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Q

GEOMETRICAL SOLUTIONS

OF THE

QUADRATURE

OF THE

CIRCLE.

BY

PETER FLEMING, CIVIL ENGINEER,

AUTHOR OF A SYSTEM OF LAND SURVEYING AND A METHOD OF MEASURING A BASE LINE BY ANGULAR OBSERVATION.

MONTREAL:

PRINTED FOR THE AUTHOR.

1850.

PREFACE.

THE title to the following pages, must indicate that the mind of the writer is actuated either by the most vain presumption or by the strongest conviction,—for that which is here now assumed to be discovered, remained through all preceding ages undiscovered.

The Problem of the Quadrature of the Circle passed under the view of Newton, Playfair and Leslie of Great Britain, and of Euler, Lagrange, Legendre and Laplace of the European Continent, whose discoveries and applications of Mathematical Science, have opened up, even to familiarity, the most profound laws of nature.—Nevertheless, they, and all others before them, have left without any finite solution, the ancient Problems—namely, the Quadrature of the Circle, the Trisection of the Angle, and the Duplication of the Cube.—Therefore, it must appear to be a great boldness in him who would even think of, and still more hope, to accomplish that, which those celebrated men, may be considered to have laid aside, or passed over as a hopeless undertaking.—Yet it may be supposed that the attempts if any made by them, to solve these ancient Problems were very limited, for the solutions of them if attained, would have led to no great or useful result; while time for this, that might have been uselessly spent, was occupied in the advancement of Mathematical Science, and in researches directed towards the more splendid discoveries regarding the Phenomena of the Physical world, and the laws of the universe.—Such have been those discoveries that the Laws of the stability and movements of the heavenly bodies are demonstrated to be only one, and that one universal, which alone regulates their motions and retains them in their orbits. Also, the mutual distances, masses and densities of them have been measured, their periods exactly determined, their anomalies accounted for, and all finally and rigorously demonstrated.

From the above view, it may appear that the modern advancements made in the Mathematical Sciences, may have rather retarded than advanced farther attempts to solve those ancient Problems, and which may be said to have been laid aside by those most proficient in these sciences; for the past had shown only ever failing attempts; while at the same time another and inexhaustible field of discovery and usefulness lay open before them,—hence it is not to be expected that the time, which could be applied to the latter, would be expended in seeking that, which the greatest of improbability made doubtful to be possible.

Legendre, in the fourth Note to his Geometry, has demonstrated that the ratio of the circumference and diameter of a Circle are irrational numbers; and all [illegible] attempts heretofore made by immense labour of calculation to find that relation in entire or rational numbers, avail nothing, more than obtaining a useful approximation by a decimal of six or seven figures. But that which may be incommensurable by numbers, it is well known may be commensurable in Geometry. Therefore the ancient Problems are still open to investigation by geometry; *for when the length of the circumference of a given Circle can be found or resolved into a straight line*, the quadrature of the Circle is accomplished. It is now the solution of this Problem which the writer presumes to lay before the public.

Whether this attempt succeeds or fails, according as it may be determined by the many eminent Geometricians of the present time—the author in treating it by a number of Lemmas and Propositions has afforded the opportunity for detecting any deficiency that may exist in the demonstrations. The figures, except the first, second and third, have all been drawn to the same scale—and instead of the Lines of the given Circle being taken from the construction of the fourth figure, as referred to in the text, they are taken for construction, from calculation of the tabular natural Sines and tangents of the angles, for each polygonal perimeter,—and for the convenience of those who may think of constructing the figures after the third, so as to avoid inaccuracy by Geometrical drawing, the numbers used for those figures, are here given :

The numerical lengths of perimeters of Polygons used in the construction of all the figures after the third.

NO. OF SIDES.	INC'D. POLYGON.	CIRC'D. POLYGON.
Radius of given Circle Fig. 4......	1,000	
3 — the half equal to Radius AC of all the figures after the third		5,196
4	5,657	8,000
8	6,133	6,627
16	6,243	6,365
32	6,273	6,303

PETER FLEMING.

Montreal, June, 1850.

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GEOMETRICAL SOLUTIONS

OF THE

QUADRATURE OF THE CIRCLE.

LEMMA 1.

With any radius CA, describe the semicircle ABDE, and make the distances AB and BD each equal to the radius AC, also describe from A through C the arc BC, and from B the arc ACD, and bisect the arc CD in the point D′, and join A and D. Then from any number of points a, b, c, d, e, f, g, h, of the arc BD make on the arc BC,—the arc Ba′ equal to the arc Ba,—the arc Bb′ equal to Bb, Bc′ equal to Bc, &c.—and from the point A as a center with the radius Aa, describe the arc am, andf rom C with the distance Ca′ describe the arc a′m, intersecting the arc a m in the point m. In the same manner from the centers A and C, through the points b and b′,—c and c′, &c.—describe the intersections n, o, p, q, r, s, t, &c.—and through the points B, m, n, o, &c., draw the line Bmnopqrst, &c.—which let be granted is drawn through infinity of points of intersection, and cutting the arc C D—It shall be a curved line passing through the point D′. FIG. 1.

For make the arc BI equal to the arc BB′; but the arc CD′ is by construction equal to the arc CB′, and BB′ is equal to CD′—hence the point D′, must be on the intersection of the arcs B′D′ and ID′—consequently the point D′ must be on the lne Bmnopqrst, &c., but the radii of each intersection are unequal, and, each intersection is of different radii—therefore the line Bmnopqrst, &c., must be a curved line, cutting CD in D′.

LEMMA 2.

From the point A as a center through B, describe the arc BCA″ and from D′ as a center describe through B, the arc IBA′A″ meeting the arc BCA″ in the point A″—and from A″ as a center describe the arc AD′, cutting the arc BA′ in the point C′, and the arc BC in the point B′, and join AI, passing through the intersections B and H by construction. Next make the arcs Ba and Ba′ equal to each other, and Bb′ equal to Bb,—Bc′ equal to Bc,—and Bd′ equal to Bd, &c. Also make D′a′ equal to D′a,—D′b′ equal to D′b,—D′c′ equal to D′c, and D′d′ equal to D′d, &c., and through the points a and a′,—b and b′,—c and c′, and d and d′, &c., describe from C and A as centers, the intersections m, n, o, p, &c. In the same manner from C′ and A″ as centers describe the intersections m′n′o′p′—it is evident, that a line drawn through the points D′, m′, n′, o′, p′, &c., and B, m, n, o, p, &c., must meet in a common point G, on the line A B′ H—for the arcs B H and B B′ are symmetrical with the arcs D′ H and D′ B′ to the straight line A B′ H. The points B, m, n, o, p, ... G, and the points D′, m′, n′, o′, p′, ... G shall be on the arc of a circle. FIG. 2.

For let G be the point of meeting of the curve lines Bmnop and D′m′n′o′p′, &c., on the straight line A′B′H and take any point K* on the line GH, at less distance from C than the half of GH, or near to G—and through the points B, K, and D′, describe the arc BKD′,—also make the distance GL, equal to GK, and through the points B, L, and D′, describe the arc BLD′,—Next from the center A, with the distance AG, describe the arc Gr, meeting the arc BH in the point r and cutting the arc BK in the point g′,—also from C, describe the arc Gn, and from A through any point of the arc Br, describe the arc pn, cutting the arc BK in the point n′—Again make Bm equal to Bp, and from C as a center, describe the arc mn, cutting the arc BL in the point n″, and intersecting the arc pn in the point n—also from C as a center through the points g′ and n′, describe the arcs g′x and n′s, and from A as a center through the points g and n″, describe the arcs gq and n″o. *FIG. 3.

Now let KG and LG, each be bisected, and through the points of each bisection and the points B and D′, describe circular arcs—the one arc must be between the arc BGD′ and BKD′ and the other arc between the arcs BGD′ and BLD′, hence it is evident that the first arc must intersect the arc rG between the points g′ and G, and the other arc must intersect the arc nG, between the points g and G. In the same manner describe other arcs *in infinitum*, through each point of bisection of the distance of the last bisection from G on LG and KG—and the points B and D′—it is evident that the ultimate arc must pass through the point G, and be the arc BGD′—also that the arcs g′G and gG, must continually be diminished and equally vanish or become nothing—but the arcs n′n and n″n also must equally vanish, when the arcs g′G and gG vanishes—for when the arc xg′ coincides with the arc nG, and qg, coincides with rG, the arc sn′ must coincide with the arc mn, and the arc on″ must coincide with the arc pn—and the point n must be on the ultimate arc BGD′—but by construction Bp is equal to Bm—consequently the points BnGn′D′, are on the arc of a circle—and every point of intersection described between the point B and G, and D′ and G through points in the same manner equally distant from B and D′, must be on an arc of the same circle.

Or let the point G move along the circular arc BG, then the points n and r would equally meet in the point B, for by construction Bn and Br are equal to one another, and the arcs rG and nG, Br and Bn would equally vanish—consequently their ultimate ratios are equal*—therefore the arc Bm must be equal to the arc Bp—and the intersection n must be on the circular arc BGD′.

LEMMA 3.

FIG. 5. Let AC be equal to one half of the perimeter of the circumscribing triangle abd of the given circle ABD, Fig. 4, and from the point C as a center, describe the semicircle ABDE and make the chords AB and BD, each equal to the radius AC—next make the chord or distance A1, equal to the perimeter of the inscribed square efgh,—the distance A2, equal to the perimeter of the inscribed polygon of eight sides—the distance A3, equal to the perimeter of the polygon of sixteen sides, &c., of the circle ABC, Fig. 4,—also make A1′, equal to the perimeter of the circumscribing square e′f′g′h′, and A2′ equal to the perimeter of the polygon of eight sides—A3′ equal to the perimeter of the polygon of thirty-two sides, &c., hence the points 1, 2, 3, are the first three of the infinite series of the inscribed-polygonal perimeters A1, A2, A3, &c., and the points 1′, 2′, 3′, 4′, are the first four of the infinite series of the circumscribing polygonal perimeters of the given circle ABD Fig. 4,—and let it be granted the distance An, is equal to the perimeter of the polygon of an infinite number of sides, or is equal to the circumference of the given circle ABD, Fig. 4.

Again from the point B as a center, with the distances B1′, B2′, B3′, &c., and from the point D, as a center, with the distance D1, D2, D3, &c., describe the intersections l, a, b, c, &c., and through the points l, a, b, c, &c., draw the line labc .. n,—by construction the points l, a, b, c, .. n, are on a curved line concave to the arc nD, and meeting the arc BD in the point n. Also from the point B as a center, with the distances B2′, B3′, B4′, &c.—and from D as center with the disances D1, D2, D3, &c., describe the intersections a′, b′, c′, &c., and draw through the points a′, b′, c′, ... n the line a′b′c′ .... n—by construction the points a′, b′, d′, ... n are on a curved line, concave to the arc nB and shall meet the arc BD in the point n.

For as the distances of the points 1, 2, 3, &c., and 2′, 3′, 4′, &c., from A, of each series approach nearer to the distance An, the nearer must the intersections approach on the curve line a′b′c′, &c., to the point n—hence the ultimate intersection must become infinitely near to, or coincide with the point n—consequently the line through the intersections a′, b′, c′, &c., must meet the arc BD in the point n.

LEMMA 4.

FIG. 5. Let it be granted that the curve lines abc ... n and a′b′c′ ... n, are drawn through an infinite number of intersections, and suppose the chords an and a′n drawn—and that the points 1 and 2, are variable ;—and that the point 1 moves or approaches towards

* Newton's Mathematical Principles of Natural Philosophy, Book 1, Section 1.

the point 2, and the point 2 approaches towards the point 3, in the ratio of the arcs 12 and 23, or that the point 1 would arrive at the point 2, equally that the point 2 would arrive at the point 3, and by construction the varying intersection a′, would arrive at the point a, at the same time that the varying intersection b′ would arrive at the point b. There shall be a point of variation between the points 1 and 2, on the arc 12, and a point of variation between the points 2 and 3, through which by intersections described from B and D, the curve line a′b′c′ ... n shall be varied and shall coincide with its chord line a′n.

For let the point 1 arrive at or coincide with the point 2, and the point 2 arrive at or coincide with the point 3—it is evident that the intersections a′, b′, c,′ ... n, must also coincide with the intersections a, b, ... n, for the intersection a′ must have moved along the arc a′a2′, to the intersection a, and in the same time the intersection b′ must have moved along the arc b′b3′ to the point b—hence the curve a′b′ ... n and its chord a′n, must coincide with the curve line ab ... n and its chord an; but by this variation the point 1, must have passed over a point between 1 and 2, and the point 2, must have passed over a point between 2 and 3, through which were intersections described in the same manner from B and D, by this variation of the points a′ and b′ these intersections, would be on a straight line passing through the point n,—for the curve line a′b′ ... n must have changed its convexity to the opposite side of its chord a′n—before it could coincide with the curve line ab ... n—or that its chord a′n could coincide with the chord an—therefore on this point of change the varied points a′ and b′ and the point n must be one straight line.

In the same manner it is demonstrated, by supposing the point 2 to vary towards the point 1, and the point 3 to vary towards the point 2,—that the points through which the intersections would make the points a and b vary, till a and b would come to be on a straight line with the point n, must be the same points of variation, as would be by varying 1 towards 2, and 2 towards 3.

