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THE CARDIOIDE

AND

SOME OF ITS RELATED CURVES.

INAUGURAL-DISSERTATION

der mathematischen und naturwissenschaftlichen Facultät

der

KAISER-WILHELMS-UNIVERSITÄT STRASSBURG

zur Erlangung der Doctorwürde.

Vorgelegt von

RAYMOND CLARE ARCHIBALD

aus NOVA-SCOTIA, Canada.

STRASSBURG i. E.
JOSEF SINGER, BUCHHANDLUNG
1900.

ANALYTIC INDEX.

CHAPTER I:

CHAPTER II:

Cardioide Pedals.

Abbreviations.

A. E. Acta Eruditorum Lipsiensis.

A. Gr. Grunert's Archiv der Mathematik und Physik.

E. T. The Educational Times (London).

E. T. R. Reprint of Mathematical Questions from E. T.

I. M. L'Intermédiaire des Mathématiciens (Paris).

J. S. Journal de Mathématiques spéciales (Paris).

N. A. Nouvelles Annales de Mathématiques (Paris).

(n) n^{th} series.

ERRATA.

Page 8, line 7-8, instead of, *Its shape and position are indicated by the dotted form* ... read: *a similar and similary placed nephroid is indicated by the* —·—·— *form* ···

„ 8, „ 13, omit *theore.*

„ 8, „ 15, instead of *(Fig. I)* read: *figure I.*

„ 13, „ 20, after, $x = m$ insert: (Fig. 2, p. 14).

„ 14, Fig. 2, instead of W. read: ω.

„ 24, „ 5, the line CD, cuts the circle in O.

CHAPTER I.

INTRODUCTORY.

The heart-shaped curve commonly known as the *Cardioide* is an exceedingly interesting one from many points of view.

Its form, length, area, Cartesian equation and generation as an epicycloid, seem to have been first indicated by JACOB OZANAM in 1691.* Early in 1692 JAMES BERNOULLI showed the curve to be a catacaustic of a circle and further, that the catacaustic of the cardioide for a luminous cusp is the (so called) "two-cusped" epicycloid or nephroid; in the *Lectiones* (1691—92) of his brother, JOHN BERNOULLI, the cardioide was again treated as a catacaustic.

A presentation of previous results was given by DE L'HOPITAL (a pupil of JOHN BERNOULLI) in his *Analyse d. Inf. Petits etc.* 1696 p. 113-117. Similarly, we may speak of an article in 1703 by LOUIS CARRÉ, who wrote the first complete work on the integral calculus.** In 1705 CARRÉ discussed a portion of a curve which he derived in the manner of a circular conchoide, but it was DE REAUMUR, 1708, who first pointed out that CARRÉ had thus generated a part of the cardioide. The cardioide was, however, first, both named and generated in its completeness as the conchoide of a circle, by DE LA HIRE in 1707 (Mém. acad. franc.); here too, it is pointed out for the first time that the cardioide is a particular case of a Pascal Limaçon. The next additions to the cardioide properties were made by COLIN MACLAURIN. In the *Philos. Trans.* 1718 and his *Geometria Organica* 1720, he showed that the cardioide is a pedal of a circle; that the parabola is the fourth, and the cubic of Tschirnhausen (§ 32) the fifth, negative pedal of a cardioide with respect to its cusp. He showed further, that particular cases of curves defined by the equation

* To this and other dates more exact references are given in the body of the paper.

** *Méthode pour la mesure des surfaces, la dimension des solides, leurs centres de pesanteur, de percussion, et d'oscillation par l'application du calcul intégral. Paris 1700.*

$r^m = a^m \cos m\Theta$ (sinusoidal Spirals, § 33) are: the cardioide and all its cuspidal pedals (positive and negative), the parabola (fourth negative pedal), straight line (third negative pedal), circle (first negative pedal), rectangular hyperbola and lemniscate of **Bernoulli.** He also found interesting expressions for the areas and lengths of such curves and the laws of force under which a particle will describe them.

During the next hundred years, practically nothing further was added to our knowledge of the cardioide, so we will pass over less known names and merely note, that some old results were given by Johannis Castilleoneus in 1741; by **Euler** *Introductio analys.* 1748 II 224-225 and by Cramer *Introd. à l'analyse des lignes courbes* 1750.

In the present century, 1823, Quetelet showed an interesting use for the cardioide in graphical astronomy [cf. note § 7]; and a few years later, that the cardioide could, by two stereographic projections be transformed into a conic.

In 1832-33 Magnus showed the cardioide to be the inverse of a parabola and deduced by the method of inversion some interesting properties.

In later years further properties have been published by the physicist J. C. Maxwell, the astronomer Richard Proctor, and Prof. Wolstenholme; by MM. Weill, Laguerre, Brocard; and by Profs. E. Weyr; K. Zahradnik (who in his six papers treats the cardioide as a *unicursal quartic*), W. Jeřábek, A. Kiefer, Siebeck etc. etc.

The literature of the cardioide is very extended, including as it does, over 30 magazine articles, dissertations, and school programmes (devoted *entirely* to the cardioide), beside the hundreds of isolated theorems scattered about in the memoirs and problems of the various mathematical journals, and other mathematical works.

We have indicated above that the cardioide may be considered as:

1. An epicycloid
2. A catacaustic of a circle
3. A conchoide of a circle
4. A pedal of a circle
5. A negative pedal
6. A unicursal quartic
7. A sinusoidal spiral
8. A conic inverse
9. A limaçon

As we shall presently see, it is further:

10. A bicircular quartic
11. A tricuspidal quartic
12. A Cartesian oval
13. A curve of the third class
14. An envelope of systems of:
    (a) circles
    (b) lemniscates
    (c) rectangular hyperbolas
15. The polar reciprocal of two interesting and much studied cubics of the fourth class

etc. etc.

The object of the present paper is to set together in a connected fashion, a series of theorems (many are believed to be new) concerning "The Cardioide and some of its related Curves". The scope of the paper allowed, of course, only a small selection from such curves; those selected which have received a name, are indexed at the end of the thesis.

In the treatment, the only knowledge presupposed is of analytic geometry, the calculus, and the theory of inversion.

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1. Introductory Theorem. A well known theorem to which we shall have occasion to refer very often in the following pages is: "If a roulette is traced by the point P, of a curve rolling on a fixed curve, in the same plane, the line joining P to the point of contact of the fixed and rolling curves is normal to the roulette."

2. The terms epicycloid and hypocycloid are applied by different writers to a great variety of curves; the most common definition is as follows:

The epicycloid (hypocycloid) is the curve traced out by a point on the circumference of a circle which rolls without sliding on a fixed circle, in the same plane, *the two circles being in external (internal) contact.*

With EULER,* I will, however, adopt the following definition:

The epicycloid (hypocycloid) is the curve traced out by a point on the circumference of a circle which rolls without sliding on the circumference of a circle in the same plane, *the rolling circle touching the outside (inside) of the fixed circle.*

That this latter is the more correct definition, is proved by the fact, that while the former leads to an altogether unsymmetrical classification of the resulting curves, the latter leads to a classification perfectly symmetrical. According to the former, every epicycloid is a hypocycloid but only some hypocycloids are epicycloids; according to the latter *no* epicycloid is a hypocycloid, and no hypocycloid is an epicycloid.

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3. Since the name cardioide has been applied to several curves,** it will be well to state here, that we will use the name only in speaking of the "Cycloide geométrique"

* EULER was the discoverer of the double generation of the epicycloid and hypocycloid, and gave both analytic and geometric proofs thereof in his memoir: *"De duplici genesi tam epicycloidum quam hypocycloidum"*, *Acta acad. Petrop.* V_1, 48—59, 1781. Cf. PROCTOR, *"Geometry of the cycloids"*, 1878, Preface.

** To the *astroide* or (so called) "fourcusped" hypocycloid: *Nouv. Corresp. Math.* 1877, III 62, 123. By CESARO, N. A. 1885 (3) IV, 257.

To a particular case of the *oblique strophoide* by W. J. C. MILLER, E. T. R. XLV, 75 Ex. no. 7896. For, the equation of an oblique strophoide has been given by BROCARD *(Notes de Bibliographie des courbes géometriques* I, 140) as $r(1 + m \tan \vartheta/2) = d$, m and d being constants. The equation of Mr. Miller is the particular case of this curve for $m = 2$.

of Ozanam.* This curve is *the epicycloid generated by equal circles.* According to our definition, the cardioide is *not* then, a hypocycloide, as so often** stated.

The point P (Fig. I) on the circumference of a circle with centre C and radius a was formerly at the point S of the equal *fixed circle* or *base,* with centre O; R is the present point of contact of the circles. Let us choose S as the origin of coordinates and the line O S produced, as axis of X. If $SP = r$ and $\angle PSX = \Theta$ (since $\angle ROS = \angle RCP = \angle PSX$) we find at once from the geometry of the figure, that the polar equation of the cardioide traced by P may be written:

$$r = 2a(1 - \cos\Theta) \tag{1}$$

a form first given by Euler.†

4. If the line O R C be produced to meet the generating circle in T_1, we see (introductory theorem) that PT_1, P E (E being the point where P R again meets the base) are respectively tangent and normal to the cardioide at P. Whence, *the tangent to a cardioide at any point makes with the radius vector an angle equal to half the polar angle of the point: and, the cardioide has a cusp at S.*

If a is variable, the equation $r = 2a(1 + \cos\Theta)$ evidently represents a family of cardioides: indeed those obtained by turning (1) around the cusp thro an angle of 180°. Hence, the orthogonal trajectories of a family of cardioides is another family of cardioides.

5. The line S O produced meets the base again in B and the cardioide in the *vertex* A. The line S A is the *axis* of the curve. From (1) it is at once evident that *all cuspidal chords of the cardioide are of constant length, double the diameter of the base,* and further, since the lines P S, E O produced meet in a point b on the base, and the figure P C O b is a parallelogram, *these cuspidal chords are all bisected by the base;* from § 4 we see also that, *the tangents to the cardioide at the extremities P, P' of any cuspidal chord intersect at right angles in T_2 on a circle with centre O and radius O A; while the normals to the curve at P, P' intersect orthogonally on the base in E.*

The normals P E, P' E are evidently bisected at R and J respectively, points which are the ends of a diameter of the base.

6. It is now easy to see that the circle thro the points P, T_2, P', is tangent to the base at E. Hence, *the second mode of generation of the cardioide as an epicycloide, the radius of the base being one half that of the generating circle.* It is further evident, that,

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* JACOB OZANAM, *Dictionaire Mathématiques ou Idée générale de Mathématiques,* Amsterdam 1691, p. 102-104.

** E. g.:- M. SIMON, *Analytische Geometrie der Ebene,* 1900, p. 299.

† EULER, *Introductio in analysin infinitorum,* 1748 II 225.

if the points T_1, T_2 travel uniformly and in the same direction around the circle with centre O and radius O A, the angular velocity of T_2 being twice that of T_1, the line $T_1 T_2$ envelops the cardioide whose base has the same centre, and radius one third, that of the given circle. Or, as this result may be stated (since $\angle A T_1 O = \angle O T_1 T_2$): *the catacaustic of a circle for rays radiating from a luminous point, A, on its circumference, is a cardioide whose vertex is at the luminous point and whose base has the same centre, and radius one third that of the given circle.**

7. Yet another point of view.

If P C be produced to meet the generating circle again in Q, the line B Q meets the base in E and the generating circle in F. We see that the line P F is perpendicular to S P and B F, and that B F = B S. Hence *our cardioide is the pedal of its axial circle, with respect to a point, S, upon it.***

Since the circles on the focal radii of a parabola as diameters are tangent to the tangent at the vertex of the parabola, we can now infer the theorem first stated by QUETELET:*** *The locus of the vertices of the parabolas with a common focus, and passing thro a fixed point, is a cardioide with vertex at the fixed point and cusp at the common focus.* Conversely, *if the vertex of a parabola lies on a cardioide whose cusp is at the focus of the parabola, the parabola will pass thro the cardioide's vertex.*

8. From the results of § 5 we infer the well known fact first stated by DE LA HIRE,† that the cardioide is a *conchoide of a circle:* the reason of the name is, that the mode of generation is analogous to that for the *conchoide of Nicomedes* which has a straight line instead of a circle, as base. From b along S b produced, we measure in both directions, lengths b P, b P', each equal to the diameter of the base circle; the locus of P, P' is a cardioide.††

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* Theorem first stated by JAMES BERNOULLI, A. E. June 1692, p. 291-296.

