Technical and Bibliographic Notes / Notes techniques et bibliographiques

The Institute has attempted to obtain the best original copy available for filming. Features of this copy which may be bibliographically unique, which may alter any of the images in the reproduction, or which may significantly change the usual method of filming are checked below.

- [x] Coloured covers / Couverture de couleur
- [ ] Covers damaged / Couverture endommagée
- [ ] Covers restored and/or laminated / Couverture restaurée et/ou pelliculée
- [ ] Cover title missing / Le titre de couverture manque
- [ ] Coloured maps / Cartes géographiques en couleur
- [ ] Coloured ink (i.e. other than blue or black) / Encre de couleur (i.e. autre que bleue ou noire)
- [ ] Coloured plates and/or illustrations / Planches et/ou illustrations en couleur
- [ ] Bound with other material / Relié avec d'autres documents
- [ ] Only edition available / Seule édition disponible
- [ ] Tight binding may cause shadows or distortion along interior margin / La reliure serrée peut causer de l'ombre ou de la distorsion le long de la marge intérieure.
- [ ] Blank leaves added during restorations may appear within the text. Whenever possible, these have been omitted from filming / Il se peut que certaines pages blanches ajoutées lors d'une restauration apparaissent dans le texte, mais, lorsque cela était possible, ces pages n'ont pas été filmées.
- [ ] Additional comments / Commentaires supplémentaires:

L'Institut a microfilmé le meilleur exemplaire qu'il lui a été possible de se procurer. Les détails de cet exemplaire qui sont peut-être uniques du point de vue bibliographique, qui peuvent modifier une image reproduite, ou qui peuvent exiger une modification dans la méthode normale de filmage sont indiqués ci-dessous.

- [ ] Coloured pages / Pages de couleur
- [ ] Pages damaged / Pages endommagées
- [ ] Pages restored and/or laminated / Pages restaurées et/ou pelliculées
- [x] Pages discoloured, stained or foxed / Pages décolorées, tachetées ou piquées
- [ ] Pages detached / Pages détachées
- [x] Showthrough / Transparence
- [ ] Quality of print varies / Qualité inégale de l'impression
- [ ] Includes supplementary material / Comprend du matériel supplémentaire
- [ ] Pages wholly or partially obscured by errata slips, tissues, etc., have been refilmed to ensure the best possible image / Les pages totalement ou partiellement obscurcies par un feuillet d'errata, une pelure, etc., ont été filmées à nouveau de façon à obtenir la meilleure image possible.
- [ ] Opposing pages with varying colouration or discolourations are filmed twice to ensure the best possible image / Les pages s'opposant ayant des colorations variables ou des décolorations sont filmées deux fois afin d'obtenir la meilleure image possible.

This item is filmed at the reduction ratio checked below /
Ce document est filmé au taux de réduction indiqué ci-dessous.

10x	12x	14x	16x	18x	20x	22x	24x	26x	28x	30x	32x
					✓						

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)

1.0 1.1 1.25 1.4 1.6 1.8 2.0 2.2 2.5 2.8 3.2 3.6 4.0 4.5 5.0

APPLIED IMAGE Inc

1653 East Main Street
Rochester, New York 14609 USA
(716) 482 - 0300 - Phone
(716) 288 - 5989 - Fax

PRACTICAL AND THEORETICAL

GEOMETRY

PART III

BY

A. H. McDOUGALL, B.A

Principal, Ottawa Collegiate Institute

TORONTO

THE COPP, CLARK COMPANY, LIMITED

1908

CONTENTS.

PART III.

CHAPTER I.

Exercises on Loci.

1. Construct the locus of a point such that the ⊥s from it to two intersecting st. lines are in the ratio of two given line-segments.

2. A fixed point O is joined to any point A on a given st. line which does not pass through O. P is a point on OA such that the ratio of OP to OA is constant. Find the locus of P.

3. A fixed point O is joined to any point A on the circumference of a given circle, P is a point on OA such that the ratio of OP to OA is constant. Find the locus of P.

Find the locus when P is on AO produced.

4. A fixed point O is joined to any point A on a given st. line which does not pass through O. P is a point on OA such that the rect. OP.OA is constant. Show that the locus of P is a circle.

Find the locus when P is on AO produced.

5. Through a fixed point O within an ∠ YXZ draw a line-segment MON, terminated in the arms of the ∠, and such that the rect. OM.ON has a given area.

6. Find the locus of a point such that the sum of the squares on its distances from the arms of a given rt. ∠ is equal to the square on a given line-segment.

7. The locus of a point, such that the sum of its distances from two given intersecting st. lines equals a given line-segment, consists of the sides of a rectangle; and the locus of a point such that the difference of its distances from the intersecting st. lines equals the line-segment, consists of the produced parts of the sides of the rectangle.

8. Given the base of a △ and the ratio of the other two sides, find the locus of the vertex.

9. AB is a fixed chord and AC a variable chord of a given circle; find the locus of the middle point of BC.

10. Find the locus of the points from which tangents drawn to two concentric circles are ⊥ to each other.

11. Construct the locus of the centre of the circle of given radius which intercepts a chord of fixed length on a given st. line.

12. Find the locus of the centre of a circle of radius a which cuts a given circle at an ∠ A.

13. A circle rotates about a fixed point in its circumference. Show that the locus of the points of contact of tangents drawn ∥ to a fixed st. line consists of the circumferences of two circles.

14. In △ ABC, two circles touch AB at B and AC at C respectively and touch each other. Find the locus of their point of contact.

15. △s are described on a given base and having a given vertical ∠. Find the loci of the middle points of their sides.

16. In a 4-gon ABCD, AB is fixed in position, AC, BC and AD are given in length.

(*a*) Find the locus of the middle point of the other diagonal.

(*b*) Find the locus of the middle point of the line segment joining the middle points of the two diagonals.

17. What is the locus of the point P when the line segment MN which joins the feet of the ⊥s PM, PN drawn to two fixed lines OX, OY is of given length.

18. BAC is any chord passing through a fixed point A within a given circle with centre E. Circles described on BA, AC as chords touch the given circle internally at B, C respectively and cut each other at D. Show that the locus of D is a circle described on AE as diameter.

19. AB, CD are two chords of a circle, AB being fixed in position and CD of given length. Find the loci of the intersections of AD, BC and of AC, BD.

NOTE:—Use Exercises 5 and 6, § 174, Part II.

20. A and B are the centres of two circles which intersect at C; through C a st. line is drawn terminated in the circumferences at D and E. DA, EB are produced to meet at P. Find the locus of P.

21. A transversal cuts the sides BC, CA, AB of a given △ ABC at D, E, F respectively. The circumscribed circles of the △s AFE, CED cut again at P. Find the locus of P.

22. From C, any point on the arc ACB, CD is drawn ⊥ AB; with centre C and radius CD a circle is described. Tangents from A and B to this circle are produced to meet at P. Find the locus of P.

23. Two similar △s ABC, AB′C″ have a common vertex A, and the △ AB′C″ rotates in the common plane about the point A. Show that the locus of the point of intersection of CC″ and BB′ is the circumscribed circle of △ ABC.

24. If a △ ABC remains similar to itself while it turns in its plane about the fixed vertex A and the vertex B describes the circumference of a circle, find the locus of C.

25. OX, OY are two fixed st. lines and from them equal successive segments are cut off; AC, CE, etc. on OX; BD, DF, etc. on OY. Show that the middle points of AB, CD, EF, etc. lie on a st. line ∥ to the bisector of the ∠ XOY.

