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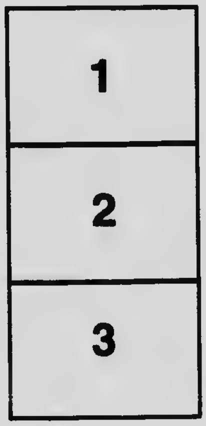
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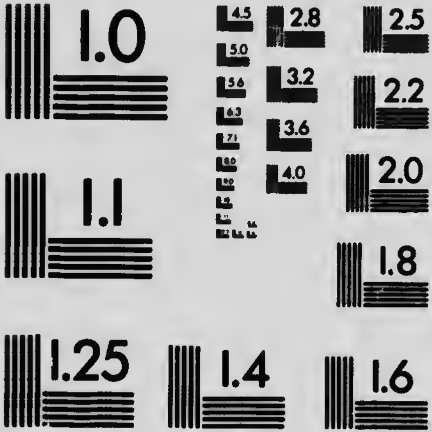

AN

ALGEBRA

FOR

HIGH SCHOOLS

AND

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"All economical and practical wisdom is an extension or variation of the following arithmetical formula: $2+2=4$. Every philosophical proposition has the more general character of the expression $a+b=c$. We are mere operatives, empirics, and egotists, until we learn to think in letters instead of figures."

—*The Autocrat.*

BY

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PREFACE

THE little work on algebra now offered to those interested in secondary education has its source in a course of lectures given by me some years ago at the Ontario Normal College. Since that time not a few teachers who had followed the course have urged me to prepare a work on algebra along the lines suggested in my lectures, insisting that the ideas then advanced are, in the main, in accord with the general movement in the direction of reform in mathematical teaching. In the hope of contributing to that reform this work was undertaken.

While the book must speak for itself, a few words as to the point of view and the plan may not be out of place.

The outstanding fact in elementary algebra is the equation. With the equation algebra began, and the greater number of the topics ordinarily dealt with in the subject stand in close relation to the equation, gaining in meaning as this relation is realized. Thus the simple rules are incidental to the equation, resolution into factors is the central idea in it, while the concept of functionality, in a certain sense embracing that of the equation, sheds light upon it. So also elimination, so important in almost every branch of mathematics, is a study in equations. It follows then that a satisfactory treatment of the subject must give prominence to the equation—to its significance and theory as well as to methods of solution.

Again, algebra, growing as it did out of arithmetic, and generalizing it, is, if it may be so put, morally bound to accept the fruits of the generalization. The step first in importance is the introduction of the *general arithmetical number-symbol*—the symbol that denotes *any* number of arithmetic, and in doing so implies *every* number. From this and the generalizing of the operations, flows the extension of the meaning to be attached to the word number. In this relation also the importance of the equation appears, for having virtually brought the fraction into arithmetic, it introduces successively the negative number which gives to algebra its special character, the irrational number, and the imaginary number. A further important generalization is that connected with the index-notation and the related concept. In the appreciation of these developments consists a great part of the educational value of the subject.

Concerning the method of treatment only slight comment need be made.

The special notation of the subject is introduced naturally, great care being taken to assure that the learner think rightly when in the presence of a general number symbol, as a, x.

The idea of the equation is then introduced, with the fact that, through it, a number-symbol, otherwise general, is assigned a definite value. The method of finding this value is briefly dealt with, while the further treatment is suspended, on economic grounds, until the student acquires a certain readiness in handling simple expressions.

The negative number appears as an invention to make it possible to speak of a difference as $a-b$ without any restriction on the generality of a and b. The laws governing the operations with arithmetical numbers are taken as applying to the system made up of positive and negative numbers, and in this way operations as multiplication and division are defined, and come to have a meaning for negative numbers.

After the simple rules are treated, methods of resolution into factors are considered and brought into close relation with the direct processes that suggest them. This part of the subject is treated at length on account of its importance in relation to the equation, and to afford a variety of exercise in acquiring readiness in manipulating expressions.

In Chapter XII are given certain methods and results of great importance yet of some difficulty. It seemed best not to give these here and there through the book. The teacher will naturally treat them as seems best to him, deferring the chapter or parts of it until such time as they may best be taken up.

The study of the equation being resumed, the idea of the function is introduced, and the circumstances under which a symbol as a or x is to be regarded as a general number, a certain definite number, or a variable, are considered. No apology for giving the function special treatment is offered, all who have considered the matter being of one opinion, and those most qualified to speak insisting on its early recognition. Closely connected with this idea is that of the graphic representation of the function. The course followed is not the conventional one. The graph renders vivid the march of the function or the relations of the concerned variables, yet it cannot be constructed until the bolder of these relations are grasped. It then exhibits them in a striking way, and suggests perhaps further relations that might have remained unnoticed. The important thing then is the study of the function precedent to the graph. Thus, in the case of the linear function, the student finds his greatest advance in seeing why the graph must be a straight line—and this is not so easy a matter at this stage—rather than in lightly accepting this fact and employing it to solve a few somewhat artificial problems. The principal care has therefore been to shew what the graph is or must be, and to employ it to throw light on the important notions of variable, function, equation.

The formal treatment of the equation is very complete, in the sense that the ideas relating thereto are developed somewhat exhaustively.

The theory of indices is developed in the same way as are the extensions of number, *i.e.*, by supposing or assuming the laws to persist and thereby assign meanings to the symbols introduced, not by assigning meanings and then seeing whether or not the laws apply.

The theory of the limit is not touched upon, a brief treatment being attended with danger, and a somewhat full treatment being out of place. On this account the question of what becomes of one root of a quadratic equation, as the first coefficient becomes zero, or of the two roots as the first two coefficients become zero, is not raised. This can be considered best in connection with the more advanced problems which suggest it and give it a natural illustration.

The exercises, it is hoped, will be found sufficiently numerous, varied and fresh. While some have been taken from examination papers, the great number have been constructed for the book. They have been arranged with a view to presenting the theory in new light, to suggesting new topics and to preparing the student for new theory. To afford further practice in the work of the earlier chapters, if desired, a supplementary set of exercises has been given at the end of the book.

To conclude, the book supposes the teacher, but makes it possible for the pupil to recover what he may have forgotten or complete what he may not have quite understood, thus encouraging him to rely on himself. The object has been to develop thoughtfulness and resource, and not merely skill on the more mechanical side of the subject. The student who proceeds with his studies will find that he will have little to unlearn.

ALFRED T. DELURY.

CONTENTS

ELEMENTARY ALGEBRA

CHAPTER I

NOTATION

1. **Number and Measurement.** As a rule, number presents itself as the measure of some quantity, and a unit is implied. Thus, if the length of a line is 3 inches, the unit is one inch and the measure is the number **3**. With advance in studies that have to do with number, the student will see that more and more he has to deal with number merely, without regard to what the unit may be. However it is well frequently to form concrete illustrations of the numbers and results.