** Theorem due to MACLAURIN, *Philos. Trans.* 1718, p. 803-812.

*** *Nouv. mém. acad. Bruxelles,* 1826, III 116 [Memoir read Feb. 1823]; QUETELET returns to the theorem a second time later in the same volume p. 169-171 "*Mémoire sur quelques constructions graphiques des orbites planétaires*". After three observations a parabolic orbit is, from the above theorem at once determined in its plane, since its vertex is the point of intersection of three cardioides.

† *Mém. de l'acad. royale d. sc.,* 10. Dec. 1707, p. 50-54. CANTOR *(Vorlesungen über die Geschichte d. Math.,* III, 1898, p. 772-778) gives as the history of the cardioide that it was named, and generated as a conchoide, by Castilleoneus *(Phil. Trans.* no 461, 1741, p. 778–881): that the curve was in part imagined by Carré *(Mém. acad. d. sciences,* 28. Feb. 1705, p. 56-61) who attributed the knowledge to "Koenersma". The name Koenersma is incorrect; it should be Koërsma. The cardioide was imagined at least a dozen times in its completeness before Castilleoneus: in fact by CARRÉ himself *(Mém. acad. d. sc.,* 24. July 1703, p. 183). For others see the "Historical Sketch" p. 1-2.

†† It is well known that if the lengths b P, b P' be measured equal to some other constant length than the diameter of the base circle, the points P, P', trace a limaçon of Pascal.

A second curve of the Greeks with which the cardioide has an analogous mode of generation is the *Cissoide of Diocles*. For, from equation (1) it is at once evident that points on the cardioide we are considering, may be found by means of the circles with diameters BD, SD. Any chord thro S, cuts the smaller circle in S_1 and the larger in P_1. From S measure the length $SP = S_1P_1$: the locus of P is the cardioide.

Now for the cissoide of Diocles we take a circle with diameter SD, and the tangent to it at D (which may be considered as another circle, of infinite radius). Any line drawn thro S meets these two circles in $\overline{S}_1$, $\overline{P}_1$ respectively; from S measure $S\overline{P} = \overline{S}_1\overline{P}_1$: the locus of $\overline{P}$ is the cissoide of Diocles whose equations* are at once found to be

$$(2) \qquad y^2(2a - x) = x^3 \qquad \text{or} \qquad r = 2a\sin\Theta\tan\Theta.$$

9. It will be interesting at this point to seek the locus of the point of intersection $\overline{P}(r'\,\Theta')$, of corresponding tangents to the cardioide and its generating circles. Since the arcs SR, PR are equal, the tangent to the base at R bisects SP perpendicularly in p. We have then $2\Theta' = \Theta$ and $r/2 = r'\cos\Theta' = a(1 - \cos\Theta)$ or $r' = 2a\sin\Theta'\tan\Theta'$ the very cissoide we have above considered. Hence, *the locus of the point of intersection of the tangents at corresponding points of the cardioide and its generating circles is a cissoide of Diocles.*

To the conjugate points P, P' of the cardioide will correspond the pair of conjugate points $\overline{P}$, $\overline{P}'$ of the cissoide of Diocles, subtending a right angle at its cusp: M. Cazamian has shown** that the envelope of the line $\overline{P}\,\overline{P}'$ is an hyperbola with vertex S.

Since the line $\overline{P}S$ when produced passes thro J we can infer the following theorem:

If the base angle, J, of an isosceles triangle $J\overline{P}\,T_1$ moves round a given circle with centre O, while the middle point of the base (which always passes thro O) also moves on the circle: if, furthermore, the side bordering the tracing angle always passes thro a fixed point S, of the circle, the vertex of the triangle traces out a cissoide of Diocles, while the free side envelops a cardioide. If $OS = a$ and S be taken as origin, if further, r', r be the radii vectores of the cissoide and cardioide respectively for a given Θ, $r' : r = \tan\Theta : \tan\Theta/2$.

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* It may be worth noting that, referred to the point, (- a, o), as origin the equation of this cissoide assumes (in polar coordinates) the graceful form $\frac{r}{a} = \frac{1 + (\tan\frac{1}{2}\vartheta)^{2/3}}{1 - (\tan\frac{1}{2}\vartheta)^{2/3}}$. The equation of the pedal of the curve with respect to the origin may be written:

$$\vartheta = 2\tan^{-1}\frac{\sqrt{9a^2 - r^2}}{2r} - \tan^{-1}\frac{\sqrt{9a^2 - r^2}}{r} \quad \text{or} \quad \frac{r}{3a} = \frac{1 - (\tan{}^1/{}_2\,\vartheta)^{2/3}}{1 + (\tan{}^1/{}_2\,\vartheta)^{2/3}}.$$

** N. A. 1894 (3) XIII, 303.

Suppose now, that instead of two *circles*, as above, we take as ground curves the cardioide and cissoide, we get the interesting result: *Thro S is drawn any chord cutting the cardioide in P and the cissoide of Diocles in P''; from S measure off in this chord SL = PP'', the locus of L is the conchoide of Nicomedes with the same cusp and asymptote as the cissoide of Diocles, its equation being* $r = 2a\left(\frac{1}{\cos\Theta} - 1\right)$.

10. The angle $SRO = \angle ORE$; hence (last theorem § 6) *the evolute of a cardioide is a cardioide of one third the linear dimensions, whose base has the same centre as that of the original curve and whose vertex lies at the cusp of the original curve.*

Observing that FBS is an isosceles triangle, perpendiculars on whose side and base are SE, BR respectively, we can now state the theorem: *One side, BS, of an isosceles triangle FBS is held fast while the other turns about the vertex B; the line joining the feet of the perpendiculars from S and B on the opposite sides, envelops a cardioide.*

Consider B as origin and (ϱ, Φ) the coordinates of the point of intersection of SE, BR: then $BE/\varrho = 2a\cos\Theta/\varrho = \cos\Theta/2$. But $\Phi = \Theta/2$ whence *the locus of the point of intersection of SE, BR is the curve defined by the equation* $\varrho = 2a\frac{\cos 2\Phi}{\cos\Phi}$ i. e., *a right strophoide.*

Similarly BSP_1 being an isosceles triangle, the locus of the point of intersection of FS and Bb is a strophoide, whose equation may be witten: $\varrho = -2a\frac{\cos 2\Phi}{\cos\Phi}$.

Further, if the radii vectores of this strophoide (S origin), be produced the constant length 2a, we get the sextic curve $r = 2a\left(1 - \frac{\cos 2\Theta}{\cos\Theta}\right)$ *which is also found by producing the cardioide radii vectores, SP, the length SP'' equal to the corresponding cissoide radii vectores.*

11. Let us again refer to our triangle FBS and seek the locus of the centre of its *nine-point circle.* If its centre be $f(r, \Theta)$ we easily find for its equation $r = BR - fR = 2a\cos\Theta - \frac{a}{2\cos\Theta}$ which defines a *Trisectrix of Maclaurin* with double point at B and asymptote $x = -a/2$: a curve of great elegance and, as we shall see, related to the cardioide in a variety of ways. Similarly, *the locus of the centre of the nine-point circle of the triangle P_1SB is the Maclaurin trisectrix* $r = \frac{a}{2\cos\Theta} - 2a\cos\Theta$, *whose double point is at the cusp of the cardioide and whose asymptote is the cardioide's double tangent.*

This curve is indicated in Fig. III.

12. Let us again turn to our circle with centre O and passing thro a luminous point A, and suppose a ray emanating from A to be reflected from the circle n times, instead of once, it is a well known theorem, given for example by Cayley,* that the envelope of the n^{th} ected ray, or the n^{th} catacaustic of the circle, is an epicycloid whose base has for radius $\frac{3a}{2n+1}$ and the radius of whose generating circle is $\frac{3na}{2n+1}$.

For $n=1$ we evidently have the cardioide treated in § 6. For $n=2$ we find the (so called) "two-cusped" epicycloid or nephroid.** Its shape and position are indicated by the dotted form of Fig. I.

For $n = 11$ we get the epicycloide of "eleven cusps" which is the curve enveloped by the line joining the hands of a watch, supposing the hands to be of equal length.

13. If S, be a luminous point of the cardioide, from which the ray SP (parallel to OC) emanates, this ray will be reflected from the curve in the direction PC, since $\angle SPR = \angle RPC$. Hence, *the catacaustic of a cardioide for a luminous cusp is the envelope, of the diameter (thro the tracing point), of its generating circle i. e. a nephroid* and indeed the very nephroid of figure I; with the same base as that of the cardioide, and generating circle with diameter RC. This interesting, tho little known theorem is due to James Bernoulli.***

14. With centre O and radius OC describe a circle. The line thro C, parallel to the X-axis makes with OC the angle θ. Hence the theorem due to Huyghens,† 1678. *The nephroid we have been considering is the catacaustic of a circle (with centre O and radius OC) for rays parallel to the X-axis.*

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* *"A memoir on caustics"*, Philos. Trans. 1857, vol. 147.

** I have followed Proctor (*Geometry of Cycloids* 1878) in using this name. It is to be noted however, that T. J. Freeth (*Proc. London Math. Soc.* 1879, X 228) so names the curve $r = a(1 + 2\sin \vartheta/2)$ a curve of which he makes use to describe a regular heptagon in a circle. The polar equation of Proctor's nephroid may be written $\left(\frac{r}{a}\right)^{2/3} = (\sin \tfrac{1}{2} \vartheta)^{2/3} + (\cos \tfrac{1}{2} \vartheta)^{2/3}$.

*** A. E. June 1692, p. 291-296.

† *Traité de la lumière*, 1690, p. 123-124. As is well known this work was written in 1678 and in that year known to members of the French Academy. The theorem was first published by Tschirnhausen, A. E. Nov. 1682, p. 364-365 but was known to him still earlier (17. April, 1681) as we see in one of his letters to Leibnitz (Vgl. *Math. Schriften v. Leibnitz herausg. v. Gerhardt* IV 484, 1859). The curve was further studied, and for the first time drawn in its completeness by Tschirnhausen A. F. April 169?, p. 169-172; Feb. 1690, p. 68-73.

15. If F_1 be the foot of the perpendicular from the cusp S of the cardioide on its tangent at any point P, the locus of F_1 is the *first positive cuspidal pedal* of the cardioide. I have named this curve a *Cayley Sextic*, for reasons which will appear later; it will evidently be tangent to the cardioide at its vertex, will pass thro its cusp, and have a double point where the double tangent of the cardioide meets its axis produced.

Suppose now that $S F_1$ be produced to F' so that $S F_1 = F_1 F'$; it is not hard to show, on making use of our introductory theorem, that $F' P$ is normal at F' to the curve traced by F'; and is, moreover the direction of the reflection from the cardioide of any ray SP emanating from S. But the locus of F', is simply the above Cayley Sextic doubled in its linear dimensions. Hence, (§ 13), *the evolute of this Cayley sextic is a nephroid* and indeed the one considered in the last two paragraphs.

16. This nephroid is, however, the evolute* of another nephroid whose cusps coincide with the vertices of the one under consideration; we can therefore state of the sextic of which it is the evolute: *If at each point P of the Cayley sextic, we take a length $PQ = 3a$ along the normal at P, toward the centre of curvature, the locus of Q will be an epicycloid (nephroid) generated by a circle of radius a rolling on a fixed circle of radius 2a, the centre of this fixed circle coinciding with the centre of the cardioide's base.*

The cardioide's pedal and its nephroid evolute are indicated in Fig. II. A more particular study of the sextic will be made in the next chapter.

