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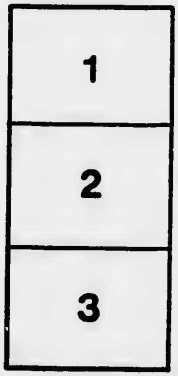
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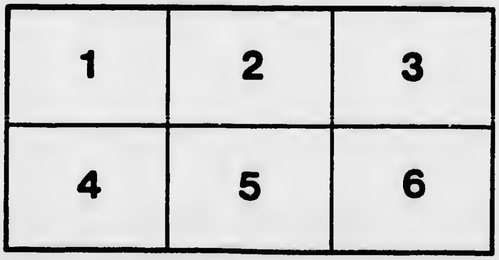
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The W. J. Gage Co.'s Mathematical Series.

# PUBLIC SCHOOL ALGEBRA

ON THE

# INDUCTIVE METHOD,

*INTENDED AS AN INTRODUCTORY SERIES*

OF

DEVELOPMENT LESSONS

*To form a guide to oral teaching and a thorough introduction to larger works.*

BY

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AND FORMERLY

*Principal of the Provincial Model School, Toronto.*

*SCHOLAR'S EDITION.*

THE W. J. GAGE COMPANY (LIMITED),

TORONTO.

# INTRODUCTION.

As this book is intended to be an aid to oral teaching, all definitions, and all explanations of merely mechanical matters, are omitted. The author is convinced that in a first book of algebra, printed explanations are comparatively useless and are never read by the pupil. The exercises are the only parts of much consequence to the learner, and accordingly this book contains almost nothing but exercises. The pupils' previous knowledge of arithmetic is a sufficient basis without postulating a series of abstract definitions. Upon this basis it is possible to begin, and the pupil may be lead by a proper set of questions to find out the facts and the generalisations of algebra for himself. THE GUIDING PRINCIPLES ARE TO FOLLOW THE LINE OF LEAST RESISTANCE, TO SEEK PRACTICAL APPLICATIONS FROM THE COMMENCEMENT, AND TO POSTPONE ALL MATTERS THAT ARE ABSTRUSE to a second and more advanced course. Within these lines it is quite practicable to commence algebra much earlier than is usual at present, and the pupil's progress in arithmetic is assisted by the light thrown on the rules of interest, discount, alligation, proportion, etc.

No teacher who is the slave of the text-book will be likely to approve of this one, which contains no provision for mere memory work and aims to appeal to reason and intelligence at every step. Very few such teachers, however, will probably teach algebra, and therefore the author has hope that these exercises will find favor. To be useful is his highest hope; to have accomplished that end will be of itself a reward.

# ERRATA.

*The following changes can easily be made with a pen. The author would be glad to have any other slips of the press pointed out for correction :—*

Page 43, Ex. 58, question 1 :—For $A^4 -$ etc., read $A^4 +$.

" 100, No. 1, question 1 :—For $-3\sqrt{b^3c}$, read $3-\sqrt[3]{b^3c}$.

" 110, No. 17, question 3 :—For $-4x^3z^3$, read $-4x^3z^2$.

" 112, No. 20, question 3 :—For $(x-y)^2$, read $(x-1)^2$.

" 117, No. 27, question 7 (ii) :—For $\frac{x}{5}$, read $\frac{y}{5}$.

" 129, last line :—For $\frac{1}{a+x}$, read $\frac{1}{a-x}$.

thousand? In estimating population by the million?

(2) Show by examples of your own choosing that several smaller units may be taken together to make a larger unit. What is the unit in measuring length by inches? By feet? By yards? By rods? By miles? By leagues?

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# PUBLIC SCHOOL ALGEBRA.

## FUNDAMENTAL CONCEPTIONS.

NOTE.—Throughout this book of exercises the student is constantly required to use the knowledge he has acquired in arithmetic and to generalise it by means of the language of algebra. The object of the first exercise is to establish in a natural manner the connection between the notation of arithmetic and the more general notation of algebra. The *unit* is the same in each; but the characters of arithmetic express definite, invariable groups of units, while the notation of algebra symbolises groups that may vary from time to time. The bridge between the particular and the general conception requires to be carefully built, and it should be crossed and re-crossed until the learner has a clear idea of the meaning of the simplest algebraical expressions. Exercise 1 should therefore be repeated by the teacher in different forms.

### Exercise 1.

(1) **Unit means One.** What is the unit in counting a flock of sheep? In counting boots by pairs? In counting eggs by the dozen? In buying baskets by the hundred? Bricks by the thousand? In estimating population by the million?

(2) Show by examples of your own choosing that **several smaller units may be taken together to make a larger unit.** What is the unit in measuring length by inches? By feet? By yards? By rods? By miles? By leagues?

(3) If unit means *one*, can this one in any case be divided or decomposed into smaller ones or units? If a line is four feet five inches long, what are the units employed? How many of the small units make one large unit?

(4) If a purse contains six dollars and five cents, what are the two units employed to count the money? How many of the small ones make one large unit in this case?

(5) A square yard contains precisely nine square feet, an acre contains four thousand eight hundred and forty square yards, what are the units mentioned here? How many of the small units make a large unit, how many large ones make a largest unit?

(6) **Arithmetical Figures denote a precise, definite number of units of any kind,** as \$5, 3 lbs., 8 cents. \$5 = \$1 + \$1 + \$1 + \$1 + \$1; 3 lbs. = 1 lb. + 1 lb. + 1 lb. What does 8 cents mean when expressed in full? 9 horses? 6 bushels? 4? 7? 2?

(7) In one flock there are 67 sheep, in another 59, and a third flock is too distant to be counted exactly. How can we represent the number of sheep in this flock? Will arithmetical figures do? **Are there any more general characters?** Can we say for the present that the third flock contains $x$ sheep?

(8) The distance from $A$ to $B$ is 4 miles and a lofty mountain is visible from both places, but its distance has never been measured yet. Will it do for the present to say that the mountain is $x$ miles from $A$ and $y$ miles from $B$?

(9) If three towns lie in a straight line and the second is $x$ miles from the first, and the third $y$ miles from the second, find the distance between the first and the third in the following cases:—When $x=5$, $y=3$; when $x=17$, $y=13$; when $x=435$, $y=287$; $x=777$, $y=888$; $x=5555$, $y=6666$ miles.

(10) At a concert there were $x$ men, $y$ women, and $z$ children; set down an expression to denote the number of the people present. If $x=11$, $y=19$, $z=17$, how many persons were present? If $x=13$, $y=18$, $z=27$? If $x=58$, $y=29$, $z=44$? If $x=1$, $y=7$, $z=100$?

(11) In every school there are $x$ boys and $y$ girls; find the total number of pupils in three different schools, when $x=27$ and $y=33$ in the first; $x=35$, $y=39$ in the second; $x=19$, $y=23$ in the third.

(12) Find the value of $x+y+z+k+w$, when $x=2$, $y=3$, $z=4$, $k=5$, $w=6$.

NOTE.—As soon as the learner has attained a clear conception of the meaning of the algebraic symbols $a$, $x$, $y$, $w$, etc., by a sound process of induction, he may begin to extend his arithmetical notion of multiplication. The object of Exercise 2 is to lead up to a definite and precise idea of multiplication and the algebraic ways of representing a group of units that contains in itself a certain smaller group repeated a given number of times. A *product* is merely the sum of a series of addends which are all equal; and *multiplication* is only a concise method of finding this sum with the least trouble. The connection between addition and multiplication needs to be thoroughly impressed. The difference in the mode of expressing the product in arithmetic and in algebra is only in the final step; thus $4\times5$, and $ab$ are the same; but we have no algebraic expression corresponding to the 20, in $4\times5=20$, unless we give $a$ and $b$ particular values.

## Exercise 2.

(1) If 2 means $1+1$, 3 means $1+1+1$; 4 means $1+1+1+1$, and so forth, then $x$ means $1+1+1+1+$etc. How many repetitions of the unit does $x$ represent?

(2) If $2\times3=2+2+2$; $2\times4=2+2+2+2$; what does $a\times5$ mean? $b\times3$? $c\times9$?

(3) If $6a$ means the same as $6\times a$, what does $2y$ mean? $4x$? $119z$? $43m$? $61t$?

(4) If $12\times13$ expresses the area of a room 12 ft. wide and 13 ft. long, what does $xy$ represent? $ab$? $cx$? $pq$?

(5) **If two algebraic expressions are written together without any sign between them, as +, −, ×, or ÷, what is meant? What does *cd* mean?**

(6) If $18\times12\times6$ represents the number of cubic feet in a pile of wood 18 ft. long, 12 ft. wide, and 6 ft. high, what does $abc$ represent? $xyz$?

(7) What is the price of $x$ horses at \$51 apiece?

(8) What is the price of $x$ horses at \$$y$ apiece?

(9) How many pounds are there in $x$ tons?

(10) Find the value of 4 bushels at $x$ cents a bushel, and 7 bushels at $y$ cents a bushel.

(11) John goes $x$ miles a day for 5 days, $y$ miles a day for the next 3 days, and $z$ miles a day for the next 7 days; how far has he travelled?

(12) In a store there are $a$ counters, in each counter $b$ compartments, in each compartment $c$ drawers, in each drawer $d$ divisions; express the total number of divisions.

(13) In a block there are $w$ houses, each house has $x$ rooms, and in each room $y$ people; how many people live in the block?

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NOTE.—Experience shows that almost every learner requires a large amount of drill on the resemblances and the differences that exist between *coefficients* and *exponents*. Exercise 3 is only a specimen of the kind of drill required to establish a clear line of demarcation between the meanings of these symbols. If the teacher has the time, he should give several more lessons on this topic, taking extreme and critical examples to re-inforce the impression of Ex. 3. It is far cheaper to prevent confusion than to spend time in correcting it afterwards. If the fundamental ideas are not grasped thoroughly the pupil will not make satisfactory progress in the later stages.

### Exercise 3.

(1) In $5a$, how many factors? **What are co-factors?**

(2) Is the 4 in $4y$ an arithmetical or an algebraic factor?

(3) Is the 6 in $6x$ **a literal or a numerical factor?**

(4) What are the co-factors in $xyz$? In $abc$?

(5) Is COEFFICIENT **the technical word for co-factor?**

(6) How many coefficients are there in each of the quantities $ab$, $cde$, $wxyz$, $abcxy$?

(7) In $6ab$, is the 6 a *literal* or *numerical coefficient?* What do you call the $a$ and the $b$?

(8) How many factors in $6 \times 6$? In $7 \times 7 \times 7$? In 8.8.8.8?

(9) Does 9.9.9 mean the same as $9 \times 9 \times 9$?

(10) Will this do for shortness $9 \times 9 \times 9 = 9.9.9 = 9^3$?

(11) If $2^4 = 2.2.2.2$, what does $a^2$ mean? $a^3$? $a^4$?

(12) How many factors are there in $a^{99}$? $x^{53}$? $m^{21}$? $k^x$?

(13) If one side of a square room is 19 feet, write the algebraic expression for the area with three figures only.

(14) If $18^2$ expresses the area of a square, will it do to read for shortness "**eighteen square**"?

(15) If "seven square" is written $7^2$, how would you read $6^2$, $11^2$, $125^2$, $a^2$, $c^2$, $x^2$, $y^2$, $z^2$?

(16) If the area of a room is $a^2$, what is the length of the side?

(17) How many factors are there in $a^2$? In $ab$?

(18) If $a=b=c$, is $a^3=abc$? **If all the factors in a product are equal, how do we write that product** for shortness?

(19) If the side of a cube is 4, is the solid content of the cube expressed by $4 \times 4 \times 4$? Can it be written with two figures alone? Will it do for shortness to call $4^3$ "**four cube**"?

(20) Find the value of $x^2+y^3+z^4$, when $x=5$, $y=6$, $z=7$.

(21) If we write $7 \times 7 \times 7 \times 7$ with two figures thus $7^4$, will it do for shortness to read $7^4$ in words, "seven to **the fourth power,**" or simply "seven to the fourth"?

(22) If 4 is raised to **the sixth power,** how many factors would the product contain? Would these factors be equal factors?

(23) In $5^8$ how many equal factors? What kind of factors does **an Exponent,** like 8 in $5^8$, indicate?

(24) May **we say that a power is the product of a certain number of equal factors,** and that **an exponent is used to show the number of these equal factors?**

(25) If $5a=a+a+a+a+a$, and $a^5=a \times a \times a \times a \times a$, what is **the difference between a Coefficient and an Exponent?**

(26) If $5a^5=a^5+a^5+a^5+a^5+a^5$, and $a^5$ itself$=a.a.a.a.a$, how many $a$'s will be required to write out in full the value of $6a^6$? $3a^{11}$? $11a^3$?

(27) If the coefficient of $x$ is 19 and its exponent is 8 and $x$ itself is $=2$, what is the numerical value of the expression? Write your calculation in the shortest form.

(28) If 4 is multiplied by six equal factors each of which is $m$, express the product by using three characters. Give the technical name of each character.

(29) If $1 \times 2=2^1$,
$1 \times 2 \times 2=2^2$,
$1 \times 2 \times 2 \times 2=2^3$, etc., explain fully what is meant by $a^3$, $a^2$, $a^1$.

(30) If $a^3=1 \times a \times a \times a$; $a^2=1 \times a \times a$; $a^1=1 \times a$; **what does** $a^0$ **mean?** If $a=6$, find the value of $a^3$, $a^2$, $a^1$ and $a^0$. If $a^0=1$, what is the value of $5x^0$?

NOTE.—**The Equation** has been used already in Arithmetic; and it is highly desirable that the pupil should learn to make some practical applications of algebra as soon as possible. The simple equation has only one mystery, viz., *transposition*, and this is easy to explain. It is better in all the first stage of work to avoid the doctrine of negative quantities as much as possible, to proceed, in fact, along the line of least resistance and learn the simplest things first. The examples chosen should, therefore, have positive roots only. **The pupil should be encouraged to make simple equations and problems leading to simple equations for himself. He should also be trained to verify his solutions by substituting in the given equation the root,** and he will thus get a practical knowledge of the *identity* as distinguished from the equation. Only one particular value will turn the equation into an identity.

## Exercise 4.

(1) If $5x=60$, what must be the value of $x$?

(2) If $15x-45=0$, what does this expression become when 45 is added to each side of **the equation**? What is the value of $x$?

(3) If $6x=3a+2b$, what is the value of $x$ when $a=4$, $b=6$?

(4) When $a=6$, $b=3$?

(5) In the equation $9x+15=8x+19$, subtract 15 from the equal quantities, and next subtract $8x$ from each side. What is the value of $x$?

(6) In the equation $2x-7=x-4$, reduce the expression to the form shown in question 1, and thus find the value of $x$.

(7) In the equation $3x+8+7x=4x+56$, **transpose** the numerical quantities to the right and the literal quantities to the left and thus find the value of $x$.

(8) In the equation $3x-21=0$, how is the sign of $-21$ affected by **transposition**? Find $x$.

(9) Show that $2x+24=100$ is the algebraic translation of this question;—"If my money were doubled and $24 added I should have $100, how many dollars have I?" Find the sum $=x$.

(10) Show that $2x+24=100$ is the algebraic shorthand to express the following problem:—"If I live twice as long as I have already lived and 24 years more I shall be 100 years old, what is my present age?" **Solve the equation** and find the number.

(11) By means of an equation find a number such that if 24 be added to the double of it the sum will be 100.

(12) A son's age is $x$, his father is three times as old, and both their ages added together make 64, find $x$.

(13) A piece of cloth contains $x$ yards, and for $100 I can buy this roll and another of the same length. The price of the first piece is $3 a yard, and of the other $2 a yard. Find $x$ by an equation.

(14) A cistern can be fillled in $x$ minutes by two pipes, which carry 8 barrels and 12 barrels of water per minute. If the cistern holds 100 barrels, form an equation and find $x$.

(15) Multiply every term of the equation $x+\frac{1}{2}x+\frac{1}{3}x=11$ by 6, and thus find the value $x$.

(16) What multiplier will **clear of fractions the equation** $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x=26$? What is the value of $x$?

(17) Multiply every term of the equation
$\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x+45=448$ by some number that will clear it of fractions, and find $x$.

(18) If 45 be added to half a certain number, $x$, and one-third and one-fourth of the number be added to the sum, the result is 448; find $x$.

(19) One-third of my money exceeds one-fourth of it by $16; if $x$ = the number of dollars I have, find $x$ by an equation.

(20) A fish was cut into three pieces; the tail weighed 9 pounds, the body $2x$ pounds, and the head as much as the tail and half the body, and the body as much as the head and tail together. Find the weight of each part and of the whole fish.

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NOTE.—This is a review exercise in which special emphasis is laid on the meaning of the sign +. The exercises in substitution will confirm the pupil's knowledge of the connection between arithmetic and algebra. The teacher should, if time permits, change numerical values given, and thus repeat this exercise a number of times. Spend time here, and save it. All progress in the higher stages depends on a full mastery of the grammar of the new language the pupil is learning.

## Exercise 5.

(1) Express in one line by means of **the sign Plus** the sum of 1, 2, 3, 4, 5, 6, 7, 8, 9.

(2) Write down in one line the sum of $a$, $b$, $c$, $d$, $e$.

(3) If $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, find the numerical value of $2a+3b+4c+5d+6e$.

(4) With the same values find the number equal to the sum of $a^2$, $b^3$, $c^4$, $d^5$, $e^6$.

(5) With the same values find the numerical expression for $2ab+3cd+4bd+5bc$.

(6) With the same values find the number equal to

$$a^b + b^c + c^d + e^f, \text{ where } f=0.$$

(7) With the same values find the number equal to

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{e}{a}.$$

(8) With the same values find the sum of

$$2a^2+3a^3+4a^4+5a^5+6a^6+7a^7+8a^8+9a^9; \text{ and of}$$

$$9b^2+8b^3+7b^4+6b^5+5b^6+4b^7+3b^8.$$

(9) What is the value of $abc+bcd+cde+eab$ with the same numbers as before?

(10) Express $x$ pounds + $y$ ounces in ounces.

(11) Express $5x+3y$ dollars in cents.

(12) Find the numerical value of the product of $\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}\times\frac{d}{e}$. Write this expression in a shorter form using some other symbol for $\times$.

(13) John is $x$ years old to-day, in five years more he will be half as old as his father, find his father's present age, if $x=16$, if $x=18$, if $x=20$.

(14) Solve the equation $\frac{1}{2}x+\frac{1}{3}x=x-3$.

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NOTE.—In a first lesson on the meaning of the sign −, and on the use of the bracket, it would be highly imprudent to plunge the learner into the difficult and somewhat unsettled doctrine of negatives quantities. Impatience spoils all good teaching. Exercises 6 and 7 give the pupil a working knowledge of brackets and of the meaning of the minus sign. Let him learn the simple parts first, and the more difficult things last. Rapid progress can take place only when this rule is strictly observed. The case of $a-b$ when $b$ is greater than $a$ must be deferred at present.

## Exercise 6.

(1) Use **the sign Minus** to express the final result when 1, 2, 3, 4, 5, 6, 7, 8, and 9 are successively subtracted from 50.

(2) Use the sign – to express the final result when $a$, $b$, $c$, $d$, $e$, and $f$ are successively subtracted from $x$.

(3) If $a=50$, $b=1$, $c=2$, $d=3$, $e=4$, $f=5$, find the numerical value of $a-b-c-d-e-f$.

(4) What is the value of $15-8$? Of $15-(5+3)$?

(5) How much is $(8+7)-(5+3)$? **What is the meaning of a Bracket enclosing two or more numbers?**

(6) If $15-8=7$, how much is $15-(6+2)$?

(7) How should $15-(6+2)$ be written if the bracket be not used? Is $15-6+2$ correct?

(8) In the expression $15-(6+2)$ how much are we to subtract from 15? If we write $15-6$ have we subtracted as much as is required? How much more should we subtract to get the correct remainder? Write the correct expression when the bracket is removed.

(9) If the bracket be removed from $11-(5+3)$, what will be the sign of the 3?

(10) If the bracket be removed from $11-(5-3)$, what will be the sign of the 3? Is it correct to write $11-5-3$?

(11) **If a minus sign stands before a bracket, what must be done with all the signs within the bracket when we remove the bracket?**

(12) Express $a-(b+c)$; $a-(b-c)$; $a-(b+c+d)$; $a-(a-b-c)$; $a-(b-c+d)$ without brackets.

## Exercise 7.

(1) Remove the brackets and find the remainder in the expression $(2a+3b)-(a+b)$.

(2) Simplify by adding or subtracting the **like terms** in $(3a^2+4a^3)-(2a^2-5a^3)$.

(3) When a man has walked eastward 5 miles, suppose we say he has travelled + 5 miles; what ought we to say of another person who has gone 6 miles westward?

(4) If a boy has walked 11 miles southward and has returned 5 miles northward, express his distance from the starting point by means of a minus sign.

(5) If a man owns \$6,000 in cash, and owes \$2,000, express his present worth by means of a minus sign.

(6) If a train goes forward $x$ miles and then backs up $y$ miles, how far is the train from the point of departure?

(7) If Mr. Johnson has $x$ dollars in the bank and owes $y$ dollars to different people, how can we express what he would have when his debts are all paid?

(8) If you give me $a$ dollars and I pay you back $y$ dollars in change, how much have I received?

(9) Make up the balance in the following cash book:—On hand \$$(25x+37y)$; 1st day, received \$$16x$ and paid \$$11y$; 2nd day, received \$$18y$ and paid \$$21x$; 3rd day, received \$$41x$ and paid \$$75y$; 4th day, received \$$100y$ and paid \$$37y$; 5th day, received \$$18x$ and paid \$$111y$; 6th day, received \$$253x$ and paid \$$41y$.

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NOTE.—The student who has carefully worked the preceding exercises has already learned enough algebra to enable him to proceed without very much assistance. His previous knowledge of arithmetic comes to his aid at every step, and he will soon be prepared to use this new language to find general solutions of problems formerly solved by the particular analysis of arithmetic. Exercises 8 to 14 supply further practice in translation of arithmetic into algebra, and give a little wider knowledge of the simple equation and easy applications to practical questions. The method of expressing the relations of the numbers involved in a problem by means of an equation has already been partly exhibited. Exercise 9 gives more examples. **The pupil should be encouraged to make new problems to fit the same equation.** Most of the equations in Exercise 12 may be used as an exercise of this kind, and a good deal of the oral teaching should be directed to this translation of problems into equations and translation of equations into problems. The pupil can easily solve the equations without much help.

## Exercise 8.

Find the sum or aggregate of the following quantities:—

(1) $2a+3b+5a-2b+2a-3b+a-3b-4a-5b-6a+11b+19a$.

(2) $5x^4+3x^3+2x^2-3x^4-3x^3-5x^2+11x^4-6x^2+7x^3-4x^4-3x^3-2x^2$.

(3) $4ab+5ac+6bc-7bc-6ab-2ac+8bc+25ac-11bc+6ab$.

(4) $14a^4 + 15b^4 + 16c^4 - 12c^4 - 11b^4 - 9a^4 + 8b^4 - 9a^4 + 21c^4 + 25a^4$.

(5) $(a+b+c)+(2a-2b-2c)-(3a-3b-3c)+(4a+4b+4c)$.

(6) $5(a+b)+6(a+b)-4(a+b)-3(a+b)+11(a+b)$.

(7) $\frac{1}{2}a - \frac{1}{3}b - \frac{1}{4}c + \frac{1}{4}a + \frac{1}{2}b + \frac{1}{3}c - \frac{1}{3}a - \frac{1}{4}b + \frac{1}{2}c$.

(8) $\frac{1}{2}x - \frac{1}{3}y + \frac{1}{4}z + \frac{1}{2}y - \frac{1}{3}z + \frac{1}{4}x + \frac{1}{2}z - \frac{1}{2}x + \frac{1}{4}y$.

(9) $a - 2b + 3c + a - \frac{1}{2}b + \frac{1}{3}c + \frac{1}{2}a + 2b + \frac{1}{3}c - \frac{1}{4}a$.

(10) \$$x$ + £$y$ + $z$ shillings + $w$ cents + $a$ dimes + \$$135b$, taking £1 = \$4, 1s. = 20 cents. Give the answer in cents.

(11) $\cdot 25ab + c + \cdot 99b + 3ab + 2c + \cdot 01b + \cdot 75ab + b$.

(12) $A$ ft. + $B$ yd. + $C$ in. Give the answer in inches.

(13) From $6x^3 + 4x + 7$ take the sum of $2x^3 + 4x^2 + 9$, and $4x^3 - x^2 + 4x - 2$.

## Exercise 9.

(1) If $x + 45 = 4x$, find the value of $x$.

(2) If I had \$45 more than I now have in my purse, I should have four times as much as I have at present. How many dollars are there in my purse?

(3) If $\frac{1}{4}x + \frac{1}{7}x = 176$, what is the value of $x$?

(4) A quarter of my money added to one-seventh of my money just makes \$176. How many dollars have I?

(5) A horse cost three times as much as the buggy, and the rig was worth \$300, including a set of fine harness worth \$48. Find the price of the horse.

(6) If I pay my men \$$2\frac{1}{2}$ per day, I shall make \$10 per day on the contract, but if I pay them \$3 per day, I shall lose \$18 per day. How many workmen have I, and how much do I receive per day?

(7) The owners of a steamboat make \$30 per day each, but if the number of partners were decreased by five, the profits would be \$40 each per day. How many owners are there?

## Exercise 10.

(1) A man walked 71 miles in three days. He walked 3 miles more the second day than on the first, and 5 miles more the third day than on the second. How far did he travel the first day?

(2) Divide \$6,900 among four sons, so that each shall have \$150 more than his next younger brother.

(3) Two fields together measure 50 acres; the smaller contains 10 acres less than half the larger. Find the size of each.

(4) Ten people bought a bicycle on shares, but four of them backed out of the bargain, and consequently the rest had to pay \$8 apiece more than they intended; what did the machine cost?

(5) Solve the equation $2x+3(6x-5)-5=x-1$.

(6) In 1892 Jones had twice as much capital as Brown; but in 1893, Jones made \$500, Brown lost \$300, by the price of wheat declining to 60 cents per bushel. Jones is now worth three times as much as Brown. What capital had each in 1892?

(7) A father is now three times as old as his son, but 8 years ago he was seven times as old. How old will each be when the father is twice as old as his son?

## Exercise 11.

(1) Evaluate $\frac{x-1}{x+1}+\frac{x+3}{x-3}-2.\frac{x+2}{x-2}$, when $x=5$.

(2) If $x=19$, find the numerical value of the expression

$$\frac{7x+5}{23}+\frac{9x-1}{10}-\frac{x-9}{5}+\frac{2x-3}{15}.$$

(3) When $a=4$, $b=3$, $c=2$, ascertain what number corresponds to the expression

$$3a^2b+\frac{a^2+b^2}{c}+abc+\frac{2ab}{3}+\frac{a^2}{b^3}+\frac{a-b}{a^3+b^3+c^4}.$$

(4) Express in shortest form $\frac{1}{2}a-\frac{1}{3}b+\frac{1}{6}c+\frac{1}{4}a-\frac{1}{5}b-\frac{1}{8}c+\frac{1}{3}a+\frac{1}{4}b+\frac{1}{2}c$.

(5) Add together

$$a^2-3ab-\frac{11}{21}b^2+2b^2-\frac{2}{3}b^3+c^2+ab-\frac{1}{3}b^2+b^3+2ab-\frac{1}{3}b^2.$$

(6) A boy being told to divide one-half of a certain number by 4 and the other half by 6, and to add the two quotients, attempted to get the sum by dividing the whole number by 5, but his answer was too small by 2. Find the number that was given.

## Exercise 12.

Find the value of $x$ in each of the following equations:—

(1) $6x+2(11-x)=3(19-x)$.

(2) $3(x+1)+2(x+2)=32$.

(3) $3x-2(5x+4)=2(4x-9)$.

(4) $6(3-2x)=24-4(4x-5)$.

(5) $45-4(x-2)=5(x+2)$.

(6) $\frac{x}{2}-\frac{x}{3}-\frac{x}{4}+\frac{4}{3}=\frac{3}{4}$. N.B.—Multiply both sides by 12.

(7) $\frac{x}{5}+\frac{x}{4}+\frac{x}{3}-\frac{x}{2}=17$. Multiply through by 60.

(8) $\frac{3x}{14}-\frac{2x}{21}+\frac{1}{3}=\frac{x}{4}-4\frac{1}{4}$.

(9) $\frac{14x}{3}-\frac{8x}{5}=10\frac{1}{3}+\frac{2x}{1\frac{1}{3}}-3\frac{2}{5}$.

(10) $\frac{x}{4}-4\frac{1}{2}+\frac{x}{5\frac{1}{2}}+\frac{x}{2}=\frac{16\frac{1}{4}}{5\frac{1}{2}}$.

(11) $\frac{1}{4}(x+6)-\frac{1}{12}(16-3x)=4\frac{1}{6}$.

(12) $\frac{1}{16}(3x+3)+\frac{1}{15}(7x-4)-\frac{1}{20}(7x+1)=2$.

## Exercise 13.

(1) Divide a rope 33 feet long into four parts, so that the second piece is 18 inches longer than the first, the third 30 inches longer than the second, and the fourth $3\frac{1}{2}$ feet longer than the third.

(2) The ages of a man and his wife amount to 80 years, and 20 years ago the woman was two-thirds the age of her husband. What was the sum of their ages at that time?

(3) Find a number such that if it be divided into *two* equal parts or into *three* equal parts, the continued product of the parts shall be the same.

(4) A butcher bought a number of sheep for \$376, but 7 of them were lost on the way to market. He sold one-fourth of the remainder at cost for \$80. How many had he left?

(5) Divide \$5,000 into two sums such that, being placed at simple interest, the first for 6 years at 10%, and the second for 8 years at 5%, the amounts may be equal.

(6) My farm cost \$75 an acre; I reserved 10 acres and sold the rest at \$80 an acre, clearing \$40 more than the cost of the whole. How many acres were there in the farm?

(7) $\frac{1}{14}(3x+\frac{2}{3})-\frac{1}{7}(4x-6\frac{2}{3})=\frac{1}{2}(5x-6)$, find $x$. (Multiply by 14.)

NOTE.—Exercises 14 to 21 inclusive are intended to develop more completely the doctrine of multiplication. There is no difficulty in doing this so long as the − sign is not used. But when this sign occurs it becomes necessary to notice that algebra has carried us beyond the ordinary arithmetical multiplication, and that new cases have arisen to which we must give some meaning not contradictory of the arithmetical meaning already agreed upon. Analogy shows that + multiplied by − gives a product with a − sign, and that − multiplied by − gives a product with a + sign. We therefore extend the meaning of the signs to include these cases. A *convention* of this kind becomes necessary whenever a sign formerly used in arithmetic is employed in new cases that did not occur in arithmetic. There is no proper *proof* of these extensions beyond showing by examples what must be the consistent interpretation in the wider sense, referring always to the original meaning for guidance.

## Exercise 14.

(1) The sign × is read **into** and means that the product of two numbers is to be taken; thus $4 \times 3 = 12$.

(2) The product of $a$ multiplied by $b$ is written $a \times b$, or $a.b$, or simply $ab$. What does $xy$ mean? When two algebraic expressions are written close together, what operation is indicated?

(3) Find the value of $ab$ when $a = 1234$, and $b = 5678$.

(4) Express in shortest form $a+a+a+a+a+a+a+$etc., when there are $x$ of these $a$'s. Find the value when $a=1$, $x=2$; when $a=777$, $x=888$; when $a=21$, $x=23$.

(5) If $km$ means the product of $k$ multiplied by $m$, what does $k(a+b)$ mean? What does $(a+b)(c+d)$ mean?

(6) Express $k(a+b)$ without the bracket. Express $w(a+b)$ without the bracket. Express $(k+m)(a+b)$ without the brackets.

(7) Multiply $x+y$ by $z+k$.

(8) Multiply $6x$ by $y$; by $2y$; by $3y$; by $7xy$; by $2x^2$.

(9) Multiply $3a+2b$ by $7a+5b$; also by $9a+8b$.

(10) What is the product of $3a$, $2a$, and $a$?

(11) Find the price of $a+b$ horses at \$$x$ apiece; at \$$5x$ apiece.

(12) Multiply $a^2+2ab+b^2$ by $a^2+3ab+b^2$.

## Exercise 15.

(1) If $k(a+b)=ka+kb$, what does $k(a-b)$ represent?

(2) If $k(a-b)=ka-kb$, what does this become when $k=c+d$?

(3) If $(a-b)(c+d)=ac+ad-bc-bd$, what must be the product in each of the following cases, $(+a)(+c)$, $(+a)(+d)$, $(-b)(+c)$, $(-b)(+d)$?

(4) If two *plus* quantities are multipled, what is the *sign* of their product?

(5) If a *plus* quantity and a *minus* quantity are multiplied, what is the *sign* of their product?

(6) If $k(a-b)=ka-kb$, what does this become when $k=c-d$?

(7) If $(c-d)(a-b)=ac-bc-ad+bd$, what must be the product in the following cases, $(+c)(+a)$, $(+c)(-b)$, $(-d)(+a)$, $(-d)(-b)$?

(8) **If two Minus quantities are multiplied, what is the Sign of their product?**

(9) If a man walks *east* $(+5)$ miles for 6 successive days, express his distance from the starting point.

(10) If a man walks *west* $(-5)$ miles per day for 6 days, express his distance from the starting point.

(11) Multiply $a-b-c$ by $d-e-f$.

## Exercise 16.

(1) **Positive quantities** have the sign + before them, either expressed or understood. **Negative quantities** have − before them, and it is always written, not in any case understood. Name each **term** of the expression $a-b+c-d-e+f$ as positive or negative.

(2) Review Exercise 15 and from the examples make a rule for the signs of **the product of algebraic quantities with Like Signs,** and also for **the product of quantities with Unlike Signs.**

(3) If I walk five yards north, and then turn back and walk five miles south, how far am I from the starting point?

(4) If I gain $5 and lose $5, how much richer am I?

(5) If $x$ people come in and $x$ people go out, how many more persons are there in the room than before?

(6) What is the sum of $(+5)$ and $(-5)$? Of $+x-x$?

(7) If $a-a=0$, multiply both sides by $b$, and also subtract $ab$ from both sides. What does this prove about $(-a)(+b)$?

(8) If $a-a=0$, multiply both sides by $(-b)$, and also add $ab$ to both sides. What does this prove about $(-a)(-b)$?

(9) Prove that $(a)(-b)=(-a)(+b)$, and that $(-a)(-b)=(+a)(+b)$.

## Exercise 17.

(1) $a^2=a.a$, $a^3=a.a.a$; express in full $a^2.a^3$.

(2) If $a^2.a^3=a.a.a.a.a$, express this product by means of one letter and one figure.

(3) If $x^7$ is the product of 7 equal factors, and $x^{11}$ is the product of 11 such equal factors, how many factors must there be in the product of $x^7.x^{11}$? What is the short way of writing the product of 18 factors, each equal to $x$?

(4) If $2x$ multiplied by $3y=2.x.3.y=6xy$, what is the product of $8x$ multiplied by $9y$? Of $8x$ multiplied by $-9y$? Of $-8x$ multipled by $-9y$? Of $8a^2\times 9a^3$? Of $-8a^2\times 9a^4$? Of $-8x^{21}\times -9a^{31}$?

(5) If $a^3.a^7=a^{7+3}$, and $a^{125}.a^{628}=a^{125+628}$, etc.; express the product of $a^x.a^y$; of $x^a.y^b$; of $a^m.a^n$.

(6) In $4x^{12}.5x^{15}$, point out the *coefficients*, the *exponents* and the *literal factors*. State what is done with each when we write down the product.

**(7) Make a rule about coefficients and exponents when Like quantities are multiplied together.**

## Exercise 18.

(1) Find the product of $a+5$ multiplied by $-5$.

(2) A certain quantity has two factors, viz., $a-b$ and $c-d$, what is the quantity?

(3) Multiply $a+b$ by $a+b$, and this product by $a+b$ again.

(4) What is the value of $(a+b)^2$ and of $(a+b)^3$ when written in full?

(5) If $(a+b)^2=a^2+2ab+b^2$, find the square of $(a+1)$ without actually multiplying. N.B.—Put $b=1$.

(6) Multiply $a^3+3a^2b+3ab^2+b^3$ by $a^2+2ab+b^2$. What is the expanded form of $(a+b)^5$?

(7) Find the expanded values of $(x+y)^2$, $(x+y)^3$, $(x+y)^4$, $(x+y)^5$ and $(x+y)^6$. **Learn these results by heart so that you can set them down by copying from memory.**

(8) Multiply $4a^4+3a^3+2a^2+a$ by $4a^4-3a^3-2a^2+a$.

(9) Multiply $263a^{128}+259$ by $1251a^{1728}+427$.

(10) Multiply $-a^{777}$ by $777a$.

(11) Multiply $-555a^x$ by $xa^{555}$.

(12) What is the product of $-2x^2$ and $2x^{-2}$? Of $a\times a^{-1}$?

(13) Simplify by expansion

$$(a^{1234567890}+b^{9876543210})(a^{123}+b^{456}).$$

## Exercise 19.

(1) Find the price of $25x$ sheep at $5 apiece.

(2) Make up the following account:—$a$ pounds of pepper @ $b$ cents per pound, $b$ pounds of coffee @ $a$ cents per pound, $c$ pounds of figs @ $d$ cents per pound, $d$ pounds of raisins @ $c$ cents per pound, and 5 pounds of tea @ 25 cents per pound.

(3) I met $x$ men, each man had $x$ bags, each bag held $x$ cats, and each cat had $x$ kittens. How many kittens were there? Find the price of the cat skins at $y$ cents apiece for the old cats and $z$ cents apiece for the kittens.

(4) A certain old couple had $a$ sons and $b$ daughters; each of their sons had $c$ sons and $d$ daughters, and each of their daughters had $x$ sons and $y$ daughters. In the third generation each grandson had $w$ boys and 5 girls, and each grand-daughter had $z$ boys and 6 girls. Tell quickly how many great-grand children this happy old couple had at their golden wedding which took place in July, 1886.

(5) Said Dick to Harry, "My money and my purse are together worth \$$d$, and the money is worth $k$ times the price of the purse." Find the value of the purse.

## Exercise 20.

Multiply the following pairs of factors:—

(1) $3a^2-4ab+5b^2$ and $a^2-2ab+3b^2$.

(2) $x^4-3x^2+2x+1$ and $x^3-2x-2$.

(3) $a^2-ab-ac+b^2-bc+c^2$ and $a+b+c$.

(4) $3x^2-2x-5$ and $2x-5$.

