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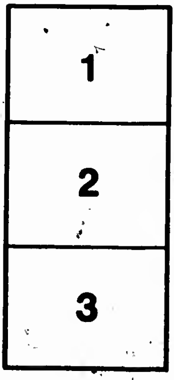

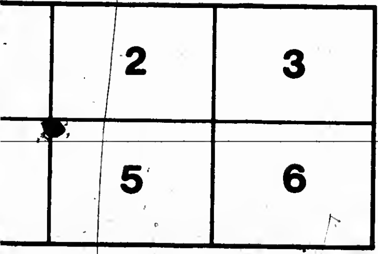

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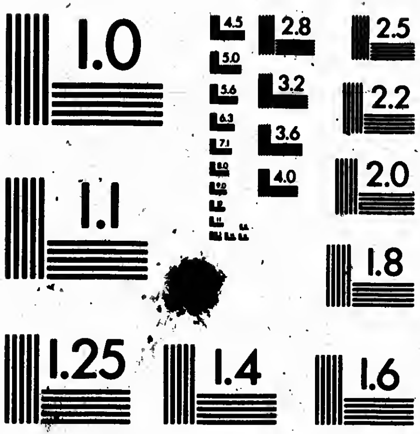

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RUDIMENTARY ALGEBRA.

Price Twenty-five Cents.

A

RUDIMENTARY

ALGEBRA.

Designed for the Use of Canadian Schools.

BY B. HENSMAN,

OF THE MIDDLE TEMPLE, BARRISTER AT LAW.

Montreal:
PUBLISHED BY R. & A. MILLER,
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1862.

PREFACE.

The object of this Elementary Work is not to displace any of the valuable Treatises on Algebra generally used in schools, nor does it assume to rank with them. It is intended simply as an introduction to the study of this most interesting science, and as a first book so to initiate the pupil that he may in a very short space of time enter upon the most complete and advanced text-books on the subject, undeterred by any apprehensions of great difficulties to be encountered.

The scholar who has duly attended to his instruction in Arithmetic will find that there is nothing difficult to comprehend in the principles of Algebra; he will see that there is nothing occult to master, but that his arithmetical knowledge may be applied and exercised upon a study of progressive interest and satisfaction.

The Author has endeavored to make his Treatise as far as it extended *démonstrative*, and thus to abbreviate the teacher's labor in explanation, as well as to fix the mind of the pupil on the principles upon which algebraic rules are founded.

With so humble an object in view as here indicated, it would be out of place to enlarge upon the benefits to be derived from the study of Algebra, but those to whom the education of youth is entrusted, conversant of these benefits, will hardly fail to welcome a book having for its object initiation and guidance, if it be found to answer its design.

Montreal, July 1, 1862.

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RUDIMENTARY ALGEBRA.

CHAPTER I.

DEFINITIONS.

1. We learn by Arithmetic how to calculate with numbers. Algebra teaches us how to perform calculations by means of the letters of the alphabet.

2. Numbers possess a particular and a relative value, while letters have no value in particular or in relation to one another. Since then letters possess no particular value, if we can perform calculations with them the results obtained will admit of general application. For if, calculating with letters, we arrive at a certain result, our calculation will apply to any value we may assign to the letters. For example, we shall presently see that if we multiply the difference between a and b by the sum of a and b, the product is equal to the difference between the square of a and the square of b. Now, as we may use a and b to represent any quantities we choose, we have, by a simple algebraical operation, arrived at a

very important result, for we learn from it that if we take any two numbers whatever and multiply their difference by their sum the product will be the difference between their squares.

3. It is then very desirable to ascertain how to perform calculations which give us results admitting of universal application. Before proceeding, however, to make algebraic calculations, we must become familiar with the meaning of different signs or symbols which are used for the purpose of abbreviation.

4. $=$ is the *sign of equality*, and is read "equal to." It indicates that the quantities between which it stands are equal to one another. Thus $4 =$ twice 2 means that 4 is equal to twice 2. $a = 7$ means that the value of a is 7 in some particular problem to which the statement relates.

5. $+$ is the *sign of addition*, and is read "plus." It signifies that the quantity before which it stands is to be added. Thus $4 + 3 = 7$ means that 4 added to 3 is equal to 7. $a + b$ means that the quantity represented by a is to be added to that represented by b. If a is equal to 2 and b to 1 then $a + b = 3$.

6. $-$ is the *sign of subtraction*, and is read "minus." It signifies that the quantity before which it stands is to be subtracted. Thus $4 - 3 = 1$ means that 3 deducted from 4 is equal to or gives a difference of 1. $a - b$ means a less b, or with b subtracted. If a is equal to 6 and b to 4 then $a - b = 2$.

7. $\times$ is the *sign of multiplication*, and is read "into." It signifies that the quantities between which it occurs are to be multiplied together, thus $7 \times 3 = 21$, means that 3 times 7 are 21. Multiplication is also indicated by a dot between the quantities, or (the more usual

way) by writing the quantities together; thus $a \times b$ or $a \,.\, b$ or (the usual mode of expression) ab all mean a multiplied by b, and if $a = 2$ and $b = 3$, then $a \times b$ or $ab = 2 \times 3 = 6$.

8. $\div$ is the *sign of division*, and is read "by" or "divided by." It signifies that the quantity after which it occurs is to be divided by that which follows it. Thus $6 \div 3 = 2$ means that 6 divided by 3 is equal to 2. Division is also indicated by writing the quantities in the form of a fraction. Thus we may express the division of x by y, thus $x \div y$ or thus $\frac{x}{y}$. Each expression means x divided by y, and if $x = 6$ and $y = 3$, then $x \div y$ or $\frac{x}{y} = \frac{6}{3} = 2$. So $\frac{a+b}{c}$ means a and b added together and their sum divided by c; while $\frac{a}{b-c}$ means a divided by the difference between b and c.

9. $\therefore$ is an abbreviation for *therefore* and $\because$ for *because*.

10. ( ) { } [ ] *brackets*, indicate that the quantities enclosed by them are to be dealt with collectively and as forming but one quantity. The same is sometimes indicated by — written over the quantities. Thus $2 \times (4-1)$ or $2 \times \overline{4-1}$ means that 1 is to be subtracted from 4 and the difference 3 multiplied by 2. The result is 6, but it would have been different if there had been no bracket, for $2 \times 4 - 1 = 7$. So $x - y - z$ and $x - (y-z)$ are different in value, for the former means x with both y and z subtracted from it, while the latter means x with only the difference between y and z subtracted from it.

The fractional line has also the same effect as a bracket, since $\frac{a-b}{c}$, for example, means the division of the entire quantity $a-b$ by c.

11. When a quantity is multiplied by itself any number of times the product is termed a *power* of the quantity and is expressed by writing the *index* or *exponent* of the power, or figure denoting the number of times it is repeated, above the quantity. 3^2 means 3×3 or the second power or square of 3, and a^3 means the third power or cube of a, or $a \times a \times a$.

12. The *root* of a quantity is that quantity which multiplied by itself a certain number of times according to the index of the root will produce the quantity of which the root is sought. Roots are indicated by the symbol $\sqrt{}$ called the *radix*, with a small figure written to the left (the index), expressing the root to be extracted. Thus $\sqrt[3]{8}$ means the cube root of 8. $\sqrt{x}$ means the square root of x, for where $\sqrt{}$ occurs with no small figure written to the left it always indicates the *square root*.

The root of a quantity is also indicated by writing a small fraction with the index of the root for denominator above the quantity; thus $x^{\frac{1}{2}}$ and $\sqrt{x}$ are equivalent expressions.

13. Having now become acquainted with algebraic signs we must investigate the nature of algebraic quantities and we shall then be able to pass on to algebraic calculations.

14. If no sign is prefixed to a quantity $+$ is understood. All quantities to which $+$ is prefixed or which have no sign prefixed are called *positive* or additive quantities, and all quantities to which — is prefixed are

called *negative* or subtractive quantities. In the expression $8-6$, 8 is positive but -6 is negative, for -6 means 6 subtracted. In $a-b$, a is positive, $-b$ is negative.

15. The *coefficient* is the number prefixed to an algebraic quantity. In the expression $8xy$, 8 is the coefficient, and the expression denotes 8 times xy; when no coefficient is expressed 1 is understood; thus x means once x.

16. A quantity not connected with any other by the sign $+$ or $-$ is called a *simple quantity*. Thus ab, $-a$, x^2y are all simple quantities. But if coupled with any other quantity by the sign $+$ or $-$ the whole expression is called a *compound quantity*; thus $ab+2c$ is a compound quantity, consisting of the simple quantities ab and $2c$ added together. The several simple quantities which make up a compound quantity are called its *terms*; thus the expression $x+2y$ is a compound quantity; x is one of its terms and $2y$ the other. A quantity which consists of one term only is called a *monomial*; if it consist of two terms a *binomial*; and if of more than two terms a *multinomial*.

17. Simple quantities often consist of more than one letter; these letters are called the *factors* which make up the quantity. We have seen that ab means a multiplied by b; a and b then are the factors which form the quantity ab; and whenever a compound quantity is composed of two or more quantities multiplied together, these quantities are similarly called factors, the term factors being employed to represent any quantities, simple or compound, that are multiplied together.

18. The value of a simple quantity remains the same in whatever order its factors be written; ab is just the same as ba, for both mean the product of a and b. It

is, however, usual to place the factors of a quantity in order of alphabetical precedence.

19. The value of a compound quantity remains the same in whatever order its terms are written, so long as the terms are prefaced by the signs which belong to them. $8ab - 2xy + y$ is the same expression as $-2xy + 8ab + y$.

20. Like quantities are those that consist of the same letter or are composed of similar factors. *Unlike* quantities are those that consist of dissimilar letters or factors. Thus $3ab$ and $2ab$ are like quantities, and so $8ax$ and $7xa$ are like quantities, for each is composed of a certain number of times the product of a and x. But x^2y and xy^2 are unlike quantities, for though the letters that enter into their composition are similar, one is composed of the factors x, x and y, and the other of the factors x, y and y.

21. The Exercise which follows is intended to familiarize the student with algebraic signs and the nature of algebraic quantities. The value of the different letters being given, he need only substitute these values in the expression, and careful attention to the signs will enable him to find the value of the whole expressions in the several examples.

EXAMPLE.—If $a = 2$, $b = 3$, and $x = 5$, what is the value of $x^2 - 2b + 2ab$?

Here by substituting the value of each letter in the different quantities we find their values. x^2 we find is 5×5 or 25; $2b = 2 \times 3$ or 6; and $2ab = 2 \times 2 \times 3$ or 12; we then substitute these values for the several quantities in the whole expression and obtain $x^2 - 2b + 2ab = 25 - 6 + 12 = 31$.

If these letters have the same value as before, what is the value of $\frac{a^2b^2+3x}{3} - 2b + bx - x - (a+b) + \sqrt{2a}$?

$$\frac{a^2b^2+3x}{3} - 2b + bx - x - (a+b) + \sqrt{2a} = \frac{36+15}{3} - 6 + 15 - 5 - 5 + 2 = \frac{51}{3} - 6 + 15 - 5 - 5 + 2 = 17 - 6 + 5 + 2 = 18.$$

If these letters have the same value as before what is the value of $a^2 + \{2bx - 3(b-a)\} - \sqrt{2a^2x + 4bx}$.

$$a^2 + \{2bx - 3\,(b-a)\} - \sqrt{2a^2x + 4\,bx} = 4 + (30-3) - \sqrt{40+60} = 4 + 27 - 10 = 21.$$

In this last example we find a double bracket, the first indicating that $2bx - 3\,(b-a)$ is all to be regarded as one quantity, and the second that $b-a$ is also to be regarded as constituting one quantity. We find $a^2 = 4$, and we put down 4 as its value; $2bx = 30$ and $3(b-a) = 3$; we therefore substitute $(30 - 3)$ for the value of $2bx - 3(b-a)$ and obviously we no longer require to use the double bracket, since we have ascertained the value of $b-a$, the quantity enclosed within the inner bracket. Then substituting $-\sqrt{40+60}$ for $-\sqrt{2a^2x+4bx}$ we complete the expression, and by continuing the simplification ascertain its numerical value.

Exercise I.

1. In the expression $a + 2ab - 8y^2 + y^2$, which of the terms are positive and which negative? Which are like and which are unlike?

2. In the expressions a^2, x^2a, x^2+y^2, $3b^2+2c^2$, which are compound and which simple quantities? Of what terms are the compound quantities composed? What are the factors of the simple quantities?

3. If $a = 7$ and $b = 5$ what is $a + b$ equal to?

4. If $x = 2$ and $y = 3$ what is $y - x$ equal to?

5. Write down the equivalent of $a + b + ab$, where $a = 2$ and $b = 3$.

6. $a = 2$ and $x = 3$: what is a^2x^2 equal to?

7. What is the value of $\sqrt{ax}$ where $a = 18$ and $x = 2$?

In the following exercises $a = 5$, $b = 2$, $c = 8$, $x = 2$, and $y = 4$.

8. Find the value of $2a + b - c + xy$.

9. Find the value of $a^2 + \sqrt{y}$.

10. Find the value of $3bx + ab - 2\sqrt{y} + ax$.

11. Find the value of $8b - cx + 4ab - ac$.

12. Find the value of $-2abcx$.

13. Find the value of $abcxy \times 2 \div 4$.

14. Find the value of $2(ab - c + 2xy)$.

15. What is the value of $8a - 3b + cx$?

16. What is the value of $\sqrt{cx} + 2ab - \sqrt{y}$?

17. Find the value of $2(x + y) - \sqrt{y}$.

18. Find the value of $2bx - c + b^3 - c$.

19. Find the value of $2ab + 2a - 2xy + (x + y)$.

20. Find the value of $\dfrac{a^2b^2 - 2ac}{y}$.

21. Find the value of $\dfrac{b^3}{x} + \dfrac{c - 2x}{b}$.

22. Find the value of $2a^2b^2 + \sqrt[3]{xy} - 8\dfrac{ac - cy}{x}$.

23. What is the value of $\dfrac{xy - b}{2} + \dfrac{abc}{ay} + \dfrac{c^2}{y^2} + \dfrac{c}{y}$?

24. What is the value of $\dfrac{c^2 - b^2}{x^2} - \sqrt{bc} + (cx)^{\frac{1}{2}}$?

CHAPTER II.