17. In closing this chapter it may be well to add a few notes of which some, will be of use later.

From (1) we immediately find for the cardioide equation:

$$(3) \qquad \begin{cases} x = a(2\cos\theta - \cos 2\theta) - a \\ y = a(2\sin\theta - \sin 2\theta) \end{cases}$$

and

$$(4) \qquad (x^2+y^2+2ax)^2 = 4a^2(x^2+y^2)$$

which on transformation to the origin O, become:

$$(5) \qquad \begin{cases} x_1 = a(2\cos\theta - \cos 2\theta) \\ y = a(2\sin\theta - \sin 2\theta) \end{cases}$$

and

$$(6) \qquad (x_1^2+y^2-a^2)^2 = 4a^2(\overline{x_1-a}^2 + y^2) \quad \text{where } x_1 = x+a.$$

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* **Kiefer**, *Zwei Brennlinien des Kreises*, Frauenfeld 1892.

From (3) we find $dx = 4a\sin\frac{\theta}{2}\cos\frac{3\theta}{2}\,d\theta$; if s be the length of the arc of a cardioide measured from the cusp and if ψ be the angle which any tangent makes with the X-axis, $\frac{dx}{ds} = \cos\psi = \cos\frac{3\theta}{2}$; therefore $ds = 4a\sin\frac{\theta}{2}\,d\theta$, or

(7) $$s = 8a\left(1 - \cos\frac{\theta}{2}\right)$$

or

(8) $s = 8a\left(1 - \cos\frac{\psi}{3}\right)$ which is Whewell's intrinsic equation of the cardioide.

It will be noted that (7) is exactly the form for the length of the arc of a cycloid. This is not chance but a special case of a theorem of Steiner.*

"If any curve roll on a straight line, the length of the arc of a roulette described by any point in the plane of the curve, is equal to that of the corresponding arc of the curve's pedal, taken with respect to the generating point as origin."

Hence, if a circle roll along a straight line, any point in its circumference traces a cycloid whose length is equal to the pedal of the circle with respect to the point i. e. (§ 7), a cardioide.

18. We have already shown (§ 4) that the cardioide has one real cusp; if in (4) we substitute $1/x \backsim x$, $1/y \backsim y$ and seek the nature of the curve at the origin, defined by the resulting equation, we find that the cardioide has a cusp at each of the circular points at infinity. From the mode of generation in § 6 it is not hard to show that the cardioide is of the third class. Hence, *the cardioide is a tricuspidal, bicircular quartic of the third class.*

The term "one-cusped epicycloid" as frequently** used in speaking of the cardioide, is therefore incorrect.

Finally, results indicated by Weyr, Laguerre and Kiefer*** may be thrown into the following form:

The cardioide has a single focus and that a triple one at the centre, O, of its base.

We may consider the cardioide as a particular case ($c = o$, $a = b$), of a Cartesian Oval which is defined by the equation $r^2 - 2(a + b\cos\theta)r + c^2 = o$, and possesses three distinct foci.†

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* Steiner, *Crelle's Jl.* 1840, XXI 35.

** E. g.: L. Orlando, Mathesis 1899, (2), IX 112.

*** Emil Weyr "Uréováni nekonečně vzdálených prvků utvarů geometrických" Casopis pro pěstávani mathematiky a fisiky (Prag) 1872 I 161-186 "Kardioida" p. 183-185. Laguerre N. A. Feb. 1878 (2) XVII 55-69 "*Sur la Cardioide*". A. Kiefer "*Ueber zwei specielle Brennlinien des Kreises*" (Progr. d. Thurgauischen Kantonschule) 1892 [Cardioide p. 1-26]. The cardioide foci are also treated by A. Fuchs "*Untersuchung der Brennpunkteigenschaften höherer algebraischen Curven*" (Diss. Marburg), 1887; Cardioide p. 33. This author's results are incorrect since he finds that the cardioide has *no* foci.

† Salmon-Fiedler, "*Analytische Geometrie der höheren ebenen Curven*", Leipzig 1873, p. 312.

CHAPTER II.

CARDIOIDE PEDALS.

Some Curves of the Family $r^m = a^m \cos m\,\theta$.

Positive Cuspidal Pedals.

19. S A is a given line of length $4a$ and the variable angles P_0SA, PSP_0, F_1SP are taken equal (Fig. 1). The perpendicular from A on SP_0 meets it in P_0; from P_0 on SP, in P; from P on SF_1 in F_1. The locus of P is a cardioide (§ 7) with cusp S and axis SA. The locus of F_1 is its first

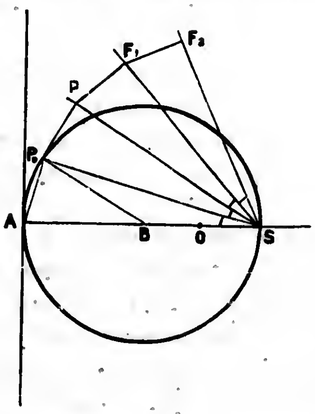

Fig. 1.

positive cuspidal pedal (§ 4), a Cayley sextic. The equations of cardioide and sextic are respectively $r^{1/2} = (4a)^{1/2} \cos \frac{\pi - \theta}{2}$ and $r^{1/3} = (4a)^{1/3} \cos \frac{\pi - \theta}{3}$. From this last equation it may be at once found that the tangent to the sextic at F_1 viz.: F_1F_2, makes with its radius vector SF_1 an angle $\frac{\pi}{2} - \left(\frac{\pi - \theta}{3}\right)$ i. e. if we add a fourth angle $F_1 S F_2 = F_1 S P =$ etc. and draw the perpendicular $F_1 F_2$ on $S F_2$, the locus of F_2 is the second cuspidal pedal of the cardioid or the third pedal of a circle with respect to a point upon it. The equation of this pedal is evidently $r = 4a \cos^4 \frac{\pi - \theta}{4}$.

The above geometrical construction can be continued; and the n^{th} positive cuspidal pedal of the cardioide $r = 2a(1 - \cos\theta)$ may be found to have the equation:

(9) $$r^{\frac{1}{n+2}} = (4a)^{\frac{1}{n+2}} \cos\frac{\pi - \theta}{n+2}$$

The corresponding pedal equation to the cordioide $r = 2a(1 + \cos\theta)$ is:

(10) $$r^{\frac{1}{n+2}} = (4a)^{\frac{1}{n+2}} \cos\frac{\theta}{n+2}$$

20. In § 7 we noted that a parabola with focus S and vertex P was tangent to $P_0 A$ at A. We can now extend the idea and deduce the following theorems: — *The locus of the vertices of the parabolas with a common focus, tangent to a fixed circle thro the focus is a Cayley sextic.**

A series of parabolas with a common focus S, are tangent to a circle thro the focus, and with centre O. The envelope of, (a): the tangents at their vertices is a cardioide with cusp S; (b): their directrices is a cardioide with cusp S and focus O. And in general: *A series of parabolas with a common focus are tangent to the n^{th} cuspidal pedal of a cardioide, (the pole being at the common focus); their vertices lie on the $(n+2)^{nd}$ pedal of the cardioide with respect to its cusp, while the tangents at the vertices envelop its $(n+1)^{st}$ pedal; their directrices envelop a curve similar and smilarly situated to the $n+1^{st}$ pedal, but of double the linear dimensions.*

21. From the construction of the cardioide and its pedals in § 19 the following theorems are evident. *The envelope of the circles described on the radii vectores of the circle $r + 4a\cos\theta = 0$ as diameters is the cardioide $r^{1/2} = (4a)^{1/2} \sin \theta/2$ whose base is the locus of the centres of the radii vectores. Conversely, the locus of the centres of circles tangent to a cardioide and passing thro its cusp, is a circle, the cardioide's base.*

The envelope of the circles described on the radii vectores of the cardioid $r^{1/2} = (4a)^{1/2} \sin\theta/2$ as diameters is its first cuspidal pedal $r^{1/3} = (4a)^{1/3} \sin\frac{\theta + 1 . \frac{\pi}{2}}{3}$; and more generally the envel-

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* This theorem is due to STREBOR N. A. 1848, VII, 45; geometrical solution by EMERY (ibid) p. 194-195. The connection of the curve with the cardioide is not noted.

ope of the circles on the radii vectores of the n^{th} cuspidal pedal of the cardioide $r^{1/2}=(4a)^{1/2}\sin\Theta/2$ *is its* $(n+1)^{st}$ *cuspidal pedal* $r^{\frac{1}{n+3}}=(4a)^{\frac{1}{n+3}}\sin\dfrac{\Theta+(n+1)\frac{\pi}{2}}{n+3}$. *The limiting curve of all cuspidal pedals of the cardioide* $r^{1/2}=(4a)^{1/2}\sin\Theta/2$ *is the circle* $r=4a$.

22. We saw (Fig. 1) that the locus of P was a cardioide on S A as axis and tangent to PF_1 at P. Similarly, the cardioide on SP^0 as axis passes thro F_1 and is tangent to F_1F_2 at F_1: and so on. Hence, *the envelope of the cardioides on the radii vectores of the circle* $r-4a\cos\Theta=0$ *as axes, is the Cayley sextic* $r^{1/3}=(4a)^{1/3}\cos\Theta/3$. *The envelope of the cardioides on the radii vectores of the cardioide* $r^{1/2}=(4a)^{1/2}\cos\Theta/2$ *as axes, is its second cuspidal pedal* $r^{1/4}=(4a)^{1/4}\cos\Theta/4$. *In general, the envelope of the cardioides on the radii vectores of the nth cuspidal pedal of the above cardioide, as axes, is the cardioides* $(n+2)^{nd}$ *cuspidal pedal:*

$$r^{\frac{1}{n+4}}=(4a)^{\frac{1}{n+4}}\cos\frac{\Theta}{n+4}.$$

23. The Cayley Sextic $r=4a\cos^3\dfrac{\pi-\Theta}{3}$ **or** $4(x^2+y^2+ax)^3=27a^2(x^2+y^2)^2$.

Altho this curve was first found (and indeed as a cardioidal pedal) by Maclaurin,* I have so named it, because the nature of the curve, geometrical constructions for the same, and its place in the theory of catacaustics were first treated by Cayley* who did not however, indicate any connections with the cardioide.

In § 19 we have indicated one geometrical construction for the curve; this will now be followed by two others.

We have a circle of radius a whose centre ω is the origin, and a line Δ whose, equation is $x=m$. A tangent at any point Q of the circle meets Δ in T and the perpendicular from Q on Δ meets it in M. The perpendicular MN on QT is produced to P so that MN=NP. Let us find the locus of P (x, y) whose ordinate is PR. If $\angle Q\omega X=\Theta$ we find $MN=\frac{1}{2}MP=(m-a\cos\Theta)\cos\Theta$. Whence without difficulty

$$x=m-2(m-a\cos\Theta)\cos^2\Theta \qquad y=a\sin\Theta-2(m-a\cos\Theta)\sin\Theta\cos\Theta$$

or

$$x=2a\cos^3\Theta-m(2\cos^2\Theta-1) \qquad y=a\sin\Theta(2\cos^2\Theta+1)-2m\sin\Theta\cos\Theta$$

or

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* MACLAURIN, *Philos. Trans.*, 1718, no. 355. — CAYLEY, "*A supplementary memoir on caustics*", *Phil. Trans.*, 1867, Vol. 157, p. 7-16. Collected Math. Papers, V, 454-464; E. T. R. IV, 70-71, Question 1771; p. 107, Question 1812. — PLAGGE (Progr. Recklinghausen) 1868, p. 18-21. — W. J. C. MILLER, E. T. R. 1894, LX, 69. — BARISIEN, *I. M.* Sept. 1895, II, 376-877. — Prof. V. RETALI, J. S., Feb. 1897, p. 32-35 "*Note sur une courbe du sixième ordre*"; I. M. July, 1900, VII, 244-245.

