s-2\kappa}$$

$$\nu = 2, 4, 6 \ldots \text{etc.}$$

If any two consecutive rows be considered, for which $(\nu = \mu)$, the remainder arising from them will contain a term

$$\lambda \,.\, P_s^{(\nu)} \,.\, \theta_{s-2\kappa}^{(2\kappa-\mu-3)}$$

found nowhere else, because all rows preceding these have $P_s^{(\nu)}$ as a factor where $\nu < \mu$, and rows following them have a remainder in which the index of $\theta_{s-2\kappa}$ cannot be as great as $(2\kappa - \mu - 3)$. This remainder is

$$\left[\frac{\mu\,!}{\mu - 1\,!\,2\,!}(3+\mu)\,P_s^{(\mu-1)} + AP_s^{(\mu-2)} + BP_s^{(\mu-3)} + \ldots + \right] \sum_{\kappa=\frac{\mu+3}{2}}^{\kappa=\left[\frac{s-2}{2}\right]} n_{2\kappa}\theta_{s-2\kappa}^{(2\kappa-\mu-3)}$$

$$+ P_s^{(\mu)}\left[\sum_{\kappa=\frac{\mu+3}{2}}^{\kappa=\left[\frac{s-2}{2}\right]} n_{2\kappa} \sum_{m=0}^{m=r-1} \left\{\binom{r}{m}(s-2\kappa) + \binom{r}{m-1}\right\} \theta_{s-2\kappa}^{(m)} \,.\quad r = 2\kappa - \mu - 2\right]$$

$$+ \left[\frac{\mu+1\,!}{\mu\,!\,2\,!}(4+\mu)\,P_s^{(\mu)} + \ldots + \text{terms of lower dimension}\right] \sum_{\kappa=\frac{\mu+4}{2}}^{\kappa=\left[\frac{s-2}{2}\right]} q_{2\kappa}\theta_{s-2\kappa}^{(2\kappa-\mu-3)}$$

$$+ P_s^{(\mu+1)}\left[\sum_{\kappa=\frac{\mu+4}{2}}^{\kappa=\left[\frac{s-2}{2}\right]} q_{2\kappa} \sum_{m=0}^{m=r-1} \left\{\binom{r}{m}(s-2\kappa) + \binom{r}{m-1}\right\} \theta_{s-2\kappa}^{(m)} \,.\quad r = 2\kappa - \mu - 3\right].$$

Equating the coefficient of the term $P_s^{(\mu)}\theta_{s-2\kappa}^{(2\kappa-\mu-3)}$ to zero we obtain

$$\frac{\mu+1}{\mu\,!\,2\,!}(4+\mu)\,q_{2\kappa} = -\left\{\left\{\binom{2\kappa-\mu-2}{2\kappa-\mu-3}(s-2\kappa) + \binom{2\kappa-\mu-2}{2\kappa-\mu-4}\right\} n_{2\kappa}\right\}. \quad (25)$$

In this x is any number and μ any of the values of ν, so that the coefficients $q_{2\kappa}$ of any row may be expressed in terms of those of the preceding row, viz. $n_{2\kappa}$.

(25) when simplified gives

$$\frac{\mu+1}{2}(4+\mu)\,q_{2\kappa} = -\frac{(2x-\mu-2)(2s-2x-\mu-3)}{2}\,n_{2\kappa}.$$

Making $\mu = 0, 1, 2, 3 \ldots$ this gives

$$\begin{aligned}
4.1.b_{2\kappa} &= -(2x-2)(2s-2x-3)\,a_{2\kappa}\\
5.2.c_{2\kappa} &= -(2x-3)(2s-2x-4)\,b_{2\kappa}\\
6.3.e_{2\kappa} &= -(2x-4)(2s-2x-5)\,c_{2\kappa}\\
&\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\\
(\mu+1)(4+\mu)\,q_{2\kappa} &= -(2x-\mu-2)(2s-2x-\mu-3)\,n_{2\kappa}.
\end{aligned}$$

Equating the product of the right members to the product of the left gives

$$q_{2\kappa}.\frac{\mu+1\,!\,\mu+4\,!}{3\,!} = (-1)^{\mu+1}\frac{2x-2\,!\,2s-2x-3\,!}{2x-\mu-3\,!\,2s-2x-\mu-4\,!}\,a_{2\kappa}. \quad (26)$$

The q's being coefficients in the row multiplied by $P_2^{(\mu+1)}$ it is seen that the coefficient of any term of the form $P_2^{(\delta)}\theta_{s-2\kappa}^{(2\kappa-\delta-2)}$ may be expressed in terms of the a's. Writing this coefficient, for brevity, $(\delta)_{2\kappa}^{(2\kappa-\delta-2)}$, then

$$\left.\begin{aligned}
&(\delta)_{2\kappa}^{(2\kappa-\delta-2)} = (-1)^{\delta+1}\\
&\frac{2x-2\,!\,2s-2x-3\,!\,s\,!\,s-2\,!\,2s-4x\,!\,.\,\theta_{(\kappa)}}{\delta\,!\,\delta+3\,!\,2x-\delta-2\,!\,2s-2x-\delta-3\,!\,2s-3\,!\,s-2x\,!\,s-2x\,!\,2}\\
&\hphantom{\frac{2x-2\,!\,2s-2x-3\,!\,s\,!\,s-2\,!\,2s-4x\,!\,.\,\theta_{(\kappa)}}{\delta\,!\,\delta+3\,!\,2x-\delta-2\,!\,2s-2x-\delta-3\,!\,2s-3\,!\,s-2x}}\frac{6}{n+1}
\end{aligned}\right\}. \quad (27)$$

There still remain terms of the form

$$P_2^{(\alpha)^a}P_2^{(\beta)^b}P_2^{(\gamma)^c}P_2^{(\delta)^d}P_2^{(\epsilon)^e}\,(\alpha^a\beta^b\gamma^c\delta^d\epsilon^e)_{2\kappa}^{(m)}\theta_{s-2\kappa}^{(m)}.$$

Here a, b, c, d, etc., are indices expressing powers of the factors to which they are attached. $(\alpha^a\beta^b\gamma^c\delta^d\epsilon^e)_{2\kappa}^{(m)}$ is the coefficient of the term having such indices, powers and suffix $s - 2x$.

Throughout the whole invariant the order of the factors will be taken so that

$$\alpha \overline{\overline{<}} \beta \overline{\overline{<}} \gamma \overline{\overline{<}} \delta \overline{\overline{<}} \epsilon, \text{ etc.} \tag{28}$$

$$\left.\begin{aligned} 2\kappa = m + a(\alpha + 2) + b(\beta + 2) + c(\gamma + 2) \\ + d(\delta + 2) + e(\epsilon + 2) + \ldots \end{aligned}\right\}. \tag{29}$$

The numerical value of $(\alpha^a\beta^b\gamma^c\delta^d\epsilon^e)^{(m)}_{2\kappa}$ is found by equating the coefficient of $P_2^{(\alpha)a}P_2^{(\beta)b}P_2^{(\gamma)c}P_2^{(\delta)d}P_2^{(\epsilon)e-1}\theta^{(m)}_{s-2\kappa}$ in the remainder to zero.