26. AB is the diameter of a given circle, E the centre and C any point on the circumference. Produce BC to D making CD = BC. Find the locus of the point of intersection of AC and ED.

27. A rectangle inscribed in a given △ ABC has one of its sides on BC. Show that the locus of the point of intersection of its diagonals is the line joining the middle point of BC to the middle point of the ⊥ from A to BC.

28. Any secant ABD is drawn from a given point A to cut a given circle at B and D. Through A, B and A, D respectively two circles are drawn to touch the given circle; find the locus of their second point of intersection.

29. Any chord BAC is drawn through a fixed point A within a circle. On BC as hypotenuse a rt. ∠d △ BPC is described such that A is the projection of P on BC. Find the locus of P.

30. Any circle is drawn through the vertex of a given ∠. Find the loci of the ends of that diameter which is ∥ to the line joining the points where the circle cuts the arms of the ∠.

31. Through C, a point of intersection of two given circles, a st. line ACB is drawn terminated in the circumferences at A and B. Find the locus of the middle point of AB.

32. From a fixed point P, two st. lines PA, PB, at rt. ∠s to each other, are drawn to cut the circumference of a fixed circle at A and B. Find the locus of the middle point of AB.

33. A ||gm is inscribed in a given 4-gon ABCD with sides || AC and BD. The locus of the point of intersection of the diagonals of the ||gm is the st. line joining the middle points of the diagonals of the 4-gon.

MAXIMA AND MINIMA.

2. **Definition:** —If a magnitude, such as the length of a line-segment, an angle, or an area, varies, subject to given conditions, it is said to be a **maximum** when it has its greatest possible value; and a **minimum** when it has its least possible value.

Cases of maxima and minima values have been given in Part II: see § 43; § 47, Ex. 10 and Ex. 11; § 92, Ex. 15; § 128, Ex. 18; § 140, Ex. 6 and Ex. 9; § 150, Ex. 10; § 174, Ex. 4 and Ex. 16; § 190, Ex. 5.

3.—Exercises

1. Give examples showing that if a magnitude vary [illegible]inuously, a maximum value is in a position where the magnit[illegible]s greater than in the positions close to it on either side; and a minimum value is in a position where the magnitude is less than in the positions close to it on either side.

2. Give examples showing that if a magnitude vary continuously, there must be between any two equal values of the magnitude at least one maximum or minimum value.

3. A and B are two fixed points and P is any point. Find the position of P for which $PA^2 + PB^2$ is a min.

4. Find the max. and min. st. lines from a given point to a given circle.

5. Of all chords drawn through a given point within a circle, that which is bisected at the point cuts off the min. [illegible]

6. Given two intersecting st. lines and a point within the [illegible] formed by them, of all st. lines drawn through the point and terminated in the st. lines that which is bisected by it cuts off the min. area.

7. Of all [illegible]s on the same base the isosceles [illegible] has

(a) Min. perimeter when the area is given,

(b) Max. area when the perimeter is given.

8. Of all [illegible]s having a given area, the equilateral has min. perimeter.

9. Of all rt. [illegible]d [illegible]s on the same hypotenuse the isosceles [illegible] has the max. perimeter.

10. Find a point in a given st. line such that the sum of the squares of its distances from two given points is a min.

11. A and B are two given points on the same side of a given st. line; find the point in the line at which AB subtends the max. [illegible]

12. Two towns are on opposite sides of a canal, unequally distant from it, and not opposite to each other. Where must a bridge be built, ⊥ to the sides of the canal, that the distance between the towns, by way of the bridge, may be a min.?

13. A [illegible]gm is inscribed in a [illegible] by drawing from a point in the base st. lines [illegible] to the sides. Find the position of the point for which the area of the [illegible]gm is a max.

14. The max. rectangle inscribed in a given [illegible] equals half the [illegible]

15. A is the centre of a given circle and B is a point without the circle. Draw a st. line BCD cutting the circle at C and D and such that the area of the [illegible] ACD is a max.

16. A, B are fixed points within a given circle. Find a point P on the circumference such that when PA, PB produced meet the circumference at C, D respectively, CD is a max.

17. Find the point in a given st. line from which the tangent drawn to a given circle is a min.

18. Through a point [illegible] draw the max. line segment [illegible].

(Note. Draw AB [illegible] segment FAE [illegible] point of intersection and are similar [illegible].)

RADICAL AXIS

4. **Definition**: The locus of the points from which tangents drawn to two circles are equal to each other is called the **radical axis** of the two circles.

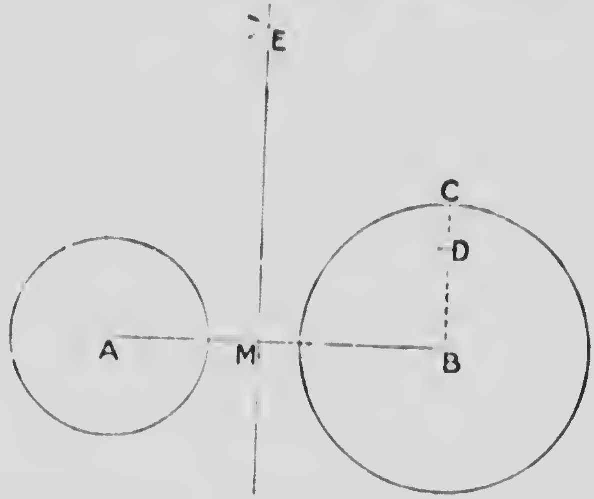
5. **Theorem**: **If two circles cut each other, their common chord produced is the radical axis.**

[*Proof left for the pupil.*]

6. A, B are the centres of two circles of radii R, r respectively and P is any point on the radical axis. Draw PM ⊥ AB. Show that $AM^2 - BM^2 = R^2 - r^2$, and thence that the radical axis is a st. line ⊥ AB.

7. Problem—To Draw the radical axis of two non-intersecting circles.

Let A, B be the centres of the two circles.



Join AB. Through B draw BC ⊥ AB cutting the circle with centre B at C, and cut off BD equal to the radius of the other circle.

With centre A and radius AD describe an arc, and with centre B and radius AC describe another arc cutting the first at E. Draw EM ⊥ AB.

Show that the st. line EM is the radical axis.

Give another method of drawing the radical axis, by describing a circle to cut the two given circles.

8. Exercises.

1. Draw two circles, radii 1 inch and 2 inches, with their centres 4 inches apart. Find a point whose tangents to the two circles are each $1\frac{1}{2}$ inches in length.

2. The radical axis of two circles bisects their common tangents.

3. Prove that the radical axis of any three circles taken two and two together meet in a point.

NOTE:—This point is called the **radical centre** of the three circles.

4. O is a fixed point outside a given circle; find a st. line such that the tangent drawn from any point P in the line to the given circle equals PO.

5. If the square on the distance between the centres of two circles equals the sum of the squares on their radii, the tangents to the circles at a common point are at right angles to each other.

NOTE:—Circles which cut each other so that the tangents at a common point are at right angles to each other are said to be **orthogonal.**

6. Two given circles intersect each other. Draw a system of circles coaxial with the given circles.

7. Draw two non-intersecting circles with centres A and B. Draw their radical axis PO cutting AB at O. From O draw a tangent OE to either circle. With centre O and radius OE describe a circle cutting AB at C and D.

The circle CED cuts the two given circles orthogonally.

Show that any circle which cuts one of the first circles orthogonally, and has its centre in PO, cuts the other orthogonally and passes through C and D.

On the circle CED take any point F, draw the tangent at F to the circle CED and let it cut AB at G. Show that the circle with centre G and radius GF is coaxial with the first two circles; and that, in this manner, a system of circles may be drawn coaxial with two given non-intersecting circles.