In the same manner it is demonstrated, that by making only the points 2′ and 3′ vary, that is 2′ towards the point 3,′ and 3′ towards the point 4′—it is evident that before the intersections a and b, could coincide with the intersections a′ and b′—there must be points between 2′ and 3′ and 3′ and 4′, through which the intersections described from the points B and D, will bring a and b to be on a straight line passing through the point n; for the intersection a must move along the arc ab′2, and the intersection b must move along the arc bc′3, and when the curve line ab ... n coincides with b′c′ ... n, the curve line ab .. n must have changed its concavity to the opposite side of its chord an, and consequently must have passed points of intersection on a straight line with the point n.

LEMMA 5.

From the point A as a center, describe through the points 1, 2, 3, &c., of the arc BD, the arcs 1T, 2S, and 3R, meeting the arc CD in the points T, S and R. Also through the points 2′, 3′ and 4′, describe the arcs 2′K, 3′O, and 4′Q, meeting the arc CD in the points K, O and Q. Next on the arc BC make the arc B1″, equal to the arc B1,—B2″ equal to the arc B2,—B3″ equal to the arc B3. Also make the arc B2‴ equal to B2′—B3‴ equal to B3′ and B4‴ equal to B4′—then from C as a center through the points 1″, 2″ and 3″ describe the arcs 1″T′, 2″S′ and 3″R′, meeting the arc CD in the points T′, S′ and R′.—Also describe the arcs 2‴K′, 3‴O′, and 4‴Q′ through the points 2‴, 3‴ and 4‴.—Again with the radius AC, through the points B and K, and B and K′ describe the arcs BK and BK′—the arc BK intersecting the series of arcs 1K in the points d, s, r, ... q, o, and K, and the arc BK′, intersecting the series of arcs 1″K′, in the points t, u, v ... q′, o′ and K′—then from the point B, as a center, with the distances BK, Bo, and Bq, and from the point K as a center with the distances Kd, Ks and Kr, describe the intersections V, p and q—also from B, with the distances K′t, K′u, and K′v, describe the intersections V′, p′ and q,′ and draw through Vp and q and through V′p′ and q′—the curved lines Vpq and V′p′q′—which by construction are concave to their chords Vn′ and Vn″, on the opposite side to that of the curved line a′b′c′ ... n. Therefore there shall be an arc of the radius AC, to be described through B, that upon intercepted part of which arc in the same manner, the intersections V, p, q, &c., may be again described from B and the point of the arc meeting the arc CD—so that these intersections will be upon a straight line with the ultimate intersection n′ ; which arc shall meet the arc CD, between the points K and D.—Also there shall be another arc of the radius A through B, upon which the intersections in the same manner described, shall be in a straight line with the ultimate intersection n″—which arc must meet the arc CD, between the points K′ and C.

For the points 1, 2, 3 ... 4′3′2, have varied by construction to be the points d, s, r, ... q, o, K on the arc B K, and the intersection, a′b′c′ ... n to be the intersections V, p, q, ... n′, but in this variation the curve a′b′c ... n, has passed over to the opposite side of its chord and become the curve Vpq ... n′, and therfore (Lem. 4.) there must be an arc of the radius AC through B, between the arcs BK and BD, upon which the chord and curve must be on a straight line—also by the same, there must be another arc between the arc BK′ and BC, on which the curve and its chord must be on one straight line.

Note.—The side of the chord on which may be the convexity or concavity, of a curve line drawn through intersections as above described, must evidently depend upon the law of the differences or distances of the points, from the point of ultimate intersec-n, and also the distance of the point n from the centers B and D, and as these may be varied, the chord line may be on the one side or the other of its curve, or the concavity of the curve changed to the opposite sides. For the points of variation upon which the chord and curve would be on one straight line, are unknown, and therefore the side of the curve on which the chord is, can only be known by construction.

LEMMA 6.

FIG. 6. Draw through the points of intersections B, d, e, f ... h, i, k, D′, the curved line Bdef ... hikD′, and from the point B as a center with the distances Bk, Bi and Bh, &c.,—and from the point D′ as a center, with the distances D′d, D′e, and D′f, describe the intersections a, b, and c &c., and through a, b, c, &c., draw the line abc ... n, and let it be granted that the curve line abc ... n is through an infinite number of points, meeting the curve line Bdef ... hikD′ in the point n, and by construction is concave to the arc nB, and consequently the chord an must be on the opposite side of the curve line abc ... n, to what that of the chords Vn′ and Vn″ (Fig. 5,) is to their curve lines Vpq ... n′ and V′p′q′ ... n″ ; There shall be an arc or curve line similar and equal to the curve line Bdef ... hikD′ through B, between the curve Bdef ... hikD′ and the arc or curve line Bd′sr ... qoK (Fig. 5,) so that on its intercepted arc between the arcs 2′K and 1′T, the curve line Vpq ... n′, and its chord Vn′, will be on the same straight line,—also there shall be an arc or curve line similar and equal to the curve line Bdef ... hikD′, through B, between the arc or curve line Btuv ... q′o′K′, (Fig. 5,) on its intercepted arc, between the arcs 2K′ and 1T′, so that the curve line V′p′q′ ... n,″ and its chord V′n″ will come to be on the same straight line.

For it is evident, by the construction that the curve line Bdef ... hikD′ is the only common arc which can be described intersecting the series of arcs 1K and 1″K′, therefore there cannot be any common intercepted arc but dk—but there must be two intercepted arcs (Lem. 5.) on which the curves Vpq ... n′ and V′p′q′ ... n″ would coincide with their chords—therefore the one must be on an arc through B, between Bdef ... hikD′ and Bd′sr ... qoK, and the other through B, must be on an arc between the arc Btuv ... q′o K′ and the curve Bdef ... hikD′.

LEMMA 7.

FIG. 6 From the point B as a center, with the distances Bk, Bi, Bh, and from the point D′ as a center, with the distances D′e, D′f, D′g, describe the intersections a′, b′, and c′,* and draw the line a′b′c′ ... n′ and join an′ (Lem. 4)—but the line a′b′c′ ... n is a curve line by construction, and concave to the arc n′D′, and therefore upon the opposite side of its chord n′n, to that of the line abc ... n to its chord an.—There shall be points on

* The intersection c′ cannot be introduced on the figure, without describing it on a much larger scale.

the arc de, and *ef*, through which, if arcs be described in the same manner, from the points B and D′ as centers, and intersecting the arcs ka, and *i*b, that the intersections a, b, and c, or the curve line abc ... n, shall be on the chord an.

For let the point d move on the arc de towards the point *e*, and the point *e* move towards *f*, in like manner the point of intersection a will move towards a′, and the intersection b will move towards b′—and let the point d arrive at the point *e*, in the same time the point *e* arrives at the point *f*—the point a, must equally arrive at the point a′ and the arcs da and *e*b, coincide with the arcs *e*a′, and *f*b′; but the points b and c have changed sides to the chord an, for the curve line abc ... n, will now coincide with the curve line a′b′c′ ... n—consequently there must be points of variation passed on the arcs d*e*, and *ef*, through which if intersections be described from the centers D′ and B, that the curve line abc ... n will become a straight line, or that the intersections a, b, c, will be on a straight line with the point n, and coincide with the chord an.

LEMMA 8.

From the center C of the semicircle ABDE, describe through the point of intersection *s*, the arc *ss*′, meeting the arc 1T, in the point *s*′, and from the same center describe through the intersection r, the arc rr′, meeting the arc 2S, in the point r′—and from K as a center with the distances Ks′ and Kr′, describe the arcs s′s″ and r′r″.—The point s″, shall be the greatest variation of the point t′ towards the point s, on the intercepted arc 1″K of the arc BK,—and the point r″, shall be the greatest variation of the point s towards the point r, and the arcs ts″ and sr″, shall be proportional to the arcs 12, and 23.* FIG. 7.

For by construction, the arc 1 2 3 ... 4′3′2′ on the arc BD, is the least intercepted arc by the series of arcs 1K, and the arc 1′K is the greatest of all the intercepted arcs of the radius AC, which can be described through B, between the arcs BD and BC, meeting the arc CD,—therefore the arcs t′s″, and sr″, are the greatest variations of the points t and r, because the greatest intercepted arc is t′K.

From the point A as a center, through the point of intersection t of the arcs 1″T′ and BK′, describe the arc tu′ meeting the arc 2″S′ in the point u′ and from the same center through the point of intersection u, describe the arc uv′—and from K′ as a center describe through the points u′ and V′—the arcs u′u″, and v′v″.—The arcs tu″ and uv″, must be the greatest variations of the point t towards u, and of u towards v, because the greatest intercepted arc is tK′

Now let the arc BK, move on the point B as a center, until it coincides with the arc BD, it is evident that the variable curvilinear triangles ts′s, and sr′r, must vanish or at the same time or become nothing; because the arcs s′s, and r′r, must always remain concentric to the arc BD,—and therefore the variations t′s″ and sr″, equally vanish, when the arcs s′s and r′r coincide with the arcs 12 and 23, on which arcs, the variations come to be nothing. In the same manner when the arc BK′ comes to coincide with the arc BC, the variations tu″ and uv″ vanish; for the arcs tu′ and uv′, are each concentric to the arc BC; consequently the ultimate ratios of the variations on the arcs BK and BK′ are equal; and the variation ts″ is to the variation sr″, as the variation tu″, is to the variation uv″; also the ultimate ratios, of ts and sr, are equal, and therefore t′s is to sr as the arc 12 is to the arc 23; but the ultimate ratios of t′s″ and sr″, and the ultimate ratios of t′s and sr are equal; for when the arcs ts and sr coincide with the arcs 12 and 23, the variations t′s″ and sr″ vanish at the same time; therefore the variation t′s″ is to the variation sr″, as the arc 12 is to the arc 23.

LEMMA 9

Bisect the arc CD, in the point D′, and through the points of intersection (Lem. 1,) B, d, e, f, ... h, i, k, D′, draw the curve line Bdef ... hikD′, and let the lines Btuv .. q′o′K′, and Bt′sr ... qoK, be each described similar and equal to the line Bdef ... hikD′—and bisect the distance BD′, by the straight line AH, intersecting the curve line in the point G.—Next from the point D′, with the distance D′*e*′ and D′*f*″, describe the arcs *ee*″ and *ff*″; The point *e*″ shall be the variation of the point d towards the point *e*, and the point *f*″, shall be the variation of the point *e* towards the point *f*, on the intercepted arc dk, which is common to the series of arcs 1K and 1″K—t′s″ shall be to sr″, as tu″ is to uv″, and as de″ is to ef″. FIG. 7.

* This is, supposing that the arcs BD, BK′, BK and BC, are similar and equal to each other, or of the same radius AC.

For let the curve lines BK′ and BK be supposed each to move on the point B as a center equally towards each other, they must meet on, and coincide with the curve line BGD′—because the arcs KD′ and K′D′, are equal by construction, and the curvilinear triangles ts′s and sr′r, and the curvilinear triangles tu′u and uv′v, will coincide with the curvilinear triangles *de′e* and *ef′f*—also the arcs of variation t′s″ and tu″, will each coincide with the arc de″—and the arcs sr″and uv″ each coincide with the arc *ef″*—hence their ultimate ratios are equal (Lem. 8) ; consequently the variation t′s″ is to the variation sr″,—as the variation tu″ is to the variation uv″,—as the variation de″ is to the variation *ef″*; therefore de″ and *ef″* must be the arcs of variation, on the common curve line, or arc dk, of the point d towards *e*, and of the point *e* towards *f*.

LEMMA 10.

FIG. 7. From the point B as a center, with the distances Bk′ and B*i*′, describe the arcs k′k″ and *i*′*i*″,—the arc kk″, shall be the variation of k towards *i*, and the arc *ii*″ shall be the variation of *i* towards h.

For from the point A, with the distances Ao′ and Aq′, describe the arcs o*x*′ and qq′; and from the point B as a center, with the distances Bq′ and B*x*′, describe the arcs q′q″ and *x*′*x*″ ; Also from the point C as a center, with the distances CK and Co, describe the arcs Ky′ and op,′ and from the point B, with the distances Bp′ and By′, describe the arcs p′p″ and y′y″.

Now let the curve line BK and BK′, each move upon the point B as a center, towards the curve line BD′, they must each coincide with BD′, (Lem. 9,)—hence the triangles qq′o′ and o′*x*′K′ and the triangles qp′o, and oy′K, will coincide, and be similar and equal to the triangles h*i*′*i*, and *i*k′k ; consequently their ultimate ratios are equal, and the variation Ky″, is to the variation op″, as the variation K′x″, is to the variation o′q″, as the variation kk″, is to the variation *ii*″, (Lem. 9,)—therefore kk″ must be the variation of the point k towards *i*, and *ii*″, the variation of the point *i* towards h.

THE SOLUTIONS.

Case First.—*When the arc Bdef ... hikD′, is a curve line, through intersections described through points of equal distances on the arcs BD and BC*, (*Lem.* 1, *Fig.* 1.)

SOLUTION FIRST.

PROPOSITION 1.

THEOREM.

FIG. 7. From the point B with the distances Bk and B*i*, and from the point D′, with the distances D′*e*″ and D′*f*″, describe the intersections a and b, and through the points a and b, draw the straight line abn meeting the curve line BGD′ in the point n.—The straight line abn, shall be that upon which, by variation of the points d and *e*, to the points *e*″ and *f*″ the curve line abc ... n (fig. 6), shall coincide with its chord an.