(5) $a-b+3c$ and $a+2b$.
(6) $2a^2+4b^2-3ab$ and $-5a^2+4b^2+3ab$.
(7) $3x^4-x^2-1$ and $2x^4-3x^2+7$.
(8) $a^2+b^2+c^2+ab+ac-bc$ and $a-b-c$.
(9) $x^2+4y+3z^2$ and $x^2-2y^2-3z^2$.
(10) $a^2+b^2+9-3a+3b+ab$ and $a-b+3$.
(11) $4x^2+9y^2+z^2-6xy-2xz-3yz$ and $2x+3y+z$.
(12) Find the fourth power of $2a+5b$.
(13) $x^5+5x^4+15x^3+15x^2+5x+1$ and
$x^4-5x^3+10x^2-5x+1$.
(14) $42a^4+105a^3b+23a^2b^2+5ab^3+b^4$ and $2a^2-5ab+b^2$.
(15) $x^5+5x^4+15x^3+30x^2+24x+21$ and
$x^4-5x^3+10x^2-5x+1$.

## Exercise 21.

Find the continued product of the following sets of factors:—

(1) $(x+1)\ (x+2)\ (x+3)$.
(2) $(x+2)\ (x+3)\ (x+4)$.
(3) $(x+4)\ (x+5)\ (x+6)$.
(4) $(x+5)\ (x+7)\ (x+9)$.
(5) $(x+a)\ (x+b)\ (x+c)$.
(6) $(x-a)\ (x-b)\ (x-c)$.
(7) $(x-10)\ (x+1)\ (x+4)$.
(8) $(x+1)\ (x+2)\ (x+3)\ (x+4)$.
(9) $(x-5)\ (x+6)\ (x-7)\ (x+8)$.
(10) $(x^2+ax+a^2)\ (x^2-ax+a^2)\ (x^4-a^2x^2+a^4)$.
(11) $(x+8)\ (x+5)\ (x+3)\ (x-3)\ (x-5)\ (x-8)$.
(12) $(1+\frac{1}{2}a+\frac{1}{3}b)\ (1-\frac{1}{2}a+\frac{1}{3}b)$.
(13) $(x^2-\frac{1}{2}x+\frac{2}{3})\ (\frac{1}{3}x+2)$.
(14) $(ab-ac+bc-bd)\ (ab-ac-bc+bd)$.
(15) Simplify

$$\tfrac{1}{6}\{x(x+1)\ (x+2)+x(x-1)\ (x-2)\}+\tfrac{2}{3}(x-1)\ (x+1)x.$$

**N.B.—The student would do well to go over Exercises 14 to 21 at least three times before proceeding to the next series of examples.**

NOTE.—It is now time that the pupil should begin to learn a little of the power of algebra to save time and labor as compared with Arithmetic. Algebraic solutions are true for all possible cases, because the language is general, whereas arithmetical solutions are true only for certain particular numbers. Five extremely useful formulas are taken in the following exercises, 22 to 25 inclusive, and the pupil is required to write down the results when the details are varied in sign or in value. He must do this without actual multiplication. The algebraic form is unchangeable throughout each case, no matter how much the items may vary. When the pupil has thoroughly appreciated this he has already entered on the great highway of mathematics, and has made a great stride of progress in abstraction and generalisation. Part of 25 and the whole of 26 are review exercises.

## Exercise 22.

(1) If $(A+B)^2 = A^2 + 2AB + B^2$, write down the expansion of $(a+x)^2$ without actual multiplication.

Of $(a+y)^2$; $(x+y)^2$; $(a+1)^2$; $(a+2)^2$.

(2) Find without actual multiplication the square of each of the expressions, $3a+2b$; $4a+3b$; $8a+6c$.

(3) What is the square of $125x+225y$? Of $343m^2+512n^2$?

(4) Find the square of $729x^3+512x^2$, and of $17a^{17}+18a^{18}$.

(5) If $(x+1)(x+2)=(x-1)(x-2)+54$, show that $x$ must $=9$.

(6) If $(A-B)^2 = A^2 - 2AB + B^2$, write down the expansion of $(a-x)^2$ without actual multiplication.

Of $(a-y)^2$; $(x-y)^2$; $(a-1)^2$; $(a-2)^2$.

(7) Find without actual multiplication the square of each of the expansions, $3a-2b$; $4a-3b$; $8a-6c$.

(8) What is the square of $125x-225y$? Of $343m^2-512n^2$?

(9) Find the square of $729x^3-512x^2$, and of $17a^{17}-18a^{18}$.

(10) Show that $x(x+1)(x+2)(x+3)+1=(x^2+3x+1)^2$.

(11) If $(3x-17)^2+(4x-25)^2-(5x-29)^2=1$, show that $x$ must $=6$.

(12) If $5(x-2)^2+7(x-3)^2=(3x-7)(4x-19)+42$, show that $x$ must $=4$.

(13) Prove $(a-b)^2+(b-c)^2+(c-a)^2$
$=2[(a-b)(a-c)+(b-a)(b-c)+(c-a)(c-b)]$.

N.B.—Show each side $=2(a^2+b^2+c^2-ab-bc-ca)$.

## Exercise 23.

(1) If $(A+B)(A+C)=A^2+A(B+C)+BC$, write down without actual multiplication the product of $(a+4)(a+5)$.

(2) Set down the product of

$(x+4)(x+3)$; $(x+1)(x+4)$; $(x+2)(x+3)$.

(3) Of $(x+7)(x-2)$; $(x+6)(x-4)$; $(x+9)(x-2)$; $(x+5)(x-3)$.

(4) Of $(x-7)(x+2)$; $(x-6)(x+4)$; $(x-9)(x+2)$; $(x-5)(x+3)$.

(5) Of $(x-7)(x-2)$; $(x-6)(x-4)$; $(x-9)(x-2)$; $(x-5)(x+3)$.

(6) Of $(2a+4)(2a+5)$; $(3a+4)(3a+3)$; $(4a+1)(4a+4)$.

(7) Of $(5x+2)(5x+3)$; $(6a+7)(6a-2)$; $(7x+6)(7x-4)$.

(8) Of $(9z+8w)(9z-11w)$; $(25a-100b)(25a+64b)$.

(9) Of $(3a^2+2ab)(3a^2+6ab)$; $(217ab+a^5)(217ab-2a^5)$.

(10) Solve the equation

$$(x+15)(x-3)-(x-3)^2=30-15(x-1).$$

(11) Find the value of $x$ from the equation

$$2(x+1)(x+3)+8=(2x+1)(x+5).$$

(12) $(x-1)(x+2)+(x-3)(x-4)=x(2x-4)$, find $x$.

(13) $(x+3)(x+4)(x+5)=x(x+7)^2-2(x-12)^2$.

(14) $(x-2)^2+(x-3)^2=2(x+1)(x+3)-101$.

(15) What are the factors of $a^2+a(b+c)+bc$?

(16) What are the factors of $a^2+9a+20$?

## Exercise 24.

(1) $(A+B+C)^2=A^2+B^2+C^2+2AB+2BC+2CA$, write down the square of $(x+y+z)$ without actual multiplication.

(2) Set down without multiplication the expanded values of $(a+2b+c)^2$; $(a+b+2c)^2$; $(a+2b+3c)^2$; $(2a+3b+c)^2$.

(3) $(12a+13b+14c)^2$; $(21x+22y+51)^2$; $(49x^3+36y^2+25z^4)^2$.

(4) $(A+B-C)^2=A^2+B^2+C^2+2AB-2BC-2CA$; write down the squares of $(x+y-z)$; $(x+2y-z)$; $(x+y-3a)$.

(5) Expand

$$(2a+3b-c)^2;\ (12a-13b+14c)^2;\ (21x+22y-51)^2.$$

(6) $(A-B-C)^2 = A^2+B^2+C^2-2AB+2BC-2CA$; write down the squares of $(x-y-z)$; $(a-b-2c)$; $(x-y-3z)$.

(7) Expand $(ax+by+cz)^2$; $(ax-by+cz)^2$; $(ax-by-cz)^2$.

(8) Expand $(ab+bc-ca)^2$; $(xy-yz+zx)^2$; $(ab-bc-ca)^2$.

(9) Expand $(2ab+3bc-4ca)^2$; $(3xy-4yz-5zx)^2$; $(ab-2bc-3ca)^2$.

(10) Expand $(a^2b+b^2c+c^2a)^2$; $(a^2bc+b^2ca+c^2ab)^2$; $(2a^{81}-3a^{64}+4a^{36})^2$.

(11) Expand $(a^{19}-a^{18}-a^{17})^2$; $(21a^{21}-22a^{22}-23a^{23})^2$; $(a^{2x}-b^{2y}-c^{2z})^2$.

(12) Expand $(a^{41}-41b-41)^2$; $(555a^{555}+5b^5+5)^2$; $(a^2+a^{-2}-1)^2$.

## Exercise 25.

(1) $(A+B+C+D)^2 = A^2+B^2+C^2+D^2+2AB+2AC+2AD+2BC+2BD+2CD$; write down the expansion of $(x+y+z+w)^2$.

(2) $(a+2b+3c+4d)^2$; $(2x+3y+4z+5w)^2$; $(3a+4b+5c+6d)^2$.

(3) $(x-y+z+w)^2$; $(a+b-c+d)^2$; $(x-y-z+w)^2$.

(4) $(a-b-c-d)^2$; $(a-2b-2c-2d)^2$; $(2a-3b-4c-5x)^2$.

(5) $(ax+by+cz+dw)^2$; $(ab+bc+cd+da)^2$; $(x-a+y+b)^2$.

(6) $(\frac{1}{2}a+\frac{1}{3}b)^2$; $(\frac{1}{2}a+\frac{1}{3}b+\frac{1}{4}c)^2$; $(\frac{1}{2}a+\frac{1}{3}b^2+\frac{1}{4}c^2)^2$; $(\frac{1}{2}a^2+\frac{1}{3}a+\frac{1}{4})^2$.

(7) $(\frac{1}{2}a+\frac{1}{3}b+\frac{1}{4}c+\frac{1}{5}d)^2$; $\frac{1}{2}ab+\frac{1}{3}bc+\frac{1}{4}ca)^2$; $(\frac{1}{2}xy-\frac{1}{3}yz-\frac{1}{4}zx)^2$.

(8) $\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2$; $\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a}{d}\right)^2$; $\left(\frac{a}{b}-\frac{b}{c}-\frac{c}{a}\right)^2$.

(9) Multiply $4x^2-\frac{2}{5}x+\frac{1}{25}$ by $2x+\frac{1}{5}$.

(10) Multiply $(x^2+xy+y)^2$ by $(x-y)^2$.

(11) Multiply $\frac{1}{4}x^2+\frac{1}{3}xy+\frac{2}{9}y^2$ by $\frac{1}{4}x^2-\frac{1}{3}xy+\frac{2}{9}y^2$.

(12) Simplify $(1+x)(1-x)(1+x^2)(1+x^4)(1+x^8)$.

(13) Multiply $6x^6-x^5+2x^4-2x^3+2x^2+19x+6$ by $3x^2+4x+1$.

(14) Multiply $x^3+5x^2-16x-1$ by $x^3-5x^2-16x+1$.

(15) Multiply $1+x+x^2+x^3+x^4$ by $1-x+x^3-x^7+x^8-x^{12}+x^{13}$.

### Exercise 26.

(1) When $x=2$, $y=3$, $z=4$, find the value of $3z^x + 3x^y$.

(2) What quantity added to $5a^2 - 3a + 12$ will give $9a^2 - 7$?

(3) When $a=0$, $b=1$, $c=2$, $x=8$, $y=3$, and $z=4$, find the value of $6ax + (2by - cz) - (2ax - 3by + 4cz) - (cx + az)$.

(4) Multiply $\frac{3}{2}a^2 - \frac{1}{2}ab + b^2$ by $a^2 + \frac{1}{3}ab - \frac{2}{3}b^2$, and prove your result correct by putting $a=2$, $b=3$.

(5) Show that if $5(x-3) - 7(6-x) + 3 = 24 - 3(8-x)$, then $x$ must $=6$.

(6) Solve the equation $(x+1)(2x-1) - 5x = (2x-3)(x-5) + 47$. Verify your solution by putting $x=7$, and thus reducing the equation to an **Identity.**

(7) Solve $5x - (4x-7)(3x-5) = 6 - 3(4x-9)(x-1)$.

(8) I spent $52\frac{1}{2}$ cents in eggs, and 9 of them cost as much over 12 cents as 15 cost under 24 cents. How many did I buy?

(9) $A$ and $B$ have \$42; $B$ and $C$, \$45; $C$ and $A$, \$53. How much has each?

(10) If $x=b+c$; $y=c-a$; $z=a-b$; find the value of

$$x^2 + y^2 + z^2 - 2xy - 2zx + 2yz.$$

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NOTE.—When the beginner has done the preceding exercises and has well understood them, **Division follows easily as the reverse process of Multiplication. The Rule of Signs** and **the Rule of Exponents** are precisely the same for both, and are best learned from particular examples. The following exercises, up to Exercise 31, are intended to develop the subject pretty rapidly, the pupil's previous experience making fewer simple questions necessary. **Great stress needs to be laid on methods of verifying and testing.** A long complicated process can often be tested in a few seconds, and in every case where it is possible the learner should test his own work and thus acquire confidence in his results. It is as yet premature to introduce the short forms of multiplication and division. These should be deferred until **skill, accuracy, and rapidity are ensured by constant practice in writing out the full forms**—otherwise they will be learned mechanically and without any proper grasp of their real meaning. If generalisation proceeds too rapidly the chances are that the pupil will fail to apprehend the steps.

## Exercise 27.

N.B.—*Read $a \div b$ thus: $a$* BY *$b$. Avoid mathematical* SLANG.

(1) $a^2 \times a^3 = a^5$. If $a^5$ is divided by $a^3$ what is the quotient?

(2) $a^4 = aaaa$; $a^3 = aaa$; what is the quotient of $a^7 \div a^4$?

(3) If $a^6 \div a^3 = a^{6-3}$; $a^7 \div a^4 = a^{7-3}$; $a^8 \div a^2 = a^{8-2}$; what is the quotient when $a^x \div a^y$? Of $a^m \div a^n$? Of $a^w \div a^k$?

(4) If $2a \times 3b = 6ab$; what is the value of
$6ab \div 2a$? Of $6ab \div 3b$?

(5) If $(+2a)(-3b) = -6ab = (-2a)(+3b)$; find the quotients when $6ab$ is divided by $+2a$, $-3b$, $-2a$, and $+3b$ respectively.

(6) Find the price of $4x$ horses at \$$6y$ apiece. Tell how many horses at \$$6x$ apiece can be bought for \$$24xy$. How many can be bought at \$$4y$ apiece?

(7) Choose an example and **show that every question in multiplication will give two examples of division.**

(8) From the example $a^{11} \div a^9 = a^2$, make a rule for the exponents in a case of division. See whether your rule holds good in these cases: $a^{11} \div a^{11}$; $a^m \div a^m$.

(9) Divide $a^{29}$ by $a^{17}$; $a^{111}$ by $a^{74}$;
$a^{123456789}$ by $a^{98765432}$.

(10) Divide $111x^{111}$ by $37x^{37}$; $512x^{196}$ by $256x^{189}$;
$1331a^{1331}$ by $121a^{121}$.

## Exercise 28.

(1) Multiply $a+b$ by $c$. Verify the result by division.

(2) Multiply $a+2b+3c$ by $4a$. Verify the result by division.

(3) Multiply $a^2+2ab+b^2$ by $a+b$. Divide the product by $a+b$, and by $a^2+2ab+b^2$.

(4) Write down the expansion of $(a+b)^4$. Divide the product by $a+b$, by $a^2+2ab+b^2$, and by $a^3+3a^2b+3ab^2+b^3$, three answers.

(5) Set down the expansion of $(a+b)^6$.
Divide this by $a^3+3a^2b+3ab^2+b^3$,
divide this quotient by $a^2+2ab+b^2$.

(6) Multiply $a^3+3a^2b+3ab^2+b^3$ by $a^3-3a^2b+3ab^2-b^3$ and divide the product by $a^4-2a^2b^2+b^4$.

(7) Multiply the expansion of $(a+b)^4$ by the expansion of $(a-b)^4$ and divide the product by $a^6-3a^4b^2+3a^2b^4-b^6$.

(8) Divide $1+2x$ by $1-3x$ to 4 terms of the quotient.

(9) Divide $27a^4-6a^2+\frac{1}{3}$ by $3a^2+2a+\frac{1}{3}$.

(10) Divide $6-2\frac{3}{4}x+4\frac{1}{8}x^2-1\frac{1}{2}x^3+\frac{1}{3}x^4$ by $1-\frac{5}{6}x+\frac{2}{3}x^2$.

## Exercise 29.

(1) Find the quotient when $24x^2-65xy+21y^2$ is divided by $8x-3y$. Prove the result by multiplication.

(2) One factor of $16x^3-46x^2+39x-9$ is $2x^2-5x+3$, find the other factor.

(3) Arrange $12+a^4-7a^2-2a^3+8a$ and $-a+a^2-6$ in proper order of the powers of $a$ and find the quotient. Verify the result.

(4) Arrange $4a^3-a-a^4-5a^2+6a^5-15$ and divide it by $-a+3+2a^2$.

(5) Find the remainder when $x^3+0-7x+5$ is the divisor and $23x^2-2x^4-4x^3+12+x^5-31x$ is the dividend. Verify your answer by multiplication.

(6) If $12x^4+5x^3-33x^2-3x+16$ is divided by $4x^2-x-5$, what is the remainder? Confirm your answer by a check calculation.

(7) $15a^4+22-32a^3-30a+50a^2\div(3-4a+5a^2)$.
Remainder = ?

(8) $x^8+1$ divided by $x^3+x^2+x+1$. Find the remainder. Give proof.

(9) $a^9+0+0+0+0-6x^4-8x^3+0+0-1\div(a^3-2a-1)$.
Quotient?

(10) $3a^2+8ab+4b^2+10ac+8bc+3c^2\div(a+2b+3c)$. Quotient?

## Exercise 30.

(1) When $8x^5-11+x^7+3x-10x^6-7x^3$ is divided by $x^2+4-5x$, what is the remainder?

(2) Multiply $a^3+2a^2-3a-6$ by $a^3-2a^2-3a+6$, and divide the product by $a^4-7a^2+12$. Put $a=1$ throughout and thus verify the quotient.

(3) Prove by multiplication that

$$(x^2+3x+1)^2=a(a+1)\ (a+2)\ (a+3)+1 \text{ when } x=a.$$

(4) Show that $x^3+y^3+z^3-3xyz=2(a^3+b^3+c^3-3abc)$ when $x=b+c$, $y=c+a$, $z=a+b$.

(5) If the product of $4x^3+6x^2-18x+27$ and another factor is equal to $4x^6+11x^4+81$, find the second factor.

(6) Simplify the expression

$$(a+b+c)(a+b+d)+(a+c+d)(b+c+d)-(a+b+c+d)^2.$$

(7) $(x+y+z)^3+(x-y-z)^3+(y-z-x)^3+(z-y-x)^3$.

Reduce this and prove your result by putting $x=y=z=1$.

(8) What number must be subtracted from $2a^6+2$ to make it exactly divisible by $a^3+2a^2+2a+1$?

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NOTE.—Exercises 31 to 36 supply **a review** of all the work passed over. The same simple principles are here applied in somewhat more difficult combinations, but all really hard questions are excluded. In numerous examples **methods of testing are suggested,** and these should receive very careful attention. In all written examinations they are of first-rate importance. **Pupils should be encouraged to invent new problems** like those in Exercise 32, and to select questions from their arithmetics which may easily be solved by means of simple equations. As a rule **the algebraic analysis of a problem is easier than the arithmetical analysis,** and the learner will soon find that he can do questions in percentage, discount, interest, and general analysis that would be beyond his power by pure arithmetic. He should be led by **the path of least resistance** independently of any formal arrangement of the subject as a science. If he learns the easiest things first, difficulties will melt away as he approaches them.

N.B.—**Algebraic solutions are fully accepted at all examinations in Ontario.**

## Exercise 31.

(1) Divide $a^4+4b^4$ by $a^2-2ab+2b^2$.

(2) $(3x-8)(3x+2)-(4x-11)(2x+1)=(x-3)(x+7)$, find $x$.

(3) $A$, $B$ and $C$ have \$168 among them; $C$'s share is three-fourth of $A$'s share, and $A$'s is \$8 more than $B$'s share. What has each?

(4) $A$ is now three times as old as $B$, but only two years ago he was five times as old as $B$ was four years ago. Find $A$'s present age.

(5) What value of $x$ will make the product of $x+3$ and $2x+3$ exceed the product of $x+1$ and $2x+1$ by 14?

(6) If $W=5a+4b-6c$, $X=-3a-9b+7c$, $Y=20a+7b-5c$, and $Z=13a-5b+9c$, evaluate $W-X-Y+Z$. Verify the result by putting $a=b=c=1$.

(7) A man rides $a$ miles in a buggy and $b$ miles in a train; the buggy goes at 7 and the train at 25 miles per hour. How long does the journey take? Verify your answer by taking $a=35$ and $b=150$.

(8) A farmer sells 9 horses and 7 cows for \$375, and he values 6 horses and 13 cows at the same price. Find the value of one cow.

## Exercise 32.

(1) A tank capable of holding 648 gallons may be filled by two pipes in 18 minutes, and the second pipe runs in 6 gallons per minute more than the other. How many gallons per minute did the first pipe convey?

(2) A boy spent half his money and half a dollar more, then half the remainder and half a dollar more, and lastly half the remainder and half a dollar more. He had now \$2 left, find how much he spent.

(3) A marker standing behind the target hears the bullet strike and $\frac{3}{10}$ of a second afterwards he hears the report of the rifle. If the bullet travelled at 1,430 feet per second and sound at 1,100 feet, find the length of the range.

(4) In a battle the general lost $\frac{1}{6}$ of his army in killed and wounded, besides 4,000 taken prisoners. He receives a reinforcement of 3,000 men, but in retreating loses again $\frac{1}{4}$ of his troops, leaving him only 18,000. Find the strength of his army at first.

(5) An investor places \$13,000 at interest, partly at 4% and partly at 5%, and his income is \$550. How much does he invest at each rate?

(6) Forming his men into a solid square, an officer finds 34 men over; but when he increases the side of the square by one man, he lacks 39 men to complete the square. How many men were on parade?

(7) $A$ and $B$ are 154 miles apart on a river, $A$ rows 3 miles in 2 hours on the average, including stoppages, and $B$ 5 miles in 4 hours. In how many hours will they meet?

(8) $\frac{1}{14}(8-2x)+\frac{1}{7}(6x-5)=\frac{3}{14}(2x+6)-\frac{1}{7}(2x)$, find $x$.

(9) $\frac{1}{2}x-\frac{2}{3}x+\frac{3}{4}x-\frac{1}{6}x+\frac{7}{8}(x-3)=0$, find $x$.

(10) Show that $ax+by+cz=(x+y+z)(a+b+c)$, if
$x=a^2-bc,\ y=b^2-ca,\ z=c^2-ab$.

(11) If $x$, $y$ and $z$ have the same values as in No. 10, prove that $bc(x^2-yz)=ca(y^2-zx)=ab(z^2-xy)$.

## Exercise 33.

(1) Find the numerical value of the following expression

$$\frac{x^2+y^2-z^2}{x-y+z}-\frac{x^2-y^2-z^2}{x+y+z} \text{ when } x=4,\ y=5,\ z=6.$$

(2) Show that when $x=5,\ y=3,\ z=4$
$(x^2+y^2-z)(x^2-y^2)$ is equal to $x^4-x^2z+y^2z-y^4$.

(3) Express in shortest form the sum of
$a^3x+a^2x^2-2ax^3$; $x^4+15+2ax^3-3a^2x^2$;
$2a^3x-4ax^3+3x^4+a^3x$; and $15+5ax^3-2a^3x+4ax^3-10$.

(4) Combine into one sum $\frac{1}{3}x^2-\frac{1}{4}x^2y+\frac{2}{3}y^2$; $2x^2y-x^2+\frac{2}{3}x^2$; and $\frac{1}{2}x^2y-\frac{1}{3}y^2+3x^2+2y^2$.

(5) When $x=2,\ y=3,\ z=4$, find the numerical value of

$$\frac{xyz}{2x+2z}-\frac{3x^2+y^2-z^2}{x^2y^2z-139}+2xyz-x+y+z.$$

(6) Simplify $(a-2)^3-2(a-2)^2+3(a-2)-4$.

(7) Solve the equation $6(x-1)+8(x+2)=27(x-3)$. Verify your answer by substitution.

## Exercise 34.

(1) Divide $8x^4-2ax^3+3a^2x^2-2a^3x+a^4$ by $2x^2+ax+a^2$, and prove that your quotient is accurate by letting $x=1$ and $a=1$.

(2) Find the product of $a-2b+1$; $2a+b-1$; and $2a+3b+1$. Prove the result by putting $a=2$, and $b=1$.

(3) Subtract $2b^2x^2+6bcx^3-12cx^5-12x^6$ from the product of $2bx+3cx^2-4x^3$ and $2bx+4x^3$.

(4) $1+xy+x^2y^2+x^3y^3$; multiply this by $1-xy+x^4y^4-x^5y^5$. Test your product by putting $x=y=1$.

(5) If two algebraic expressions are written close together without any sign between them, what operation is intended?

(6) In the expression $(m+n)^3-3(m+n)\ (m-n)+2(m+n)+m-n$, substitute $x$ for $m+n$, and $y$ for $m-n$. Suppose $m=n$, what is the value of $y$? What is the value of the whole expression?

(7) $(a+b)\ (a-b)=a^2-b^2$. Apply this principle to find the product of $m-n+p-q$ and $m-n-p+q$ without actual multiplication.

(8) Get the product of $x^3-m^2y+y^3$ and $x^3-m^2y-y^3$ without actual multiplication. Test your result. See Exercise 23.

## Exercise 35.

(1) Is $x^2+(x+1)^2+(x^2+x)^2$ greater than, equal to, or less than the square of $x^2+x+1$. Put $x=1$ and test your answer.

(2) Divide $3a^{3x}$ by $\frac{1}{3}a^{\frac{1}{3}x}$; and $9a^{9a}$ by $3a^{3a}$; and $11^{11}$ by $11^7$.

(3) Divide $x^4+2x^3y+xy^3+2y^4-3x^3z-3y^3z$ by $x+2y-3z$.

(4) Divide $x^7+2,187$ by $x+3$.

(5) Show that $x^2(y+z)+y^2(z+x)+z^2(x+y)+2xyz$ is another form of the expression $(x+y)\ (y+z)\ (z+x)$. Test the truth of your result by making $x=y=z=1$.

(6) Prove that $(9x^2-4y^2)\ (81x^4+16y^4+36x^2y^2)$ gives the same product as $27x^3+8y^3$ multiplied by $27x^3-8y^3$. Verify your answer by putting $x=y=1$.

(7) Divide $(x+y)^2+3(x+y)z+2z^2$ by $x+y+z$ without expansion in this way:—Put $m$ for $x+y$; find the quotient; then write $x+y$ for $m$ in this quotient.

(8) In $9(x+3)^2+16(x+1)^2=(5x+9)^2$, find the value of $x$.

(9) Is $(a^2+c^2)\ (b^2+d^2)$ greater or less than $(ab+cd)^2$, when $ad=bc$?

## Exercise 36.

(1) If I take 5 steps toward the north, then 6 steps, then 7 steps, express by means of the sign + my distance from the starting point.

(2) If I turn round and reverse my steps express by means of a sign the *distance* I have walked and also the *direction* (south) in which I have walked.

(3) If $+a$ means $1+1+1+1+$etc., what does $-a$ mean?

(4) If $(+3)\ (+4)$ means $+3+3+3+3$, or $+12$, what does $(-3)\ (+4)$ mean? What does $(+3)\ (-4)$ mean?

(5) State *the Rule of Signs* in multiplication. Show that the same rule applies to division.

(6) If $(-3)(-4)$ means $-3$ *subtracted* 4 times, or $=(-3)-(-3)-(-3)-(-3)$, remove the brackets and reduce the expression to $+12$. Explain the meaning of $(-a)(-b)$.

(7) Distinguish clearly the difference in meaning between *arithmetical* figures as 1, 2, 3, etc., and *algebraical* characters used to represent numbers, as $a$, $b$, $x$, $y$, etc.

(8) What is the name given to an expression like $6x+11=31$?

(9) How would you get rid of fractions in the equation $\frac{1}{3}x+\frac{1}{5}x+\frac{1}{7}x+\frac{1}{9}x+\frac{1}{11}x=2187$? State the **axiom** used?

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NOTE.—In these exercises the *formal*, *scientific* arrangement of the subject as usually presented has been discarded for the sake of presenting the subject in the form suited best to the requirements of a beginner, viz., that order which will introduce him first to the easiest parts of the science, next to those somewhat difficult, and last of all to the topics that require most maturity of thought and most skill in the application of abstract principles. **To connect the new with the old, to proceed gradually from the pupil's knowledge of numbers to the unknown abstractions of algebra, to repeat the old in new forms, to select the easiest things first—this is surely the common sense of good teaching.** Guided by experience, we therefore introduce the simple equation of one unknown among the first things to be learned. **The simple equation of two unknown quantities is also easy to learn and is here introduced with Exercise 37.** The pupil thus acquires an insight into the practical applications of algebra that formal treatises defer to about page 150. All really difficult questions are excluded as foreign to our purpose, and with a little patience and care the pupil can do the exercises without much help.

## Exercise 37.

(1) An equation containing *only one* unknown quantity, as $x$, is a **Simple Equation.** If there are several equations which contain *more than one* unknown quantity, as $x+y=6$, and $2x+3y=16$, with common values, as $x=2$, $y=4$, these equations are called **Simultaneous Equations.**

(2) Solve the simultaneous equations $x+y=6$; $2x+3y=16$ in this way:—Multiply the 1st by 2 and subtract the product from the 2nd. Next multiply the 1st by 3 and subtract the product from the 2nd.

(3) Find the values of $x$ and $y$ in the following equations:— $2x-y=9$; $2y-x=3$.

(4) $x+y=3$; $2x+y=4$.

(5) $7x-3y=19$; $4x+7y=37$.

HINT.—Multiply 1st by 4, 2nd by 7.

(6) $7x-9y=5$; $13x+4y=30$.

(7) $4x+y=11$; $x+4y=14$.

(8) $2x+3y=21$; $3x+5y=34$.

(9) $3x=23-2y$; $10+2x=5y$.

HINT.—Transpose $x$ and $y$ to one side.

(10) $\frac{x}{2}+\frac{y}{3}=7$; $\frac{x}{3}+\frac{y}{2}=8$. HINT.—Clear of fractions first.

(11) $3x-2y=3(6-x)$; $3(4x-3y)=7y$.

## Exercise 38.

(1) Find two numbers whose difference is 7 and sum 33.

(2) Divide 150 into two parts such that three times the greater part exceeds seven times the less by 15. N.B. $x+y=150$, etc.

(3) I can get 7 pounds of tea and 5 pounds of coffee for \$5.50, or I can buy 6 pounds of tea and 3 pounds of coffee for \$4.20. What is the price per pound of each?

N.B. $7x+5y=550$; $6x+3y=420$.

(4) $A$ and $B$ have together \$570; but if $A$ had three times as much and $B$ five times as much as at present, they would have altogether \$2,350. Find the amount each now has.

(5) A man bought 100 acres for \$2,450, paying \$20 an acre for one part and \$30 an acre for the remainder. How many acres were there in each part?

(6) $A$ and $B$ have \$9,800. $A$ invests $\frac{1}{6}$ of his money and $B$ $\frac{1}{5}$ of his, and then each has the same sum left. Find their capitals.

(7) A man invested \$4,400, partly in bonds paying 3% interest and partly in bank stock paying $2\frac{1}{2}$% interest. His income was the same from each; find the amount of the bonds and of the stock, which were both bought at par.

## Exercise 39.

(1) Solve the equations $28x-23y=22$, and $63x-55y=17$ in this way:—L. C. M. of 28 and $63=252$; $\therefore$ multiply 1st by 9 and 2nd by 4.

(2) Solve $13x-9y=46$, and $11x-12y=17$. N.B.—L. C. M. of 9 and 12 is 36. Multiply 1st by 4, 2nd by 3, and thus employ the smallest numbers possible.

(3) Solve $14x+13y=35$, and $21x+19y=56$.
N.B.—Multiply by 3; 2.

(4) Solve $11x+13y=7$, and $13x+11y=17$.
N.B.—Add; divide by 24, and $x+y=1$, $\therefore 11x+11y=11$, etc.

(5) 15 pounds of tea and 17 pounds of coffee cost \$7.86, and 25 pounds of tea and 13 pounds of coffee cost \$10.34. What is the price of each per pound?
N.B.—Multiply 1st equation by 5, 2nd by 3, etc.

(6) A man invested \$100,000, partly at 5% and partly at 4%. His interest amounted to \$4,640 per annum. Find how much he placed at each rate.

(7) Bought a number of eggs at 2 for a penny, and as many more at 3 for a penny. Sold the whole at 5 for 2 pence, and lost 4 pence by the transaction. How many at each price did I buy? N.B.—Let $60x=$ number bought.

## Exercise 40.

(1) A boy bought 400 oranges, some at 3 for 2 cents, and some at 2 for 3 cents. The whole lot cost him 1 cent apiece. How many did he buy at each price?

(2) *A* can do a piece of work in 12 days; *B* and *C* together can do it in 4 days; *A* and *C* can do it in half the time that *B* alone can do it. How long will *B* and *C* each take to do the job?

(3) *A* and *B* have equal sums when they begin to gamble. At the end of the play it is seen that if *A* had \$6 more he would have three times as much as *B* has, and if *A* had \$16 less he would have twice what *B* has. How much does *A* win? What had each at first?

(4) Sold 9 horses and 7 cows for \$300, and 6 horses and 13 cows at the same price for the same sum. Find the selling price of each.

(5) Bought two grades of sugar; 7 pounds of the first cost the same as 9 pounds of the second, and 9 pounds of the first and 7 pounds of the second cost together 65 cents. Find the price per pound of each grade.

(6) A room is twice as long as it is broad, and it takes the same quantity of carpet as another room 10 feet shorter and 9 feet broader. Find the length of the room.

(7) Bought 12 cows and 20 lambs for \$335, also 10 cows and 26 lambs for the same sum, paying \$2 more apiece for the cows and 75 cents more for each lamb. Find the price of each cow and lamb in the first lot.

(8) If Tom gives Harry \$10, the latter will have three times as much as Tom has left. If Harry gives Tom \$10, Tom will then have twice as much as Harry. How much has each?

(9) How much tea at 72 cents and at 40 cents respectively must be taken to make a mixture of 50 pounds worth 60 cents per pound?

(10) I have \$12,750 to invest. I can buy 3% bonds at 81 and 5% bonds at 120. How much must I spend on each kind so that the income may be the same for each?

(11) At simple interest a certain sum amounted to \$$a$ in $m$ years, and to \$$b$ in $n$ years. Find the sum and the rate.

## Exercise 41.

(1) If $\frac{x}{a}+\frac{x}{b}=c$, multiply both sides by $ab$, and thus show that $x=abc\div(a+b)$.

(2) The sum of two number is 85, and three times their difference is 81. Show that three times the smaller number is 87.

(3) Divide \$42 among three persons, so that $B$ may have \$5 more than $A$, and $C$ may have as much as $A$ and $B$ together.

(4) How many pounds of tea at 24 cents per pound, and at 42 cents per pound, must be mixed to produce 100 pounds worth 30 cents per pound.

(5) Loaned \$1,000, part at 4% and the rest at 5%; interest on the whole was \$44. How much was loaned at 4%?

(6) A hare takes 4 leaps in the same time that a hound takes 3 leaps, but 3 of the hare's take her only the same distance as 2 of the dog's. The hare has made 50 leaps before the dog is loosed; how many springs will the dog make on a straight course to

catch puss? N.B.—Let $4x$ and $3x$ be the number of leaps taken by hare and hound, and $2y$ and $3y$ be the length of each leap, hence hare goes over $8xy$ while dog goes over $9xy$, *i.e.*, he gains $y$ in going $9y$, etc.

(7) Solve $\frac{x-5}{x-7}=\frac{x+3}{x+9}$. N.B.—Multiply by $(x-7)(x+9)$.

## Exercise 42.

(1) Two men own together 175 shares in a company. They divide equally. *A* keeps 85 shares and *B* 90; *B* gives *A* \$100. Find the value of a share.

(2) If 8 gold coins and 9 silver coins are worth as much as 6 gold coins and 19 silver ones, find the number of silver coins equal in value to one gold coin.

(3) Brown sets outs from X to Z at $3\frac{1}{2}$ miles per hour, and forty minutes later Jones starts from Z to X at $4\frac{1}{2}$ miles per hour. When they meet Jones has gone half a mile beyond the middle point between X and Z. Find the distance between the two places.

(4) An express train going at 32 miles per hour makes a certain trip in 2 hours and 15 minutes less than the time required by a freight train going at 14 miles per hour. Find the distance.

(5) Solve the equations $17x-13y=144$; $23x+19y=890$.

(6) The sum of two numbers is 13; the difference of their squares is 39; find the numbers.

(7) Bought a certain number of shares for \$2,625; sold them all but five at double the cost price for \$4,000. How many shares?

(8) Divide \$7,500 so as to give each of two sons twice as much as each of their three sisters, and their mother \$500 more than all her children.

(9) If \$1,024 amounts to \$1,156 in two years at compound interest, what is the rate of interest per annum?

(10) The sum of \$145 is distributed to 17 boys and 15 girls; and at the same rates it might have been given to 13 boys and 20 girls. Find the share of each boy and each girl.

(11) In 9 hours a horse travels one mile more than a train does in 2 hours, but in 3 hours the train goes 2 miles more than the horse goes in 13 hours. Find the rate of each per hour.

(12) $19x+17y=7$; $41x+37y=17$; solve the equations.

(13) $13x-17y=11$; $29x-39y=17$; solve for $x$ and $y$.

NOTE.—**Exercises 43 to 50** are intended to sum up and test what the pupil has now mastered. They **are of the nature of a written examination.** He is recommended to **do them over three times** before entering on the SECOND STAGE:—The **first time** to work them out accurately and to master them, the **second time** to go through them thoroughly at moderate speed without any break, **and finally to work the whole at high speed** to make a record against time. This practice in writing out solutions at high speed is **a fine piece of mental gymnastics** and soon produces a firm, rapid "touch," and gives the student the indomitable courage and "attack" that secure success when he has to face new problems on the official examination papers. **The habit of doing all lessons in a certain limited time** is of the greatest assistance in every branch of study, but particularly in mathematics. It economises time wonderfully and is in fact a useful stimulant to the growing powers of the mind. **Try it and see.**

### Exercise 43.—GENERAL REVIEW.

(1) If $a=1$, $b=3$, $c=5$, find the value of :—

$$\frac{a^2b^2+1}{a^2+b^2}-\frac{1+a^2c^2}{a^2+c^2}+\frac{4a+b^2+b^2c^2}{b^2+c^2}-\frac{a^2+2ab+b^2}{b^2-2bc+c^2}.$$

(2) Shew that

$\{(b+c)-(d-a)\}^2+\{(c+d)-(b-a)\}^2+\{(b+d)-(c-a)\}^2$
$+(b+c+d-a)^2=4(a^2+b^2+c^2+d^2)$, when
$a=1$, $b=2$, $c=3$, and $d=4$.