NOTE:—No circle of the coaxial system has its centre between C and D, and consequently these points are called the **limiting points** of the system.

8. If from any point P tangents be drawn to two circles, the difference between their squares equals twice the rectangle contained by the $\perp$ from P on the radical axis of the two circles and the distance between their centres.

9. The tangent drawn from a limiting point to any circle of a coaxial system is bisected by the radical axis.

10. Find the locus of the centre of a circle the tangents to which from two given points are respectively equal to two given line-segments.

11. If O be the orthocentre of △ABC, the circles described on AB and CO as diameters are orthogonal.

12. With a given radius describe a circle to cut two given circles orthogonally.

13. If circles are described on the three sides of a △ as diameters, their radical centre is the orthocentre of the △.

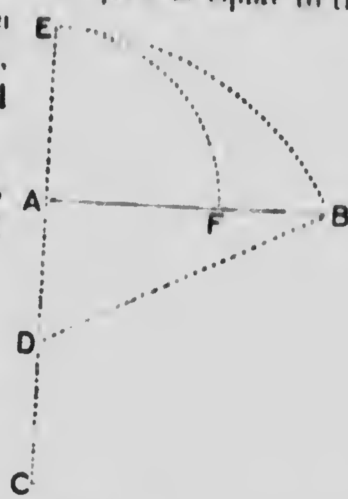
14. XYZ is the pedal △ of △ABC; YZ, BC meet in L; ZX, CA meet in M; XY, AB meet in N. Show that L, M, N are on the radical axis of the circumscribed and N. P. circles of △ABC.

(Note.—△s MAZ, MXC are easily shown to be similar.)

CHAPTER II.

Medial Section.

9. **Definition:**—When a line-segment is divided into two parts such that the square on one part is equal to the rectangle contained by the given line-segment and the other part, it is said to be divided in **medial section.**



10. **Problem:—To divide a given line-segment in medial section.**

Let AB be the given line-segment.

Draw AC ⊥ AB and = AB. Bisect AC at D. With centre D and radius DB describe an arc cutting CA produced at E. With centre A and radius AE describe an arc cutting AB at F.

Then AB is divided in medial section at F.

$$DA^2 + 2\,DA.AE + AE^2 = DE^2. \qquad \text{(II, § 86.)}$$
$$= DB^2.$$
$$= DA^2 + AB^2.$$

$$\therefore AF^2 = AB^2 - AB.AE. \qquad (2DA = AB.)$$
$$= AB\,(AB - AF).$$
$$= AB.FB.$$

11.—Exercises.

1. If a line-segment AB be divided at F so that $AF^2 = AB \cdot BF$, show that $AB : AF = AF : BF$.

Give a general statement of this result.

2. Draw a line-segment AB. Through A draw AC ⊥ AB and = AB. Bisect AC at D. With centre D and radius DB describe an arc cutting AC produced through C at F. With centre A and radius AF describe an arc cutting BA produced through A at G. Show that $AG^2 = AB \cdot BG$.

3. A given line-segment AB is to be divided in medial section. Let F be the point of section, a the length of AB, x the length of AF.

Then, by the definition of medial section $x^2 = a(a - x)$

or $x^2 + ax - a^2 = 0$.

Solving this quadratic equation $x = \dfrac{-a \pm a\sqrt{5}}{2}$.

Show that the construction in § 10 is suggested by the root $\dfrac{-a + a\sqrt{5}}{2}$; and the construction in Ex. 2 by the root $\dfrac{-a - a\sqrt{5}}{2}$.

4. Divide a st. line 4 inches in length in medial section. Measure the length of each part, and test the results by calculation.

5. The difference of the squares on the parts of a line-segment divided in medial section equals the rectangle contained by the parts.

6. On a given line-segment as hypotenuse describe a rt. ∠d △ such that the square on one side equals the rectangle contained by the hypotenuse and the other side.

7. If AB be divided at C so that $AC^2 = AB \cdot BC$, $AB^2 + BC^2 = 3AC^2$.

---

12. Problem:—To describe an isosceles triangle having each of the argles at the base double the vertical angle.

Draw a line-segment AB and divide it at H so that $AH^2 = AB \cdot BH$. (§ 10.)

Describe arcs with centres A, B and radii AB, AH respectively and let them cut at C.

Join AC, BC.

ABC is the required △.

Join HC.

∠ B is common to the △s ABC, CBH, and since

$$\frac{AB}{AH} = \frac{AH}{BH};\quad \frac{AB}{BC} = \frac{BC}{BH}$$

∴ these △s are similar and ∠ BCH = ∠ A.

Again $\frac{AC}{BC} = \frac{AB}{AH} = \frac{AH}{HB}$

∴ CH bisects ∠ ACB. (II, § 115.)

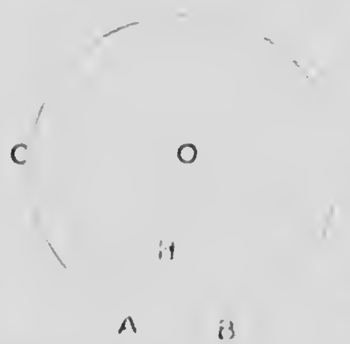
∴ ∠ ACB = twice ∠ BCH = twice ∠ A,

and also ∠ ABC = twice ∠ A.

13.—Exercises.

1. Express the ∠s of the △ ABC (fig., § 12) in degrees.

2. Show that AHC (fig., § 12) is an isosceles △ having the vertical ∠ three times each of the base ∠s.



3. In a given circle CAB draw any radius OA. Divide AO at H so that $OH^2 = AO \cdot AH$. Place the chord AB = HO.

Join BH and produce BH to cut the circumference at C. Join AC.

Show that AB is a side of a regular decagon inscribed in the circle; and that AC is a side of a regular pentagon inscribed in the circle.

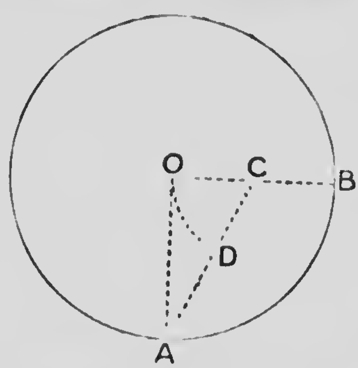
4. Show that the side of a regular decagon inscribed in a circle of radius r is $\frac{r}{2}(\sqrt{5} - 1)$.

5. The square on a side of a regular pentagon inscribed in a circle equals the sum of the squares on a side of the regular inscribed decagon and on the radius of the circle.

6. Show that the side of a regular pentagon inscribed in a circle of radius r is $\frac{r}{2}\sqrt{10-2\sqrt{5}}$.

7. In a circle of radius 2 inches inscribe a regular decagon by the method of Ex. 3. Measure a side of the decagon and check your result by calculation.

8. In a circle of radius 3 inches inscribe a regular pentagon by the method of Ex. 3. Measure a side of the pentagon and check your result by calculation.



9. In a given circle draw two radii OA, OB at rt. ∠s to each other. Bisect OB at C. Join AC and cut off CD = CO.

Show that AD is equal to a side of a regular decagon inscribed in the circle.

The regular inscribed pentagon may be drawn by joining alternate points obtained by placing successive chords each equal to AD.