For the present a number will be taken to mean some integer or fraction, and when quantities are in question it will be supposed that they can be measured by integers or fractions.

2. **Algebraic Number Symbols.** Suppose that a given straight line is 3 inches long, so that, the unit being 1 inch, the measure of the length is 3. Then 3 and all such symbols as

$$1, 2, \ldots., \; 0, \; \ldots., \; 13, \; \ldots. \tfrac{3}{4}, \; \tfrac{5}{7}, \; \ldots. 0{\cdot}5, \; 2{\cdot}037 \ldots.$$

are called *arithmetical number-symbols* or *arithmetical numbers*.

Let now AB be any given straight line, and let a unit

A ———————————— B

of length be taken. Without actually measuring the line, we can say that it has a certain length, and we may denote the measure by a, in other words we may use a letter to denote a number. So, too, the length of a different straight line may be given by the number b, or the weight of a piece of lead by the number x. The symbols

$$a, \ b, \ \ldots., \ l, \ \ldots., \ x, \ y, \ z,$$

or the letters of another alphabet, when employed to denote numbers, are called *algebraic number-symbols*.

In its beginnings, algebra is simply arithmetic with *literal* number-symbols in addition to the number-symbols of ordinary arithmetic.

3. **Operations and Signs of Operation.** Since the letters a, b, c, ..., denote numbers not unlike the numbers with which arithmetic deals, the simple rules—addition, subtraction, multiplication, and division—have at once a meaning with respect to them. Further, such laws as:

(*a*) *The sum of several numbers is independent of the order in which they are taken;*

(*b*) *The product of several numbers is independent of the order in which they are taken;*

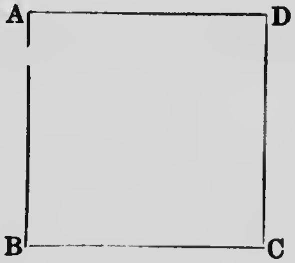
(*c*) *A series of additions and subtractions may be made in any order, provided the subtractions are always possible;*

(*d*) *A series of multiplications and divisions may be made in any order;*

are seen to hold for numbers denoted by letters, for the same reasons as in ordinary arithmetic.

The signs of arithmetic, as those of the elementary operations, $+$, $-$, $\times$, $\div$, the method of writing a fraction (or quotient) as $\frac{3}{4}$ or 3/4, the root symbol $\surd$, brackets ( ), { }, signs of abbreviation as $\therefore$ and $\because$ will be employed in algebra as in arithmetic. There are additional symbols and modes of writing results some of which will now be introduced.

Let ABCD be a square, and let the measure of the length of AB be a.



Then, since AB, BC, CD, DA are of equal length, the perimeter is given by

$$a+a+a+a.$$

This may be denoted by $a\times 4$, but this product is generally written $4a$,—read *four a*. In this product 4 is called the *coefficient* (*i.e. co-factor*) of a, or a may be called the coefficient of 4. The multiplication is sometimes indicated by a point, thus $4.a$.

The area of the square would now be given by any of the following:

$$a\times a, \quad a.a, \quad aa.$$

This product is usually denoted by a^2,—read *a square*, or *a to the second*. It is called the *second power of a*, or, *the square of a*. The 2 in a^2, giving the number of the factors

a multiplied together, is called the ***index*** or the ***exponent of the power*** or simply the ***index*** or the ***exponent.***

Next let ABCD be a rectangle: let the unequal sides AB, BC have a and b as measures of their lengths, different lengths being denoted by different letters.

The perimeter is given by

$$a+b+a+b,$$

or, since additions may be made in any order, by

$$2a+2b,$$

or, brackets indicating that the included sum is to be taken as one number, by

$$(a+b)\times 2 \text{ or } 2(a+b).$$

The area is given by any of the following:

$$a\times b, \quad b\times a, \quad a.b, \quad b.a, \quad ab, \quad ba,$$

and preferably in practice by ab (or ba). In ab, the coefficient of b is a; it is called a *literal coefficient* whereas the 4 in $4a$ is called a *numerical coefficient.*

In like manner for a cube with edge of measure h (in length) the student should see that

(i) The complete length of edge is given by $12h$;

(ii) The area of the surface is given by $6h^2$;

(iii) The volume is given by h^3,—read ***h cube***, or ***h to the third***, and called the ***third power of h***, or ***the cube of h***;

and for a block, in shape of a box, with h, k, l, as the measures of the lengths of the unequal edges, that

(i) The complete length of edge is given by

$$4h+4k+4l \text{ or } 4(h+k+l);$$

(ii) The area of the surface is given by

$$2kl+2lh+2hk, \quad \text{or} \quad 2(kl+lh+hk);$$

(iii) The volume is given by hkl.

It is to be noted that such an arithmetical symbol as 35 continues to mean *thirty five*, *i.e.* $3 \times 10 + 5$, and is therefore not to be confused with the product of 3 and 5.

Further, it is to be kept in mind that in finding the value of *expressions* or *quantities* as

$$2a+2b, \quad 2(a+b), \quad bc-hk, \quad a\times b\div c\times d+\frac{m}{n},$$

the operations are to be performed in the same order as agreed upon in arithmetic. Thus $bc-hk$ means the product of b and c, diminished by the product of h and k; and $a\times b\div c\times d+\frac{m}{n}$ means that a is to be multiplied by b, the product is to be divided by c, the quotient then multiplied by d, and the result added to the result of dividing m by n.

However, in such an expression as $6ab\div 3cd$, the product $3cd$ is to be regarded as one number, so that $6ab\div 3cd$ equals $6ab\div 3\div c\div d$.

It may also be pointed out, as a matter of contrast with ordinary arithmetic, that operations are frequently and of necessity only indicated. Thus, the sum of a and b is $a+b$ and we *think of the aggregate*, whereas the sum of 5 and 4 is 9, for we do not stop with the mere indication $5+4$.

Exercises I

1. Find the value of

(1) m acres of land at r dollars an acre.
(2) l yards of cloth at b cents a yard.
(3) mn pounds of tea at a cents a pound.
(4) h^2 tons of coal at k^2 dollars a ton.

In each case assign to the letters involved arbitrary values and work out the results.

2. Find

(1) The number of cents in x dollars.
(2) The number of cents in h dollars and k cents.
(3) The number of inches in x yards, y feet and z inches.
(4) The number of centimetres in l metres, m decimetres and n centimetres.

3. Read the following expressions,

$$ma + nb, \quad \frac{ma}{nb}, \quad \frac{a+b}{m+n},$$

and state, in each case, what is to be done to find the value.