For as the curves V′p′q ... n″, and Vpq ... n′, are each upon the opposite side of their chords, to the curve a′b′c′ ... n on the arc BD (fig. 5)—it is evident that intersections described through the points of variation u″ and v″, and s″ and r″, (fig. 7,) must give the curve lines on the same side of their chord as Vpq ... n′ and Vpq ... n″—and therefore be also on the opposite side to the curve line a′b′c′ ... n (fig. 5)—because the variations tu″ and uv″ are proportional to tu and uv—and ts″ and sr″ are proportional to ts and , and therefore as the curve line described by intersections through the points u″ and v″ shall vary, the arc BK′ will move towards BC on the center B, and that described through the points s″ and r″, the arc BK, must move towards BD on the center B, (Lem. 10)—so that the curves of intersections described through the points of variations u″ and v″, and s″ and r″, will come to be, each in one straight line with their chords; Now it is evident from their ultimate ratios being equal, that the distances K′u″ will become equal to Ks″, and Kv″ equal to Kr″, and *BK*″ equal to *BK*, and *Bo* equal to *Bo*′; but this can only be possible on the curve line BD′, on which is the intercepted arc dk common to both of the series of arcs 1K and 1″K′, and upon dk are the common variations de″ and *ef*″, (Lem. 9,) ; consequently it is through the points *e*″ and *f*″ and h and *i*, in this case, that the intersections described from the points B and D′, forming the curve line ab ... n will only coincide with the chord an.

PROPOSITION 2.

THEOREM.

Join the points A and n ; the distance An shall be equal to the perimeter of the FIG. 7. polygon of an infinite number of sides, or equal to the circumference of the given circle ABD, (fig. 4.)

For with the distance An, from A as a center describe through the point n, the arc *n*nN, meeting the arc BD in the point *n*′nN′, and CD in the point N. Also from the point C, as a center, with the distance Cn, describe through n, the arc *n*′nN′ meeting the arc BC in the point *n*′, and CD in the point N′ ; because the curve line Bdef, ... hikD′ is through intersections of equal arcs on the arcs BD and BC (Lem. 1,) the arc B*n*′, must be equal to the arc B*n* ; but the point n is the point of ultimate intersection by construction on the curve line Bdef ... hikD′ which is through intersections of the infinite series of arcs through the points 1, 2, 3, ... n, and through the infinite series of points 2′, 3′, 4′, ... n ; hence the point n, must be on the arc nN (fig. 5.) Also the point n, is the point of ultimate intersection of the infinite series of points 1″, 2″, 3″ ... n′ and of the infinite series of points 2′, 3′, 4′ ... n′, each intersecting the curve line Bdef ... hikD′ ; hence the point n must be on the arc n′N′, (fig. 5), and consequently the distance An, must be equal to A*n* (fig. 5), and equal to the circumference of the circle ABD, (fig. 4) ; but the part BG of the curve line Bdef ... hikD′ is the arc of a circle (Lem. 2, fig. 3,) having its center on the right line ABG, in the point L, hence the point n is on the arc of a circle ; and the distance An is the determinate length of the circumference of the circle ABD (fig. 4).

SOLUTION SECOND.

PROPOSITION 3.

THEOREM.

From the point D′ with the distances D′*e* and D′*f*, and from B, with the distances FIG. 7. Bk″ and B*i*″, describe the intersections a′ and b′, and through the points a′ and b′, draw the straight line a′b′n, meeting the circular arc Bd*ef* ... h*i*G, (Lem. 2), in the point n. The point n shall be on the intersections of the arcs nN, n′N, BG and the right line abn,—and the distance An, shall be equal to the length of the circumference of the given circle ABD, (fig. 4.)

For the point k″, is the variation of k towards *i*, and the point *i*″ the variation of *i* towards h, upon the common intercepted arc dk, (Lem. 10) ; and in the same manner as demonstrated, (prop. 1), that the curve of intersections, described through the points y″ and p″ and r″ and q″ may be in one straight line with their chords, the arc BK must move towards BD, and BK′ must move towards BC, (Lem. 10) ; then it is evident that the distance B*x*″ must come to be equal to By″, and Bq″ must be equal to Bp--and K′u must be equal to K′s, and K′v must be equal to Kr, for their ultimate ratios are equal ; but this can only be possible on the common arc dk---and therefore, (prop. 1), it is only through the points of variation k″ and i″, and the points *e* and *f*, that the curve line through the intersections n′b′ ... n, in this case, will come to be on the same straight line with its chord a′n ; but the point n is by construction the ultimate intersection of the line n′b′ ... n on the curve line Bd*ef* ... h*i*kD′ of the intersections described from B and D′, through the infinite series of points *e*, *f* ... n, and of k″, *i*″ ... *n* on the common arc through the intersections, of the series of arcs 1K, and 1″K ; hence the point n, must be on the arc *n*N, and also on the arc *n*′N′, and consequently on the intersection of *n*N and *n*′N′ ; but the part Bd*ef* ... hiG, of the curve line Bd*ef* ... h*i*kD′ is a circular arc, (Lem- 2.) Therefore the distance An, must be the determinate length of the circumference of the circle ABD, (Fig. 4.)

SOLUTION THIRD.

PROPOSITION 4.

THEOREM.

Let the straight line abn, be drawn through the points of intersection a and b, FIG. 7. and n′b′n′ drawn through the points a′ and b′, (prop. 3), intersecting each other in the point n. The distance An, shall be the determinate length of the circumference of the circle ABD, (Fig. 4.)

For the point n is the ultimate intersection of the intersections of the lines ab ... n and a′b′ ... n meeting upon the curve line Bdef ... hikD′, (prop. 2 and 3.) Now from the points A, describe through n the arc *an*, meeting the arc BD in the point *n*—it is evident that the distance An is equal to the distance A*n*, (Fig. 5.) Therefore the distance An will be equal to the determinate length of the circumference of the circle ABD, (Fig. 4.)

The intersection of the straight lines ab and a′b′ in the point D, is independent of the nature of the curve line or arc BD′—for let BD′, be any curve line described between the circular arc BD′, and the curve line Bdef ... hikD, and describe the arcs BK′ and BK, similar and equal to such curve line, it is evident that when the curve lines BK′ and BK, come each to coincide with the similar and equal curve line BD′—the variations de″ and *ef*″—kk″ and *ii*″, must be the variations of the intercepted arc dk of the curve line BD′—and as the intersection n, of ab and ab′, must be the ultimate intersection on d*k*, the point n consequently must be upon the intersection of the arc *nN*, (Fig. 5), and the curve line BD′; therefore this solution must be independent of the nature of the curve line BD′.

Case Second.—*When the arc Bdef ... hikD′, is the arc of a circle of the radius AC, described, through the points B and D′—intersecting either of the series of arcs 1K or 1″K only*, (*Lem.* 3, *Fig.* 5.)

SOLUTION FOURTH.

PROPOSITION 5.

THEOREM.

FIG. 8. Bisect the arc CD in the point D′, and with the radius AC, describe through the points B and D′ the arc BD′, intersecting the arcs 1T, 2S, 3R, 4′Q, 3′O and 2′K, in the points d*ef* ... h*ik*. Then from the points C, with the distance C*e* and C*f* describe the arc *ee*′ meeting the arc 1T, in the point *e*′ and the arc 2S, in the point *f*′. Then from C as a center, with the distances Ck, Ci, Ch, &c., describe the arcs 2′K′, 3′O′, and 4′Q′, &c.,—and from the same center with the distances Cd, C*e*, and C*f*, &c., describe the arcs 1″T′, 2″S′, and 3″R′, &c.—Then through the points B and K′, and B and K, describe the arcs BK′ and BK, meeting the arc CD in the points K′ and K; also from the point D′ as a center with the distances D′*e*′, and D′*f*, describe the arcs *e*′*e*″ and *ff*; ″the point *e*″ shall be the variation of d towards *e*″, and *f*″ the variation of *e* towards *f*″.

For in the same manner we have by (Fig. 7,)—The arc tu″ is the variation of t towards n, and nv″ is the variation of n towards v on the arc BK′—and t′s″ is the variation of t′ towards s, and sr″ is the variation of s towards r, on the arc BK. Now let the arc BK′, move on the center B towards the arc BC, till it coincides with the arc BD′—the variation tu″ must coincide with the arc of variation d*e*″, and also the arc of variation nv″ must coincide with the arc of variation *ef*″; also let the arc BK, move on the center B towards the arc BD till it coincides with the arc BD′; the variation t′s″ must coincide with d*e*″, and sr″ must coincide with *ef*″; hence the arcs de″ and *ef*″, are the variations on the common arc of intersections dk of the arc BD′.

PROPOSITION 6.

THEOREM.

FIG. 8. From the point D′, with the distances D′*e*″ and D′*f*″, and from the point B with the distances Bk, and B*i*, describe the intersections a and b; and through the points a and b, draw the straight line abn, meeting the circular arc BD′ in the point n, and join A and n; the distance An, shall be the determinate length of the circumference of the circle ABD, (Fig. 4.)

For it has been demonstrated, (prop. 1,) that the point n, must be the point of ultimate intersection of the line ab, &c., or of all the intersections described through the series of points of variation *e*″ and *f*″, &c., and through the points k and i, &c.—through which the curve line ab .. n, will coincide with its chord an, (Lem. 10.) Therefore the point n must also be on the intersection of the arc *n*N, (Lem. 3.), and the circular arc BD′; and An must be equal to A*n*, and equal to the circumference of the circle ABD, (Fig. 4.)

SOLUTION FIFTH.

PROPOSITION 7.

THEOREM.

In the same manner as demonstrated, (Lem. 10,) the arc Kk″ is the variation of the point K′ towards o′, and oq″ is the variation of o′ towards q′, and Ky″ is the variation of K towards o, and op″ is the variation of o towards q, and that K′x″ and K′y″, will each become the variation kk″—that o′q″ and op″ will become the variation *ii*″ (Lem. 10.)—Then from the point B as a center, with the distances Bk″ and Bi″, and from D′ as a center, with the distances D*e*′ and D′*f*, describe the intersections a′ and b′, and through the points a′ and b′, draw the straight line a′b′n meeting the circular arc BD′ in the point n,—the distance An shall be equal to the length of the circumference of the circle ABD, (Fig. 4.) FIG. 8.

For it is evident, (Lem. 3), that n is the ultimate intersection of all the intersections described through the series of points k″ and *i*″ &c. and through the series of points *e* and *f* &c., and that the curve line a′b′ ... n must be on the same straight line with its chord a′n. (prop. 1) ; consequently the point n must be on the intersection of the arcs *n*N and BD′—and the distance An equal to A*n* (Fig. 5.)—and equal to the circumference of the circle ABD, (Fig. 4.)

SOLUTION SIXTH.

PROPOSITION 8.

THEOREM.

The straight line abn drawn through the points of intersections a and b, and the straight line a′b′n drawn through the points a′ and b′, intersecting each other in the point n—The distance An shall be the determinate length of the circumference of the Circle ABD, Fig. 4. FIG. 8.

The demonstration for this Solution is the same as that of Solution Third.

Case Third.—*When the arc Bdef ... hikD′, is the arc of a circle of the radius AC, described through the points B and D′, intersecting the series of arc 1″K′ only,* (*Lem. Fig.* 3, 5.)

SOLUTION SEVENTH.

PROPOSITION 9.

THEOREM.

Make the arc B1″, B2″, B3″, &c., and B2′, B3′, B4′, &c., on the arc BC, equal to the arcs B1, B2, B3, &c., and B2′, B3′, B4′, &c., of the arc BD, of Fig. 5—and from C as a center with the distances C1″, C2″, C3″, &c., and C2′, C3′, B4′, &c., describe the arcs 1″T′, 2″S′, 3″R′, &c., and 2′K′, 3′O′, 4′Q′, meeting the arc CD in the points T′, S′, R′, &c., and K′, O′, Q′. Then bisect the arc CD in the point D′, and with AC as a radius, through the points B and D′ describe the arc BD′, intersecting the arc 1″T′ in the point d—the arc 2″S′ in the point e—the arc 3″R′ in the point *f*, &c.—and the arc 2′K′ in the point k—the arc 3′O′ in the point *i*—and the arc 4′Q′ in the point h.—Next from the point A as a center, with the distances Ad, A*e*, A*f*, &c., and Ak, A*i*, and Ah, &c,, describe the arcs 1T, 2S, 3R, and the arcs 2′K, 3′O, and 4′Q, meeting the arc BD in the points 1, 2, 3, &c., and 2′, 3′, 4′, &c., and meeting the arc CD in the points T, S, R, &c., and K, O, and Q, &c. ; also through the points B and K′, and B and K, with AC for a radius, describe the arcs BK′, and BK.—Again from A as a center with the distances At, Au, describe the arcs tu′ and uv′—and from K′ as a center with the distances K′u′ and K′v′ describe the arcs u′u″ and v′v″ ; also from C as a center with the distances Cs and Cr describe the arcs ss′ and rr′, and from K as a center with the distances Kr′ and Ks′, describe the arcs r′r″ and s′s″. The arc tu″ is the variation of t towards u, and the arc uv″ is the variation of u towards v on the arc tK′ ; also the arc t′s″ is the variation of t′ towards s, and the arc sr″ is the variation of s towards r, on the arc t′K—and from D′ as a center with the distances D′*e*″ and D′*f*″ describe the arcs *e*′*e*″ and *f*′*f*″—the arc d*e*″ is the variation of d towards *e*, and *ef*″ is the variation of *e* towards *f* on the arc dk, (Lem. 9.) The demonstration of this is the same as for Proposition 5. FIG. 9.