ADDITION AND SUBTRACTION.

1. The quantities xy and $2xy$ as we have seen are like quantities, and we can readily add them together. They mean once xy and twice xy and their sum is three times xy or $3\,xy$.

2. The quantities just added together are both positive. If they were both negative, $-xy$ and $-2xy$, they would indicate once xy to be subtracted from some quantity and twice xy also to be subtracted, and their sum would be $3xy$ to be subtracted, or $-3xy$.

3. Hence when the quantities are like and the signs are like also, algebraic quantities are added by the following

Rule.

Add together the coefficients and set down the sum, prefixing the sign and annexing the quantity.

EXAMPLES.

(1)

$$\begin{array}{r} 8ax \\ 4ax \\ 6ax \\ ax \\ \hline 19ax \end{array}$$

(2)

$$\begin{array}{r} 8a-7by+\ \ (x+y) \\ a-2by+2(x+y) \\ 9a-4by+3(x+y) \\ 4a-7by+\ \ (x+y) \\ \hline 22a-20by+7(x+y) \end{array}$$

In the first example we add together the several coefficients, 1 (for ax means 1 ax) and 6 and 4 and 8, and set down their sum 19, annexing the quantity ax; we do not prefix any sign, because in the case of a quantity by itself with no sign prefixed + is understood.

In the second example we begin by adding together the coefficients of a, the sum, 22, we write down, annexing the quantity a. We proceed to add the coefficients of by, which we find to be 20; we set down $20by$, prefixing the sign −, and then dealing with the quantity $(x+y)$ which being within brackets is to be regarded as constituting one quantity, we complete the addition.

4. In setting down compound quantities for addition we so place their terms that like quantities come under one another, and we are thus able the more easily to find their sum.

Exercise II.

1. Add together $8b$, $4b$, $6b$, and b.
2. Add together $-6xy$, $-8xy$, $-7xy$, $-6xy$, and $-2xy$.
3. Add together $4ab-3by$, $8ab-2by$, $ab-by$ and $2ab-2by$.
4. Add together $3x^2y^2+x^2y$, $4x^2y^2+2x^2y$, and $x^2y^2+x^2y$.
5. What is the sum of $3a^2b-2cd$, $2a^2b-3cd$, and $4a^2b-3cd$?
6. What is the sum of $xyz+xy-yz$, $3xyz+2xy-3yz$, $2xyz+3xy-2yz$ and $xyz+4xy-4yz$?
7. What is the sum of $3ax+3ac+f$, $9ax+7ac+2f$, $2ax+4ac+2f$?
8. Add together $2ax^2+3y+8$, ax^2+2y+4, $3ax^2+y+5$, and $ax\ \ +4y+9$.
9. Add together $8a-4b+y-z$, $3a-2b+2y-4z$, $8a-b+3y-z$, and $2a-b+y-z$.

5. Let us now take two like quantities with unlike signs, and proceed to add them together. $2x$ is a positive quantity, and $-x$ is a negative or *subtractive* quantity; that is, it is to be deducted. Their sum is twice x to be added and once x to be subtracted, which is the same as x to be added. The addition then of $2x$ to $-x$ results in x. Hence when the quantities are like and the signs unlike we add by the following

RULE.

Add together the positive coefficients and also add together the negative coefficients; deduct one sum from the other, and set down the difference, annexing the quantity, and prefixing the sign which belongs to the greater coefficient.

EXAMPLES.

(1)

$$\begin{array}{r} 2ax \\ -4ax \\ 2ax \\ -3ax \\ \hline -3ax \end{array}$$

(2)

$$\begin{array}{rrr} 8ax & +4b^2 & -8xy \\ 7ax & -2b^2 & +6xy \\ -3ax & +b^2 & -8xy \\ \hline 12ax & +3b^2 & -10xy \end{array}$$

In the first example the positive coefficients amount to 4, the negative to 7; the difference 3 we set down, annexing the quantity ax, and prefixing a minus sign, since the negative coefficient is the greater of the two.

In the second example, commencing with the quantity ax, the sum of the positive coefficients is 15, from which we deduct 3, the negative coefficient, setting down the difference $12ax$ in the answer; we proceed to deal with the other quantities similarly.

Exercise III.

1. Add together $8ab$, $-4ab$, $-7ab$, and $5ab$.

2. Add together $7x + 4y$, $8x - 2y$ and $-x - 3y$.

3. Add together $-4ax + 2bx + 3cx$, $-2ax - 4bx + cx$, and $8ax + 2bx - 8cx$.

4. Add together $8xyz$, $-3xyz$, $6xyz$, $-7xyz$, $-9xyz$, $4xyz$, and $-5xyz$.

5. What is the sum of $6ab - xy$, $2ab + 3xy$, $-4ab - 8xy$, $-ab - xy$, and $6ab - xy$?

6. Add together $4x^2y^2 + 2xy - 3$, $-x^2y^2 - xy - 1$, $3x^2y^2 + 4xy - 3$, $7x^2y^2 - xy + 8$.

7. Add together $3a + ab + ac$, $4a - 2ab - ac$, and $-6a + ab - ac$.

8. Add together $a^2 + 3bx + 2\sqrt{y}$, $2a^2 - bx + \sqrt{y}$, and $-a^2 - 2bx - \sqrt{y}$.

9. Add together $2ab + 3xy - x^2y^2 + z$, $8ab - xy + x^2y^2 - z$, $ab - xy + 2x^2y^2 + 4z$, and $-7ab - x^2y^2 + 2xy - z$.

---

6. The quantities x and y are unlike; and evidently their sum will be neither $2x$ nor $2y$; it can only be expressed as $x + y$. So the sum of x and $-ab$ is $x - ab$. When therefore we have unlike quantities to add we proceed by the following

Rule.

To the sum of such quantities as are like (obtained by the preceding Rules) annex the unlike quantities with their proper signs.

EXAMPLE.

$$\begin{array}{rrrrr}
x^2y^2 & + 8xy & - 3b & & \\
2x^2y^2 & - 7xy & + 2b & & \\
3x^2y^2 & - 2xy & & - 8c & \\
-4x^2y^2 & & + b & & + yz \\
\hline
2x^2y^2 & - xy & & - 8c & + yz
\end{array}$$

In this example having disposed of the quantities x^2y^2 and xy, by the preceding rules, we find the sum of the coefficients of b, which are positive, to be 3, and that of the negative to be 3 also; this leaves no difference (since $3b$ to add and $3b$ to subtract cancel one another), and consequently nothing to set down under the quantity b; then $-8c$ having no like quantity in the sum is set down in the answer, and similarly $+yz$.

Exercise IV.

1. Add together $a+b$, $2a-b$, and $8a+9b$.
2. Add together $x^2y-2xy-z$, $8x^2y+9xy+y$, and $x^2y-7xy-y$.
3. Add together $8a^2b-9xy+8x^2y^2$, $7a^2b+xy-xyd$, and $10xy+7x^2y^2+abc$.
4. Find the sum of $3a^2+3b^2+3ab$, $2a^2+9b^2+8bc$, $-3a^2-9ab+x^2y^2$, and $5bc-8b^2+9a^2$.
5. Find the sum of $4+xy+x^2y^2+xy^2$, $8-3x^2y^2+x^2y$, $-7+5xy+8xy^2+x^2y$, $9xy+10x^2y^2+xy^2+xyz$.
6. Find the sum of $9ab+ab^2+a^2b^2+ac^2$, $10ab-8ab^2+bc$, $-8ab-2ab^2+8ac^2$, $7ab+a^2b^2+8bc$.
7. Find the sum of $2ax+yz$, $5ax+4yz$, $2yz-8d$, $8-8yz$, $12ax+4$, $5yz-8$, $8ax+8yz+8d$.
8. What is the sum of $x+3xy-7y$, $8x-xy+4z$, and $-x+4xy+y-z$?
9. What is the sum of $a^2+2ab+2bc+2cd$, $8a^2+4bc-2cd+d$, $-7a^2+2ab+8bc+4cd$, $4a^2+8ab+7bc$, $a^2+ab+bc+cd+d$?
10. Add together $8ax^2+7by+abcd+24$, $7ax^2+8by+2abcd$, $5ax^2+9by-abcd+16$, $4ax^2-abcd+8+8ax^2$.
11. Find the sum of $x^2+2xy+y^2$, $x^2-2xy+y^2$, $x^2+2xy-y^2$, $x^2-2xy-y^2$, $2xy+y^2-x^2$, $-2xy+y^2-x^2$ $2xy-y^2-x^2$.

12. Add together $-3ab+2bc-yz$, $-ab+2bc+yz$, $2ab-2bc+xy$ and $5ab-6bc+yz$.

13. Add together $2abc-8ax+2axd-bc$, $2abc-4ax+3bc$, and $8abc-8axd+y$.

14. Add together $a^2+2ab+2a^2b^2+b^2$, $3a^2-ab+3b^2$, $4a^2-2ab+a^2b^2$, and $3a^2+ab-a^2b^2$.

15. Add together $x^2y+3ab+4ax-b$, $2x^2y-2ab+x$, $3x^2y-ax+b$, and $4x^2y^2-2ax+2b$.

16. Add together $3\sqrt{xy}+5bc-8$, $2xy-3bc-xy^2$, $2\sqrt{xy}-xy+bc+d$, and $-\sqrt{xy}+x^2y^2+2bc-xy^2$.

17. Add together $2ab-\sqrt{(a+x)}+2(a+x)+xy$ $3ab+4(a+x)-2x^2$, $2\sqrt{(a+x)}-(a+x)+xy$, and $ab+3(a+x)+x^2$.

18. Add together $x^2y+8ax+3\sqrt{x^2y}-4$, $2xy-5ax+\sqrt{x^2y}$, $7-2x^2y+\sqrt{x^2y}+xy$, $3xy^2+5ax+\sqrt{x^2y}+5$, and x^2y-xy^2-ax-2.

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7. If we want to take b from a we express the result by writing the minus sign before b, thus $a-b$, because the sign minus indicates subtraction. In subtracting b from a then we perform the same operation as in adding, only we change the sign of the quantity to be subtracted, and so, if we take a from $2a$ the result is a, just as it would be if we changed the sign of the a to be subtracted and added it. Again if we want to take $-a$ from $a+b$ the result is $2a+b$. For $a+b=2a-a+b$, and if $-a$ be taken away or subtracted there remains $2a+b$. The same result is attained by changing the sign of $-a$ and adding it.

Rule for Subtraction.

Change the sign of the quantity to be subtracted, or imagine it to be changed, and proceed as in addition.

EXAMPLE.

$$\begin{array}{rrrrr} 8ax + 6b^2y^2 & - & 7abc - 8a^2b^2 & \\ 6ax - 2b^2y^2 & + & 5abc - 8a^2b^2 & - b \\ \hline 2ax + 8b^2y^2 & - & 12abc & + b \end{array}$$

Here we take the $6ax$ and changing the sign to $-6ax$ we proceed as in addition by deducting the smaller coefficient 6 from the larger 8 and setting down the difference $2ax$. Passing to the next term in the sum we change the sign of $2b^2y^2$, and then adding the $2b^2y^2$ to the $6b^2y^2$ set down the result $8b^2y^2$, with the plus sign prefixed; then $5abc$ with the sign changed is $-5abc$, and $-5abc$ and $-7abc$ are $-12abc$, which we set down; then $-8a^2b^2$ with the sign changed becomes $8a^2b^2$ which cancels $-8a^2b^2$, for $8a^2b^2$ added to $-8a^2b^2$ is equivalent to $8a^2b^2 - 8a^2b^2$; then $-b$ with the sign changed becomes $+b$, and there being no other similar quantity we set down the result with its sign prefixed, and thus complete the answer.

EXERCISE V.

1. From $8ab + 4bc + 7$ take $6ab - bc - 4$.
2. From $5y^2 - 4y + a$ take $6y^2 - 4y - a$.
3. From $x^3 + 2x^2 - 6x$ take $-x^3 + 3x^2 + 4x$.
4. From $9x^2y^2 + 7ab - 2a^2y - 6$ take $-7x^2y^2 + 5ab + 2a^2y - 5$.
5. From $-8x + abc + 2d^2 - 4ax$ take $6x - abc + 2d^2 + ax - 8$.
6. From $x^2y - xy^2 - 8x^2y^2 + 2bc$ take $2x^2y + x^2y^2 - xy^2 + 7bc - x$.
7. Take $7ab + 8bc - 9cd - 10dex$ from $8ab - 7bc - 10cd + 8ex + 9ax$.

8. Take $6ab + 2xy + 8x^2y + y^2$ from $9y^2 - 2xy + 2x^2y - bc$.

9. Take $8ax + 7xy + 4yz + yz$ from $9ax - 6xy + 8yz + zy$.

10. From $108 + 6a - 9b + 10xy + 8z$ deduct $7z + 6a + 8b + xy$.

11. Take $7rst + 8bcy + 8y - 9yz$ from $8rst - 7cby + 6y - 8z$.

12. From $10ax - 10bx + 10cx + 8ayz + b^3$ take 7[illegible] $cx + 7bx + ayz$.

13. From $2a^2x - 2ax^2 - 2xy + y^3$ take $a^2x + ax^2 + y + y^3$.

14. Take $8b^2 + 2ay + bx + d$ from $2ay + 9b^2 - bx - c$.

15. Take $2\sqrt{x} - xy + 7ab - 2cd + x$ from $2xy - 3ab + cd - y$.

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8. Subtraction we have seen is performed by changing the sign of the quantity to be subtracted. If we want to subtract $(b - c)$ from a, since $(b - c)$ being within brackets is to be regarded as one quantity, we write the result $a - (b - c)$. But if we desire to remove the brackets and so break up the quantity $(b - c)$ into the simple quantities composing it, then bearing in mind that $(b - c)$ is to be subtracted, that is, both b and $-c$ are to be subtracted, we must change the signs of the several terms, and write the result $a - b + c$. Hence the removal of brackets where preceded by a minus sign necessitates the changing the signs of all the terms which were in the brackets.