(11) $$\begin{cases} x = \frac{3}{2} a \cos \theta - m \cos 2\theta + \frac{a}{2} \cos 3\theta \\ y = \frac{3}{2} a \sin \theta - m \sin 2\theta + \frac{a}{2} \sin 3\theta \end{cases}$$

CAYLEY proved that these equations also define the envelop of the circles with their centres on the above circle and tangent to Δ. Now, I say, eqns. (11) define our Cayley

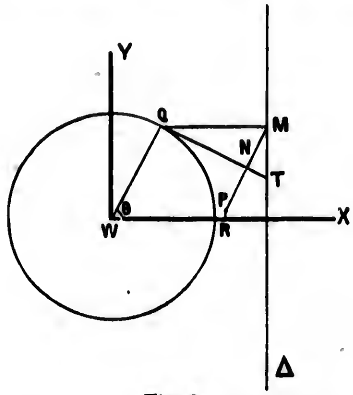

Fig. 2.

sextic referred to the middle point of O S (Fig. II) as origin, if $m = \frac{3a}{2}$; i. e. Δ is the sextic's double tangent. For:- from the general equations of an epicycloid* we find the equation of the nephroid with origin at the centre of its base (which is of radius a), to be

(12) $\quad x = \frac{3a}{2} \cos \theta + \frac{a}{2} \cos 3\theta \qquad y = \frac{3a}{2} \sin \theta + \frac{a}{2} \sin 3\theta$

where the X-axis passes thro the *vertices* of the nephroid and θ denotes the angle between the centres of the generating circles. Further, from general forms* we find that any tangent to this nephroid makes an angle $(2\theta + \frac{\pi}{2})$ with the X-axis. Hence if m denote the distance between the nephroid and any parallel curve, the equation of the parallel curve is got by adding $-m \cos 2\theta$, $-m \sin 2\theta$ to x and y respectively in (12). Whence (§ 16), the equation of our sextic may be written

(13) $$\begin{cases} x = \frac{a}{2} (3 \cos \theta - 3 \cos 2\theta + \cos 3\theta) \\ y = \frac{a}{2} (3 \sin \theta - 3 \sin 2\theta + \sin 3\theta) \end{cases}$$

* CARR, "Synopsis of pure mathematics", 1886, p. 721.

It was shown by CAYLEY that the curve is of the sixth degree and the fourth class; that it has six cusps, four double points and three double tangents. A study of the curves defined by (11) will make clear that the apparently simple point S (Fig. II), and the two circular points at infinity, are *triple* points formed by the union of two cusps and a double point; with the double point, O′, all singularities are then accounted for. The position of the conjugate imaginary double tangents is treated by Retali.

It may be noted in closing this paragraph, that if (Fig. 1) we measure off in SP_0, lengths $SF = SF_1$, the locus of F will be the curve $r = 4a\cos^3\Theta$, none other than the *simple folium*;* pedal of a *Steiner hypocycloide* or *tricuspide* with respect to a cusp.

Negative Cuspidal Pedals.

24. Returning to fig. 1 we see, that since the perpendiculars at the ends P of the cardioide radii vectors envelop the axial circle, this circle is the *first negative pedal* of the cardioide (traced by P) with respect to the cusp S; so also the *second* is the point A; the third, the line thro A perpendicular to SA, while *the fourth negative pedal is the parabola* $r^{1/2}\sin\Theta/2 = (4a)^{1/2}$, *traced by* N_4, *(Fig. IV) which is the inverse*† *of the fundamental cardioide with respect to a circle of radius 4a about its cusp.*

It will be found more convenient if we choose a parabola whose parameter is one quarter of the above, i. e. whose equation is $r(1-\cos\Theta) = 2a$. This curve will be the inverse of the cardioide with respect to a circle of radius 2a; and conversely.

Its position is indicated in Fig. I: S is the focus. The directrix of the parabola inverts into the cardioide's base, while the tangent at the vertex inverts into its axial circle. The curves cut orthogonally on the line L_1L_2 perpendicular to the axis at S.

25. Any cuspidal chord PSP' of the cardioide when produced meets the circle of inversion in P_1, P_2 and the parabola in Q', Q''. Evidently $SP.SQ' = SP'.SQ'' = (2a)^2 = SP_1^2 = SP_2'^2$. Hence, *the point-pairs (Q′P), (P′Q″) form harmonic ranges with* P_1, P_2.

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* First studied by VIVIANI *Quinto libro di Euclide, overo Scienze universale delle proporzioni spiegate colle dottrine de Galileo de Vincenzo Viviani,* Firenze 1647. — Later treated by MACLAURIN, *Geometria Organica,* 1720, p. 113. ED. BARTL, "*Ueber aie Eilinie*", (Progr. Kaadner) 1873; (Progr. Prag) 1879, p. 15-17. — G. DE LONGCHAMPS, J. S. 1886, p. 273-275; *Géom. de la Règle,* 1890, p. 126-127. — DR. ARMIN WITTSTEIN, "*Notiz über das eigentliche Oval*", A. Gr., 1895, XIV, 109-111: 241.

† First pointed out by MAGNUS, Crelle 1832, IX, 135-138; *Aufgaben u. Lehrsätzen aus d. analytischen Geometrie der Ebene,* 1833, p. 292. Properties of the cardioide have been obtained by the method of inversinn by: J. K. INGRAM, J. W. STUBBS, *Dublin Phil. Soc. Trans.,* I 1842-43: *Phil. Mag.* vol. 23, Nov. 1843. — H. M. TAYLOR, *Messenger of Math.,* April 1866. — CHAS. TAYLOR, *Ancient and modern Geometry of Conics,* 1881, p. 354-358. — WEILL, N. A. April 1881, p. 160-171, "*Note sur la cardioide et la Limaçon de Pascal*". — WEINMEISTER, "*Die Herzlinie*", Leipzig (Teubner) 1884. — RIGGS, "*On Pascal's Limaçon and the Cardioide*", Kansas Univ. Quarterly, 1892, I, 89-94.

26. Further, let us *seek the locus of the fourth harmonic point, P'', to P', S, P.* If the coordinates of this point are (r', Θ') we have,

$$2a(1-\cos\Theta') : 2a(1+\cos\Theta') = r' - 2a(1-\cos\Theta') : r' + 2a(1+\cos\Theta') \quad \text{or}$$

$r' = 2a \sin\Theta' \tan\Theta'$ *the Cissoide of Diocles of § 8 Equation (2).*

If $Q' Q''$ be any focal chord of a parabola, the fourth harmonic point to Q', S, Q'', is the point Q''' on the directrix.

The Cissoide of Diocles may therefore be considered as the directrix of the cardioide.

27. It will be interesting, and useful later, to give some applications of the theory of inversion.

Given two fixed lines cutting in Q, and a fixed point S not in either of them. Thro Q and S a circle is described cutting the fixed lines in M_1, M_2. The envelope of the line $M_1 M_2$ is a parabola with focus S. The point where $M_1 M_2$ touches the parabola is determined by the intersection of the circles thro S tangent to the fixed lines at M_1, M_2. The angular points of the triangle formed by three tangents to a parabola lie on a circle thro the focus.

Two circles intersect in Q and S. Any straight line thro Q intersects the circles again in M_1, M_2. (a) *The envelope of the circles thro S, M_1, M_2, is a cardioide with cusp S.* (b)* *The tangents to the circles in $M_1 M_2$ intersect on the same cardioide.*

From § 21 we further see that *the centres of the fixed and variable circles lie on the base of the cardioide.*

Any three circles thro the cusp S of a cardioide and tangent to the curve meet again in three points M_1, Q, M_2, which lie in a line.

Now it may be shown by inversion that: — If three circles passing thro a point, S, again intersect in three points on a line (M_1, Q, M_2), the centres of these circles lie on a circle thro S. Hence these results may be thrown into the following form: *about the triangles formed by drawing any four lines in a plane, circles are described. These circles meet in a point S and their four centres lie on a circle, $\varkappa$, thro S. Let us form a system of circles, Σ, passing thro S and with their centres on $\varkappa$. The envelope of the system Σ, is a cardioide with cusp S and base, $\varkappa$.* (Cf. the first theorem of § 21.)

28. A slight investigation of the geometry of the figure in the last paragraph leads to: — *The vertex of an angle equal to the angle of intersection of two circles, and whose sides always touch the circles, traces a cardioide.* And finally a form of P. Mansion: —** *The*

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* First shown by Siebeck from an entirely different point of view. *Crelle Jl.* 1866, vol. 66, p. 361.

** *Nouv. Corresp. Math.* 1876, II, 189; Neuberg, II, 358.

triangle SM_1M_2 whose vertex, S, is fixed, turns about the vertex, varying its size but always remaining similar to itself. If the side M_1M_2 passes thro a fixed point Q, the envelope of the circle circumscribing the triangle is a cardioide.

29. If the radius vector S P of a parabola be produced thro the focus three times its own length to K_0, K_0 is a point on the circle of curvature at P.

If one produces the radius vector PS of any point P of a cardioide, thro the cusp S to K_0, a distance $\frac{1}{3}PS$, K_0 is a point on the circle of curvature at P.

Hence *the chord of curvature thro the cusp of a cardioide is $\frac{4}{3}r$, and the radius of curvature $\frac{8a}{3}\sin\frac{\theta}{2}$.*

The centre of curvature, K, of the cardioide may then be found geometrically in a number of ways. I: In R E (Fig. I) measure off R K equal to one third of P R. II: K is the point of intersection of Q O with the normal. III: The points, P, R, K, E form a harmonic range.

30. The chord of a parabola subtending a constant angle at the focus envelops a conic having the same focus and directrix.

If a circle pass thro the cusp of a cardioide and cut off from it an arc subtending a constant angle at the cusp, it's envelope is a Limaçon with the same pole and fixed circle.

31. Finally from § 7 we infer: —

If a series of co-cuspidal cardioides intersect in a fixed point, the locus of their vertices is a parabola whose focus is the common cusp and whose vertex is the fixed point. Conversely, *If the vertices of a series of co-cuspidal cardioides lie on a parabola with focus coinciding with the common cusp, the cardioides intersect one another at the vertex of the parabola.*

32. It is a well known fact* that the first negative pedal of the parabola $r^{1/2}\sin\frac{\theta}{2}=(4a)^{1/2}$ with respect to its focus, is the curve defined by the equation:

$$r^{1/3}\cos\frac{\theta-\pi}{3}=(4a)^{1/3} \quad \text{or} \quad (-x+16a)^3=108a(x^2+y^2).$$

Since the perpendicular at P to the radius vector of any point P of the parabola, makes the same angle with the normal at P as the perpendicular from P on the parabola's

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* E. g. SALMON, Higher Plane Curves 3rd Ed 1879, p. 107. Cf. also p. 184.

axis, this cubic is the catacaustic of the parabola for parallel rays perpendicular to the axis. This fact was first shown by TSCHIRNHAUSEN*; hence I have called the curve *Tschirnhausen's Cubic;* it has been called *cubique de l'Hopital,*** and as one of a class of curves which we shall consider in the next paragraph, has received several other names.

A geometrical construction for points of the curve is at once indicated by its polar equation. For (Fig. IV) draw N_4N_5 perpendicular to SN_4 where $\angle N_5SN_4 = \angle N_4SN_3 = \angle N_3SA$. The locus of N_5 is the cubic.