(3) Find the sum of

$x^4+3xy^3-xz^3+x^3y+x^3z+3x^2y^2+3x^2z^2$
$+3xy^2z-3xyz^2-6x^2yz-x^3y+y^4-yz^3-3x^2y^2+3x^2yz$
$-3xy^3-3xyz^2-3y^3z+3y^2z^2-6xy^2z-x^3z+3y^3z+z^4$
$+3x^2yz-3x^2z^2+3xy^2z+xz^3-3y^2z^2+yz^3+6xyz^2$.

(5) From $a^5-4a^3b^2-8a^2b^3-17ab^4-12b^5$ subtract successively $a^5-2a^4b-3a^3b^2$ ; $2a^4b-4a^3b^2-6a^2b^3$ ;
$3a^3b^2-6a^2b^3-9ab^4$ ; and $4a^2b^3-8ab^4-12b^5$.

(6) Reduce to simplest form $\{2a-(3b+c-2d)\}$
$-\{(2a-3b)+(c-2d)\}+\{2a-(3b+c)-2d\}$
$-\{(2a-3b+c)-2d\}$.

### Exercise 44.

(1) Put brackets round the quantities containing like powers of $x$, in the expression

$$ax^3-bx^2-cx-bx^3+cx^2-dx+cx^3-dx^2-ex.$$

Put $x$ outside each bracket.

(2) Multiply $x^2+y^2+z^2+xy-xz+yz$ by $x-y+z$.

(3) Multiply $a^2+4b^2+9c^2+2ab+3ac-6bc$ by $a-2b-3c$.

(4) Multiply $a^4-2a^3b+3a^2b^2-2ab^3+b^4$ by $a^2+2ab+b^2$.

(5) Multiply $a^2+ax+x^2$ by $a^2-ax+x^2$.

**Learn this product by heart.**

(6) Multiply $a-2b+3c+d$ by $a+2b-3c+d$.

(7) Divide $1+6x^5+5x^6$ by $1+2x+x^2$.

(8) Divide $a^6-6a+5$ by $a^2-2a+1$.

(9) Divide $x^6-2x^3+1$ by $x^2-2x+1$.

(10) Divide $a^6+2a^3b^3+b^6$ by $a^2+2ab+b^2$.

(11) Divide $a^2+ab+2ac-2b^2+7bc-3c^2$ by $a-b+3c$.

(12) Find the remainder when $x^3-px^2+qx-r$ is divided by $x-a$.

(13) Divide to four terms of quotient and give the remainder in $1\div(1-2x+x^2)$.

(14) Solve the equation $\frac{1}{2}(3x^2+x)-\frac{1}{3}(2x^2+x)+\frac{1}{4}(x^2+x)-2\frac{3}{20}$ $=x^2+\frac{2}{15}+\frac{1}{6}(x^2+x)-\frac{1}{12}(x^2+5x)$.
N.B. – Multiply through by 60.

## Exercise 45.

(1) $\dfrac{2x+3}{11}-\dfrac{x-12}{3}+\dfrac{3x+1}{4}=5\frac{1}{3}+\dfrac{4x+3}{12}$, find $x$.

HINT.—Multiply through by 12; collect all like terms and get $\frac{12}{11}(2x+3)+x=16$, etc.

(2) $\frac{1}{12}(2x-3)-\frac{1}{6}(3x-2)=\frac{1}{8}(4x-3)-3\frac{5}{24}$, find $x$.
N.B.—Use 24.

(3) $\frac{5}{7}(x-9)+\frac{7}{9}(x-5)=\frac{9}{5}(x-7)+1\frac{2}{3}$. N.B.—Use 9, then 35.

(4) $\frac{1}{8}(7x+20)-\frac{3}{16}(3x+4)=\frac{1}{10}(3x+1)-\frac{1}{20}(29-8x)$.
N.B. $\frac{1}{8}=\frac{2}{16}$; $\frac{1}{10}=\frac{2}{20}$; add each side separately; multiply through by 4, and get $\frac{1}{4}(5x+28)=\frac{1}{5}(14x-27)$.

(5) $\frac{1}{16}(2x-1)-\frac{1}{16}(3x-2)=\frac{1}{18}(x-12)-\frac{1}{24}(x+12)$.
N.B.—Use 144, and get $\frac{5}{48}(2x-1)=$ etc.

(6) Divide 620 marbles among $A$, $B$ and $C$. Give $A$ 4 for every 3 to $B$, and $A$ 6 for every 5 to $C$.
N.B.—Let $12x=A$'s share, etc.

(7) $(x+1)(x+3)(x+5)=(x+7)(x+9)(x-7)$, find $x$.

(8) Water expands 10% when it turns to ice; how much per cent. does ice contract when it turns to water?
N.B.—Let $100x=$ volume of water, $\therefore$ $110x=$ etc., etc.

## Exercise 46.

(1) Write down the square of $1+2x-x^2-\frac{1}{2}x^3$ without actual multiplication.

(2) Multiply $(a+b+c)\,(a+b-c)$ by $(a-b+c)\,(b+c-a)$.

(3) Divide $1-\frac{1}{2}x$ by $1-\frac{1}{3}x-\frac{1}{4}x^2$ to five terms of quotient.
HINT.—Multiply both divisor and dividend by 12 before dividing.

(4) Remove the brackets and simplify the expression
$$\tfrac{1}{4}\{\tfrac{1}{6}a-(b-a)\}-\tfrac{1}{2}[(b-\tfrac{1}{5}a)-\tfrac{2}{3}\{a-\tfrac{3}{4}(b-\tfrac{4}{5}a)\}].$$

(5) Expand and simplify the expression
$$\{a-(b-c)\}^2+\{b-(c-a)\}^2+\{c-(a-b)\}^2.$$
HINT.—Remove the inner brackets, then write down the expansions under one another and add.

(6) Test the accuracy of your result in No. 5 by putting $a=b=c=1$.

(7) If $x=\frac{1}{2}$, and $x+y=x+y+z=0$, find the value of the expression $(y^2-z^2)\{y^2+z^2-y(x-z)\}$.

(8) Divide \$5,000 among $A$, $B$, $C$, so that $B$ shall get twice as much as $C$, and $A$ three times as much as $B$, after paying \$20 to $D$. N.B.—Let $x=C$'s.

(9) If $a-b=x=3$, and $a+b+x=2$, what is the value of
$$(a-b)\,\{x^3-2ax^2+a^2x-(a+b)b^2\}?$$

(10) $\frac{1}{13}(6x+18)-4\frac{5}{6}-\frac{1}{36}(11-3x)$
$=5x-48-\frac{1}{12}(13-x)-\frac{1}{18}(21-2x)$.
N.B.—Multiply through by 36, and then collect terms, next multiply by 13.

## Exercise 47.

(1) Find the difference between these two expressions:—
$(n+2)\,(n+3)\,(n+4)$ and
$24\{n-\frac{1}{2}(n-1)\}\,\{n-\frac{2}{3}(n-2)\}\,\{n-\frac{3}{4}(n-1\frac{1}{3})\}$.
HINT.—In the 2nd quantity $24=2\times3\times4$. Multiply the first bracket by 2, the next by 3, and the last by 4, and reduce.

(2) What multiplier will give with $5x+4$ the product $60x^2+53x+4$?

(3) Find the coefficient of $x$ in the product of
$(x+2)\,(x+10)\,(x-5)\,(x-6)$.

(4) $A$ leaves Seaforth at 6 a.m. at 3 miles per hour, at 8 a.m. $B$ follows him at 4 miles per hour and overtakes him at Stratford. Find the distance between the places.

(5) Two vessels contain a mixture of water and wine; in $A$ there are 2 of water to 3 of wine, in $B$ 3 of water to 7 of wine. It is required to form a mixture containing 5 gallons of water and 11 of wine. How many gallons must be taken from $A$ and $B$ respectively? N.B.—Let $50x$ and $50y$ be the numbers from each, $\therefore\ 4x+3y=1$, etc.

(6) A pound of tea and 3 pounds of sugar cost 60 cents, but when tea is 10% dearer and sugar 50% dearer they cost 70 cents; find the prices.

## Exercise 48.

(1) If $2a=3b$ find the numerical value of $\dfrac{a-b}{a+b}$ and of $\dfrac{a^2-b^2}{a^2+b^2}$.

N.B.—Multiply the terms of the fractions by 2, and by 4 respectively before substitution.

(2) $\frac{1}{14}(3x+7)-\frac{1}{21}(2x-7)=\frac{1}{4}(x-4)-2\frac{3}{4}$, find $x$.
N.B.—Add each side separately and reduce.

(3) A stage makes the trip in 5 hours, but if the speed were 3 miles an hour greater the trip could be made in $3\frac{1}{2}$ hours; what is the length of the journey?

(4) If $a$ horses are worth $b$ cows, and $c$ cows are worth $d$ sheep, find the value of a horse when a sheep sells for $10.

(5) $32x+81y=43$, $28x-39y=1$, find $x$ and $y$. *See Note.*

(6) $93x+75y=102$, $92x+80y=101$, find $x$ and $y$.
N.B.—In 5 and 6 subtract the equations first, then multiply, etc.

(7) If $a$ gallons at $p$ shillings, $b$ gallons at $q$ shillings, and $c$ gallons at $r$ shillings are mixed, what will be the value of one gallon of the mixture?

(8) Bronze contains 70 pounds of copper to 30 pounds of tin. how much copper must be added to 112 pounds of bronze to make the mixture contain 16% of tin?

## Exercise 49.

(1) Simplify $(2x-1)(2x^2+\frac{1}{2})(2x+1)$.

(2) Reduce $a^2(b+c)+b^2(c+a)+c^2(a+b)-(a+b)(b+c)(c+a)$.
Test your results in No. 1 and No. 2 by putting $x$, $a$, $b$, $c$, each $=1$.

(3) Show that $x=2$ is one solution of the equation,
$$x^3+8=2x^2+11x-14.$$

(4) What is the L. C. M. of $3x^2y$; $2xy^3$; $5xyz$; and $4x^2y$?

(5) $21(x-2)+28(2x-6)=14(x-6)$, find $x$.

(6) What is the G.C.M. of $84x^4y^5z^6$; $60x^3y^8z^9$; $132x^8y^4z^{10}$?

(7) Without multiplying out the brackets, show that
$$(p+q+r)^2-(p^2+q^2+r^2)=p(q+r)+q(p+r)+r(p+q).$$

(8) Give the L. C. M. of $(a-b)(a-c)$; $(b-a)(b-c)$; and $(c-a)(c-b)$ as the product of three factors.

(9) The populations of 5 towns are $a+b$; $b+c$; $c+d$; $d+e$; and $e+a$. Find the average population of each town.

(10) Sold a cow for \$42 at a loss; if I had sold her for \$57 the gain would have been four times my present loss. Find the cost of the cow.

(11) If $x=a+b$; $y=a-b$; find the value of
$$(x^2-y^2)\div(x^2+y^2).$$

## Exercise 30.

(1) Show that $(x+1)(x+2)(x+3)(x+4)-24$
$$-(x+4)(x+2)(x^2+5x+8)=0.$$

(2) Show that $(x+1)(x+3)(x+5)(x+7)+15$ divided by
$$(x+2)(x+6)(x^2+8x+10)=1.$$

(3) Solve the equations, $45x+8y=350$; $21y-13x=132$.

(4) Find the values of $x$ and $y$ from the equations,
$$\tfrac{1}{8}x+\tfrac{1}{9}y=42;\ \tfrac{1}{9}x+\tfrac{1}{8}y=43.$$

(5) Three pounds of one sort of tea cost as much as four pounds of a second kind; nine pounds of the first and eight pounds of the second kind cost together \$5.40. What is the price per pound of each kind?

(6) I have two bars of metal; the first weighs 30 pounds and contains 80% of pure gold; the second contains 90% of gold. How many pounds of the second must I fuse with the first bar to cast a third bar containing 87% of pure gold?

(7) A boy swam half a mile down stream in 10 minutes; in still water it would have taken him 15 minutes. How long will he take to swim back again?

# SECOND STAGE.

## FACTORING.

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### Exercise 51.

(1) If $x+a$ multiplied by $x=x^2+ax$, what are the factors of $x^2+ax$? Of $2a^2-3a$? $a^3-a^2b$? $3m^2-6mn$? $p^3-2p^2q$?

(2) Factor $x^5-5x^2$; $12x+48x^2y$; $27-162x$; $17x^2-51x$.

(3) What are the factors of $2a^3-a^2+a$? Of $6x^2y^2z^2+3xy$?

(4) Factorise $3x^3+6a^2x^2-3a^3x$; and $7p^2-7p^3+343p^4$.

(5) Factor $4b^5+6a^2b^3-2b^2$; and $26a^3b^5+39a^4b^2$.

(6) Separate $x^3y-x^2y^2+2xy$ into two factors.

(7) Separate $5a^2bx^3-15abx^2-20b^3x^2$ into two factors.

(8) Factor $35m^3y+28m^2y^2-14my^3$.

(9) Factor $12x^6y^4-24x^4y^2+36x^3y^3-12x^2y^2$.

(10) Factor $3x^3yz^2+6x^4yz^3-15x^5y^2z^3+18x^6y^3z$.

(11) $x^6-5x^4+5x^2-3$ is one factor of $x^{10}-17x^6+7x^4-9$, find the other factor by division. Test your answer by putting $x=1$.

(12) $x^4-6x^3-9x^2+94x-120=(x-2)\ (x-3)\ (x+4)\text{F}$, find F. Verify your answer.

### Exercise 52.

(1) $\text{A}^2-\text{B}^2=(\text{A}+\text{B})(\text{A}-\text{B})$; what are the factors of $p^2-q^2$?

(2) Factor $121-16y^2$; $121-36a^2$; and $4y^2-25$.

(3) Apply the formula in No. 1 to reduce the following arithmetical expressions: $39^2-31^2$; $51^2-49^2$; $275^2-225^2$; $329^2-171^2$; and $936^2-64^2$.

(4) Factor $100x^2y^2-121a^2b^2$; and $(3x+5)^2-(5x+3)^2$.

(5) Factor $(a-b)^2-c^2$; and $x^2-(a-b)^2$.

(6) Find the factors of $(a+b)^2-(c-d)^2$.

(7) Factor $(ax+by)^2-1$; and $2ab+a^2+b^2-c^2$.

(8) Factor $a^4-b^4$; and $a^8-b^8$ into three and four factors respectively.

(9) Express $x^{16}-y^{16}$ in five factors.

(10) Factor $a^2+12bc-4b^2-9c^2$.

HINT.—Take the last 3 terms together.

(11) Factor $x^2+y^2-z^2-d^2-2xy-2dz$.

HINT.—Take the terms 3 together.

(12) Factor $x^2-y^2+z^2-a^2-2xz+2ay$.

(13) Factor $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$.

## Exercise 53.

(1) $A^2+2AB+B^2=(A+B)^2$, find the factors of
$x^2+2xy+y^2$; $4a^2+4ab+b^2$; $9a^2+12ab+4b^2$.

(2) $x^2+12xy+36y^2$; $4a^4x^2+20a^2x^3y+25x^4y^2$.

(3) $x^2+18x+81$; $81a^4+36a^2+16$.

(4) $4x^4-80x^2y^2+400y^4$; $36x^4y^4+12x^2y^2+1$.

(5) $36a^2+84ab+49b^2$; $m^2+162m+6561$.

(6) $121a^2+220ab+100b^2$; $225x^2+360xy^2+144y^4$.

(7) $A^2-2AB+B^2=(A-B)^2$, find the factors of
$x^2-2xy+y^2$; $4a^2-4ab+b^2$; $9a^2-6ab+b^2$.

(8) $4x^8-12x^4y^2+9y^4$; $25m^4n^2-30m^2nx^2+9x^4$.

(9) $81b^4c^2-126b^2cd^3+49d^6$; $400m^2-1200mn^2+900n^4$.

(10) $49m^6+140m^3n^2+100n^4$; $4a^4x^2-20a^2x^3y+25x^4y^2$.

(11) $169x^2y^2+364xy^2z+196y^2z^2$;
$256a^2b^2-544ab^2c+289b^2c^2$.

(12) $324m^6+432m^3n^4+144n^8$.

(13) $361a^{2m}-684a^{m+n}+324a^{2n}$.

NOTE.—The student would do well to work Exercises 51, 52 and 53 over again several times before proceeding farther.

## Exercise 54.

(1) $A^2+A(B+C)+BC=(A+B)(A+C)$;
$a^2+7a+10=(a+5)(a+2)$; find the factors of $x^2+3x+2$.

(2) Factor $a^2+15x+56$; $x^2+14x+40$; $a^2-7a+10$.

(3) $a^2+11a+24$; $x^2+21x+110$; $m^2+19m+48$.
(4) $x^2-29x+190$; $x^2-9x-190$; $x^2+29x+190$.
(5) $x^2+9x-190$; $y^2z^2-28abyz+187a^2b^2$.
(6) $x^2y^2-27xy+26$; $a^4b^6-11a^2b^3+30$.
(7) $c^2d^2-30abcd+221a^2b^2$; $a^2c^2-24acz+143z^2$.
(8) $x^2+5x-14$; $x^2-34x+225$; $4a^2-12ax+9x^2$.
(9) $a^2x^2-b^2y^2$; $(x+3y^2)^2-(x-3y^2)^2$.
(10) $(a+b+c)^2-2(a+b+c)(b+c)+(b+c)^2$.
(11) $x^2+x(y+z)+yz$; $x^2-x(y+z)+yz$.
(12) Apply $(a+b)(a-b)=a^2-b^2$ to find the products of $79\times 81$ and $118\frac{1}{4}\times 121\frac{3}{4}$.
(13) What are the factors of $a^2+2ab+b^2+a+b$?
(14) Find the common factor of
$x^2-17x-38$; and $x^2+11x-570$.

## Exercise 55.

(1) $(A+B)(C+D)=AC+(AD+BC)+BD$. **Observe that the product of the extreme terms, viz. AC and BD, is equal to the product of the two middle terms, viz. AD and BC, each product being ABCD.**

(2) Multiply $(3x+4)(2x+3)$ and get the product $6x^2+(8x+9x)+12$. Observe that $6x^2\times 12=8x\times 9x$.

(3) Factor $6x^2+17x+12$. Separate $17x$ into two parts whose product shall be $=6x^2\times 12$, or $72x^2$. These parts can only be $8x$ and $9x$.
$(6x^2+9x)+(8x+12)=3x(2x+3)+4(2x+3)=(2x+3)(3x+4)$.

(4) Factor $4x^2+13x+3$. HINT. $4x^2\times 3=12x^2=ABCD$ $13x=12x+x$; thus $4x^2+(12x+x)+3=4x(x+3)+(x+3)$.

(5) Factor $8x^2+34xy+21y^2$.
HINT. $34xy=6xy+28xy$; $168x^2y^2=ABCD$.

(6) Factor $2x^2+5xy+2y^2$. HINT. $5xy=4xy+xy$; $4x^2y^2$.

(7) Factor $8a^2+14ab-15b^2$. HINT. $ABCD=-120a^2b^2$, hence split $14ab$ into $+20ab-6ab$, and proceed as before.

(8) Factor $6x^2+5x-4$. HINT. $ABCD=24x^2$. Split $5x$ into two parts whose product shall be $-24x^2$, *i.e.* $8x-3x$. Proceed as before.

## Exercise 56.

(1) Factor $12x^2+37x+21$.

HINT. $12\times 21=4\times 3\times 7\times 3=28\times 9$, hence split $37x$ into $28x+9x$.

(2) Factor $12x^2-37x+21$.

HINT. $-37x=-28x-9x$, $ABCD=+252x^2$.

(3) Factor $56x^2+137x-27885$. *(Matriculation, Toronto University.)* HINT. $27885=15\times 11\times 13^2=165\times 169$; $56=7\times 8$; and $8\times 165-7\times 169=137$. Hence say $137x=8\times 165x-7\times 169x$, and write $56x^2+8\times 165x-7\times 169x-165\times 169$ and factor.

(4) Factor $12x^2-5x-2$; $12x^2-7x+1$; $12x^2-x-1$.

(5) $3x^2-2x-5$; $3x^2+4x-4$; $6x^2+5x-4$; $4x^2+13x+3$.

(6) $6y^2+7yz-3z^2$; $11a^2-23ab+2b^2$; $6x^2+19xy-7y^2$.

(7) $12x^2-x-1$; $4x^2+8xy+3y^2$; $6b^2-7bx-3x^2$.

(8) $7x^2+123x-54$. HINT. $ABCD=7.3.3.3.2$; $126-3=123$.

(9) $5x^2-29xy+36y^2$; $16c^2-48c+35$; $x^2-10x-39$.

(10) $5x^2-38x+21$; $3x^2-x-2$; $3x^2-7x-6$; $3x^2+11x-20$.

(11) $2x^2+11x+12$; $16y^2+24yz-7z^2$; $4x^2-14xy+10y^2$.

(12) $25y^2-25yz+6z^2$; $49x^6+14x^3y-15y^2$.

(13) $731x^2-20xy-319y^2$; $413a^2-606ab-299b^2$.

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NOTE.—It is easily seen that the formula of Exercise 55 includes those given in Exercise 52, 53, 54. **The student should pause here and apply the formula of Exercise 55 to the examples given in Exercises 52, 53, 54.** If a given expression is the product of any two *binomial* factors, like $x+y$ and $w+z$, the factors can be found by this method. For example $a^2-b^2$ may be written $a^2+0.ab-b^2$, and we require to divide 0 into two parts whose product is $-1$, for $ABCD=-a^2b^2$. These parts are evidently $+1-1$, and we have $a^2+ab-ab-b^2$, which is $=a(a+b)-b(a+b)$, or $(a+b)(a-b)$, as in Exercise 52. **This method is therefore more general than the others and deserves careful attention.** As algebra is the soul of mathematics, so factoring is the soul of algebra. The subject cannot be treated exhaustively in a small book of elementary exercises, but what is here given will be found of the greatest service in the advanced course where very difficult examples often occur. The methods of Symmetry and Substitution, and the Theory of Divisors will be comparatively easy if the student will patiently master these easy cases of factoring thoroughly.

### Exercise 57.

(1) $A^3+B^3=(A+B)(A^2-AB+B^2)$. Prove this.

(2) What are the factors of $x^3+y^3$; $m^3+n^3$; $w^3+z^3$?

(3) Factor $8a^3+b^3$. N.B. $8a^3=(2a)^3$.

(4) Factor $8a^3+27b^3$. N.B. This is $(2a)^3+(3b)^3$.

(5) Factor $125x^3+343y^3$; $216m^3+512n^3$; $729p^3+343s^3$.

(6) Factor $a^6+b^6$. N.B. This is $(a^2)^3+(b^2)^3$.

(7) Factor $a^9+b^9$. N.B. This is $(a^3)^3+(b^3)^3$.

(8) Factor $x^{12}+y^{12}$; $x^{18}+y^9$; $1331x^{15}+1728y^{18}$.

(9) $A^3-B^3=(A-B)(A^2+AB+B^2)$. Prove this.

(10) What are the factors of $x^3-y^3$; $m^3-n^3$; $w^3-z^3$?

(11) Factor $8a^3-b^3$; $8a^3-27b^3$.

(12) $125x^3-343y^3$; $216m^3-512n^3$; $729p^3-343s^3$.

(13) $a^6-b^6$; $a^9-b^9$; $x^{12}-y^{12}$; $x^{18}-y^9$; $1331x^{15}-1728x^{18}$.

(14) Factor $2197x^{21}-2744y^{18}$; $2197x^{18}+2744y^{21}$.

### Exercise 58.

(1) $A^4+A^2B^2+B^4=(A^2+AB+B^2)(A^2-AB+B^2)$.
Prove this.

(2) $x^4+x^2y^2+y^4=(x^4+2x^2y^2+y^4)-x^2y^2$
$=(x^2+xy+y^2)(x^2-xy+y^2)$. See Exercise 52.

(3) $x^8+x^4y^4+y^8$. Add $x^4y^4$ to this and subtract $x^4y^4$ from it, and get $(x^4+x^2y^2+y^4)(x^4-x^2y^2+y^4)$, by applying $a^2-b^2=(a+b)(a-b)$. Separate the first bracket into two more factors.

(4) Factor $x^4+9x^2+81$ by adding $9x^2-9x^2$.

(5) Factor $16a^4+4a^2x^2+x^4$ by adding $4a^2x^2-4a^2x^2$.

(6) Factor $81a^4+36a^2+16$, and $x^4+16x^2+256$.

(7) $a^4+4a^2+16$; $m^8+25m^4+625$; $p^8+p^4+1$.

(8) Factor $4x^4-37x^2y^2+9y^4$ by adding $25x^2y^2-25x^2y^2$.

(9) Factor $64x^4+128x^2y^2+81y^4$ by adding $16x^2y^2-16x^2y^2$.

(10) Factor $81a^4 - 28a^2b^2 + 16b^4$ by adding $100a^2b^2 - 100a^2b^2$.
(11) Factor $9a^4 + 21a^2x^2 + 25x^4$.
(12) Factorise $16x^4 - 17x^2 + 1$.
(13) Factor $49a^4 - 15a^2b^2 + 121b^4$
(14) Factor $x^4 - 11x^2y^2 + y^4$.
(15) Factor $x^4 - 63x^2y^2 + 81y^4$.

## Exercise 59.

(1) Factor $x^3 + 2bx^2 + 3ax + 6ab$. Hint.—**Select any letter in the expression that appears in only One and the Same power throughout,** *e.g.* $a$ or $b$. Group these together first. Thus, arrange $(3ax + 6ab)$ + etc., or $(2bx^2 + 6ab)$ + etc.

(2) Factor $2x^2 + 5xy - 3y^2 - 4xz + 2yz$. N.B.—Here $z$ is the only letter that has the *same power* throughout, ∴ arrange thus $(2x^2 + 5xy - 3y^2) - (4xz - 2yz)$ and factor each bracket.

(3) Factor $a^2 - x^2 - ab - bx$. N.B.—Pick out $b$, and group.

(4) $x^2 - y^2 - xz + yz$. N.B.—Pick out $z$, and group.

(5) $ab - ac - b^2 + bc$; $2x^2 + 4xy + 2y^2 + 2ax + 2ay$.

(6) $2x^2 + 2xy - 12y^2 + 6xz + 18yz$.

(7) $(a + 3b)^2 - 9(b - c)^2$. *See Exercise 52.*

(8) $x^2 - xy - 6y^2 - 4xz + 12yz$.

(9) $18x^2 - 24xy + 8y^2 + 9xz - 6yz$.

(10) $6a^4 - 10b^4 + 11a^2b^2 - 25c^2 + 10b^2 + 25b^2c^2 - 15a^2 + 10a^2c^2$.
N.B.—Take $c^2$, and split $11a^2b^2$ into $15a^2b^2 - 4a^2b^2$; group in three brackets, factor.

(11) $72a^2 - 8b^2 + 55ab + 20bc - 160ac + 20c - 9a + 8b$.
N.B. $55ab = 64ab - 9ab$; group in 3 brackets and factor.

## Exercise 60.

(1) Expand $(a - 3b + c)(a + b - 3c)$.

(2) $a^2 - 3b^2 - 3c^2 + 10bc - 2ca - 2ab$. Factor this expression.

Note.—**If an expression like this contains only terms like $a^2$, $b^2$, $c^2$ and like bc, ca, ab, it is usually easy to factor,** *i.e.* if it can be expressed as the product of two trinomials like those in No. 1. If all the signs are + it is only necessary to factor the terms like $a^2$, $b^2$, $c^2$, and so arrange them by trial as to get the terms like $bc$, etc. This can generally be done by inspection after a little practice. It is not quite so easy in case some of the terms are negative. Observe that in No. 1 the final product must contain $(a-3b)(a+b)$; $(a+c)(a-3c)$; and $(-3b+c)(b-3c)$. Hence group all the $a$ and $b$ terms of No. 2, all the $b$ and $c$ terms and all the $c$ and $a$ terms, and factor each group separately thus:

$$a^2-2ab-3b^2=(a-3b)(a+b);$$
$$a^2-2ca-3c^2=(a+c)(a-3c);$$
$$-3b^2+10bc-3c^2=(-3b+c)(b-3c).$$

This shows plainly that $-3b$ goes with $+a$ and $+c$, *i.e.* $a-3b+c$ is one factor. Similary, it is readily seen that $-3c$ goes with $+a$ and $+b$, and that $a+b-3c$ is the other factor, and the result can be tested by multiplication.

(3) Factor $2y^2-5xy+2x^2-ay-ax-a^2$. N.B.—Grouping $y$ and $x$ we get $(2y-x)(y-2x)$; grouping $y$ and $a$, $(2y+a)(y-a)$; which is enough to show that $2y-x+a$ is one factor, and $y-2x-a$ the other factor. There can be no others, for the given quantity contains no term of higher *Dimensions* than the second power.

(4) $2a^2-7ab-22b^2-5a-35b+3$. N.B. $(2a+1)(a-3)$ and $(2a-11b)(a+2b)$ are sufficient to show the factors.

(5) $2a^2-5ab+2b^2-17a+13b+21$.

(6) $6x^2-37ax+6a^2-5x-5a-1$.

(7) $6m^2-5my-6y^2-m-5y-1$.

(8) $5k^2-8km+3m^2+7k-5m+2$.

(9) $6k^2+ky-y^2-3kz+6yz-9z^2$.

(10) $6x^2-7xy+y^2+35kx-5yk-6k^2$.

(11) $2x^2-xy-3y^2-5yz-2z^2$.

(12) $6p^2-13pq+6q^2+12ap-13aq+6a^2$.

(13) $6x^2-7xy+2xz-20y^2+64yz-48z^2$.

## Exercise 61.

(1) $A^2+B^2+C^2+D^2+\text{etc.}+2AB+2AC+\text{etc.}$
$+2BC+2BD+\text{etc.}+2CD+\text{etc.}=(A+B+C+D, \text{etc.})^2$.

**Hence, if any given expression contains only perfect squares, like $A^2$, $B^2$, $C^2$, etc., and Double Products like 2AB, 2BC, 2CD, etc., the expression may be a perfect square** in itself and is easily factored. The method given in Exercise 60 will generally apply, but the following is shorter: **Take the square root of each square term and connect these with the + or − signs required by the double products.**

(2) Factor $x^4+2x^3+3x^2+2x+1$. N.B. $x^4$ and 1 are *perfect squares;* $2x^3$ and $2x$ are *double products;* and $3x^2=x^2+2x^2$, *i.e.* a perfect square + a double product, and we see that the expression is of the form $a^2+b^2+c^2+2ab+2ac+2bc$, which is $(a+b+c)^2$.

(3) Factor $4a^4-4a^3-3a^2+2a+1$. N.B. $-3a^2=-4a^2+a^2$, and all the other terms are either *perfect squares* or *double products.*

(4) $a^6+2a^5b+3a^4b^2+4a^3b^3+3a^2b^4+2ab^5+b^6$.
N.B. $3a^4b^2=2a^4b^2+a^4b^2$; similarly for $3a^2b^4$;
$4a^3b^3=2a^3b^3+2a^3b^3$.

(5) $49x^4+56x^3y+30x^2y^2+8xy^3+y^4$. N.B. $7x^2$ and $y^2$ *must* be two of the terms; their *double product* is $14x^2y^2$, hence make $30x^2y^2=14x^2y^2+16x^2y^2$, the latter being a *square term.*

## Exercise 62.

(1) Factor $a^2+12ab+36b^2$; $x^2-8xy+16y$;
$4a^2x^2y^2-4abxy+b^2$.

(2) $x^2+y^2+2xy+z^2+2zx+2yz$.

(3) $a^2+12bc+4b^2+6ac+9c^2+4ab$.

(4) $9x^2+16yz+4y^2-24zx+16z^2-12xy$.

(5) $81a^2+9b^2+c^2-54ab+18ac-6bc$.

(6) $x^4+4x^3+6x^2+4x+1$.

(7) $a^4-2c^2d+b^6+2b^3d+c^4-2b^3c^2+d^2+2a^2d-2a^2c^2$
$+2a^2b^3$.

(8) $m^4+8m^3+18m^2+8m+1$.

(9) $4m^2-12mn+20my+9n^2-30ny+25y^2$.

(10) $x^6-6x^5+13x^4-22x^3+34x^2-20x+25$.

(11) $a^6+6a^5+15a^4+20a^3+15a^2+6a+1$.

(12) $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

(13) $\frac{9}{16}x^2+\frac{9}{4}y^2+\frac{49}{16}z^2-\frac{9}{4}xy+\frac{21}{8}xz-\frac{21}{4}yz$.

(14) $\frac{1}{4}a^2+a^4b^2+\frac{1}{4}b^2+a^3b+\frac{1}{2}ab+a^2b^2$.

(15) $m^4+m^3+\frac{5}{4}m^2+\frac{1}{2}m+\frac{1}{4}$.

(16) $25x^2+9y^4-{\cdot}0625z^2+30xy^2-2{\cdot}5xz-1{\cdot}5y^2z$.

(17) $3(3a^2-2ab+b^2)(a^2+3b^2)+b^2(a+4b)^2$.

## Exercise 63.

(1) Find the factor that is common to both the quantities $2x^2-16x+14$ and $x^2-5x-14$. N.B.—Factor each expression.

(2) $3a^2+14a+8$, and $4a^2+19a+12$.

(3) $x^3-y^3$, and $x^2-2xy+y^2$.

(4) $a^2-9$; $a^2-3a-18$; and $a^2+11a+24$.

(5) $p^2-3p-28$; $p^2-11p+28$; $p^2-15p+56$.

(6) $n^2+6n+9$; $n^3-n^2-12n$; $n^2-4n-21$.

(7) $k^4-m^4$; $k^3+k^2m-km^2-m^3$; $k^4-2k^2m^2+m^4$.

(8) $a^4-2a^2b^2+b^4$ and $a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

(9) $a^3+3a^2b+3ab^2+b^3$; $4a^2b^2+12ab^3+8b^4$; and $a^2-b^2$.

(10) $x^4+5x^3+6x^2$; $x^3+3x^2+3x+2$; and $3x^3+8x^2+5x+2$.

HINT.—Factor the 1st quantity; find by division whether any of these three factors will divide the 2nd and the 3rd quantities. Observe that it is evidently useless to try $x^2$ or $x+3$.

(11) $16mx^3-20mx^2+10mx-6m$, and $3mx^2-15mx+12m$.

(12) $3k^2+21k-132$ and $6k^3y+54k^2y-138ky-66y$.

(13) $3x^2+11x+6$; $2x^2+11x+15$; and $x^5+4x^4+4x^3+4x^2+4x+3$.

N.B. $3x+2$; $2x+5$ are evidently of no use.

## Exercise 64.

(1) **Every MEASURE of A and B will also measure A+B and A−B.** Take the two numbers A=28, B=21 having a measure M=7, and show that this will exactly divide 28+21, or 28−21. Give A and B five other values each and prove that the proposition is true.

(2) Suppose $A=mx$, $B=nx$; what is the *common measure* of A and B? If $A=kmx$, $B=lnx$; what is the C. M. of A and B?

(3) In Question 2, show that $x$ in each case measures $A\pm B$ (read A *plus* or *minus* B).

(4) **Every MEASURE of A and B will also measure mA±nB.** Let A=9, B=6, M=3; show that M will exactly divide $7A\pm4B$; $225A\pm159B$; $29A\pm23B$; $1331A\pm729B$.

(5) In Question 4, suppose $A=mx$, $B=nx$; show that $x$ is a factor of $pA\pm qB$ where $p$ and $q$ do not contain $x$.

(6) If $x$ is a factor of $PA\pm QB$ and is not a factor of P and Q, show that $x$ must be an exact divisor of A and B. Test this by putting P=7, Q=5, A=39, B=26; and by giving these letters five other values each.

(7) If $A=2x^2-16x+14$, $B=x^2-5x-14$. Find A+B and factor this sum. Determine now the common factor of A and B, if they have any.

(8) If $A = 3a^2 + 14a + 8$, $B = 4a^2 + 19a + 12$; find the value of $3A - 2B$ and factor this sum. Can $a$ be a factor of A and B? What is the common factor, if they have any? Test them for $a + 4$.

(9) If $A = x^3 - y^3$, $B = x^2 - 2xy + y^2$; find $A - Bx$ and factor this difference. Can $xy$ be a factor of A and B? Can $2x + y$? What is the C. F. of A and B if they have one? Is $x - y$ a factor?

(10) If $A = a^2 - 3a - 18$, $B = a^2 + 11x + 24$; what is $B - A$? What are its factors? Which one is a C. F. of A and B?

(11) In Question 10, find $B - A$ as before, also find $3B + 4A$. What is the common *binomial* factor of these two quantities? Since $7a$ cannot measure A and B, what must be their common factor?

(12) If $A = 3k^2 + 21k - 132$, $B = 6k^3y + 54k^2y - 198ky - 66y$, can the $y$ in B be a factor of A? Can the 6 in B and the 3 in A have any common factor? Strike out the factor 3 in A and $6y$ in B, and set aside their common factor, 3. Call the quotients C and D and find their C. F., $x + 11$. What must be the C. F. of A and B?

## Exercise 65.

(1) When $A = 3x^2 - 2x - 1$, $B = 6x^2 - x - 1$, show that $B - A = x(3x + 1)$, and that $B - 2A = 3x + 1$, and hence infer *the highest common factor* of A and B.

(2) When $A = x^3 - 3x + 2$, $B = x^3 + 2x - 3$, show that $B - A$ is $x - 1$, and that $2B - 3A$ is $5x(x + 1)(x - 1)$ and hence find the H. C. F. of A and B.

(3) If $A = 3x^3 - 2x^2 - 2x + 1$, $B = x^5 - x^4 - 3x^2 + 4x - 1$, find their H. C. F.

N.B. $A = 3x^2(x - 1) + (x^2 - 2x + 1)$;
$B = x^3(x^2 - 1) - (3x^2 - 4x + 1)$; *i.e.*
$A = 3x^2(x - 1) + (x - 1)^2$;
$B = x^3(x^2 - 1) - (3x - 1)(x - 1)$.

(4) If $A = 6a^4b + a^3b^2 - ab^4$; $B = 4a^3 - 6a^2b - 4ab^2 + 3b^3$, find the H. C. F. of A and B. N.B.—Strike out from A the simple factor $ab$ which is not contained at all in B; call the quotient $A_1$. Then $A_1 + 3B = (22a^2 - 3ab - 4b^2)a$, of which $a$ cannot be part of the common factor; $\therefore$ *either* $11a + 4b$ or $2a - b$, or *both* or *neither* will be the H. C. F. It is plainly useless to try the first; $\therefore$ try $2a - b$.