9. To show that $a - (b - c)$ is equal to $a - b + c$, it is only necessary to observe that the expression signifies not that b is to be subtracted from a but b lessened by c. Now if we subtract b we subtract too much by c,

and we must add c to make the result correct. Thus it becomes $a-b+c$. An arithmetical illustration shows this more plainly still. To subtract $(4-2)$ from 8 we must subtract not 4, but 4 less 2, or 2. The difference is 6, and will be found to be so if the signs are changed as directed. Thus $8-(4-2)=8-4+2=6$.

CHAPTER III.

MULTIPLICATION AND DIVISION.

1. We have seen that ab denotes a multiplied by b, and therefore if we wanted to multiply a by b we should express the result as ab.

2. If we want to multiply $2a$ by b we require to add $2a$ b times; the result is $2ab$. But if we want to multiply $2a$ by $-b$ we in fact require to subtract $2a$ b times, and the result is $-2ab$. If again we require to multiply $-2a$ by $-b$ we in fact want to subtract $-2a$ b times; but we know that the subtraction of $-2a$ would be expressed by $2a$, so the result of the multiplication is $2ab$. Hence in multiplying algebraic quantities we have not only to regard the quantities themselves but the signs which precede them, and it must be carefully noted that *like signs produce plus and unlike signs minus.*

3. a multiplied by a is a square, or a to the 2nd power, which we have seen is expressed thus, a^2; and a^4 means $a\times a\times a\times a$, which may be otherwise expressed as $a^2\times a^2$, or $a^3\times a$. Where therefore we have the same letters in both multiplicand and multiplier, we express the result by placing the letter in the

answer, affixing as the index of its power the sum of its indices in the multiplier and multiplicand.

4. To multiply simple quantities then we proceed by the following

RULE.

If the signs in multiplier and multiplicand are like, the product will be positive, but if they are unlike a minus sign must be prefixed to the product.

Multiply the arithmetical coefficients and affix the several letters composing the quantities.

When the multiplier and multiplicand contain powers of the same letter, add the exponents or indices of such letter for its exponent in the product.

EXAMPLES.

Multiply $7ab$ by $2ax^2$: $-8xy$ by $2z$: and $3ab$ by $2(x-y)$.

(1)	(2)	(3)
$7ab$	$-8xy$	$3ab$
$2ax^2$	$2z$	$2(x-y)$
$14a^2bx^2$	$-16xyz$	$6ab(x-y)$

In the first example we multiply 7 by 2 and set down the product 14; we then find a both in multiplier and multiplicand, and therefore add together its indices, which are 1 in each case, and give 2 for the index in the product, or a^2, to this we append the letters bx^2 which remain in the multiplier and multiplicand. No sign need be prefixed to this, since the signs of the multiplier and multiplicand are similar, and consequently the product is positive. In multiplying, in the second example, $-8xy$ by $2z$ the signs are unlike, and it becomes necessary to prefix $-$ to the product. In multiplying in the last example, we regard the $(x-y)$ as one quantity, and affix it in the product as we should any other quantity.

Exercise VI.

1. Multiply $8ab^2x$ by $6aby$.
2. Multiply $3xyz$ by $-4x^2z$.
3. Multiply $8abcd$ by $2xyz$.
4. What is the product of $-3a^2b^2x$ by $-2bc$?
5. What is the product of $8(a-x)$ by $2b$?
6. Multiply $-7(bx-y)$ by $-8ac$.
7. Multiply $3a^2x$ by $3ax^2$.
8. Multiply $-2x^2y^2$ by $-3ay^2$.
9. Multiply $2a^2bx$ by aby.
10. Multiply $-3a^2x^2y^2$ by $-2axy$.

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5. As compound quantities consist of an aggregate of simple quantities, we must when we have to multiply a compound quantity by a simple one, multiply each term of the multiplicand by the multiplier; and to multiply a compound quantity by a compound quantity we must multiply each term of the multiplicand by each term of the multiplier. Hence the multiplication of compound quantities is regulated by the following

Rule.

Multiply each quantity in the multiplicand by each quantity in the multiplier, according to the rule already given, and add the several partial products together for the product of the entire multiplication.

Examples.

Multiply $7ax-2y^2$ by a^2; $4a^2-3y^2$ by a^2-y^2; and $a+b$ by $a-b$.

(1)	(2)	(3)
$7ax-2y^2$	$4a^2-3y^2$	$a+b$
a^2	a^2-y^2	$a-b$
$7a^2x-2a^2y^2$	$4a^4-3a^2y^2$	a^2+ab
	$-4a^2y^2+3y^4$	$-ab-b^2$
	$4a^4-7a^2y^2+3y^4$	$a^2 \quad -b^2$

In the first example we multiply $7\,ax$ by a^2, and set down the result; we then multiply $-2y^2$ by a^2 and append the result, and thus form the product of the multiplication of the whole quantity $7ax-2y^2$ by a^2.

In the second example we multiply the whole quantity by a^2 and set down the product; we then multiply the whole quantity by $-y^2$ and set down the product; the addition of the two partial products gives us the product of the multiplication of the two compound quantities.

In the last example we find on adding the partial products that $+ab$ and $-ab$ cancel one another, and consequently that a^2-b^2 is the product of the multiplication of $a+b$ by $a-b$.

Exercise VII.

1. Multiply $7ab+2x+3$ by $8a$.
2. Multiply $a^2-2ab+b^2$ by $a-b$.
3. Multiply $2x^2+x$ by $x+2$.
4. What is the product of a^2-ax+x^2 by $a+x$?
5. Multiply $-7b+cx$ by $2a-cx$.
6. Multiply x^2-8x+2 by $x-4$.
7. Multiply $x^2-2xy+y^2$ by $x-y$.
8. Multiply $4a-4b-4c$ by $x+y$.
9. What is the product of $a^2b^2+9xy+6$ by $5+xy$?

10. What is the product of $9x^2 - 3xy + y^2$ by $3x + y + 2$?

11. Multiply $5a^2 + 8ax - 2x^2$ by $3a - x$.

12. Multiply $a^4 - 4a^3x + 9a^2x^2 - 4ax^3 - x^4$ by $a^2 - 2ax - x^2$.

13. Multiply $2ax + b + c$ by $ax - b + c$.

14. Multiply $2a^2 - 2b^2 + c^2$ by $a^2 - b^2$.

15. Multiply $x^4 + x^2y^2 + y^4$ by $x^2 - y^2$.

16. Multiply $a^2 + ab + b^2$ by $a^2 - ab + b^2$.

17. Multiply $x + y$ by $x + y$ and the product by $x + y$.

18. Multiply $x^2 + 2xy + y^2$ by $x^2 - 2xy + y^2$.

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6. Since in multiplication like signs produce plus and unlike minus, it follows that in division where the signs of dividend and divisor are similar, the sign of the quotient will be plus, but where they are unlike it will be minus.

7. If we have to divide $8ab$ by $2b$ we require to ascertain how often $2b$ is contained in $8ab$. Evidently $4a$ times, since $4a \times 2b = 8ab$. We attain the answer then by dividing the coefficient of the dividend by that of the divisor, and then dividing the letters of the one by the other, by cancelling any letter that is contained in both dividend and divisor. If now we have to divide $2ax$ by b we are unable to proceed as in the above example, for the divisor consists of b only, and there is no b in the quantity to be cancelled and thus effect the division. In this case we can only indicate the division by writing the quantities in fractional form, thus $\frac{2ax}{b}$.

8. a^2 divided by a gives a for $a \times a = a^2$. Hence $a^2 \div a = a$, and therefore when the divisor and dividend

contain different powers of the same letter we subtract the smaller index from the greater, and place the difference as the index of the letter, either above or below the fractional line, according as the dividend or divisor contains the higher power.

9. Hence the division of simple quantities is performed by the following

RULE.

If the signs of the divisor and dividend are like, the quotient will be positive, but if the signs are unlike the quotient will be negative, and must be prefaced by a minus sign.

Write the divisor under the dividend, in fractional form. Divide the coefficient of the dividend by that of the divisor, or reduce the coefficients of both divisor and dividend by dividing both by the highest number that will go into each without a remainder.

Cancel any letters that are common to both divisor and dividend.

Where powers of the same letter are contained in divisor and dividend, subtract the lesser index from the greater and the difference will be the index for the letter in the dividend or divisor, whichever has the higher power.

EXAMPLES.

Divide $8a^2x$ by $2ax$, and $-10ab$ by $4y$.

$$\frac{8a^2x}{2ax} = 4a \qquad \frac{-10ab}{4y} = -\frac{5ab}{2y}$$

In the first example the signs of both divisor and dividend are similar, and the quotient is therefore positive. We write the dividend and divisor in fractional form, and then find that the coefficient of the dividend is exactly divisible by that of the divisor, and gives 4 for the quotient. We find x both in dividend and divisor, and therefore it becomes cancelled. a is contained in both dividend and divisor, and subtracting the index of a in the divisor from that in the dividend gives a as the quotient. We thus obtain $4a$ as the result of the division of $8a^2x$ by $2ax$.

In the second example the signs are dissimilar, and the quotient requires to have a minus sign prefixed. We reduce the coefficients by dividing by 2, and as there are no letters in the divisor that are contained in the dividend, we can only express the quotient as $-\frac{5ab}{2y}$.

EXERCISE VIII.

1. Divide $4a^2b^2$ by $2ab$.
2. Divide $8xy$ by $3z$.
3. Divide $-16a^3bc$ by $2ab^2$.
4. Divide $9x^2y$ by $-3x$.
5. Divide $2a^2b$ by abc.
6. Divide $6abx$ by $-3xy$.
7. Divide $16a^3b^2x$ by $2b^2y$.
8. Divide $3a^2x^2y^2$ by axy.

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10. Since compound quantities consist of an aggregate of simple quantities, we must, in order to divide a compound quantity, divide each of its terms by the divisor. Hence where the dividend is a compound quantity and the divisor a simple one, we proceed by the following

RULE.

Divide each term of the dividend by the divisor according to the preceding rule, and prefix to each term in the quotient its proper sign.

EXAMPLE.

Divide $8a^2x - 7y$ by $-2a$.

$$\frac{8a^2x-7y}{-2a} = -4ax + \frac{7y}{2a}.$$

Placing the quantities in fractional form, we find that $8a^2x$ divided by $-2a$ gives $-4ax$, and that $-7y$ divided by $-2a$ gives $+\frac{7y}{2a}$. Hence the quotient is $-4ax + \frac{7y}{2a}$.

EXERCISE IX.

1. Divide $8a^2x - 8xy^2$ by $4ax$.
2. Divide $3abc + 3bcd + 3cd$ by $-3bc$.
3. Divide $7bxy^2 - 8x^2y$ by $2by$.
4. Divide $-4a^2x^2 + 8a^2x$ by a^2x.
5. Divide $2a^2b^2 + 2ab + 2a^2b$ by $-ab$.
6. Divide $5x^2 + 5y^2 + 15xy$ by $3xy$.
7. Divide $10ab + 10a^2b^2$ by $2ab$.
8. Divide $12a + 6ax + 9a^2x^2$ by $3a$.

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11. Where the divisor is a compound quantity, since each quantity in the dividend must be divided by each quantity in the divisor, we divide by the following

RULE.

Place the divisor on the left hand of the dividend, and arrange the several quantities in both divisor and dividend, so that the different powers of one letter common to both may succeed each other in the order of their indices.

Having ascertained how often the first term of the divisor is contained in the first term of the dividend, set the result in the quotient ; multiply the whole divisor by the quotient *figure; subtract, bring down as many fresh terms as are necessary for the next division, and continue the operation as long as practicable.*

If there be any remainder place it in the form of a fraction in the quotient with the divisor for its denominator.

EXAMPLE 1.

Divide $6x^4 - 96$ by $-6 + 3x$.

$$\begin{array}{l} 3x - 6)6x^4 - 96(2x^3 + 4x^2 + 8x + 16 \\ \quad\quad 6x^4 - 12x^3 \\ \hline \quad\quad\quad 12x^3 - 96 \\ \quad\quad\quad 12x^3 - 24x^2 \\ \hline \quad\quad\quad\quad 24x^2 - 96 \\ \quad\quad\quad\quad 24x^2 - 48x \\ \hline \quad\quad\quad\quad\quad 48x - 96 \\ \quad\quad\quad\quad\quad 48x - 96 \end{array}$$

The terms of the divisor require to be transposed so as to bring x first, as we have placed $6x^4$ first in the dividend. The first term of the dividend divided by the first in the divisor results in $2x^3$; we put this in the quotient, multiply the whole divisor by it, subtract the product, bringing down the next term of the dividend; then $3x$ will go into $12x^3$ $4x^2$ times. We put $+ 4x^2$ in the quotient, multiply the divisor by it and proceed as before.

EXAMPLE 2.

Divide $-8ax + 4x^2 + 4a^2 + 2b$ by $-2x + 2a$.

$$\begin{array}{l} 2a - 2x)4a^2 - 8ax + 4x^2 + 2b(2a - 2x + \dfrac{2b}{2a - 2x}. \\ \quad\quad\quad \underline{4a^2 - 4ax} \\ \quad\quad\quad\quad -4ax + 4x^2 \\ \quad\quad\quad\quad \underline{-4ax + 4x^2} \\ \quad\quad\quad\quad\quad\quad\quad 2b \end{array}$$

We here arrange the terms of the divisor and dividend according to the indices of a; having then divided as in the last example we find a remainder $2b$, which is not divisible; we therefore write this remainder in the quotient in the form of a fraction, with the divisor for denominator, thus $\frac{2b}{2a-2x}$

EXAMPLE 3.