It is

$$\left.\begin{aligned} &\frac{e}{6}(n+1)(\alpha^a\beta^b\gamma^c\delta^d\epsilon^e)^{(m)}_{2\kappa} \\ &\quad +(\alpha^{a-1}\beta^b\gamma^c\delta^d\epsilon^{e-1}\epsilon+\alpha+2)^{(m)}_{2\kappa}\frac{\epsilon+\alpha+2!}{\alpha!\,\epsilon+3!}(2\epsilon+6+\alpha) \\ &\quad +(\alpha^a\beta^{b-1}\gamma^c\delta^d\epsilon^{e-1}\epsilon+\beta+2)^{(m)}_{2\kappa}\frac{\epsilon+\beta+2!}{\beta!\,\epsilon+3!}(2\epsilon+6+\beta) \\ &\quad +(\alpha^a\beta^b\gamma^{c-1}\delta^d\epsilon^{e-1}\epsilon+\gamma+2)^{(m)}_{2\kappa}\frac{\epsilon+\gamma+2!}{\gamma!\,\epsilon+3!}(2\epsilon+6+\gamma) \\ &\quad +(\alpha^a\beta^b\gamma^c\delta^{d-1}\epsilon^{e-1}\epsilon+\delta+2)^{(m)}_{2\kappa}\frac{\epsilon+\delta+2!}{\gamma!\,\epsilon+3!}(2\epsilon+6+\delta) \\ &\quad +(\alpha^a\beta^b\gamma^c\delta^d\epsilon^{e-2}2\epsilon+2)^{(m)}_{2\kappa}\frac{2\epsilon+2!}{\epsilon!\,\epsilon+3!}(3\epsilon+6) \\ &\quad +(\alpha^a\beta^b\gamma^c\delta^d\epsilon^{e-1})^{(\epsilon+2+m)}_{2\kappa}\frac{m+\epsilon+2!}{m!\,\epsilon+3!} \\ &\qquad\qquad \{(\epsilon+3)(s-2\kappa)+m\} \end{aligned}\right\} = 0. \tag{30}$$

$$\frac{n+1}{6}\Sigma(\epsilon)^{(\pi)}_{\pi+\epsilon+2}\Sigma(\alpha^a\beta^b\gamma^c\delta^d\epsilon^{e-1})^{(r)}\frac{\pi!}{r!\,\pi-r!}$$
$$(\alpha^{a_1}\beta^{b_1}\gamma^{c_1}\delta^{d_1}\epsilon^{e_1})^{(v)}_{2\kappa-\pi-\epsilon-2}.$$

$$\left.\begin{aligned} \pi - \epsilon + 2\left[\frac{\epsilon}{2}\right] = 0.2.4.6\ldots 2\kappa \\ -2(a+b+c+d+e) - 2\epsilon + 2\left[\frac{\epsilon}{2}\right] \end{aligned}\right\}. \tag{31}$$

$(\epsilon)^{(\pi)}_{\pi+\epsilon+2}$ is the numerical coefficient of $P_2^{(\epsilon)}\theta^{(\pi)}_{s-\pi-\epsilon-2}$. $a_1b_1c_1d_1e_1$ take all values consistent with $e_1 < e$, and

$$a_1 + b_1 + c_1 + d_1 + e_1 = \text{the constant } (a+b+c+d+e-1).$$

$(a^a\beta^b\gamma^c\delta^d\epsilon^{e-1})^{(r)}$ stands for the numerical coefficient of

$$(P_2^{(a)a}P_2^{(\beta)b}P_2^{(\gamma)c}P_2^{(\delta)d}P_2^{(\epsilon)e-1})$$

in
$$\frac{d^r}{dx^r}(P_2^{(a)a_1}P_2^{(\beta)b_1}P_2^{(\gamma)c_1}P_2^{(\delta)d_1}P_2^{(\epsilon)e_1}).$$

$$r = y + \pi - m.$$

$$y = 2\kappa - 2(a_1 + b_1 + c_1 + d_1 + e_1 + 1)$$
$$- \pi - a_1 a - b_1\beta - c_1\gamma - d_1\delta - (e_1 + 1)\epsilon.$$

In the coefficient $(a^{a_1}\beta^{b_1}\gamma^{c_1}\delta^{d_1}\epsilon^{e_1})^{(y)}_{2\kappa-\pi-s-r}$, s is to be changed to $s - \epsilon - \pi - 2$. When $s - 2\kappa = 2$ the terms that must be added are easily recognized.

For an example, let us find the coefficient of $P_2^6\theta^{(2\kappa-12)}_{s-2\kappa}$. In this

$$a = b = c = d = \epsilon = 0, \qquad e = 6,$$
$$\pi = 0.2, 4, \ldots 2\kappa - 12, \qquad r = 0, \qquad y = 2\kappa - 12 - \pi.$$

Then

$$\frac{6}{6}(n+1)(0^6)^{(2\kappa-12)}_{2\kappa} + 0 + 0 + 0 + 0 + (0^4 2)^{(2\kappa-10)}_{2\kappa}\frac{6}{3}$$
$$+ (0^5)^{(14)}_{2\kappa}\frac{2\kappa - 10 . 2\kappa - 11}{6}(3s - 4\kappa - 12)$$
$$+ \frac{n+1}{6}[(0)_2(0^5)^{(2\kappa-12)}_{2\kappa-2} + (0)^{(2)}_4(0^5)^{(2\kappa-14)}_{2\kappa-4} + (0)^{(4)}_6(0^5)^{(2\kappa-16)}_{2\kappa-6}$$
$$+ \ldots + (0)^{(2\kappa-12)}_{2\kappa-10}(0^5)_{10}] = 0.$$

This states that

$n + 1$ times the coefficient of $P_2^6\theta^{(2\kappa-12)}_{s-2\kappa}$
+ twice the coefficient of $P_2^4 P_2^{\prime\prime}\theta^{(2\kappa-12)}_{s-2\kappa}$
$+ \frac{2\kappa - 10 . 2\kappa - 11}{6}(3s - 4\kappa - 12)$ times the coefficient of $P_2^5\theta^{(12)}_{s-2\kappa}$
$+ \frac{n+1}{6}$ times a number of terms $= 0$.

Any one of these last terms, as $(0)^{(4)}_6(0^5)^{(2\kappa-16)}_{2\kappa-6}$, is written in full thus: The coefficient of $P_2\theta^{(4)}_{s-6}$ times the coefficient of $P_2^5\theta^{(2\kappa-16)}_{s-2\kappa+6}$ in the invariant $\theta^{(4)}_{s-6}$.