10. Draw a regular pentagon on a given line-segment.

11. Circumscribe a regular pentagon about a given circle.

12. ABCDE is a regular pentagon. Show that AD, BD trisect ∠ CDE.

13. ABCDE is a regular pentagon. Show that AC, BD divide each other in medial section.

14. Construct a regular 5-pointed star. What is the measure of the ∠ at each vertex?

15. Construct a regular decagon by cutting off the corners of a regular pentagon.

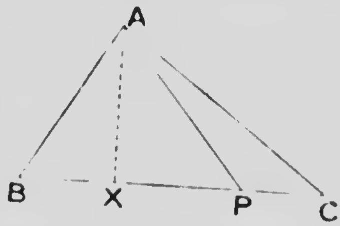
16. On a line-segment 2 inches in length describe a regular pentagon. Measure a diagonal of the pentagon and check your result by calculation.

MISCELLANEOUS THEOREMS.

14. **Theorem:—ABC is a triangle and P is a point in BC such that $\frac{BP}{PC} = \frac{n}{m}$. It is required to show that**

$$mAB^2 + nAC^2 = (m + n)\ AP^2 + mBP^2 + nCP^2.$$

Draw AX ⊥ BC.



From △ ABP,

$$AB^2 = AP^2 + BP^2 - 2BP.PX.$$

(II, § 127.)

From △ APC,

$$AC^2 = AP^2 + CP^2 + 2CP,\ PX.\quad \text{(II, § 128, Ex. 1.)}$$

Multiplying both sides of the first of these equations by m, both sides of the second by n, adding the results and using the condition $mBP = nPC$, we obtain

$$mAB^2 + nAC^2 = (m + n)\ AP^2 + mBP^2 + nCP^2.$$

15. What does the result in § 14 become when $m = n$?

In a △ ABC, $a = 77$ *mm*, $b = 90$ *mm* and $c = 123$ *mm*. Find the distances from C to the points of trisection of AB.

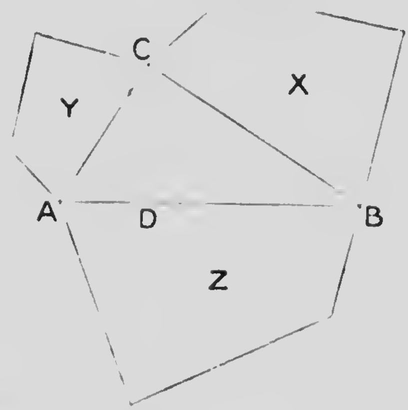
16. **In a right-angled triangle a rectilineal figure described on the hypotenuse equals the sum of the similar and similarly described figures on the other two sides.**

ABC is a △ rt.-∠d at C and having the similar and similarly described figures X, Y, Z on the sides.

It is required to show that X + Y = Z.

Draw CD ⊥ AB.

From the similar △s ABC, ACD,



AB : AC = AC : AD

$$\therefore \frac{Z}{Y} = \frac{AB}{AD} \qquad \text{(II, § 198.)}$$

$$\text{and } \therefore \frac{Y}{Z} = \frac{AD}{AB}$$

$$\text{Similarly } \frac{X}{Z} = \frac{DB}{AB}$$

$$\therefore \frac{X + Y}{Z} = \frac{AD + DB}{AB} = 1.$$

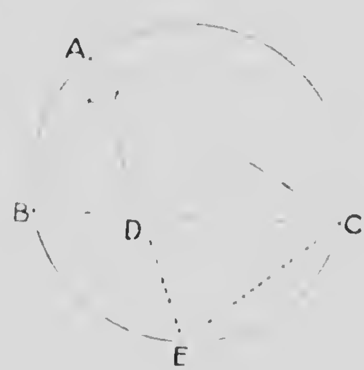
and consequently X + Y = Z.

Ex.—Prove this theorem by using §§ 197 and 121 of Part II.

17. **Theorem: If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base together with the square on the straight line which bisects the angle.**

ABC is a △ and AD the bisector of ∠ A.

It is required to show that the rect. AB.AC = rect. BD.DC + AD².



Circumscribe a circle about the △ ABC. Produce AD to cut the circumference at E. Join EC.

In △s BAD, EAC

∠ BAD = ∠ EAC, ∠ ABD = ∠ AEC.

∴ ∠ ADB = ∠ ACE and the △s are similar;

hence $\dfrac{BA}{AD} = \dfrac{EA}{AC}$

and ∴ BA.AC = AD.EA.

But AD.EA = AD (AD + DE)

= AD² + AD.DE

= AD² + BD.DC (II, § 165.)

∴ rect. BA.AC = rect. BD.DC + AD².

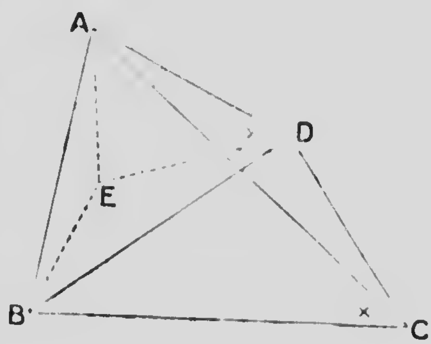
18.—Exercises.

1. If the exterior vertical ∠ A of △ ABC be bisected by a line which cuts BC produced at D, rect. AB.AC = rect. BD.CD − AD².

2. Draw △ ABC having a = 81 mm., b = 60 mm., c = 30 mm. Bisect the interior and exterior ∠s at A and produce the bisectors to meet BC and BC produced at D and E. Measure AD, AE; and check your results by calculation.

---

19. **Theorem:**—The sum of the rectangles contained by the opposite sides of a quadrilateral is not less than the rectangle contained by the diagonals.



ABCD is a 4-gon, AC, BD its diagonals.

Required to show that AB.DC + AD.BC is not less than AC.BD.

Make ∠ BAE = ∠ CAD and ∠ ADE = ∠ ACB. Join EB.

△s BAC, EAD are similar,

$\therefore \frac{BA}{AE} = \frac{CA}{AD}$ (II, § 109.)

and ∠ BAE = ∠ CAD,

∴ △s BAE, CAD are also similar. (II, § 112.)

From the similar △s BAE, CAD

$\frac{AB}{BE} = \frac{AC}{CD}$, and ∴ AB.CD = AC.BE.

From the similar △s BAC, EAD,

$\frac{BC}{AC} = \frac{ED}{AD}$, and ∴ BC.AD = AC.ED.

Consequently AB.CD + BC.AD = AC(BE + ED);

but BE + ED is not < BD;

∴ AB.CD + BC.AD is not < AC.BD.

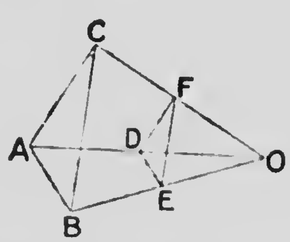
20. **Theorem**: **The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the rectangles contained by its opposite sides.**

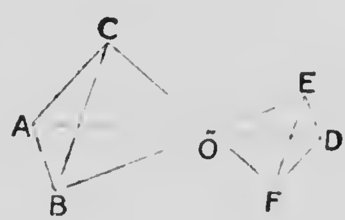
[*Proof left for the pupil.*]

---

21. **Theorem**:—If two similar triangles have their corresponding sides parallel, the st. lines joining corresponding vertices are concurrent. (See II, § 111, Ex. 13).

Let ABC, DEF be two similar △s having the sides BC, CA, AB respectively ∥ to the corresponding sides EF, FD, DE.





Prove AD, BE, CF concurrent.

Distinguish the two cases shown by the diagrams.

22. **Theorem**:—**If two similar polygons have the sides of one respectively parallel to the corresponding sides of the other, the straight lines joining corresponding vertices are concurrent.**

[*The proof is left for the pupil.*]

Distinguish two cases.