Taking m, n, a, b as 3, 4, $\frac{1}{2}$, $\frac{1}{3}$, find the values.

4. Find the average of each of the following sets of three numbers:

(i) 7, 8, 11; (ii) $10a$, $13a$, $19a$; (iii) a, b, c; (iv) $2a, 3b, 4c$.

5. The sides of a triangle measure a, b, c; write down the expression for the perimeter and for the semi-perimeter.

6. A merchant mixes m pounds of tea worth a cents a pound with n pounds worth b cents a pound; find the value of one pound of the mixture.

7. The sides of a rectangle measure $3a$ and $4b$; find the measure of the area.

Make a figure shewing this rectangle and the rectangle whose sides measure a and b, and compare the areas.

8. The sides of a rectangle measure ma and nb; find the measure of the area and compare it with that of the rectangle whose sides measure a and b.

9. By merely varying the order of the letters, write in as many ways as possible

$$x+y+z, \quad lmn.$$

10. Express in the symbols of algebra,

(1) The sum of any two numbers.
(2) The difference of any two numbers.
(3) The sum of any two fractions.

11. If n is taken to be any integer whatever, including zero, write down the expression for any even integer and for any odd integer.

What must n be taken to be to give the even number 86, the odd number 53?

12. If A can do a piece of work in m days and B in n days, write down the part of the work that could be done by each and by both in 1 day.

13. Write down the integer, of two figures, whose tens digit is m and units digit is n.

14. Write in as simple a way as possible,

(1) $13a+5a-9a+3a-7a$.
(2) $18x+13y-7x-5y+2x-3y$.
(3) $ab \times ab \times ab$.
(4) $3ab \times 2ab \times 5ab$.

15. Compute the values of the following expressions involving x, for $x=0, 1, 2, 3, 4, 5$:

$$2x, \quad 3x+7, \quad x^2, \quad x^2+4x+5.$$

16. Compute the values of the following expressions involving x, for $x=0, 2, 4, 6, 8, 10$:

$$\frac{x+1}{x+2}, \qquad \frac{x^2+1}{x+1}, \qquad \frac{x^2+x+1}{x^2-x+1}.$$

4. Index Laws. The following examples are given with a view to leading up to certain rules known as the *index laws*. Formal proofs are not given, but the essential features of the proofs are present:

(1) To shew that

$$a^4 \times a^5 = a^9,$$

read, *a to the fourth into (or multiplied by) a to the fifth equals a to the ninth.*

The product $a^4 \times a^5$ is plainly that of 4 factors a and 5 factors a, *i.e.*, in all, of $5+4$ or 9 factors a, and this may be written a^9.

In the same way the product $a^5 . a^6 . a^3$ is seen to be that of $5+6+3$ or 14 factors a and is therefore a^{14}, and the following law may be stated: *The product of two or more powers of a number is a power of that number whose index is the sum of the indices of the given powers.*

(2) To shew that

$$a^9 \div a^5 = a^4,$$

read, *a to the ninth by (or divided by) a to the fifth equals a to the fourth.*

This follows from what has just been shewn, or may be seen directly by noting that the 5 factors a of the divisor *cancel* or *divide out* 5 of the 9 factors a of the dividend leaving still $9-5$ or 4 factors a to be multiplied together, *i.e.* a^4.

The following law may now be stated: *The quotient of a higher power of a number by a lower power, is a power of that number with index the difference of the indices of the given powers.*

(3) To shew that

$$(a^4)^3 = a^{12},$$

read, *a to the fourth, raised to the third power equals a to the twelfth.*

From (1) and the meaning of the notation it follows that

$$\begin{aligned}(a^4)^3 &= a^4 \times a^4 \times a^4 \\ &= a^{4+4+4} \\ &= a^{4\times 3} \\ &= a^{12}.\end{aligned}$$

Thus it appears that: *A power of a number raised to a given power is a power of that number with index the product of the indices of the given powers.*

EXERCISES II

1. Write out in full the following products, and write the expressions in the simplest form:

$$x^3 \times x^4; \quad y^2 \times y^3 \times y; \quad l^5 \times l^7.$$

2. Write out in full the following quotients and write the results of division in the simplest form:

$$x^8 \div x^5; \quad h^9 \div h^4; \quad a^3b^4 \div a^2b^2.$$

3. Apply the laws stated to find in the simplest form,

$$a^7 \times a^5 \times a^9; \quad a^9 \times a^8 \div a^6; \quad (a^2b^3)^2 \times a^2b^5;$$
$$(x^2y^4)^3 \div x^4y^7; \quad (2m^2n^2)^4 \div 4m^5n^5; \quad (12x^2y^3)^2 \div 36xy.$$

4. Examine into the meaning and value of

$$(ab)^3, \quad x^7 \div x^7, \quad y^3 \div y^5.$$

5. If m and n are integers state what is understood by

$$x^m, \quad x^n, \quad x^m \times x^n, \quad (x^m)^n.$$

5. **Concerning Algebraic Number-Symbols.** It has been seen that the letters

$$a, \quad b, \quad \ldots . l \ldots . x, \quad ..$$

denote numbers that are not different in character from those denoted by the number-symbols of arithmetic, as

$$7, \quad \tfrac{1}{3}, \quad 0, \quad \ldots . 23 \cdot 5, \quad \ldots .$$

in that, like them, they can be added, subtracted, multiplied, and divided, according to the same laws. But in employing the two kinds of number-symbols the student has probably come to feel that there is a certain difference between them. This difference can be illustrated by considering a square with side of measure 5 (in length) and a square with side of measure a. In the former case, we say that the perimeter is given by 5×4 or 20, and we think of a certain definite square; in th latter case we say that the perimeter is given by $4a$, and i so saying we feel that the statement applies not to one square rather than another, but *to any square whatever*, or *equally to all squares*.

In like manner we can say

(i) The cost of m pounds of tea at v cents a pound is mv cents, *whatever be m and v*.

(ii) $a(b+c)=ab+ac$, *whatever be a, b, c*.

In this way we come to regard the numbers denoted by letters as *general*, in contrast with the numbers of arithmetic which may be spoken of as *particular*, and when a number as x or l is mentioned we think of it as being *any number, i.e., any integer or fraction whatever*, unless something is stated to limit it in some way.

Thus, algebra, in employing literal number-symbols, lends itself to the expression, by number-symbols, of statements of general truth, statements which in the symbols of ordinary arithmetic may be made only for particular cases.

Exercises III.

1. A man is now 40 years old; how old was he x years ago, and how old will he be in m years?

2. State what is to be done to find the value of

$$2ab+5hk-7yz\div mn$$

and read the expression.

3. A man is m years of age; how old was he n years ago, how old will he be in l years, and in how many years will he be twice as old as he is now?