PROPOSITION 10.

THEOREM.

FIG 9. From the point D′, with the distances D′e″ and Df″, and from the point B, with the distances Bk and Bi, describe the intersections a and b, and through the points a and b, draw the straight line abn, meeting the circular arc BD′ in the point n; then from C as a center with the distance Cn, describe through n the arc n′nN′, and on the arc BD make the arc Bn equal to the arc Bn′—and join A and n.—The distance An is equal to the circumference of the circle ABD, (Fig. 4.)

For it is demonstrated (prop. 1.) that the point n, is the point of ultimate intersection of the line ab &c., or of all the intersections described through the series of points of variation e″ and f″, &c., and through the points k and i &c—through which the curve line ab ... n shall coincide with its chord an (Lem. 3.); therefore the point n must be in the arc n′N′ (Fig. 5.); but Bn is equal to Bn′ (Fig. 5.)—hence An (Fig. 9.), is equal to the circumference of the circle ABD (Fig. 4.)

SOLUTION EIGHTH.

PROPOSITION 11.

THEOREM.

FIG. 9. From the point A as a center with the distances Aq and Ao′ describe the arcs qq′ and o″x, and from B as a center with the distances Bq′ and Bx′ describe the arcs q′q″ and x′x″—also from C as a center with the distances CK and Co, describe the arcs Ky′ and op′—and from B as a center with the distances By′ and Bp′, describe the arcs y′y″ and p′p″—and with the distances Bk′ and Bi′, describe the arcs k′k″ and i′i″—then from B as a center with the distances Bk″ and Bi″, and from D′ as a center with the distances De and Df, describe the intersections a′ and b′, and through the points a′ and b′ draw the straight line a′b′n meeting the circular arc BD′ in the point n; then from C as a center through the point n, describe the arc n′nN′, and on the arc BD, make the arc Bn equal to the arc Bn′, and join A and n.—The distance An is equal to the circumference of the circle ABD (Fig. 4.)

The demonstration of this evidently must be the same as prop. 10.

SOLUTION NINTH.

PROPOSITION 12.

THEOREM.

Let the straight line abn be drawn through the points of intersections a and b,

FIG. 9. and the straight line a′b′n drawn through the points a′ and b′, intersecting each other in the point n—also from the point C as a center, through n describe the arc n′nN′, meeting the arc BC in the point n′, and the arc CD in the point N′,—and on the arc BD make the arc Bn equal to the arc Bn′, and join A and n.—The distance An shall be equal to the determinate length of the circumference of the Circle ABD, Fig. 4.

For the point n is the ultimate intersection of the intersections of each of the lines ab ... n and a′b′ ... n, upon the circular arc Bdef ... hikD′ (prop. 2 and 3), and the points d,e,f ... h,i,k, are the intersections of the infinite series of arcs 1″T′, 2″S′, and 3″R′, &c., and of the infinite series of arcs 2′K′, 3′O′ and 4′Q′, &c.—hence the point n must be upon the arc n′nN′, Fig. 5, but Bn′ is by construction equal to Bn, Fig. 5—hence Bn, Fig. 9, must be equal to Bn Fig. 5, and An Fig. 9 is equal to the circumference of the Circle ABD, Fig. 4.

FINIS.

Pl. 1.

Fig. 4.

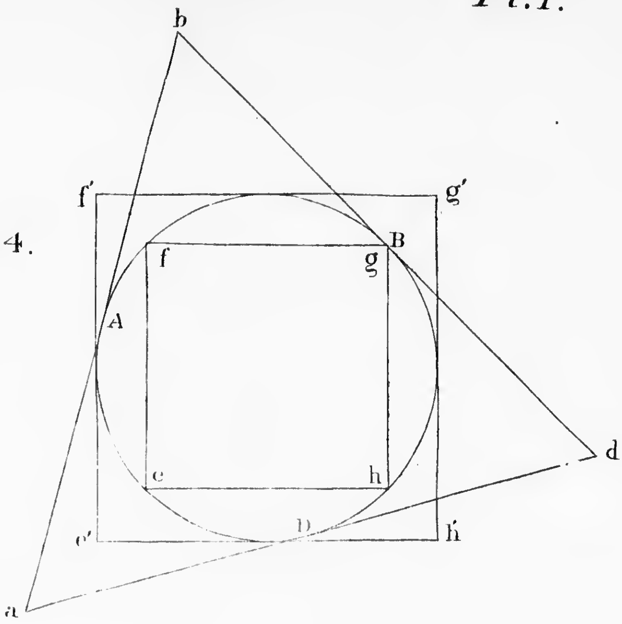
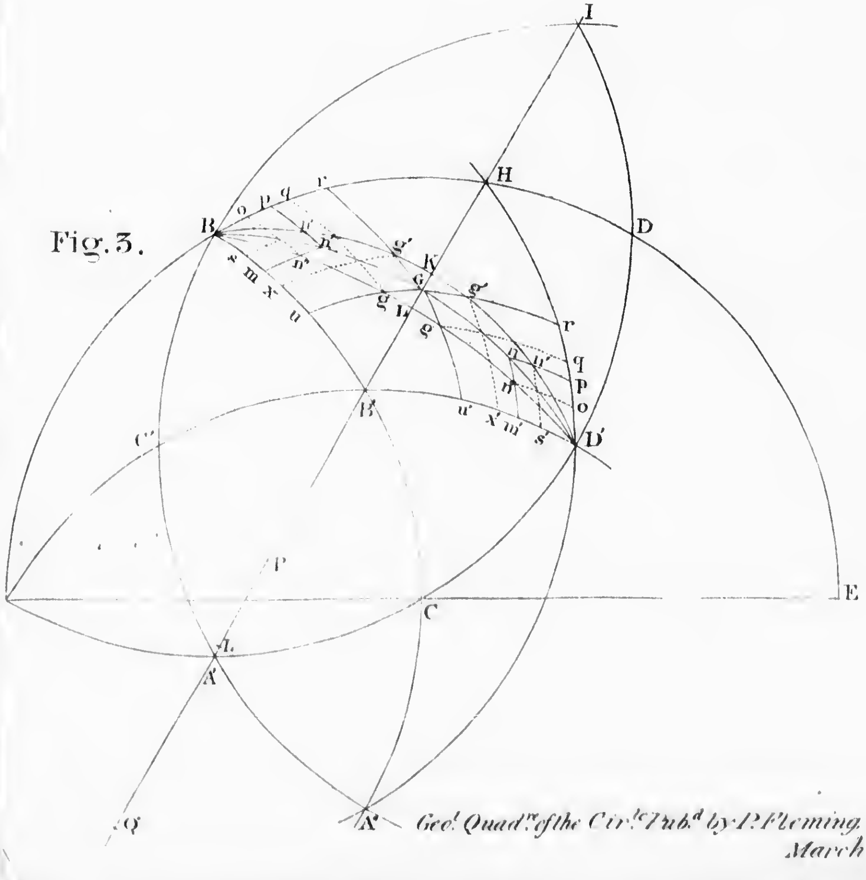
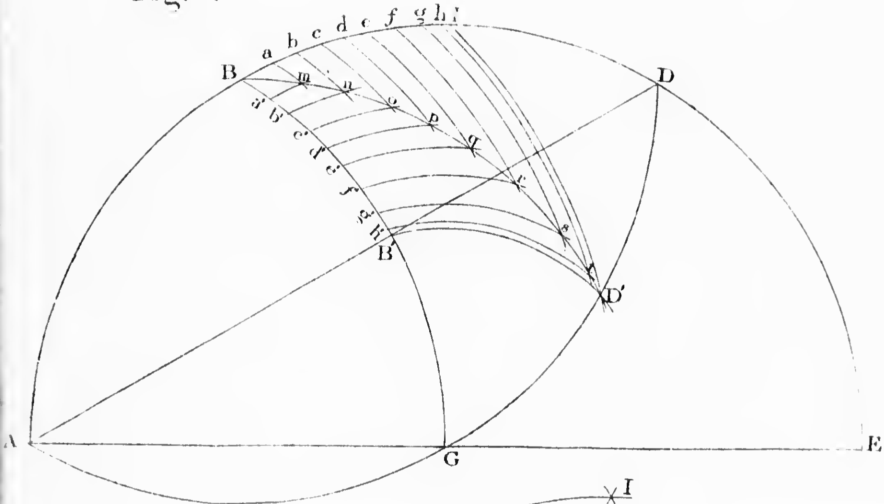


Fig. 3.



Geo.l Quad.re of the Cir.le Pub.d by P. Fleming. March 1850.

Fig. 1.



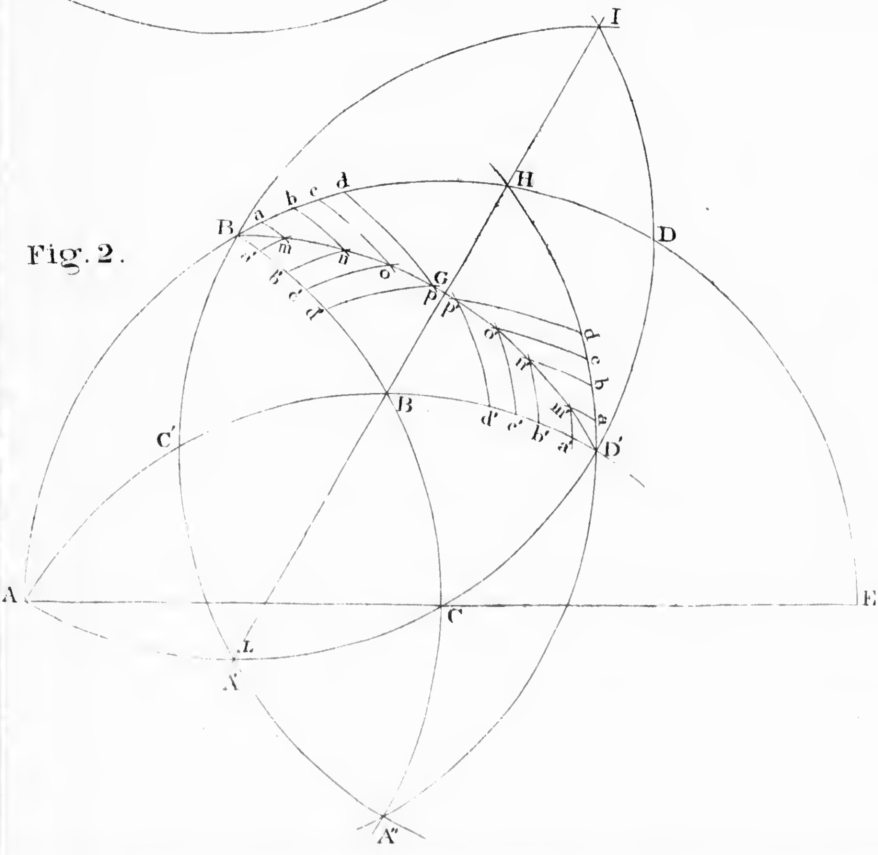


Fig. 2.

Fig. 3

Pl. 1.

Fig. 4.