23. Exercises.

1. Inscribe a square in a given [illegible]. Show that there are three solutions.

2. In a given [illegible] inscribe a rectangle similar to a given rectangle. Show that there are six solutions.

3. In a given semi-circle inscribe a square.

4. In a given semi-circle inscribe a rectangle having its sides in a given ratio.

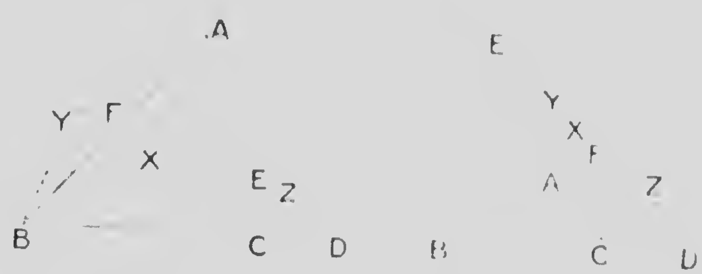
5. In a given [illegible] inscribe a [illegible] having its sides [illegible] to three given st. lines.

CHAPTER III

Theorems of Menelaus and Ceva.

21. **Menelaus' Theorem: If a transversal cut the sides BC, CA, AB of the triangle ABC in the points D, E, F respectively, AF.BD.CE = FB.DC.EA.**

(Note: The transversal must cut two sides and the third side produced, or cut all three produced.)



Draw AX, BY, CZ ⊥ to the transversal.

From similar △s

$$\frac{AF}{FB} = \frac{AX}{BY},$$

$$\frac{BD}{DC} = \frac{BY}{CZ},$$

and
$$\frac{CE}{EA} = \frac{CZ}{AX}.$$

By multiplication,

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1.$$

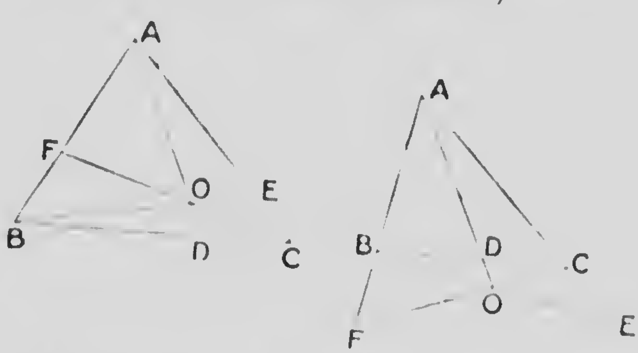
and ∴ AF.BD.CE = FB.DC.EA

25. **Converse of Menelaus' Theorem**:—**If, in △ ABC, on two of the sides BC, CA, AB and on the third produced, or if on all three produced, points D, E, F respectively be taken so that AF.BD.CE = FB.DC.EA, the points D, E, F are collinear.**

[*The proof is left for the pupil.*]

26. **Ceva's Theorem**: **If from the vertices A, B, C of △ ABC concurrent straight lines AO, BO, CO be drawn to cut BC, CA, AB at D, E, F respectively, AF.BD.CE = FB.DC.EA.**

(NOTE:—D, E and F must be on the three sides, or on one side and on the other two produced.)



FOC is a transversal of △ ABD,

∴ AF.BC.DO = FB.CD.OA. (§ 24.)

BOE is a transversal of △ ADC,

∴ AO.DB.CE = OD.BC.EA.

By multiplication, and division by DO, OA and BC,

AF.BD.CE = FB.DC.EA.

(For another proof of this theorem see II, § 101, Exercises 11 and 13.)

27. **Converse of Ceva's Theorem**:—*If, in* **△ ABC, on the three sides BC, CA, AB, or if on one of these sides and on the other two produced, points D, E, F respectively be taken so that AF.BD.CE = FB.DC.EA, the lines AD, BE, CF are concurrent.**

[*The proof is left for the pupil.*]

28. Exercises.

1. Show, by using the converse of Ceva's Theorem, that:

(*a*) The medians of a are concurrent.

(*b*) The bisectors of the ∠s of a are concurrent.

(*c*) The bisector of one ∠ at one vertex of a and the bisectors of the exterior ∠s at the other two vertices are concurrent.

(*d*) The ⊥s from the vertices of a to the opposite sides are concurrent.

(*e*) The st. lines joining the vertices of a to the points of contact of the opposite sides with the inscribed circle are concurrent.

(*f*) The st. lines joining the vertices of a to the points of contact of the opposite sides with any one of the escribed circles are concurrent.

2. If AB, CD, EF be three line segments, and AC, BD meet in N, CE, DF meet in L, and EA, FB meet in M, then L, M, N are collinear.

3. If one escribed circle of ABC touch AC at F and BA produced at G, and another escribed circle touch AB at H and CA produced at K, FH, KG produced cut BC produced in points equidistant from the middle point of BC.

4. O is a point within the ABC and AO, BO, CO produced cut BC, CA, AB at D, E, F respectively. The circle through D, E, F cuts BC, CA, AB again at P, Q, R. Show that AP, BQ, CR are concurrent.

5. If two s are so situated that the st. lines joining their vertices in pairs are concurrent the intersections of pairs of corresponding sides are collinear.

State and prove the converse.

6. The inscribed circle of △ ABC touches the sides BC, CA, AB at D, E, F respectively; EF, FD, DE, produced meet BC, CA, AB respectively in L, M, N. Show that L, M, N are collinear.

7. The points where the bisectors of the exterior angles at A, B, C of △ ABC meet BC, CA, AB respectively are collinear.

8. Tangents to the circumcircle at A, B, C meet BC, CA, AB respectively in collinear points.

Harmonic Ranges and Pencils.

29. **Definition:** A set of collinear points is called a **range.**

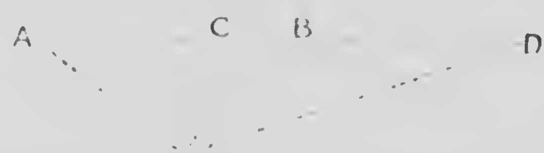
30. **Definition:** A set of concurrent straight lines is called a **pencil.**

The lines are called the **rays of the pencil**; and their common point is called the **vertex of the pencil.**

31. **Definition:** When three magnitudes are such that the first has the same ratio to the third that the difference between the first and second has to the difference between the second and third, the differences being taken in the same order, the magnitudes are said to be in **harmonic proportion.** (H. P.)

Thus, if a, b and c represent three numbers such that $a : c = b - a : c - b$, a, b and c are in H. P.

32. Take any point C in a line-segment AB.



Find the point D in AB produced such that AD : DB = AC : CB.
(H, § 107, Ex. 1.)

show that the following [illegible] line-segments are in H. P.

(*a*) AC, AB, AD

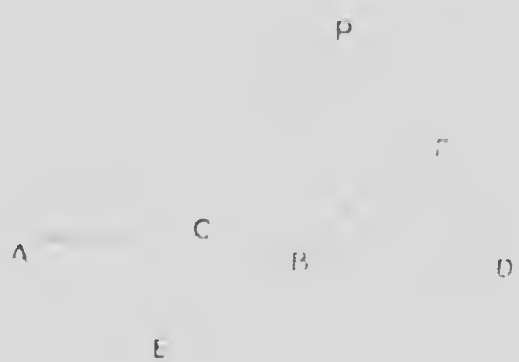
(*b*) DB, DC, DA

When four collinear points A, C, B, D are such that AC : CB = AD : DB, they form a **harmonic range**.

State and prove a converse.

Definition: If any point P be joined to the four points of a harmonic range, the joining lines form a **harmonic pencil**.