As another example, find the coefficient of $P_2^2 P_2^{(1)2} P_2^{(3)2}\theta^{m}_{s-2\kappa}$. Here

$$2\kappa = m + 23, \qquad a = 2, b = 3, c = 2,$$
$$a = 0, \beta = 1, \epsilon = 3, \qquad \pi = 1.3.5 \ldots 2\kappa - 17.$$

Then

$$
\left.
\begin{aligned}
&\frac{n+1}{3}(0^2 1^4 3^2)^{(m)}_{2\kappa} + (0^2 1^2 3\,.\,5)^{(m)}_{2\kappa}(2\,.\,6+0) \\
&\quad + \frac{6!}{1\,.\,6!}(3\,.\,6+1)(0^2 1^2 3\,.\,6)^{(m)}_{2\kappa} \\
&\quad + \frac{8!}{3!\,6!}(2\,.\,6+3)(0^2 1^2 8)^{(m)}_{2\kappa} \\
&\quad + \frac{m-5!}{m!\,6!}(6s-10\kappa-23)(0^2 1^2 3)^{s+m+2}_{2\kappa} \\
&\quad + \frac{n+1}{6}\Big[(3)^{(1)}_{6}\Big\{(0^2 1^2 3)^{(m-1)}_{2\kappa-6} + \left(\frac{1}{r}\right)C_r(0^2 1^2 3)^{(m)}_{2\kappa-6} \\
&\quad + (0^2 1^2 2)^{(m)}_{2\kappa-6}\Big\} + (3)^{2}_{6}\Big\{(0^2 1^2 3)^{(m-2)}_{2\kappa-6} \\
&\quad + \left(\frac{3}{r}\right)C_r(0^2 1^2 3)^{(m-2)}_{2\kappa-6} + \left(\frac{3}{r}\right)C_r(0^2 1^2 2)^{(m-2)}_{2\kappa-6} \\
&\quad + \left(\frac{3}{r}\right)C_r(0^4 1 3)^{(m-1)}_{2\kappa-6} + \left(\frac{3}{4}\right)C_r(0^2 1^2 2)^{(m-1)}_{2\kappa-6} \\
&\quad + \left(\frac{3}{4}\right)C_r(0^2 1^4)^{(m-1)}_{2\kappa-6} + \left(\frac{3}{r}\right)C_r(0^5 3)^{(m)}_{2\kappa-6} \\
&\quad + \left(\frac{3}{r}\right)C_r(0^4 1 2)^{(m)}_{2\kappa-6} + \left(\frac{3}{r}\right)C_r(0^2 1^5)^{(m)}_{2\kappa-6}\Big\} \\
&\quad + (3)^{(6)}_{10}\Big\{(0^2 1^2 3)^{(m-5)}_{2\kappa-10} + \left(\frac{5}{r}\right)C_r(0^2 1^2 3)^{(m-4)}_{2\kappa-10} \\
&\quad + \left(\frac{5}{r}\right)C_r(0^2 1^2 2)^{(m-4)}_{2\kappa-10} + \left(\frac{5}{r}\right)C_r(0^4 1 3)^{(m-3)}_{2\kappa-10} \\
&\quad + \left(\frac{5}{r}\right)C_r(0^2 1^3 2)^{(m-3)}_{2\kappa-10} + \left(\frac{5}{r}\right)C_r(0^2 1^4)^{(m-3)}_{2\kappa-10} \\
&\quad + \left(\frac{5}{r}\right)C_r(0^5 3)^{(m-2)}_{2\kappa-10} + \left(\frac{5}{r}\right)C_r(0^4 1 2)^{(m-2)}_{2\kappa-10} \\
&\quad + \left(\frac{5}{r}\right)C_r(0^2 1^5)^{(m-2)}_{2\kappa-10} + \left(\frac{5}{r}\right)C_r(0^5 2)^{(m-1)}_{2\kappa-10} \\
&\quad + \left(\frac{5}{r}\right)C_r(0^4 1^2)^{(m-1)}_{2\kappa-10} + \left(\frac{5}{r}\right)C_r(0^6 1)^{m}_{2\kappa-10}\Big\} \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad (3)^{(2\kappa-18)}_{2\kappa-14}\{(0^6 1)^{(2)} + (0^6)^{(1)}\} + (3)^{(2\kappa-17)}_{2\kappa-18}\{(0^6)\}\Big]
\end{aligned}
\right\} = 0. \quad (33)
$$

In this r varies, being $= y + \pi - m$ always, and C_r also varies. The term $(3)^{9}_{14}\binom{9}{r}(0^{5}2)^{(m-5)}_{3\kappa-14}$ means the coefficient of $P_3^{(3)}\theta^{(9)}_{s-14}$ times the coefficient of $P_3^{5}P_3^{\prime\prime}\theta^{(m-5)}_{s-3\kappa+14}$ in the invariant $\theta^{(9)}_{s-14}$ multiplied by $\binom{9}{r}C_r$. $r = m - 5 + 9 - m = 4$, and C_r is the numerical coefficient of $P_3^{3}P_3^{(1)^{8}}P_3^{(8)}$ in

$$\frac{d^4}{dx^4}(P_3^{5}P_3^{(3)}) \text{ and } \binom{9}{r} = \frac{9!}{4!\,5!}.$$

Thus every term in the invariant θ_s has been considered, and by (23), (24) and (27) every coefficient has been expressed by simple formulae in terms of s and n excepting those represented by (30), and they are expressed in terms of preceding coefficients.

Section III.

Associate Equations and Associate Variables.

In the memoir previously referred to, Mr. Forsyth shows that in connection with any differential equation A_1 of order n there are $n-2$ other equations, $A_2, A_3, A_4, \dots A_{n-1}$, whose variables are formed as follows: Let $u_1, u_2, u_3, \dots u_n$ be solutions of A_1, then if we take any two u_λ, u_μ, the determinant

$$\begin{vmatrix} u_\lambda & u_\mu \\ u'_\lambda & u'_\mu \end{vmatrix}$$

is a solution of A_2. Generally if we take any κ of the u's and form a determinant

$$\begin{vmatrix} u_\alpha & u_\beta & u_\gamma & \dots u_\nu \\ u'_\alpha & u'_\beta & u'_\gamma & \dots u'_\nu \\ u''_\alpha & u''_\beta & u''_\gamma & \dots u''_\nu \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ u_\alpha^{(\kappa-1)} & u_\beta^{(\kappa-1)} & u_\gamma^{(\kappa-1)} & \dots u_\nu^{(\kappa-1)} \end{vmatrix} \equiv a_\kappa,$$

where $\alpha, \beta, \gamma \dots \nu$ are any κ of the numbers $1, 2, 3 \dots n$, then a_κ will be a solution of A_κ. As there are $\left(\frac{n}{\kappa}\right)$ combinations of n things κ at a time, there will be $\left(\frac{n}{\kappa}\right)$ variables a_κ satisfying an equation A_κ of order $\left(\frac{n}{\kappa}\right)$. A_κ will be called the $(\kappa-1)$th associate equation, and the variables a_κ the $(\kappa-1)$th associate variables. These variables a_κ are particular and linearly independent solutions of A_κ. A_{n-1} is the Lagrangian adjoint equation. a_κ may be written $(\alpha\beta'\gamma'' \dots \nu^{(\kappa-1)})$, or, as we are not concerned with which suffixes are taken, $\overline{0123\dots(\kappa-1)}$, then

$$a_2 \equiv (\alpha\beta') \equiv \overline{01}, \quad a_4 \equiv (\alpha\beta'\gamma''\delta''') \equiv \overline{0123}.$$