4. State what is to be done to find the value of

$$\frac{3x}{5y} - \frac{5m}{7n}$$

and read the expression.

5. The area of a rectangle measures a and its length measures l; find the measure of its width.

6. Compute the values of the expressions

$$2x+3,\ 2x^2,\ 3x-2x^2$$

for

$$x=0,\ 0{\cdot}1,\ 0{\cdot}2,\ 0{\cdot}3,\ 0{\cdot}4,\ 0{\cdot}5,\ 0{\cdot}6,\ 0{\cdot}7,\ 0{\cdot}8,\ 0{\cdot}9,\ 1.$$

7. Express in the symbols of algebra:

(1) The product of the sum of any two numbers and the difference of the same two numbers;

(2) The excess of the sum of any two numbers over the difference of any other two numbers.

8. Taking n to signify any integer whatever, write down the general expression for

(1) Three consecutive integers;

(2) Three consecutive even integers;

(3) Three consecutive odd integers.

9. A man starts from a place P on a highway running east and west and walks

(1) 13 miles east and then 8 miles back;

(2) 8 miles east and then 13 miles back;

(3) a miles east and then b miles back;

at what point does he, in each case, find himself?

10. A has m marbles of one kind and B has n of another kind; A gives x of his marbles to B in exchange for y of his. Find how many marbles are now owned by each.

11. If n means any integer whatever, state the arithmetical fact expressed by the relation

$$(2n-1)+(2n+1)=2(2n).$$

12. Write down the number whose hundreds, tens, and units digits are l, m, n.

13. If π denotes the definite number (approximately 3·1416) which gives the ratio of the lengths of the circumference and the diameter of a circle and if the measures of the radius, circumference, diameter, and area of a circle are denoted by r, c, d, A, state in words the meaning of the following statements:

(i) $d=2r$
(ii) $c=2\pi r$
(iii) $A=\pi r^2$

14. Multiply

(1) 57 by 38;

(2) The number of two figures whose digits in order (left to right) are a, b, by the number of two figures whose digits in order are h, k.

15. State the numerical fact expressed by the relation

$$\frac{a}{b}=\frac{ma}{mb}$$

16. Write down the integer of two figures, of which the tens and units digits are m, n.

Write down also the integer formed by reversing the order of the digits.

Add the two integers thus formed and comment on the result.

Illustrate by particular examples.

17. Find in how many different ways, by merely varying the order of the letters, each of the following may be written :—

(i) $a+b+c+d$; (ii) $abcd$; (iii) $x(y+z)$; (iv) $\frac{ab}{cd}$

18. State what facts are expressed by the following relations:—

(1) $a+b+c=a+c+b=b+c+a$;
(2) $abc=bca=acb$.

19. State what facts are expressed by the following relations:

(1) $a(b+c)=ab+ac$;

(2) $\frac{x}{a}+\frac{y}{a}=\frac{x+y}{a}$.

20. Shew that the number of three figures whose digits in order from left to right are a, b, c is equal to

$$9(11a+b)+(a+b+c).$$

21. "Think of a number, double it and add twelve; add the number thought of and divide by 3; add 3 times the number thought of, divide by 4 and subtract 1; give the result." The result is the number thought of.

Take n as the number and follow the work out.

6. **Brackets**. Already brackets have been employed, as in arithmetic, to indicate that the quantity enclosed is to be regarded as one number. The following simple illustrations are given to bring out certain important rules, a knowledge of which cannot too early be acquired.

(1) *To add the sum of two numbers.*

$$5+(3+4)=5+3+4=12.$$
$$a+(b+c)=a+b+c.$$

Illustrate by taking a, b, c as the measures of lengths of straight lines.

(2) *To subtract the sum of two numbers.*

$$7-(3+2)=7-3-2=2.$$
$$a-(b+c)=a-b-c.$$

Illustrate as in (1).

(3) *To add the difference of two numbers.*

$$7+(5-3)=7+5-3=9.$$
$$a+(b-c)=a+b-c.$$

Illustrate as in (1).

(4). *To subtract the difference of two numbers.*

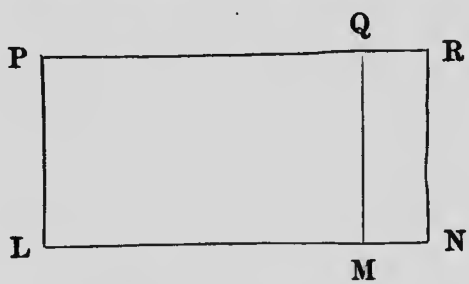
$$7-(5-2)=7-5+2.$$
$$a-(b-c)=a-b+c.$$

Illustrate as in (1).

(5) *To multiply the sum of two numbers by a third number.*

$$(3+2)\times 4=3\times 4+2\times 4=12+8=20.$$
$$(a+b)c=ac+bc.$$
$$c(a+b)=ca+cb.$$

To illustrate, take LM, MN, LP, as in the figure, of measures a, b, c in length.



Then $(a+b)$ is the measure of LN. The rectangles PM, QN, PN, have, as measures of their areas, ca, cb, $c(a+b)$ and it is seen that

$$c(a+b)=ca+cb.$$

(6) *To multiply the sum of two numbers by the sum of two other numbers.*

$$(5+3)(4+7)=(5+3)\times 4+(5+3)\times 7$$
$$=5\times 4+3\times 4+5\times 7+3\times 7$$
$$=20+12+35+21=88.$$
$$(a+b)(x+y)=(a+b)x+(a+b)y$$
$$=ax+bx+ay+by.$$

Illustrate as in (5).

EXERCISES IV.

1. State tersely the reason for saying
$$a+(b-c)=a+b-c;\quad a-(b-c)=a-b+c.$$

2. Write without using brackets the following:
$$a-(b+c+d);\quad a+(b-c-d);\quad (x+y)-(z+w).$$

3. Employ brackets to write as the sum of two numbers:
$$a+b+c+d;\quad x+y-z-u;\quad x-y+z;\quad h-k+l-m.$$

4. Employ brackets to write as the difference of two numbers:
$$a-b-c;\quad x+y-z-u;\quad x+y-z;\quad h-k-l-m.$$

5. Find the value of, *i.e.*, write without brackets,
$$(a+b)(x-y).$$
Illustrate by a figure.

6. Find the value of:
$$x(a+b+c);\quad (a+b+c)(x+y);\quad (a+b+c)(x+y+z).$$

7. Write as the product of two numbers:
$$mx+nx;\quad mx-nx;\quad ax+bx+cx.$$

8. Without changing the order of the letters, write as the sum of two numbers, in as many ways as possible, the following:
$$a+b+c;\quad x+y+z+w;\quad h+k+l+m+n.$$

9. What does the relation
$$(a+b)(x+y)=ax+bx+ay+by$$
become if x is a and y is b?

Illustrate by a carefully drawn figure.