In fact, this construction for the cardioide's negative pedals may be at once generalized, and the equation of the nth negative pedal found to be:

$$r^{\frac{1}{n-2}} \cos \frac{\theta - \pi}{n-2} = (4\,a)^{\frac{1}{n-2}}$$

It is observed that the nth negative cuspidal pedal of the cardioide is the inverse of the (n—4)th positive cuspidal pedal with respect to a circle of radius 4a. In particular,

Tschirnhausen's Cubic $r^{1/3} \cos \frac{\theta - \pi}{3} = (4a)^{1/3}$ *benig the inverse of Cayley's Sextic* $r^{1/3} = (4\,a)^{1/3} \cos \frac{\theta - \pi}{3}$, *is the polar reciprocal of the cardioide* $r^{1/2} = (4\,a)^{1/2} \cos \frac{\theta - \pi}{2}$ *with respect to the circle of radius* $4a$ *about the cusp.*

The polar reciprocal curve with respect to a circle of radius $2a$ is indicated in Fig. III. Its equation is $r^{1/3} \cos \frac{\theta - \pi}{3} = a^{1/3}$ and is evidently the negative pedal of the cardioide's inverse with respect to the same circle. The curve is a very interesting one.

Any tangent to it makes with the radius vector of the point of contact, an angle $\frac{\pi}{6} + \frac{\theta}{3}$. Whence, *any chord thro S cuts the curve in three points the tangents at which form an equilateral triangle.*

Tschirnhausen's cubic is connected with the *semi-cubical* or *Neil's parabola*. For, by the method of § 15, it can be shown that *the catacaustic of the cubic for the luminous point S is a semicubical parabola* [since the evolute of a parabola is a semicubical parabola].

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* A. E. 1690, p. 68-73. The curve has been also treated by: JOHN BERNOULLI, Opera, 1742, III 471-472. — L'HOPITAL, *Inf. Petits.*, 1696, § 119, p. 109-112. — CARRÉ, *Mem. acad. d. Sc.*, 1703, p. 194. — HAYES, *Fluxions*, 1704, p. 238-240. — FUSS, *Nova Acta Petrop.*, 1790, VIII, 182-200. — MILLER, BOOTH, E. T. R. 1872, XVI, 77-82; E. T., Dec. 1862; March, Nov. 1863. — *N. A.* 1863, p. 97-104. — MOESSARD, BARBIER, LUCAS, N. A. 1866, p. 21-31; 1878, p. 240; 1895, p. 5*-8*. — HAMERLE, *Le Catacaustiche della parabola, Trieste* 1877, p. 5-8. — LORD M'LAREN, Monthly Notices Roy. Astr. Soc., 1887, XLVII, p. 396 etc.; Proc. Roy. Soc. Edin. 1891, XVIII, 85. — PEESCHKE, *Die negativen Fusspunkten-Kurven der Kegelschnitte dargestellt als Rollcurven,* (Diss. Rostock), 1890, p. 7, 25, 27. — Etc.

** CAZAMIAN, N. A. 1894, p. 307.

Further, we may mention what was pointed out by Fuss, 1790; the arc $ON_5 = NP_2 + P_2N_5$ where to any point P_2 (the foot of whose ordinate is N) of the parabola, corresponds the point N_5 of the cubic.

Tschirnhausen's cubic is of the third order, the fourth class, with a single real double point and three points of inflexion (one real) at infinity. This and other properties are found by reciprocation but we will not continue the method further.

From § 20, by inversion we have: *the locus of the vertices of co-cuspidal cardioides tangent to a given line is a Tschirnhausen Cubic.*

And finally, according to the geometrical construction for the cubic which we have above indicated we deduce the theorem :

If a parabola always has its vertex on a given parabola with the same focus, the envelope of its directrix is a Tschirnhausen Cubic.

Inversion of this, gives a theorem at once evident from §§ 21, 22.

33. Now *all* the cardioid cuspidal pedals which we have been considering are but particular cases of the curves defined by the equation $r^m = a^m \cos m\theta$. If Φ be the angle which a tangent to one of these curves makes with the radius vector of the point of tangency $\Phi = \frac{\pi}{2} + m\theta$. Whence the name of Laquière* as applied to the curves: *spirale à inflexion proportionelle.* Allegret** gave the name *orthogénide*, while the name *spiral sinusoïde,* now used almost entirely, was given by Haton de Goupillière.***

Beside the cardioide pedals we get a lemniscate for $m=2$; a circle for $m=1$; a cardioide itself for $m={}^1/_2$; a parabola (i.e. a pedal) for $n=-{}^1/_2$; a rectangular hyperbola for $m=-2$.

34. I will now prove the following theorem which has several special cases of interest: *If O be the pole and P any point of the curve $r^n = a^n \cos m\theta$, and if with O for pole and P for vertex a curve similar to $r^n = a^n \cos n\theta$ be described, the envelope of all such curves is* $r^{\frac{mn}{m+n}} = a^{\frac{mn}{m+n}} \cos\frac{mn}{m+n}\theta$.

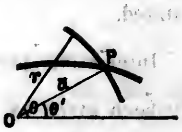

Fig. 3.

We have (Fig. 3) $\bar{a} = a\cos^{\frac{1}{m}} m\theta'$

$$r = \bar{a}\cos^{\frac{1}{n}} n(\theta-\theta') = a\cos^{\frac{1}{m}} m\theta' \cos^{\frac{1}{n}} n(\theta-\theta')$$

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* N. A. 1883, p. 118.

** *Annales de l'école normale supérieure,* (2), II, 167, 1873.

*** *"Thèse de Mecanique: sur le mouvement d'un corps* etc.", 1857, p. 33.

Differentiating and setting $dr/d\theta'$ equal to zero, we get $\sin(\overline{m+n}\,\theta' - n\theta) = 0$ whence $\theta' = \frac{n}{m+n}\theta$ and hence the equation of the envelope.

The theorems of §§ 21, 22 are at once deduced as special cases. Further, if $m = -\,{}^2/_3$ or 2 $n = 2$ or ${}^2/_3$ we have:— *The envelope of a series of lemniscates whose axes are the radii vectores of the curve* $r^{2/3} = a^{2/3}\cos\frac{2\theta}{3}$ *is a cardioide. Or, if a series of curves* $r^{2/3} = a^{2/3}\cos\frac{2\theta}{3}$ *similar tho the central pedal of the lemniscate* $r^2 = a^2\cos 2\theta$ *be described on the radii vectores of the lemniscate as diameter their envelope is a cardioide.* Similarly,

The envelope of a series of rectangular hyperbolas $r^2\cos 2\theta = a^2$, *with a common centre and vertices on the curve* $r^{2/5} = a^{2/5}\cos\frac{2\theta}{5}$ *is a cardioide. Or, the envelope of the curves* $r^{2/5} = a^{2/5}\cos\frac{2\theta}{5}$ *on the radii rectores of the hyperbola* $r^2\cos 2\theta = a^2$ *as axes is a cardioide.*

On the radii vectors of the curve $r^{3/4} = a^{3/4}\cos\frac{3\theta}{4}$ *as axes, are described the curves* $r^{3/2} = a^{3/2}\cos\frac{3\theta}{2}$. *Their envelope is a cardioide* — or, turned about, as above.

On the radii rectores of the curve $r^{3/2}\cos\frac{3\theta}{2} = a^{3/2}$ *as axes, are described the curves* $r^{3/8} = a^{3/8}\cos\frac{3\theta}{8}$; *their envelope is a cardioide:* and turned about, as above.

The envelope of the cardioides on the radii vectores of the Bernoullian Lemniscate as axes is the lemniscate's second central pedal; etc.

And in general:—

The envelope of the cardioides similar to $r^{1/2} = a^{1/2}\cos\frac{\theta}{2}$ *on the radii vectores of the curve* $r^{\frac{2}{\lambda}} = a^{\frac{2}{\lambda}}\cos\frac{2\theta}{\lambda}$ *as axes is the curve* $r^{\frac{2}{\lambda+4}} = a^{\frac{2}{\lambda+4}}\cos\frac{2\theta}{\lambda+4}$, where λ is a positive integer.

These are, of course but a selection of the infinite number of theorems which could be stated.

Interesting properties of the curve $r^{3/2} = a^{3/2}\cos\frac{3\theta}{2}$ and its inverse $r^{3/2}\cos\frac{3\theta}{2} = a^{3/2}$ have been given by W. Roberts.*

35. We will close this section, in pointing out one more theorem for spiral sinusoides, and a special case:

* *Journal de Math. pures et appliquées,* (*1*) XII, 447.

The polar reciprocal of the curve $r^m = a^m \cos m\Theta$ *with regard to the hyperbola* $r^2 \cos 2\Theta = a^2$ *is* $r^{\frac{m}{m+1}} \cos \frac{m}{m+1}\,\Theta = a^{\frac{m}{m+1}}$.

For, it is not hard to show that the equation of any tangent to the curve $r^m = a^m \cos m\Theta$ is, $x \cos \overline{m+1}\,\Theta + y \sin \overline{m+1}\,\Theta = a \cos^{\frac{m+1}{m}} m\Theta$. The equation of the polar of the point (x_1, y_1) with respect to the hyperbola $r^2 \cos 2\Theta = a^2$ is $xx_1 - yy_1 = a^2$. If these lines are idential we get $x_1 = a \cos \overline{m+1}\,\Theta / \cos^{\frac{m+1}{m}} m\Theta$, $y_1 = -a \sin \overline{m+1}\,\Theta / \cos^{\frac{m+1}{m}} m\Theta$, or

$$x_1 = a\frac{\cos\Phi}{\cos^{\frac{m+1}{m}}\frac{m\Phi}{m+1}} \qquad y_1 = a\frac{\sin\Phi}{\cos^{\frac{m+1}{m}}\frac{m\Phi}{m+1}} \quad \text{where } \Phi = -\overline{m+1}\,\Theta. \quad \text{Hence etc.}$$

Hence, (§ 32) *the polar reciprocal of the cardioide* $r^{1/2} = a^{1/2} \cos \Theta.2$ *with respect either to the hyperbola* $r^2 \cos 2\Theta = a^2$, *or the circle* $r = a$, *is the same Tschirnhausen Cubic* $r^{1/3} \cos \frac{\Theta}{3} = a^{1/3}$.

The first focal pedal of a cardioide and its inverse.

36. Taking the focus of the cardioide as origin, we see that if (ϱ, Φ) be the coordinates of the point of intersection of a perpendicular from the origin on any tangent, $T_1 T_2$, $\varrho = 3a \sin \frac{\Theta}{2}$ and $\Phi = \frac{3\Theta}{2} - \pi/2$ or $\Theta = \frac{2\Phi + \pi}{3}$; $\therefore \varrho = 3a \sin \frac{\Phi + \pi/2}{3}$. Setting for (ϱ, Φ), (r, Θ), the equation of the first *tangential* focal pedal may be written:

(14) $$r = 3a \sin \frac{\Theta + \pi/2}{3} = 3a \cos \frac{\Theta - \pi}{3}.$$

Since the focus of the cardioide is the focus of its evolute, we see (§ 10 first theorem) that the equation of the first *normal* (i. e. the normal is treated as the tangent just above) focal pedal is:

(15) $$r = a \cos \Theta/3.$$

From § 29 we further find (since the normal makes an angle $\frac{3\Theta}{2} - \frac{\pi}{2}$ with the axis) that the *Radial Curve** of the cardioide has for equation:

* The Radial Curve or Radial of a given curve, is the locus of the ends of the vectors drawn from a given point parallel and equal to the curve's radii of curvature. These curves were first named and defined by R. Tucker, E. T. Feb. 1863 etc.

(16) $$r = \frac{8a}{3} \sin \frac{\pi/2 + \Theta}{3} = \frac{8a}{3} \cos \frac{\Theta - \pi}{3}.$$

From (14) (15) (16) we infer: *the radial curve of a cardioide, and its focal, tangential and normal pedals are similar Rhodonea.* The polar reciprocal of the cardioide with respect to a circle of radius $\sqrt{\frac{3}{2}}\,a$ about the focus, is the Epi† whose equation is* $r \cos \frac{\pi - \Theta}{3} = a/2$. [For, the polar reciprocal of a curve is the inverse of a pedal of this curve].