(5) $11x^4 - 9ax^3 - a^2x^2 - a^4$, and $13x^4 - 10ax^3 - 2a^2x^2 - a^4$.

N.B. $B - A = x^2(2x + a)(x - a)$, and it is plainly useless to try the first two factors; $\therefore$ try the third.

## Exercise 66.

NOTE.—In Exercise 64 it is shown that when any two quantities have a *measure* or *common factor*, this measure or common factor will also divide the sum or the difference of any multiples of the two given quantities. **This principle is the foundation of the ordinary arithmetical process of finding the G. C. M. of two numbers,** which, as the student knows, is conducted somewhat like long division. The first remainder, say $R_1$, is the difference between two multiples of A and B, the two given numbers; hence every common factor of A and B must divide $R_1$. Then $R_1$ and A are taken as two numbers which have a common factor, and $R_2$, the second remainder, is found; hence every common factor of A and $R_1$, *i.e.* every common factor of A and B will divide $R_2$. At the end a remainder, say $R_n$, is found which divides the last pair exactly and is their G. C. M., and is therefore the G. C. M. of A and B. Precisely in a similar way the H. C. F. of two algebraic expressions may be found, although it is not generally the most concise method. Thus in Exercise 65, question No. 1, we may proceed in this way:—

$$
\begin{array}{l}
3x^2-2x-1 \,\Big)\, 6x^2-x-1 \,\Big(\, 2 \\
\qquad\qquad\quad \underline{6x^2-4x-2} \\
\qquad\qquad\qquad\quad 3x+1 \,\Big)\, 3x^2-2x-1 \,\Big(\, x-1 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{3x^2+x} \\
\qquad\qquad\qquad\qquad\qquad\qquad -3x-1 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{-3x-1}
\end{array}
$$

And $3x+1$ is the H. C. F. as before. The work may be arranged more conveniently, as may the work in arithmetic, by writing the quotients alternately in the right and left margins thus:—

$$
\begin{array}{r|r|r|l}
 & 3x^2-2x-1 & 6x^2-x-1 & \\
 & & \underline{6x^2-4x-2} & 2 \\
x & \underline{3x^2+x} & 3x+1 & \\
 & -3x-1 & & \\
-1 & \underline{-3x-1} & &
\end{array}
$$

A modification of this plan with detached coefficients (*i.e.* the coefficients alone being used) is perhaps the most rapid method of proceeding with large expressions that are not easily factored. In using the above plan the student should strike out from $R_1$, $R_2$, $R_3$, etc., any simple factor which is plainly no divisor of A and B. The operation is not mere division; **the purpose and**

**end in view is to find the H. C. F. of A and B,** and if any one of the remainders contains some factor manifestly not contained in A and B this factor is of no present use and may be set aside. Also, a simple factor may be introduced at any stage without changing the final result. For instance, if the preceding example contained $4x^2$ instead of $3x^2$, we might multiply $6x^2-x-1$ by the simple factor 2 so as to make it exactly divisible by $4x^2$ and thus save a line or two of work. The student may apply this method to find the H. C. F. in the following examples.

(1) $x^2-7x+10$ and $4x^3-25x^2+20x+25$.

(2) $8y^2+14y-15$ and $8y^3+30y^2+13y-30$.

(3) $2a^4+a^3-20a^2-7a+24$ and $2a^4+3a^3-13a^2-7a+15$.

(4) $2m^5-8m^4+12m^3-8m^2+2m$ and $3m^5-6m^3+3m$.

(5) $3a^6+15a^5b-3a^3b^2-15a^2b^3$ and
$10a^5-30a^4b-10a^2b^2+30ab^3$.

(6) $3x^2+(4a-2b)x-2ab+a^2$ and
$x^3+(2a-b)x^2-(2ab-a^2)x-a^2b$.

(7) $300a^3+265a^2+50a+24$ and $60a^2+53a+4$.

(8) $x^4+7x^3+7x^2-15x$ and $x^3-2x^2-13x+110$.

(9) $(b-c)x^2+2(ab-ac)x+a^2b-a^2c$
and $(ab-ac+b^2-bc)x+(a^2c+ab^2-a^2b-abc)$.

(10) $20a^4+a^2-1$ and $25a^4+5a^3-a-1$.

(11) $a^5+3a^4-8a^2-9a-3$ and $a^5-2a^4-6a^3+4a^2+13a+6$.

(12) $2x^4+9x^3+14x+3$ and $3x^4+15x^3+5x^2+10x+2$.

After applying the method of division to these exercises the student is recommended to work them over again, applying the principles of factoring as far as possible, and at any stage. Thus in question 1 and 2 factor the first expression, in 4 and 7 the second expression, in other cases factor the sum or the difference as the particular case seems to require.

**One of the most useful exercises that can be devised is to search out various ways of solving the same problem.**

## Exercise 67.

(1) What is the H. C. F. of $ax$ and $bx$? What is their product? What is their L. C. M.?

(2) Divide the product of $ax$ and $bx$ by their H. C. F.; how does this differ from their L. C. M.?

(3) Divide $ax$ by their H. C. F. and multiply the quotient by $bx$; what is the result? Give some other way of finding their L. C. M.

(4) Find the L C. M. of $abc$, $bcd$, $cde$, and $def$. What is their H. C. F.?

(5) The H. C. F. of two quantities is $x$, their L. C. M. is $abx$, find the product of the two expressions.

(6) If the L. C. M. × H. C. F. of two quantities is $cdy^2$, what is the product of the two quantities? If one of them is $cy$ what is the other?

(7) Find the L. C. M. of $ax$, $bx^2$, $cx^3$, $dx^4$, $ex^5$, $fx^6$; H.C.F. =?

(8) Find the L. C. M. of $abc$, $bca$, $cab$; H. C. F. =?

(9) Find the L. C. M. of $x(a+b)$, $y(a-b)$, $z(a^2-b^2)$.

(10) Find the L. C. M. of $x(a+b)$, $x^2(a^2-b^2)$, $x^3(a^3+b^3)$.

## Exercise 68.

(1) The H. C. F. × L. C. M. of two quantities is $(a^2-b^2)(a^3-b^3)$; what is the quotient when their product is divided by $a^2+ab+b^2$?

(2) Find the L. C. M. of $x^2+5x+4$; $x^2+2x-8$; $x^2+7x+12$.

(3) What is the lowest quantity that will contain each of the following expressions as an exact divisor, viz.:

$$x-1;\ x^2-1;\ x-2;\ \text{and } x^2-4?$$

(4) L. C. M. of $3x^2-11x+6$; $2x^2-7x+3$; and $6x^2-7x+2=$?

(5) L. C. M. of $6x^2-13x+6$; $12x^2-5x-2$; and $15x^2+2x-8=$?

(6) L. C. M. of $6x^3-11x^2+5x-3$, and $9x^3-9x^2+5x-2=$?

(7) Each of the following quantities will divide a vast number of expressions without remainder, find the lowest of these expressions, the quantities being $1+a$; $1-a$; $1+a+a^2$; and $1-a+a^2$.

(8) L. C. M. of $x^2+3x+2$; $15(x+1)$; $20(x+2)=$?

(9) $x^3-7x+6$; $x^2-2x-3$; $x^2-x-6$; L. C. M. =?

(10) $8x^3-14x+6$; $4x^2+4x-3$; $4x^2+2x-6$; L. C. M. =?

(11) $x^3+x^2-2$; $x^3+2x^2-3$.

(12) $x^5-2x^4-6x^3+4x^2+13x+6$; $3x^4+4x^3-6x^2-12x-5$.

# FRACTIONS.

## Exercise 69.

(1) In the fractions $\frac{3}{4}$ and $\frac{x}{y}$ change the denominators to 20 and $yz$ respectively without altering the values of the fractions.

(2) In the same fractions find what the denominators become when the numerators are changed to 21 and $wx$ respectively.

(3) Reduce $\frac{96975}{110025}$ to lowest terms, and do the same with the fraction $\frac{abcde}{fcdeb}$.

(4) Convert $\frac{21}{17}$ into a mixed number, and express in that form the fraction $\frac{a^2+2ab+b^2}{a-b}$.

(5) Express $27+\frac{11}{13}$ in the form of a fraction, and put $3a-\frac{11b}{c}$ into the fractional form.

(6) $\frac{x^2-6x+8}{4x^2-23x+28}$. Factor both terms of this fraction and then reduce it to its lowest terms.

## Exercise 70.

Reduce the following fractions to their lowest terms.

(1) $\frac{a^3+3a^2-4a}{7a^3-18a^2+6a+5}$. HINT. Factor the N.

(2) $\frac{2a^3+3a^2-1}{a^4+2a^3+2a^2+2a+1}$. HINT. $3a^2=2a^2+a^2$.

(3) $\dfrac{a^4 - a^3 - a + 1}{a^4 - 2a^3 - a^2 - 2a + 1}$.

(4) $\dfrac{10x^4 - 7x^3 + x^2}{4x^4 - 2x^3 - 2x + 1}$.

(5) $\dfrac{1 - 2c + 2c^2 + 2c^3 - 3c^4}{1 - c + 2c^2 + c^3 + 3c^4}$.

HINT. In N., $2c^2 = 3c^2 - c^2$; factor.

(6) $\dfrac{x^4 + 3x^2 + 6x + 35}{x^4 + 2x^3 - 5x^2 + 26x + 21}$.

HINT. $D - N. = (x - 1)(x^2 - 3x + 7)$.

(7) $\dfrac{5m^4 - 4m^3 - 1}{m^5 - m^4 - m + 1}$. HINT. $-4m^3 = -5m^3 + m^3$; factor.

(8) $\dfrac{8x^7 - 29x^3 + 21}{21x^7 - 29x^4 + 8}$. HINT. $29 = 21 + 8$; factor N. and D.

## Exercise 71.

(1) Reduce $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ to the same denominator and find their sum.

(2) Reduce $\frac{1}{a}$, $\frac{1}{b}$ to the same denominator and find their sum.

(3) Reduce $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and also $\frac{a}{b}$, $\frac{b}{c}$, $\frac{c}{d}$ to their least common denominator, and find the sum of each set of fractions.

(4) What is the sum of $\frac{5a}{6} + \frac{3a}{4} + \frac{2b}{9}$?

(5) What is the difference of $\frac{1}{2} - \frac{1}{3}$? Of $\frac{1}{a} - \frac{1}{b}$?

(6) Find the sum and also the difference of $\frac{a}{b}$ and $\frac{c}{d}$.

(7) Make $\frac{x}{x-1}$ and $\frac{y}{x+1}$ have the same denominator and thus find their sum.

(8) Find the sum of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

(9) Add together $\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}$.

(10) Add $\frac{1}{x+7}+\frac{1}{x-3}$; and also $\frac{x}{x+a}+\frac{a}{x-a}$.

(11) $\frac{x}{ab}+\frac{y}{ca}+\frac{z}{bc}$; $\frac{a-b}{ab}+\frac{b-c}{bc}+\frac{c-a}{ca}$.

(12) $\frac{a}{2a-2b}+\frac{b}{2b-2a}$; $\frac{x^2+xy+y^2}{x^2-xy+y^2}+\frac{x^2-xy+y^2}{x^2+xy+y^2}$.

## Exercise 72.

(1) What is the product of $\frac{4}{5}$ and $\frac{3}{7}$? Of $\frac{a}{b}$ and $\frac{c}{d}$?

(2) Multiply $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$; also $\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \frac{d}{e}$.

(3) Multiply $1\frac{1}{2}$ by $2\frac{1}{3}$; also $a+\frac{1}{b}$ by $2a+\frac{1}{c}$.

(4) $\frac{ax^2}{by^3} \cdot \frac{cy^2}{dz^2} \cdot \frac{dm^2}{an^2} \cdot \frac{bn^2}{cx^2}$; reduce to simplest form.

(5) $\frac{a^2-x^2}{x^4-y^4} \cdot \frac{x^2+y^2}{a+x}$.

(6) $\frac{x^2-6x+8}{x^2-4x+3} \times \frac{x^2-5x+6}{x^2-2x+8} \times \frac{x-1}{x^2-4x+4}$.

(7) $\frac{x^2+2x-3}{x^2+5x+6} \times \frac{4x^2-12x-40}{3x^2-18x+15} \times \frac{a^2-b^2}{a^3-b^3}$.

(8) $\frac{(x-a)^2-b^2}{(x-b)^2-a^2} \times \frac{x^2-(b-a)^2}{x^2-(a-b)^2}$.

## Exercise 73.

(1) Divide $\frac{4}{5}$ by $\frac{3}{7}$; and $\frac{a}{b}$ by $\frac{c}{d}$.

(2) Divide $\left(\frac{2}{3}\times\frac{3}{4}\right)$ by $\left(\frac{5}{4}\times\frac{6}{5}\right)$; and $\frac{a}{b}\cdot\frac{b}{c}$ by $\frac{d}{c}\cdot\frac{e}{d}$.

(3) Divide $\frac{3}{2}$ by $\frac{3}{7}$; and $\frac{3x}{2y}$ by $\frac{3x}{7y}$.

(4) Reduce $\frac{acx^3y^2}{bdy^2z^2}\div\frac{bdm^2n^2}{acn^3x^2}$.

(5) $\frac{x^2-11x+30}{x^2-6x+9}\div\frac{x^2-5x}{x^2-3x}$.

(6) $\frac{x^2-4x+3}{x^2-5x+4}\times\frac{x^2-9x+20}{x^2-10x+21}\div\frac{x^2-5x}{x^2-7x}$.

(7) $\frac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}\div\frac{(a-c)^2-(d-b)^2}{(a-b)^2-(d-c)^2}$.

(8) $\frac{a^2-2ab+b^2-c^2}{a^2+2ab+b^2-c^2}\div\frac{a-b+c}{a+b-c}$.

## Exercise 74.

Reduce the following expressions to their simplest form:—

(1) $\frac{x-1}{x-2}+\frac{x-2}{x-3}+\frac{x-3}{x-4}$; $\frac{1+x}{1+x+x^2}-\frac{1-x}{1-x+x^2}$.

(2) $\left(\frac{x^2}{a^2}-\frac{bx}{ay}+\frac{b^2}{y^2}\right)\left(\frac{x^2}{a^2}+\frac{bx}{ay}+\frac{b^2}{y^2}\right)$

N.B.—See page 35, No. 5, and Exercise 58.

(3) $\frac{x^4-9x^3+7x^2+9x-8}{x^4+7x^3-9x^2-7x+8}$. HINT.—Factor N ÷ D.

(4) $\left(\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right)\div\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right)$.

(5) $\left(\frac{a^2-ab}{a^3-b^3}\right)\left(\frac{a^2+ab+b^2}{a+b}\right)+\left(\frac{2a^3}{a^3+b^3}-1\right)\left(1-\frac{2ab}{a^2+ab+b^2}\right)$.

(6) $\frac{1}{x+2}-\frac{3}{x+4}+\frac{3}{x+6}-\frac{1}{x+8}$.

N.B.—Add (1st + 4th) + (2nd + 3rd).

(7) $\frac{1}{x-2a}-\frac{4}{x-a}+\frac{6}{x}-\frac{4}{x+a}+\frac{1}{x+2a}$.

N.B.—Add 1st + 5th; 2nd + 4th, etc.

(8) $\frac{b-c}{a^2-(b-c)^2}+\frac{c-a}{b^2-(c-a)^2}+\frac{a-b}{c^2-(a-b)^2}$.

N.B.—L. C. M. $=(a+b-c)\ (-a+b+c)\ (a-b+c)$.

## Exercise 75.

(1) Find the value of $\frac{x-3}{2x+2}+\frac{x-2}{3x+9}+\frac{x+3}{x^2-1}$, when $x=5$.

(2) Express $s$ shilling $+d$ pence $+q$ farthings as a fraction of £1.

(3) Find the numerical value of the fraction,

$\frac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}$, when $x=b+c$, $y=c+a$, $z=a+b$.

HINT. Factor N. and D.; observe that $x+y+z=2(a+b+c)$, and that $x^2+y^2+z^2-xy-yz-zx=\frac{1}{2}\{(x-y)^2+(y-z)^2+(z-x)^2\}$, etc., and therefore $x^2+y^2+\text{etc.}=a^2+b^2+c^2-ab-\text{etc.}$

(4) Find the value of the following expression when $y=\cdot 1$;

$$y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\frac{y^5}{5}.$$

N.B. $y=\cdot 1$, $y^2=\cdot 01$, $y^3=\cdot 001$, etc., $\therefore \frac{1}{5}y^5=\cdot 000002$, etc.

(5) Divide $1-2a$ by $1-a$ to 5 terms, and write down the 120th term.

(6) If $x+\frac{1}{y}=1$ and $z+\frac{1}{x}=1$, prove that $y+\frac{1}{z}=1$.

HINT. $x=\frac{y-1}{y}$, $\therefore z+\frac{y}{y-1}=1$, etc.

## Exercise 76.

(1) $\frac{1}{6}[a(a+1)\ (a+2)+a(a-1)\ (a-2)]+\frac{2}{3}(a+1)\ (a-1)a$.

Test your answer by putting $a=1$.

(2) $\frac{3a+2x}{3a-2x}-\frac{3a-2x}{3a+2x}-\frac{16x^2}{9a^2-4x^2}$.

Test your answer by putting $a=x=1$.

(3) $\frac{1}{x-1}-\frac{1}{2(x+1)}-\frac{x+3}{2(x^2+1)}$.

Put $x=2$ and verify your result.

(4) $\left(\frac{a+x}{a-x}-\frac{a-x}{a+x}\right)\div\left(\frac{a+x}{a-x}+\frac{a-x}{a+x}\right)$.

Put $a=2$, $x=1$, and verify your result.

(5) $(ab+bc+ca)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$.

Put $a=b=c=1$ and verify your answer.

(6) $\frac{a(b+c)}{b+c-a}+\frac{b(c+a)}{c+a-b}+\frac{c(a+b)}{a+b-c}$.

Put $a=b=c=1$ and verify your answer.

(7) $\frac{1}{a^2+3a+2}+\frac{2a}{a^2+4a+3}+\frac{1}{a^2+5a+6}$. Test your result.

## Exercise 77.

(1) $\frac{a+c}{a+x}+\frac{a-c}{a-x}=\frac{2b^2}{a^2-x^2}$, find the value of $x$.

(2) $\frac{x+1}{x-1}+\frac{x+2}{x-2}=\frac{2(x+3)}{x-3}$.

find $x$ and verify the value by substitution.

(3) $\frac{x+a}{x+b}=\frac{1+a}{1+b}$, find $x$ and verify the value.

(4) $\frac{x-a}{a^2-b^2}=\frac{x-b}{c(a+b)}+\frac{x-c}{c(a-b)}$. Put $a=2$; $b=c=1$; and verify.

(5) $\frac{4x-17}{x-4}+\frac{10x-13}{2x-3}=\frac{8x-30}{2x-7}+\frac{5x-4}{x-1}$.

HINT. Divide each N. by its D.; cancel quotients; add the remainders, taking each pair separately, and get $2x^2-11x+12=2x^2-9x+7$.

(6) $\frac{x}{x-2}+\frac{x-9}{x-7}=\frac{x+1}{x-1}+\frac{x-8}{x-6}$.

(7) Verify the results in 5 and 6 by substituting the values of $x$.

NOTE.—In several of the preceding exercises **the Hints** have pointed out to the student the advantage that frequently arises from making partial additions or subtractions in dealing with a number of fractions, instead of attempting to perform the whole operation at one step. **Very often an important simplification and reduction is possible by grouping the terms of an equation or of a series of fractions so as to avoid a L. C. D. of high dimensions or of complicated form.** One great benefit of mathematical studies is to give continuous exercise to the judgment of the student in choosing the simplest method of accomplishing the end in view. **The power of thought *versus* brute force** is constantly illustrated in these studies, **the simplest solution of a problem** being **always the result of the most perfect knowledge** and the most thorough grasp of the abstract thinking necessary to produce the result sought for at the least expense of mechanical work. As **the student** proceeds he will find that Algebra gives almost unlimited scope for Invention, and he **should decide never to leave any problem or question till he has mastered the simplest solution** he can produce.

The following test exercises will supply a GENERAL REVIEW of the subject to the end of fractional equations. In a few cases fractional equations of *two* unknown quantities have been introduced in order to keep up the supply of new ideas and to exercise the original invention of the student. But in all cases the examples will be found within the limits of his power and capable of solution, and the new matter here and there interspersed with the review of the old should only serve to whet his interest and encourage him with the consciousness of growth in intellectual power.

## Exercise 78.

(1) What is the H. C. F. of $6x^5-4x^4-11x^3-3x^2-3x-1$ and $4x^4+2x^3-18x^2+3x-5$?

(2) What multiplier with $3-5x+7x^2$ will give the product $12+82x^2+106x^4-70x^5-112x^3-38x$?

(3) Find the factors of $20x^2+21xy-27y^2$; and the factors of $a^3-3a^2-9a+27$.

(4) How much must I add to $(2x+1)(2x+3)(2x+5)(2x+7)$ to make it equal to $(4x^2+16x+11)^2$?

(5) One factor of $a^3+8b^3+c^3-6abc$ is $a^2+4b^2+c^2-2ab-2bc-ca$, find the other factor and apply some kind of test to your answer.

(6) $\dfrac{2}{(x^2-1)^2} - \dfrac{1}{2x^2-4x+2} - \dfrac{1}{1-x^2}$. Simplify and test.

(7) If $\left(x+\dfrac{1}{x}\right)^2=3$, prove that $x^3+\dfrac{1}{x^3}=0$.

HINT. Expand bracket and transpose; factor second expression, and substitute for one factor.

### Exercise 79.

(1) Divide $60x^3-17x^2-4x+1$ by $5x^2+9x-2$, and give the remainder reduced to lowest terms.

(2) Divide the product of $2x^2+x-6$ and $6x^2-5x+1$ by $3x^2+5x-2$. HINT. Factor and cancel.

(3) A certain mortgage would amount in $x$ months to $\$y$, and in $z$ months to $\$w$. Find the face of the mortgage and the rate of simple interest per annum.

N.B. Let $m$ = mortgage, $r$ = rate per cent. per annum; then in 1st case $m+m\cdot\dfrac{r}{100}\cdot\dfrac{x}{12}=y$, etc.; multiply first equation by $w$, 2nd by $y$, and subtract.

(4) Reduce $\dfrac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}}$ to simplest form.

N.B. Multiply the N. and the D. throughout by $x^2-1$.

(5) Divide \$100 among 3 men, 5 women, 4 boys, and 3 girls, so that each man shall have as much as a woman and a girl, each woman as much as a boy and a girl, and each boy half as much as a man and a girl.

N.B. $m=w+g$; $w=b+g$; $2b=m+g$; whence $b=3g$; $5:4:3:1$.

### Exercise 80.

(1) Factorise $x^2+4x-21$; $12x^4+11x^2y^2-15y^4$; and $a^4+a^2b^2+b^4$.

(2) $\frac{x^2-(y-z)^2}{(x+z)^2-y^2}+\frac{y^2-(z-x)^2}{(y+x)^2-z^2}+\frac{z^2-(x-y)^2}{(y+z)^2-x^2}$. Sum?

N.B. Test your answer by putting $x=y=z=1$.

(3) From $(a-b)^3+(b-c)^3+(c-a)^3$ subtract

$3\{a^2(c-b)+b^2(a-c)+c^2(b-a)\}$

Put $a=2$, $b=1$, $c=0$; test.

(4) $\frac{1}{7}(2x+1)-\frac{1}{10}(3x-1)+\frac{1}{11}(5x+3)=8$, find $x$.

(5) $\frac{(x+a)\,(x+b)}{x+a+b}=\frac{(x+c)\,(x+d)}{x+c+d}$, find $x$.

(6) $\frac{x}{3}+\frac{3}{x}=\frac{x}{5}+\frac{5}{x}$, find $x$.

(7) $3x-5y=x+9y+4=166$. Verify your results for $x$ and $y$.

(8) If each side of a square plot were enlarged by 10 yards it would contain $1\frac{1}{2}$ acres more. Find the side in yards.

## Exercise 81.

(1) Add together $a+b-2c$; $8a+4c+2b$; $3c-2a-6b$; and $2b-2c+3a$. Subtract $7a-5b$ from the sum; and multiply the remainder by $3a-6b+5c$.

(2) Multiply $x^2+y^2+z^2-2yz$ by $x^2-y^2-z^2+2yz$.

(3) Find the factors of $9x^2-3xy-42y^2$.

(4) Find the L. C. M. of $2x^4-4x^3-6x^2+16x-8$,
and $3x^4-15x^3+60x-48$.

(5) Find a fraction that is equal to these three together:—

$$\frac{x+3}{x^2-7x+10},\quad \frac{x+2}{x^2-8x+15},\quad \frac{x+5}{x^2-5x+6}.$$

(6) Reduce to lower terms the fraction

$$\frac{(15x^2-7x-2)\,(6x^2+7x+2)}{15x^2+13x+2}.$$

(7) $\frac{2}{x+y}+\frac{2}{x-y}-\frac{4x}{x^2+y^2}+\frac{4y}{x^2-y^2}$

HINT. Add 1st and 2nd; from this subtract 3rd; then add the 4th.

## Exercise 82.

(1) Solve the equation $\frac{1}{7}(27-x)+\frac{1}{5}(3x-4)-\frac{1}{9}(5x-2)=2$.

(2) Find the three factors of $256b^4-81a^4$.

(3) One factor of $4a^3-9a^2b-16ab^2+36b^3$ is $a-2b$, find the other two factors.

(4) If $a^2+2ab+9$ is divisible by $a+b$, what must be the numerical value of $b$?

(5) If $a+b+c=0$, show that $a^2+b^2+c^2=-2(ab+bc+ca)$.

(6) $\frac{x+3}{8}-\frac{x-3}{9}=\frac{3x+7}{7}-\frac{5x-6}{11}$, find $x$ and verify.

(7) If 27 men and 60 boys earn \$134 in 17 days, how much will 18 men and 24 boys earn in $23\frac{1}{2}$ days, supposing that 3 boys earn as much as 2 men?

HINT. Let $3x=$ a man's daily earning; $\therefore 2x=$ a boy's wages.

$$\therefore 201x=\$\frac{134}{17};\quad x=\$\frac{2}{51}, \text{ etc.}$$

## Exercise 83.

(1) $A$ can do a piece of work in $x$ days which $B$ can do in $y$ days; how many days will they take working together?

HINT. $A$ does 1 piece in $x$ days, or $y$ pieces in $xy$ days.
$B$ does 1 piece in $y$ days, or $x$ pieces in $xy$ days.
$\therefore$ $A$ and $B$ do $(x+y)$ pieces of work in $xy$ days, etc.

(2) Divide $x^4+\frac{9}{4}x^3+\frac{21}{8}x^2+\frac{33}{16}x+\frac{5}{16}$ by $x^2+\frac{3}{2}x+\frac{1}{4}$.

HINT. Multiply both quantities by 16.

(3) Find the remainder when $a^2+ab+c$ is divided by $a-x$.

(4) $8x-4y=5$; $9x=2y+10$, find $x$ and $y$, and test.

(5) A ship carries provisions enough for $m$ days, but if the number of people was $p$ more than at present, there would be food sufficient for only $n$ days. Find the number of the crew and passengers on board.

(6) Solve the equations $32x+81y=45$, and $28x-39y=369$.

HINT. Subtract; multiply difference by 7.

## Exercise 84.

(1) Find the H. C. F. of $x^3-3x^2-13x+15$ and $x^3+9x^2+11x-21$. N.B. Factor B − A.

(2) $\frac{1}{5}(x-3)+\frac{1}{3}(x-7)=\frac{1}{7}(x-9)+9$, find $x$.

(3) $\frac{x}{3}+\frac{y}{8}=41$; $4y-3x=0$, find $x$ and $y$.

(4) If $x^2+ax+576$ is a perfect square, determine the numerical value of $a$. N.B. $(a+b)^2=?$

(5) $\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}$, find the sum.

Test your answer by putting $a=2$, $b=1$, $c=0$.

(6) If $x=\frac{a+b}{a-b}$, $y=\frac{a-b}{a+b}$, find the value of $\frac{x^2+y^2}{x^2-y^2}$.

Test your answer by putting $a=2$, $b=1$.

(7) $\frac{1}{8}(5x-9)+\frac{1}{10}(7x-1)=\frac{1}{15}(11x+7)+6$. Test your result.

## Exercise 85.

(1) Factorise $x^2+3x-10$; $x^6-1$; $9x^2+15x-14$.

(2) $\frac{1}{10}(3x+2)=\frac{1}{4}(5y-11)-2\frac{1}{2}$ and

$\frac{3}{2}x+\frac{5}{4}y=25\frac{1}{4}$, find $x$ and $y$.

(3) A row of trees is planted with trees 20 feet apart; if they had been 15 feet apart the cost would have been increased by \$5 at 25 cents for each tree. Find the number of trees in the row and the length of the street in feet.

(4) Find the value of $(a+b)^2+(b+c)^2+(c+a)^2$, when $a=-1$; $b=-2$; $c=-3$.

(5) Divide $(a+b+c)^2+(a-b)^2+(b-c)^2+(c-a)^2$ by $a^2+b^2+c^2$.

(6) $\frac{2}{x}+\frac{3}{y}=31$; $\frac{5}{y}-\frac{7}{x}=31$, find $x$ and $y$.

(7) Reduce $\frac{x+15x^3-8x^2}{x-x^2-20x^3}$ to lowest terms.

(8) Simplify $\left(\frac{2x-y}{x+2y}-\frac{x-2y}{2x-y}\right)\div\frac{x^2+y^2}{x^2+2xy}$.

Put $x=1$, $y=0$, and test your answer.

## Exercise 86.

(1) $\frac{1}{21}(5x-13)-\frac{5}{14}(x+3)+4+\frac{1}{6}(5-x)=0$, find $x$.

(2) Reduce $\frac{39(x^4-4x^3+4x^2-1)}{65(x^4+4x^2-4x-1)}$.

(3) $\frac{1}{7}(3x-2y)+\frac{1}{6}(2x-3y)=2\frac{4}{6}$; and $31y-30x+111=0$; find $x$ and $y$.

(4) Resolve the expression $(1+x)\ (1+y)-xy(1-x)\ (1-y)$ into two factors.

(5) A pound of salt is added to 31 pounds of pure water; how much pure water must be added to the solution so that 32 pounds of the new mixture may contain 2 ounces of salt?

(6) A capitalist receives an income of \$4,640 from \$100,000, the greater part being invested at 5% and the rest at 4%; what is the amount at each rate?

(7) Divide a number $a$ into three parts so that the second shall be $n$ times and the third $m$ times the first.

## Exercise 87.

(1) Prove that if $c$ be a common measure of $a$ and $b$, it will also measure $ma \pm nb$.

(2) Find the H. C. F. of $6x^3-10x^2+10x-4$ and $9x^3+3x^2+3x-6$ by applying the principle of Q. 1.

(3) Examine $81x^4+108x^3-24x+4$, and divide it into two equal factors by comparing it with the formula of Exercise 24, page 20.

(4) Divide $\left(\frac{1}{1+x}+\frac{x}{1-x^2}\right)$ by $\left(\frac{1}{1-x^2}-\frac{x}{1+x}\right)$.

(5) Twenty-four persons subscribed equally to pay for a new boat, but four of them failed to pay up and consequently each of the others had to increase his subscription by \$2. What did the boat cost?

(6) A square floor would contain 17 square yards more if each side were a yard longer. Find its area.

(7) Compare $9a^4-12a^3b+34a^2b^2-20ab^3+25b^4$ with Exercise 24, page 20, and thus find the factors.

## Exercise 88.

(1) Factorise $x^2-y^2+z^2-a^2-2zx+2ay$.

(2) Simplify $\frac{3}{1-2x}-\frac{7}{1+2x}-\frac{4-20x}{4x^2-1}$.

(3) $2x+3y=22$; $3x+4y=31$, find $x$ and $y$.

(4) $\frac{1}{3}x+\frac{1}{2}y=\frac{7}{6}$; $\frac{3}{7}x-\frac{2}{3}y=\frac{4}{21}$.

(5) I paid a certain number of dollars for 16 yards of cloth; but if I had received a yard less for the money the cloth would have cost me 25 cents per yard more. Find the price per yard.

(6) $A$, $B$, and $C$ can dig a trench in 30 hours. $A$ can dig half as much again as $B$, and $B$ two-thirds as much again as $C$. In how many hours would $C$ alone perform the work?

(7) Find the H. C. F. of $x^4+x^3-2x^2+3x-3$ and $x^5-4x^3-2x^2+3x+2$. (Factor B – A.)

(8) Divide $x^3-8y^3-z^3-6xyz$ by $x-z-2y$.

## Exercise 89.

(1) A man walking $\frac{1}{3}$ of a mile per hour above his ordinary rate gains $\frac{3}{4}$ of an hour in going 39 miles. Find his ordinary rate per hour.

(2) Two ships sail from the same place, the first due north at 9 miles an hour, the other due east at 12 miles an hour. In how many hours will they be 60 miles apart? (See Euclid I. 47.)

(3) If a traveller had gone $\frac{1}{2}$ a mile an hour faster he would have arrived at his destination in $\frac{4}{5}$ of the time he spent on the journey; but if he had gone $\frac{1}{2}$ a mile an hour slower he would have been $2\frac{1}{2}$ hours longer on the road. Find the distance and the actual rate.

(4) When $A$ is 18 miles from Toronto he finds that at his present rate he will be $\frac{1}{2}$ hour too late to catch the Montreal boat. He quickens his speed by $\frac{1}{2}$ mile per hour and arrives just in time. Find his rate at first.

N.B. $2x^2+x-36=0$, when $x=$ 1st rate,
*i.e.* $(x-4)(2x+9)=0$; hence either $x-4=0$, or $2x+9=0$.
If $x-4=0$, $x=4$; if $2x+9=0$, $x=-4\frac{1}{2}$, which would have no meaning here.

## Exercise 90.

(1) If a certain class-room were 7 feet shorter and 5 feet wider it would be an exact square and of the same area as at present. Find the length and breadth of the class-room.

(2) $A$ and $B$ can pull 225 square rods of flax in 3 days; $B$ and $C$ 223; and $A$ and $C$ 230 in the same time. How many square rods can each do in one day?

(3) A man invested $330 in two parts; on one part he gained 15%, on the other he lost 8%. He received from both $345; find the amount of each investment.

(4) A Mississippi steamer takes 160 minutes less time to go from $A$ to $B$ than from $B$ to $A$. In still water she could make 14 miles per hour, but the river runs at the rate of $1\frac{3}{4}$ miles per hour. Find the distance from $A$ to $B$.

(5) When $A$ and $B$ dissolved partnership, $A$ got $2,070, $B$ $1,920; the capital of the firm was $3,400, and $A$'s money was put in for 12 months and $B$'s for 16 months. Find each man's share of the profits.

## Exercise 91.

(1) Prove that $a^0=1$; and that $a^m \times a^n = a^{m+n}$.

(2) $\frac{x}{a}-\frac{y}{b}=1$; $\frac{x}{b}+\frac{y}{a}=\frac{a}{b}$; find $x$ and $y$.

(3) A speculator paid $3,000 for some railroad shares; he reserved 10 shares, and made $4 a share on the remainder by selling them for $2,700. How many shares did he purchase?

(4) If a lake steamer could go $\frac{1}{2}$ mile an hour faster she could make a trip of 420 miles in 2 hours less time than she now makes it. What is the steamer's usual rate?

(5) Bought a number of cattle and made 20% on the cost; invested the proceeds in lambs and made 25% profit; spent the proceeds in horses and realized a gain of 16%. If my profit on the horses was $300, how much did I spend in cattle?

(6) If $a^2+b^2=c^2$,

simplify $(a+b+c)(a+b-c)(a+c-b)(b+c-a)$.

## Exercise 92.

(1) Angus starts from $C$ at the same time that Bert starts from $D$; when they meet, Angus has travelled 30 miles more than Bert. Angus arrives at $D$ in 4 days and Bert at $C$ 9 days after they passed each other. Find the distance from $C$ to $D$.

N.B. Let $2x$ = distance; $\therefore x+15$, $x-15$ are distances travelled by each before they meet; $A$ goes $\frac{1}{4}(x-15)$, $B$ goes $\frac{1}{9}(x+15)$ per day, etc.

(2) Find the H. C. F. of $ab+am$, $bn+am$, $b^2n-m^2n$.

(3) $A$ and $B$ perform a piece of labor in 20 days; but if $A$ had worked twice as fast and $B$ half as fast they would have completed the job in 15 days. How many days would each require alone? N.B. Let $2x=A$'s time, $y=B$'s time;

then in II. case $x=A$'s and $2y=B$'s time; $\frac{1}{2x}+\frac{1}{y}=\frac{1}{20}$, etc.

(4) Factor $x^2-2x-255$; and $21x^2-13xy-20y^2$.

(5) Compare $9x^4-6x^3+43x^2-14x+49$ with

$a^2+b^2+c^2+2ab+2ac+2ca$, and thus find its *square root, i.e. one of its two equal factors.*

N.B. $a^2=9x^4$, $c^2=49$, $2ab=-6x^3$, $2bc=-14x$, etc.

## Exercise 93.

(1) Prove that

$$(x-y)^3+(y-z)^3+(z-x)^3=3(x-y)(y-z)(z-x)$$

(2) Find the value of $a^3+b^3-c^3+3abc$, when $c=a+b$.

(3) A person invests $500 partly at 5% and partly at 3%, and he makes $4\frac{1}{2}$% on the whole. Find the amount of each investment.

(4) $\frac{x-2}{x-3}-\frac{x-3}{x-4}=\frac{x-5}{x-6}-\frac{x-6}{x-7}$. N.B. Add each side separately, $(x-3)(x-4)=(x-6)(x-7)$, etc.

(5) If I sell my horse for \$200 I shall lose money, but if I get \$250 for him my gain will be three-fourths of the former loss. Find the true value of the horse.

(6) Between 2 and 3 o'clock, I mistook the hour hand for the minute hand and therefore thought it was 55 minutes earlier than the true time. Find the correct time.

(7) A mass of lead and tin weighs 180 pounds in the air, but if it is suspended in water it weighs 21 pounds less. On experiment it is found that 37 pounds of tin weigh 32 pounds when immersed in water, and that 23 pounds of lead weigh only 21 pounds in water. Find the number of pounds of tin and of lead in the given mass.