CHAPTER II

EQUATIONS

7. Algebraic Solutions. A few simple problems in arithmetic will now be considered with a view to shewing that the employment of the number-symbols of algebra may win a certain directness in the solution or may reduce the difficulty. The student should work the problems, avoiding the symbols of algebra, and compare the two treatments.

(1) *Divide \$72 among A, B and C, so that B may have 3 times, and C may have 5 times as much as A.*

Let $x =$ the number of dollars in A's share.
$\therefore\ 3x =$ " " " B's "
and $5x =$ " " " C's "
$\therefore\ x + 3x + 5x$, *i.e.*, $9x =$ " " " the shares of all.

$$\therefore\ 9x = 72.$$
$$\therefore\ x = 8.$$

$\therefore$ A's share $=$ \$8, B's $=$ \$8 $\times$ 3 or \$24, and C's share $=$ \$8 $\times$ 5 or \$40.

(2) *B's age is twice A's age, and in 10 years the sum of their ages will be 50 years; find their ages.*

Let $x =$ the number of years in A's age.
$\therefore\ 2x =$ " " B's "

$\therefore\ x + 10$ and $2x + 10$ give their respective ages 10 years hence.

$$\therefore\ (x + 10) + (2x + 10) = 50.$$
$$\therefore\ x + 10 + 2x + 10 = 50.$$
$$\therefore\ 3x + 20 = 50.$$
$$\therefore\ 3x = 30,\ (i.e.,\ 50 - 20).$$
$$\therefore\ x = 10.$$

$\therefore$ A's age is 10 years, and B's is 20 years.

(3) *The length of a rectangular field is twice its breadth, and its area is 45 acres; find the dimensions of the field.*

Let x = the number of rods in the breadth.

$\therefore$ $2x$ = the number of rods in the length.

$\therefore$ $2x.x$ or $2x^2$ = the number of square rods in the area.

But the area is 45 acres, or (160×45) square rods.

$$\therefore\ 2x^2 = 160 \times 45.$$

$$\therefore\ x^2 = 80 \times 45, \text{ or } 3600.$$

$$\therefore\ x = 60.$$

$\therefore$ The field is 60 rods by 120 rods.

8. Meaning of Equation. In each of the solutions just given, the letter x has been employed as a number-symbol; in contrast with the letters employed in Chapter I., in no case does it represent *any number we please* or *any number whatever*, but rather a certain definite number whose value is for a time *unknown*. The solution of each problem culminates in a certain *statement of equality*, namely, in order,

$$(1)\ 9x = 72,$$

$$(2)\ (x + 10) + (x + 20) = 50,$$

$$(3)\ 2x^2 = 160 \times 45,$$

and, reasoning from these statements, we find that they, each for its own problem, lead to, or determine the value x must have.

The statements of equality thus [illegible]t are called *equations;* the letter x, appearing in th[illegible], is called the *unknown* (of the equation); the finding of the value of x that will satisfy the equation is called *solving the equation;* and the value thus found is called the *root* or the *solution* of the equation.

Thus an equation in x implies that x does not stand for any number whatever or equally for all numbers, but denotes a certain particular number, an unknown, whose value is to be found by reasoning from the equation.

It is scarcely necessary to say that any other letter than x might just as well be taken, but as a rule the unknowns, whose values are given by equations, are denoted by the later letters of the alphabet.

Exercises V

1. Divide $565 among A, B and C, so that B may have $10 more than twice as much as A, and C may have $10 more than twice as much as B.

2. A has three times as much money as B; he gives $10 to B, and then finds that he has only twice as much as B. Find how much each at first had.

3. A is three times as old as B, but in 15 years he will be only twice as old; find the age of each.

4. The sum of two numbers is 37, and one of them exceeds the other by 11; find the numbers.

5. The units digit of an integer expressed by two figures is twice the tens digit: the integer formed by reversing the digits exceeds the first integer by 27. Find the integer.

6. A merchant bought a certain number of pounds of tea at 30 cents a pound and twice as much at 40 cents a pound and mixed the two kinds. By selling the mixture at 50 cents a pound he gains $60. Find the number of pounds of each kind bought.

7. The length of a certain rectangular field is three times its breadth. A second field of the same perimeter is square, and contains 10 acres more than the first field. Find the dimensions of the fields.

8. A man borrows $400 dollars at a certain rate per cent., and $300 at a rate higher by 1 per cent., and the yearly interest on the two sums is $31. Find the rates.

9. In connection with certain problems the following statements present themselves:

$$(1)\ x\,(a+b)=ax+bx;$$
$$(2)\ 2x+13=37.$$

Contrast the two statements.

CHAPTER III

POSITIVE AND NEGATIVE NUMBERS

9. Extension of the Meaning of Number. Let a and b be any two algebraic number-symbols, or, for brevity of statement, *any two numbers;* then, it is plain that it is always possible to find the result of adding b to a, of multiplying a by b, and of dividing a by b. It is otherwise, however, if b is to be subtracted from a, *i.e.*, if we have to find the value of

$$a - b.$$

Thus, if, as may well be, a is 3 and b is 5, we have to find the value of

$$3 - 5,$$

and there is standing in the way the fact that in subtraction we have required the number to be taken away, to be not greater than the other; accordingly, if we continue to look upon subtraction in this way, we cannot in an expression as

$$a - b,$$

suppose that a and b can be any numbers whatever.

A like difficulty arose in very elementary arithmetic, for, though we could say $14 \div 7 = 2$, when it came to a division like $19 \div 7$ we said that the quotient was 2 with a remainder 5, *i.e.*, a 5 that could not be divided by 7. When, however, the fractional number was introduced, and our meaning of number widened, we gave as the quotient $2\frac{5}{7}$.

So here we shall seek an extension of the meaning of number, in order that we may speak of the difference

$$a - b$$

whatever values a and b may have. Taking the difference

$$3 - 5,$$

we see that, in accordance with what we know of numbers, it may be looked upon as

$$3 - (3 + 2) \text{ or } 3 - 3 - 2.$$

Here $3 - 3$ is zero, and the value appears as 2 to be subtracted, with nothing from which to subtract. We shall say that the result is the *negative number* 2, or *minus* 2, and denote it by the symbol

$$(-2), \text{ or simply } -2.$$

The sign $-$ in this symbol indicates the negative character, but, since in a number as $5 - 2$ it indicates a subtraction, it might seem that confusion would arise. But, just as in arithmetic no trouble springs from the fact that the one symbol $\frac{2}{3}$ means the quotient of 2 by 3, or the fraction two-thirds, no serious difficulty will come of it if, at the outset, a little care is taken.