37. Referring to the cusp of the cardioide as origin the equations of this polar reciprocal assume other forms. The equation $r \cos \frac{\Theta}{3} = a/2$ becomes $2r\left(4 \cos^3 \frac{\Theta}{3} - \cos\Theta\right) = 3a$, or $-r\cos\Theta + 4r\left(\frac{a}{2r}\right)^3 = \frac{3a}{2}$ or $-2r^3 \cos\Theta + a^3 = 3ar^2$ or $(x^2 + y^2)(2x + 3a) - a^3 = 0$. If for x we substitute $-x$, we have the Cartesian equation of the polar reciprocal, which on transformation to the cusp becomes: $(\overline{x + a}^2 + y^2)(2x - a) + a^3 = 0$ or

(17) $$(x^2 + y^2)(2x - a) + 4ax^2 = 0 \qquad \text{or}$$

(18) $$2r \cos\Theta = a(1 - 4\cos^2\Theta).$$

This last equation we at once recognise as that of the *Trisectrix of Maclaurin* treated in § 11.

Therefore, *the polar reciprocal of a cardioide with respect to a circle about its focus, is a Maclaurin Trisectrix.*

Equation (17) may be thrown ints the more usual form

(19) $$y = x\sqrt{\frac{3a + 2x}{a - 2x}}$$

Let us turn the axis about the origin thro an angle of 225°, and the equation of the trisectrix may then be written:

$$\left(\frac{x}{\sqrt{3}}\right)^3 + y^3 = 3(a2\sqrt{?})y.$$

---

* Name first applied by GUIDO GRANDI to curves of the family $r = a \cos k\vartheta$, because of the fancied resemblance to roses. They are epitrochoids and can be easily constructed geometrically. Fig. 4. The arcs AN, AH are in a given proportion k. In OH take OM equal to the perpendicular from N on OA $= a$; the locus of M is the curve $r = a \cos k\vartheta$ if ϑ be measured from a perpendicular to OA.

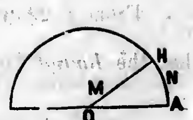

Fig. 4

† Name given by AUBRY (J. S. 1895, p. 2[illegible]1) to the inverse of the Rhodoneae: $r \cos k\vartheta = a$.

Hence, *the Trisectrix of Maclaurin passes, by the very simple transformation* $x=\sqrt{3}\,x'$, $y=y'$ *over into the Folium of Descartes*, a fact first shown by MACLAURIN*.

Equation (18) may be written:

$$r=-a\frac{\sin 3\theta}{\sin 2\theta}$$

and in the next chapter, § 45, we will see that the tangents to the trisectrix at the origin make an angle of 120° with one another. Whence,

The Trisectrix of Maclaurin is an Araneïde†: its inverse with respect to a circle about the double point is an hyperbola whose asymptotes are parallel to the tangents to the trisectrix at the double point.

The right strophoide (§ 10) is the inverse of a *rectangular* hyperbola with respect to a circle about the vertex.

38. *The Cissoide of Diocles, strophoide, and Maclaurin Trisectrix are respectively pedals of a parabola with respect to its vertex, the foot of its directrix, and the symmetrical point from the focus with regard to the directrix.*

This theorem is easily shown from the following geometrical construction for the three curves. *Given a circle with centre B and diameter OA: BO is bisected in C and produced to D and E so that* $BO=CD=OE$ *(Fig. 5). The lines* $\Delta_1, \Delta_2, \Delta_3, d_3, d_2$ *are drawn*

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* MACLAURIN, Fluxions 1742, I 262-263, French ed. 1749, p. 198. It was here that the curve was studied for the first time; and it was for this reason, (and the fact that if P_1 be any point on the curve (Fig. III), $\angle P_1OA=3\angle P_1SA$), that the curve is so named. It has been further studied by:— WASSERSCHLEBEN, "*Zur Teilung des Winkels*", A. Gr. LVI, 335-336, 1874. — WOLSTENHOLME, *Math. Problems*, 1878, no. 1840. WOLSTENHOLME, NASH TOWNSEND E. T. 1881, XXXV, 65. — B. SPORER, "*Beitrag zur Trisection des Winkels*" A. Gr. 1883, LXIX, 224. — P. S. SCHOUTE, J. S. 1885, p. 221-222; Archives Nèerlandaises XX, 76-78, 1885:— "*Sur la Construction des Courbes Unicursales par points et par tangents*". — G. DE LONGCHAMPS, "*Trisectrice de Maclaurin*", J. S. 1885, p. 176-179; *Supplément au cours de mathématique spéciales*, Ire Ed., p. 159. — *L'annuaire de l'association française, congrés de Grenoble*, 1885. — HABICH, Gaceta cientifica 1885 no. 9-12, p. 248 etc, "*Division de une angulo en Parties iguales*". — D'ALMEIDA LIMA, Journal de Sciencias Mathematicas e astronomicas (Coimbre, Portugal) 1885, p. 13, "*Sobre une curva do terciò gras*". — J. KOEHLER, "*Exercises de geometrie*", Ire Partie 1886, p. 309. — G. DE LONGCHAMPS, J. S. 1886, p. 204-206. — D'OCAGNE, J. S. 1886. p. 255-256; 1887, p. 193-199, "*Note sur la cardioide et la trisection de Maclaurin*". — G. DE LONGCHAMPS, Comptes Rendus 7. Mars 1887, CIV, 676-678: "*Sur la rectification de la trisectrice de Maclaurin au moyen des transeandantes elliptiques*". — CATALAN, Ertract of a letter dated, Feb. 10th 1888 relative to the Cardioide and Trisectrix, J. S. 1888, p. 116-119. — SVÉCHNICOFF, "*Sur la polaire reciproque de l'epicycloïde [Cardioide]*", J. S. 1890, p. 169-170. — G. DE LONGCHAMPS. *Géometrie de la régle*, 1890, p. 102-104. — H. BROCARD, J. S. 1891, p. 245-246. — LEMAIRE, MALO, N. A. 1892, (3), XI, 49-70. — DEPREZ, Mathesis 1893, Question 852, p. 274. — AUBRY, J. S. 1896, p. 82-83. — H. BROCARD, I. M., 1898, p. 104. — V. JARABEK, Mathesis 1899, p. 61-63: "*Sur la Trisectrice de Maclaurin*". — E. N. BARISIEN, J. S. June 1899, p. 139-140 [If. $a=2R$].

† Name given by Prof. W. HEYMANN to the curves $r=c\frac{\sin n\vartheta}{\sin(n-1)\vartheta}$, $r=c\frac{\sin n\vartheta}{\sin(n+1)\vartheta}$. (Schlömilch Zeitschrift "Ueber Winkeltheilung mittelst Araneïden" Nov. 1899.

perpendicular to AE thro the points A, B, C, D, E respectively. Any chord $O\delta$ of the circle, cuts Δ_1 in δ_1; Δ_2 in δ_2; Δ_3 in δ_3. Measure off $OC_1 = \delta\delta_1$; $OC_2 = \delta_2\delta$; $OC_3 = \delta_3\delta$. The locus of C_1 is the cissoide of Diocles with asymptote Δ_1; of C_2, the strophoide with asymptote d_2: of C_3 the trisectrix of Maclaurin with asymptote d_3; the double points of all three curves are at O.

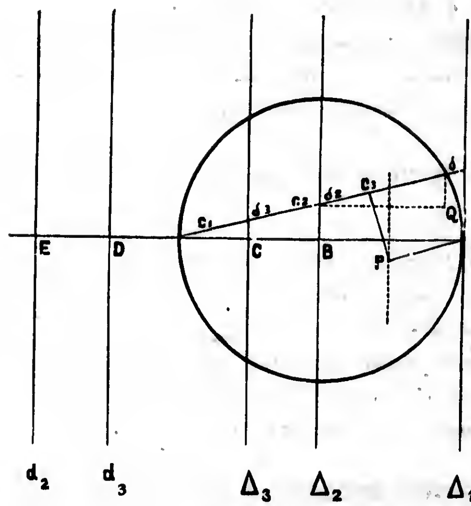

Fig. 5.

For, consider for example the trisectrix (locus of C_3); we would have to prove that the perpendicular C_3P to $O\delta$ at C_3 is tangent to the parabola with focus A and directrix Δ_2; in other words, to show that the point of intersection of C_3P and the perpendicular thereto from A, is on the tangent at the vertex of the parabola. This fact is evident, since $O\delta_3 = C_3\delta$. Suppose that $Q(x_1, y_1)$ is the point of intersection of the lines thro δ_2 and δ parallel to OA and Δ_2 respectively. If $\angle \delta O A = \theta$ and θ be chosen as origin we have $x_1 = 2a\cos^2\theta$ $y_1 = a\tan\theta$ or on eliminating θ, $y_1^2 x_1 + a^2(x_1 - 2a) = 0$ which defines the cubic commonly known in England as the *Witch of Agnesi.**

39. On expressing the condition that the line $x + iy = p$, $[i = \sqrt{-1}]$ is tangent to our cubic (17), we find that, the curve has a double focus at the origin (i. e. the cusp

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* This curve was named the *Versiera* by Agnesi in her "Instituzioni analitiche etc.", 1748. The curve was also studied by Gregory "Geom. pars universalis", 1667; Barrow "Lectiones geometricae", 1672; Newton, Fluxions.

of the cardioide see Fig. III) and two single foci at the points $-a$, $3a$. Let r_2, r_3, r_1, be the distances of any point on the curve from these three foci respectively. Then we find

$$r_2^2 = -\frac{4ax^2}{2x-a} \quad ; \quad r_3^2 = -\frac{a^3}{2x-a} \quad ; \quad r_1^2 = -a\frac{(4x-3a)^2}{2x-a}$$

Whence the vector equation of the curve:

$$r_1 \pm 2r_2 + 3r_3 = 0 \tag{20}$$

the positive sign being taken for the sinuous branch; the negative for a point on the oval. See Fig. III.

For other properties and several elegant geometrical constructions of the curve, consult the bibliography above.

CHAPTER III.

TANGENTS, NORMALS.

40. The equation of a tangent to the cardioide $r=2a(1-\cos\theta)$ at the point (r,θ) is: —

(21) $$x\sin\frac{3\theta}{2}-y\cos\frac{3\theta}{2}=4a\sin^3\frac{\theta}{2}$$

of a normal:

(22) $$x\cos\frac{3\theta}{2}+y\sin\frac{3\theta}{2}=4a\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}$$

From Equation (21) and the fact that the cardioid is of the third class, we have the theorem: — *In any direction three parallel tangents can be drawn to a cardioide: and the points of contact of such tangents subtend angles of 120° at the cusp.*

If these radii of contact be produced to cut the curve a second time, the tangents at these points will be parallel to one another, and perpendicular to the above three tangents. We see then, that there are two sets of tangents to the cardioide, which cut one another at right angles. (1) Those whose points of contact subtend angles of 180° at the cusp. (2) Those whose points of contact subtend angles of 60° at the cusp. The locus of the points of intersection of the first set, we found (§ 5) to be the circle with centre at the focus of the cardioide, and tangent at its vertex. The locus of the points of intersection of the second set is the Limaçon of Pascal whose equation is: —

(23) $$r=\frac{3a}{2}(\sqrt{3}+2\cos\theta)$$

and which is an epitrochoid, generated by the rolling of a circle of radius $\frac{3\sqrt{3}a}{4}$, on an equal circle, the point of generation being distant $3a/2$ from the centre of the rolling circle. Hence, *the complete orthoptic curve of a cardiode is, a circle and a Pascal Limaçon.**

Referred to the centre of its base, O, as origin, the Cartesian equation of the Limaçon may be written (after Woistenholme):

$$(24) \qquad 8\,(x^2+y^2-9\,a^2)^2+54\,a^2\,(x^2+y^2-9\,a^2)+81\,(2x-3\,a)\,a^3=o.$$

The curve (which is indicated in Fig. II) will evidently have a node at the point where the axis of the cardiode produced, meets the double tangent; and, the points R_1, R_2 where the double tangent meets the tangents parallel to the axis are evidently common to both parts of the orthoptic curve. It is also obvious that the Limaçon has double contact with the cardioide, at the points where the normals to the cardioide are also tangents. From (21), (22) these two points are determined from the equation $\sin^3\frac{\theta}{2}=\sin^2\left(\frac{\theta}{2}-\frac{\pi}{6}\right)\cos\left(\frac{\theta}{2}-\frac{\pi}{6}\right)$ which reduces to $\tan\frac{\theta}{2}=\pm\frac{\sqrt{3}}{5}$.