33. **Theorem: If in the harmonic pencil P (A, C, B, D), a straight line through B parallel to PA cut PC, PD at E, F respectively, BE = BF.**



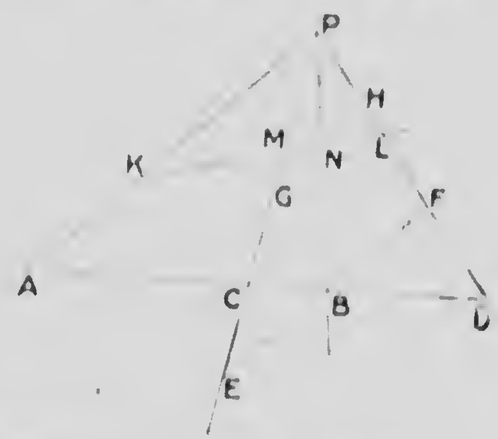
[*The proof is left for the pupil.*]

34. State and prove a converse to the theorem of § 33.

35. Give a method of finding the fourth ray of a harmonic pencil when three rays are given.

36. Theorem: Any transversal is cut harmonic by the rays of a harmonic pencil.

A transversal cuts the rays PA, PC, PB, PD of harmonic pencil P(A, C, B, D) at K, M, N, L respectiv



It is required to show that K, M, N, L is a harmoni range.

Through B, N respectively draw EF, GH ∥ PA.

[*The proof is left for the pupil.*]

Prove this theorem when the transversal cuts the ray produced through the vertex.

37. **Definition:** If A, C, B, D is a harmonic range A and B are said to be **harmonic conjugates** with respect to C and D; and C and D are said to be **harmonic conjugates** with respect to A and B.

38.—Exercises.

1. In the △ ABC the bisectors of the interior and exterior ∠s at A cut BC and BC produced at D, E respectively. Show that B, D, C, E is a harmonic range.

2. A, C, B, D is a harmonic range and P is any point on the circle described on AB as diameter. Show that PA, PB respectively bisect the exterior and interior vertical ∠s of △ CPD.

(NOTE: Draw EBF ∥ AP cutting PC, PD at E, F respectively.)

3. A, C, B, D is a harmonic range and O is the middle point of AB. Show that $OB^2 = OC \cdot OD$.

State and prove a converse.

4. A, C, B, D is a harmonic range. Show that the circles described on AB, CD as diameters cut each other orthogonally.

5. The inscribed circle of $\triangle$ ABC touches BC, CA, AB at D, E, F respectively, and DF meets CA produced at P. Show that C, E, A, P is a harmonic range.

(NOTE:—Use Menelaus' Theorem.)

6. The diameter AB of a circle is $\perp$ to a chord CD. P is any point on the circumference. PC, PD cut AB, or AB produced at E, F. Show that A, E, B, F is a harmonic range.

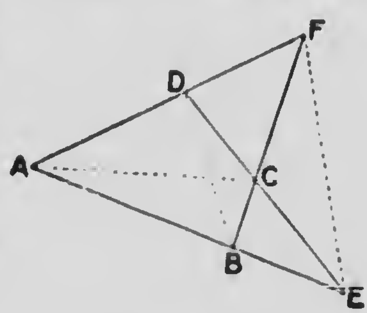
7. Through E, the middle point of the side AC of the $\triangle$ ABC, a transversal is drawn to cut AB at F, BC produced at D, and a line through B $\parallel$ CA at G. Show that G, F, E, D is a harmonic range.

8. A common tangent of two given circles is divided harmonically by any circle which is coaxial with the given circles.

9. In a circle AC, BD are two diameters at rt. $\angle$s to each other, and P is any point on the quadrant AD. Show that PA, PB, PC, PD constitute a harmonic range.

---

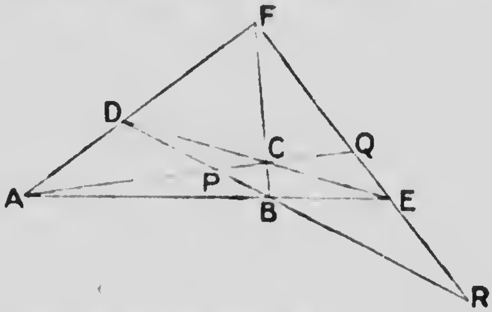
39. **Definition**:—The figure formed by four straight lines which meet in pairs in six points is called a **complete quadrilateral**.



The figure ABCDEF is a complete quadrilateral, of which AC, BD and EF are the three diagonals.

40. **Theorem:—In a complete quadrilateral each diagonal is divided harmonically by the two other diagonals, and the angular points through which it passes.**

ABCDEF is a complete quadrilateral having the diagonal AC cut by DB at P and by EF at Q.



It is required to show that A, P, C, Q is a harmonic range.

In △ ACF, AB, CD, FQ are concurrent at E and ∴, by Ceva's Theorem,

$$FD.AQ.CB = DA.QC.BF.$$

The transversal DPB cuts the sides of the △ ACF and ∴, by Menelaus' Theorem,

$$FD.AP.CB = DA.PC.BF.$$

Hence, by division,

$$\frac{AQ}{AP} = \frac{QC}{PC},$$

or,

$$\frac{AP}{PC} = \frac{AQ}{QC};$$

and ∴ A, P, C, Q is a harmonic range.

From the above result it is seen that FA, FP, FC, FQ is a harmonic pencil, and consequently, by § 36, D, P, B, R is a harmonic range.

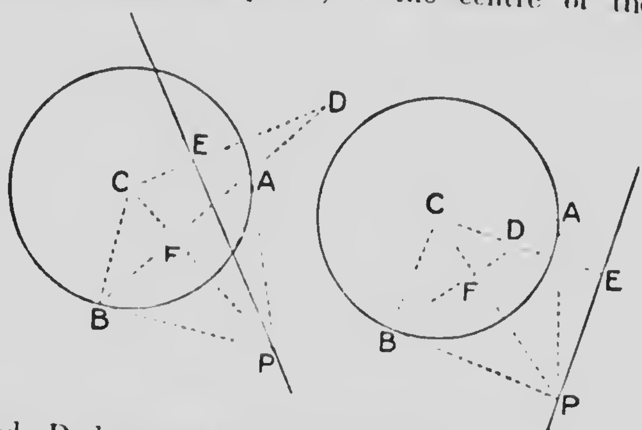
Show, in the same manner that F, Q, E, R is a harmonic range.

POLES AND POLARS.

41. **Definition**: If through a fixed point a line be drawn to cut a given circle and at the points of intersection tangents be drawn, the locus of the intersection of the tangents is called the **polar** of the fixed point; and the fixed point is called the **pole** of the locus.

42. **Problem: To find the polar of a fixed point with respect to a given circle.**

Let D be the fixed point; C the centre of the given circle.



Through D draw any st. line cutting the circle at A and B. At A, B draw tangents to the circle intersecting at P.

P is a point on the required locus.

Join CD and from P draw PE ⊥ CD.

Join CP cutting AB at F. Join CB.

∠s DEP, DFP are rt. ∠s and consequently D, E, F, P are concyclic,

and ∴ $CE.CD = CF.CP = CB^2$;

then since CD and CB are constant,

CE must also be constant.

∴ the locus of P must be the st. line ⊥ CD through this fixed point E; that is, the st. line PE is the polar of the point D.

43.—Exercises.

1. P is a point at a distance of 4 cm. from the centre of a circle of radius 6 cm. Construct the polar of P.

2. P is a point at a distance of 7 cm. from the centre of a circle of radius 5 cm. Construct the polar of P.

3. Draw a st. line at a distance of 7 cm. from the centre of a circle of radius 4 cm. Construct the pole of the line.

4. When the point P is within the given circle, the polar of P falls without the circle; and when P is without the circle, the polar of P cuts the circle.