## Exercise 94.

(1) If the sign $\sqrt{}$ placed over a quantity means "the square root," so that $\sqrt{4}=2$, $\sqrt{9}=3$, $\sqrt{16}=4$, etc.; simplify the following expressions, $\sqrt{81}-\sqrt{25}$; $\sqrt{a^2}-\sqrt{b^2}$; $\sqrt{(a^2+2ab+b^2)}$.

(2) If $\sqrt[3]{}$ denotes the cube root of a quantity, so that $\sqrt[3]{8}=2$, $\sqrt[3]{27}=3$, $\sqrt[3]{64}=4$, etc.; simplify the expressions $\sqrt[3]{a^6}-\sqrt[3]{b^{12}}$; $\sqrt{81}-\sqrt[3]{125}$; $\sqrt[3]{(a^3+3a^2b+3ab^2+b^3)}$; $\sqrt[3]{(27x^3+27x^2+9x+1)}$.

N.B. In the last example compared with the preceding observe that $b=1$, and $a=3x$.

(3) If $(a+b)^2=a^2+2ab+b^2$, what is $\sqrt{(a^2+2ab+b^2)}$? Of $x^2+4x+4$?

(4) Reduce $\sqrt{(a^2+b^2+c^2+2ab-2bc-2ca)}$ to simplest form.

N.B. Review Exercise 24 and Exercise 61, and do this question by *inspection* alone.

(5) What is the value of $\sqrt[4]{81}-\sqrt[4]{16}$? Of $\sqrt[3]{343}+\sqrt[4]{256}$?

(6) If $9\,.\,9\,.\,9=729$, express 729 with the **Radical Sign** ($\sqrt{}$) so that its value may $=9$; 1331 so that its value may $=11$.

(7) Find the value of $5\sqrt{(62+3x)}-\frac{1}{2}\sqrt{(95\frac{2}{3}-5x)}$, when $x=6\frac{1}{3}$.

(8) Evaluate $\frac{x}{y}-\sqrt{\frac{1+x}{1-y}}$, when $x=\frac{1}{4}$, $y=\frac{1}{5}$.

## Exercise 95.

(1) Find the value of $[\sqrt{(a^2+b^2)}+c]\,[\sqrt{(a^2+b^2)}-c]$, when $a=4$, $b=5$, $c=6$.

(2) Evaluate $\sqrt{(1-a)}+\sqrt{[1-a+\sqrt{(1+a)}]}$ when $a=\frac{24}{25}$.

(3) Find the numerical value of $\frac{a-b}{a+b}+\frac{a+b}{a-b}$.
when $a=2+\sqrt{3}$, and $b=2-\sqrt{3}$.

(4) If $a^b=b^a$ and $a=2b$, then $b=2$. Prove this theorem.

(5) Find the square root of $\frac{a^2}{9}-\frac{2ax}{21}+\frac{x^2}{49}-\frac{ay}{6}+\frac{xy}{14}+\frac{y^2}{16}$.

(6) If $x=9a^2+12ab$, and $y=2b^2+6ab$, find in terms of $a$ and $b$ the square root of $x^2-2xy+4y^2$.

(7) Find the least number that will give a square number for product when it is multiplied by 650.

(8) The difference between the squares of any two consecutive numbers is equal to the sum of the numbers. Prove this.

## Exercise 96.

Prove the following theorems :—

(1) $(a+b)^3=a^3+b^3+3ab(a+b)$. **Learn this by heart.**

(2) $(a+b+c)^3=a^3+b^3+c^3+3a^2(b+c)+3b^2(c+a)+3c^2(a+b)+6abc$.

(3) $(a+b+c)^3=a^3+b^3+c^3+3ab(a+b)+3bc(b+c)+3ca(c+a)+6abc$.

(4) $(a+b+c)^3=a^3+b^3+c^3+3(a+b+c)\ (ab+bc+ca)-3abc$.

(5) $(a+b+c)^3=a^3+b^3+c^3+3(a+b)\ (b+c)\ (c+a)$.

(6) In question 2 substitute $c+d$ for $c$ throughout and thus prove $(a+b+c+d)^3=a^3+b^3+c^3+d^3+3a^2(b+c+d)+3b^2(c+d+a)+3c^2(d+a+b)+6(abc+bcd+cda+dab)$.

(7) Write out from memory the preceding theorems ten times; next read them off from memory in the same order; and again write them out, using $x$, $y$, $z$, $w$ instead of $a$, $b$, $c$, $d$.

(8) Show that $a^3+b^3+c^3-3abc=$
$(a+b+c)\ (a^2+b^2+c^2-ab-bc-ca)$

(9) Show that $a^3+b^3+c^3-3abc=$
$(a+b+c)\ \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$.

(10) Show that $a^3+b^3+c^3-3abc=$
$-(a+b+c)\ [(a-b)\ (b-c)+(b-c)\ (c-a)+(c-a)\ (a-b)]$.

(11) Factor $a^3+b^3-c^3+3abc$; $a^3-b^3+c^3+3abc$; $a^3-b^3-c^3-3abc$.

(12) In $a^3+b^3+c^3-3abc$ substitute $c+d$ for $c$ and arrange the result in a form similar to the original expression.

## Exercise 97.

(1) Prove that $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$.

(2) From question (1) find without actual multiplication the squares of $-a+b+c$; $a-b+c$; $a+b-c$; and $a-b-c$.

(3) If $x^2+bx+c^2$ is a perfect square, find the value of $b$ in terms of $x$ and $c$.

(4) The difference between the squares of any two odd numbers is always exactly divisible by 8. N.B. Let $x$ and $y$ be any two numbers, then $2x$ and $2y$ are even numbers, and $2x+1$ and $2y+1$ are odd numbers. Then if $x$ and $y$ are even, $x-y$ is even and divisible by 2; if $x$ and $y$ are odd numbers $x+y$ is even; if one is odd and the other even $x+y$ is odd and $x+y+1$ is even.

(5) The square of every odd number with 1 subtracted is divisible by 8.

(6) The area of a rectangular field is 6 acres 960 yards, and its length is three times its breadth. Find the distance round the field.

(7) Find the diagonal of a common field 143 yd. by 116 yd.

(8) Two boys start from a street corner, one west, the other north, the one at 3 miles per hour, the other at 4 miles per hour; find the length of the straight line between them at the end of two hours and three-quarters. (See Euclid I. 47.)

## Exercise 98.

(1) $A$ earns twice as much and $B$ three times as much per day as $C$ earns. $A$ works 7 days, $B$ 4 days, and $C$ 10 days, and their total earnings amount to \$72. Find the daily wages of each.

(2) A span of horses is worth \$460, and the carriage is worth \$210. When the better horse is in the carriage the rig is worth three times as much as the second horse together with 10% of the value of the first horse. Find the value of each horse.

(3) A bicyclist set out from home at 8 miles per hour, and kept up that speed until his wheel broke down. He was then compelled to walk back at $3\frac{1}{2}$ miles per hour and arrived at home after an absence of $11\frac{1}{2}$ hours. How far did he ride before the accident?

(4) After paying out $\frac{1}{m}$ and $\frac{1}{n}$ of my money, I had only $\$b$ left. How much had I at first?

(5) Between 11 and 12 o'clock the minute hand is in a straight line with the hour hand. Find the exact time.

## Exercise 99.

(1) Brown leaves Stratford and Jones leaves Seaforth at the same time, as they both suppose on consulting their watches. The distance is 24 miles and they travel towards each other, Brown driving 2 miles in the same time that Jones drives 3 miles. They meet at Mitchell, which is half-way between the other towns, and on comparing notes to find out how this can have happened, they discover that Brown's watch is 15 minutes too fast, and Jones' watch 15 minutes too slow. At what rates per hour did each of them drive the distance?

N.B. Let $2x$ and $3x$ be the rates; equate the times.

(2) Two sums invested at 3% and 9% respectively produce an income of $240; but if the rates of interest were interchanged, the income would be doubled. Find the sums.

(3) Archer and Banker start from two towns at the same time, and meet in 7 hours, *A* riding 2 miles an hour faster than *B*. But if *B* had gone 1 mile an hour faster, and *A* at half the speed he did go, they would have taken 2 hours longer to meet. Find the distance between the towns.

(4) Find the numerical value of $a$ that will make the expression $x^4 - x^3 - x^2 - ax$ exactly divisible by $x^2 + x$.

(5) A railway train is timed to make 45 miles an hour between two stations, X and Y. From X to an intermediate station, Z, it goes up grade at 40 miles an hour; from Z to Y it goes down grade at 50 miles an hour, and arrives precisely on time. Compare the distance from X to Z with the distance from Z to Y.

N.B. Let $a$ = distance in miles from X to Z; $b$ = distance from Z to Y; then $(a+b) \div 45$ = schedule time, etc.

(6) A liquor dealer buys proof spirits and mixes a certain quantity of water with it. He sells the mixture at 2 shillings a gallon more than he paid for the spirits and thus gains $23\frac{3}{4}\%$ on his outlay. But if he had used twice the quantity of water that he did use, he would have gained $37\frac{1}{2}\%$ on his money. Find the proportion of water in the mixture he sold.

N.B. Let $s =$ No. gallons of spirits; $y$ shillings $=$ cost per gal.; and $x =$ No. gallons of water added to form the mixture; cost $= sy$; selling price $= \frac{99}{80}sy$; supposed selling price $= \frac{110}{80}sy$; divide the equations.

## Exercise 100.

(1) Find the H. C. F. of $1 + x + x^3 - x^5$ and $1 - x^4 - x^6 + x^7$.

(2) If $a + b = 1$, prove that $(a^2 - b^2)^2 = a^3 + b^3 - ab$.

(3) Solve the equation $(x+2)^2 + (x+5)^2 = (x+7)^2$.

(4) Divide $x^3 + 8y^3 - 27z^3 + 18xyz$ by $x + 2y - 3z$, and test your answer by substituting $x = 5$, $y = 4$, $z = 3$ throughout.

(5) Find the square root of

$$36x^2 - 120ax - 12a^2x + 100a^2 + 20a^3 + a^4.$$

(6) Find two equal factors of the expression

$$(x^2 + y^2)(x^2 + z^2) + 2x(x^2 + yz)(y + z) + 4x^2yz.$$

(7) Show that $(x-y)^5 + (y-z)^5 + (z-x)^5$ is equal to

$$5(x-y)(y-z)(z-x)(x^2 + y^2 + z^2 - xy - yz - zx).$$

(8) Show that

$$(a+b)^3 + (a+c)^3 + (a+d)^3 + (b+c)^3 + (b+d)^3 + (c+d)^3$$
$$= 3(a+b+c+d)(a^2 + b^2 + c^2 + d^2).$$

(9) Show that $8(a+b+c)^3 - (a+b)^3 - (b+c)^3 - (c+a)^3$

$$= 3(2a+b+c)(a+2b+c)(a+b+2c).$$

(10) Show that $(3a-b-c)^3 + (3b-c-a)^3 + (3c-a-b)^3$

$$- 3(3a-b-c)(3b-c-a)(3c-a-b)$$
$$= 16(a^3 + b^3 + c^3 - 3abc).$$

# THIRD STAGE.

## TYPE SOLUTIONS.

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NOTE.—As this book is primarily intended to be an aid to oral teaching, very little space has been given up to exhibit the mechanical arrangement of solutions. No one ever learns much of that from a text-book in his first course. The teacher's work will be the model in any case. The following pages are, however, intended to teach something *incidentally* about the clear and concise arrangement of the work as well as to illustrate certain methods of working. A great deal depends on rapidity and accuracy at the written tests given at the various examinations, and rapidity and accuracy are nearly impossible achievements for a pupil who has not first learned neatness, order, and method. To write out in perfect form solutions of exercises already mastered is a fine piece of training, and many of the best men at the universities who intend to compete for the highest honors and prizes regularly spend one hour a day for many months in succession in writing out demonstrations and solutions against time, and in this way they acquire extraordinary speed and wonderful accuracy in placing their work on paper. The amount of work which an average pupil can do in one hour can be more than trebled by this kind of training in the course of a single year. Whenever it is practicable he should correct his own work two or three days after writing it and then hand it in to the teacher for final revision. With the limited time at command in most public schools, a very good plan is to get pupils to copy down on slips of paper two or three exercises on each slip, and then distribute these slips at random to the class, keeping a record of the time each pupil consumes in writing out his solutions in perfect form, and giving no credit for work that is not completely accurate. If the blackboard space is sufficient the whole class should frequently do this work with the crayon instead of the pen. The following exercises should all be reproduced by the pupil independently of the book a considerable number of times. If they are written on the blackboard, each pupil may in turn read over his work aloud and demonstrate the solution, the remaining members being required to cross-examine and "quiz" him at the conclusion. In this connection the teacher may occasionally write down false or inaccurate solutions and require pupils to detect the fallacy or the mistake and to correct the work. To secure thorough, solid progress, review work of some kind should be allowed about one-half of all the time that can be allotted to algebra on the time-table, and the rapid reproduction of former work is the essential part of review.

(1) Express $[a-\{b-c-(d-e)-f\}-g]$ in simplest form.

SOLUTION. $+\quad -\quad +\quad +\quad -\quad +\quad -$

$$a-b+c+d-e+f-g.$$

N.B. In the first line write down the signs only, changing those that require change. In writing out this line follow out the changes caused by each minus sign separately. The line is to be written piece by piece, not continuously, thus the signs in this question are to be written as here numbered, 1, 2, 3, 4, 6, 7, 5.

(2) Simplify the expression $2x^3+\{4x^2-(7x^3-2x)\}-[8x^2-4x-\{5x^3+(6x^2-\overline{4x-7})-11x\}]$.

SOLUTION. $+\quad +\quad -\quad +\quad -\quad +\quad +\quad +\quad -\quad +\quad -$

$$+2x^3+4x^2-7x^3+2x-8x^2+4x+5x^3+6x^2-4x+7-11x.$$

Result $=2x^2-9x+7$.

N.B. The line written over $4x-7$ is called **the vinculum** and has the same meaning as a bracket.

(3) Multiply $x^6-5x^4+5x^2-3$ by $x^4+5x^2+3$.

SOLUTION.

| | |
|---|---|
| | $1+0-5+0+5\quad +0-3$ |
| $1$ | $1+0-5+0+5\quad +0-3$ |
| $+0$ | |
| $+5$ | $+5+0-25\quad +0+25\quad +0-15$ |
| $+0$ | |
| $+3$ | $+3\quad +0-15\quad +0+15+9-9$ |
| | $x^{10}+0+0+0-17x^6+0+7x^4+0+0+0-9$ |

N.B. Zeros are introduced to mark the places of the missing powers, and the **detached coefficients** are used to save the trouble of repeating $x$'s. At the end, $x^6$ multiplied by $x^4$ shows that the first term of the product is $x^{10}$ and the other powers follow in regular order 9, 8, 7, 6, 5, 4, 3, 2, 1, those having zero coefficients being omitted.

(4) Multiply $x^5+5x^4+15x^3+30x^2+24x+21$
by $x^4-5x^3+10x^2-5x+1$. Question 15, Ex. 20.

SOLUTION.

| | |
|---|---|
| | 1 + 5 + 15 + 30 + 24 + 21 |
| 1 | 1 + 5 + 15 + 30 + 24 + 21 |
| −5 | −5 − 25 − 75 − 150 − 120 − 105 |
| +10 | +10 + 50 + 150 + 300 + 240 + 210 |
| −5 | −5 − 25 − 75 − 150 − 120 − 105 |
| +1 | 1 + 5 + 15 + 30 + 24 + 21 |
| | $x^9+0+0+0+0+131x^4+0+120x^2-81x+21$ |

N.B. Each partial product is written opposite the figure that produces it, and on adding up, we know that the first term is $x^9$ since $x^5 \times x^4 = x^9$, and the zero coefficients show that $x^8$, $x^7$, $x^6$, $x^5$ and $x^3$ are to be omitted in the product.

(5) Find the continued product of $x+4$, $x+10$, $x-7$, $x-9$, and $x+2$.

SOLUTION.

| | |
|---|---|
| | 1 + 4 |
| 1 | 1 + 4 |
| +10 | 10 + 40 |
| | 1 + 14 + 40 |
| 1 | 1 + 14 + 40 |
| −7 | − 7 − 98 − 280 |
| | 1 + 7 − 58 − 280 |
| 1 | 1 + 7 − 58 − 280 |
| −9 | − 9 − 63 + 522 + 2520 |
| | 1 − 2 − 121 + 242 + 2520 |
| 1 | 1 − 2 − 121 + 242 + 2520 |
| +2 | 2 − 4 − 242 + 484 + 5040 |
| | $x^5+0-125x^3+0+3004x+5040$ |

N.B. This arrangement is known as **Horner's Synthetic Multiplication.**

(6) Explain by an example **Horner's Synthetic Division.**

DEMONSTRATION. As division is simply the *un-multiplying* of a product of which one factor is given we require to reverse the process of multiplication in each particular, *i.e.* to subtract where we formerly added and to divide where we multiplied. Let us take the factors $a^2+2ab+b^2$ and $a+b$ and first form the product, and then divide the product back by $a+b$. The quotient must be $a^2+2ab+b^2$. The operation will stand thus:—

EXAMPLE (*a*)

$$\begin{array}{r|l}
 & 1+2+1 \\
1 & 1+2+1 \\
+1 & \quad\; 1+2+1 = \text{partial product } \textit{added.} \\ \hline
1 & 1+3+3+1 = \text{the product.} \\
-1 & \quad -1-2-1 = \text{partial product } \textit{subtracted.} \\ \hline
 & 1+2+1 \qquad = \text{quotient.}
\end{array}$$

In order to change the signs of the partial product as required for subtraction we change the sign of every term of the divisor except the first term. The quotient is found by dividing the sum of each column by the first term. A second example will make this more apparent. Let us take $a^2+2ab+b^2$ and $2a+3b$ and proceed as before. The operation will stand thus:—

EXAMPLE (*b*)

$$\begin{array}{r|l}
 & a^2+2ab+b^2 \\
2a & 2a^3+4a^2b+2ab^2 \\
+3b & \qquad +3a^2b+6ab^2+3b^3 = \text{partial product } \textit{added.} \\ \hline
2a & 2a^3+7a^2b+8ab^2+3b^3 = \text{product.} \\
-3b & \qquad -3a^2b-6ab^2-3b^3 = \text{partial prod'ct } \textit{subtracted.} \\ \hline
 & a^2+2ab+b^2 \qquad = \text{quotient.}
\end{array}$$

The last two lines are found thus:—Dividing $2a$ into $2a^3$ we get $a^2$, the *first term of the quotient.* Multiply this $a^2$ into $3b$ with its sign changed we get $-3a^2b$ and place it under $+7a^2b$; the sum is $4a^2b$, which we divide by $2a$ and get $2ab$, the *second term of the quotient.* Multiply this by $-3b$ and we have $-6ab^2$, and the sum is $2ab^2$ in the third column. Dividing this by $2a$ we get $b^2$, the *third term of the quotient.* Again multiplying by $-3b$, adding and we find the sum $=0$, which ends the operation and shows that there is no remainder. Using detached coefficients only the division would stand thus:—

$$\begin{array}{r|l}
2 & 2+7+8+3 \\
-3 & \quad -3-6-3 \\ \hline
 & 1+2+1 \qquad = a^2+2ab+b^2.
\end{array}$$

(6) Divide $12a^4 - 26a^3 + 10a^2 + 8a - 4$ by $2a^2 - 3a + 1$.

| | | |
|---|---|---|
| **Solution.** 2 | $12 - 26 + 10$ | $+ 8 - 4$ |
| $+3$ | $+18 - 12$ | $-12$ |
| $-1$ | $-6$ | $+4 + 4$ |
| | $6 - 4 - 4$ | $0 + 0 = 6a^2 - 4a - 4$. |

The second vertical line marks the separation between quotient and remainder. Its position is fixed by counting off from the end of the dividend as many terms as there are in the divisor, less one. The next example contains a remainder to illustrate this point more clearly.

(7) Divide $5x^5 - 18x^3 - 8x^2 + 20x - 5$ by $x^3 + 2x^2 - 3$.

| | | |
|---|---|---|
| **Solution.** 1 | $5 + 0 - 18$ | $- 8 + 20 - 5$ |
| $-2$ | $-10 + 20$ | $- 4$ |
| $+0$ | $+ 0$ | $- 0 + 0$ |
| $+3$ | | $+15 - 30 + 6$ |
| | $5 - 10 + 2$ | $+ 3 - 10 + 1$ |

Quotient $= 5x^2 - 10x + 2$; remainder $= 3x^2 - 10x + 1$.

The line separating the quotient and remainder is drawn after the third term of the dividend, for $x^5 \div x^3 = x^2$, that is the quotient will be of *two* dimensions and will therefore contain *three* terms.

(8) Find the numerical value of $10x^4 - 7x^3 - 19x^2$, when $x = 3$.

**Solution.** If $x = 3$, then $x - 3 = 0$.

Divide the given expression by $x - 3$.

| | | |
|---|---|---|
| 1 | $10 - 7 - 19 + 0$ | $+ 0$ |
| $+3$ | $+30 + 69 + 150$ | $+450$ |
| | $10 + 23 + 50 + 150$ | $+450$ |

This shows that the given quantity is =

$(x - 3)(10x^3 + 23x^2 + 50x + 150) + 450$, and as $x - 3 = 0$, the value of the given expression must be 450.

(9) Find the numerical value of

$x^6 - 102x^5 + 100x^4 + 102x^3 - 99x^2 - 201x$, when $x = 101$.

| | | |
|---|---|---|
| **Solution.** 1 | $1 - 102 + 100 + 102 - 99 - 201$ | $+0$ |
| $+101$ | $+101 - 101 - 101 + 101 + 202$ | $+101$ |
| | $1 - 1 - 1 + 1 + 2 + 1$ | $+101$ |

$\therefore$ expression $= (x - 101)(x^6 - x^5 - x^4 + \text{etc.}) + 101$,

and $\therefore$ value $= 101$.

In all cases of this kind therefore the remainder is the numerical value of the expression.

(10) What value of $x$ will make $x^2-2x+3$ an exact divisor of the expression $7x^4-3x^2+1$?

SOLUTION. By division we find the remainder $= -34x-11$.
To make the division exact we must have this $=0$,
whence $x=-\frac{11}{34}$.

(11) Find the value of $x$, when $10x^{10}+10x^6+10x^3-1000$ is exactly divisible by $x^3(x^4+1)-(x-1)$.

SOLUTION. Dividing by $x^7+x^3-x+1$, we get the quotient $10x^3$ and a remainder of $10x^4-1000$, and this must $=0$, as in examples 8 and 9;
*i.e.* $10x^4=1000$; $x^4=100$; $x^2=10$; $x=\sqrt{10}=3{\cdot}1622777$.

(12) Find the value of $x^3-4x+3$, when $x=3$.

SOLUTION. Since $x^3=x^2.x=x^2.3$, $x^3=3x^2=3x.x=9x$.
Take in $-4x$, and $x^3-4x=5x=15$.
Add $+3$, and $x^3-4x+3=18$.

The work may be arranged as in Horner's division, and is then an exemplification of that method. Instead of bringing down $-4x$ and $+3$, carry up $9x$ and $15$ thus:—

| | | | |
|---|---|---|---|
| | $x^3+0$ | $-4x$ | $+3$ |
| $3$ | $3x^2$ | $+9x$ | $+15$ |
| | $3x$ | $+5$ | $+18$ |

(13) Divide $x^3+y^3+z^3-3xyz$ by $x+y+z$.

SOLUTION.

| | | |
|---|---|---|
| $x$ | $x^3+\;0.x^2\;-\;x(3yz)$ | $+(y^3+z^3)$ |
| $-(y+z)$ | $-x^2(y+z)+\;x(y+z)^2$ | $-(y^3+z^3)$ |
| | $x^2-x(y+z)+(y^2-yz+z^2)$ | |

Here the dividend is arranged in descending powers of $x$ and all the coefficients are put in brackets.

(14) Reduce to simplest form
$(a+b+c+d)^2+(a-b-c+d)^2+(a-b+c-d)^2+(a+b-c-d)^2$

SOLUTION. The brackets are alike except the signs; and for every $a$ there is a $b$, a $c$ and a $d$, *i.e.* **the expressions are symmetrical.** Each bracket is a square, and therefore the expansion will contain only *two* kinds of terms, viz. *squares* like $a^2$, and *products* like $2ab$. Following $a^2$ through we get $4a^2$, and we therefore know that $4a^2+4b^2+4c^2+4d^2$ is part of the answer. Next following $2ab$ through the four brackets, we get 0, each pair cancelling, and we therefore know that $2ac$, $2ad$, etc., will also cancel. Hence $4(a^2+b^2+c^2+d^2)$ is the result.

(15) Simplify the expression

$$(a+b+c)^2+(a+b-c)^2+(a+c-b)^2+(b+c-a)^2.$$

SOLUTION. This expression is **Cyclo-Symmetrical,** for it remains the same when we change $a$ into $b$, $b$ into $c$, and $c$ into $a$ in each bracket. The first bracket becomes $b+c+a$ and is the same; the second becomes $b+c-a$, which is the same as the fourth; the third becomes $b+a-c$, which is the same as the second; and the fourth becomes $c+a-b$, which is the same as the third. This is a simple **Test of Cyclo-Symmetry,** from which we can write down the whole of the result by calculating one part of it, thus: $a^2+a^2+a^2+a^2=4a^2$

$$2ab+2ab-2ab-2ab=0$$

Result $=4(a^2+b^2+c^2)$, by symmetry.

(16) Simplify $(a+b-2c)^2+(a+c-2b)^2+(b+c-2a)^2$.

SOLUTION. The expression is symmetrical; hence we get,

$$a^2+a^2+4a^2=6a^2;$$

$$2ab-4ab-4ab=-6ab.$$

$\therefore$ result $=6(a^2+b^2+c^2-ab-bc-ca)$, by symmetry.

(17) Simplify $(ax+by+cz)^2+(ax+cy+bz)^2+(bx+ay+cz)^2$
$+(bx+cy+az)^2+(cx+ay+bz)^2+(cx+by+az)^2$.

SOLUTION. The symmetry is apparent; wherever there is an $ax$ there is a corresponding $bx$ and $cx$, etc., throughout. Take the square terms and follow $a$ through each bracket and we get: $a^2x^2+a^2x^2+a^2y^2+a^2z^2+a^2y^2+a^2z^2$,

*i.e.* $2a^2(x^2+y^2+z^2)$.

Similarly the double products with $ab$ in them are:—

$$2ax(by)+2ax(bz)+2ay(bx)+2az(bx)+2ay(bz)+2az(by)$$
$$=4ab(xy+yz+zx),$$

from which we see that part of the result is

$$2a^2(x^2+y^2+z^2)+4ab(xy+yz+zx),$$

and therefore, by symmetry, the whole result must be

$$2(a^2+b^2+c^2)(x^2+y^2+z^2)+4(ab+bc+ca)(xy+yz+zx).$$

(18) Simplify

$$(a+b+c)^3+(a+b-c)^3+(a+c-b)^3+(b+c-a)^3.$$

SOLUTION. The symmetry is plain; each succeeding bracket is formed from the first by changing one sign. Now a complete cube *(see Exercise 96)* contains *three* sorts of terms, 1st those like $a^3$, 2nd those like $3a^2b$, and 3rd those like $6abc$. Following $a^3$ throughout, we get $a^3+a^3+a^3-a^3=2a^3$, hence $2(a^3+b^3+c^3)$ is one part of the result. Following $3a^2b$, we have $3a^2b+3a^2b-3a^2b+3a^2b=6a^2b$. Hence part of the answer must be $2(a^3+b^3+c^3)+6(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)$, and the remaining part must be of the same type as $abc$. To find this part, put $a=b=c=1$, which makes the given expression $=30$, and the part found $=42$; and from this we see that $-12abc$ is the remaining part of the result, and the whole expression is $=2(a^3+b^3+c^3)+6(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)-12abc$.

(19) Simplify

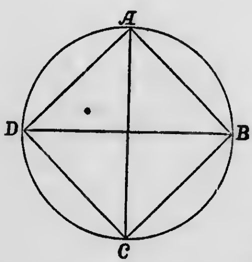
$$(a+b+c)^3-(a+b-c)^3-(a+c-b)^3-(b+c-a)^3.$$

SOLUTION. The expression is symmetrical. Reasoning as in No. 18, $a^3-a^3-a^3+a^3=0$; which shows that $b^3$ and $c^3$ also vanish. $3a^2b-3a^2b+3a^2b-3a^2b=0$; and all similar products vanish. Thus $abc$ is the type of the only term left. To find its coefficients put $a=b=c=1$, and the given expression $=27-1-1-1=24$, and thus the given quantity $=24abc$.

(20) Expand $(a+b+c+d)^3$ without actual multiplication.

SOLUTION. As in Ex. 96 $(a+b+c)^3$ is shown to be equal to $a^3+b^3+c^3+3a^2(b+c)+3b^2(c+a)+3c^2(a+b)+6abc$, we have by symmetry

$$\begin{aligned}(a+b+c+d)^3 &= a^3+b^3+c^3+d^3\\ &+3a^2(b+c+d)+6bcd\\ &+3b^2(c+d+a)+6cda\\ &+3c^2(d+a+b)+6dab\\ &+3d^2(a+b+c)+6abc\end{aligned}$$




N.B. If we write $a, b, c, d$ in a circle and draw all possible straight lines from $a$ to $b$, $a$ to $c$, $a$ to $d$, $b$ to $c$, $b$ to $d$, $c$ to $d$, and so on if there are more than four letters, then the straight lines will indicate the double products in the square, the triangles will show the products like $abc$ in the cube, the quadrilaterals, the products like $abcd$, etc. Thus with three letters there is only one product like $abc$ for only one triangle can be drawn; but with four letters there are four triangles, with five letters there are ten triangles, etc. Let the student form the cube of $a+b+c+d+e$ by symmetry.

(21) What is the value of $a^3+b^3+c^3-3abc$ when $c+d$ is substituted for $c$? See Exercise 96, question 12.

SOLUTION. The given expression is symmetrical for $a$, $b$, $c$, and contains all their cubes and all their products three and three together. The required expression must contain the same two types with four letters, hence in the light of No. 20 we can write $a^3+b^3+c^3+d^3-3bcd-3cda-3dab-3abc$. Let the reader form the expression when $c+d+e$ is substituted for $c$.

N.B. By noticing the symmetry it is generally easy to correct an error, *e.g.* every term of a square will be of *two* dimensions only, hence if we found $2a^2b$ or $2abc$ which are of three dimensions, we should perceive that there must be a mistake. Every product, quotient, remainder will preserve its symmetry.

(22) Reduce to simplest form the expression

$$(a+b+c)(x+y+z)+(a+b-c)(x+y-z)$$
$$+(a-b+c)(x-y+z)+(-a+b+c)(-x+y+z).$$

Solution. The symmetry is manifest, the last three products being formed from the first by changing the sign in each bracket; thus the expansions can be derived in the same way, viz. the 2nd can be formed from the 1st by changing $c$ into $-c$ and $z$ into $-z$, and similarly for the rest. Thus we get

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1st term | $=+ax$ | $+ay$ | $+az$ | $+bx$ | $+by$ | $+bz$ | $+cx$ | $+cy$ | $+cz$ |
| 2nd " | $=+$ | $+$ | $-$ | $+$ | $+$ | $-$ | $-$ | $-$ | $+$ |
| 3rd " | $=+$ | $-$ | $+$ | $-$ | $+$ | $-$ | $+$ | $-$ | $+$ |
| 4th " | $=+$ | $-$ | $-$ | $-$ | $+$ | $+$ | $-$ | $+$ | $+$ |
| Sum | $=4ax$ | | | | $+4by$ | | | | $+4cz.$ |

(23) Factor $a^3+b^3+c^3-3abc$.

Solution (1). Looking to the symmetry we can arrange the expression

$$\begin{vmatrix} +a^3-abc \\ +b^3-abc \\ +c^3-abc \end{vmatrix} = \begin{vmatrix} +a(a^2-bc) \\ +b(b^2-ca) \\ +c(c^2-ab) \end{vmatrix}$$

Now $a^2-bc$ is evidently an incomplete expression of which the full form is $a^2+b^2+c^2-ab-bc-ca$, and we therefore complete each of these brackets by adding the missing terms and subtracting the same from each; we thus get

$$\begin{aligned} &a(a^2+b^2+c^2-ab-bc-ca)-a(b^2+c^2-ab-ca) \\ &+b(\quad\text{"}\qquad\text{"}\qquad\text{"}\quad)-b(c^2+a^2-bc-ab) \\ &+c(\quad\text{"}\qquad\text{"}\qquad\text{"}\quad)-c(a^2+b^2-ca-bc) \end{aligned}$$

N.B. The second and the third lines are formed from the first by changing $a$ into $b$, $b$ into $c$ and $c$ into $a$ successively.

Adding up, the second row of terms cancels out, and we have $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$, the sum of the first row.

SOLUTION (2). Looked at as a symmetrical expression $a^3+b^3$ is incomplete, and we add the missing term $3ab(a+b)$; see Exercise 96, No. 1; and subtract the same term at the end. This gives us $(a+b)^3+c^3-3ab(a+b+c)$, of which $a+b+c$ is evidently one factor. The quotient is $(a+b)^2-(a+b)c+c^2-3ab$, which reduces to $a^2+b^2+c^2-ab-bc-ca$, the other factor.

SOLUTION (3). Looking at $a^3+b^3+c^3-3abc$ and comparing it with Exercise 96, No. 4, we see that it is symmetrically incomplete, and we add and subtract the missing term

$3(a+b+c)(ab+bc+ca)$ and get

$(a+b+c)^3-3(a+b+c)(ab+bc+ca)$, of which $a+b+c$ is one factor and $(a+b+c)^2-3(ab+bc+ca)$,

or $a^2+b^2+c^2-ab-bc-ca$ is the other factor as before.

Let the student observe how the clue to the solution in each case is found by studying the symmetrical form of the given expression.

(24) Prove that

$(a^2+b^2+c^2)^3+2(ab+bc+ca)^3-3(a^2+b^2+c^2)(ab+bc+ca)^2$ is a perfect square and find its square root.

SOLUTION. Let $a^2+b^2+c^2=x$; and $ab+bc+ca=y$, and *substitute* $x$ and $y$ in the given quantity. We get in this way $x^3+2y^3-3xy^2$, which $=(x-y)^2(x+2y)$ by adding and subtracting $2x^2y$. Restore the values of $x$ and $y$ and we get

$$(a^2+b^2+c^2-ab-bc-ca)^2(a^2+b^2+c^2+2ab+2bc+2ca)$$
$$=(a^2+\text{etc.}\quad\text{''}\quad\text{''}\quad)^2(a+b+c)^2$$
$$=(a^3+b^3+c^3-3abc)^2.$$

The square root of this perfect square is therefore

$$a^3+b^3+c^3-3abc.$$

(25) Express $8(a+b+c)^3-(a+b)^3-(b+c)^3-(c+a)^3$ in factors.

SOLUTION. Let $a+b=x$; $b+c=y$; $c+a=z$; and therefore by addition $x+y+z=2(a+b+c)$. By *substitution*, the given quantity becomes $(x+y+z)^3-(x^3+y^3+z^3)$, and by Exercise 96, No. 5, this $=3(x+y)(y+z)(z+x)$. Restore the values of $x$, $y$, $z$, and this $=3(a+2b+c)(a+b+2c)(2a+b+c)$.

(26) Simplify the expression

$$(3a-b-c)^3+(3b-c-a)^3+(3c-a-b)^3$$
$$-3(3a-b-c)(3b-c-a)(3c-a-b).$$ Ex. 100, No. 10.

| | |
|---|---|
| SOLUTION. Let $x=3a-b-c$ | hence $x+y+z=a+b+c$ |
| $y=3b-c-a$ | $x-y=4(a-b)$ |
| $z=3c-a-b$ | $y-z=4(b-c)$ |
| | $z-x=4(c-a)$ |

Substituting in the given quantity we get

$x^3+y^3+z^3-3xyz=(x+y+z)\times\frac{1}{2}[(x-y)^2+(y-z)^2+(z-x)^2]$; Ex. 96, No. 9.

$=(a+b+c)\times\frac{1}{2}[16(a-b)^2+16(b-c)^2+16(c-a)^2]$

$=16(a+b+c)\times\frac{1}{2}[(a-b)^2+\text{etc.}]$

$=16(a^3+b^3+c^3-3abc)$. See Exercise 96, No. 9.

(27) Find the factors of the expression

$$(a^2+2bc)^3+(b^2+2ca)^3+(c^2+2ab)^3$$
$$-3(a^2+2bc)(b^2+2ca)(c^2+2ab).$$

SOLUTION. The expression is evidently symmetrical. Let $x=a^2+2bc$, $y=b^2+2ca$, $z=c^2+2ab$; $\therefore\ x+y+z=(a+b+c)^2$. Substituting we get $(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$, of which the first factor $=(a+b+c)^2$. To find the second factor we have to expand $(a^2+2bc)^2+$etc. Take the first term and the fourth together, viz. $(a^2+2ab)^2-(a^2+2bc)(b^2+2ca)$, expand and write down the other two terms by changing $a$ to $b$, $b$ to $c$, $c$ to $a$ in succession, and we get

$$a^4+4a^2bc+4b^2c^2-a^2b^2-2ca^3-2b^3c-4abc^2$$
$$b^4+4b^2ca+4c^2a^2-b^2c^2-2ab^3-2c^3a-4bca^2$$
$$c^4+4c^2ab+4a^2b^2-c^2a^2-2bc^3-2a^3b-4cab^2.$$

The second column and the last column taken together $=0$, and we have left $a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2$

$$+2(a^2b^2+b^2c^2+c^2a^2-a^3b-a^3c-ac^3-ab^3)$$

which is plainly $=(a^2+b^2+c^2-ab-bc-ca)^2$.

Hence the whole expression

$$=(a+b+c)^2(a^2+b^2+c^2-ab-\text{etc.})^2=(a^3+b^3+c^3-3abc)^2.$$

(28) If $2s=a+b+c$, express in simplest form

$$(s-a)^3+(s-b)^3+(s-c)^3+3abc.$$

SOLUTION. Let $x=s-a$, $y=s-b$, $z=s-c$, and add these, $\therefore\ x+y+z=3s-(a+b+c)=3s-2s=s$. (A) Substituting these values the given expression becomes $x^3+y^3+z^3+3abc$. Add to this $3(x+y)(y+z)(z+x)-3(x+y)(y+z)(z+x)$, which is simply $=0$, and $(x+y+z)^3-3(x+y)(y+z)(z+x)+3abc$ is the result. (B) Exercise 96, No. 5. But from the values assumed for $x$, $y$ and $z$ we get by addition $x+y=2s-(a+b)=c$, $y+z=a$, $z+x=b$, so that $3(x+y)(y+z)(z+x)=3abc$, and the whole given expression (B) becomes $(x+y+z)^3$, *i.e.* $s^3$ from (A); result $s^3$.