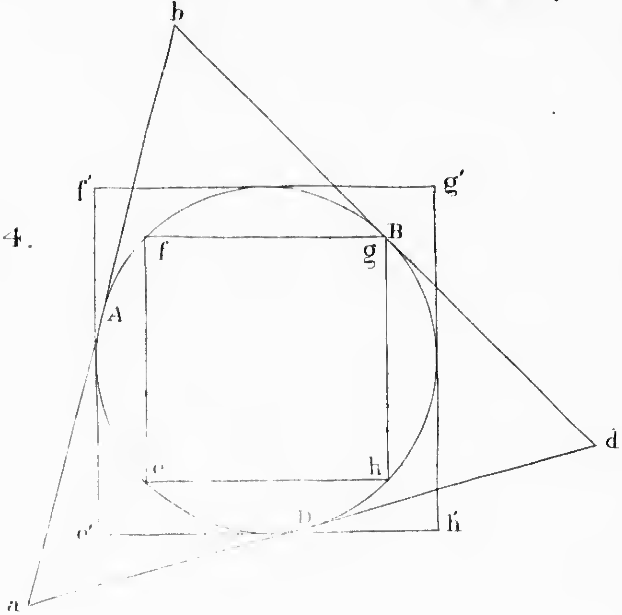
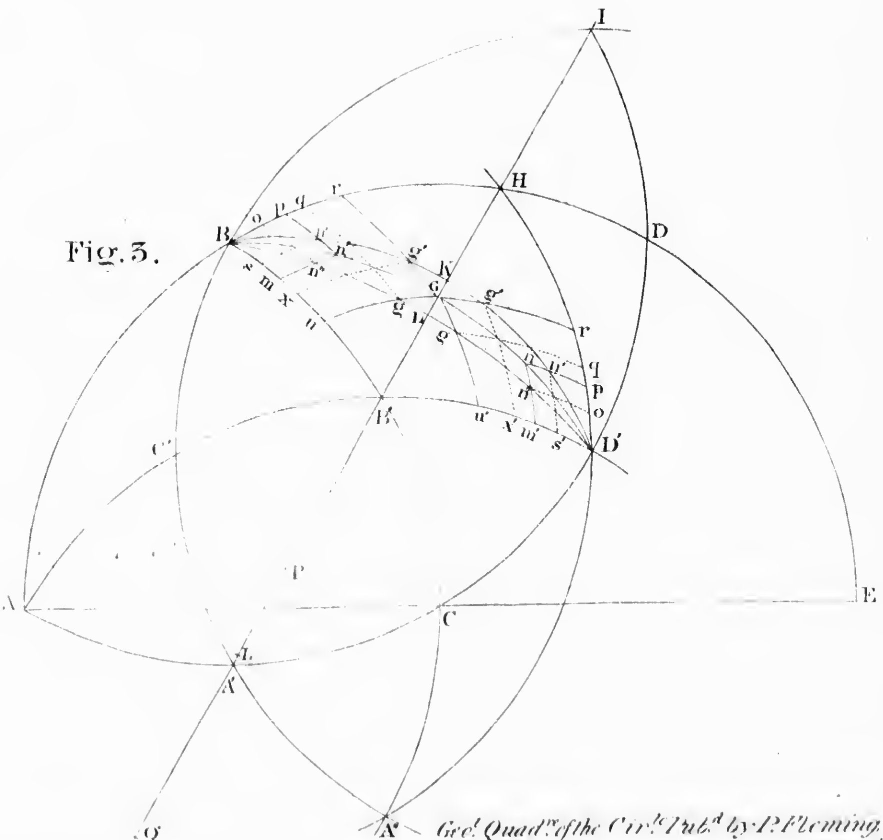
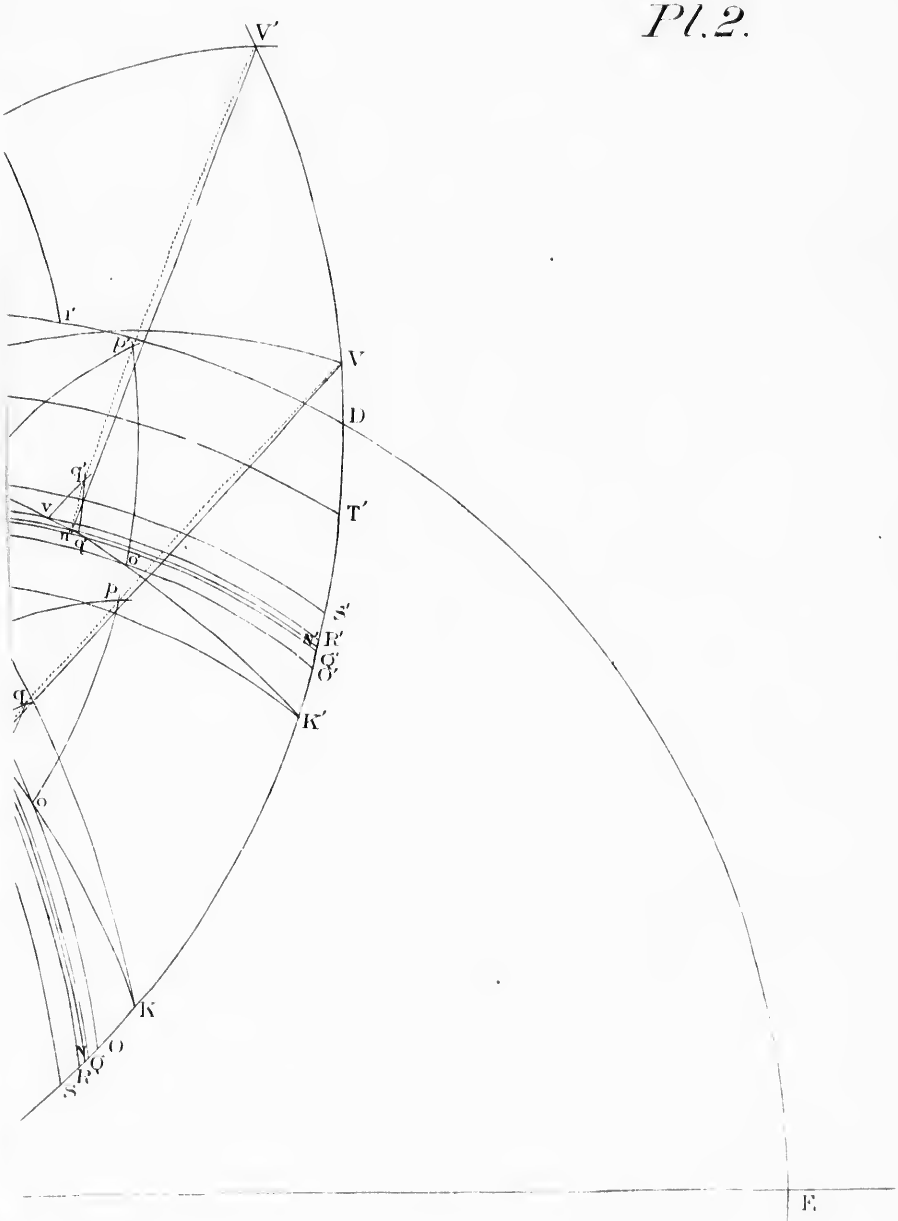


Fig. 3.



Geo. Quad. of the Cir. Pub. by P. Fleming.
March 1850.

Pl. 2.



Geo.l Quad.r of the Cr.le Pub.d by P. Fleming
March 1850.

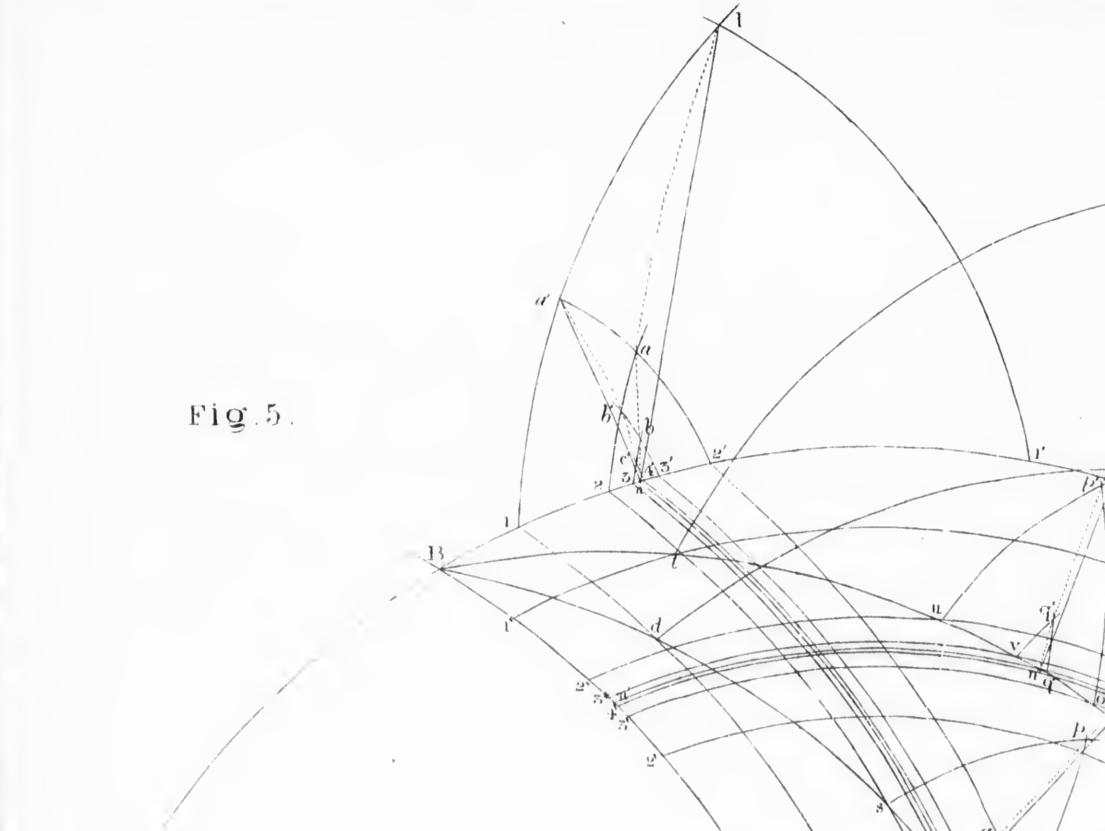
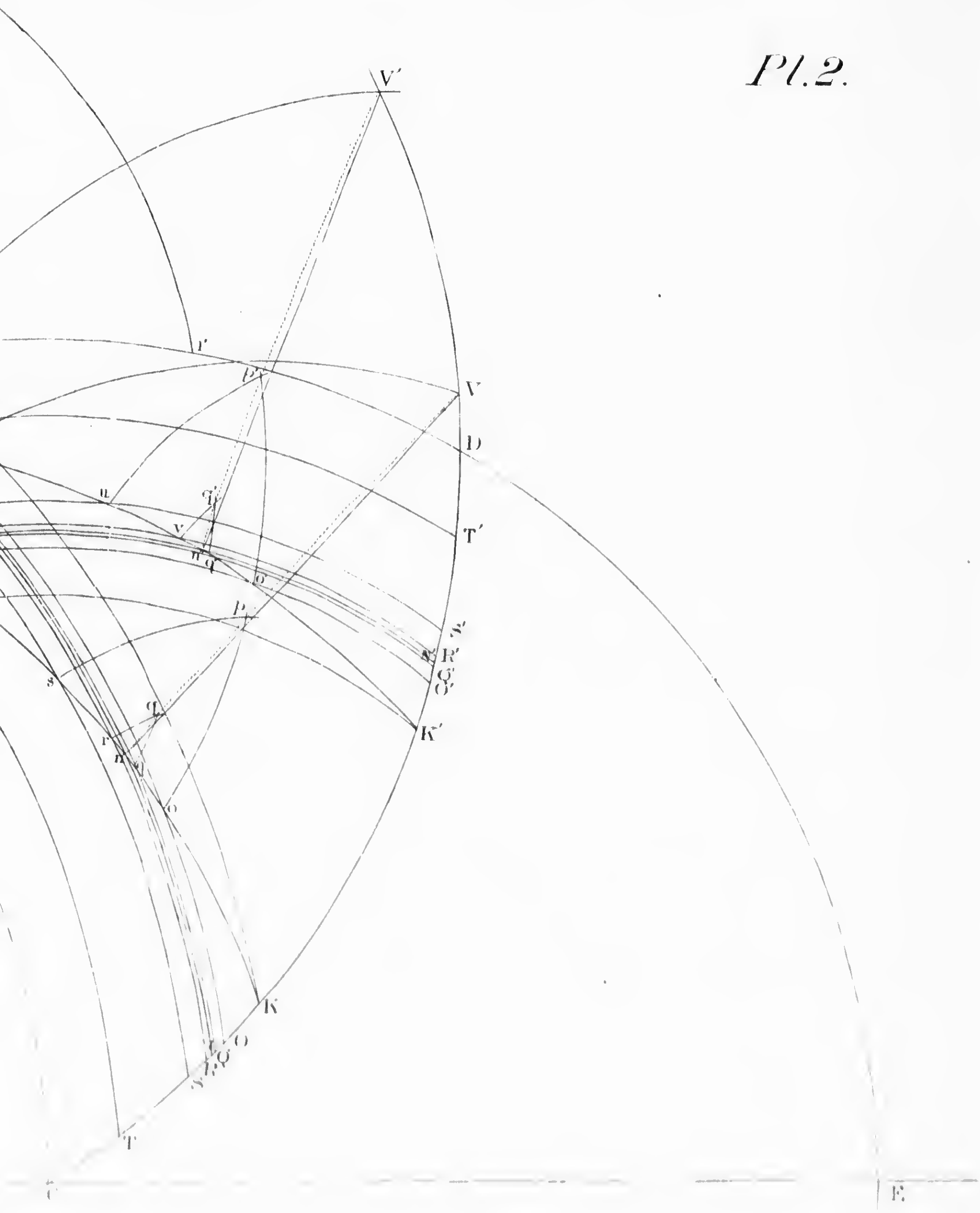


Fig. 5.

Pl. 2.