5. The polar of a point on the circumference is the tangent at that point.

6. P is a point without a given circle and the polar of P cuts the circle at A. Show that PA is a tangent to the circle

Give a general statement of this theorem.

7. **If a point A lies on the polar of a point B with respect to a given circle, then B lies on the polar of A.**

8. If any number of points are collinear, their polars with respect to any circle are concurrent.

9. Any number of lines pass through a given point; find the locus of their poles.

10. If A and B are two points such that A lies on the polar of B with respect to a circle, and consequently, B lies on the polar of A, and if C be the intersection of the polars of A and B, then the line joining A and B is the polar of C.

Definition:—A △ such that each side is the polar of the opposite vertex is said to be **self-conjugate.**

11. If a st. line PAB cut a circle at A, B and cut the polar of P at C, and if D be the middle point of AB,

$$PA.PB = PC.PD.$$

12. Two circles ABC, ABD cut orthogonally. Show that the polar of D, any point on the circle ABD, with respect to the circle ABC passes through E, the point diametrically opposite to D.

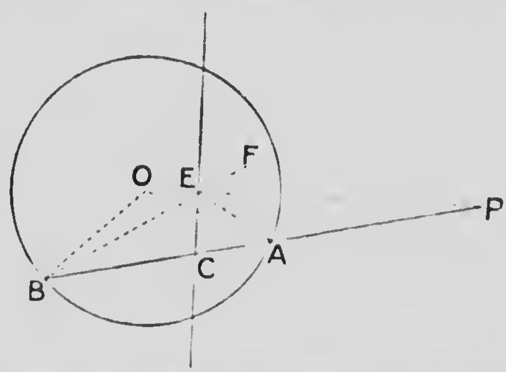
13. A is a given point and B any point on the polar of A with respect to a given circle. Show that the circle described on AB as diameter cuts the given circle orthogonally.

14. ABC is a △ inscribed in a circle, and a ∥ to AC through the pole of AB meets BC at D. Show that AD = CD.

44. Theorem :—Any straight line which passes through a fixed point is cut harmonically by the point, any circle, and the polar of the point with respect to the circle.

P is the fixed point, O the centre of the circle, PACB any line through P cutting the circle at A, B and the polar EC of P with respect to the circle at C.

It is required to show that B, C, A, P is a harmonic range.



Join BO, OA, BE, EA and produce BE to F.

$PO.OE = OA^2$, $\therefore$ $PO : OA = OA : OE$,

and $\angle$ POA is common to the $\triangle$s POA, AOE,

$\therefore$ these $\triangle$s are similar,

and consequently $\angle$ OEA = $\angle$ OAP,

$\therefore$ $\angle$ PEA = $\angle$ OAB = $\angle$ OBA.

Similarly, from $\triangle$s POB, BOE,

$\angle$ OBA = $\angle$ OEB = $\angle$ FEP.

$\therefore$ $\angle$ FEP = $\angle$ PEA;

and since PEC is a rt. $\angle$, $\angle$ BEC = $\angle$ CEA.

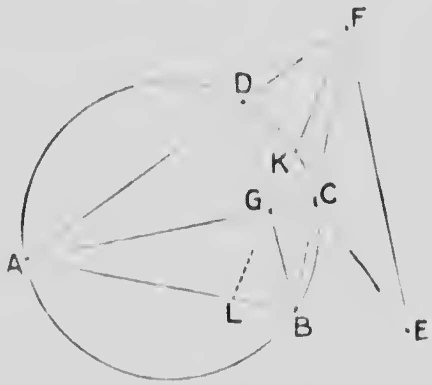
Hence $\frac{BC}{CA} = \frac{BE}{EA} = \frac{BP}{PA}$;

and B, C, A, P is a harmonic range.

Prove this theorem when the line PAB passes through the centre of the circle.

Prove this theorem when the point P is within the circle.

15. ABCD is a 4-gon inscribed in a circle. AB, DC are produced to meet at E; BC, AD to meet at F, forming the complete 4-gon ABCDEF.



AC cuts BD at G, FG cuts AB at L.

From the complete 4-gon FDGCAB, A, L, B, E and D, K, C, E are harmonic ranges, (§ 40.)

∴ L and K are points on the polar of E; (§ 41.)
that is, GF is the polar of E.

Similarly, GE is the polar of F.

Hence FE is the polar of G; and the △ EFG is self-conjugate with respect to the circle ABC.

46.—Exercises.

1. Tangents AB, AC are drawn to a circle. The tangent at any point P cuts BC, CA, AB at X, Y, Z respectively. Show that X, Z, P, Y is a harmonic range.

2. If PM, QN be respectively drawn ⊥ to the polars of Q, P with respect to a circle whose centre is O, PM : QN = OP : OQ.

MISCELLANEOUS EXERCISES.

47.—Theorems.

1. **If p be the length of the $\perp$ from A to BC and d the length of the diameter of the circumscribed circle of $\triangle$ ABC, then $pd = bc$.**

2. Show that the area of $\triangle$ ABC $= \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a + b + c)$.

3. If R be the radius of the circumscribed circle and $\triangle$ the area of $\triangle$ ABC, then 4R $\triangle = abc$.

Ex.—$a = 65$ mm., $b = 70$ mm. $c = 75$ mm.: find R.

Ex.—$a = 7$ cm., $b = 8$ cm., $c = 9$ cm; find R.

4. If L, M, N be the centres of the escribed circles of $\triangle$ ABC, the circumscribed circle of $\triangle$ ABC is the N.—P. circle of $\triangle$ LMN.

5. In Ex. 4, if I be the centre of the inscribed circle, P the point where the circumscribed circle cuts IL and PH be $\perp$ AC, AH equals half the sum and CH half the difference of b and c.

6. If I be the inscribed centre, S the circumscribed centre and O the orthocentre of $\triangle$ ABC, AI bisects $\angle$ SAO.

7. If O be the orthocentre of $\triangle$ ABC, A, B, C, O are the centres of the circles which touch the sides of the pedal $\triangle$.

8. The Simpson's line of a point P bisects PO, where O is the orthocentre of $\triangle$ ABC.

9. CA, CB are two tangents to a circle; E is the foot of the $\perp$ from B on the diameter AD; prove that CD bisects BE.

10. The $\perp$ from the vertex of the rt. $\angle$ on the hypotenuse of a rt.-$\angle$d $\triangle$ is a harmonic mean between the segments of the hypotenuse made by the point of contact of the inscribed circle.

11. The side of a square inscribed in a $\triangle$ is half the harmonic mean between the base and the $\perp$ from the vertex to the base.

12. The circumscribed centre of a $\triangle$ is the orthocentre of the $\triangle$ formed by joining the middle points of its sides; and the two $\triangle$s have a common centroid.

13. ABC is a △. Describe a circle to touch AC at C and pass through B. Describe another circle to touch BC at B and pass through A. Let P be the second point of intersection of these circles. Show that ∠ ACP = ∠ CBP = ∠ BAP; and that the circumscribed circle of △ APC touches BA at A. Find another point Q such that ∠ QBA = ∠ QAC = ∠ QCB.

14. O is the orthocentre of △ ABC, AX, BY, CZ are the ⊥s from A, B, C on the opposite sides, BD is a diameter of the circumscribed circle. Show that:—

(a) DC = AO;

(b) $AO^2 + BC^2 = BO^2 + CA^2 = CO^2 + AB^2$ = the square on the diameter of the circumscribed circle.