Divide a^3-x^3 by $a-x$.

$$a-x)a^3-x^3(a^2+ax+x^2$$
$$\underline{a^3-a^2x}$$
$$a^2x-x^3$$
$$\underline{a^2x-ax^2}$$
$$ax^2-x^3$$
$$ax^2-x^3$$

EXERCISE X.

1. Divide $a^2-2ab+b^2$ by $a-b$.
2. Divide $a^2+2ax+x^2$ by $a+x$.
3. Divide $27a^3-12a^2x+28ax^2-3x^3$ by $9a-x$.
4. Divide x^2+4x+3 by x^2-2x+3.
5. Divide x^3+a^3 by $a+x$.
6. Divide $-8+y^3+2y-4y^2$ by $y-4$.
7. Divide $2ax-2bcx-3ay+3bcy+cd$ by $2x-3y$.
8. Divide $2ab+8ac-2ab^2-16abc+3y$ by $a-2ab$.
9. Divide $2ax+x^3-2ay-x^2y-y^3$ by $x-y$.
10. Divide $-8a^2+8a^3b+16a^2b^2$ by $2ab+2a$.
11. Divide x^4-y^4 by $x-y$.
12. Divide $a^3-3a^2b+3ab^2-b^3$ by $a-b$.
13. Divide $48x^3-76ax^2-64a^2x+48a^3$ by $2x-4a$.
14. Divide $3a^4-8a^2b^2+3a^2c^2+5b^4-3b^2c^2$ by a^2-b^2.

12. Having now ascertained how to add, subtract, multiply, and divide algebraic quantities, we may note some points before proceeding. As in the expression $8ab$, 8, one of the factors, is called the coefficient, so a, another of the factors, is sometimes called a *literal coefficient*. $8ab$ means 8 times ab or a times $8b$ or b times $8a$. Any factor of a quantity then may be regarded as a coefficient. If we had to add $8ax$ and $2bx$, or subtract $8ax$ from $2bx$, according to the rule, the quantities being unlike, we should express the results as $8ax + 2bx$ and $2bx - 8ax$. But if we regard a and b as literal coefficients, the result is expressed as $(8a + 2b)x$ and $(2b - 8a)x$.

13. We are obliged to place $8a + 2b$ within brackets because they are both regarded as coefficients of the quantity x, and the result of the addition is $8a + 2b$ times x. So if we removed the bracket from the expression $(8a + 2b)x$ it would be necessary to multiply each term in the bracket by x.

14. We must be very careful in removing brackets to remember that anything affecting the quantity within brackets affects the whole quantity, and therefore affects each term when the bracket is removed. If the whole quantity is to be multiplied then each term must be multiplied on removing the bracket, and if the whole quantity is to be subtracted the signs of each term must be changed on its removal.

15. The following results, the truth of which may be ascertained by actual multiplication, should be here noted as general formulæ of considerable practical value:

$$(a+b)\times(a+b)=a^2+2ab+b^2.$$
$$(a-b)\times(a-b)=a^2-2ab+b^2.$$
$$(a-b)\times(a+b)=a^2-b^2.$$

CHAPTER IV.

GREATEST COMMON MEASURE AND LEAST COMMON MULTIPLE.

1. A *measure* of a quantity is any quantity which will divide it and leave no remainder; in other words one of its factors. A *common measure* of two or more quantities is any quantity which will divide all of them without a remainder; in other words a factor common to all of them. The *greatest common measure* of two or more quantities is the greatest quantity which will divide all of them without a remainder; or, in other words, the product of the highest common factors.

2. Thus a is a measure of ab, for ab is composed of the factors a, and b; so it is a measure of a^2b which is composed of the factors a, a, and b; x is a common measure of x^2yz and x^2y, for they are respectively composed of the factors x, x, y, z, and x, x, y, and it is apparent that x is a factor which is common to or contained in both quantities. The greatest common measure of x^2yz and x^2y is x^2y, for the highest factors that are common to both are x^2 and y, and their product x^2y is the highest factor which is contained in both the quantities.

3. We ascertain the G. C. M. of simple quantities by inspection, for upon a glance we are able to perceive what are the highest factors that are common, and the product of the highest factors is the G. C. M. If the quantities have numerical coefficients we must ascertain their G. C. M., and prefix it to the result.

If, for example, we require to ascertain the G. C. M. of a^2b^2, a^3bc, and a^2b, a moment's inspection shows us that a^2, and b are the highest factors that are common, and that consequently a^2b is the G. C. M. of the three quantities.

4. But if the quantities of which the G. C. M. is to be ascertained are compound, we must proceed as in arithmetic, by dividing one by the other, treating the remainder after the first division as a new divisor, and the former divisor as the new dividend, and thus continuing till there is no remainder. The last divisor used will be the G. C. M. If we have to ascertain the G. C. M. of more than two compound quantities, we first ascertain the G. C. M. of any two of them, and then ascertain the G. C. M. of another quantity, and the G. C. M. already found, and so on.

5. Where any of the quantities contains a factor common to all its terms, we may simplify the quantity by eradicating or striking out the factor. While, however, we may eradicate factors common to the different terms in one quantity, and factors common to those in another, we must when we strike out the *same factor* from all the quantities be careful to note that it will form a part of the G. C. M., and that the last divisor used must be multiplied by it to obtain the correct G. C. M.

6. Whenever we have a remainder brought into use as a divisor in the course of ascertaining the G. C. M., we may strike out any common factor that its terms contain.

7. Whenever in the course of ascertaining the G. C. M. the coefficient of the first term of the dividend is not exactly divisible by the first term of the divisor, we

may multiply the dividend by such a number as will make it so divisible.

EXAMPLE.

What is the G. C. M. of $9a^2b - 25b$ and $9a^2 + 3a - 20$?

$$9a^2b - 25b)9a^2 + 3a - 20(1$$
$$9a^2 - 25 \quad 9a^2 - 25$$
$$3a + 5)9a^2 - 25(3a - 5$$
$$9a^2 + 15a$$
$$-15a - 25$$
$$-15a - 25$$

We find that $9a^2b - 25b$ contains a factor (b) common to all its terms; this we eradicate, and simplify the quantity to $9a^2 - 25$. If, however, the other quantity were $9a^2b + 3ab - 20b$, it would be necessary to note when eradicating the b from the divisor and dividend, that b would form a part of the G. C. M., and that the G. C. M. found by division would require to be multiplied by b to obtain the true G. C. M.

We now divide one quantity by the other, and obtain as a remainder $3a + 5$, which we make a divisor, placing the preceding divisor as the new dividend. We find that $3a + 5$ divides it exactly, and that it is consequently the G. C. M. of the two quantities.

If instead of seeking the G. C. M. of $9a^2b - 25b$ and $9a^2 + 3a - 20$ we had to ascertain the G. C. M. of $9a^2 - 25$, $9a^2 + 3a - 20$, and $6ab + 10b$, we should, having ascertained the G. C. M. of the two first quantities to be $3a + 5$, take the two quantities $3a + 5$ and $6ab + 10b$, and proceed to ascertain their G. C. M. This we should find to be $3a + 5$, which would consequently be the G. C. M. of the three quantities.

8. A *multiple* of a quantity is any quantity that contains it as divisor, or as one of its factors. A *common multiple* of two or more quantities is any quantity that contains all of them as divisors, in other words that has all the quantities in it as factors. The *least common multiple* of two or more quantities is the lowest quantity that contains all of them as factors.

9. The least common multiple of two quantities is ascertained by finding their G. C. M., dividing one of them by it, and multiplying the quotient by the remaining quantity. Or we may strike out the factors that are common to any two of the quantities of which we desire to ascertain the L. C. M.; multiply the quantities so simplified together and the product by the factors struck out.

10. In seeking the L. C. M. of quantities we are endeavouring to find the lowest quantities that contain them all as measures; obviously then all we have to do is to resolve each quantity into its factors, and the L. C. M. will be composed of all the factors peculiar to each quantity and of the factors common to any two or more of them.

EXAMPLE.

Find the L. C. M. of a^2b^2c, $8abc$, and $2d$.

Here striking out the factors abc, common to the first and second quantity, and 2, common to the second and third, the quantities are reduced to ab, 4, and d; their product, $4abd$, multiplied by the factors struck out, $2abc$, gives $8a^2b^2cd$ as the L. C. M.

11. For ascertaining the G. C. M. of two quantities we may use the following

RULE:

If the quantities be simple find by inspection the greatest common factors, and their product will be the G. C. M. The G. C. M. of the numerical coefficients must be prefixed.

If the quantities be compound, divide one by the other, treating the remainder as a new divisor, and the former divisor as a new dividend, and the divisor which leaves no remainder will be the G. C. M. of the quantities dealt with.

A factor common to all the terms of one of the quantities may be struck out, but if it is common to all the terms of all the quantities, the last divisor must be multiplied by it to obtain the true G. C. M.

12. To find the L. C. M. we proceed according to the following

RULE.

Strike out the factors that are common to any two of the quantities; multiply together the quotients and the factors struck out.

If the quantities be compound and the factors common to any two of them be not perceived by inspection, find them by ascertaining the G. C. M. of the two quantities.

EXERCISE XI.

1. Find the G. C. M. of $9a^2b^2$, $3ab$ and $27b^3$.
2. " " $2x^2y^2z^2$, xyz, and $x^3y^3z^3$.
3. " " $b^3 - a^2b$ and $b^2 + 2ab + a^2$.
4. " " $a^2 - a - 2$, and $a^2 - 3a + 2$.
5. " " $3x^4 - 15x^3 + 24x^2 - 12$ and $2x^3 - 10x^2 + 12x$.
6. " " $a^2 + ab - 2a^2b - 2ab^2$ and $a^3x - 2a^2bx$.
7. " " x^2y^2 and $x^2 + 2xy + y^2$.

8. Find the G. C. M. of $a^2(a^3 - x^3)$ and $a^2x + ax^2$.
9. " " $3x^5 + x^3 + 9x^2 - 2x - 6$ and $3x^5 + 3x^4 - 2x^3 + x^2 - 2$.
10. What is the L.C.M. of $6a^2b^2$, $3a^2b$, $3ab^2$ and $6ab$?
11. " " $x + y$, $x^2 + 2xy + y^2$, $(x+y)^2$, $3x + 3y$?
12. " " $2(a+b)$ and $3(a^2 - b^2)$?
13. " " $6(x^2y + xy^2)$, $9(x^3 + x^2y)$, $4(y^3 + xy^2)$?
14. " " $8xy$, $16x^2y^2$, $4bxy$, and $9axy^2$?
15. " " axy and $a(xy - y^2)$?

CHAPTER V.

FRACTIONS.

1. By the arithmetical expression $\frac{1}{2}$ we mean one half or one divided by 2, and in like manner we have seen that the expression $\frac{a}{b}$ means the division of a by b, and is an algebraic fraction.

2. The quantity above the line is called the *numerator*, that below the *denominator*; and both together constitute the *terms of the fraction*.

3. By multiplying the numerator or dividing the denominator of a fraction, we in effect multiply the fraction; by dividing the numerator or multiplying the denominator we in effect divide the fraction.

4. Since if we multiply the numerator and denominator of a fraction by the same quantity we in effect multiply and at the same time divide the fraction by

the same quantity, it follows that multiplying both numerator and denominator of a fraction by the same quantity leaves its value unaltered, and similarly that dividing both numerator and denominator by the same quantity leaves the value of the fraction unaltered.

5. To reduce a fraction to its lowest terms :—

RULE.

Divide the terms of the fraction by their G. C. M.

EXAMPLES.

Reduce $\frac{a^2b^2x}{aby}$ and $\frac{2a^2+4ab+2b^2}{3x(a+b)}$ to their lowest terms.

The G. C. M. of a^2b^2x and aby is ab; dividing the terms of the fraction by this we obtain $\frac{abx}{y}$ as the lowest terms of the fraction. The value of the fraction is unaltered, for we have divided both numerator and denominator by the same quantity.

In the second example the fraction may be expressed as $\frac{2(a^2+2ab+b^2)}{3x(a+b)}$, but since $(a+b)^2=a^2+2ab+b^2$, we further simplify it to $\frac{2(a+b)(a+b)}{3x(a+b)}$, then striking out $a+b$, which is a factor common to both terms of the fraction, and which is the G. C. M. of the terms, we reduce the fraction to its lowest terms $\frac{2(a+b)}{3x}$.

6. It is therefore apparent that we need not ascertain the G. C. M. of the terms of a fraction if we are able to split up or resolve them into their several factors, for by cancelling or dividing the terms of the fraction by the common factors we reduce it to its lowest terms.

7. A mixed quantity, that is a quantity consisting of a whole quantity and a fraction, may be reduced to fractional form by multiplying the whole quantity by the denominator of the fraction and connecting the product with the fraction, placing underneath the denominator.

EXAMPLE.

Reduce $2b - \frac{a^2 - b^2}{b}$ to fractional form.

$2b$ multiplied by b gives $2b^2$, and annexing the fraction we obtain $\frac{2b^2 - (a^2 - b^2)}{b}$ or $\frac{2b^2 - a^2 + b^2}{b}$ or $\frac{3b^2 - a^2}{b}$ as the equivalent fraction.

8. Where the denominator of a fraction will divide the numerator, or divide it leaving a remainder, we can reduce the fraction to a whole quantity or a mixed quantity (as the case may be.) Thus $\frac{x^2 + 2xy + y^2}{x + y}$ can at once be reduced to the whole quantity $x+y$, and $\frac{2a^2b + 8bx}{a}$ can be reduced to the mixed quantity $2ab + \frac{8bx}{a}$. The student will perceive that this is only applying the rules of division in cases where the numerator of the fraction is divisible by the denominator, or divisible leaving a remainder.

9. To reduce fractions to a common denominator :—

RULE.