[illegible]
March 1850.


a'
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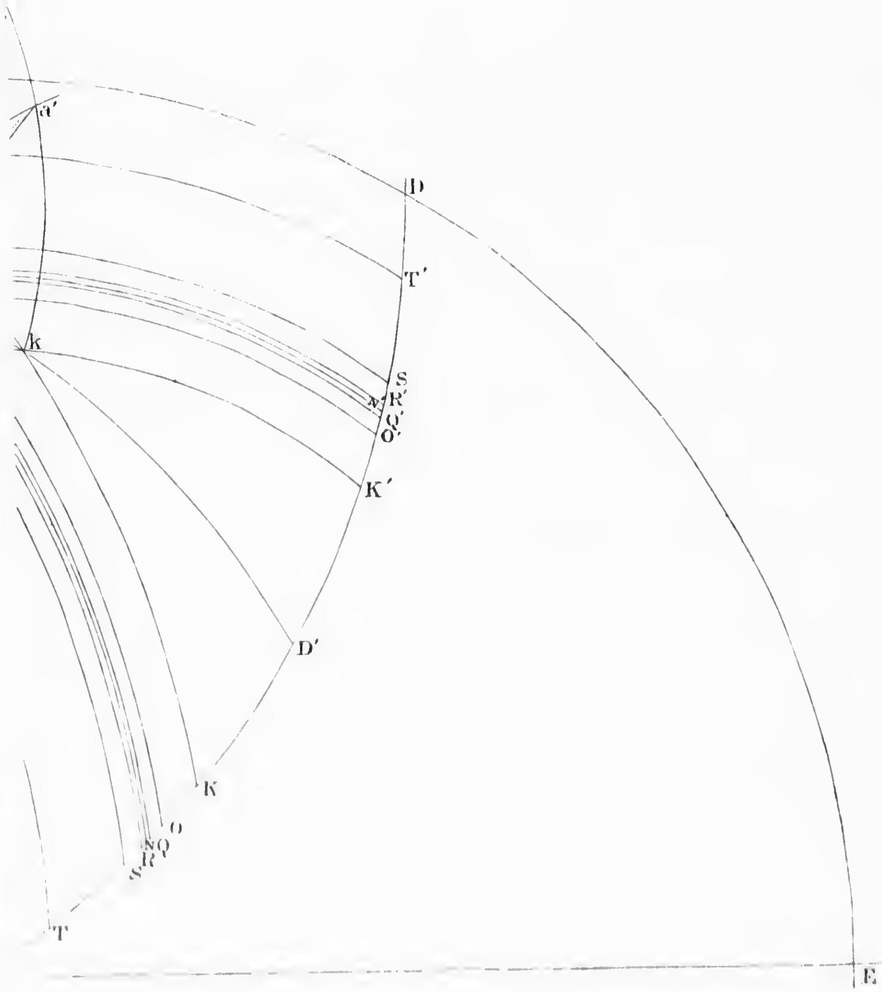
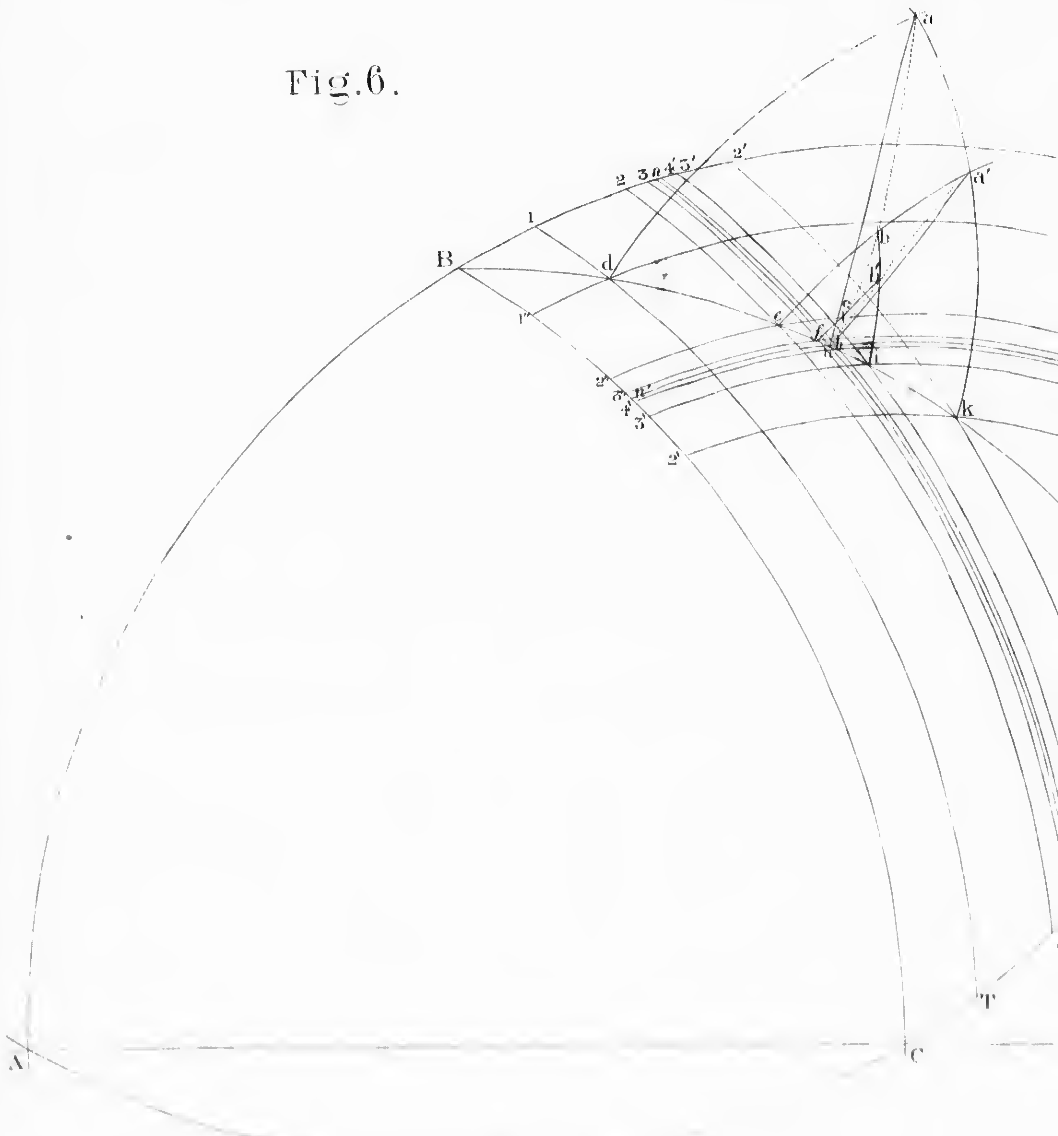
Pl. 3.

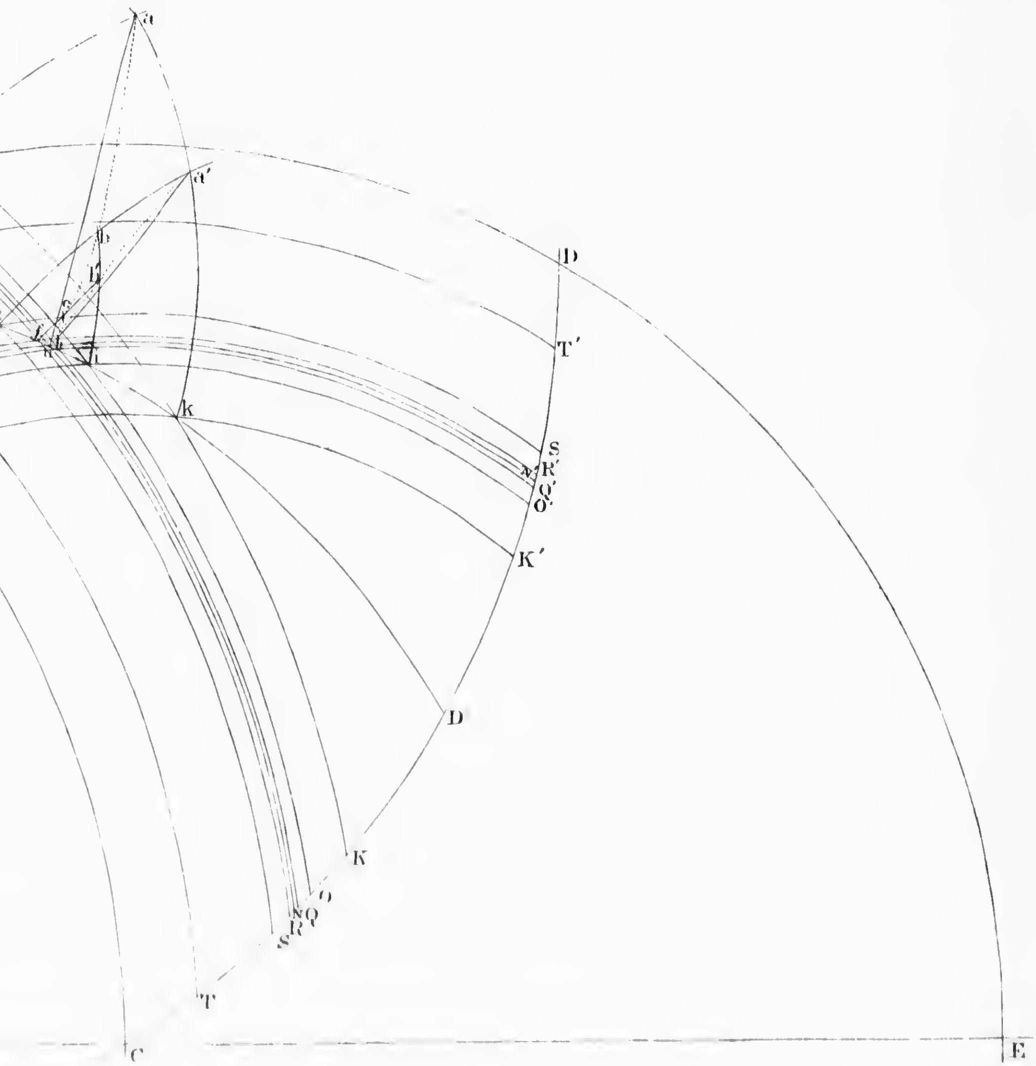


Geo.l Quad.e of the Cir.le Pub.d by P Fleming.
March 1850.

Fig.6.

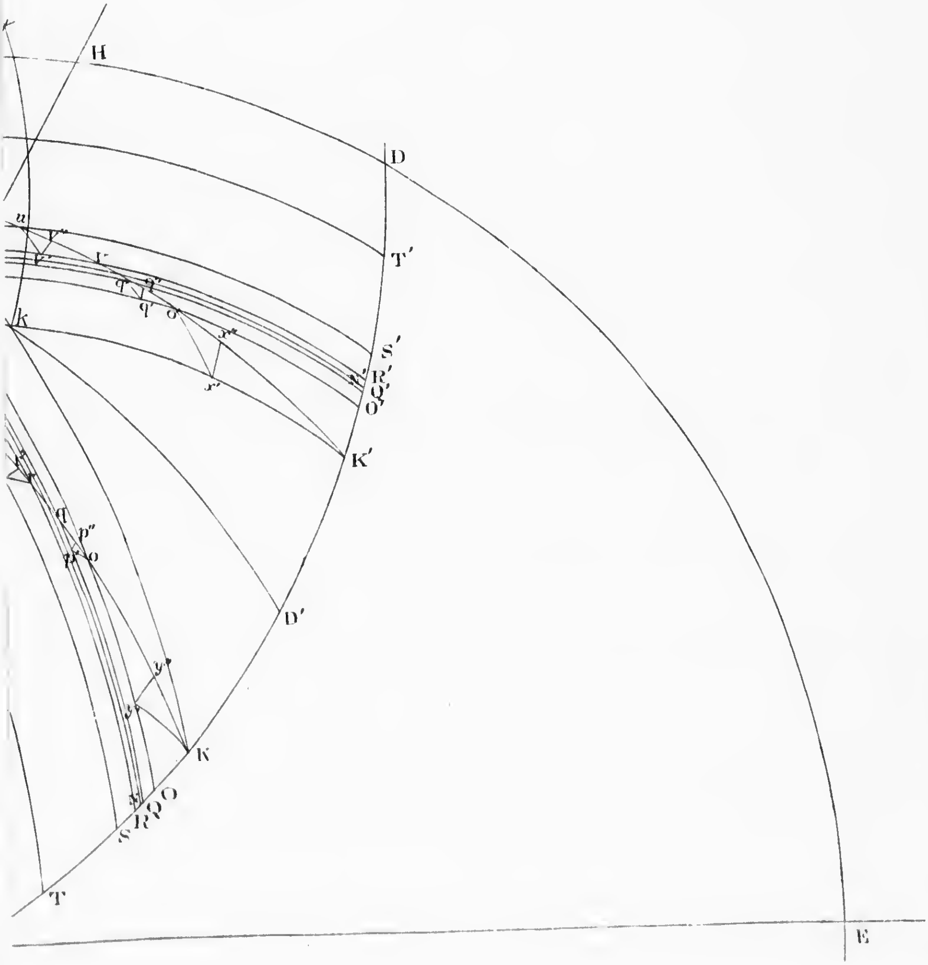


Pl. 3.