Take next the difference

$$8 - 10,$$

which we should now say is equal to -2. We can say, in accordance with our way of regarding numbers,

$$\begin{aligned} 8 - 10 &= (3 + 5) - (5 + 5), \\ &= 3 + 5 - 5 - 5, \\ &= 3 - 5, \end{aligned}$$

so that the -2 that comes from $8 - 10$ is the same as that which comes from $3 - 5$. Hence while -2 may arise from many subtractions it is a definite number, just as $\frac{2}{3}$ is a definite number, though it may come from different divisions, as 2 by 3, 4 by 6, and so on.

Next we have

$$2-3=-1,$$

and
$$\begin{aligned}4-6&=(2+2)-(3+3)\\&=2+2-3-3\\&=(2-3)+(2-3)\\&=(-1)+(-1).\end{aligned}$$

But
$$4-6=-2.$$

$$\therefore (-2)=(-1)+(-1).$$

Hence it appears that negative numbers as -2, -3, etc., are built up from -1, just as the numbers of arithmetic are built up from 1, and it follows that

$$(-3)+(-5)=(-8) \text{ or } -8.$$

In the statement

$$3-5=-2,$$

the numbers 3 and 5 on the left are the numbers of arithmetic. When we think of them in relation to the negative numbers now introduced we call them *positive numbers*, and denote this by writing the sign + before them; we may then make such a statement as

$$5-3=(+2),$$

or simply,

$$5-3=+2.$$

Now,
$$\begin{aligned}(5-3)+(3-5)&=5-3+3-5\\&=0.\end{aligned}$$

$$\therefore \quad (+2)+(-2)=0.$$

In like manner,

$$(+1)+(-1)=0,$$
$$(-5)+(+5)=0,$$

and we have the essential property of positive and negative numbers:

$$(+a) + (-a) = 0,$$

or *the sum of a positive number and the corresponding negative number is zero.*

Further, we see that

$$(+5) + (-3) = (+2) + (+3) + (-3)$$
$$= +2,$$

and the student can easily discover how to find the value of expressions as

$$(+5) + (-3) + (-7) + (+9),$$
$$(-a) + (-3a) + (+7a) + (-5a).$$

Next let us compare the two numbers

$$5 - 3 \text{ and } (+5) + (-3).$$

We see that each equals $+2$, so that the two expressions are equivalent. In like manner

$$3 - 5 \text{ and } (+3) + (-5)$$

are seen to be equal, and we see that an expression as

$$a - b$$

can be regarded as *the subtraction of b from a*, or *the addition of the negative number b to the positive number a.*

Hence it may be said that a principal difference between arithmetic and algebra consists in this extension of the idea of number—in other words that in arithmetic we have to do with the numbers, integral and fractional,

$$0, \ldots, \tfrac{1}{2}, \ldots., \tfrac{3}{5}, \ldots., 2, \ldots.$$

and in algebra with the numbers,

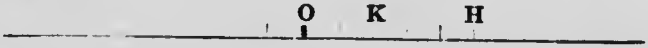
$$\ldots -2, \ldots -\tfrac{3}{5}, \ldots -\tfrac{1}{2}, \ldots 0, \ldots +\tfrac{1}{2}, \ldots +\tfrac{3}{5}, \ldots +2 \ldots,$$

the former extending indefinitely *to the right*, the latter extending indefinitely *in both directions*. It has been seen also, on account of the equality of $a-b$ and $(+a)+(-b)$, that any statement may be regarded arithmetically or algebraically.

10. Illustrations. Some illustrations will now be given to shew that negative numbers have, or may have, a meaning in relation to actual quantities somewhat as the fraction $\frac{2}{3}$ was found to have a definite meaning in the expressions $\frac{2}{3}$ yard, \$$\frac{2}{3}$.

Illustration 1. From a point O on a straight road, running east and west, a man walks 5 miles east and then 3 miles in the opposite direction. How far is he from the starting point?

The answer is plainly $(5-3)$ or 2 miles, and the relation between this computation and the representation on a straight line is at once seen.



If now we look on $5-3$ as $(+5)+(-3)$, and take, as has been done, $+5$ in the direction OH, then the -3 is represented by HK measured in the opposite direction. Hence any expression as

$$5-7+3-5+2-7,$$

or, the same thing,

$$(+5)+(-7)+(+3)+(-5)+(+2)+(-7),$$

admits a natural interpretation.

Next suppose the man to walk 3 miles east and then 5 miles in an opposite direction. We are led to say that he is $3-5$ miles from where he started, *i.e.*, -2 miles, and the meaning of the -2 miles is 2 miles in a direction opposite to that in which the supposed positive measurements were taken.

Thus the oppositeness of character expressed by a relation as

$$(+3)+(-3)=0$$

has been illustrated. Further, it is seen that the numbers of algebra, in addition to the quantitative element, have also a *sense* or *direction* feature. The numbers -3 and $+3$ are quantitatively the same, but the numbers are not the same, having a certain oppositeness of character.

Illustration 2. A merchant in the course of a day gains \$130 on one set of transactions, and loses \$120 on another set. Find the gain for the day.

The answer is at once \$130 − \$120, or \$10.

So, too, if the gain had been \$$a$ and the loss \$$b$ we should have said his gain = \$$(a-b)$.

If now $a=20$ and $b=30$, this result would be \$$(20-30)$, or -10 dollars. But it is plain that this means a loss of \$10, and it appears that a *negative gain* is a *positive loss*, and in like manner that a *negative loss* would be a *positive gain*. Hence loss and gain in such cases are related to each other as the positive and negative numbers of algebra.

Illustration 3. The thermometer stands at 10° at 9 o'clock, and in three hours the temperature falls 17°; find the temperature at the end of that time.

We should say at once 7° *below zero*. But we might also say $10°-17°$ or $-7°$, so that the 7° below zero serves as an interpretation of our $-7°$, and the $-7°$ faithfully represents the 7° *below zero*.

On account of the relation of a number, as -2, to zero, it is often said that -2 *is* 2 *less than zero*. This is not a very accurate use of language, but is permitted if we keep in mind that *less than* refers to the *relation* or *order* of the two numbers, but not, in such a case, to the quantitative feature. However, in the case of the numbers $+3$ and $+5$, it is plain that $+3$ is less than $+5$ in both senses.

The foregoing illustrations will help the student to feel that positive and negative quantities of algebra are not mere curiosities of a purely artificial character. Some exercises are now given, and the student is recommended to give them representation by a drawing or diagram. For this purpose the use of squared paper will be a gain, as the measurements can be made without the aid of a rule, and on one sheet many diagrams can be made and compared.

Exercises VI

1. A man starts from a point O on a road running east and west and walks 13 miles east, and then 15 miles in the opposite direction. Write down the expression that gives the distance he is from the starting point, and find this distance.