Multiply each numerator by the denominator of the other fractions, and all the denominators together for a common denominator.

EXAMPLE.

Reduce $\frac{a^2}{x}$, $\frac{2b}{ay}$ and $\frac{3}{z}$ to a common denominator.

Multiplying the first numerator a^2 by the denominators of the other fractions we obtain a^2yz for the numerator of the first fraction, and similarly $2bxz$ for the numerator of the second, and $3axy$ for that of the third; then multiplying all the denominators together we obtain $axyz$ for the common denominator, and the fractions become $\frac{a^3yz}{axyz}$, $\frac{2bxz}{axyz}$, $\frac{3axy}{axyz}$.

10. The common denominator of any fractions is not necessarily their *least* common denominator; this is obtained by finding the L. C. M. of the several denominators, and the fractions may be reduced to their least common denominator by multiplying the numerator of each by the quotient obtained by the division of the least common denominator by its denominator.

Thus to reduce $\frac{x}{2ab}$ and $\frac{2y}{3b}$ to their least common denominator, we find the L. C. M. of $2ab$ and $3b$, which is $6ab$; $2ab$ will go into $6ab$ 3 times, and we multiply the numerator x by 3; $3b$ will go into $6ab$ $2a$ times, and we multiply the numerator $2y$ by $2a$; and thus obtain $\frac{3x}{6ab}$ and $\frac{4ay}{6ab}$ for the fractions reduced to their least common denominator.

11. To add or subtract fractions we observe the following

RULE.

Reduce the fractions to a common denominator. Add or subtract the numerators (as the case may be) for a new numerator, under which place the common denominator.

EXAMPLE.

Add together $\frac{x}{x-2}$ and $\frac{x}{x+2}$ and subtract $\frac{1}{x+y}$ from $\frac{1}{x-y}$.

$$\frac{x}{x-2}+\frac{x}{x+2}=\frac{x^2+2x}{x^2-4}+\frac{x^2-2x}{x^2-4}=\frac{x^2+2x+x^2-2x}{x^2-4}$$

$$=\frac{2x^2}{x^2-4}$$

$$\frac{1}{x-y}-\frac{1}{x+y}=\frac{x+y}{x^2-y^2}-\frac{x-y}{x^2-y^2}=\frac{x+y-x+y}{x^2-y^2}=\frac{2y}{x^2-y^2}$$

It must be noted in the last example that when we bring the fractions together under the common denominator it is necessary to change the signs of both terms of $x-y$ to $-x+y$. For the entire quantity is to be subtracted, and may be regarded the same as if it was within brackets.

12. To multiply fractions :—

RULE.

Multiply the numerators together for a new numerator, and the denominators together for a new denominator.

EXAMPLE.

Multiply $\frac{7ab}{x}$ by $\frac{3a^2}{xy}$.

$$\frac{7ab}{x}\times\frac{3a^2}{xy}=\frac{7ab\times 3a^2}{x\times xy}=\frac{21a^3b}{x^2y}$$

13. The division of fractions is performed by inverting the terms of the divisor and then multiplying.

EXAMPLE.

Divide $\frac{ab}{c}$ by $\frac{x}{y}$.

$$\frac{ab}{c} \div \frac{x}{y} = \frac{ab}{c} \times \frac{y}{x} = \frac{aby}{cx}$$

14. In multiplying fractions we may cancel any factor that is common to either of the numerators and either of the denominators.

For example if we required to multiply $\frac{2a}{a-b}$ by $\frac{a^2-b^2}{6}$ by placing the fractions in order for multiplication, thus $\frac{2a \times (a^2-b^2)}{(a-b) \times 6}$ we find one of the factors in the numerator, a^2-b^2, may be resolved into the factors $(a+b)$ $(a-b)$; then placing these factors in place of a^2-b^2 we obtain $\frac{2a \times (a+b)\ (a-b)}{(a-b) \times 6}$; we find the factor $a-b$ common to both numerator and denominator, and therefore cancel it, thus reducing the fraction to $\frac{2a(a+b)}{6}$; cancelling the common factor, 2, we obtain $\frac{a\,(a+b)}{3}$ or $\frac{a^2+ab}{3}$ for the answer.

EXERCISE XII.

1. Reduce $\frac{6a^2b^2+12a^3b}{3a^2bx}$ to its lowest terms.

2. Reduce $\frac{x-y}{x^2-y^2}$ to its lowest terms.

3. Reduce $\frac{27a^2bc}{18a^2c+9acd}$ to its lowest terms.

4. Reduce $a - \frac{bx + a^2}{b}$ to fractional form.

5. Reduce $\frac{a^2 - b^2}{a + b}$ to a whole quantity.

6. Reduce $\frac{x - 4}{2}$ and $\frac{x - a}{a}$ to a common denominator.

7. Reduce $\frac{3ab - 2}{2ab}$ and $\frac{2b}{8a}$ to a common denominator.

8. Reduce $\frac{8x - y}{x^2y}$ and $\frac{x - y}{x}$ to their least com. denom.

9. Add together $\frac{5y}{2}$, $\frac{3y}{5}$, and $\frac{y}{x}$.

10. Add together $\frac{a + b}{2a}$ and $\frac{a - b}{2b}$.

11. Add together $\frac{ax - x^2}{a}$ and $\frac{2x}{a}$.

12. Add together $\frac{3y + 1}{5}$, $\frac{y - 2}{y}$, and $\frac{y - 3}{4}$.

13. Subtract $\frac{3y}{2}$ from $\frac{5y}{a}$.

14. Subtract $\frac{3x + 4}{5}$ from $\frac{5x + 8}{3}$.

15. Subtract $\frac{ax - x^2}{a}$ from $\frac{2x}{b}$.

16. Subtract $\frac{2x - 5}{x} + \frac{x - 1}{x}$ from $\frac{7x - 4}{5}$.

17. Multiply $\frac{ab}{a - 2}$ by $\frac{b}{7}$.

18. Multiply $\frac{x - y}{x}$ by $\frac{3ax}{x - y}$.

19. Multiply $\frac{x^2-a^2}{4}$ by $\frac{3b}{a+x}$.

20. Multiply together $\frac{8a}{a-x}$, $\frac{7b}{ax}$, and $\frac{8c}{a^2}$.

21. Divide $\frac{8x}{3}$ by $\frac{4x}{6}$.

22. Divide $\frac{8y^2-y}{2}$ by $\frac{y}{3}$.

23. Divide $\frac{8a-3b}{2a}$ by $\frac{4a-4b}{3b}$.

CHAPTER VI.

INVOLUTION AND EVOLUTION.

1. Involution is the process of raising quantities to any required power, and is performed by multiplying the quantity into itself as many times (less one) as there are units in the index of the required power.

2. The involution of simple quantities is generally performed by multiplying the index of the quantity by that of the required power, and prefixing the result of the involution of the coefficient (if any) obtained by actual multiplication. For since a^2 raised to the 3rd power $= a^2 \times a^2 \times a^2 = a^6$, the result is evidently more simply obtained by multiplying the index of a (2) by the index of the power (3), thus a^2 to the 3rd power $= a^{2\times3} = a^6$. Thus we see that in the case of simple quantities by the process above mentioned we obtain the same result as if we multiplied the quantity into itself as many times (less one) as there are units in the index of the required power.

3. But in the case of compound quantities we must proceed by actual multiplication.

4. Since $x^2 \times x^2 = x^4$ it is evident that we may in some measure abbreviate the process of involving compound quantities to high powers. For since the square of a quantity multiplied by itself gives the 4th power, we may obtain the 4th power by first squaring the quantity and then multiplying the square by itself. Similarly since $x^3 \times x^3 = x^6$ we may obtain the 6th power by multiplying the cube by itself, &c.

5. In the case of a fraction we must involve the numerator and also the denominator to the required power, and the results will be the terms of the fraction raised to the required power.

6. In the case of simple quantities we must note that where they are negative, the powers whose index is odd will be negative, while those whose index is even will be positive.

EXAMPLES.

What is the square of $2a^2x$ and the cube of ax^2?

$$(2a^2x)^2 = 2^2a^{2\times2}x^{1\times2} = 4a^4x^2$$
$$(ax^2)^3 = a^{1\times3}x^{2\times3} \quad = a^3x^6$$

or by actual multiplication,

$$(2a^2x)^2 = 2a^2x \times 2a^2x = 4a^4x^2$$
$$(ax^2)^3 = ax^2 \times ax^2 \times ax^2 = a^3x^6$$

What is the 4^{th} power of $a - 2x$?

Here we multiply $a - 2x$ by itself, and thus obtain the square, $a^2 - 4ax + 4x^2$, and multiplying this again by itself we obtain the 4^{th} power required, $a^4 - 8a^3x + 24a^2x^2 - 32ax^3 + 16x^4$, as shown on the next page.

$$
\begin{array}{l}
a-2x \\
a-2x \\
\hline
a^2-2ax \\
\quad -2ax+4x^2 \\
\hline
a^2-4ax+4x^2 \\
a^2-4ax+4x^2 \\
\hline
a^4-4a^3x+4a^2x^2 \\
\quad -4a^3x+16a^2x^2-16ax^3 \\
\qquad\qquad 4a^2x^2-16ax^3+16x^4 \\
\hline
a^4-8a^3x+24a^2x^2-32ax^3+16x^4
\end{array}
$$

7. We may therefore, for the involution of algebraic quantities, proceed by the following

RULE.

In the case of simple quantities involve the coefficient to the required power and append the quantity with the indices of its several letters multiplied by the index of the required power. If the quantity be negative and the index of the power odd, the product or power must have a negative sign prefixed.

In the case of compound quantities multiply the quantity into itself as many times (less one) as there are units in the required power abbreviating, however, if possible, the number of actual multiplications, as shewn in section 4.

In the case of fractions involve the numerator and also the denominator for the terms of the fraction raised to the required power.

8. Evolution is the extraction of the roots of quantities. Since $(x^3)^2 = x^6$, it follows that the square root of x^6 is x^3. And hence to obtain the required root of a simple quantity, we must first extract the root of the

numerical coefficient (if any) and then divide the index of the quantity by the index of the root. But if it should happen that we cannot do this, the index of the quantity not being exactly divisible by that of the root, or if the quantity have no index greater than unity (as ax) then the root required cannot be extracted, and the quantity must be written down with its radical sign prefixed. This expression is called a surd.

Thus the 5th root of a^3 can only be expressed thus, $\sqrt[5]{a^3}$, and this is called a surd. So the cube of $2x^3 = \sqrt[3]{2} \times \sqrt[3]{x^3}$. But the cube root of x^3 is x, for the index of the quantity 3, divided by the index of the root to be extracted, 3, is 1. Therefore the cube root of $2x^3$ is $x\sqrt[3]{2}$, and this is similarly called a surd.

9. We know that + multiplied by + gives +, and that − multiplied by − gives + also. + is produced, therefore, both by the intermultiplication of positive and of negative quantities. It follows, therefore, that the square root, or any root whose index is even, of a positive quantity, may be either positive or negative; and this is expressed by writing the result thus $\sqrt{x^2} = \pm x$.

Hence, the even roots of a positive quantity may be positive or negative, and are expressed by $\pm$

No negative quantity can have an even root.

The odd root of a quantity will have the same sign as the quantity itself.

This last position is evident, for if the quantity is positive, every power of it will be positive also; but if it be negative, while the second power would be positive, the third power (and similarly every other odd power) would be immediately produced by multiplying a positive by a negative quantity, necessarily producing a negative quantity as the result.

10. Hence to extract the roots of simple quantities we have the following

RULE.

If the root to be extracted be even, the result may be positive or negative, but if odd prefix the sign of the quantity itself.

Extract the required root of the coefficient, and append the letters composing the quantity, dividing their indices by that of the root for the indices to place in the root.

Extract the square root of $9a^4x^4$; and the cube root of $-8a^3b^6$.

$$\sqrt{9a^4x^4} = 3a^2x^2$$
$$\sqrt[3]{-8a^3b^6} = -2ab^2$$

In the first example we find the square root of 9 to be 3, and the square root of a^4x^4 is obtained by dividing the index of each letter by 2, the index of the root required, and we thus obtain $3a^2x^2$, which should strictly be expressed $\pm 3a^2x^2$, since $3a^2x^2$ may be positive or negative.

In the second example we have to extract an odd root, and it will therefore have the same sign as the quantity itself or $-$. The root of the quantity is extracted in a similar manner to the preceding example.

11. If we multiply $a+b$ by $a+b$ we obtain $a^2+2ab+b^2$, and if $a-b$ be multiplied by $a-b$ the result is $a^2-2ab+b^2$. That is the square of a quantity of two terms consists of the square of each and twice their product, added or subtracted (as the case may be). From this formula we find the rule for the extraction of the square root of compound quantities,

RULE.

Extract the square root of the first term in the quantity, and place that root in the quotient; square the term placed in the root, deduct it and bring down the remainder; multiply the term in the quotient by 2, and find how often it will go into the first term of the remainder, and place the result in the quotient with its proper sign; also couple the quantity last placed in the root to the divisor, and multiply the whole divisor by the term last placed in the root.

If after this is done there is still a remainder, proceed as before, multiplying all the terms already in the root for the first part of the new divisor.

EXAMPLE 1.

Extract the square root of $4x^2-4xy+y^2$.

$$\begin{array}{rl} & 4x^2-4xy+y^2\,(2x-y \\ & 4x^2 \\ \hline 4x-y) & \quad -4xy+y^2 \\ & \quad -4xy+y^2 \end{array}$$

We extract the square root of the first term and find it is $2x$; place it in the quotient, square it, and subtract it from the quantity, bringing down the remainder $-4xy+y^2$; we now multiply the terms of the quotient by 2, and obtain $4x$ for the first part of our divisor; $4x$ will go into $-4xy$ $-y$ times; we place $-y$ in the quotient, and also complete our divisor with it; now multiplying the divisor by the term last placed in the root, we obtain $-4xy+y^2$, which deducted leaves no remainder. Therefore $2x-y$ is the square root of $4x^2-4xy+y^2$.