(29) Find the relation between $r$ and $q$ which is necessary to make $x^5 - 5qx + 4r$ exactly divisible by $(x-m)^2$.

SOLUTION. Divide $x^5$ – etc. by $(x-m)^2$ and put the remainder $= 0$.

| | | |
|---|---|---|
| $1$ | $1 + 0 + 0 + 0$ | $-5q + 4r$ |
| $+2m$ | $2m + 4m^2 + 6m^3$ | $+8m^4$ |
| $-m^2$ | $- m^2 - 2m^3$ | $-3m^4 - 4m^5$ |
| | $1 + 2m + 3m^2 + 4m^3$ | $(5m^4 - 5q) + (4r - 4m^5)$. |

It is therefore necessary that each of the last two columns $= 0$, separately. $\therefore\ 5m^4 = 5q$; $m^4 = q$; $m^{20} = q^5$, from the first of the two; and $4r = 4m^5$; $r = m^5$; $r^4 = m^{20}$, from the last of the two. Hence the relation must be $r^4 = q^5$.

(30) Find the H. C. F. and the L. C. M. of

$$apx^2 + (aq + bp)x + bq. \quad \text{(A)}$$
$$\text{and } aqx^2 - (ap - bq)x - bp. \quad \text{(B)}$$

SOLUTION. $A = (ax + b)\ (px + q)$

$B = (ax + b)\ (qx - p)$

$\therefore$ H. C. F. $= ax + b$;

and L. C. M. $= (ax + b)\ (px + q)\ (qx - p)$.

(31) In the expression $(x-5)\ (x^2 + 3x + 7) + 11$, if we suppose that $x = 5$, then $x - 5 = 0$, and the expression $= 11$. Now 11 is plainly the remainder when the expression is divided by $x - 5$, and thus we see that one way of finding the remainder when we divide $x^3 - 2x^2 - 8x - 24$ by $x - 5$, is to put $x - 5 = 0$, and call the result the remainder, thus:

$$5^3 - 2.5^2 - 8.5 - 24 = 125 - 50 - 40 - 24 = 125 - 114 = 11,$$

as before, and as we may verify by division. It is also plain that if the remainder in any case should come out $= 0$, that $x - 5$ would be an exact divisor of the expression. For instance, we can tell without division that $x - 5$ will exactly divide

$$x^3 - 2x^2 - 8x - 35, \text{ because } 5^3 - 2.5^2 - 8.5 - 35 = 0,$$

when we put $x - 5 = 0$, *i.e.* $x = 5$. Again, for example, we can find without division that $abc$ is a factor of the expression

$$(a + b + c)\ (ab + bc + ca) - (a + b)\ (b + c)\ (c + a);$$

for, put $a = 0$ and we have $(b + c)bc - b(b + c)c$, which $= 0$, and this is the remainder. As the expression is symmetrical, $b$ and $c$ must also be factors if $a$ is one, therefore $abc$ will divide the expression. Now each term of the expression if multiplied out would contain *three* letters and no more, like $aab$, $abc$, $caa$, etc.; it is therefore evident that there cannot be any other *literal* factor besides $abc$, for in that case we must have terms of *four* dimensions. The only other possible factor is therefore some number like 3, 4, 5, etc. If we wish to ascertain without expansion and division what this number is we can easily do so. Call this number N and say

$$(a+b+c)(ab+bc+ca)-(a+b)(b+c)(c+a)=N.abc,$$

and suppose $a=b=c=1$, and we get

$$(1+1+1)(1.1+1.1+1.1)-(1+1)(1+1)(1+1)=N.1.1.1,$$
$$\text{or } (3)(3)-(2)(2)(2)=N=9-8=1.$$

Therefore the whole expression $=1.abc$, or $abc$.

N.B. In giving values to $a$, $b$, $c$, we must take great care not to assume them such that either side $=0$, because in that case N will vanish altogether and we shall fail to determine its value. For example here it would not do to take $a$, or $b$, or $c=0$; but we might take $a=1$, $b=2$, $c=3$, or any other values that will not make the expression $=0$. To illustrate further, suppose we wish to prove that

$$(a+b+c)(ab+bc+ca)-abc=(a+b)(b+c)(c+a).$$

Put $a+b=0$, *i.e.* $a=-b$, and therefore change every $a$ into $-b$, and get $(-b+b+c)(-b^2+bc-cb)+b^2c$, *i.e.* $-cb^2+b^2c$, which $=0$, and shows that $a+b$ leaves no remainder and is therefore a factor of the expression. By symmetry $b+c$ and $c+a$ are [factors], that is their product, $(a+b)(b+c)(c+a)$ is an exact div[isor]. There can be no other literal factor, because the expression has only *three* dimensions. If there is any other factor it is some number, hence

$$(a+b+c)(ab+bc+ca)-abc=N(a+b)(b+c)(c+a).$$

Now in this case, we can put $a=1=b$, $c=0$, for these values do not cause $(a+b)(b+c)(c+a)$ to become $=0$ and thus cause $N(a+b)(b+c)(c+a)$ to $=0$ and make N disappear altogether. Taking these values we have

$$(1+1+0)(1+0+0)-0=N(1+1)(1+0)(0+1)$$
$$i.e.\ 2=N(2);\ \text{or } N=1, \text{ as required.}$$

(32) Show that

$$(a+b+c)^4-(b+c)^4-(c+a)^4-(a+b)^4+a^4+b^4+c^4$$
$$=12abc(a+b+c) \text{ without actual expansion and division.}$$

SOLUTION. Put $a=0$, and the left hand member becomes $(b+c)^4-(b+c)^4-c^4-b^4+b^4+c^4=0$. Hence $a$ is an exact divisor, and by symmetry $b$ and $c$ are also factors, *i.e.* $abc$. This gives *three* dimensions, but the expression contains *four* throughout, therefore there must be another *literal* factor of *one* dimension yet to be found. This factor must obey symmetry, must have $a$, $b$, $c$ similarly involved and must therefore be $a+b+c$, for this is the only expression of *one* dimension that is symmetrical for $a$, $b$, and $c$.

Hence $(a+b+c)^4-(b+c)^4$ – etc. $= Nabc(a+b+c)$.

Put $a=b=c=1$ and $3^4-2^4-2^4-2^4+1^4+1^4+1^4=N.3$;

*i.e.* $81-16-16-16+3=3N=36$;

hence $N=12$, and $12abc(a+b+c)$ is the reduced expression as may be verified by expanding the terms and adding them up.

(33) Find the factors of $(x-y)^5+(y-z)^5+(z-x)^5$.
Exercise 100, No. 7.

SOLUTION. Test for $x-y$ by putting $x-y=0$; *i.e.* $x=y$, Write $y$ instead of $x$ throughout and get $0+(y-z)^5+(z-y)^5$. which $=0$ and shows that $x-y$ leaves no remainder. Hence by symmetry $(x-y)(y-z)(z-x)$ is a factor of the expression, and accounts for *three* out of the *five* dimensions. The remaining part must be a single factor of *two* dimensions, because any factor like $x$, $x+y$, or $x-y$, would also give two more by symmetry, and produce a product of *six* dimensions. Now a factor of two dimensions contains only two kinds of terms, viz. square terms like $x^2$, $y^2$, $z^2$, etc., and products like $xy$, $yz$, $zx$, etc.; therefore its general form must be $N(x^2+y^2+z^2)+P(xy+yz+zx)$ in this case. Hence we may say $(x-y)^5+(y-z)^5+(z-x)^5$
$=(x-y)(y-z)(z-x)[N(x^2+y^2+z^2)+P(xy+yz+zx)]$
where N and P are some numbers like 3, 5, 7, 8, etc. To find these numbers put $z=0$ throughout and get

$$(x-y)^5+y^5-x^5=(x-y)(y)(-x)[N(x^2+y^2)+P(xy)];$$

$$i.e.,\ 5x^4y+10x^3y^2-10x^2y^3+5xy^4= \\ -xy(x-y)[N(x^2+y^2)+P(xy)],$$

$$\text{or } -xy(x-y)[5(x^2+y^2)-5(xy)]= \\ -xy(x-y)[N(x^2+y^2)+P(xy)],$$

from which it is plain that $N=5$, $P=-5$; and therefore the last factor of two dimensions which we are seeking must be $5(x^2+y^2+z^2-xy-yz-zx)$; and all the factors are
$=5(x-y)(y-z)(z-x)(x^2+y^2+z^2-xy-yz-zx)$.

The student may apply the preceding method to the following examples, which will enable him to test his knowledge.

(A) $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

(B) $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

(C) $a^4(b-c)+b^4(c-a)+c^4(a-b)$.

(D) $(a+b+c)^3-(a^3+b^3+c^3)$.

(E) $(a+b+c)^5-(a^5+b^5+c^5)$.

(F) $(a+b+c)^3-(a+b-c)^3-(a+c-b)^3-(b+c-a)^3$.

(G) $a(b+c)\ (b^2+c^2-a^2)+b(c+a)\ (c^2+a^2-b^2)$
$\qquad +c(a+b)\ (a^2+b^2-c^2)$.

(H) $a^3(b+c-a)^2+b^3(c+a-b)^2+c^3(a+b-c)^2$
$\qquad +abc(a^2+b^2+c^2)$
$\quad +(a^2+b^2+c^2-bc-ca-ab)\ (b+c-a)\ (c+a-b)\ (a+b-c)$.

(K) $(a+b+c+d)^4-(a^4+b^4+c^4+d^4)-(a+b+d)^4$
$\quad -(a+c+d)^4-(a+b+c)^4-(b+c+d)^4+(a+b)^4+(b+c)^4$
$\quad +(c+d)^4+(a+d)^4+(b+d)^4+(a+c)^4$.

(L) $(a+b+c+d)^5-(a^5+b^5+c^5+d^5)-(a+b+d)^5$
$\quad -(a+c+d)^5-(a+b+c)^5-(b+c+d)^5+(a+b)^5+(b+c)^5$
$\quad +(c+d)^5+(a+d)^5+(b+d)^5+(a+c)^5$.

RESULTS. (A) $-(a-b)\ (b-c)\ (c-a)$.

(B) $-(a-b)\ (b-c)\ (c-a)\ (a+b+c)$.

(C) $-(a-b)\ (b-c)\ (c-a)\ (a^2+b^2+c^2+ab+bc+ca)$.

(D) $3(a+b)\ (b+c)\ (c+a)$.

(E) $5(a+b)\ (b+c)\ (c+a)\ (a^2+b^2+c^2+ab+bc+ca)$.

(F) $24abc$. (G) $2abc(a+b+c)$. (H) $2abc(ab+bc+ca)$.

(K) $24abcd$. (L) $60abcd(a+b+c+d)$.

HINTS. In A and B, $N=-1$. In C, $N=-1$ and $P=-1$.
In E, $N=5$, $P=5$. In H, $N=0$, $P=2$.

(34) Find the H. C. F. of $2x^4+9x^3+14x+3$ and
$3x^4+15x^2+5x^2+10x+2$. Exercise 66, No. 12.

SOLUTION. Apply the principle of Exercise 64, No. 4, in this way :—**Take multiples of the two expressions such that when added or subtracted the first terms will cancel; next take multiples such that the last terms will cancel;** then use the two remainders in the same way; and the next two remainders in the same way, until the operation is completed. Thus in the given example call the first expression A and the second B. Take $2B-3A$, call the remainder C. Take $3B-2A$, call the remainder D. Hence every measure of A and B will measure C and D. Take $3D-5C$, call the remainder E. Take $2C+5D$, call the remainder F, and observe that F and E are the same.

*Argument:* Every measure of A and B will measure C and D; every measure of C and D will measure E and F; hence every measure of A and B will measure E and F. But E and F are equal and are their own greatest common measure, therefore E and F are the H. C. F. of A and B.

*Operation:* Using detached coefficients the work may be exhibited in the following form:—

| | | | |
|---|---|---|---|
| A | | 2+ 9+ 0+ 14+ 3 | 1 |
| B | | 3+15+ 5+ 10+ 2 | 2 |
| 3A | | 6+27+ 0+ 42+ 9 | 3 |
| 2B | | 6+30+ 10+ 20+ 4 | 4 |
| 2B−3A | | 3+ 10− 22− 5=C | 5 |
| 2A | | 4+18+ 0+ 28+ 6 | 6 |
| 3B | | 9+45+ 15+ 30+ 6 | 7 |
| 3B−2A | | 5+27+ 15+ 2 =D | 8 |
| 5C | | 15+ 50−110−25 | 9 |
| 3D | | 15+ 81+ 45+ 6 | 10 |
| 3D−5C | 31 | 31+155+31 | 11 |
| | | 1+ 5+ 1=E | 12 |
| 2C | | 6+ 20− 44−10 | 13 |
| 5D | | 25+135+ 75+10 | 14 |
| 2C+5D | 31 | 31+155+ 31 | 15 |
| | | 1+ 5+ 1 =F | 16 |

But E = F, $\therefore x^2+5x+1=$ H. C. F. of A and B.

*Explanation:* In line 1 a *zero* is put in where $x^2$ is missing. In line 8 the quantity written in full would be

$$5x^4+27x^3+15x^2+2x=x(5x^3+27x^2+15x+2),$$

but as $x$ is plainly no factor of A and B it is dropped, and in line 10 we take $3D=3(5x^3+27x^2+\text{etc.})$ instead of

$$3(5x^4+27x^3+\text{etc.})$$

In line 11 the remainder $=31(x^2+5x+1)$, and 31 is evidently no part of the H. C. F. for which we are searching, and therefore we drop it. Lines 13, 14, 15, 16 are *necessary*, because if A and B had *no common factor*, it would be necessary to show that E and F had none. In line 15 the numerical factor 31 is struck out for the same reason as in line 11.

*Remark:* If the principle of Exercise 64, No. 4, has been comprehended, actual experience shows that the youngest pupils can apply it intelligently by the preceding method to the most difficult examples of this kind that can be proposed with numerical coefficients. The next example can be understood by junior pupils who have mastered Exercise 64, page 47.

(35) Reduce to lowest terms $\dfrac{8x^7 - 377x^3 + 21}{21x^7 - 377x^4 + 8}$.

*Matriculation Examination, Toronto University.*

SOLUTION.

| | | |
|---|---|---|
| A | $8x^7 - 377x^3 + 21$ | |
| B | $21x^7 - 377x^4 + 8$ | |
| A + B | $29x^7 - 377x^4 - 377x^3 + 29$ | Divide by 29. |
| | $x^7 - 13x^4 - 13x^3 + 1 = C.$ | |
| B − A | $13x^7 - 377x^4 + 377x^3 - 13$ | Divide by 13. |
| | $x^7 - 29x^4 + 29x^3 - 1 = D.$ | |
| C + D | $2x^7 - 42x^4 + 16x^3$ | Divide by $2x^3$. |
| | $x^4 - 21x + 8 = E.$ | |
| C − D | $16x^4 - 42x^3 + 2$ | Divide by 2. |
| | $8x^4 - 21x^3 + 1 = F.$ | |
| E + F | $9x^4 - 21x^3 - 21x + 9$ | Divide by 3. |
| | $3x^4 - 7x^3 - 7x + 3 = G.$ | |
| F − E | $7x^4 - 21x^3 + 21x - 7$ | Divide by 7. |
| | $x^4 - 3x^3 + 3x - 1 = H.$ | |
| G + 3H | $6x^4 - 16x^3 + 2x$ | Divide by $2x$. |
| | $3x^3 - 8x^2 + 1 = K.$ | |
| (G − 3H) ÷ 2 | $x^3 - 8x + 3 = L.$ | |
| (3L − K) ÷ 8 | $x^2 - 3x + 1 = M.$ | |
| (3K − L) ÷ 8x | $x^2 - 3x + 1 = N.$ | |
| | = H. C. F. of A and B. | |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 8 | + 0 | + 0 | + 0 | − 377 | + 0 | + 0 | + 21 |
| + 3 | | 24 | + 72 | + 192 | + 504 | + 189 | + 63 | |
| − 1 | | | − 8 | − 24 | − 64 | − 168 | − 63 | − 21 |
| | $8x^5 + 24x^4 + 64x^3 + 168x^2 + 63x + 21$ | | | | | | = reduc'd numerat'r | |
| 1 | 21 | + 0 | + 0 | − 377 | + 0 | + 0 | + 0 | + 8 |
| + 3 | | 63 | + 189 | + 504 | + 192 | + 72 | + 24 | |
| − 1 | | | − 21 | − 63 | − 168 | − 64 | − 24 | − 8 |
| | $21x^5 + 63x^4 + 168x^3 + 64x^2 + 24x + 8$ | | | | | | = red. denominator | |

*Compare Exercise 70, No. 8, page 52, and apply this method to solve it.*

(36) If $x+a$ is the H. C. F. of the quantities $x^2+px+q$ and $x^2+rx+s$, prove that $a=(q-s)\div(p-r)$.

SOLUTION. If $x+a$ divides each quantity exactly, the remainders are each $=0$. But, as in the examples on pages 76 and 77, the remainders are found by putting

$$x+a=0, \text{ i.e. } x=-a; \text{ hence}$$
$$R_1=a^2-pa+q=0=a(a-p)+q$$
$$R_2=a^2-ra+s=0=a(a-r)+s$$

Subtracting these equations we get $a(p-r)-(q-s)=0$

$$\text{i.e. } a(p-r)=q-s$$
$$\text{or } a=(q-s)\div(p-r).$$

(37) Show that $x+a$ is a common factor of the quantities $x^2+qx+1$ and $x^3+px^2+qx+1$, provided that

$$(p-1)^2-q(p-1)+1=0.$$

SOLUTION. Finding the remainders as in the preceding example we get $R_1=a^2-qa+1=0$

$$R_2=-a^3+pa^2-qa+1=0.$$

Subtract and $a^3-a^2(p-1)=0$;

$$\text{i.e. } a-(p-1)=0, \text{ and } a=p-1.$$

Substitute this value of $a$ in $R_1$, and we get

$$(p-1)^2-q(p-1)+1=0.$$

(38) Find the sum of the fractions $\dfrac{ax}{a-b}+\dfrac{bx}{a+b}+\dfrac{2abx}{b^2-a^2}$.

SOLUTION. Set aside the $x$ which is common to the numerators; and change $b^2-a^2$ into $a^2-b^2$, and *therefore* change the sign + into − before the 3rd fraction. Next multiply the terms of the 1st by $a+b$, the terms of the 2nd by $a-b$, and take the sum of all the numerators, viz:

$$a(a+b)+b(a-b)-2ab, \text{ which is } a^2-b^2.$$

The C. D. is also $a^2-b^2$, hence sum $=1$. Restore $x$ and sum $=x$.

(39) Find the sum of the fractions

$$\frac{1}{(x-3)(x-4)}+\frac{2}{(x-2)(4-x)}+\frac{1}{(2-x)(3-x)}.$$

SOLUTION. Restore the symmetry, so that $(x-2)(x-3)(x-4)$ is the C. D. This makes the 2nd fraction *negative*, but the 3rd remains *positive*. Hence numerator of the sum

$$=(x-2)-2(x-3)+(x-4)=0.$$

Therefore whole sum $=0\div(x-2)(x-3)(x-4)=0$.

(40) Simplify the expression

$$\frac{y+z}{(y^2-zx)(z^2-xy)}+\frac{z+x}{(z^2-xy)(x^2-yz)}+\frac{x+y}{(x^2-yz)(y^2-zx)}.$$

—*Second Class, 1887.*

SOLUTION. The C. D. $=(x^2-yz)(y^2-zx)(z^3-xy)$; hence the numerator of the sum will be

$$\left.\begin{array}{l}(y+z)(x^2-yz)\\+(z+x)(y^2-zx)\\+(x+y)(z^2-xy)\end{array}\right| = \left|\begin{array}{l}+y(x^2-yz+z^2-xy)\\+z(x^2-yz+y^2-zx)\\+x(y^3-zx+z^2-xy)\end{array}\right| = 0.$$

Hence the whole sum $=0$, as in the preceding example.

(41) Simplify the expression

$$\frac{x}{(x+y)(x+2y)}+\frac{2y}{(x+y)(x+3y)}+\frac{x}{(x+2y)(x+3y)}-\frac{1}{x+3y}.$$

—*Third Class, 1887.*

SOLUTION. The C. D. is $(x+y)(x+2y)(x+3y)$. Hence the numerator of the sum is

$$x(x+3y)+2y(x+2y)+x(x+y)-(x+y)(x+2y)$$
$$=(x+y)(x+2y).$$

$$\therefore \text{ whole expression} = \frac{(x+y)(x+2y)}{(x+y)(x+2y)(x+3y)}=\frac{1}{x+3y}.$$

TEST. Put $x=0$, $y=1$, and $\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$

(42) Solve the equation

$$\frac{a-x}{a^2-bc}+\frac{b-x}{b^2-ca}+\frac{c-x}{c^2-ab}=\frac{3x}{ab+bc+ca}.$$

SOLUTION. Transpose the right hand side part by part, and add each fraction on the left to one of the parts. Thus for the first fraction on the left $\frac{a-x}{a^2-bc}-\frac{x}{ab+bc+ca}$.

The numerator of the sum of these two is,

$$a(ab+bc+ca)-x(ab+bc+ca)-x(a^2-bc),$$

which is $=a(ab+bc+ca)-x(a+b+c)a$. By symmetry the the other two are $b($ " " " $)-x($ " " $)b$, and

$c($ " " " $)-x($ " " $)c$.

Each fraction has $ab+bc+ca$ in its denominator. Strike this out of the denominator, and clear of fractions, and we get

$$\left.\begin{array}{l}[a(ab+bc+ca)-x(a+b+c)a]\,(b^2-ca)\,(c^2-ab)\\ +[b(\text{`` `` ``})-x(\text{`` ``})b]\,(c^2-ab)\,(a^2-bc)\\ +[c(\text{`` `` ``})-x(\text{`` ``})c]\,(a^2-bc)\,(b^2-ca)\end{array}\right\}=0.$$

Collecting terms and bracketing, we have by transposition

$$(ab+bc+ca)\left|\begin{array}{l}a(b^2-ca)\,(c^2-ab)\\ +b(a^2-bc)\,(c^2-ab)\\ +c(a^2-bc)\,(b^2-ca)\end{array}\right|=$$

$$x(a+b+c)\left|\begin{array}{l}a(b^2-ca)\,(c^2-ab)\\ b(a^2-bc)\,(c^2-ab)\\ c(a^2-bc)\,(b^2-ca)\end{array}\right|$$

Divide through by the common factor which is contained in the vertical vinculums and we get $(ab+bc+ca)=x(a+b+c)$,

*i.e.* $x=(ab+bc+ca)\div(a+b+c)$.

(43) Solve the equation

$$(x+a+b)^4-(x+a)^4-(x+b)^4+x^4-(a+b)^4+a^4+b^4$$
$$=12ab\{x^2+(a+b)^2\}.$$

SOLUTION. For $a+b$ write $w$ throughout, and expand thus

$$\left.\begin{array}{llllll}x^4 & +4x^3w & +6x^2w^2 & +4xw^3 & +w^4 & \\ -x^4 & -4x^3a & -6x^2a^2 & -4xa^3 & -a^4 & \\ -x^4 & -4x^3b & -6x^2b^2 & -4xb^3 & -b^4 & \\ x^4 & & & (-w^4 & +a^4 & +b^4)\end{array}\right|=12ab(x^2+w^2)$$

$$0+0+6x^2(w^2-a^2-b^2)+4x(w^3-a^3-b^3)+0=12ab(x^2+w^2)$$

*i.e.* $6x^2(2ab)+(4x)(3ab)(w)=12ab(x^2+w^2)$.

Divide by $12ab$ and $x^2+wx=x^2+w^2$; $\therefore\ x=w=a+b$.

(44) Solve the equation

$$\frac{3abc}{a+b}-\frac{bx}{a}+\frac{a^2b^2}{(a+b)^3}=3cx-\frac{b^2x}{a}\cdot\frac{2a+b}{(a+b)^2}.$$

SOLUTION. Transpose $\dfrac{bx}{a}$ and factor, thus

$$\frac{ab}{a+b}\left\{3c+\frac{ab}{(a+b)^2}\right\}=x\left\{3c+\frac{b}{a}\left[1-\frac{2ab+b^2}{(a+b)^2}\right]\right\}$$
$$=x\left\{3c+\frac{b}{a}\cdot\frac{a^2}{(a+b)^2}\right\}$$
$$=x\left\{3c+\frac{ab}{(a+b)^2}\right\}$$

$$\therefore\ \frac{ab}{a+b}=x.$$

(45) Solve the equation $\left(\frac{x-a}{x-b}\right)^3 = \frac{x-2a+b}{x-2b+a}$.

SOLUTION. Let $x-a=m$; $x-b=n$; $\therefore\ n-m=a-b$; and, on substituting these values throughout, we get

$$\frac{m^3}{n^3} = \frac{(x-a)-(a-b)}{(x-b)+(a-b)} = \frac{m-(n-m)}{n+(n-m)} = \frac{2m-n}{2n-m}$$

Clearing of the fractions and transposing, we get

$$m^3(2n-m) - n^3(2m-n) = 0;$$
$$i.e.\ 2mn(m^2-n^2) - (m^4-n^4) = 0,$$
$$\text{or } (m^2-n^2)\{2mn-(m^2+n^2)\} = 0.$$

Hence one of the two brackets must $=0$, since their product $=0$.

Select the first, and $(m+n)(m-n)=0$.

Hence $m+n=0$, or $m-n=0$.

If $m+n=0$, $2x-a-b=0$, and $x=\frac{1}{2}(a+b)$.

N.B. It will be found on examination that the other two conditions, viz. $m-n=0$, and $2mn-(m^2+n^2)=0$, are the same; and from each we get $n-m=a-b=0$, or $a=b$, which gives no value for $x$, but is **The Equation of Condition among the Coefficients** which reduces the original **Equation** to an **Identity.**

(46) A person goes from Hamilton to Toronto by boat at the rate of 13 miles per hour, remains an hour and a half in Toronto, and returns by rail at the rate of 26 miles per hour. He is gone altogether six hours; find the distance from Hamilton to Toronto.—*Third Class, 1887.*

SOLUTION. Let $x$ = the distance expressed in miles.

$$\therefore\ \frac{x}{13}+\frac{x}{26} = 4\tfrac{1}{2};\quad \therefore\ x = 39 \text{ miles.}$$

ARITHMETICAL SOLUTION. He is 9 half-hours on the road. The *rates* are as $1:2$; hence the *times* are as $2:1$. Divide 9 into two parts in this proportion, viz. 6 and 3.

$$\text{Distance} = 3 \times 13 = 39 = 1\tfrac{1}{2} \times 26.$$

(47) A number consists of two digits; if these digits be reversed, the number thus formed is less than the first number by twice the greater digit; also, four times one digit exceeds three times the other by unity. Find the digits.

SOLUTION. Let $10x+y=$ No.; $\therefore\ 10y+x=$ No. with digits reversed. $\therefore\ 9x-9y=2x$; *i.e.* $7x=9y$.

Also $4y=3x+1$; whence $x=9$, $y=7$. No. $=97$.

(48) A compound of tin and lead weighs 10·43 times as much as an equal bulk of water, while tin weighs 7·44 times, and lead 11·35 times as much as equal bulks of water. Find the number of pounds of each metal in the compound.

SOLUTION. Let $x$ be the weight of the tin, and $y$ the weight of the lead in a given quantity of the compound.
$\therefore x \div 7{\cdot}44 =$ weight of water equal in bulk to the tin in compound
and $y \div 11.35 =$ " " " " lead "

Also $x + y =$ weight of the compound;
$\therefore (x+y) \div 10{\cdot}43 =$ weight of water equal in bulk to the compound. Thus we get the equation

$\frac{x}{744} + \frac{y}{1135} = \frac{x+y}{1043}$. From this we can get $\frac{x}{y} = \frac{744 \times 4}{1135 \times 13} = \frac{2976}{14755}$.

Therefore the proportions of tin and lead are as 2976 : 14755 in any given quantity.

(49) If 76 men and 59 boys can do as much work in 299 days as 40 men and 33 boys can do in 557 days, how many men will do as much work in a day as 15 boys can do?

SOLUTION. Let $x$ stand for a day's work by a man.
Let $y$ " " " " boy.
Then $299(76x + 59y) = 557(40x + 33y)$,
*i.e.* $22{,}724x + 17{,}641y = 22{,}280x + 18{,}381y$;
or $444x = 740y$; or $6x = 10y$;
Therefore $9x = 15y$. ANS. 9 men.

(50) A man divides $1,300 into two sums and lends them at different rates of interest. He finds the incomes from them to be equal. If he had loaned the first at the rate of the second he would have received $36, and the second at the rate of the first he would have obtained $49. Find the rates of interest.—*Junior Matriculation, Toronto University, 1890.*

SOLUTION. Several interpretations of the meaning of this problem are possible, according to the answers returned to the following questions: Simple or compound interest? For the same time or for different times? Is $36 the income from the whole investment or from the first sum alone? Is $49 the income from the whole, etc.? Taking one set of answers: Let $P_1$ and $P_2$ be the sums, so that $P_1 + P_2 = 1300$; and let $r_1$ and $r_2$ be the rates at simple interest for equal times on $1. Then we have $P_1 r_1 = P_2 r_2$; and $P_1 r_2 = 36$; $P_2 r_1 = 49$, from which

$$\frac{P_1}{P_2} = \frac{r_2}{r_1} = \frac{36 r_1}{49 r_2}; \text{ whence } \frac{r_2}{r_1} = \frac{6}{7} = \frac{P_1}{P_2}.$$

Hence $P_1 = \$600$; $P_2 = \$700$; and the rates must be 7% on the first part, 6% on the second part.

(51) *A* lends one-half of his money to *B* at 5% per annum simple interest, and the remaining half he invests in the three per cents at 90. *B* pays the interest regularly during the first five years, but afterwards neglects to do so till other five years' interest is due, when *A* calls in all his money, and *B* becomes a bankrupt paying 10*s.* in the pound. *A* sells out when the funds are at 81, and then finds that the whole sum he has received as principal and interest in the ten years exceeds the sum he originally possessed by £34 13*s.* 4*d.* How much did he lend *B*?—*Cambridge, England.*

SOLUTION. Let $x =$ the number of pounds lent to $B =$ sum invested in 3 per cents; then in 10 years, £90 in the 3 per cents yields £30; *i.e.* $x$ yields $\frac{1}{3}x$, and this stock is finally sold out for $\frac{9}{10}x$. Hence total proceeds of stock $= \frac{1}{3}x + \frac{9}{10}x = \frac{37}{30}x$.

Again, $B$ pays in interest for 5 years $5 \times \frac{x}{20} = \frac{x}{4}$; and at the end of 10 years he owes $A$, $x + \frac{1}{4}x = \frac{5}{4}x$; and he pays only $\frac{5}{8}x$ instead of the whole debt. Hence the whole sum received from $B$ in 10 years is $= \frac{1}{4}x + \frac{5}{8}x = \frac{7}{8}x$. Thus we have the equation

$$\frac{37}{30}x + \frac{7}{8}x = 2x + 34\tfrac{2}{3}, \text{ from which } x = £320.$$

(52) A man starts to walk at uniform speed from C to M and back without stopping, and at the same time another man starts to walk in the same manner from M to C and back. They meet $1\frac{1}{2}$ miles from M; and again, an hour after, 1 mile from C. Find their rates of walking per hour, and the distance from C to M.

SOLUTION. Let $x =$ number of miles from C to M.

Then $A$ goes $x - 1\frac{1}{2}$ miles while $B$ goes $1\frac{1}{2}$ miles;
and $A$ " $2x - 1$ " " $B$ " $x + 1$ miles.

$\therefore x - 1\frac{1}{2} : 2x - 1 = 1\frac{1}{2} : x + 1$;

*i.e.* $(2x - 3)(x + 1) = 3(2x - 1)$;

or $2x^2 - x = 6x$; *i.e.* $x = 3\frac{1}{2}$ miles.

$A$'s rate per hour $= 1\frac{1}{2} + x - 1 = 4$ miles.
$B$'s " " $= x - 1\frac{1}{2} + 1 = 3$ miles.

(53) Alfred, Edward, and Herbert come each with his pail to a well; when a question arises about the quantity of water in the well; but, as no one of them knows how much his pail will hold, they cannot settle the dispute. Luckily Mary comes up

with a pint measure, by aid of which they discover that Alfred's pail holds half a gallon more than Edward's and a gallon more than Herbert's; but before the precise content of any pail is found out an accident happens and Mary's measure is broken. They are now however in a position to ascertain the quantity of water in the well; for they find that it fills each pail an exact number of times; and that the number of times it fills Edward's is greater by eight than the number of times it fills Alfred's, and less by forty than the number of times it fills Herbert's. How much water was there in the well?—*Cambridge, England.*

SOLUTION. Let $x$ = number of quarts in Herbert's pail, therefore $x+2$ and $x+4$ are the number of quarts in the pails of Edward and Alfred; and let $y$ = number in the well. From the data given we have therefore the equations

$$\frac{y}{x+2}=\frac{y}{x+4}+8=\frac{y}{x}-40.$$

Whence $\dfrac{2y}{(x+2)(x+4)}=8$; and $\dfrac{2y}{x(x+2)}=40$.

And from these two we get $\dfrac{x+4}{x}=5$, or $x=1$.

Also $y=4(x+2)(x+4)=60$ quarts, or 15 gallons.

(54) Each of three cubical vessels, $A$, $B$, $C$, whose capacities are as $1:8:27$, respectively, is partially filled with water, the quantities of water in them being as $1:2:3$, respectively. So much water is now poured from $A$ into $B$, and and so much from $B$ into $C$, as to make the depth of the water the same in each vessel. After this, $128\frac{4}{7}$ cubic feet of water is poured from $C$ into $B$, and then so much from $B$ into $A$, as to leave the depth of water in $A$ twice as great as the depth of water in $B$. The quantity of water in $A$ is now less by 100 cubic feet than it was originally. How much water did each of the vessels originally contain?—*Cambridge, England.*

SOLUTION. Let $6x$ = total number of cubic feet of water in the three vessels, viz. $x$ cubic feet in $A$, $2x$ in $B$ and $3x$ in $C$. Then since the volumes of the cubes are as $1:8:27$, their sides are as $1:2:3$, and their square bases as $1:4:9$. Hence when all are filled to the same height, the number of cubic feet in each will be as $1:4:9$; and each vessel will contain

$$A,\ \frac{6x}{14},\ i.e.\ \frac{3x}{7};\quad B,\ \frac{12x}{7};\quad C,\ \frac{27x}{7}.$$

We also see that when the depth of water in $A$ is twice the depth in $B$, the number of cubic feet in each will be as $2:4$ or as $1:2$. But $A$ contains $x-100$, hence $B$ contains $2(x-100)$; and by the question $C$ contains $\frac{27x}{7}-128\frac{4}{7}$; and the sum of these must be $6x$. Thus we get the equation $3(x-100)+\frac{27x}{7}-128\frac{4}{7}=6x$; and from this $x=500$, $2x=1000$, $3x=1500$ cubic feet, which are the original quantities, as may easily be verified.

(55) A fraudulent merchant uses his *false* balance both in buying and selling a certain article, thereby gaining 11% more on his outlay than he would gain were the balance *true*. If, however, the scale-pans in which the article is weighed when bought and sold respectively, were interchanged, he would neither gain nor lose by the transaction. Find the legitimate gain per cent. on the article.—*Cambridge, England.*

SOLUTION. Let $w$ = apparent weight in pounds of article when *bought*.
Let $w_1$ = apparent weight in pounds of article when *sold*.
Let $c$ = *cost price* of one pound of the article.
Let $x$ = *legitimate gain* per cent. on the cost price.

Then an article which cost $cw$ is sold for $w_1\left(c+\frac{cx}{100}\right)$.

Therefore we have $w_1\left(c+\frac{cx}{100}\right)-cw=\frac{(x+11)cw}{100}$ . . (1)

Secondly, cost of article $=cw_1$; and it sells for

$cw\left(1+\frac{x}{100}\right)$, and therefore $cw_1=cw\left(1+\frac{x}{100}\right)$ . . (2)

From (1) we get on dividing through by $c$ and transposing $-w$,

$$w_1\left(1+\frac{x}{100}\right)=w\left(1+\frac{x+11}{100}\right);$$

and from (2) $w\left(1+\frac{x}{100}\right)=w_1$. Multiply these two and divide through by $ww_1$, and $\left(1+\frac{x}{100}\right)^2=1+\frac{x+11}{100}$;

expanding and transposing

$$x^2+100x-1100=0=(x-10)(x+110).$$

Hence $x-10=0$, or else $x+110=0$, since the product $=0$.
$\therefore x=10$, or $-110$. The former root is the proper answer, 10%.

(56) Simplify $\dfrac{(a-b)^2-(b-c)^2}{a^2+ab-bc-c^2}$ + two similar fractions.

SOLUTION. 1st fraction $=(a-2b+c)\div(a+b+c)$.

$\therefore$ 2nd " $=(b-2c+a)\div(b+c+a)$, by symmetry

3rd " $=(c-2a+b)\div(c+a+b)$, " "

$\therefore$ Sum $=\quad 0 \quad \div(a+b+c)=0$.

(57) Solve the equations

$$\frac{x}{b+c}+\frac{y}{c-a}=a+b.$$

$$\frac{y}{c+a}+\frac{z}{a-b}=b+c.$$

$$\frac{x}{b-c}+\frac{z}{a-b}=c+a.$$

SOLUTION. Adding up the equations as they stand, we get

$$(2b)\frac{x}{b^2-c^2}+(2c)\frac{y}{c^2-a^2}+(2a)\frac{z}{a^2-b^2}$$
$$=2b+2c+2a.$$

From this it is plain that each of the fractions must $=1$; *i.e.* $x=b^2-c^2$; $y=c^2-a^2$; $z=a^2-b^2$, which is easily verified.