The number of these is $\left(\frac{n}{4}\right)$.

$$a_{n-1} \equiv (12'3''4'''5^{\mathrm{IV}} \ldots n-1^{(n-2)}) \text{ or } \overline{01234 \ldots (n-2)},$$

while $\qquad (12'3''4''' \ldots n^{n-1})$ or $\overline{1234 \ldots (n-1)}$

is the non-vanishing constant Δ. To illustrate what follows I shall first take a particular case, $n = 5$. Then A_1 will be

$$u^{(5)} + 10\varphi_2 u'' + 5\varphi_4 u' + \varphi_5 u = 0. \tag{34a}$$

u_1, u_2, u_3, u_4, u_5 are the five independent solutions; then $a_2 = \overline{01}$. 0 and 1 being the differential indices of the diagonal of the determinant formed with any two of the u's and their first derivatives, then

$$\frac{da_2}{dx} = a_2' = \overline{02},$$
$$a_2'' = \overline{03} + \overline{12},$$
$$a_2''' = \overline{04} + 2 . \overline{13},$$
$$a_2^{\mathrm{IV}} = 3 . \overline{14} + 2 .. \overline{23} + \overline{05}.$$

Substituting for u^{V} in $\overline{05}$ its value from (35),

$$a_2^{(4)} = 3 . \overline{14} + 2 . \overline{23} - 10\varphi_2\overline{02} - 5\varphi_4\overline{01},$$

or

$$a_2^{\mathrm{IV}} + 10\varphi_2\overline{02} + 5\varphi_4\overline{01} = 3 . \overline{14} + 2 . \overline{23} = s_4, \text{ say.}$$

Differentiating,

$$5 . \overline{24} + 3 . \overline{15} = s_4' = 5 . \overline{24} - .3 . 10\varphi_2\overline{12} + 3\varphi_5\overline{01},$$

$$s_4' - 3\varphi_5 a_2 \equiv s_5 = 5 . \overline{24} - 30\varphi_2\overline{12},$$

$$s_5' = 5 . \overline{34} - 30(\varphi_2'\overline{12} + \varphi_2\overline{13}) + 5 . \overline{25}$$
$$= 5 . \overline{34} - 30(\varphi_2'\overline{12} + \varphi_2\overline{13}) + 5\{5\varphi_4\overline{12} + \varphi_5\overline{02}\},$$

$$s_5' - 5\varphi_5 a_2' = 5 . \overline{34} + (25\varphi_4 - 30\varphi_2')\overline{12} - 30\varphi_2\overline{13} = s_6, \text{ say.}$$

$$s_6' = (25\varphi_4' - 30\varphi_2'')\overline{12} + (25\varphi_4 - 30\varphi_2')\overline{13}$$
$$- 30\varphi_2(\overline{14} + \overline{23}) + 5 . \overline{35}$$
$$= (25\varphi_4' - 30\varphi_2'')\overline{12} + (50\varphi_4 - 60\varphi_2')\overline{13}$$
$$- 30\varphi_2\left(\overline{14} + \frac{s_4 - 3 . \overline{14}}{2}\right) + 5\varphi_5(a_2'' - \overline{12})$$
$$+ 25\varphi_2(s_4 - 3 . \overline{14}),$$

$$s_6' - 10\varphi_2 s_4 - 5\varphi_5 a_2'' = -(5\varphi_5 - 25\varphi_4' + 30\varphi_2'')\overline{12}$$
$$+ (50\varphi_4 - 60\varphi_2')\overline{13} - 60\varphi_2\overline{14}.$$

Let

$X \equiv -(5\varphi_3 - 25\varphi_3' + 30\varphi_3''),\quad Y \equiv (50\varphi_4 - 60\varphi_3'),\quad Z \equiv -60\varphi_3.$

and

$$s_6' - 10\varphi_3 s_4 - 5\varphi_3 a_2'' = s_7.$$

Then

$$s_7 = X\overline{12} + Y\overline{13} + Z\overline{14}, \tag{35}$$

$$\begin{aligned} s_7' &= X'\overline{12} + (X + Y')\,\overline{13} + (Y + Z')\,\overline{14} + Y\overline{23} + Z(\overline{24} + \overline{15}) \\ &\doteq \left(X' + \frac{Z^2}{15}\right)\overline{12} + \left(Z' - \frac{Y}{2}\right)\overline{14} + (X + Y')\,\overline{13} \\ &\qquad + \frac{Y}{2}\, s_4 + \frac{Z}{5}\,(s_6' + 2\varphi_3 a_2), \end{aligned}$$

$$\left.\begin{aligned} &s_7' - \frac{Y}{2}\, s_4 - \frac{Z}{5}\,(s_4' + 2\varphi_3 a_2) \\ &\quad = s_8 = \left(X' + \frac{Z^2}{15}\right)\overline{12} + (X + Y')\,\overline{13} + \left(Z - \frac{Y}{2}\right)\overline{14} \end{aligned}\right\}, \tag{36}$$

$$\begin{aligned} s_8' &= \left(X'' + 2\,\frac{ZZ'}{15}\right)\dot{\overline{12}} + \left(2X' + Y'' + \frac{Z^2}{15}\right)\overline{13} \\ &\qquad + \left(X + \frac{Y'}{2} + Z''\right)\overline{14} + (X + Y')\,\overline{23} \\ &\qquad + \left(Z' - \frac{Y}{2}\right)(\overline{24} + \overline{15}) \\ &= \left(X'' + 3\,\frac{ZZ'}{15} - \frac{YZ}{30}\right)\overline{12} + \left(Y'' + 2X' + \frac{Z^2}{15}\right)\overline{13} \\ &\qquad + \left(Z'' - \frac{2Y' + X}{2}\right)\overline{14} + \frac{X + Y'}{2}\, s_4 \\ &\qquad + \frac{\left(Z' - \dfrac{Y}{2}\right)}{5}\,(s_4' + 2\varphi_3 a_2). \end{aligned}$$