41. The more general problem of finding the *isoptic* curve of the cardioide is best treated analytically; the method is obvious, so I will merely give the final result:—

The isoptic curve of the cardioide is a Pascal Limaçon whose generating and fixed circles have the same radius $\frac{3a}{2}\frac{\sin\frac{2}{3}a}{\sin a}$, *the generating point of the moving circle being distant* $3a\frac{\sin\frac{a}{3}}{\sin a}$ *from its centre,* where a is the angle thro which one tangent (always in contact

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* This theorem was first annonnced by Prof. J. WOLSTENHOLME *Proc. London Math. Soc.* April 1878, IV 327-330 "*On the locus of the point of concourse of perpendicular tangents to a cardioid*"). His proof is analytic: a geometric one was given by W. W. BASSET (E. T. R. 1890, LIII, 45). The theorem was the subject of numerous articles in *Nouv. Corresp. Math.* Dec. 1876, II, 401: 1877, III, 58-63: 123-125: 408-410. The locus is very rarely given in its completeness. On the one hand A. CAZAMIAN (N. A. July 1894, XIII, 307) states that the locus is a *circle*, incorrectly referring for the result to LAGUERRE. On the other hand LOUCHEUR (N. A. 1892 p. 374-384) starting from a theorem of CHASLES, declares the locus to be a Pascal Limaçon. The theorem of CHASLES referred to, is given without proof in his *Aperçu historique*, 1837, p. 125: "Si d'une épicycloïde, engendrée par un point d'une circonférence de "cercle qui roule sur un autre cercle fixe, on circonscrit des angles tous égaux entre eux, leurs "sommets seront situés sur une épicycloïde allongée ou raccourcie". The particular case of the *cardioide* or hoptic curve shows that this theorem is inaccurate. One may, of course regard the orthoptic curve as *two* epitrochoids, if the circle be thought of as generated by two infinitely small circles, the tracing point of the rolling circle being distant $3a$ from its centre.

with the curve) would turn, in passing into the position of the other. When $a = \pi 2$ we get the Limaçon of the last section.

42. It may be worth while to indicate some other Limaçons which are connected with the cardioide.

The coordinates (ξ, η) of the centre of the circle on a radius of curvature of the cardioide as diameter, one easily finds (Fig. I) (O origin) to be $\xi = a/_3(4\cos\theta - \cos 2\theta)$, $\eta = a/_3(4\sin\theta - \sin 2\theta)$ since P is the point [Eqn. (5)]: $[a(2\cos\theta - \cos 2\theta), a(2\sin\theta - \sin 2\theta)]$ and R the point $[a\cos\theta, a\sin\theta]$. Whence we can deduce that the circle in question cuts the base orthogonally. For $\xi^2 + \eta^2 = a^2 + \left(\frac{4a}{3}\sin\frac{\theta}{2}\right)^2$, (§29).

The tangents from the focus of a cardioid to the circles on the radii of curvature of the curve as diameters, are of constant length and equal to the radius of the base. The centres of these circles lie on a Pascal Limaçon whose fixed and rolling circles have equal radius $\frac{2a}{3}$, the generating point of the rolling circle being distant $\frac{a}{3}$ from its centre.

43. Suppose (Fig. I) PR be divided in any constant ratio $\lambda : \mu$ instead of the ratio $1:2$ above; we find for the coordinates of such a point of division:

$\xi = \frac{a}{\lambda+\mu}[(2\lambda+\mu)\cos\theta - \lambda\cos 2\theta]$ $\quad \eta = \frac{a}{\lambda+\mu}[(2\lambda+\mu)\sin\theta - \lambda\sin 2\theta]$. Hence *the locus of the point which divides in a constant ratis $(\lambda:\mu)$ the portions of the normal to the cardioide, intercepted between it and its base, is an epitrochoide whose generating circles have equal radius $\frac{a(2\lambda+\mu)}{2(\lambda+\mu)}$ and the tracing point of the rolling circle being distant $a\lambda/(\lambda+\mu)$ from its centre.*

We can evidently extend this theorem and say: —

The locus of the points which divide the radii of curvature of a cardioide in a constant ratio is a Pascal Limaçon.

44. The focus, O, of a cardioide has some interesting properties in connection with the parallel tangents to the curve.

I. If the cusp S be origin and the tangent at any point (r, θ), meet the double tangent in p_1, the angle $p_1OS = \theta/2$. For, (Fig. I) the equation of the double tangent is $x = a/2$ and $OT_1 = 3a$; if then we erect a perpendicular to OT_1 at it's middle point Z, say, it passes thro p_1 and we have the three triangles T_1Zp_1, ZOp_1, and the triangle formed by p_1O, the axis and the double tangent, equal in all respects. So that $\angle p_1T_1O = \angle T_1Op_1 = \angle p_1OS = \theta/2$. Whence (first part of § 40) the theorem: —

If the three points p_1, p_2, p_3 where any three parallel tangents to a cardioide, cut the double tangent, are joined to the focus O, the angles p_1Op_2, p_2Op_3 are each equal to 60°.

II. We have observed, that if the tangent at any point P_1 of a cardioide meet the double tangent in p_1, $\angle p_1P_1S = \angle p_1OS = \theta/2$ Hence a circle can be described thro p_1, P_1, O, S. Conversely, if $\angle p_1P_1S = \angle p_1OS = \theta/2$ (where p_1 is a point on the double tangent, and P_1 a point on the cardioide whose polar angle is θ), the line p_1P_1, is tangent to the cardioide at P_1. But the circle cuts the double tangent a second time in p_2 and the cardioide a second time in P_2; hence p_2P_2 is tangent to the cardioide at P_2. Now by inversion of a property of the parabola* we can show that the line P_1P_2 is also tangent to the cardioide. We have then the following theorem:— *Any circle thro the cusp and focus of the cardioide cuts the curve agin in two (real) points P_1, P_2, and its double tangent in p_1, p_2. Then, the lines P_1P_2, P_1p_1, P_2p_2, are tangent to the cardioide. Conversely, any tangent to a cardioide meets the curve again in P_1, P_2. If the tangents to the curve at P_1, P_2, meet the double tangent in p_1, p_2, the points P_1, P_2, p_1, p_2, lie on a circle thro the cusp and focus of the cardioide.*

III. It was shown by Kiefer† that *the area of the triangles formed by joining the points of contact of parallel tangents to the cardioide $r = 2a(1-\cos\theta)$, is constantly equal to $9\sqrt{3a}\,4$; further, these triangles have a common centre of gravity, O, the centre of the base.*

45. The analytic work of finding the envelop of the sides of the triangles mentioned in part III of the last paragraph is very long; the final result, is however worth stating:—

The envelope of the sides of the triangles formed by joining the points of contact of parallel tangents of the cardioide $r = 2a(1-\cos\theta)$ is the Trisectrix of Maclaurin, of Chapters I and II, whose equation is $r\cos\frac{\theta-\pi}{3} = \frac{a}{2}$.

In particular, the tangents to the trisectrix at its double point make an angle of 120° with one another. For, the tangents are the lines joining the points of contact of the cardioide tangents parallel to the axis.

46. We will close this chapter in noting yet other Limaçons connected with the car-

* Given a parabola $y^2 = 4ax$ with focus S, and the fixed point $(-3a, o)$. If any line thro the fixed point cuts the parabola in P_1, P_2, the circle SP_1P_2 is tangent to the parabola.

† Kiefer. "Ueber zwei Brennlinien des Kreises" (Progr. d. Thurgauischen Kantonschule) 1892.

dioide. *The locus of the middle points* (ξ,η) *of the chords,* Δ, *of a cardioide subtending a constant angle, a, at the cusp is a Limaçon.*

For, from Equation (3),

$$2\xi = a\,[2\cos\overline{\theta - a} - \cos 2\,\overline{\theta - a} - 1 + 2\cos\theta - \cos 2\theta - 1]$$

$$2\eta = a\,[2\sin\overline{\theta + a} - \sin 2\,\overline{\theta + a} + 2\sin\theta - \sin 2\theta].$$

Hence, *the centres of the sides of the triangles enveloping the Maclaurin Trisectrix of § 45 lie on a Pascal Limaçon* and further, from § 30, *the envelope of the circles passing thro the cusp of the cardioide and the points of meeting of* Δ *with the curve, is a Limaçon.*

INDEX

of the

RELATED CURVES WITH A NAME.

No.		Cf. no.	Paragraph number
18.	$\left.\begin{matrix}\text{Radial Curve}\\\text{Rhodoneae}\end{matrix}\right\}\ r = c\cos\frac{\theta}{3}$		36.
19.	Folium of Descartes		37.
20.	Hyperbola	15.	9, 37.
21.	$\left\{\begin{matrix}\text{Witch of Agnesi}\\\text{Versiera}\end{matrix}\right\}$		38.
22.	Limaçon of Pascal	26, 27.	8, 40, 41, 42, 43, 46.
23.	Astroide		3.
24.	Positive cuspidal cardioidal pedals		19 to 23.
25.	Negative cuspidal cordioidal pedals.		24 to 32.
26.	Orthoptic Curve		40.
27.	Isoptic Curve		41.

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I, Raymond Clare, son of Abram Newcomb Archibald, was born in Nova Scotia. Canada, on the 7th of October, 1876. From the Fall of 1885 to the Spring of 1889, I was student at the *"Mount Allison Male Academy"* Sackville, New Brunswick, Canada.

In the Fall of 1889 I matriculated into the *Mount Allison University*, and received my B. A. degree with first class honors in mathematics, in the Spring of 1894; during the whole of my course I studied mathematics with Prof. Sidney W. Hunten. Further, in the Spring of 1894, I received a "*Teacher's* Diploma" from the *Mount Allison Conservatory of Music*, having completed the required three years' course in violin playing and the theory of music. In 1894-95 I was teacher of Mathematics in the Mount Allison Ladies' College and in the Spring of 1895 received an "*Artist's* Diploma" for violin playing from the *Conservatory*.

In the Fall of '95 I matriculated into *Harvard University*, Cambridge (Mass.) United States, and here, for the next three years, I continued my studies in Mathematics and Astronomy under Profs. W. E. Byerly, B. O. Peirce, J. M. Peirce, W. F. Osgood, Maxime Bôcher and Asaph Hall. At the end of my first year, 1896, I received the diploma of B. A.; at the end of the second, 1897, the diploma of M. A.; while during the third, I was pursuing advanced work.

In the Fall of 1898 I matriculated into *Berlin University* and remained two semesters, hearing lectures by Profs. Schwarz and Fuchs. Since Oct. 1899 I have studied Mathematics and Astronomy at the *University in Strassburg*, under Profs. Weber, Reye, Roth and Becker. For the invariable kindness shown to him by all, the author would like at this point to express his grateful appreciation; but especially would he like to mention, Prof. Reye who has ever been so ready with his help and counsel with regard to the accompanying thesis.