15. If a △ be formed with its sides equal to AD, BE, CF, the medians of △ ABC, the medians of the new △ will be respectively three-fourths of the corresponding sides of the original △.

16. The opposite sides of a 4-gon inscribed in a circle are produced to meet; show that the bisectors of the two ∠s so formed are ⊥ to each other.

17. AG is a median of the △ ABC. BDEF cuts AG, AC and the line through A ∥ BC at D, E, F respectively. Show that B, D, E, F is a harmonic range.

18. If A, C, B, D be a harmonic range, show that

$$\frac{2}{AB} = \frac{1}{AC} + \frac{1}{AD}.$$

19. A, B are the centres of two circles. A common tangent, direct or transverse, cuts the line of centres at S. Show that the corresponding ends of two ∥ diameters of the circles are in the same st. line with S.

20. A, B are the centres of two circles of radii R, r. The transverse and direct common tangents intersect the line of centres at P, Q. Show that

$$\frac{AP}{PB} = \frac{AQ}{QB} = \frac{R}{r}.$$

NOTE:—If A, B be the centres of two circles, and points P, Q be found in AB and AB produced such that $\frac{AP}{PB} = \frac{AQ}{QB} = \frac{R}{r}$, the points P, Q are called the **centres of similitude** of the circles.

21. If a circle touch two fixed circles, the line joining the points of contact passes through a centre of similitude of the two circles.

22. In a system of coaxial circles the two limiting points and the points in which any one circle of the system cuts the line of centres form a harmonic range.

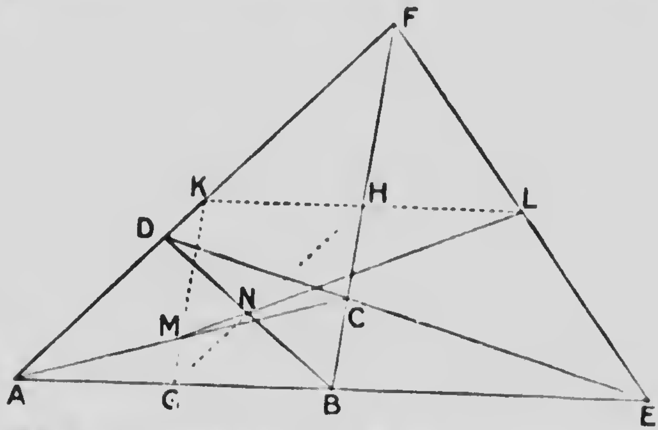
23. If two circles cut orthogonally, any diameter of one which cuts the other is cut harmonically by that other.

24. The six centres of similitude of three circles lie three by three on four st. lines.

25. The sides AB, BC, CD, DA of a 4-gon touch a circle at E, F, G, H respectively. Show that AC, BD, EG and FH are concurrent.

26. A and B are the centres of two circles and C the point where their radical axis cuts AB. Show that the locus of the centres of circles which bisect the two circumferences is a st. line ∥ to the radical axis and cutting AB at D so that AD = BC.

27. The middle points of the diagonals of a complete 4-gon are collinear.



ABCDEF is a complete 4-gon; L, M, N the middle points of its diagonals.

Draw LHK ∥ AE, KMG ∥ FB and join GH.

Prove L, M, N collinear.

28. ABCD is a 4-gon and O is a point within it such that △ AOB + △ COD = △ BOC + △ AOD; show that the locus of O is the st. line joining the middle points of the diagonals AC and BD.

29. If a 4-gon be circumscribed about a circle, the centre of the circle is in the st. line joining the middle points of the diagonals.

30. G is any point in the base BC of the △ ABC and O is a point within the △ such that △ AOB + △ COG = △ AOC + △ BOG; show that the locus of O is the st. line joining the middle points of BC and AG.

31. G is the point of contact of the inscribed circle of △ ABC with BC. It is required to show that the centre of the circle is in the st. line joining the middle points of BC and AG.

MISCELLANEOUS EXERCISES.

48.—Problems.

1. Draw a line-segment terminated in the circumferences of two given circles, equal in length to a given line-segment, and ∥ to a given st. line.

2. Through a given point on the circumference of a circle draw a chord which shall be bisected by a given chord.

3. In the hypotenuse of a rt.-∠d △ find a point such that the sum of the ⊥s on the arms of the rt. ∠ equals a given line-segment. What are the limits to the length of the given line-segment.

4. In the hypotenuse of a rt.-∠d △ find a point such that the difference of the ⊥s on the arms of the rt. ∠ equals a given line-segment.

When will there be two, one or no solutions?

5. In the hypotenuse of a rt.-∠d △ find a point such that the ⊥s on the arms of the rt. ∠ are in a given ratio.

6. Through a given point draw a line-segment terminated in the circumferences of two given circles and divided at the given point in a given ratio.

7. In a given circle inscribe a rectangle having its perimeter equal to a given line-segment.

8. In a given circle inscribe a rectangle having the difference between adjacent sides equal to a given line-segment.

9. In a given circle inscribe a rectangle having its sides in a given ratio.

10. In a circle of radius 5 cm. inscribe a rectangle having its area 22 sq. cm.

11. A and B are fixed points on the circumference of a given circle. Find a point C on the circumference such that CA, CB intercept a given length on a fixed chord.

12. A and B are fixed points on a circumference. Find a point C on the circumference such that CA, CB cut a fixed diameter at points equally distant from the centre.

13. In a given △, find a point such that the ⊥s from it to the sides are proportional to the lengths of the sides.

14. Two towns are on different sides of a straight canal, at unequal distances from it, and not opposite to each other. Where must a bridge be built ⊥ to the direction of the canal so that the towns may be equally distant from the bridge?

15. Divide a given line-segment into two parts so that the squares on the two parts are in the ratio of two given line-segments.

16. Construct the locus of a point the difference of the squares of whose distances from two points 3 inches apart is 5 square inches.

17. Two points A and B are four inches apart. Construct the locus of the point the sum of the squares of whose distances from A and B is 20.5 square inches.

18. Divide a given line-segment into two parts such that the sum of the squares on the whole line-segment and on one part is twice the square on the other part.

19. Two non-intersecting circles have their centres at A and B, and C is a point in AB. Draw a circle through the point C and coaxial with the two given circles. (Note:—Use Ex. 22, § 47.)

20. Construct a △ having one side and two medians equal to three given line-segments. (Two cases.)

21. Construct a △ having the three medians equal to three given line-segments.

22. Given the vertical ∠, the ratio of the sides containing it, and the diameter of the circumscribing circle; construct the △.

23. Given the feet of the ⊥s drawn from the vertices of a △ to the opposite sides; construct the △.

24. Draw a circle to touch a given circle, and also to touch a given st. line at a given point.

25. Draw a circle to pass through two given points and touch a given circle.

26. Draw a circle to pass through a given point and touch two given intersecting st. lines.

27. A and B are two fixed points without a given circle. Find a point P on the circumference, such that $PA^2 + PB^2$ is a minimum.

28. AB is the chord of a given segment of a circle. Find a point P on the arc such that AP + BP is a maximum.

29. Find a point O, within a △ ABC such that:—

(1) △ AOB : △ BOC : △ COA = 1 : 2 : 3;

(2) △ AOB : △ BOC : △ COA = $l : m : n$.

30. Find a point such that its distances from the three sides of a △ may be proportional to three given line-segments.

31. Through a given point within a circle draw a chord which shall be divided in a given ratio at the given point.

h a

h a

two

d a
m.

oint

of a

hich