Let

$$s_8' - \frac{X + Y'}{2}\, s - \frac{1}{5}\left(Z' - \frac{Y}{2}\right)(s_4' + 2\varphi_3 a_2) = s_9,$$

$$\left.\begin{aligned} s_9 &= \left(X'' + 3\,\frac{ZZ'}{15} - \frac{YZ}{30}\right)\overline{12} \\ &\qquad + \left(Y'' + 2X' + \frac{Z^2}{15}\right)\overline{13} + \left(Z'' - \frac{2Y' + X}{2}\right)\overline{14} \end{aligned}\right\}, \tag{37}$$

$$s_9' = \left(X''' + \frac{3Z^{(1)2} + 4ZZ''}{15} - \frac{3Y'Z + YZ' + ZX}{30}\right)\overline{12}$$
$$+ \left(Y''' + 3X'' + \frac{ZZ'}{3} - \frac{YZ}{30}\right)\overline{13}$$
$$+ \left(Z''' - \frac{3}{2}(Y'' + X') - \frac{Z^2}{30}\right)\overline{14} + \left(Y'' + 2X'\right.$$
$$\left. + \frac{Z^2}{15}\right)s_4 + \frac{1}{5}\left(Z'' - \frac{2Y' + X}{2}\right)(s_4' + 2\varphi_2 a_2),$$

$$\left.\begin{aligned} s_{10} \equiv s_9' - \left(Y'' + 2X' + \frac{Z^2}{15}\right)s_4 - \frac{1}{5}\left(Z'' - Y'\right. \\ \left. - \frac{X}{2}\right)(s_4' + 2\varphi_2 a_2) \\ = \left\{X''' + \frac{Z^{(1)2}}{5} + \frac{4ZZ''}{15} - \frac{1}{10}Y'Z - \frac{1}{30}YZ'\right. \\ \left. - \frac{1}{30}ZX\right\}\overline{12} + \left(Y''' + 3X'' + \frac{ZZ'}{3}\right. \\ \left. - \frac{ZY}{30}\right)\overline{13} + \left(Z'' - \frac{3Y''}{2} - \frac{3X'}{2} - \frac{Z^2}{30}\right)\overline{14} \end{aligned}\right\} . \quad (38)$$

Now we have four equations, (35), (36), (37), (38), by which (12), (13) and (14) can be eliminated, leaving

$$\begin{vmatrix} s_7, & X, & Y, & Z \\ s_8, & X' + \dfrac{Z^2}{15}, & Y' + X, & Z' - \dfrac{Y}{2} \\ s_9, & X'' + \dfrac{ZZ'}{5} - \dfrac{YZ}{30}, & Y'' + 2X' + \dfrac{Z^2}{15}, & Z'' - Y' - \dfrac{1}{2}X \\ s_{10}, & \left[\begin{aligned} X''' + \frac{4ZZ''}{5} - \frac{ZY'}{10} \\ - \frac{6Z^{(1)2} + YZ' + ZX}{30} \end{aligned}\right], & \left[\begin{aligned} Y''' + 3X'' + \frac{ZZ'}{3} \\ - \frac{ZY}{30} \end{aligned}\right], & \left[\begin{aligned} Z''' - \frac{3Y''}{2} - \frac{3X'}{2} \\ - \frac{Z^2}{30} \end{aligned}\right] \end{vmatrix} = 0. \quad (39)$$

an equation in a_2, its derivatives, and functions derived from the coefficients of (34a). It is of the tenth order and linear, and is the first associate of (34a). To obtain the second associate, let w represent the second associate variables. Then

$$
\begin{aligned}
w &= \overline{012},\\
w' &= \overline{013},\\
w'' &= \overline{014} + \overline{023},\\
w''' &= 2\,.\,\overline{024} + \overline{123} - 10\varphi_3\,\overline{012},\\
w''' + 10\varphi_3 w = \tau_3 &= 2\,.\,\overline{024} + \overline{123},\\
\tau_3' &= 3\,.\,\overline{124} + 2\,.\,\overline{034} + 2\,.\,5\varphi_4\,\overline{012},\\
\tau_3' - 10\varphi_4 w &= 3\,.\,\overline{124} + 2\,.\,\overline{034} = \tau_4\text{, say},\\
\tau_4' &= 5\,.\,\overline{134} + 3\,.\,\overline{125} + 2\,.\,\overline{035},\\
\tau_4' + 3\varphi_5 w - 10\varphi_4 w' &= 5\,.\,\overline{134} + 20\varphi_3\,\overline{023} = \tau_5\text{, say},\\
\tau_5' &= 5\,.\,\overline{234} + 60\varphi_3\,\overline{123} + 20\varphi_3'\,\overline{023}\\
&\qquad - 5\varphi_5 w' + 10\varphi_3\tau_3,\\
\tau_5' + 5\varphi_5 w' - 10\varphi_3\tau_3 &= 5\,.\,\overline{234} + 60\varphi_3\,\overline{123} + 20\varphi_3'\,\overline{023} \equiv \tau_6.
\end{aligned}
$$

Proceeding thus, four equations are obtained from which $\overline{024}$, $\overline{023}$ and $\overline{124}$ can be eliminated. The result is

$$
A_3 =
\begin{vmatrix}
\tau_7, & X_1, & Y_1, & Z_1\\
\tau_8, & X_1' + \dfrac{Z_1^2}{15}, & Y_1' + X_1, & Z_1' - \dfrac{Y_1}{2}\\
\tau_9, & X_1'' + \dfrac{Z_1 Z_1'}{5} - \dfrac{Y_1 Z_1}{30}, & Y_1'' + 2X_1 + \dfrac{Z_1^2}{15}, & Z_1'' - Y_1' - \dfrac{X_1}{2}\\
\tau_{10}, & \left[\begin{array}{l} X_1''' + \dfrac{4Z_1 Z_1''}{5} - \dfrac{Z_1 Y_1'}{10}\\ -\dfrac{6Z^{(1)2} + Y_1 Z_1' + Z_1 X_1}{30}\end{array}\right], & \left[\begin{array}{r} Y_1''' + 3X_1'' + \dfrac{Z_1 Z_1'}{3}\\ -\dfrac{Z_1 Y_1}{30}\end{array}\right], & \left[\begin{array}{r} Z_1''' - \dfrac{3}{2}Y_1'' - \dfrac{3X_1'}{2}\\ -\dfrac{Z_1^2}{30}\end{array}\right]
\end{vmatrix}
= 0, \qquad (40)
$$

where $X_1 = 5\varphi_5 - 20\varphi_3''$, $Y_1 = 50\varphi_4 - 140\varphi_3'$, $Z_1 = 60\varphi_3$.

(40) is also of the tenth order and linear.

The third associate is the adjoint equation. It is

$$v^{v} - 10\varphi_3 v'' + (5\varphi_4 - 20\varphi_3')\,v' - (\varphi_5 - 5\varphi_4' + 10\varphi_3'')\,v = 0 \equiv A_4. \quad (41)$$

The first associate of this adjoint equation may be obtained from (39) by writing in it

$$-\varphi_3 \text{ for } \varphi_3,$$
$$5\varphi_4 - 20\varphi_3' \text{ for } 5\varphi_4,$$
$$-(\varphi_5 - 5\varphi_4' + 10\varphi_3'') \text{ for } \varphi_5.$$

A little examination will show that these transformations among the coefficients, which change A_1 into A_4 and A_4 into A_1, also transforms A_2 into A_3 and A_3 into A_2, and in particular,

$$s_7,\ s_8,\ s_9,\ s_{10},\ X,\ Y \text{ and } Z$$

into

$$\tau_7,\ \tau_8,\ \tau_9,\ \tau_{10},\ X_1,\ Y_1 \text{ and } Z_1$$

respectively and vice versa. Then for the quintic at least it follows that the rth associate of an equation is the pth associate of the adjoint equation when

$$r + p = 3. \tag{42}$$

Preparatory to extending this theorem to the $n$thic, it will be well to consider it in a different way.