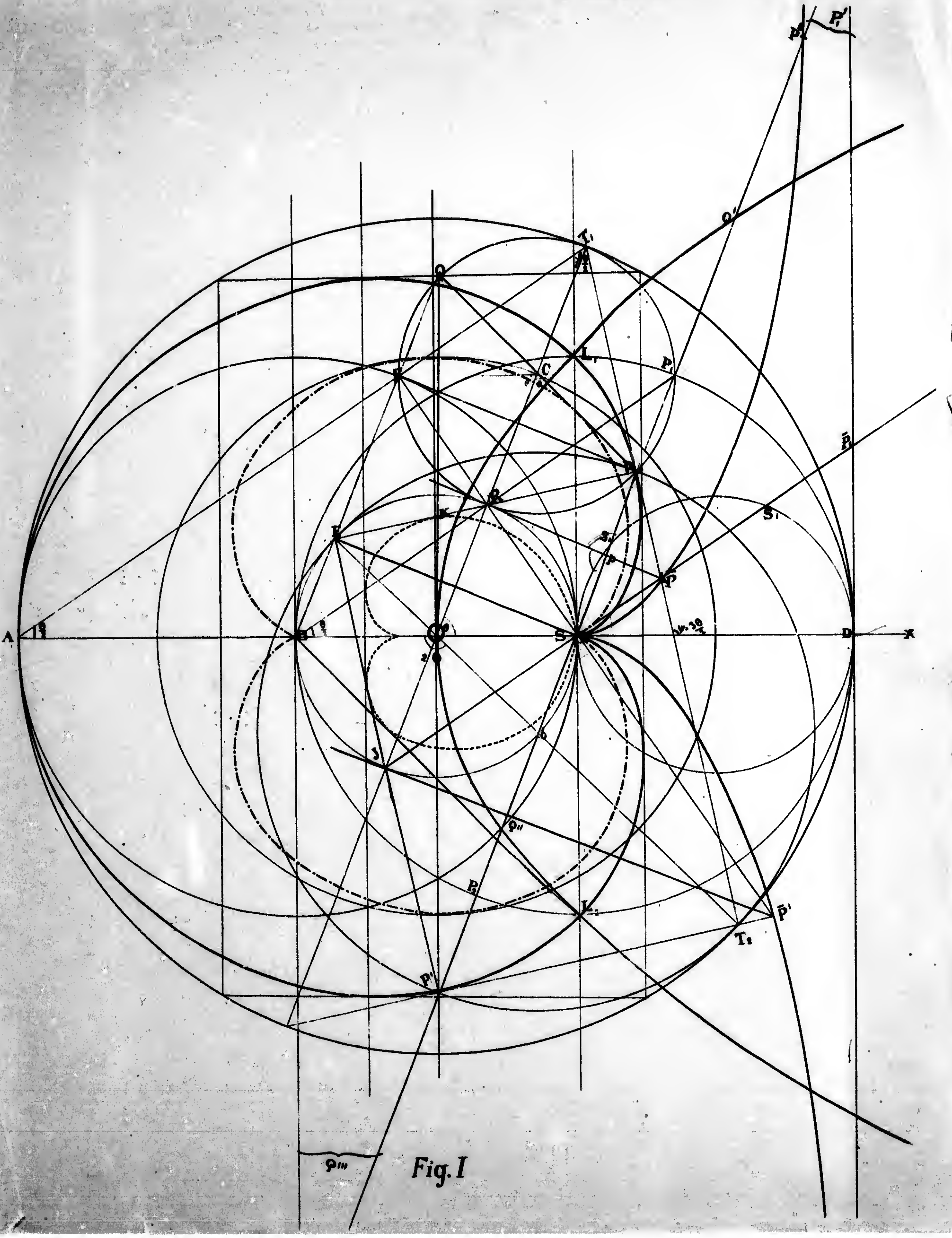


Fig. I

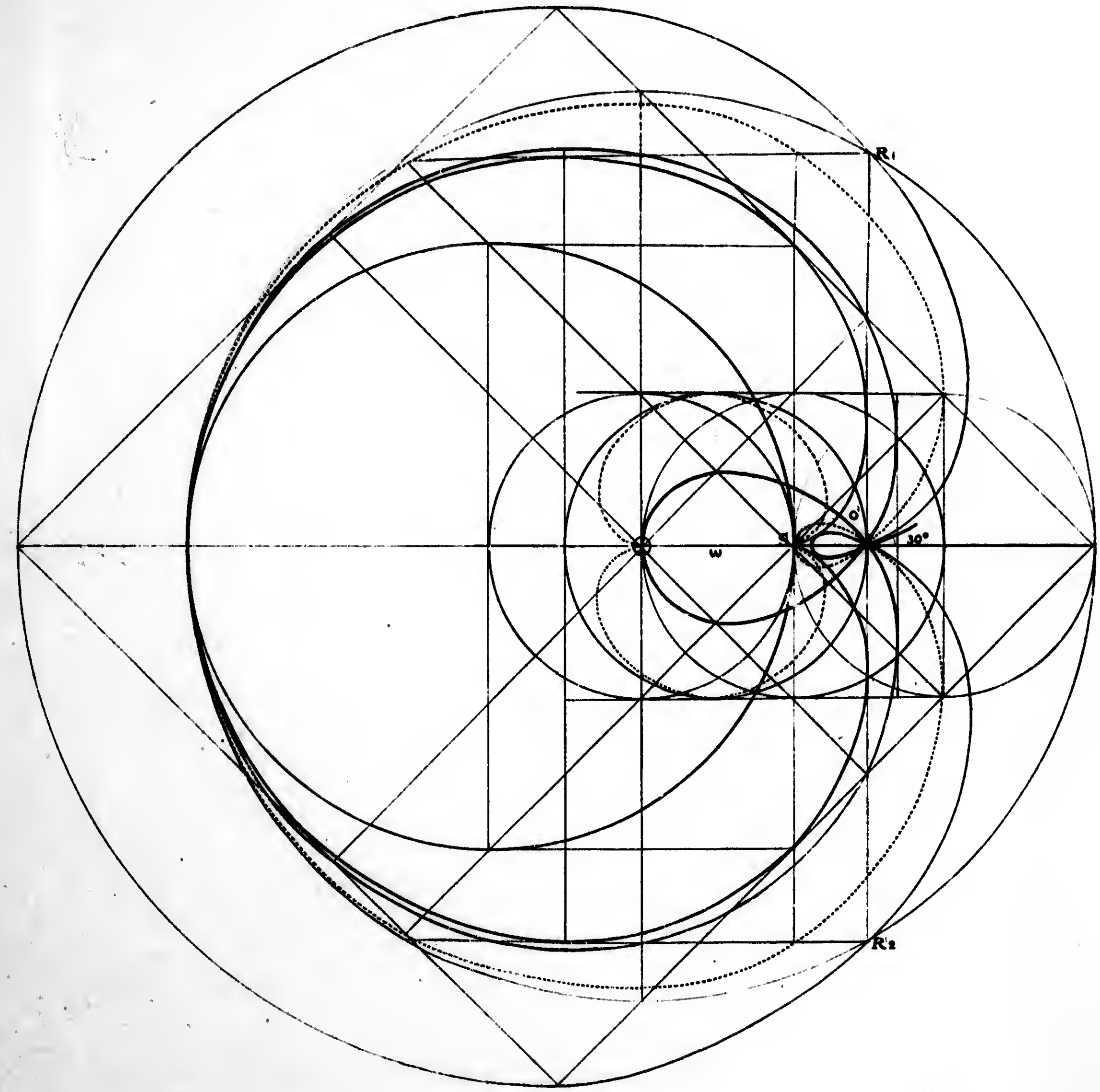


Fig. II

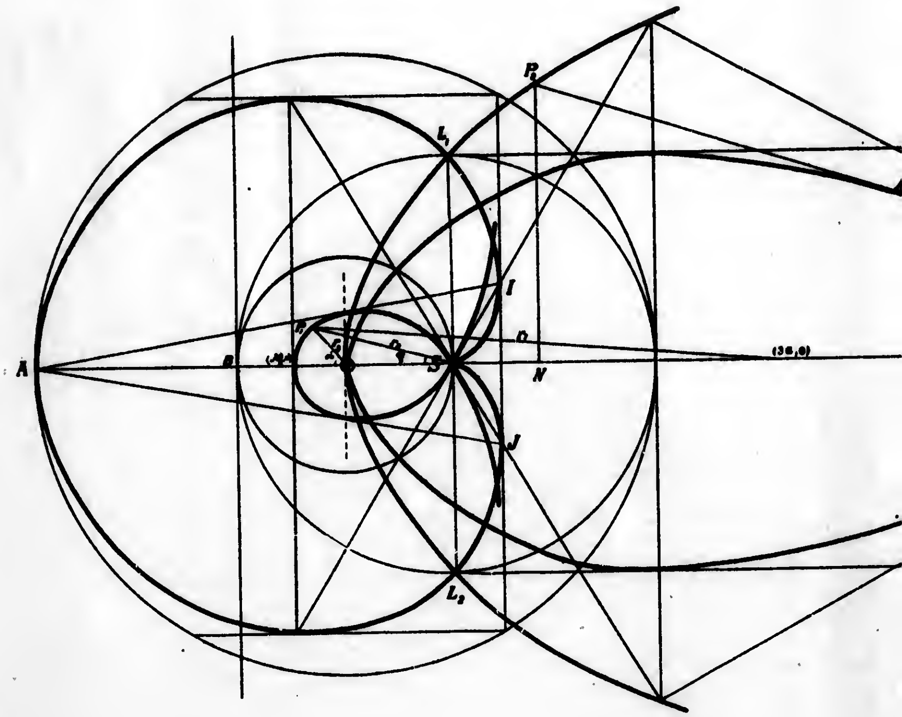
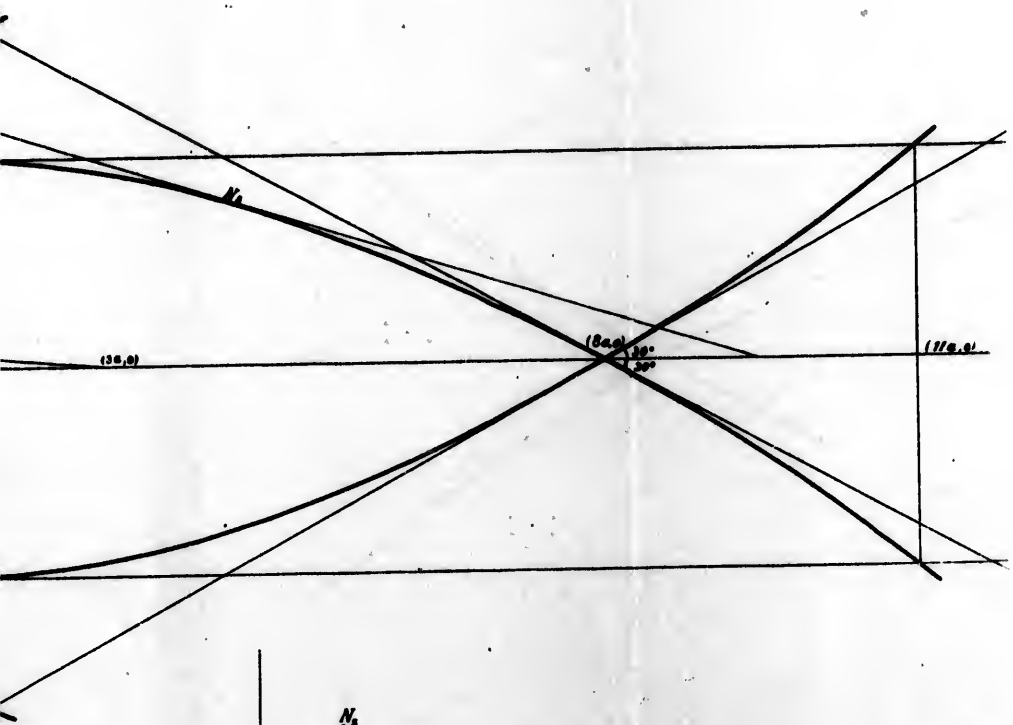


Fig. III.



ig. III.

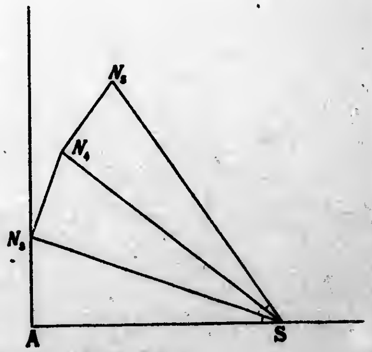


Fig. IV.