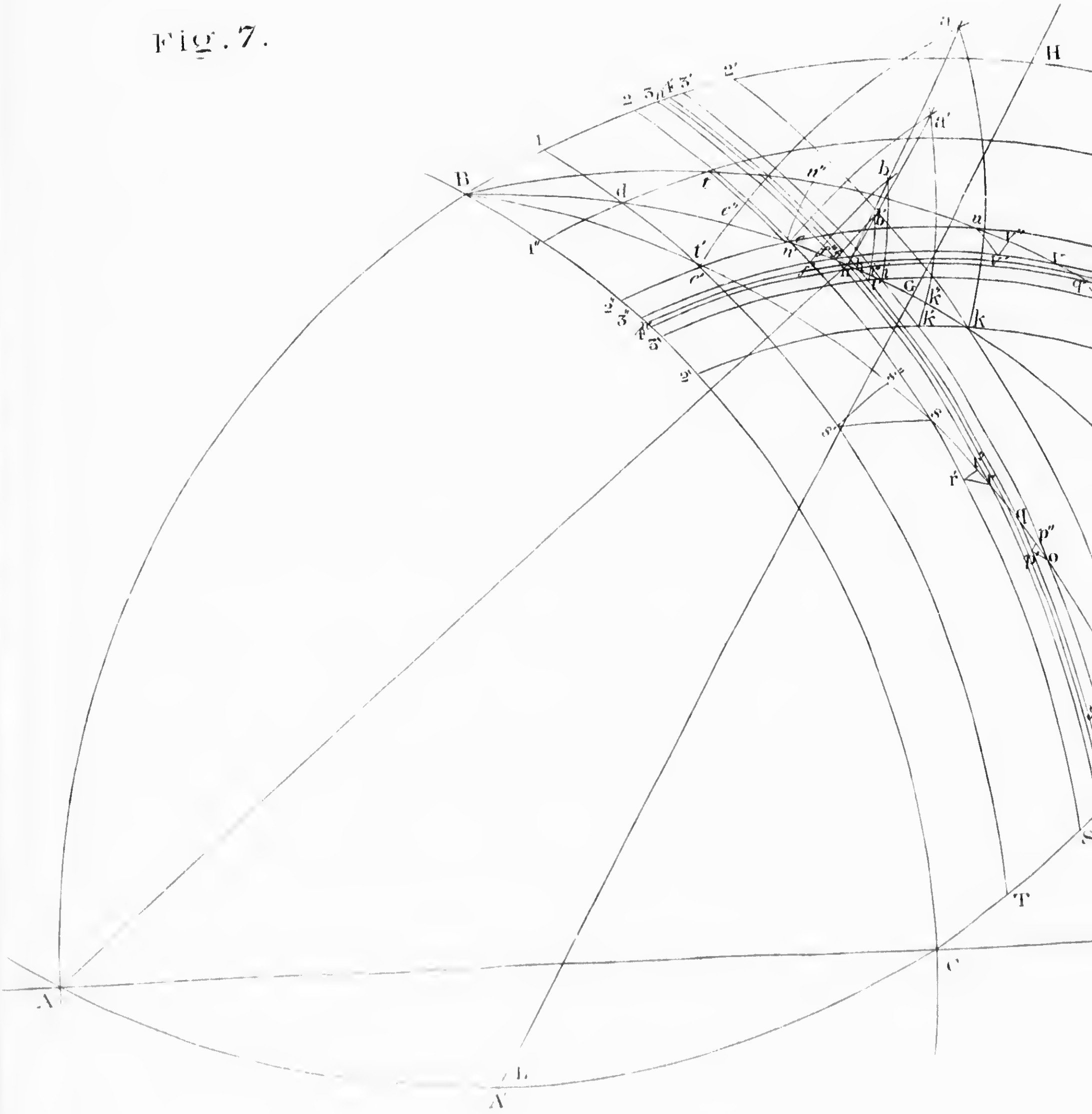
Geo[1] Quad.[e] of the Cir.[le] Pub.[d] by P. Fleming.
March 1850.

Pl. 4.

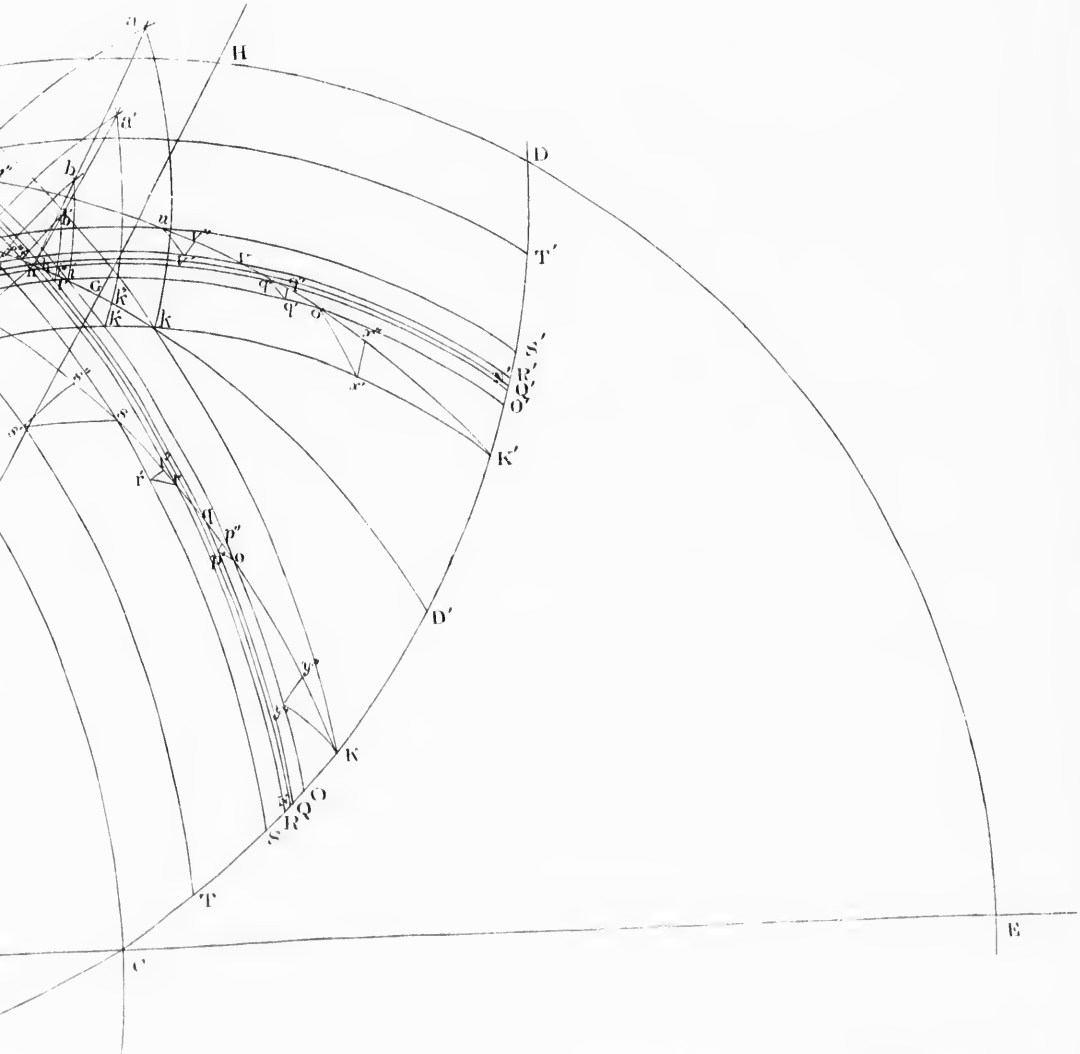


Geo! Quad! of the Cir! Pub! by P. Fleming.
March 1850.

Fig. 7.

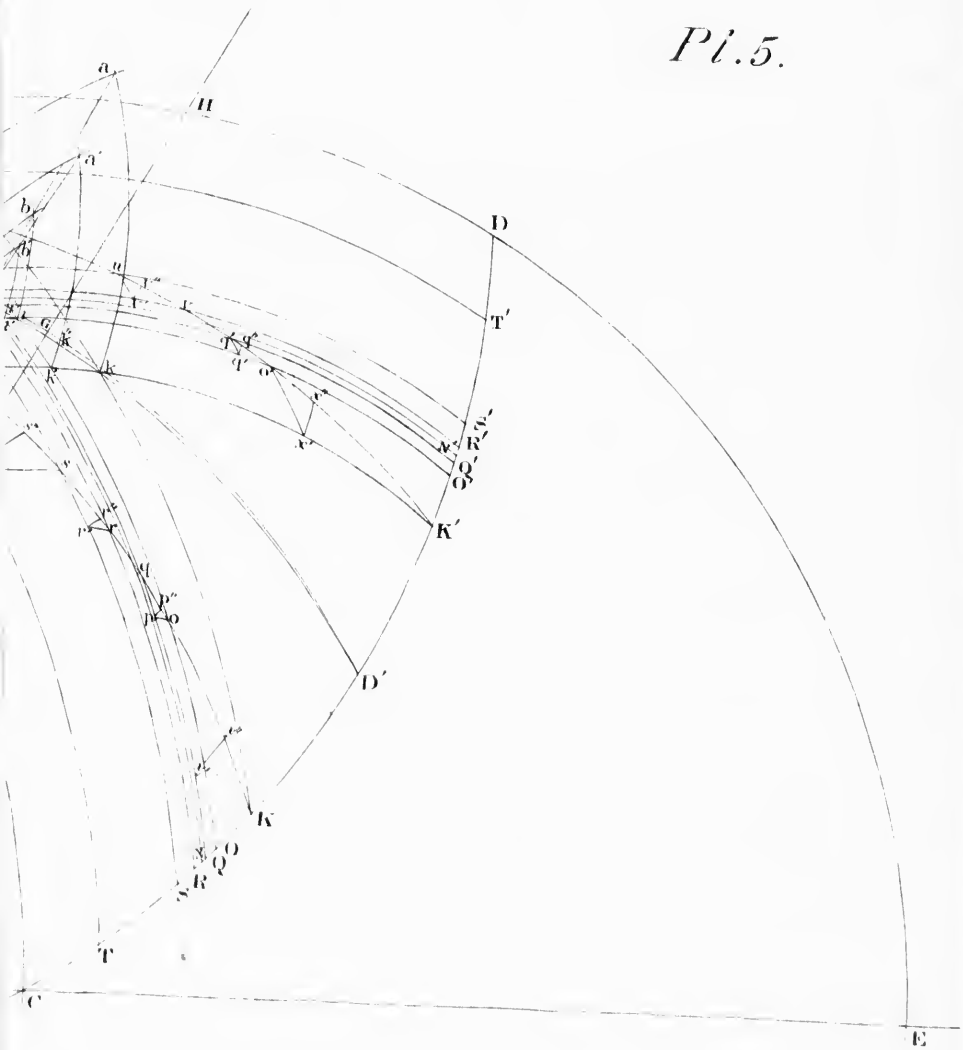


Pl. 4.



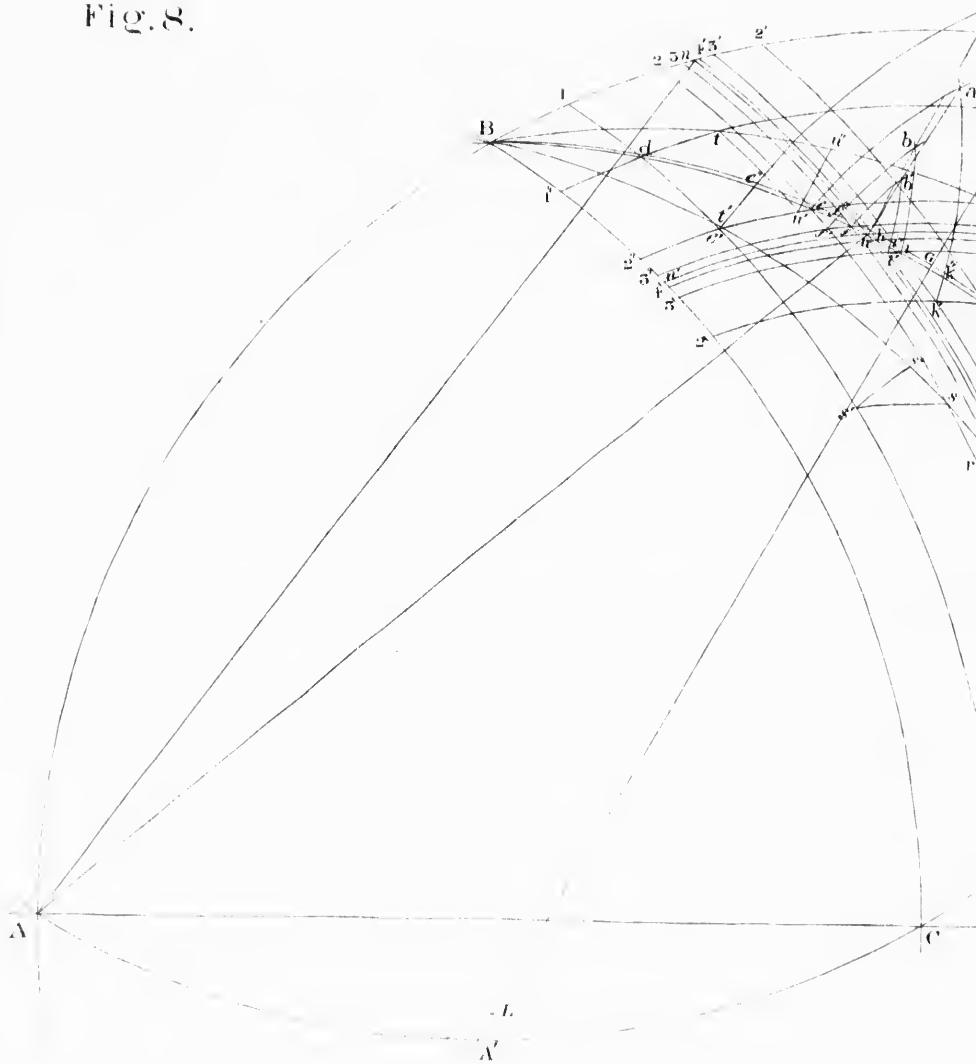
Gen.l Quad.r & the Co.l Pub.d by P. Fleming.
March 1850.

Pl. 5.