(58) Prove that

$$\frac{a-b}{1+ab}+\frac{b-c}{1+bc}+\frac{c-a}{1+ca}=\frac{a-b}{1+ab}\cdot\frac{b-c}{1+bc}\cdot\frac{c-a}{1+ca}.$$

SOLUTION. Add the 1st and the 2nd and get

$\dfrac{a-b}{1+ab}+\dfrac{b-c}{1+bc}=\dfrac{(a-c)(1+b^2)}{(1+ab)(1+bc)}$; to this add the 3rd,

thus, $(c-a)\left\{\dfrac{1}{1+ca}-\dfrac{1+b^2}{(1+ab)(1+bc)}\right\}$

$$=(c-a)\left\{\frac{(1+ab)(1+bc)-(1+b^2)(1+ca)}{(1+ca)(1+ab)(1+bc)}\right\}$$

$$=(c-a)\left\{\frac{1+bc+ab+ab^2c-1-ca-b^2-ab^2c}{(1+ab)(\quad " \quad)(\quad " \quad)}\right\}$$

$$=\frac{(a-b)(b-c)(c-a)}{(1+ab)(1+bc)(1+ca)}.$$

(59) If $\dfrac{x+y}{3a-b}=\dfrac{y+z}{3b-c}=\dfrac{z+x}{3c-a}$, prove that

$$\frac{x+y+z}{ax+by+cz}=\frac{a+b+c}{a^2+b^2+c^2}.$$

—*Matriculation, Toronto University.*

SOLUTION. Let the three equal fractions $=m$, and we get

$$\left.\begin{aligned} x+y&=m(3a-b)\\ y+z&=m(3b-c)\\ z+x&=m(3c-a)\end{aligned}\right\}, \text{ whence } x+y+z=m(a+b+c)\;..(1)$$

Also, $x-z=m(3a-4b+c)$

$$\therefore \text{ and } \left\{\begin{aligned} x&=m(a-2b+2c)\\ y&=m(b-2c+2a)\\ z&=m(c-2a+2b)\end{aligned}\right\} \text{ by symmetry, changing } a \text{ into } b, \text{ etc.}$$

$$\therefore \left\{\begin{aligned} ax&=m(a^2-2ab+2ac)\\ by&=m(b^2-2bc+2ab)\\ cz&=m(c^2-2ac+2bc)\end{aligned}\right\}, \text{ whence } ax+by+cz=m(a^2+b^2+c^2)\;..(2)$$

And (1) ÷ (2) gives $\dfrac{x+y+z}{ax+by+cz}=\dfrac{a+b+c}{a^2+b^2+c^2}$.

(60) If $\dfrac{x}{b+c-a}=\dfrac{y}{c+a-b}=\dfrac{z}{a+b+c}$, prove that

$$(a+b+c)\,(xy+yz+zx)=(x+y+z)\,(ax+by+cz).$$

SOLUTION. Let each fraction $=m$, and we get

$$\begin{aligned} x&=m(b+c-a)\\ y&=m(c+a-b)\\ z&=m(a+b-c), \text{ whence } x+y+z=m(a+b+c)\;\ldots\;(1)\end{aligned}$$

$$\text{Also, } \begin{aligned} xy&=m(by+cy-ay)\\ yz&=m(cz+az-bz)\\ zx&=m(ax+bx-cx)\end{aligned}$$

$$\text{And } \begin{aligned} zx&=m(bz+cz-az)\\ xy&=m(cx+ax-bx)\\ yz&=m(ay+by-cy), \text{ whence } xy+yz+zx=m(ax+by+cz)\;\ldots\;(2)\end{aligned}$$

(1) ÷ (2) gives $\dfrac{x+y+z}{xy+yz+zx}=\dfrac{a+b+c}{ax+by+cz}$;

*i.e.* $(a+b+c)\,(xy+yz+zx)=(x+y+z)\,(ax+by+cz)$.

(61) If $\frac{2y-z}{2b+c}=\frac{2z-x}{2c+a}=\frac{2x-y}{2a+b}$, show that

$$21(a+b+c)(x+2y+3z)=(x+y+z)(41a+38b+47c).$$

—*Matriculation, Toronto University. 1876, and Second Class. 1886.*

Solution. $2y-z=m(2b+c)$
$2z-x=m(2c+a)$
$2x-y=m(2a+b)$, $\therefore\ x+y+z=m(a+b+c)3$ .. (1)

Also, $26y-13z=m(2b+c)13$
$34z-17x=m(2c+a)17$
$24x-12y=m(2a+b)12$,
$\therefore\ 7(x+2y+3z)=m(41a+38b+47c)$ . . . (2)

Multiply these results (1) and (2) and striking out $m$ we have the relation required.

# EXAMINATION PAPERS.

## No. 1.

(1) If $a=3$, $b=4$, $c=27$, find the value of
$$3ab+ac+4\sqrt{a^2b}-3\sqrt{b^3c}+\sqrt{(a^2+b^2)}.$$

(2) Simplify the expression
$$5c-6a-\{3(2b-c)+4(a-2b)-6(2a-c)\}.$$

(3) Prove the formula $a^m \times a^n = a^{m+n}$.

(4) Multiply $x^3-x^2y-2y^3$ by $x^3+x^2y-2y^3$.

(5) Find a number such that when it is divided into 4 and into 3 equal parts, the continued product of the former shall equal 81 times the continued product of the latter.

(6) Two trucks whose wheels are of different sizes are in motion; one makes 2,002 revolutions more than the other in 7 miles, whereas if its wheels had been twice as large in circumference, it would have made 770 less than the other. Find the circumference of the wheels.

## No. 2.

(1) Simplify $7a-4b-\{5a-3[b-2(a-b)]\}$.

(2) If $x^2-ax-\frac{8}{2}$ is exactly divisible by $x-2$, what is the value of $a$?

(3) If $x^2+ax+b$, and $x^2+cx+d$ have a common measure of the form $x+e$, show that $e=(b-d)/(a-c)$.

(4) If $a+b+c=0$, show that $a^4+b^4+c^4=2(ab+bc+ca)^2$.

(5) $(x+2)(x-2)(x+8)=x(x-3)(x+16)$, find $x$.
N.B. $(5x-4)(x-8)=0$, $\therefore$ either $5x-4=0$, or $x-8=0$.

(6) $A$ and $B$ can do a piece of work in $m$ days; $A$ works $n$ days alone, when $B$ joins him, and both together finish the work in $p$ days more. How long would each require to do it singly?

## No. 3.

(1) Multiply $1-x^2+x^4-x^6$ by $1+x^2$.

(2) Find the value of $\sqrt{a^2b^3}+\sqrt[3]{a^3b^3}+a^2\sqrt[6]{b^2}$ when $a=5$, $b=25$.

(3) Find the H. C. F. of $x^4+7x^3+6x^2-32x-32$ and $x^2+9x+20$.

(4) Find the L. C. M. of $x^2-y^2$, $x^2+xy-2y^2$, and $x^2+3xy+2y^2$.

(5) What is the condition that $x^2+px+q$ may be exactly divisible by $x-r$?

(6) $16-4x+8x=5x+14$, find $x$.

(7) The breadth of an oblong space is four yards less than its length; the area of the space is 252 yards. Find the length of its sides.

(8) A square field contains 1 acre 2 roods 27 perches $23\frac{1}{4}$ square yards. An oblong field contains 6,400 square yards, and has one side as much longer than the side of the square field as the other side is shorter than it. Find the length and breadth of the oblong.

## No. 4.

(1) Express algebraically:—The fourth power of the sum of two numbers, $a$ and $b$, together with twice the product of their squares, is equal to the sum of their fourth powers together with four times the product of their product and the square of their sum. Verify your statement when $a=2$, $b=3$.

(2) Subtract $(x+y)(3a-2b)$ from $(x+y)(3a+2b)$.

(3) Divide $x^2+y^2+1-2y+2x-2xy$ by $x-y+1$.

(4) Simplify $\left(x+2+\dfrac{4}{x-2}\right) / \left(\dfrac{x^3}{x^2-4}-x\right)$.

(5) $-17\left(x-\dfrac{4-x}{3}\right)=12\left(5x-\dfrac{7+3x}{8}\right)$.

(6) $\dfrac{3x}{x+2}-\dfrac{x-1}{6}=x-9$.

N.B. $(x-10)(7x+11)=0$; $\therefore x-10=0$, or else $7x+11=0$.

(7) Prove that the difference between the sum of any two numbers and the sum of their cubes is divisible by three times their product.

## No. 5.

(1) Factorise in simplest forms $x^4-1$; $x^2-x-12$.

(2) Show that $(a-b)^3-3ab(b-a)-a^3+b^3=0$.

(3) $\dfrac{x-12}{x-5}+\dfrac{x-5}{x-12}=2\tfrac{4}{35}$.

N.B. $(2x-59)(2x+25)=0$; $\therefore 2x-59=0$, or $2x+25=0$.

(4) A purse contained only threepenny and fourpenny pieces, and the value is £2 10*s.* 7*d.*; but if the numbers of the coins were interchanged, the value would then be £2 8*s.* 7*d.* Find the number of coins of each sort.

(5) Divide $7x^2-\frac{1}{7}$ by $x+\frac{1}{7}$.

(6) Factor $x^8-y^8$; $x^3-3x^2+3x-1$; and $x^2+8x-105$.

(7) A train 200 yards long passes another train 240 yards long, running in the opposite direction on a parallel track, in 12 seconds. If the first train had been overtaking the other it would have taken one minute to be entirely clear of the second train. Find the rate of each train per hour.

## No. 6.

(1) $\frac{1}{5}(2x+7)-\frac{1}{3}(25+x)=\frac{1}{15}(7x-5)+2$, find $x$.

(2) $\dfrac{2x}{51-2x}+\dfrac{68-3x}{x}=3$.

N.B. $(x-17)(7x-102)=0$. An equation like this which gives two values for $x$ is called a **Quadratic Equation.**

(3) Show that when $a$ is less than $b$, the fraction $\dfrac{a}{b}$ is less than $\dfrac{a+1}{b+1}$, but greater than $\dfrac{a-1}{b-1}$.

N.B. Let $\dfrac{a}{b}=x$; $\therefore a=bx$; $\therefore \dfrac{a+1}{b+1}=x+\dfrac{1-x}{b+1}$, etc.

(4) Simplify

$2\{3a-(4b-5c)\}-3\{4a-(5b-2c)\}+4\{5a-3(b-c)\}$.

(5) Find the value of $\dfrac{7x^2-10xy+3y^2}{7x^2+10xy+3y^2}-\dfrac{20y}{33x}$, when $x=5$, $y=3$.

(6) Solve the equation $\dfrac{23}{x+4}-\dfrac{x+5}{3}+\dfrac{3x}{11}=0$.

(7) $A$ and $B$ have two guineas between them; and if $A$ gives to $B$ one shilling for every penny $B$ has, $A$ will then have ten shillings less than $B$ now has. How much money has each?

N B. One guinea $=21s.$; $B$ gives nothing to $A$.

## No. 7.

(1) Divide $\frac{1}{2}a^3+\frac{3}{2}a^2x-2x^3$ by $\frac{1}{2}a+x$.

(2) Find the H. C. F. of
$x^3y+2x^2y^2+2xy^3+y^4$ and $5x^5+10x^4y+5x^3y^2$.

(3) Simplify $\dfrac{1}{x-1}-\dfrac{1}{2(x+1)}-\dfrac{x-3}{2(x^2+1)}$.

(4) $\frac{1}{7}(3x-1)+\frac{1}{4}(6-x)-\frac{1}{12}(2x-4)=2-\frac{1}{28}(x+2)$.

(5) $\left.\begin{array}{l}\frac{1}{4}(x+3y)+\frac{1}{12}(3x+y)=11\\ x+y=8(y-x)\end{array}\right\}$, find $x$ and $y$.

(6) Solve the equation $\dfrac{2x-3}{x-2}-\dfrac{1}{6}+\dfrac{2x-1}{1-x}=0$.

(7) A grocer bought 224 pounds of sugar at the rate of 25*s.* for 112 pounds. The sugar having been damaged, he sold part of it at 2*d.* per pound and the rest at $2\frac{1}{2}d.$ He lost $12\frac{1}{2}\%$ on the whole transaction. How much did he sell at each price?

## No. 8.

(1) Simplify $\dfrac{a^3-b^3}{a^2+ab+b^2}-\frac{1}{2}\{a-3b-(3b-a)\}$.

(2) Determine the numerical values of $c$ and $d$ so that in the product of $x^2+x+1$ and $x^3+cx^2+dx+e$ the coefficients of $x^4$ and $x^2$ may vanish.

(3) Show that the L. C. M. of two quantities equals their product divided by their H. C. F.

(4) Find the H. C. F. and the L. C. M. of
$6a^3 - 5a^2x - 6ax^2$; $4a^3x - 12a^2x^2 + 9ax^3$; and
$6a^2x - 13ax^2 + 6x^3$.

(5) Simplify $\frac{3x-4y}{4x+5y} - \frac{4x-5y}{5x+6y} + \frac{1}{20}$.

(6) Solve $\frac{3x}{x-2} + \frac{x-1}{x-5} = 8$.

(7) The debt on a new school was paid off by three generous friends of Education. *A* paid half the debt and \$2 more; *B* paid half the remainder and \$7 more; *C* paid half the remainder and \$8 more. How much did each pay?

## No. 9.

(1) Show that the L. C. M. of two algebraical expressions is their product divided by their H. C. F.

(2) Prove that $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, where the letters denote any numbers whatever.

N.B. Let $\frac{a}{b} = x$, $\frac{c}{d} = y$, $\therefore a = bx$, etc.

(3) Solve the equation $\frac{2x+7}{26} - \frac{x-8}{39} + \frac{x}{13} = 1\frac{1}{2}$.

(4) $11x = 9y - 18$; $13x - 3y = 78$, find $x$ and $y$.

(5) Divide $2x^4 + 12x^3 + 26x^2 + 24x + 8$ by $x^2 + 3x + 2$.

(6) Solve $\frac{1}{9-2x} + \frac{4}{5} = \frac{1}{3-x}$.

(7) In a factory there are 62 workmen, and the total amount of their weekly wages is £74 6*s*. A certain number of them receive 40*s*. a week, twice as many receive 29*s*. 6*d*. a week, and the rest 17*s*. a week. Find the number receiving each rate of wages.

## No. 10.

(1) Express $21a^2 - 40x - 21$; $143x^2 + 8x - 16$; and
$x^3 - 2x^2 + 2x - 1$ in the form of factors.

(2) Prove that $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$. N.B. Let $\frac{a}{b} = x$, $\frac{c}{d} = y$. etc.

(3) Simplify $\frac{x^3+2}{(x-2)^2\,(x^2+1)} - \frac{4}{5(x-2)} - \frac{x+2}{5(x^2+1)}$.

(4) $\frac{1}{6}(x+10) - 4\frac{3}{4} - \frac{x}{4} = \frac{1}{3}(x-2) - (x-1)$. find $x$.

(5) $\left.\begin{array}{l} \frac{1}{7}(3x+1) - \frac{1}{2}(2x-y) = \frac{1}{8}(2y-x) \\ \frac{1}{3}(4x-2) - \frac{1}{2}(4y-5x) = \frac{1}{6}(x+y) \end{array}\right\}$, find $x$ and $y$.

(6) How much silver must I add to 2 pounds 6 ounces of an alloy of silver and gold containing 91·7 per cent. of pure gold, so that the resulting mixture may contain 84 per cent. of gold?

(7) *A* can do half as much work as *B*, and *B* can do half as much as *C*; how long would each separately require to do a piece of work that they can together complete in 24 days?

## No. 11.

(1) Simplify $a - [2b + \{3c - 3a - (a+b)\} + \{2a - (b+c)\}]$, using only three lines in the solution.

(2) Factor $x^2 - y^2 + z^2 - a^2 - 2xz + 2ay$.

(3) Find the H. C. F. of $5x^2(12x^3 + 4x^2 + 17x - 3)$
and $10x(24x^3 - 52x^2 + 14x - 1)$.

(4) Simplify $\frac{3}{1-2x} - \frac{7}{1+2x} - \frac{4-20x}{4x^2-1}$.

(5) Find the values of $x$, $y$ and $z$ from the equations
$10x + 15y - 24z = 41$; $15x - 12y + 16z = 10$;
and $18x - 14y - 7z = -13$.

(6) Show that three globes or spheres whose diameters are 3 inches, 4 inches, and 5 inches, respectively, contain together the same volume as a globe 6 inches in diameter.

N.B. Cubes, spheres and other regular solids have their volumes proportional to the cubes of their like dimensions.

(7) A train runs 1 hour and then stops 15 minutes to repair a break in the engine. Afterwards it runs at $\frac{3}{4}$ its former rate, and arrives 24 minutes late. If the break had happened 5 miles farther on, the train would have been only 21 minutes late. Find the usual speed of the train in miles per hour.

## No. 12.

(1) Divide $a^6 - b^6$ by $a^3 + 2a^2b + 2ab^2 + b^3$.

(2) Divide $x^4 + x^3 + 5x - 4x^2 - 3$ by $x^2 - 2x - 3$.

(3) Resolve $x^{16} - y^{16}$ into five factors.

(4) From $\frac{a}{a-x}$ subtract $\frac{ax}{a^2 - x^2}$.

(5) Multiply together $\frac{1-x^2}{1+y}$, $\frac{1-y^2}{x+x^2}$, and $1 + \frac{x}{1-x}$.

(6) $(x + \frac{5}{2})(x - \frac{3}{3}) + \frac{3}{4} = (x+5)(x-3)$; find $x$.

(7) Two passengers have together 400 pounds of baggage. One pays $1.20 and the other $1.80 for excess above the weight allowed to go free. If all the baggage had belonged to one person he would have to pay $4.50. How many pounds of baggage is allowed free with each ticket?

(8) If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a+b}{a-b} = \frac{c+d}{c-d}$;

prove also that $\frac{3a^3 + 4b^4}{2a^2 - 3b^3} = \frac{3c^3 + 4d^4}{2c^2 - 3d^3}$.

## No. 13.

*Combined Examination—Toronto Public Schools.*

(1) Simplify $15x - \{4 - [3 - 5x - (3x - 7)]\}$.

(2) Multiply $x^2 + 2ax + 3a^2$ by $x^2 - 2ax + a^2$.

(3) From $\frac{1+x}{1+x+x^2}$ take $\frac{1-x}{1-x+x^2}$.

(4) Divide $3x^2 +$ [illegible] $b^2x - 4a^2b^2$ by $2ab + x$.

(5) $\frac{1}{2}x + \frac{1}{3}x = x - 7$. Find the value of $x$.

(6) *A* can correct 70 pages for the press in $1\frac{1}{2}$ hours, *B* can correct 150 pages in $2\frac{1}{4}$ hours; how long will they be in correcting 425 pages jointly?

(7) $\frac{(2x+3)x}{2x+1}+\frac{1}{3x}=x+1$. Find $x$.

(8) $x^2-12x=-35$. Find $x$.

(9) Find the value of

$$2\sqrt{d-b}+3\sqrt{3d+2c-1}+4\sqrt{a+b+2c+d}$$

when $a=0$, $b=2$, $c=4$, and $d=6$.

(10) Find the product of $a-b$ by $a+b$.

(11) Find the difference between $x-3y+4z$ and $x+2y-6z$.

(12) Divide $x^4+y^4-z^4+2x^2y^2-2z^2-1$ by $x^2+y^2-z^2-1$.

(13) Find the value of $x$ in the equation $4x+9=8x-3$.

## No. 14.

*Provincial Model School, Toronto.*

(1) Write in words the meaning of the expression $(abc+xyz)^2$.

(2) Add together $5m+3n+p$, $3(m+n+p)$ and $5p+3n+m$, and obtain the numerical result when $p=\frac{n}{10}=\frac{m}{100}=1$.

(3) Add together $(a+b)x+(a+c)y$, $(b+c)y$, and $(c-a)x+(b-a)y$.

(4) Prove that $c-(a-b)=c-a+b$.

(5) Simplify $16-\left\{\frac{5}{4}-x-\overline{-\frac{1}{4}\left(3-\frac{x}{2}\right)}\right\}$.

(6) Prove that $a^4\times a^3=a^7$, and multiply $x^4+2x^3y+4x^2y^2+8xy^3+16y^4$ by $x-2y$.

(7) Prove that $a^6\div a^2=a^4$, and divide $(x^3-9x^2y+23xy^2-15y^3)(x-7y)$ by $x^2-8xy+7y^2$.

(8) Resolve the following expressions into factors :— $81x^4-1$, $(4x+3y)^2-(3x+4y)^2$, $12x^2-14x+2$.

(9) Solve the following equations :—

(i) $13x-21(x-3)=10-21(3-x)$.

(ii) $(2+x)(a-3)=-4-2ax$.

(10) $A$ and $B$ play together for \$5; if $A$ win, he will have thrice as much as $B$, but if he lose he will have only twice as much. What has each at first?

(11) Reduce to its lowest terms

$$\frac{y^3-(2a+b)y^2+(2ab+a^2)y-a^2b}{3y^2-(4a+2b)y+2ab+x^2}.$$

(12) Find the L.C.M. of $4(a^3-ab^2)$, $12(ab^2+b^3)$, $8(a^3-a^2b)$.

(13) Prove that $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.

(14) Explain fully the method of solving a quadratic equation, and solve $\frac{5x}{x+4}-\frac{3x-2}{2x-3}=2$.

(15)

$$\left.\begin{aligned} xy &= x+y \\ xz &= 2(x+z) \\ yz &= 3(y+z) \end{aligned}\right\}$$

## No. 15.

*Fifth Class, South Essex, 1881.*

(1) Represent the sum of

$$\frac{1}{x(x-y)(x-z)},\ \frac{1}{y(y-z)(y-x)},\ \frac{1}{z(z-x)(z-y)}.$$

(2) Simplify the expression $\frac{a-\sqrt{b}}{c+\sqrt{d}}+\frac{a+\sqrt{b}}{c-\sqrt{d}}$.

(3) Investigate a rule for finding L. C. M. of two algebraic expressions.

(4) If $\frac{1}{b}+\frac{1}{c}=\frac{4}{a}$, show that

$$(a+b-c)^3+2(b+c-a)^3+(c+a-b)^3=2(b+c)^3.$$

(5) Prove the equation $a^m\times a^n=a^{m+n}$ is true when $m$ and $n$ are integral and positive.

(6) Factor $a^{2m}-3a^mc^n+2c^{2n}$.

(7) Solve $\frac{4x+7}{4x+5}+\frac{4x+9}{4x+7}=\frac{4x+6}{4x+4}+\frac{4x+10}{4x+8}$.

(8) The sum of the digits of a number is 9; if the digits be inverted the difference between the two numbers is 9, find the number, the right hand digit being the greater.

## No. 16.

*University of Toronto—Pass Matriculation, 1881.*

(5) Multiply $b^2 + (a-b)(b-c)$ by $c^2 + (b-c)(c-a)$. Show that your answer is correct by substituting $a=2$, $b=0$, $c=-3$.

(6) Simplify (i) $\dfrac{a^2 b c^{-2}}{a^{-1} b^2 c^{-3}}$.

(ii) $\dfrac{x^2 - 2 + x^{-2}}{x^2 - x^{-2}} - \dfrac{x^2}{x^2+1}$.

(7) Resolve into factors

$a^2 - b^2$, $ab + bc + ca + b^2$,

$a(b+c)^2 + b(c+a)^2 + c(a+b)^2 - 4abc$.

$(a+b)^3 - 2b\dfrac{a^3 - b^3}{a-b} + c(a^2 - b^2) - 2ab^2$.

Find the Greatest Common Measure, and the Least Common Multiple of these *four* quantities.

(8) Solve the equations

(i) $ax + b = bx + a$.

(ii) $\dfrac{1}{x^2+3x+2} + \dfrac{1}{x^2+5x+6} = \dfrac{1}{x^2+x-2}$.

(iii) $\begin{cases} \dfrac{2}{x} - \dfrac{3}{y} = 4. \\ 2x - 3y = 2xy. \end{cases}$

(9) There are two vessels, *A* and *B*, each containing a mixture of water and wine, *A* in the ratio of 2:3, *B* in the ratio of 3:7. What quantity must be taken from each in order to form a third mixture which shall contain 5 gallons of water and 11 of wine?

## No. 17.

*University of Toronto—Matriculation, 1883.*

(1) Find the product of $(a+b)$, $(a^2+ab+b^2)$, $(a-b)$, and $(a^2-ab+b^2)$.

(2) If $a$ and $b$ are positive integers, show that
$$x^a \times x^b = x^{a+b}.$$

(3) Prove the rule for finding the G.C.M. of two quantities. Find the G.C.M. of $6x^5 + 15x^4y - 4x^3z^3 - 10x^2yz^3$ and $9x^3y - 27x^2yz - 6xyz^2 + 18yz^3$.

(4) State the rule for extracting the square root of a compound quantity.

(5) Solve the following equations

(i) $3x + z = 11,\ 2y + 3z = 16,\ 5x + 4y = 35.$

(ii) $\frac{x+a}{x-a} - \frac{x+b}{x-b} = c.$

(iii) $\frac{x}{a} + \frac{a}{x} = 2 + \frac{c}{x}.$

## No. 18.

*McGill University, Montreal—School Examination.*

(1) Multiply $1 + 2x - x^2 - \frac{1}{2}x^3$ by itself, and find the value of the result if $1 - 2x = 3$.

(2) Find the remainder when $a^5 - 4a^3b^2 - 8a^2b^3 - 17ab^4 - 15b^5$ is divided by $a^2 - 2ab - 3b^2$.

(3) Simplify $\frac{2}{3}x(x+1)\{x+2-\frac{1}{2}(2x+1)\}$; $\frac{2(x^2-\frac{1}{4})}{2x+1} + \frac{1}{2}$.

(4) Reduce the following fractions to their lowest terms:
$$\frac{a^2x+a^3}{ax^2-a^3};\quad \frac{(x^4-a^4)(x-a)}{(x^2+a^2-2ax)(ax+x^2)};\quad \frac{1+x^3}{1+2x+2x^2+x^3}.$$

(5) Find the square root of
$$x^4 + 2x^3 - x + \frac{1}{4} \text{ and of } \frac{4x^2-4x+1}{9x^2+6x+1}.$$

(6) Solve the equations

(i) $2x - \frac{x}{2} = 18$; (ii) $(m+n)(m-x) = m(n-x)$;

(iii) $2x - \frac{y-3}{5} = 4$; $3y + \frac{x-2}{3} = 9$.

## No. 19.

*College of Ottawa, Ont.—Matriculation Examination.*

(1) Clear away the parentheses, and reduce the following expression:

$$a+b-(2a-3b)-4(5a+7b)-(-13a+2b)+3\{a-6(b-a)\}.$$

(2) Give the three formulas for the expansion of $(a+b)^2$, $(a-b)^2$, and $(a+b)(a-b)$, and give an example of each formula.

(3) Divide $5x-3-4x^2+x^4+x^3$ by $-3+x^2-2x$.

(4) Find the G.C.D. and the L.C.M. of the three following expressions

$$(2x-4)(3x-6);\ (x-3)(4x-8);\ (2x-6)(5x-10).$$

(5) Simplify $\dfrac{\dfrac{m^2+n^2}{n}-2m}{\dfrac{1}{n}+\dfrac{1}{m}} \times \dfrac{m^2-n^2}{m-n}$.

(6) Solve the equations

$$2x+4y-3z=22;\ 4x-2y+5z=18;\ 5x+7y-z=63.$$

(7) Extract the square root of

$$15a^4b^2+a^6-6a^5b-20a^3b^3+b^6+15a^2b^4-6ab^5.$$

(8) Convert $\sqrt[3]{\frac{3}{4}}$ into such an expression, not a decimal, as shall not necessitate two extractions in finding the cube root of $\frac{3}{4}$.

(9) Solve the following equation, $\frac{1}{2}x^2-\frac{1}{3}x+20\frac{1}{2}=42\frac{2}{3}$.

(10) The hypotenuse of a right-angled triangle is 20 feet and the area of the triangle is 96 square feet. Find the length of the legs.

## No. 20.

*University of Oxford, England—Local Examination, Junior Candidates.*

(1) Find the value of $\dfrac{a^2-c-3ac(b-2c)}{4c(a+b)}+\sqrt{2a-\dfrac{1+b}{c}}$,
when $a=1$, $b=0$, $c=-\frac{1}{2}$.

(2) Multiply $x^4 - ax^3 + a^3x - a^4$ by $x^2 + ax + a^2$;
also divide $p^4 - 9pq^3 + 18q^4$ by $p^2 - 3pq + 3q^2$.

(3) Simplify (i) $\left(\frac{2x+y}{y} - \frac{y}{2x+y}\right)\left(\frac{x}{x+y} - \frac{x+y}{x}\right)$;

(ii) $\frac{1}{x+1} - 2\frac{x+2}{x^2-1} + \frac{x+3}{(x-y)^2}$.

(4) Find the G. C. M. of $x^3 - 6x - 4$ and $3x^3 - 8x + 8$; also the L. C. M. of $(3a^2 - 3ab)^2$, $18(a^3b^2 - ab^4)$, and $24(a^3b^3 - b^6)$.

(5) Solve the equations

(i) $\frac{5x-3}{8} - \frac{1}{3}(x - 2\frac{1}{2}) = \frac{2x-1}{10} + 1\frac{3}{4}$.

(ii) $\frac{x+3}{y+4} = 2\frac{x+1}{2y+7}$, $6(x+\frac{1}{2}) = 11(y+5)$.

## No. 21.

*University of Oxford, England—Local Examination, Junior Candidates.*

(1) Find the value of $a^3 + b^3 - c^3 + 3abc$;
when $a = \frac{1}{2}$, $b = \frac{1}{3}$, $c = \frac{1}{3}$.

(2) Multiply together $x^2 - 7x + 6$, $x^2 + 7x - 18$, $x^3 - 1$, and express the result in simple factors.

(3) Find the G. C. M. of $2x^3 + 7x^2 + 10x + 5$
and $x^3 + 3x^2 + 4x + 2$,
and the L. C. M. of $6xy^2(x+y)$, $3x^3(x-y)^2$, and $4(x^2 - y^2)$.

(4) Simplify (i) $\left(\frac{x+y}{y(x^2+y^2)} - \frac{x+y}{x(x^2+y^2)} + \frac{1}{xy}\right)\left(\frac{1}{y^2} + \frac{1}{x^2}\right)$;

(ii) $\left(1 - \frac{y^4}{x^4}\right) \div \left(\frac{x}{y} + \frac{y}{x}\right)$.

(5) Solve the equations (i) $\frac{\frac{1}{8} - x}{2} + \frac{1 - \frac{x}{2}}{\frac{3}{2}} = \frac{x}{4} + \frac{x-1}{3}$;

(ii) $9x - 8y = 1$, $12x - 10y = 1$.

(6) Into a cistern one-third full of water 31 gallons are poured, and the cistern is then found to be half full; find its capacity.

## No. 22.

*University of Oxford, England—Local Examination, Senior Candidates.*

(1) Prove that

$$(a+b)\,(a+x)\,(b+x)-a(b+x)^2-b(a+x)^2=(a-b)^2x,$$

and divide $a^3-b^3$ by $a^{\frac{3}{2}}-2ab^{\frac{1}{2}}+2a^{\frac{1}{2}}b-b^{\frac{3}{2}}$.

(2) Resolve into component factors

(i) $63x^3y-28xy^3$; (ii) $a^5-a^4b-ab^4+b^5$.

Find the remainder when $a^n+b^n$ is divided by $a-b$.

(3) Find the G. C. M. of

$x^4-6x^3+13x^2-12x+4$ and $x^4-4x^3+8x^2-16x+16$,

and the L. C. M. of $x^3-y^3$, $x^3+y^3$, $x^3-xy^2$, and $x^2y+xy^2+y^3$.

(4) Simplify the fractions (i) $\dfrac{x+2}{x+1}-\dfrac{x+1}{x+2}-\dfrac{2}{x+4}$;

(ii) $\dfrac{1}{1-\dfrac{1}{x}}-\dfrac{1}{\dfrac{1}{x^3}-\dfrac{1}{x}}-\dfrac{1}{1-\dfrac{2x}{x^2+1}}$.

(5) Solve the equations

(i) $\frac{1}{8}\{(2x-32)-(x+16)\}=\frac{1}{11}\{(x-20)-(2x-11)\}$;

(ii) $(x+5)\,(y+7)=(x+1)\,(y-9)+112$, $2x+5=3y-4$;

(iii) $x-\dfrac{x^3-8}{x^2+5}=2$.

## No. 23.

*University of Oxford, Eng.—First Examination of Women.*

(1) Find the value of $\dfrac{b^2c-2a^2b}{a^5+4b^3}$ when $a=1$, $b=-\frac{1}{2}$, $c=0$.

(2) Take $a+2b+3c-4d-5e$ from $3a-4b+c-d+e$.

(3) Multiply $a^3-a^2b+ab^2-b^3$ by $a+b$, and divide $a^4+a^2b^2+b^4$ by $a^2+ab+b^2$.

(4) Find the H. C. F. of $a^4+5a^2-6$ and $a^4+5a^2+4$, and the L. C. M. of $12(a^3-b^3)$, $15(a^3+b^3)$, $20ab(a^2-b^2)$.

(5) Simplify $\dfrac{a^2+4a+3}{a^2-a-2}$.

(6) Extract the square root of $a^4-2a^3+2a^2-a+\frac{1}{4}$.

(7) Solve (i) $7(x-1)-6(x-2)=3(x-3)$;

(ii) $\frac{x}{9}+\frac{x}{6}+\frac{x}{3}=x-7$;

(iii) $(x-3)(x-13)=(x-4)(x-9)$;

(iv) $8x+3y=74$, $9x-2y=51$.

(8) I have in my purse £1 13*s.* 9*d.* made up of a certain number of pence, twice the same number of farthings, and thrice the same number of fourpenny pieces. Find the number of each coin.

(9) *A* is thrice as old as *B*. Seven years ago *A* was four times as old as *B*. Find their ages now.

(10) *A* and *B* play at cards. *A* wins six shillings, and finds he has thrice as much as *B*. The game is continued till *A* finds he has lost twenty-four shillings, and then has a third of what *B* has. With what sum did each begin?

## No. 24.

*University of Oxford, Eng.—First Examination of Women.*

(1) Evaluate $(x-y)^2+(y-z)^2+(z-x)^2$, when $x=3\frac{1}{2}$, $y=2\frac{1}{2}$, $z=1\frac{1}{2}$.

(2) From the sum of $\frac{1}{8}(2x-3y+4z)$ and $\frac{1}{12}(4x+3y-11z)$ subtract $\frac{1}{24}(8x-9y+6z)$.

(3) Multiply $x^2+y^2+a(x-y)$ by $xy-a(x+y)+a^2$.

(4) Divide $x^3+8y^3-125z^3+30xyz$ by $x+2y-5z$.

(5) Express in factors (i) $7x^2-77x-182$;

(ii) $20x^4-60x^3y+45x^2y^2$.

(6) Find the G. C. M. of $x^2+11x+30$. $9x^3+53x^2-9x-18$.

(7) Find the L. C. M. of
$15x^2(a^2-2ax+x^2)$, $21a^2(a^2+2ax+x^2)$, $35ax(a^2-x^2)$.

(8) Simplify $\dfrac{\dfrac{a}{a-b}+\dfrac{b}{a+b}-1}{\dfrac{a}{a-b}-\dfrac{b}{a-b}+1}$, and find the value of

$\frac{x+2}{2-x}+\frac{x-2}{2+x}-\frac{4}{4-x^2}$ when $x=\frac{1}{2}$.

(9) Solve the equations

(i) $\frac{2}{3}(x-1)+3\left(\frac{x}{2}-9\right)-(x-13)=11$;

(ii) $\frac{1}{x-a}+\frac{1}{x-b}-\frac{2}{x-a-b}=0$;

(iii) $\frac{5x+7y}{7}=\frac{7x+5y}{8}=8$.

(10) The sum of three consecutive whole numbers exceeds the greatest of them by 19; what are the numbers?

## No. 25.

*University of Oxford, Eng.—First Examination of Women.*

(1) Find the value of $x(y+z)+y[x-(y+z)]-z[y-x(z-x)]$ when $x=3$, $y=2$, $z=1$.

(2) Subtract $2x^4+\frac{5}{3}x^2-x+\frac{1}{4}$ from $\frac{5}{2}x^4+x^2-\frac{1}{4}x-\frac{3}{4}$.

(3) Multiply $1+2x+3y+4x^2-6xy+9y^2$ by $1-2x-3y$.

(4) Divide $x^4-\frac{13}{6}x^3+x^2+\frac{4}{3}x-2$ by $x-\frac{3}{2}$.

(5) Resolve into factors (i) $4x^4-36x^2y^2$;

(ii) $(2x-3y)^2-(x-2y)^2$.

(6) Find the G. C. M. of $x^3-3x+2$ and $x^3+4x^2-5$.

(7) Find the L. C. M. of
$12(1-x^2)$, $15(1-x)^2$, and $20(x+x^2)$.

(8) Simplify (i) $\frac{8a^2b}{c}\times\frac{c^2d}{8a^3}\times\frac{4ab}{cd}\times\frac{bcd-cd^2}{4b^2-4bd}$;

(ii) $\left(\frac{1}{1+4x}+\frac{4x}{1-4x}\right)\div\left(\frac{1}{1-4x}-\frac{4x}{1+4x}\right)$.

(9) Solve the equations (i) $\frac{x}{3}+\frac{2x}{9}-\frac{x}{27}-\frac{x+1}{11}=23$;

(ii) $\frac{x+a}{x+b}+\frac{x+b}{x+c}=2$;

(iii) $\frac{2x+3}{y}=1\frac{3}{4}$, $\frac{2y+3}{x}=2\frac{1}{5}$.

(10) A person walks a certain distance at the rate of $3\frac{1}{2}$ miles an hour, and finds that if he had walked 4 miles an hour, he would have gone the same distance in less time by one hour; what is the distance?

## No. 26.

*University of Cambridge, England.*

(1) Prove that $(x+4)^3-(x+1)^3=9(x+1)(x+4)+27$.

(2) Simplify (i) $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{(b+c)(c+a)(a+b)}{abc}$;

(ii) $\frac{a^2b^2-a^2-b^2+1}{ab-a-b+1}$.

(3) Find the G.C.M. of $x^3-4x^2+2x+3$ and $2x^4-x^2-5x-3$.

(4) Prove that, if $m$ and $n$ be positive integers, $(a^m)^n=(a^n)^m$.

(5) Solve the equations (i) $\frac{1}{3}(5x+1)+\frac{2x-3}{7}=x+\frac{8}{15}$;

(ii) $\frac{1}{a+x}+\frac{1}{b+x}=\frac{a+b}{ab}$;

(iii) $ax+by=2$, $ab(x+y)=a+b$.

## No. 27.

*University of Cambridge, Eng.—Second Previous Examination.*

(1) Simplify $6(a-2b)(b-2a)-(a-3b)(4b-a)-12ab$, and from the sum of $(2a-b)^2$ and $(a-2b)^2$ take the square of $2(a-b)$.

(2) Define *multiplication*, *product*, and *coefficient*.
Divide $14a^4+15a^3b+33a^2b^2+36ab^3+28b^4$ by $7a^2-3ab+14b^2$.

(3) Find the value of $(a-b)^2+(b-c)^2+(a-b)(b-c)+5c^2$ when $a=1$, $b=-2$, $c=\frac{1}{2}$.

(4) Resolve into the simplest possible factors:

(i) $6x^2+5xy-6y^2$;

(ii) $x^3-13x^2y+42xy^2$;

(iii) $(a+2b+3c)^2-4(a+b-c)^2$;

(iv) $81x^4-625y^4$.

(5) Define the highest common factor of two algebraical expressions.