If a_3A_4 represent the first associate variable of the third associate equation, and a_rA_s the $(r-1)$st associate variable of the $(s-1)$st associate equation, then

$$a_3A_4 = \begin{vmatrix} (12'3''4''')(56'8''9''') \\ (12'3''4^{\text{IV}})(56'8''9^{\text{IV}}) \end{vmatrix}$$

$$= (23'4'')(45'6''8'''9^{\text{IV}}) - (13'4'')(25'6''8'''9^{\text{IV}}) + (12'4'')(35'6''8'''9^{\text{IV}}) - (12'3'')(45'6''8'''9^{\text{IV}})$$

If $n = 5$, then 6, 9, 8 will be 2, 4, 3, say, and the above becomes

$$(23'4'')(15'2''3'''4^{\text{IV}}) = -a_5(23'4''),$$

where a_5 is the non-vanishing constant. Then $a_3A_4 = Ca_5A_1$, C is a constant. Take $n = 6$. A_5 is the adjoint. Then

$$a_3A_5 = \begin{vmatrix} (12'3''4'''5^{\text{IV}}), & (12'3''4'''6^{\text{IV}}), & (12'3''5'''6^{\text{IV}}) \\ (12'3''4'''5^{\text{IV}})', & (12'3''4'''6^{\text{IV}})', & (12'3''5'''6^{\text{IV}})' \\ (12'3''4'''5^{\text{IV}})'', & (12'3''4'''6^{\text{IV}})'', & (12'3''5'''6^{\text{IV}})'' \end{vmatrix}$$

$$= a_6^2(12'3'') \quad \text{or} \quad a_6A_1\Delta^2,$$

then

$$a_3A_3 = Ca_2A_1,$$

where $C =$ the constant Δ^2. The general theorem is

$$a_\kappa A_{n-1} = \Delta^{\kappa-1} a_\lambda A_1$$

for all values of κ and λ for which $\kappa+\lambda = n$; that is, the $\kappa-1$)st associate variable of the adjoint equation is a constant multiple of the $(\lambda-1)$st associate variables of the original equation when $\lambda + \kappa = n$.

$a_\kappa A_{n-1}$ is

$$\begin{vmatrix}
(23'4''5''' \ldots n^{(n-2)}),\ (13'4''5''' \ldots n-1^{(n-3)},\ n^{(n-2)}), \\
\qquad (12'4''5'''6^{\text{IV}} \ldots n-1^{(n-3)},\ n^{(n-2)}), \ldots \\
\qquad (12'3'' \ldots \kappa-1^{(\kappa-2)},\ \kappa+1^{(\kappa-1)} \ldots n^{(n-2)}) \\
(23'4''5''' \ldots n^{(n-2)})',\ (13'4''5''' \ldots n-1^{(n-3)},\ n^{(n-2)})', \\
\qquad (\ldots\ldots)',\ \ldots (\ldots\ldots)' \\
(23'4''5''' \ldots n^{(n-2)})'',\ (13'4'5''' \ldots n^{(n-2)})'', \\
\qquad (\ldots\ldots)'', \ldots (\ldots\ldots)'' \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
(23'4''5''' \ldots n^{(n-2)})^{(\kappa-1)}, \ldots \\
\qquad (12'3'' \ldots \kappa-1^{(\kappa-2)},\ \kappa+1^{(\kappa-1)} \ldots n^{(n-2)})^{(\kappa-1)}
\end{vmatrix}$$

This is a determinant of order κ. In the third and lower rows each constituent equals the sum of a number of terms, all but one of which will contain $u^{(n)}$, and substituting for this its value from the differential equation, the terms are seen to be multiples of preceding rows and may be omitted. Each constituent becomes then a first mirror of Δ, and the conjugate determinant is

$$\begin{vmatrix}
1^{(n-1)}, & 2^{(n-1)}, & 3^{(n-1)}, & 4^{(n-1)} & \ldots & \kappa^{(n-1)} \\
1^{(n-2)}, & 2^{(n-2)}, & 3^{(n-2)}, & 4^{(n-2)} & \ldots & \kappa^{(n-2)} \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
1^{(n-\kappa)}, & 2^{(n-\kappa)}, & 3^{(n-\kappa)}, & 4^{(n-\kappa)} & \ldots & \kappa^{(n-\kappa)}
\end{vmatrix}$$

Having found a proof showing that $a_x A_{n-v} = a_{n-x} A_v \Delta^{x-1}$ was not, in general, true, I used it for the case when $v = 1$, when it is true that $a_x A_{n-1} = a_{n-x} A_1 \Delta^{x-1}$. But this follows immediately from Section 6, Chapter V, of Determinants, by R. F. Scott. Then we conclude that for all values of x the $(x - 1)$st associate variable of an equation is a constant multiple of the $(n - x - 1)$st associate variable of its adjoint equation. (45)

When A_1 is self-adjoint, $A_{n-1} = A_1$, and then

$$a_x A_1 = a_{n-x} A_1,$$

or all equations of complementary rank associate to a self-adjoint equation are equal. (46)

The associate equations A_x and A_{n-x} are said to be of complementary rank.

The question arises, does this hold for other associate equations of complementary rank, i. e. for any equation does

$$a_r A_v = a_{n-r} A_{n-v} \Delta^v .$$

Turning to equations (39) and (40), make

$$\varphi_2 = 0 \quad \text{and} \quad \varphi_3 = 5\varphi_4',$$

then (39) reduces to an equation of the ninth order, there being a linear relation between the a's. But A_2 or (40) does not reduce.

$a_2 A_2$ is now a non-vanishing constant and cannot be a solution of A_2. Therefore $a_2 A_2$ does not equal $a_1 A_3$. (47)

Section IV.

Conditions for the Self-Adjointness of Differential Equations.

Any equation is self-adjoint when its invariants with odd suffix vanish.