If after we had deducted $-4xy+y^2$ there had still been terms remaining, we should have multiplied $2x-y$

by 2 for the first part of a new divisor, and then proceeded as before, and if we found that the exact square root of the quantity could not be extracted, we should express the result as a surd. Thus a^2-3b has no exact square root, and its square root would be expressed in the form of a surd, thus, $\sqrt{a^2-3b}$.

EXAMPLE 2.

Extract the square root of $a^2-ab+\frac{b^2}{4}$.

$$
\begin{array}{r|l}
 & a^2-ab+\frac{b^2}{4}\ \left(a-\frac{b}{2}\right. \\
 & a^2 \\
\hline
2a-\frac{b}{2} & -ab+\frac{b^2}{4} \\
 & -ab+\frac{b^2}{4}
\end{array}
$$

12. If we had to extract the 4th root we could extract the square root, and then again extract the square root of the root found for $x^4=x^2\times x^2$.

13. By cubing $a+b$ and $a-b$, and investigating the composition of the product we are enabled to find, for the extraction of the cube root of compound quantities, the following

RULE.

Take the cube root of the first term and place it in the quotient; cube the first term, and deduct it from the quantity, bringing down the remainder. Multiply by 3 the square of the root already in the quotient, and place the result as the first term of the divisor. Ascertain how often the divisor will go into the first term of the dividend, and place the result with its proper sign in the quotient; complete the division by annexing three times the term previously *in the quotient to the term therein, multiplying the sum by the term last placed in the root and annexing the whole to the divisor.*

EXAMPLE.

Find the cube root of $a^3 - 3a^2x + 3ax^2 - x^3$.

$$
\begin{array}{rl}
 & a^3 - 3a^2x + 3ax^2 - x^3 \,(a - x \\
 & a^3 \\
\hline
3a^2 - 3ax + x^2) & -3a^2x + 3ax^2 - x^3 \\
 & -3a^2x + 3ax^2 - x^3
\end{array}
$$

We first extract the cube root of a^3, place the result in the quotient, cube it, subtract and bring down the remainder $-3a^2x + 3ax^2 - x^3$. Then we place 3 times the square of a or $3a^2$ as the first term of the divisor, and find it will go into $-3a^2x$, $-x$ times; we place $-x$ in the quotient; we then complete the divisor by annexing 3 times a to the term last placed in the root, $-x$, and multiply the sum $3a - x$ by $-x$, obtaining to complete the divisor $-3ax + x^2$; multiplying the completed divisor by the term last placed in the root, we obtain $-3a^2x + 3ax^2 - x^3$, which subtracted leaves no remainder. The cube root therefore of $a^3 - 3a^2x + 3ax^2 - x^3$ is $a - x$, or $\sqrt[3]{a^3 - 3a^2x + 3ax^2 - x^3} = a - x$.

14. The evolution of fractions is performed by extracting the required root of both numerator and denominator for the terms of the fraction evolved to the root required.

EXERCISE XIII.

1. What is the square of $2a - b$?
2. Raise $a^2 - 2x$ to the 3rd power.
3. What is the 10th power of $2a^2x$?

4. What is the square of $a-x-2y$?
5. Cube $a-b$.
6. Raise $3a^2x^2$ to the 4^{th} power.
7. What is the 3^{rd} power of $\frac{a^2-x}{2y}$?
8. Raise $a+2b$ to the 4^{th} power.
9. What is the square root of $a^2-2ax+x^2$?
10. Extract the cube root of $64a^6x^3y^3$.
11. Extract the square root of $a^4-4a^3x+6a^2x^2-4ax^3+x^4$.
12. What is the cube root of $a^6-6a^4x+12a^2x^2-8x^3$?
13. Find the square root of $4a^2+4b^2-8ab$.
14. Raise $2abx$ to the 6^{th} power.
15. Extract the square root of $\frac{a^2+2ab+b^2}{4x^2+4y^2+8xy}$.

CHAPTER VII.

SIMPLE EQUATIONS.

1. We have seen that the sign $=$ denotes equality. Where this sign occurs between two quantities the whole expression is termed an *equation*. If $x=a$ this is an equation. It does not of course mean that x is *always* equal to a, but that in the particular investigation we are making, either from facts we know or from deductions we have made by algebra, $x=a$. The two sides of an equation separated by the sign $=$ may consist of simple or compound quantities.

2. It is customary to denote *known quantities* by the earlier letters of the alphabet; while the last letters of the alphabet are used to represent *unknown quantities*, that is quantities the value of which we have to discover either numerically or in terms of the known quantities.

Thus if $2x = 8$ we have an equation wherein x is an *unknown quantity*; the value of x in the equation is readily found, for if $2x = 8$, x must equal 4. We have ascertained the value of the unknown quantity, and by so doing have (as it is termed) *satisfied* or *solved* the equation. So we might require to find the value of x where $2x = a$, and we should satisfy this equation or ascertain the value of x by $x = \frac{a}{2}$.

3 Every equation may be regarded as the expression of a particular problem in algebraic language. The equation $2x = 8$ may be regarded as the algebraic expression of the problem, to find such a number that when multiplied by 2 the product shall be 8, and the equation $2x = a$ may be regarded as the algebraic expression of the problem, to find a number such that when multiplied by 2 it shall equal a, and we solve the equation by finding that the number required, or x, is $\frac{a}{2}$, that is, will be half a, whatever that may be.

4. An equation or problem, then, may require us not simply to find the numerical value of the unknown quantity, but to find its value in terms of some other quantities, these quantities being for the purposes of the problem known quantities, and our object being from the given conditions of the equation to find the equivalent of the unknown quantity.

5. An equation in which the unknown quantity is of the 1st power only is called a *simple equation*; if the unknown quantity be of the 2nd power it is termed a *quadratic* equation; if of the 3rd power a *cubic* equation; if of the 4th power a *biquadratic* equation.

6. We may add to or subtract from one side of an equation any quantity we please, provided that we maintain the equality by adding to or subtracting from the other side of the equation the same quantity. The reason of this is evident, since the two sides of an equation being equal to one another, an addition to each side of the same quantity cannot affect the equality subsisting.

7. We may multiply or divide one side of an equation by any quantity, provided we maintain the equality by multiplying or dividing the other side by the same quantity.

8. Any term may be transposed from one side of an equation to the other if the sign be changed. For if $x + 7 = 8$, and in order to solve the equation we wish to transpose the 7 from the first side of the equation to the other, we in fact subtract 7 from the first side, and therefore must be careful to subtract it from the other side too, thus $x = 8 - 7$. We have in reality transposed 7 from one side of the equation to the other, changing its signs.

9. The signs of the several quantities in an equation may be changed, provided the signs of all the quantities on both sides are changed.

10. Simple equations involving one unknown quantity are solved by the following

RULE.

If there are any fractions multiply the equation by the denominator or least common denominator, so as to eradicate the fractions.

Transpose the quantities so that all those involving the unknown quantity may be on the left hand or first side of the equation and the known ones on the other.

Collect together the terms on each side, and divide the known quantities by the coefficient of the unknown quantity.

EXAMPLES.

(1)

If $x-1=\frac{x}{4}+5$, what is the value of x?

$$x-1=\frac{x}{4}+5$$

Multiplying by 4 to eradicate the fraction, $4x-4=x+20$

Transposing $4x-x=20+4$

$$3x=24$$

$$\therefore x=8$$

(2)

$\frac{x}{a}+b=\frac{x}{b}+c$: find the value of x.

To eradicate the fractions we multiply the equation by ab, the least common multiple of the denominators.

$$\therefore bx+ab^2=ax+abc$$

Transposing $bx-ax=abc-ab^2$

Or $(b-a)x=ab(c-b)$

$$\therefore x=\frac{ab(c-b)}{b-a}$$

(3)

$$\frac{3}{4}-\frac{x-2}{3}=\frac{5}{4}-\frac{x+3}{4}$$: find the value of x.

Multiplying by 12, the least common denominator of the fractions $9-4x+8=15-3x-9$

Transposing $-4x+3x=15-9-9-8$

$\therefore \quad -x=-11$

Changing the signs $x=11$

(4)

$4bx^2-9bx^2=9bx^2+2bx^2$: find the value of x.

We have here an equation with x in all its terms, and it is necessary to simplify it by dividing by the highest power of x common to all the terms; b being also a common factor we divide by bx^2 and obtain

$$4x-9=9+2x$$

Transposing $4x-2x=9+9$

$$\therefore 2x=18$$

$$\therefore x=9$$

(5)

$\sqrt{x-3}-3=4$ find the value of x.

Transposing $\sqrt{x-3}=4+3$

$$\sqrt{x-3}=7$$

Squaring, to get rid of the radical sign $x-3=49$

$$x=52$$

In this example, after transposing, we find it necessary to get rid of the radical sign; to do this we multiply one side by $\sqrt{x-3}$, and the other by its equivalent 7; in other words we square both sides of the equation. For since we may multiply both sides of an equation by the same quantity, obviously we may square each side of the equation if desirable, since in so doing we multiply each side by equivalent quantities.

Exercise XIV.

1. $2x + 8 = x + 9$; find x.
2. $\sqrt{x} - 2 = 3$; find x.
3. $\frac{x}{3} + \frac{x}{4} = 14$; find x.
4. $\frac{x}{a-c} - 1 = \frac{x}{a+c}$; find x.
5. $2ax - b = 3cx + 4a$; find x.
6. $ax - 4c = 2b - c$; find x.
7. $3(a-x) = \frac{x}{2} + 3$; find x.
8. $5ax - 2b + 4bx = 2x + 5c$; find x.
9. $x + \frac{a}{2} = \frac{ab}{3}$; find x.
10. $2x + \frac{x}{3} + \frac{x}{2} = 2 - x$; find x.
11. $3x = \frac{x+24}{3}$; find x.
12. $\sqrt{x+2} = 3$; find x.
13. $b + 2x = b - \sqrt{b^2 + x^2}$; find x.
14. $\frac{a}{bx} = a^2 - b^2 + \frac{b}{ax}$; find x.
15. $x^2 + \frac{3x^2}{4} + \frac{x^2}{4} = x + 2x^2 - 9$; find x.
16. $\frac{x+6}{4} - \frac{16-3x}{12} = \frac{25}{6}$; find x.
17. $4 - \sqrt{1+x} = 2\sqrt{1+x}$; find x.
18. $3ax + \frac{a}{2} - 3 = bx - a$; find x.

19. $x+\frac{3x}{2}+\frac{9x}{2}=a-b$; find x.

20. $3(x-a)=\frac{a}{2}+x$; find x.

21. $x+4cx-8bx=8ab+c$; find x.

22. $ax+bx=\frac{a^2+2ab+b^2}{4}$; find x.

23. $\frac{ab+3}{x}-\frac{cd+4}{x}=18$; find x.

11. The solution of equations involving more than one unknown quantity requires that as many independent equations be given as there are unknown quantities. We will now investigate the solution of equations involving two unknown quantities.

12. One method of solving these equations is to obtain the value of one of the unknown quantities in one equation in terms of the other, and of the known quantities, and then to substitute the value found for that unknown quantity in the other equation. We have then an equation with only one unknown quantity, and having solved it are enabled by substituting its value in either equation to find the value of the other unknown quantity.

13. Another method of solution is to obtain the value of one of the unknown quantities in terms of the other, and of the known quantities, and also to obtain its value similarly in the other equation. Then since things that are equal to the same are equal to one another, we make the values found constitute an equation by solving which we find the value of the other unknown quantity. The substitution of its value in either equation enables us to find the value of the other unknown quantity.

14. Another method of solution is to multiply one or both of the equations by some number that will make the coefficient of one of the unknown quantities similar in both equations. By then adding or subtracting the two equations we are enabled to get rid of or eliminate one of the unknown quantities and find the value of the other.

15. For the solution then of equations containing two unknown quantities, we have three modes of operation, and we avail ourselves of whichever mode is best adapted to the particular case, the object being to obtain an equation with only one unknown quantity, to find the value of that quantity, and then by substitution to ascertain the value of the other unknown quantity.

16. We solve equations with two unknown quantities by the following

RULE.

By one of the methods indicated in 12, 13, *and* 14, *obtain an equation with only one unknown quantity. Solve it, and having thus found the value of one of the unknown quantities, substitute it in either of the equations, and thus obtain the value of the other unknown quantity.*

EXAMPLE.

(1)

$$5x - 2y = 4$$
$$2x - y = 1 \text{ ; find } x \text{ and } y.$$

$$5x - 2y = 4$$
$$\therefore 5x = 2y + 4$$
$$\text{and } x = \frac{2y + 4}{5}$$

Substituting this value of x in the 2nd equation

$$\frac{4y + 8}{5} - y = 1$$
$$\therefore 4y + 8 - 5y = 5$$
$$\therefore -y = -3 \text{ and } y = 3$$

But $2x - y = 1$

$\therefore 2x - 3 = 1$ and $2x = 4$

$\therefore x = 2$

(2)

$x + y = 8$

$x - y = 4$; find x and y.

From the first $x = 8 - y$

and from the second $x = 4 + y$

$\therefore 4 + y = 8 - y$

$\therefore 2y = 4$ and $y = 2$

and since $y = 2$

$x + 2 = 8$

$x = 8 - 2 = 6$

(3)

$4x + 3y = 31$

$3x + 2y = 22$

Multiplying the first equation by 3, $12x + 9y = 93$

Multiplying the second " by 4, $12x + 8y = 88$

Subtracting $y = 5$

And since $y = 5$

$4x + 15 = 31$

$\therefore 4x = 16$

$\therefore x = 4$

Exercise XV.

1. $x - y = 1$

$x + y = 9$; find x and y.