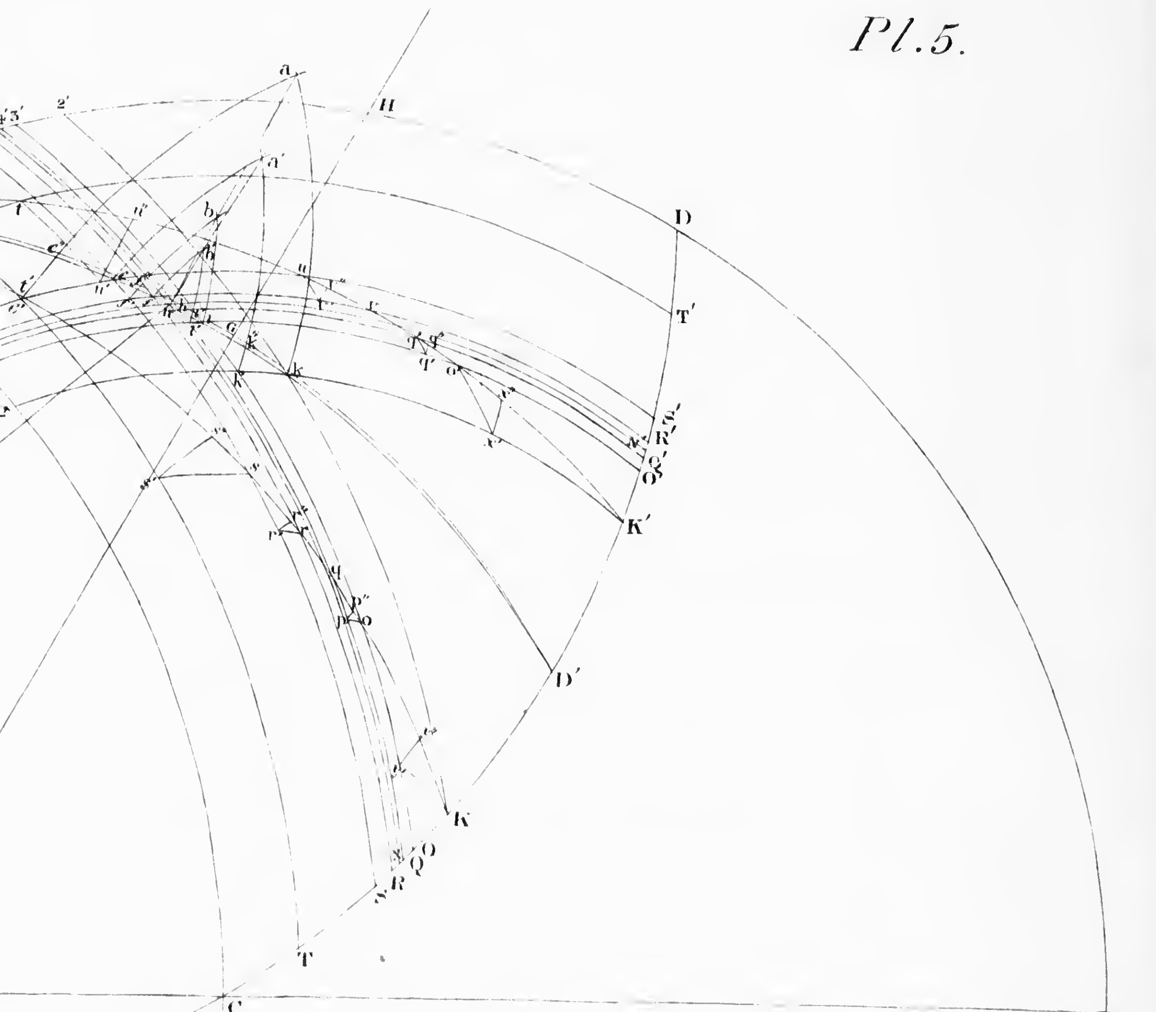


Geo.l Quad.re of the Cir.l Pub.d by J. P. Fleming
March 1850

Fig. 8.

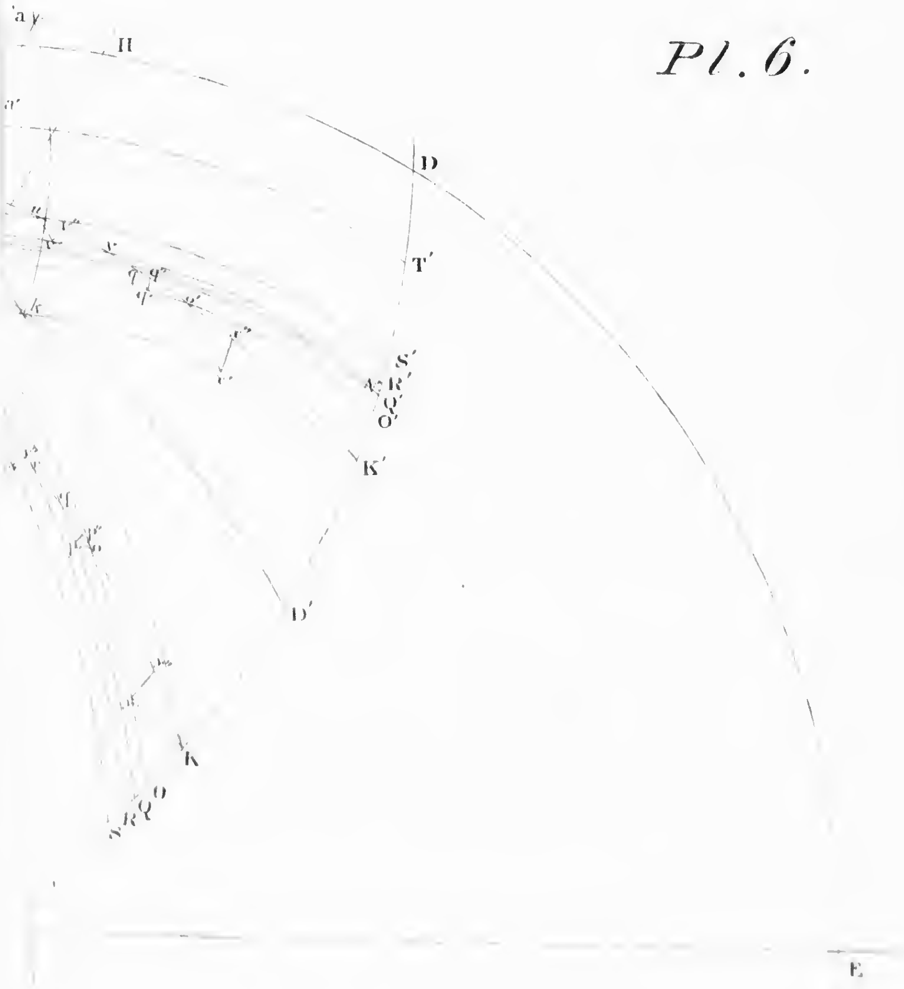


Pl. 5.



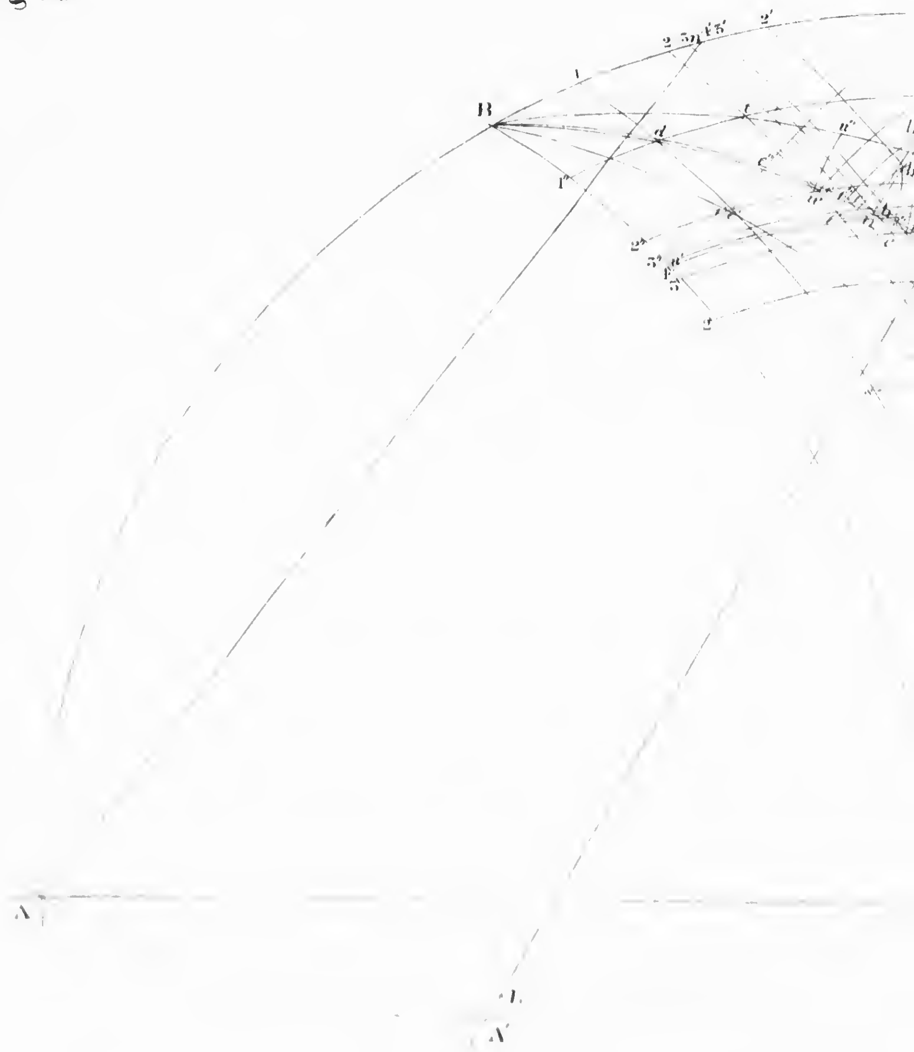
Geo! Quad^n of the Cir! Pub^d by P. Fleming
March 1850

Pl. 6.

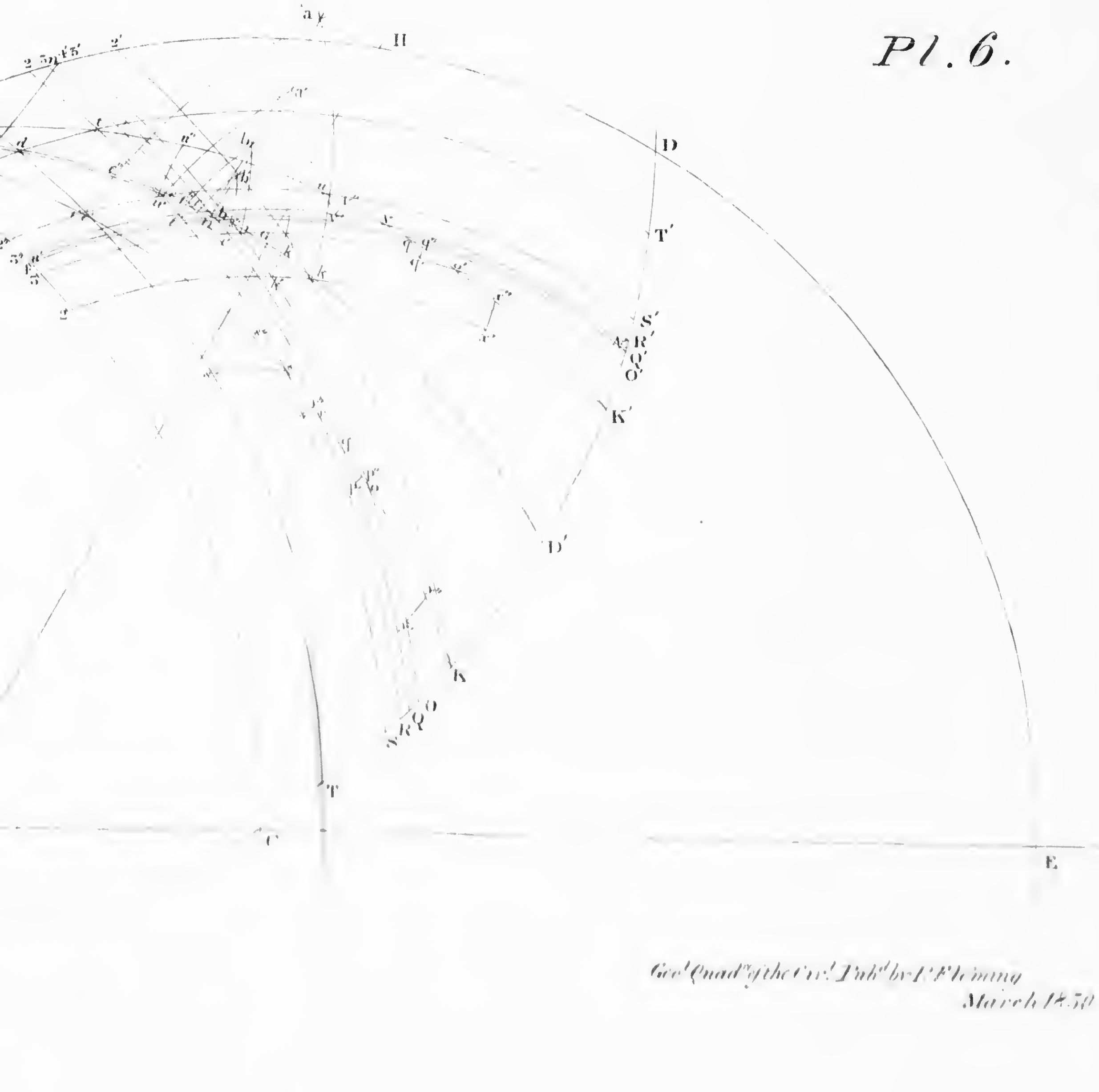


Geo.l Quad.t of the Cr.l Pub.d by P. Fleming
March 1st 59

Fig. 9.



Pl. 6.



Geo^l Quad^s of the Cr.^l Pub^d by I. Fleming
March 1850