Let r be the order of the equation. The relations which exist between the coefficients are

$$\left.\begin{aligned}(-1)^n P_n = P_n - nP'_{n-1} + \binom{n}{2} P''_{n-2} - \binom{n}{3} P'''_{n-3} \\ + \binom{n}{4} P^{\text{iv}}_{n-4} + \dots \quad n = 1, 2, 3, \dots r\end{aligned}\right\} \quad (47a)$$

These relations follow from those given by Dr. Craig in his treatise, pp. 490–493. For example, take the sextic (γ), p. 491, and (γ)', p. 492. In order that it may be self-adjoint,

$$\begin{aligned} P_2 &= P_2, \\ -P_3 &= P_3 - 4P'_2, \\ P_4 &= P_4 - 3P'_3 + 6P''_2, \end{aligned}$$

or generally,

$$(-1)^x P_{6-x} = \sum_{v=0}^{v=6-x} (-1)^v \binom{v+x}{x} P^{(v)}_{6-x-v}.$$

If the equation had been written with binomial coefficients this would become

$$(-1)^x \binom{6}{x} P_{6-x} = \sum_{v=0}^{v=6-x} (-1)^v \binom{v+x}{x} \binom{6}{x+v} P^{(v)}_{6-x-v}.$$

If we call $6 - x$, m and divide $\binom{6}{x}$ it becomes

$$\begin{aligned}(-1)^x P_m &= P_m - mP'_{m-1} + -, \text{ etc.} \\ &= \sum_{v=0}^{v=m} (-1)^v \binom{m}{v} P^{(v)}_{m-v}.\end{aligned}$$

It is not difficult to see that this will hold for any equation.

First, let n be odd, then

$$0 = 2P_n - nP'_{n-1} + \left(\frac{n}{2}\right) P''_{n-2} - \left(\frac{n}{3}\right) P'''_{n-3} + -, \text{ etc.} \quad (48)$$

$$2\theta_n = 2P_n - nP_{n-1} + \frac{n-2}{2n-3}\left(\frac{n}{2}\right) P''_{n-2}$$
$$- \frac{n-2\,!\,2n-5\,!}{n-4\,!\,2n-3\,!}\left(\frac{n}{3}\right) P'''_{3} + \ldots$$

$$\frac{n-1}{2n-3}\left(\frac{n}{2}\right)\theta''_{n-2} = \frac{n-1}{2n-3}\left(\frac{n}{2}\right)\Big[P''_{n-2}$$
$$- \frac{n-2}{2} P'''_{n-3} + \frac{1}{2}\left(\frac{n-2}{2}\right) P^{\mathrm{iv}}_{n-4} + \ldots \Big].$$

Thus it is seen that $(48) - 2\theta_n$ contains neither P_n nor P'_{n-1}, and that $(48) - 2\theta_n - \left(\frac{n}{2}\right)\frac{n-1}{2n-3}\theta''_{n-2}$ is without the first two pair of terms in P_n, P'_{n-1}, P''_{n-2}, P'''_{n-3}, and from

$$(48) - 2\theta_n - \left(\frac{n}{2}\right)\frac{n-1}{2n-3}\theta''_{n-2} - \left(\frac{n}{4}\right)\left(\frac{n-1}{3}\right)\left(\frac{3}{2n-5}\right)\theta^{(\mathrm{v})}_{n-4}$$

the first three pairs of terms disappear. By subtracting certain multiples of the invariants and their derivatives from (48) the terms continue to disappear in pairs. The multiplier of $\theta^{(2\sigma)}_{n-2\sigma}$ would be $2\left(\frac{n}{2\sigma}\right)\left(\frac{n-1}{2\sigma}\right)\left(\frac{2\sigma}{2n-2\sigma-1}\right) \equiv 2M_\sigma$, say.

From what precedes, especially (22) and (23), we know

the coefficient of $P^{(2\kappa)}_{n-2\kappa}$ in (48) is $\left(\frac{n}{2\kappa}\right)$,

the coefficient of $P^{(2\kappa)}_{n-2\kappa}$ in $2M_0\theta_n$ is

$$M_0\left(\frac{n}{2\kappa}\right)\left(\frac{n-2}{2\kappa-1}\right)\left(\frac{2\kappa-1}{2n-3}\right) = M_0C_0, \text{ say,}$$

the coefficient of $P^{(2\kappa)}_{n-2\kappa}$ in $2M_1\theta''_{n-2}$ is

$$M_1\left(\frac{n-2}{n-2\kappa}\right)\left(\frac{n-4}{2\kappa-3}\right)\left(\frac{2\kappa-3}{2n-7}\right) = M_1C_1, \text{ say,}$$

. . . .
. . . .
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the coefficient of $P^{(2\kappa)}_{n-2\kappa}$ in $2M_\sigma\theta^{(2\sigma)}_{n-2\sigma}$ is

$$M_\sigma\left(\frac{n-2\sigma}{n-2\kappa}\right)\left(\frac{n-2\sigma-2}{2\kappa-2\sigma-1}\right)\left(\frac{2\kappa-2\sigma-1}{2n-4\sigma-3}\right)=M_\sigma C\text{ , say.}$$

It will now be shown that

$$\left(\frac{n}{2\kappa}\right)=\sum_{\sigma=0}^{\sigma=\kappa}M_\sigma C_\sigma,\quad\text{i.e. } 1=\Sigma\frac{M_\sigma C_\sigma}{\left(\frac{n}{2\kappa}\right)}.$$

Let

$$\frac{M_\sigma C_\sigma}{\left(\frac{n}{2\kappa}\right)}\equiv m_\sigma c_\sigma,$$

then

$$m_0c_0=\frac{n-1\,!\,2n-2\kappa-2\,!\,2n-1\,.\,2\kappa\,!}{2\kappa\,!\,n-2\kappa-1\,!\,2n-1\,!},$$

$$m_1c_1=\frac{n-1\,!\,2n-2\kappa-4\,!\,2n-5\,.\,2\kappa\,!}{2\,!\,2n-3\,!\,2\kappa-2\,!\,n-2\kappa-1\,!},$$

generally

$$m_\sigma c_\sigma=\left(\frac{2\kappa}{2\sigma}\right)\frac{n-1\,!\,n-2\,.\,n-3\dots n-2\kappa\,.\,2n-4\sigma-1}{2n-2\sigma-1\,.\,2n-2\sigma-2\dots2n-2\sigma-2\kappa-1}.$$

When $n=1$, $m_\sigma c_\sigma$ has a zero factor in the numerator for all values of σ except $\sigma=0$. The series reduces to

$$m_0c_0=\frac{-2\kappa-1\,!}{-2\kappa-1\,!}=1.$$

For $n=2$ the series has no zero factor, if $\sigma=0$ or 1, and reduces to

$$\frac{-2\kappa-2\,!\,3}{3\,.\,2\,.\,1\,.\,2\kappa-3\,!}+\frac{2\kappa-2\,!\,2\kappa\,.\,2\kappa-1}{2\kappa-1\,!\,2}=1.$$

Similarly for $n=3, 4, 5$.