Find the highest common factor of
$7x^3-10x^2-7x+10$ and $2x^3-x^2-2x+1$.

(6) Reduce to simple fractions in their lowest terms:

(i) $\dfrac{x^2-7xy+12y^2}{x^2+5xy+6y^2} \div \dfrac{x^2-5xy+4y^2}{x^2+xy-2y^2}$;

(ii) $\dfrac{x-a}{x+a}+\dfrac{a^2+3ax}{a^2-x^2}+\dfrac{x+a}{x-a}$;

(iii) $\dfrac{a-\dfrac{ab}{a+b}}{a^2+\dfrac{a^2b^2}{a^2-b^2}} \times \dfrac{\dfrac{1}{a^2}-\dfrac{1}{b^2}}{\dfrac{1}{a}-\dfrac{1}{b}}$

(7) Solve the equations

(i) $\dfrac{x+1}{5}-\dfrac{2x-3}{9}=\dfrac{3x-2}{7}-10$;

(ii) $\dfrac{x}{6}+\dfrac{x}{5}=14, \quad \dfrac{x}{9}+\dfrac{y}{2}=24$;

(iii) $5x^2-17x+14=0$;

(iv) $x^2+y^2=5a^2+5b^2+8ab, \; xy=2a^2+2b^2+5ab$.

(8) Find the value of $x^m \times x^n$, when $m$ and $n$ are positive integers.

Simplify $a^{2p+q} \times a^{p+4q} \div a^{q-p}$.

## No. 28.

*Intermediate Examination, Ontario, December, 1878.*

(1) Multiply $4x^2-\frac{2}{5}x+\frac{1}{25}$ by $2x+\frac{1}{5}$.

Prove that

$(\frac{1}{2}x-y)^3-(x-\frac{1}{2}y)^3$ is exactly divisible by $x+y$.

(2) Express in words the meaning of the formulæ

$(x+a)(x+b)=x^2+(a+b)x+ab$.

Retaining the order of the terms, how will the right-handed member of this expression be affected by changing, in the left-hand member (i) the sign of $b$ only, (ii) the sign of $a$ only, (iii) the signs of both $a$ and $b$?

(3) Simplify $(a+b)^4+(a-b)^4-2(a^2-b^2)^2$; and show that $(a+b+c)(b+c-a)(a+c-b)(a+b-c)=4a^2b^2$ when $a^2+b^2=c^2$.

(4) Prove that $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$.

Simplify $\left(\frac{a^2+b^2}{2ab}+1\right)\left(\frac{ab^2}{a^3+b^3}\right) \div \frac{4a(a+b)}{a-ab^2+b^2}$.

(5) I went from Toronto to Niagara, 35 miles, in the steamer "City of Toronto" and returned in the "Rothesay," making the round trip in 5 hours and 15 minutes; on another occasion I went in the "Rothesay" (whose speed on this occasion was 1 mile an hour less than usual), from Toronto to Lewiston, 42 miles, and returned in the "City of Toronto," making the round trip in 6 hours and 30 minutes; find the usual rates per hour which these steamers make.

(6) Solve $\frac{3}{x} - \frac{2}{y} = \frac{1}{a}$

$\frac{2}{x} - \frac{1}{y} = \frac{2}{a}$.

(7) Find 3 consecutive numbers whose product is 48 times the middle number.

(8) If $\frac{m}{x} = \frac{n}{y}$, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; prove that

$$\frac{m^2}{a^2} + \frac{n^2}{b^2} = \frac{m^2+n^2}{x^2+y^2}.$$

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## THIRD CLASS TEACHERS, ONTARIO.

NOTE.—Here follows the complete series of papers set for **Third Class Teachers in Ontario** from 1878, when algebra was first required for this grade of certificate, down to date. Where **Notes and Hints** seem to be required they will be found **with the answers** in the **Teacher's Edition.**

### No. 29.

*July, 1878.*

(1) If $a=3$, $b=5$, $c=-9$ and $2s=a+b+c$, find the value of $s(s-a)(s-b)(s-c)$.

(2) Prove that $a^3\left(\frac{a^3+2b^3}{a^3-b^3}\right)^3 + b^3\left(\frac{b^3+2a^3}{b^3-a^3}\right)^3 = a^3+b^3$.

(3) Multiply together $\sqrt{p}+\sqrt{q}+\sqrt{r}$, $\sqrt{p}-\sqrt{q}-\sqrt{r}$, $\sqrt{p}-\sqrt{q}+\sqrt{r}$, and $\sqrt{p}+\sqrt{q}-\sqrt{r}$.

(4) $x^3+y^3$ is divided by $x^3-y^3$ and the result is divided by the quotient of $x+y$ by $x-y$, what will be the result if $\frac{1}{2}x=-\frac{2}{3}y$?

(5) Find two factors each of $x^4+x^2+1$, $x^4+1$ and three factors of $x^6+1$.

(6) Solve the equations :—

$$(a)\quad \frac{x-3\frac{1}{2}}{2}-\frac{11-x}{4}+\frac{x-16\frac{1}{4}}{5\frac{1}{2}}=0.$$

$$(b)\quad \frac{x-3}{6}-\frac{1-2x}{4}+\frac{2-x}{x}-\frac{2x+1}{3}=0.$$

(7) $A$ and $B$ could do a certain piece of work in 15 days; $B$ and $C$ could do it in 20 days; in what time could $A$ and $C$ do it supposing $A$ can do three times as much as $C$ in a given time?

## No. 30.

*July, 1879.*

(1) Find the value of

$$3x^5+54x^4+50x^3-19x^2-35x-18 \text{ when } x=-17.$$

(2) Demonstrate the identities :—

(i) $(5m^2+4mn+n^2)^2-(3m^2+4mn+n^2)^2 = 4m^2(2m+n)^2$.

(ii) $(a+b+c)(ab+bc+ca)-abc=(a+b)(b+c)(c+a)$.

(iii) $(a-b)(c-d)+(b-c)(a-d)+(c-a)(b+d)=0$.

(3) Divide $(m^2+an^2)(x^2+ay^2)-a(nx-my)^2$ by $mx+any$.

(4) Prove that if from the square of the sum of two numbers there be taken four times their product, the remainder is a square.

(5) Solve (i) $(x-1)(x-2)-(x-3)(x-4)=3$.

(ii) $\dfrac{2}{x-1}+\dfrac{3}{x-2}=\dfrac{8}{x^2-3x+2}$.

(iii) $(x-a)(b-c)+(x-b)(c-a)+(x-c)(a-b) = x-a-b-c$.

(6) What value of $x$ will make $x^2+2ax+b^2$ the square of $x+c$? What is the result when $a=b=c$?

(7) A man is thrice as old as his son, five years ago he was four times as old; how old is he?

## No. 31.

*July, 1880.*

(1) $\pi = 3 \cdot 1416$, $a = 5$ inches and $h = 7$ feet 11 inches, find the value of $2\pi(ah + a^2)$.

(2) If $x = \cdot 4$ find, correct to one decimal place, the value of

$$x^8 - 4x^6 - 2x^4 - 52x^2 + 9.$$

(3) If $x = a + d$, $y = b + d$, $z = c + d$, prove that

$$x^2 + y^2 + z^2 - xy - yz - zx = a^2 + b^2 + c^2 - ab - bc - ca.$$

(4) Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

(5) Find the factors of

(i) $15x^2 - 19xy - 10y^2$?

(ii) $15(a+b)^2 + 14(a+b)(x+y) - 8(x+y)^2$.

(iii) $x^3 - x^2y - xy^2 + y^3$.

(6) Solve (i) $(10x - 11)(11 + 2x) + (5x - 11)(11 + 3x) + (7x - 11)(11 - 5x) = 0$.

(ii) $(x - 2n + 1)^2 - (2n - 1)^2 = (x - 2n)^2$.

(7) What value of $x$ will make $x^3 + 3cx^2 + 2c^2x + 5c^3$ equal to the cube of $x + c$?

(8) If $a^2 + b^2 = c^2$ and $s = a + b + c$, prove that

$$(2s - a)^2 + (2s - b)^2 = (2s - c)^2.$$

(9) Having 75 minutes at my disposal, how far can I go in a carriage at $6\frac{2}{3}$ miles per hour, having to walk back at $3\frac{3}{4}$ miles per hour?

(10) I row $a$ miles down a stream in $b$ minutes and return in $c$ minutes; find the rate at which I row in still water and the rate at which the stream flows.

## No. 32.

*Intermediate Examination, Ontario, July, 1881.*

(1) Factor $x^3 + y^3$; and $x^3 + y^3 + z^3 - 3xyz$.
Utilize your results to show that

(i) $(x + z)^3 + (y - z)^3 - (x + y)(x - y + 2z)^2 = (x + y)(yz - zx + xy - z^2)$.

(ii) $(a^2 - bc)^3 + (b^2 - ca)^3 + (c^2 - ab)^3 - 3(a^2 - bc)(b^2 - ca)(c^2 - ab) = (a^3 + b^3 + c^3 - 3abc)^2$.

(2) If $a^2 - bc = b^2 - ca$, and $a$ be not equal to $b$, then

$$a(b^2 + bc + c^2) + b(c^2 + ca + a^2) + c(a^2 + ab + b^2) = 0.$$

(3) Show how to find the L.C.M. of two algebraic expressions. Find the conditions that $x^3 + ax^2 + b$ and $x^3 + cx + d$ may have a L. C. M. of the form $x^4 + px^3 + qx^2 + rx + s$.

(4) Simplify $\dfrac{(x+y)z^3}{(y-z)(z-x)} + \dfrac{(y+z)x^3}{(z-x)(x-y)} + \dfrac{(z+x)y^3}{(x-y)(y-z)}$.

(5) Extract the square root of

(i) $2\left(1 - \dfrac{b^2 + c^2 - a^2}{2bc}\right)\left(1 - \dfrac{c^2 + a^2 - b^2}{2ca}\right)\left(1 - \dfrac{a^2 + b^2 - c^2}{2ab}\right)$.

(ii) $x^4 + x^3 + \frac{29}{4}x^2 + \frac{7}{2}x + \frac{49}{4}$.

(6) Find the value of $x$ in

$$(x+a)(b-c) + (x+b)(c-a) + (x+c)(a-b) = 0.$$

Explain result.

(7) Find an expression for $k$ in terms of $a$, $b$, $c$, that will make

$$\frac{b^2 - c^2}{k - a} + \frac{c^2 - a^2}{k - b} + \frac{a^2 - b^2}{k - c},$$ vanish.

(8) If for every \$3 of income $A$ has, $B$ has \$2; for every \$12 $A$ spends, $B$ spends \$1; and for every \$4 $A$ saves, $B$ saves \$5; find the proportion of his income that $A$ saves.

(9) Solve the equations:

(i) $\dfrac{x+1}{5} + x(x-1) = (x-1)^2$.

(ii) $\dfrac{1}{x-a} - \dfrac{1}{x-2a} = \dfrac{1}{x-3a} - \dfrac{1}{x-4a}$.

(iii) $\dfrac{2x^3 + 2x^2 + 3x + 1}{x^2 + x + 1} = \dfrac{x^2 - x + 1}{x - 1} + \dfrac{x^4 - x + 1}{x^3 - 1}$.

(iv) $\left.\begin{array}{l} x^2 + xy + y = 25 \\ x + xy + y^2 = 31 \end{array}\right\}$.

## No. 33.

*Intermediate Examination, 1882.*

(1) Form an expression symmetrical with respect to $x$, $y$, $z$, $u$, similar to $x^3 + y^3 + z^3 - 3xyz$, and write down the quotient on dividing it by $x + y + z + u$.

(2) Factor $ax^3-(a+b)(x-y)xy-by^3$.

Deduce or find by other means the factors of

$$(a+b)^3(x+y)-(x+2y+z)(a-c)(a+b)(b+c)-(b+c)^3(y+z).$$

Obtain four different relations between the quantities $a$, $b$, $c$, $d$, for any one of which the expression

$4(ad-bc)^2-(a^2+d^2-b^2-c^2)^2$ will vanish.

(3) Find the lowest common measure, not being a fraction, of the quantities $\dfrac{x^2+5x+6}{x+4}$ and $\dfrac{x^2+7x+12}{x+5}$.

(4) Reduce to lowest terms the following fractions:—

(i) $\dfrac{6x^5-5x^4-1}{x^6-x^4-x+1}$. (ii) $\dfrac{(a-b)(b-c)(c-a)}{(a-b)^3+(b-c)^3+(c-a)^3}$.

(5) (i) If $y+z+u=a$, $z+u+x=b$, $u+x+y=c$, $x+y+z=d$, then

$$\frac{1}{1+\frac{a}{x}}+\frac{1}{1+\frac{b}{y}}+\frac{1}{1+\frac{c}{z}}+\frac{1}{1+\frac{d}{u}}=1.$$

(ii) If $ax=b+c$, $by=c+a$, $cz=a+b$, then

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=1.$$

(6) Solve the equation $ax^2+bx+c=0$.

What value of $x$ will satisfy the equation

$$\frac{b-c}{x+a}+\frac{c-a}{x+b}+\frac{a-b}{x+c}=0.$$

(7) Solve the equations

(i) $\dfrac{7x}{3}-\left\{\frac{1}{2}-\left(\dfrac{x}{3}-\dfrac{x-1}{2}\right)\right\}=\dfrac{4x-2}{5}$.

(ii) $\dfrac{28}{x-4}-\dfrac{20}{x-3}=\dfrac{9}{x-5}-\dfrac{1}{x-1}$.

(iii) $\begin{cases} x^4+x^2y^2+y^4=21. \\ x^2+xy+y^2=7. \end{cases}$

(8) Solve the equations

$$\left.\begin{aligned} x+y+z&=6 \\ 3x+2y-z&=4 \\ x+3y+2z&=13 \end{aligned}\right\} \qquad \left.\begin{aligned} 3x-2y+5z&=4 \\ x-4y+z&=1 \\ 4x-6y+6z&=5 \end{aligned}\right\}$$

(9) The edge of a cube is 3 feet. What must be taken as the unit of length, that the number expressing the sum of the areas of the faces may be the same as that which expresses the sum of the lengths of the edges?

(10) The hour, minute, and second hands of a watch are on concentric axes, the same divisions on the dial answering for both minutes and seconds. Find when first between 3 and 4 o'clock the second hand will equally divide the interval between the minute and hour hands.

## No. 34.

*Intermediate and Third Class, July, 1883.*

(1) Divide (i) $(a-b)c^3+(b-c)a^3+(c-a)b^3$ by $(a-b)(b-c)(c-a)$;

(ii) $\dfrac{x^2+y^2}{x^3y^2}-\dfrac{x^2+y^2}{x^2y^3}$ by $\dfrac{1}{x}-\dfrac{1}{y}$.

(2) What must be the values of $a$, $b$, and $c$, that $x^3+ax^2+bx+c$ may have $x-1$, $x-2$, and $x-3$ all as factors?

(3) Find the H. C. F. of

(i) $3x^4-4x^3+1$ and $4x^4-5x^3-x^2+x+1$;

(ii) $8x^3-y^3+27z^3+18xyz$ and $4x^2+12xz+9z^2-y^2$.

(4) Simplify

(i) $\left(\dfrac{4x^2}{y^2}-1\right)\left(\dfrac{2x}{2x-y}-1\right)+\left(\dfrac{8x^3}{y^3}-1\right)\left(\dfrac{4x^2+2xy}{4x^2+2xy+y^2}-1\right)$;

(ii) $\dfrac{x^3+(a+b)x^2+(ab+1)x+b}{bx^3+(ab+1)x^2+(a+b)x+1}$.

(5) Find the value of $x$ that will make $\dfrac{ac+bd+ad+bc}{x-3c+2d}$ independent of $c$ and $d$.

(6) (i) If $a+b+c=0$, then $\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}=\left\{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right\}^2$.

(ii) If $x=a^2+b^2+c^2$ and $y=ab+bc+ca$, then $x^3+2y^3-3xy^2=(a^3+b^3+c^3-3abc)^2$.

(iii) If $2a=y+z$, $2b=z+x$, $2c=x+y$, express $(a+b+c)^3-2(a+b+c)(a^2+b^2+c^2)$ in terms of $x$, $y$, and $z$.

(7) Find a value of $a$ which will make the quantities

$$\frac{(a+b)(a+c)}{a+b+c} \text{ and } \frac{(a+c)(a+d)}{a+c+d}$$ equal to one another.

(8) Solve the equations

(i) $\sqrt{x+3}+\sqrt{x+2}=5$;

(ii) $\frac{5-x}{3}+\frac{5-2x}{4}+\frac{x+1}{3}-\frac{2+5x}{2}=0$;

(iii) $(x+a+b)(c+d)=(x+c+d)(a+b)$,
where $c+d$ is not equal to $a+b$.

(9) One side of a right-angled triangle exceeds the other by 3 feet, neither being the hypotenuse, and its area is 18 square feet. What are the sides?

(10) A cistern with vertical sides is $h$ feet deep. Water is carried away from it by one pipe $\frac{5}{6}$ as fast as it is supplied by another. Find at what point in the side the former pipe must be inserted that the cistern may fill in twice the time it would did water not flow from it at all.

## No. 35.

*Intermediate and Third Class, July, 1884.*

(1) Divide $(a^4-b^4)(x^4-y^4)-4abxy(b^2x^2-a^2y^2)$
by $a^2(x^2-y^2)+b^2(x^2+y^2)+2abxy$.

(2) Simplify (i) $\dfrac{\frac{n+m}{m}-\frac{m+n}{n}}{\frac{n-m}{m}+\frac{m-n}{n}}$;

(ii) $\frac{(a-b)(b-c)(c-a)}{abc}+\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}$.

(3) Resolve into linear factors

(i) $12(3x-2y)^2-44(3x-2y)(4y-2x)-45(4y-2x)^2$;

(ii) $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$.

(4) Show that $(a^2x+ay+z)(b-c)+(b^2x+by+z)(c-a)$
$+(c^2x+cy+z)(a-b)=(a-c)(c-b)(b-a)x$.

(5) If $y+z=2a$, $z+x=2b$, $x+y=2c$, find the value of
$(x+y+z)(xy+yz+zx)-xyz$ in terms of $a$, $b$, and $c$.

(6) If $\frac{a-b}{y-x}=\frac{b-c}{z-y}=\frac{a+b+c}{2(x+y+z)}$,

prove that $\frac{a}{y+z}=\frac{b}{z+x}=\frac{c}{x+y}$.

(7) Solve (i) $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}$;

(ii) $\frac{a}{x-a}-\frac{b}{x-b}=\frac{a-b}{x+c}$;

(iii) $(x^2-9)^2-11(x^2-9)=80$.

(8) Find the values of $x$ and $y$ that will satisfy both

$$\frac{3}{x}+\frac{2}{y}=2 \text{ and } \frac{2}{x}+\frac{3}{y}=\frac{1}{2}.$$

(9) A boy has a bag of nuts. He gives three more than two-fifths of them to his sister, six more than a quarter of the remainder to his brother, and eats three-thirteenths of what then remains, and finds he has exactly two-sevenths of the original number left. How many had he at first?

## No. 36.

*Third Class, July, 1885.*

(1). Simplify $a^2+b^2+c^2-(a-b+c)(a+b-c)$
$-(b-c+a)(b+c-a)-(c-a+b)(c+a-b)$.

(2) Divide $a^4+b^4+c^4-2b^2c^2-2a^2c^2-2a^2b^2$
by $a^2+b^2-c^2+2ab$.

(3) Multiply $x^{n-3}-x^{n-6}+x^3-1$ by $x^3+1$.

(4) Find the factors of $a^2-b^2+c^2-d^2+2ac-2bd$.

(5) Find the factors of $(a+b)^2-(b-c)^2+(c+a)^2$.

(6) Simplify $\dfrac{\frac{1}{x}-\frac{2}{x+c}+\frac{1}{x+2c}}{\frac{1}{x}-\frac{3}{x+c}+\frac{3}{x+2c}-\frac{1}{x+3c}}$.

(7) Find the value of $x$ that will satisfy the equation
$m(x-m)+n(x-n)=2mn$.

(8) Determine $x$ given
$4\{(x-a)(x-b)-(x-c)(x-d)\}=(d-c)^2-(b-a)^2$.

(9) Solve the simultaneous equations

$$\frac{1}{x}+\frac{2}{y}=8,$$
$$x+2y=xy.$$

(10) A drover bought 12 oxen and 20 sheep for $1,340; he afterwards bought 10 oxen and 26 sheep for an equal sum, paying $8 each more for the oxen and $3 each more for the sheep. What was the price per ox and what the price per sheep of the first lot?

## No. 37.

*Third Class, 1886.*

(1) Divide $\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}-2\right)^2$ by $\frac{a}{x}-\frac{x}{a}$.

(2) Simplify $\frac{1}{4(x-1)}-\frac{1}{4(x+1)}+\frac{1}{(x-1)^2\,(x+1)}$.

(3) Simplify $\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right)\div\left(\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right)$.

(4) Prove that $\frac{a+b}{ab}\left(\frac{1}{a}-\frac{1}{b}\right)-\frac{b+c}{bc}\left(\frac{1}{c}-\frac{1}{b}\right)-\frac{c-e}{ce}\left(\frac{1}{c}+\frac{1}{e}\right)$ is the difference of two squares.

(5) Resolve into linear factors

$$(a^2+bc+ca+ab)\,(b^2+ca+ab+bc)\,(c^2+ab+bc+ca).$$

(6) Resolve into three factors

$$(x+y)^2\,(x^2+z^2)-(x+z)^2\,(x^2+y^2).$$

(7) Show that there is only one value of $x$ that will make

$$x^3+6x^2c+8xc^2+10c^3$$

equal to the cube of $(x+2c)$, and find that value.

(8) Solve the equation $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}$.

(9) Solve the simultaneous equations

$$\frac{2x-y}{1}=\frac{2y-z}{2}=\frac{2z-u}{4}=\frac{2u-x}{8}=15.$$

(10) Find a number less than 100 the sum of whose digits is 12, and whose digits if reversed form a number which is greater by 6 than half of the original number.

## No. 38.

*Third Class, 1887.*

(1) Show that $(x^2+2xy+3y^2)^3+(y^2-2xy+3x^2)^3$ is divisible by $4x^2+4y^2$.

(2) Find the product of

$$\frac{\frac{1}{x}+\frac{1}{y+z}}{\frac{1}{x}-\frac{1}{y+z}} \text{ and } 1+\frac{y^2+z^2-x^2}{2yz}.$$

(3) Find the G. C. M. of

$2x^3-5x^2y+6xy^2-2y^3$ and $2x^3+3x^2y-6xy^2+2y^3$.

(4) Find the factors of $abc-ab-bc-ca+a+b+c-1$; $b(b-2a)-(c^2-a^2)$; and $512x^{27}+y^{18}$.

(5) If $x$ and $y$ are the G. C. M. and L. C. M. of $a$ and $b$, show that $xy=ab$.

(6) Simplify

$$\frac{x}{(x+y)(x+2y)}+\frac{2y}{(x+y)(x+3y)}+\frac{x}{(x+2y)(x+3y)}-\frac{1}{(x+3y)}$$

(7) Solve the equations :—

(i) $p(x-q)=q(x-p)$;

(ii) $\frac{3}{5}(5x-6)+\frac{7}{8}(3-2x)=\frac{4}{5}\left(\frac{x}{2}-3\frac{1}{2}\right)$.

(8) Solve the equations :—

(i) $x+y=b,\ ax+by=b^2$;

(ii) $\frac{1}{x}+\frac{1}{y}=a$; $\frac{1}{x}-\frac{1}{y}=b$.

(9) A person goes from Hamilton to Toronto by boat at the rate of 13 miles an hour, remains an hour and a half in Toronto and returns by rail at the rate of 26 miles an hour. He is gone altogether six hours; find the distance from Hamilton to Toronto.

(10) A number consists of two digits; if these digits be reversed the number thus formed is less than the first number by twice the greater digit; also, four times one digit exceeds three times the other by unity. Find the digits.

(11) A merchant goes into business with a certain capital which he finds has doubled itself by the end of the year. He then withdraws \$1,000 to pay expenses, and the remaining capital doubles itself during the second year; he then withdraws \$1,000 as before, and so on for four years. He finds that he begins his fifth year with \$5,000; how much had he to commence with?

(12) The sum of two numbers is one-fourth of their product, and if 6 be divided by the first number and 3 by the second the sum of the quotients is 1; find the numbers.

## No. 39.

*Third Class, July, 1888.*

(1) Simplify $(a+b+c)^4+(b+c-a)^4+(c+a-b)^4+(a+b-c)^4$.

(2) (i) A cellar, $a$ feet long, $b$ feet wide, and 10 feet deep, is excavated at $c$ cents per cubic yard; find the cost (in dollars) of doing the work.

(ii) A wall, $x$ feet long, $y$ feet high, and $z$ feet thick, contains $m$ bricks each $a$ inches long and $b$ inches wide; find the thickness of each brick.

(3) Show that $x$ is a factor of $\frac{1}{2}x(y+z)(y^2+z^2-x^2)$ $+\frac{1}{2}y(z+x)(z^2+x^2-y^2)+\frac{1}{2}z(x+y)(x^2+y^2-z^2)$. Find the other factors and reduce the expression to its simplest form.

(4) Simplify $\dfrac{2p+r}{p-q}-\dfrac{q(4p+3r)-r(p+r)}{p^2-q^2}+\dfrac{(p-q+r)^2}{p^2-q^2}-1$.

(5) Divide $a^4(b-c)+b^4(c-a)+c^4(a-b)$ by $(a-b)(b-c)(c-a)$.

(6) What value of $x$ will make $\frac{5}{8}\{5(5x-4)+3(3x-2)\}$ exceed $\frac{1}{6}\{4(4x-3)+2(2x-1)\}$ by $\frac{4}{7}\{4(6x-5)+3(7x-6)\}$?

(7) Multiply $\dfrac{a^2-(b+c)^2}{a^2-(b-c)^2}\div\dfrac{b^2-(c-a)^2}{b^2-(c+a)^2}$ by $\dfrac{c^2-(a-b)^2}{c^2-(a+b)^2}$.

(8) Find $x$ (i) when $\dfrac{x}{x+b-a}+\dfrac{b}{x+b-c}=1$;

(ii) when $2{\cdot}1x-3{\cdot}4={\cdot}02x+{\cdot}3x-2{\cdot}688$.

(9) Factor $x^3-19x-30$, $ab^2+bc^2+ca^2-a^2b-b^2c-c^2a$; $abc-x^3+ax(x-b)+bx(x-c)+cx(x-a)$.

(10) The difference between the squares of two consecutive numbers is 987; find the numbers.

(11) *A* and *B* had equal sums of money. *A* then spent \$55 and *B* \$58.50. It was then found that one had two-thirds as much as the other. How much had each at first?

(12) A speculator bought a house, expended \$800 in repairs, paid one year's taxes at the rate of 15 mills on the dollar (estimated on the buying price), and then sold it for \$5,346, making 10% on his total outlay. Find the buying price.

(13) Find two numbers whose sum is 503, such that four-fifths of the greater exceeds five-eighths of the less by seven times their difference.

(14) A number consists of two digits, one of which is the square of the other. If six times the greater digit be added to the number, the digits will be inverted. Find the number.

## No. 40.

*Third Class Teachers, 1889.*

(1) (i) Define the terms quantity, unit, number, negative quantity. How is quantity measured?

(ii) Distinguish between the arithmetical sum (or difference) and the algebraical sum (or difference) of two quantities.

(2) Factor

$$(c-x)(x^2+ab)+(a+x)(x^2-bc)+(b-x)(x^2+ca).$$

What values of $x$ will make this expression $=0$?

If $a$, $b$, $c$, $x$, are all positive quantities, under what conditions will the expression be negative?

(3) If two expressions have a common factor, prove that the sum or the difference of any multiples of these expressions will have that common factor.

Find the highest common factor of the expressions:—

$$x^2(3-2y)+x(3x^2-5y^2)-(2x+5y)y,$$
$$x^2(3+2y)+x(3x^2-5y^2)+(2x-5y)y.$$

(4) Add together the following:—

$$\frac{1}{a+x}\cdot\frac{1}{(a-b)(a-c)}+\frac{1}{b-x}\cdot\frac{1}{(b-c)(b-a)}+\frac{1}{c-x}\cdot\frac{1}{(c-a)(c-b)}.$$

(5) Find all the factors of $x^4+4y^4$;
$$2a^2-b^2+ab^2-a^2b-2a-ab+2b;$$
$$a^3b^3+b^3c^3+c^3a^3-3a^2b^2c^2.$$

(6) Reduce to its simplest form
$$s^2-(s-a)(s-b)-(s-b)(s-c)-(s-c)(s-a),$$
where $2s=a+b+c$.

(7) Solve the equations :—

(i) $\frac{x-4}{x-5}-\frac{x-5}{x-6}=\frac{x-7}{x-8}-\frac{x-8}{x-9}$;

(ii) $\frac{1}{16}(9x-7)-\frac{1}{8}(x-2\frac{7}{9})-\frac{7x}{19}+\frac{22}{171}=0$.

(8) For what value of $x$ will the sum of the following fractions be 3 :—
$$\frac{(x-a)^2}{(x-b)(x-c)},\quad \frac{(x-b)^2}{(x-c)(x-a)},\quad \frac{(x-c)^2}{(x-a)(x-b)}?$$

(9) A person who has \$30,000 invested receives from part of it an income of $4\frac{3}{4}\%$ per annum, and from the remainder an income of $5\frac{1}{4}\%$ per annum. His total income is \$1,490; how much has he invested at each rate per cent.?

(10) A piece of work is done in 4 days by three men, $A$, $B$, $C$, working together. $A$ would require 5 days longer than $C$ to do the whole work; and the work done by $A$ and $B$ together in a day exceeds that done by $C$ in a day, by one-twentieth of the whole work. What time would each require to do the work by himself?

(11) A whole number, greater than 800 and less than 900, is altered by removing the left-hand digit and putting it in the units' place. The new number is three-fourths of the original one. Find the number.

(12) The difference between the cubes of two consecutive odd numbers is 218; state the equation from which these numbers may be found and carry on the solution as far as you can.

## No. 41.

*Primary Examination, 1890.*

(1) Find the value (in the simplest form) of
$$m^3(c-n^2)+n^3(m-c^2)+c^3(m-n)^2+mnc(mnc-1)+7,$$
when $n-m^2=0$.

(2) (i) Find the remainder when $9a^{13}+4a^6-27a^5+1$ is divided by $a^8+2a^4+1$.

(ii) Divide, by Horner's method,
$5y^4+\frac{7}{2}ay^3-\frac{107}{12}a^2y^2+\frac{5}{6}a^3y+\frac{7}{6}a^4$ by $\frac{5}{2}y^2+3ay-\frac{7}{3}a^2$.

(3) If $x+a$ is a common factor of $x^2+px+1$ and $x^3+px^2+qx+1$, show that $(p-1)^2-q(p-1)+1=0$.

(4) Simplify

(i) $\dfrac{(x+y+z)(x^2+y^2+z^2)}{xyz}-\left(\dfrac{y+z}{x}+\dfrac{z+x}{y}+\dfrac{x+y}{z}\right)$.

(ii) $\dfrac{(x+y)(1-xy)}{(1-xy)^2-(x+y)^2}-\dfrac{x(1-y^2)+y(1-x^2)}{(1-x^2)(1-y^2)-4xy}$.

(5) Resolve in factors:

(i) $7x-42y-2x^2+9xy+18y^2$.

(ii) $a^6-3a^5+3a^4-a^3-8$.

(iii) $(ax+by)^2+(ay-bx)^2+c^2x^2+c^2y^2$.

(6) Solve the equations:

(i) $\dfrac{ax+b-c}{ax-b+c}=\dfrac{(b-c)^2}{(b+c)^2}$;

(ii) $\dfrac{x}{6\frac{1}{4}}+\dfrac{56-2x}{5\frac{1}{2}x+5\frac{1}{4}}=x-\dfrac{7x-2}{8\frac{1}{8}}$;

(iii) $(x-2a)^3+(x-2b)^3=2(x-a-b)^3$.

(7) In paying two bills, one of which exceeded the other by one-third of the smaller, the change out of \$5 was half the difference of the bills. Find the amount of each.

(8) A man leaving his property by will to his three sons, left one-third of it and \$1,000 to the eldest; one-half of the remainder and \$1,000 to the second; and the rest to the third who found his share to be \$2,500. What was the total value of the property?

(9) The digit in the tens' place of a certain number of two digits exceeds that in the units' place by four. When the number is divided by the sum of the digits the quotient is seven. Find the number.

(10) At what time between 8 and 8.30 are the hands of a clock equally distant from the figure VI?

(11) $A$ and $B$ together own $p$ dollars, $A$ and $C$ $q$ dollars, and $B$ and $C$ $r$ dollars. How much money does each own?

(12) A train starts on a journey of 240 miles; after going 103 miles it reduces its speed by one-fifth, and in consequence is $1\frac{1}{4}$ hours late at its destination. Find the ordinary speed of the train.

## No. 42.

*Primary Examination, 1891.*

(1) (i) Show that $6x^2+13xy+6y^2+12x+18y$ is divisible by $2x+3y$.

(ii) If the product of $a$ and $b$ equal
$$x^4-2x^3y-3x^2y^2+4xy^3+2y^4,$$
and if $a=x^2-2y^2$, find $b$.

(2) Factor (i) $x^2+5ax+bx+10ab-2b^2$.

(ii) $x^7-2x^5+x^4-a^4x^3+2xa^4-a^4$.

(3) Simplify

(i) $\dfrac{(a+b)^2-4ab}{a^2-b^2}$.

(ii) $\dfrac{x-y}{z^2-(x-y)^2}+\dfrac{y-z}{x^2-(y-z)^2}+\dfrac{z-x}{y^2-(z-x)^2}$.

(iii) $\dfrac{x^3}{(x-y)(x-z)}+\dfrac{y^3}{(y-x)(y-z)}+\dfrac{z^3}{(z-x)(z-y)}$.

(4) State a rule for finding the L. C. M. of two or more algebraic expressions. Apply your rule to the finding of the L. C. M. of $x^2-4a^2$, $x^3+2ax^2+4a^2x+8a^3$, and $x^3-2ax^2+4a^2x-8a^3$.

(5) Solve the following equations:—

(i) $\dfrac{3x+1}{x+1}=\dfrac{3bx-2a+c}{b(x+1)-a}$;

(ii) $\dfrac{(x+n)(x+q)}{x+n+q}=\dfrac{(x+c)(x+a)}{x+c+a}$;

(iii) $(a+x)(b+x)-a(b+c)=\dfrac{a^2c}{b}+x^2$.

(6) A cistern holding 820 gallons is filled in 20 minutes by 3 pipes which let in water at uniform rates, the first pipe admits 10 gallons more than the third and the second 5 less than the third every minute. How much water flows through each pipe per minute?

(7) $A$ and $B$ have the same income. $A$ contracts a debt each year amounting to one-seventh of his income; $B$ lives on four-fifths of his. At the end of 10 years $B$ lends $A$ enough to pay his debts and has \$160 to spare. Find the income of each.

(8) $A$ and $B$ play for a stake of \$12. If $A$ win he will have thrice as much money as $B$. If he lose he will have twice as much. What amount of money does each possess at first?

(9) A man buys $m$ horses for \$$p$ and sells $n$ of them at a profit of 5%. At what price must he sell the remainder that he may gain 10% on the whole?

(10) $A$ travels from $C$ to $D$ at the rate of 6 miles per hour; $B$ starts from $C$ two hours after $A$, and travelling 10 miles per hour reaches $D$ four hours before $A$. Find the distance from $C$ to $D$.

## No. 43.

*Primary Examination, 1892.*

(1) Multiply

$$1+x(1-2x)+x^2(1-2x)^2+x^3(1-2x)^3+x^4(1-2x)^4+\ldots$$

by $1-x+2x^2$, carrying the product to the term containing $x^4$.

(2) The dividend is $y^2 . y^{\frac{1}{3}}+2y^2-3y-2$, the quotient is $y . y^{\frac{1}{3}}-y^{\frac{1}{3}}-1$, and the remainder is $3y^{\frac{1}{3}}-1$. Find the divisor.

(3) What must be added to $(a+b+c)\,(ab+bc+ca)$ to make it evenly divisible by $a+b$?

(4) Put $4a^2b^2-(a^2+b^2-c^2)^2$ into four factors.

(5) Put into four factors

$$(x+2)\,(x+6)\,(x+4+\sqrt{6})\,(x+4-\sqrt{6})-15.$$

(6) Find the H. C. F. of

$$2x^4+x^3-3x^2-x+1 \text{ and } x^4-2x^3+x^2+2x-2.$$

(7) Simplify

$$\frac{(1+ab)\,(1+ac)}{(a-b)\,(a-c)}+\frac{(1+bc)\,(1+ba)}{(b-c)\,(b-a)}+\frac{(1+ca)\,(1+cb)}{(c-a)\,(c-b)}.$$

(8) Find $x$ when $(x-a)^2\,(1+ax)=(x+a)^2\,(1-ax)$; and prove that the value you get satisfies the equation.

(9) How much are eggs a dozen when a rise of 20% in their price makes a difference of 50 eggs in the number sold for \$5?

## No. 44.

*Primary Examination, 1893.*

(1) (i) Divide $4a^2 + 4a(n-1)d + (n-1)^2d^2$ by $2a + (n-1)d$.

(ii) Divide $1 - x^3 - y^3 - 3xy$ by $1 - x - y$.

(2) (i) Show that the difference of the squares of any two consecutive odd numbers is equal to twice their sum.

(ii) Prove that the cube of the sum of any two positive numbers is greater than the sum of the cubes of the numbers, by three times the sum of the numbers multiplied by their product.

(3) Solve the equations :

(i) $\frac{2x+3}{5} - \frac{3x+4}{6} = 12$.

(ii) $(x+7)^2 + (5-x)(x+5) = 36x$.

(4) (i) What is the price of bread per loaf, if an increase of 25% in the price would reduce the number of loaves that could be purchased for one dollar by two?

(ii) The breadth of a field is two-thirds of its length; if the breadth is increased by 100 yards, and the length diminished by the same amount, the new area is equal to the old. Find the length of the field.

(5) (i) Factor $x^6 - 64$; $x^4 + x^2y^2 + y^4$.

(ii) Show that $x+y$ is a factor of
$\{(1-m)x + py\}^3 + \{mx + (1-p)y\}^3$.

(iii) Factor $16a^2 + 4ab - 4ac - 12b^2 + 17bc - 6c^2$.

(6) Simplify (i) $\frac{(101)^4 - (99)^4}{(101)^2 + (99)^2}$.

(ii) $\frac{(p-a)}{(a-b)(a-c)} + \frac{(p-b)}{(b-c)(b-a)} + \frac{(p-c)}{(c-a)(c-b)}$.

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