Find also the distance travelled.

2. A man starts from O on a road running north and south, and walks 17 miles north, then 23 miles south, then 9 miles north, and then 5 miles south. Write down the expression giving the distance he is now from O, and find this distance.

Find also the distance travelled.

3. If AB is a straight line and P any point in AB, then

$$AB = AP + PB.$$

If A, B are any two points in a straight line and P a third point in this line, not between A and B, explain on what grounds it is said that

$$AB = AP + PB.$$

4. A speculator in land buys on a certain day 640 acres, 320 acres, 800 acres, and sells 8 farms of 160 acres. Compare his holdings at the end of the day with those at the beginning.

5. A speculator in land buys on a certain day 4 sections of land of 640 acres each, and sells 5 farms of 160 acres each, 7 farms of 320 acres each, and 1 section of 640 acres. By how much has he *increased* his holdings?

By how much has he *diminished* them?

6. A boy adds 15 marbles to his supply of marbles, gives away 10, buys 5 more, gives away 10, buys 5 more, and gives away 10. How many has he thus added to his supply?

7. The thermometer at a certain time stands at 15°, and in the course of an hour the temperature falls 20°. Find the reading of the thermometer at the end of the hour.

8. A thermometer stands at −20°; in the course of an hour there is a fall of 5°, and in the course of the next hour a rise of 15°. Find the reading at the end of the time.

9. A man buys a horse for $150, but discovering a fault sells the horse at once for $120. Find his gain and his gain per cent.

10. A merchant buys goods and marks them at an advance of 50 per cent., but, through a change of style, in order to sell, he reduces the marked price 50 per cent. Find his gain per cent. at the new price.

11. Find the value of:

(i) $7+5-11+3-8+2$.
(ii) $-9+6-2+7-5$.
(iii) $-5+8-7+6-2$.
(iv) $3a-5a+7a-6a$.
(v) $2a-3b-5a+2b+4a-4b$.

12. Shew that:

$$(7-5)\times 3=6;\ (5-7)\times 3=-6;\ (-3)\times 5=-15.$$

13. A man starts from a point O on a road running east and west, and walks a miles east and then b miles in the opposite direction. How far is he now from the starting point, and how far has he travelled?

Illustrate for the following cases: (1) $a=9$, $b=5$; (2) $a=6$, $b=9$; (3) $a=7$, $b=7$.

14. The thermometer stands at $a°$; in the course of an hour there is a fall of $b°$ in temperature, and in the course of the next hour a rise of $c°$. Find the reading at the end of this time, and illustrate for the following cases: (1) $a=5$, $b=3$, $c=4$; (2) $a=9$, $b=15$, $c=5$; (3) $a=-2$, $b=3$, $c=5$.

11. The Simple Rules for Negative Numbers. As negative numbers have been introduced, it is well now to examine what is to be understood by addition, subtraction, multiplication, and division, in regard to them. The meaning of these operations will be reached, or assigned, by supposing certain general rules, which we know to be valid in arithmetic, to hold also when negative numbers are conconcerned.

(a) Addition. It has been seen that, when b is greater than c,

$$a+(b-c)=a+b-c.$$

Suppose this to hold, for all values of the letters involved, and let b be zero. Then

$$a+(0-c)=a+0-c,$$

or

$$a+(-c)=a-c.$$

Thus, *the addition of a negative number is to be interpreted arithmetically as a subtraction.*

This has already appeared in the two preceding paragraphs.

Exercises VII

1. A man worth \$1000 makes a gain of \$200, and then incurs a loss of \$1500. How much is he now worth?

Shew that this example illustrates the relation

$$(+1000)+(+200)+(-1500)=(-300).$$

2. Find the value of each of the following:

(1) $(+3)+(-7)+(+4)+(-3)+(-2)$.

(2) $(+a)+(-b)+(-3a)+(+4b)+(+5a)+(-5b)$.

(3) $(+2x)+(-3y)+(+2z)+(-x)+(+4y)+(-3z)$.

3. Illustrate the addition of a negative number by an example in (i) gain and loss, (ii) rise and fall of temperature, (iii) travelling in two opposite directions, (iv) buying and selling, (v) expansion and contraction of a rod of iron under heat and cold.

4. Find the value of

$$(a-b)+(c-d)$$

for

(i) $a=5,\ b=3,\ c=7,\ d=4$;

(ii) $a=6,\ b=3,\ c=5,\ d=9$;

(iii) $a=4,\ b=7,\ c=6,\ d=9$.

(b) Subtraction. It has been seen that, when b is greater than c, and a greater than their difference,

$$a-(b-c)=a-b+c.$$

Suppose this to hold whatever be a,b,c, and let b be zero.

Then

$$a - (0 - c) = a - 0 + c,$$

or

$$a - (-c) = a + c.$$

Thus *the subtraction of a negative number is to be interpreted arithmetically as an addition.*

Exercises VIII

1. Find the value of each of the following:

(1) $(+7) + (-5) - (-9)$.
(2) $(-9) + (+11) - (-3)$.
(3) $(-5) + (-9) - (-15)$.
(4) $(-7) - (-5) - (+4) - (-3)$.
(5) $(-2) - (-3) - (-4) - (-5)$.

2. The thermometer stands at 13°; there is a fall of 5° in temperature, followed by a rise of 5°. Show that the resulting temperature is given by either of the following:

(1) $13° - 5° + 5°$.
(2) $13° + (-5°) - (-5°)$.

3. A man having \$100 incurs a loss of \$20, but through an additional transaction cancels this loss. Shew that as a result he has now either of the following:

(1) \$100 − \$20 + \$20.
(2) \$100 + (−\$20) − (−\$20).

4. A man having \$75 loses in trade \$30 and then gains \$40. Write two expressions that give the amount of money he now has.

(c) Multiplication. Let a, b, x, y be four arithmetical numbers, such that a is greater than b and x greater than y. Then, as is arithmetically evident,

$$(a-b)(x-y) = (a-b)x - (a-b)y,$$
$$\therefore\ (a-b)(x-y) = (ax - bx) - (ay - by),$$
$$\therefore\ (a-b)(x-y) = ax - bx - ay + by.$$

This result has already appeared in one of the exercises. The product, *i.e.*, the result on the right, consists of four *parts*, formed by multiplying each of the numbers a, b in the factor $(a-b)$, by each of the numbers x, y in the factor $(x-y)$, with a certain disposition of signs. Suppose now that a, b, x, y are any numbers whatever, and treat them as algebraic, *i.e.*, as being positive or negative. Then, since we regard $a-b$ as the equivalent of $(+a)+(-b)$, we have

$$\{(+a)+(-b)\}\{(+x)+(-y)\}=(+ax)+(-bx)+(-ay)+(+by).$$

This relation we take as determining the rules of multiplication for positive and negative numbers, and we have

$$(+a).(+x)=+ax,$$
$$(-b).(+x)=-bx,$$
$$(+a).(-y)=-ay,$$
$$(-b).(-y)=+by,$$

and the important fundamental *rule of signs*:

The product of two numbers of like signs is positive, and the product of two numbers of unlike signs is negative.