2. $2x + 3y = 31$

$3x - 5y = 18$; find x and y

3. $\frac{x}{2} + \frac{y}{3} = 14$

$\frac{x}{3} - \frac{y}{5} = 3$; find x and y

4. $\frac{x+y}{9} + \frac{x-y}{2} = 8$

$\frac{x+y}{3} - (x-y) = 4$; find x and y

5. $ax + by = 10$

$bx + cy = 8$; find x and y.

6. $ax + by = c$

$mx - ny = d$; find x and y.

7. $\frac{x+y}{2} - 2y = 2$

$\frac{x-4y}{2} + y = 3$; find x and y.

8. $x + a = y + b$

$2x + 7 = y - 3$; find x and y.

9. $x + y = a$

$bx + cy = de$; find x and y.

10. $x + y = 20$

$x - 2y = 5$; find x and y.

11. $3x - 5y = 13$

$2x + 7y = 81$; find x and y

12. $\frac{3x-5y}{2} + 3 = \frac{2x+y}{5}$

$8 - \frac{x-2y}{4} = \frac{x}{2} + \frac{y}{3}$; find x and y.

17. If we have to find the values of three unknown quantities, we must, as we have seen, have three independent equations.

18. We solve these equations by taking two of the equations and thence obtaining an equation involving only two of the unknown quantities; we then take another two of the equations, and thence obtain an equation involving the same two quantities; thus we obtain two equations involving two unknown quantities. As we already know how to solve these we are able to ascertain the values of two of the unknown quantities. By substitution in one of the equations of the values already found, we obtain the value of the third unknown quantity.

19. It is not necessary to give a specific rule for the solution of these equations. We will proceed to show by examples how readily we may reduce these equations to those involving two unknown quantities only.

EXAMPLE.

$$5x - 2y + z = 8$$
$$3x + 2y - 2z = 4$$
$$x + y + z = 9\text{; find } x, y \text{ and } z.$$

Multiplying the first equation by 2, $10x - 4y + 2z = 16$
By the second equation, $3x + 2y - 2z = 4$
By addition, $13x - 2y = 20$

By the second equation, $3x + 2y - 2z = 4$
Multiplying the third equation by 2, $2x + 2y + 2z = 18$
By addition, $5x + 4y = 22$

We have thus eliminated z and obtained two equations involving two unknown quantities only, namely,

$$13x - 2y = 20$$
$$5x + 4y = 22$$

Solving these we find $x = 2$ and $y = 3$, and substituting these values of x and y in the first equation we obtain

$$10 - 6 + z = 8$$
$$\therefore z = 8 - 4 = 4$$
$$\therefore x = 2,\ y = 3 \text{ and } z = 4$$

Exercise XVI.

1. $2x + 3y + z = 17$
   $x + y + z = 9$
   $4x - y - z = 1$; find x, y, and z.

2. $\dfrac{2x + y}{3} + z = 12$
   $\dfrac{2y + z}{4} + 2x = 13$
   $\dfrac{2x - z}{2} + 4y = 9$; find x, y, and z.

3. $\dfrac{x + y - 2z}{2} = 2$
   $\dfrac{3y + z}{3} = x + 3$
   $8z - (x + y) = 14$; find x, y, and z.

4. $x + y + z = 29$
   $x + 2y + 3z = 62$
   $\dfrac{x}{2} + \dfrac{y}{3} + \dfrac{z}{4} = 10$; find x, y, and z.

CHAPTER VIII.

PROBLEMS PRODUCING SIMPLE EQUATIONS.

1. We can now practically apply our knowledge of equations to the solution of arithmetical problems. Certain facts being given in the question we have to find some quantity or quantities unknown, from their relation to other quantities as shown in the problem.

2. There is no general rule for the solution of these problems. The student must read carefully over the terms of the equation, and then putting x, or x and y, or x, y, and z, to represent the unknown quantities, he must express in algebraic language the relation subsisting between the known and unknown quantities in the problem. He then has an equation involving one or more unknown quantities, which he already knows how to solve.

EXAMPLES.

(1)

The sum of two numbers is 20, and one is two-thirds of the other. What are the numbers?

Let $x =$ one of the numbers.

Then by the question $\frac{2x}{3} =$ the other.

and $x + \frac{2x}{3} = 20$

$$\therefore 3x + 2x = 60$$
$$5x = 60$$
$$x = 12$$
$$\text{and } \frac{2x}{3} = 8$$

Therefore 12 and 8 are the numbers.

(2)

I spend every year nine-tenths of my income all but $40; what I save is just $20 less than one-fourth of my income. How much do I receive per annum?

Let $x =$ my income.

Then by the question $\frac{9x}{10} - 40 =$ what I spend.

and $x - (\frac{9x}{10} - 40) =$ what I save.

But by the question $\frac{x}{4} - 20 =$ what I save.

$$\therefore \frac{x}{4} - 20 = x - \left(\frac{9x}{10} - 40\right)$$
$$\frac{x}{4} - 20 = x - \frac{9x}{10} + 40$$

Multiplying by 20, $5x - 400 = 20x - 18x + 800$

$$5x - 20x + 18x = 800 + 400$$
$$3x = 1200$$
$$x = \$400, \text{ which is my income.}$$

(3)

I gave away to a poor person half the money I had in my pocket, and meeting another gave him four-fifths of the remainder; I had then but one dollar left. What sum had I originally?

Let $x =$ the sum I had.

Then I gave the first $\frac{x}{2}$ and had left $\frac{x}{2}$

To the second I gave $\frac{4}{5}$ of $\frac{x}{2}$ or $\frac{2x}{5}$

Then I had left $\frac{x}{2} - \frac{2x}{5}$

$$\therefore \frac{x}{2} - \frac{2x}{5} = 1$$

$$5x - 4x = 10$$

$x = \$10$, what I had at first.

(4)

A certain sum is to be divided amongst a certain number of individuals; if there were three more each would get a dollar less than he receives, but if there were two less each would receive a dollar more. How many persons are there, and what does each receive?

Let $x =$ the number of persons,

And $y =$ what each receives.

Then the sum to be divided $= xy$.

$$(x+3)(y-1) = xy$$
$$(x-2)(y+1) = xy$$

From the first $xy - x + 3y - 3 = xy$

$$-x + 3y = xy - xy + 3 = 3$$

From the second $xy + x - 2y - 2 = xy$

$$x - 2y = xy - xy + 2 = 2$$

and since $x - 2y = 2$

and $-x + 3y = 3$

by addition $y = 5$

and $x = 2 + 2y = 2 + 10 = 12$

There are therefore 12 persons, and each receives 5 dollars, the sum divided being 60 dollars.

(5)

There is a number consisting of two digits; the sum of the digits is equal to one-fourth of the number, and if 18 be added to the number the digits will be inverted. What is the number?

Let xy be the number, which will consequently be equal to $10x + y$

Now by the question $x + y = \frac{10x + y}{4}$

and $10x + y + 18 = 10y + x$

From the first $4x + 4y = 10x + y$

$\therefore 3y = 6x$

$y = 2x$

Substituting this value of y in the second $10x + 2x + 18 = 20x + x$

$12x - 21x = -18$

$9x = 18$

$\therefore x = 2$

and $y = 2x = 4$

Hence the number is 24.

(6)

There is a certain fraction; if 1 be added to the numerator it becomes [illegible], but if 3 be added to the denominator it becomes [illegible]. What is the fraction?

Let $\frac{x}{y}$ = the fraction,

Then $\frac{x + 1}{y} =$ [illegible]

and $\frac{x}{y + 3} =$ [illegible]

From the first $2x + 2 = y$

$$2x = y - 2 \text{ and } x = \frac{y-2}{2}$$

From the second $3x = y + 3$ and $x = \frac{y+3}{3}$

$$\therefore \frac{y-2}{2} = \frac{y+3}{3}$$

$$3y - 6 = 2y + 6$$

$$y = 12$$

$$\text{and } x = \frac{y-2}{2} = \frac{10}{2} = 5$$

and the fraction is therefore $\frac{5}{12}$.

Exercise XVII.

1. Find a number such that $\frac{1}{4}$ of it shall exceed $\frac{1}{6}$ of it by 3.

2. What number is that which being divided by 3, and 6 added to the quotient, and the sum then multiplied by 4 gives 60?

3. I bought wood at $4.50 per cord; if the amount I laid out had enabled me to purchase 10 cords more, it would have cost me only $3.00 per cord. How many cords did I purchase?

4. I paid an account amounting to $114.00 in English sovereigns (at $5.00 each), American half-dollars, and Canadian twenty cent pieces, using an equal number of each coin; what was the number?

5. The sum of two numbers is 23; one-third the greater added to the less is equal to 13. What are the numbers?

6. What two numbers are those whose sum is 14, and difference 4?

7. A man was employed for 20 days; each day he worked he received a dollar, each day he was idle he forfeited 20 cents; he received at the end of the time 14 dollars. How many days did he work?

8. Find a fraction such that 3 subtracted from the numerator makes it [illegible], but if 20 is added to the denominator it becomes [illegible].

9. A person has two horses and a sleigh worth $50. If the first horse is harnessed to the sleigh they are worth three times as much as the second horse; but if the second horse be put to the sleigh they are worth exactly the value of the first horse. What is each horse worth?

10. A number consists of two digits whose sum is 9; add 63 to the number and the digits become inverted. What is the number?

11. A and B have each a certain sum; A asked B for 15 dollars, so that what he would then have might equal 5 times what B had. B in reply asked A for 5 dollars, so that the sum each had might be equal. What sum does each possess?

12. A man purchased two building lots and a house adjoining. He paid for one of the lots twice as much as for the other, and for the house double what he paid for the building lots, while the entire property cost him $7200. What was the price of each lot and of the house?

13. [illegible] number of votes polled at a recent election was [illegible]; the successful candidate had a majority of 234; how many votes were recorded for each candidate?

14. In discharging some accounts I paid away successively [illegible], [illegible], [illegible], and [illegible] of my money. I had then but £2 left; what had I at first?

15. There are two numbers. Twice the greater is 8 less than four times the less; but if to three times the greater you add twice the less and divide the sum by 61, the quotient will be 4. What are the numbers?

16. There are three numbers; the first added to twice the sum of the other two amounts to 47; the second and twice the sum of the other two equals 45; the third and twice the sum of the other two gives 43. What are the numbers?

17. Two persons live on their private fortunes. A has $50,000 invested in stock, but has borrowed $20,000 on the security of it; while B has $60,000 invested in the same stock, on the security of which he has only borrowed $10,000, at the same rate of interest as B. A's net income is £750, while B's is £1075. What rate of interest does the stock in which they have invested yield, and what rate of interest are they paying for the money that they have borrowed?

18. A certain number consists of three digits. If 396 be added to the number its digits become inverted. The middle digit is equal to the sum of the other two divided by 2; and if the number be divided by the sum of its digits the quotient will be 24. Find the number.

19. The sum of two numbers is a; their difference b. What are they?

20. I find that I can get a book printed for me for so much a page, and I have 600 dollars which I can spare for the expense. I find, however, that the book is so extensive that it will cost me 300 dollars more than I have, and I am compelled to reduce it. This I do by cancelling 200 pages. I then find that I have 60 dollars more than is needed for the purpose. How many pages would the book have made if it had not been abbreviated, and what is the cost of printing per page?

CHAPTER IX.

QUADRATIC EQUATIONS.

1. We have seen that a quadratic equation is one in which the second power or square of the unknown quantity is involved. We design to give the student some insight into these equations here, leaving the more complete study of them to future investigation.

2. Quadratic equations are divisible into two classes, *Pure* and *Adfected*.

3. A Pure Quadratic is one in which the unknown quantity is found of the second power only, while an Adfected Quadratic is one in which the unknown quantity is found both of the second and also of the first power. Thus $x^2 = 4$ is a pure quadratic; $x^2 + 2x = 8$ is an adfected quadratic.

4. Pure quadratic equations are solved just like simple equations, excepting that when the equation shows the value of x^2 we require to extract the square root of both sides of the equation, and thus find the value of x.

EXAMPLE.

(1)

$$2x^2 + 8 = 16 \quad \text{find } x.$$
$$\therefore 2x^2 = 16 - 8 = 8$$
$$\text{and } x^2 = 4$$
$$\therefore x = \pm 2$$

(2)

$$3x^2 + 5a = 8b$$

$$\therefore 3x^2 = 8b - 5a$$

$$\text{and } x^2 = \frac{8b - 5a}{3}$$

$$\therefore x = \pm \sqrt{\frac{8b - 5a}{3}}$$

5. In the first example having obtained the value of x^2 just as in a simple equation, we extract the square root of each side; and since + may be produced either by the inter-multiplication of positive or negative quantities, the square root of x may be positive or negative, and is written $\pm x$. The second example differs from the first only in that as we cannot extract the square root of $\frac{8b - 5a}{3}$, we place the radix before it.

6. An adfected quadratic is solved by the following

RULE.

Reduce the equation to the simplest form, bringing all the terms in which x is involved to the left-hand side, and the known quantities to the other. Then divide the equation by the coefficient of x^2, if it have any numerical or literal coefficient.

Add the square of half the coefficient of x to each side of the equation, by which means the left-hand side of the equation will become a complete square.

Extract the roots of both sides of the equation, prefixing to the right-hand side the sign ±.

EXAMPLES.

(1)

$x^2 + 6x + 4 = 44$; find x.

Here transferring the known quantity $\quad x^2 + 6x = 44 - 4 = 40$

Adding the square of half the coefficient of x $\quad x^2 + 6x + 9 = 49$

Extracting the root $\quad x + 3 = \pm 7$

$x = \pm 7 - 3 = 4$ or -10

(2)

$x^2 - px = q$; find x.

Completing the square $\quad x^2 - px + \frac{p^2}{4} = q + \frac{p^2}{4}$

Extracting the root $\quad \therefore x - \frac{p}{2} = \pm \sqrt{q + \frac{p^2}{4}}$

$\therefore x = \frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}$

(3)

$x + \sqrt{4x + 1} = 11$; find x.