For $n=2\kappa$ the series is $m_\kappa c_\kappa=\dfrac{2\kappa-1\,!\,2}{2\kappa-1\,!\,2}=1.$

For $n=2\kappa-1$ the series is

$$-m_\kappa c_\kappa+m_{\kappa-1}c_{\kappa-1}=\frac{2\kappa-2\,!\,2\kappa\,!}{2\kappa-2\,!\,2\kappa-1\,!\,2}-\frac{2\kappa-2\,!}{2\kappa-3\,!\,2}=1.$$

For $n = \kappa - 1$,

$$m_0c_0 = a\frac{2\kappa!\,2\kappa-3}{2\kappa-3!\,3!}(-1)^{\kappa}, \quad a = \frac{\kappa-2!\,\kappa+1!}{2\kappa!},$$

$$m_1c_1 = a\frac{2\kappa!\,2\kappa-7\,.\,2\kappa!}{2!\,2\kappa-2!\,2\kappa-5!\,5!}(-1)^{\kappa},$$

$$m_2c_2 = a\frac{2\kappa!\,2\kappa!\,2\kappa-11}{4!\,2\kappa-4!\,7!\,2\kappa-7!}(-1)^{\kappa},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$m_{\kappa-3}c_{\kappa-3} = a\frac{2\kappa!\,2\kappa!\,2\kappa-9}{6!\,2\kappa-6!\,3!\,2\kappa-3!}(-1)^{\kappa-1},$$

$$m_{\kappa-2}c_{\kappa-2} = a\binom{2\kappa}{4}\binom{2\kappa}{1}2\kappa-5(-1)^{\kappa-1},$$

$$m_{\kappa-1}c_{\kappa-1} = m_{\kappa}c_{\kappa} = 0,$$

$$m_0c_0 - m_{\kappa-3}c_{\kappa-3} + m_1c_1 - m_{\kappa-2}c_{\kappa-2} + m_2c_2 - +, \text{ etc.}$$

forms a series which is equal to unity. This is seen by taking the coefficient of $y^{2\kappa+2}$ from each member of the equation in which $(1-y)^{2\kappa}\frac{d}{dy}(1+y)^{2\kappa}$ is written equal to its expansion

$$(1-y)^{2\kappa} = 1 - 2\kappa y + \binom{2\kappa}{2}y^2 - \binom{2\kappa}{3}y^3 - \binom{2\kappa}{4}y^4 - \binom{2\kappa}{5}y^5$$
$$+ \ldots + \binom{2\kappa}{4}y^{2\kappa-4} - \binom{2\kappa}{3}y^{2\kappa-3} + \binom{2\kappa}{2}y^{2\kappa-2} - 2\kappa y^{2\kappa-1} + y^{2\kappa},$$

$$2\kappa(1+y)^{2\kappa-1} = 2\kappa y^{2\kappa-1} + (2\kappa-1)\binom{2\kappa}{1}y^{2\kappa-2} + (2\kappa-2)\binom{2\kappa}{3}y^{2\kappa-3}$$
$$+ \ldots + 8\binom{2\kappa}{8}y^7 + 7\binom{2\kappa}{7}y^6 + \ldots + 4\binom{2\kappa}{4}y^3 + \ldots$$

The coefficient of $y^{2\kappa+2}$ in the product of the right members is

$$\left\{4\binom{2\kappa}{4} - 5\binom{2\kappa}{5}\binom{2\kappa}{1} + 6\binom{2\kappa}{6}\binom{2\kappa}{2} \qquad - 7\binom{2\kappa}{7}\binom{2\kappa}{3} + - \ldots\right.$$
$$\left. + 2\kappa\binom{2\kappa}{4} - (2\kappa-1)\binom{2\kappa}{1}\binom{2\kappa}{5} + (2\kappa-2)\binom{2\kappa}{2}\binom{2\kappa}{6} - + \ldots\right\}$$

and adding the terms in the upper line to those below them this equals

$$\binom{2\kappa}{3}(2\kappa-3)-\binom{2\kappa}{4}\binom{2\kappa}{1}(2\kappa-5)+\binom{2\kappa}{5}\binom{2\kappa}{2}(2\kappa-7)$$
$$-\binom{2\kappa}{6}\binom{2\kappa}{3}(2\kappa-9)+\text{etc.},$$

which is the series $\sum\limits_{\sigma=0}^{\sigma=\kappa}\frac{m_\sigma c_\sigma}{\alpha}(-1)^\kappa$. The coefficient of $y^{2\kappa+2}$ in $2\kappa(1-y)(1-y^2)^{2\kappa-1}$ is

$$(-1)^\kappa\, 2\kappa\binom{2\kappa-1}{\kappa+1}=(-1)^\kappa\,\frac{2\kappa\,!}{\kappa+1\,!\,\kappa-2\,!}=(-1)^\kappa\,\frac{1}{a}.$$

Therefore

$$\sum_{\sigma=0}^{\sigma=\kappa} m_\sigma c_\sigma = 1. \tag{49}$$

Then for all values of n in like manner the same result will follow, and thus the coefficient of $P^{(2\kappa)}_{n-2\kappa}$ in (48)

$$=2M_0\theta_n+2M_1\theta'_{n-2}+2M_2\theta^{\mathrm{IV}}_{n-4}+\ldots+2M_\kappa\theta^{(2\kappa)}_{n-2\kappa}. \tag{50}$$

The coefficient of $P^{(2\kappa+1)}_{n-2\kappa-1}$ in this last series is found from that of $P^{(2\kappa)}_{n-2\kappa}$ by giving σ the same values and changing 2κ to $2\kappa+1$, and therefore this also equals $\binom{n}{2\kappa+1}$.

If in (47a) n be even, the general relation between the coefficients is expressed by

$$\left.\begin{aligned}0=P'_{n-1}-\frac{n-1}{2}P''_{n-2}+(-1)^{v-1}\frac{n-1\,!}{v\,!\,n-v\,!}P^{(v)}_{n-v}.\\ v=3,4,5\ldots n-2\end{aligned}\right\}. \tag{51}$$

In a way similar to the case when n is odd, it may be shown that (51) is equal to a linear function of the invariants and their derivatives, say

$$(51)=\theta_{n-1}+N_2\theta''_{n-3}+N_4\theta^{\mathrm{IV}}_{n-5}+\ldots+N_{n-4}\theta_3^{(n-4)}. \tag{52}$$

Now, the invariants in (50) and (52) have odd suffixes. Then when the invariants with odd suffixes vanish (48) equals zero, and also (51) equals zero, and the conditions for self-adjointness are satisfied, and the proposition with which this section begins is established.

It is to be noticed, however, that an equation may be self-adjoint when its invariants with odd suffix do not vanish, but satisfy the linear relation expressed by equating the right members of (50) and (52) to zero, which is equivalent to saying that (47a) and (57) are satisfied.

Biographical.

George Frederic Metzler, the son of George Frederic and Nancy Ann (Shannon) Metzler, was born July 17, 1853, at Westbrook, County of Frontenac, Ont., Canada. His early education was received at the Odessa public schools and at different high schools. His collegiate education was received at Albert College, Belleville, Ont. (now consolidated with Victoria College and federated with Toronto College in Toronto University). At Albert College he took the degree A. B. in 1880, and the degree M. A. in 1883. He has taught going on two years in public schools, two years in high school, one year as head-master, and was called to teach in Albert College in 1881. He entered Johns Hopkins University October, 1884, remained one session, entered again 1887. He taught in Marietta College, Ohio, 1889–90. The present year he spent in Baltimore preparing for the degree Ph. D. His studies have been in mathematics, astronomy and physics.

Baltimore, Md., 1890–91.