Transposing x in order to square and thus get rid of the radix $\quad \sqrt{4x + 1} = 11 - x$

Squaring $\quad 4x + 1 = 121 - 22x + x^2$

Transposing $\quad -x^2 + 26x = 120$

$x^2 - 26x = -120$

Completing the square $\quad x^2 - 26x + 169 = 169 - 120 = 49$

Extracting the root $\quad x - 13 = \pm 7$

$\therefore x = 13 \pm 7 = 20$ or 6

7. When two unknown quantities are involved in a quadratic, the solution may be made according to the form of the equation by different modes, and the most practicable mode of solution is ascertained by careful inspection of the form. We may, as in simple equations, obtain an equation involving only one unknown quantity, and thus solve the equation, or we may proceed by a readier method, if practicable.

EXAMPLE.

(1)

$$x^2 + y^2 = 13$$
$$xy = 6; \text{ find } x \text{ and } y.$$

Here since $xy = 6$, by adding and subtracting $2xy = 12$ from the first equation, we obtain

$$x^2 + 2xy + y^2 = 25 \quad \therefore x + y = \pm 5$$
$$x^2 - 2xy + y^2 = 1 \quad \therefore x - y = \pm 1$$
$$\text{Hence } 2x = \pm 6 \text{ and } x = \pm 3$$
$$\text{and } 2y = \pm 4 \text{ and } y = \pm 2$$

EXERCISE XVIII.

1. $x^2 + 4x = 21$; find x.
2. $x^2 - 8x = 9$; find x.
3. $2x^2 - 4x + 18 = 34$; find x.
4. $\frac{x}{5} + \frac{8}{x - 18} = 8$; find x.
5. $x^2 - 2x = a$; find x.
6. $ax^2 - bx = c$; find x.
7. $x^2 + y^2 = 25$
   $xy = 12$; find x and y.
8. $x^2 - y^2 = 72$
   $x + y = 12$; find x and y.

9. What is that number from the square of which if you deduct 6 times itself the remainder is 40?

10. Find two numbers such that their difference is 8, and their product 240.

11. What two numbers are those the product of which is 24, and the sum of their squares 148?

12. Required a number such that if you take 12 from its square, the remainder shall be 11 times the number itself.

ANSWERS TO THE EXERCISES.

Ex. I.—3. 12; 4. 1; 5. 11; 6. 36; 7. 8; 8. 12; 9. 27; 10. 28; 11. 0; 12. −320; 13. 320; 14. 36; 15. 50; 16. 22; 17. 10; 18. 0; 19. 20; 20. 5; 21. 6; 22. 170; 23. 13; 24. 15.

Ex. II.—1. $19b$; 2. $-29xy$; 3. $15ab - 8by$; 4. $8x^2y^2 + 4x^2y$; 5. $9a^2b - 8cd$; 6. $7xyz + 10xy - 10yz$; 7. $14ax + 14ac + 5f$; 8. $7ax^3 + 10y + 26$; 9. $21a - 8b + 7y - 7z$.

Ex. III.—1. $2ab$; 2. $14x - y$; 3. $2ax - 4cx$; 4. $-6xyz$; 5. $9db - 8xy$; 6. $13x^2y^2 + 4xy + 1$; 7. $a - ac$; 8. $2a^2 + 2\sqrt{y}$; 9. $4ab + 3xy + x^2y^2 + 3z$.

Ex. IV.—1. $11a + 9b$; 2. $10x^2y - z$; 3. $15a^2b + 2xy + 15x^2y^2 - xyd + abc$; 4. $11a^2 + 4b^2 - 6ab + 13bc + x^2y^2$; 5. $5 + 15xy + 8x^2y^2 + 10xy^2 + 2x^2y + xyz$; 6. $18ab - 9ab^2 + 2a^2b^2 + 9ac^2 + 9bc$; 7. $27ax + 12yz + 5d + 4$; 8. $8x + 6xy - 6y + 3z$; 9. $7a^2 + 13ab + 22bc + 5cd + 2d$; 10. $32ax^2 + 24by + abcd + 48$; 11. $x^2 + 2xy + y^2$; 12. $3ab - 4bc + yz + xy$; 13. $12abc - 12ax - 6axd + 2bc + y$; 14. $11a^2 + 2a^2b^2 + 4b^2$; 15. $6x^2y + ab + ax + 2b + z + 4x^2y^2$; 16. $4\sqrt{xy} + 5bc - 8 + xy - 2xy^2 + d + x^2y^2$; 17. $6ab + \sqrt{(a + x)} + 8(a + x) + 2xy - x^2$; 18. $7ax + 6\sqrt{x^2y} + 6 + 3xy + 2xy^2$.

Ex. V.—1. $2ab+5c+11$; 2. $2a-y^2$; 3. $2x^3-x^2-10x$; 4. $16x^2y^2+2ab-4a^2y-1$; 5. $-14x+2abc-5ax+8$; 6. $-x^2y-9x^2y^2-5bc+x$; 7. $ab-15bc-cd+10dex+8ex+9ax$; 8. $8y^2-4xy-6x^2y-bc-6ab$; 9. $ax-13xy+4yz$; 10. $108-17b+9xy+z$; 11. $rst-15bcy-2y-8z+9yz$; 12. $3ax-17bx+9cx+7ayz+b^2$; 13. $a^2x-3ax^2-2xy-y$; 14. $b^2-2bx-c-d$; 15. $3xy-10ab+3cd-y-2\sqrt{x}-x$.

Ex. VI.—1. $48a^2b^3xy$; 2. $-12x^3yz^3$; 3. $16abcdxyz$; 4. $6a^2b^3cx$; 5. $16b(a-x)$; 6. $56ac(bx-y)$; 7. $9a^3x^3$; 8. $6ax^2y^5$; 9. $2a^3b^2xy$; 10. $6a^3x^3y^3$.

Ex. VII.—1. $56a^2b+16ax+24a$; 2. $a^3-3a^2b+3ab^2-b^3$; 3. $2x^3+5x^2+2x$; 4. a^3+x^3; 5. $2acx-14ab-c^2x^2+7bcx$; 6. $x^3-12x^2+34x-8$; 7. $x^3-3x^2y+3xy^2-y^3$; 8. $4ax-4bx-4cx+4ay-4by-4cy$; 9. $5a^2b^2+51xy+30+a^2b^2xy+9x^2y^2$; 10. $27x^3+y^3+18x^2-6xy+2y^2$; 11. $15a^3+19a^2x-14ax^2+2x^3$; 12. $a^6-6a^5x+16a^4x^2-18a^3x^3-2a^2x^4+6ax^5+x^6$; 13. $2a^2x^2-abx+3acx-b^2+c^2$; 14. $2a^4-4a^2b^2+a^2c^2+2b^4-b^2c^2$; 15. x^6-y^6; 16. $a^4+a^2b^2+b^4$; 17. $x^3+3x^2y+3xy^2+y^3$; 18. $x^4-4x^2y^2+y^4$.

Ex. VIII.—1. $2ab$; 2. $\frac{8xy}{3z}$; 3. $-\frac{8ac}{b}$; 4. $-3xy$; 5. $\frac{2a}{c}$; 6. $-\frac{2ab}{y}$; 7. $\frac{8a^3x}{y}$; 8. $3axy$.

Ex. IX.—1. $2a-\frac{2y^2}{a}$; 2. $-a-d-\frac{d}{b}$; 3. $\frac{7xy}{2}-\frac{4x^2}{b}$; 4. $-4x+8$; 5. $-2ab-2-2a$; 6. $\frac{5x}{3y}+\frac{5y}{3x}+5$; 7. $5+5ab$; 8. $4+2x+3ax^2$.

Ex. X.—1. $a-b$; 2. $a+x$; 3. $3a^2-ax+3x^2$; 4. $1+\frac{6x}{x^2-2x+3}$; 5. a^2-ax+x^2; 6. y^2+2; 7. $a-bc+\frac{cd}{2x-3y}$; 8. $b+8c+\frac{ab+3y}{a-2ab}$; 9. $x^2+2a-\frac{y^2}{x-y}$; 10. $8ab-4a$; 11. $x^3+x^2y+xy^2+y^3$; 12. $a^2-2ab+b^2$; 13. $24x^2+10ax-12a^2$; 14. $3a^2-5b^2+3c^2$.

Ex. XI.—1. $3b$; 2. xyz; 3. $a+b$; 4. $a-2$; 5. $x-2$; 6. $a-2ab$; 7. *They have no G. C. M.*; 8. $ax+x^2$; 9. $3x^2-2$; 10. $6a^2b^2$; 11. $3(x+y)^2$; 12. $6(a^2-b^2)$; 13. $36x^2y^2(x+y)$; 14. $144abx^2y^2$; 15. $axy(x-y)$.

Ex. XII.—1. $\frac{2b+4a}{x}$; 2. $\frac{1}{x+y}$; 3. $\frac{3ab}{2a+d}$; 4. $\frac{ab-bx-a^2}{b}$; 5. $a+b$; 6. $\frac{ax-4a}{2a}$ and $\frac{2x-2a}{2a}$; 7. $\frac{24a^2b-16a}{16a^2b}$ and $\frac{4ab^2}{16a^2b}$; 8. $\frac{8x-y}{x^2y}$ and $\frac{x^2y-xy^2}{x^2y}$; 9. $\frac{31xy+10y}{10x}$; 10. $\frac{a^2+b^2}{2ab}$; 11. $\frac{ax-x^2+2x}{a}$; 12. $\frac{17y^2+9y-40}{20y}$; 13. $\frac{10y-3ay}{2a}$; 14. $\frac{16x+28}{15}$; 15. $\frac{2ax-abx+bx^2}{ab}$; 16. $\frac{7x^2-19x+30}{5x}$; 17. $\frac{ab^2}{7a-14}$; 18. $3a$; 19. $\frac{3b(x-a)}{4}$; 20. $\frac{448bc}{a^2x(a-x)}$; 21. 4; 22. $\frac{24y-3}{2}$; 23. $\frac{9b}{8a}$.

Ex. XIII.—$4a^2-4ab+b^2$; 2. $a^6-6a^4x+12a^2x^2-8x^3$; 3. $1024a^{20}x^{10}$; 4. $a^2-2ax-4ay+x^2+4xy+4y^2$; 5. $a^3-3a^2b+3ab^2-b^3$; 6. $81a^8x^8$; 7. $\frac{a^6-3a^4x+3a^2x^2-x^3}{8y^3}$; 8. $a^4+8a^3b+24a^2b^2+32ab^3+16b^4$; 9. $a-x$; 10. $4a^2xy$; 11. $a^2-2ax+x^2$; 12. a^2-2x; 13. $2a-2b$; 14. $64a^6b^6x^6$; 15. $\frac{x+b}{2x+2y}$.

Ex. XIV.—1. 1; 2. 25; 3. 24; 4. $\frac{a^2-c^2}{2c}$; 5. $\frac{4a+b}{2a-3c}$; 6. $\frac{2b+3c}{a}$; 7. $\frac{16a-6}{17}$; 8. $\frac{2b+5c}{5a+4b-2}$; 9. $\frac{2ab-3a}{6}$; 10. $\frac{12}{23}$; 11. 3; 12. 7; 13. $\frac{b}{\sqrt{3}}$; 14. $\frac{1}{ab}$; 15. 9; 16. 8;

17. $\frac{7}{9}$; 18. $\frac{6-3a}{6a-2b}$; 19. $\frac{a-b}{7}$; 20. $\frac{7a}{4}$; 21. $\frac{3ab+c}{1+4c-8b}$; 22. $\frac{a+b}{4}$; 23. $\frac{ab-cd-1}{18}$.

Ex. XV.—1. $x=5$, $y=4$; 2. $x=11$, $y=3$; 3. $x=18$, $y=15$; 4. $x=10$, $y=8$; 5. $x=\frac{10c-8b}{ac-b^2}$, $y=\frac{8a-10b}{ac-b^2}$; 6. $x=\frac{bd+cn}{an+bm}$, $y=\frac{cm-ad}{an+bm}$; 7. $x=10$, $y=2$; 8. $x=a-b-10$, $y=2a-2b-10$; 9. $x=\frac{de-ac}{b-c}$, $y=\frac{ab-de}{b-c}$; 10. $x=15$, $y=5$; 11. $x=16$, $y=7$; 12. $x=12$, $y=6$.

Ex. XVI.—1. $x=2$, $y=3$, $z=4$; 2. $x=5$, $y=2$, $z=8$; 3. $x=4$, $y=6$, $z=3$; 4. $x=8$, $y=9$, $z=12$.

Ex. XVII.—1. 36; 2. 27; 3. 20; 4. 20; 5. 15 and 8; 6. 9 and 5; 7. 15; 8. $\frac{4}{5}$; 9. \$100 and \$50; 10. 18; 11. \$35 and \$25; 12. \$800, \$1600 and \$4800; 13. 759 and 525; 14. £240; 15. 60 and 32; 16. 7, 9, and 11; 17. 8 and 5 per cent; 18. 216; 19. $\frac{a+b}{2}$ and $\frac{a-b}{2}$. 20. 500 pp. and \$1.80.

Ex. XVIII.—1. 8 or -7; 2. 9 or -1; 3. 4 or -2; 4. 38; 5. $1\pm\sqrt{a+1}$; 6. $\frac{b}{2a}\pm\sqrt{\frac{c}{a}+\frac{b^2}{4a^2}}$; 7. 4 and 3; 8. 9 and 3; 9. 10; 10. 20 and 12; 11. 12 and 2; 12. 12.

THE END.

JOHN LOVELL, PRINTER.

$\frac{ab + c}{-4c - 8b}$;

$= 3$; 3. $\frac{10c - 8b}{ac - b^2}$,

nd $\overline{bm}$;

$2b - 10$;

11. $x =$

, $y = 2$,

, $z = 12$.

15 and

; 10. 18;

13. 759

and 11;

nd $\frac{a-b}{2}$.

or -2;

7. 4 and

and 2;