In a way this is what we, in some measure, may have expected, for we have had such statements as

$$3\times4=12,\quad -5\times3=-15,$$

which point to the algebraic statements:

(a) *The product of two positive numbers is positive.*

(b) *The product of a negative number by a positive number is negative.*

If now we carry the known rule of arithmetic into algebra and say

$$(-5)\times(+3)=(+3)\times(-5)$$

we have further:

(c) ***The product of a positive number by a negative number is negative.***

Next, since

$$(+3)+(-3)=0,$$

let us propose to multiply the two numbers, here stated to be equal, by -5. Then, taking as granted that the product of 0 and any number is to be zero in algebra as in arithmetic, we have

$$(+3)(-5)+(-3)(-5)=0.$$

Therefore, from what we have just seen,

$$(-15)+(-3)(-5)=0.$$

Thus $(-3)(-5)$ is the number which taken, in addition, with -15 yields zero, so that $(-3)(-5)$ must be $+15$, and we are led to the fourth statement:

(d) ***The product of a negative number by a negative number is a positive number.***

The student may now ask what is the meaning of a multiplication as $(+3)\times(-5)$. He should recall that in arithmetic a multiplication as 3×4 meant, and still means, 3 taken 4 times, but that when it came to a product as

$$\tfrac{2}{3}\times\tfrac{5}{7}$$

he had in some way to change his notion of multiplication, and it was difficult to say what was meant by multiplication. So here, we choose to introduce multiplication by positive and negative numbers, and we do so by supposing the general rules of arithmetic to persist; it would be difficult, as it is unnecessary, to give a formal statement of the meaning of the operation.

Thus the product of any two numbers, with signs indicating their character, is found as if they were two numbers of arithmetic, the sign being determined by the rule stated. From the product of two such numbers it is easy to pass to the product of any number of such, and, in particular, to the powers of such numbers.

EXERCISES IX

1. Find the following products:

$(+3)(+7)$; $(+13)(-5)$; $(-12)(+7)$; $(-9)(-11)$.

2. Find the following products:

$(+h)(+k)$; $(-2y)(+3z)$; $(-5m)(-3n)$; $(+5p)(-8z)$.

3. Find the value of

$$(-5)^2,\ (-3)^3,\ (-4)^4,\ (-2)^5,\ (-3)^6.$$

4. Find the powers:

$$(-a)^2,\ (-p)^7,\ (-x)^{13},\ (-y)^{20},\ (-z)^{99}.$$

5. Find the products:

$(-2)(+3)(-5)$; $(-x)(-y)(-z)$; $(-2x)(-3y)(+5z)$.

6. If n is an arithmetical integer, explain what is meant by $(-x)^n$, and find the value of

$$(-x)^{2n},\ (-x)^{2n+1}$$

7. Find the value of

$$(-x)^3(-y)^4(-z)^5;\quad (+2h)^3(-3k)^2(-4l).$$

(d) Division. As in arithmetic, we treat division in the case of positive and negative numbers as the inverse of multiplication. Thus:

(*a*) Since the product of $+3$ and $+5$ is $+15$, the quotient of $+15$ by $+5$ is $+3$.

(*b*) Since the product of $+3$ and -5 is -15, the quotient of -15 by -5 is $+3$.

(*c*) Since the product of -3 and $+5$ is -15, the quotient of -15 by $+5$ is -3.

(*d*) Since the product of -3 and -5 is $+15$, the quotient of $+15$ by -5 is -3.

Hence the rule: ***The quotient of two numbers of like signs is positive, and the quotient of two numbers of unlike signs is negative.***

Thus the quotient of any two numbers, with signs indicating their character, is found as if they were two numbers of arithmetic, the sign being determined by the rule stated.

Exercises X

1. Find the following quotients:

$$(+15)\div(+3);\ (-21)\div(-7);\ (-35)\div(+5);\ (+45)\div(-9).$$

2. Find the following quotients:

$$(+2)\div(+3);\quad (-3)\div(-5);\quad (-3)\div(+4);\quad (+2)\div(-3).$$

3. Find the value of

$$(-\tfrac{2}{3})^2\times(+\tfrac{3}{4})^3\times(-2)^3;\quad (-\tfrac{5}{7})^3\times(-\tfrac{7}{11})^2\times(-\tfrac{77}{25}).$$

4. Find the value of

$$\left(-\frac{a}{b}\right)^2\left(-\frac{b}{c}\right)^3\left(-\frac{c}{a}\right)^4;\quad \left(-\frac{p}{q}\right)^3\left(-\frac{x}{y}\right)^3\left(-\frac{a}{b}\right).$$

12. Further Note on Positive and Negative Numbers. The number a, or b, or x, ... is in general thought of as signifying any number. With the introduction of negative numbers we shall then have to think of it as signifying, in general, any positive number, or any negative number, or zero. If the letter a, or b, or x, ... without any attached sign denotes a negative number, it is said to be ***implicitly negative***, whereas if the negative character of a number is indicated by the sign $-$ before the letter, the number is said to be ***explicitly negative***.

In an expression such as

$$2ab - 3lm + 5pq - 7yz,$$

the signs + and −, regarded as signs of addition and subtraction, break up the number signified into parts. Each of these parts with the sign just preceding it, is called a *term* of the expression. In the given expression, $2ab$ has no sign before it; in such a case the sign + is understood and the terms of the expression are, in order,

$$+2ab, \quad -3lm, \quad +5pq, \quad -7yz,$$

and we think of the expression as

$$(+2ab) + (-3lm) + (+5pq) + (-7yz).$$

An expression of two terms is called a *binomial* (expression), of three terms a *trinomial*, while the name *polynomial* is used generally to denote any expression of two or more terms. By an extension of language an expression as a, or $5x$, or $2ab$, which is not broken into parts by the signs +, or −, is called a *monomial* or an expression of one term.

We are now to suppose that the general rules of arithmetic as those stated in Section 3, and the rules for brackets, stated in Section 6, hold for all the numbers of the extended system of numbers. Among these numbers there is one of special importance, the number zero. Its properties, in operations, are given by the relations

$$a + 0 = a = 0 + a,$$
$$x.0 = 0 = 0.x,$$

i.e., as an addend 0 does not affect the result, as a factor it requires the product to be zero. *There is no such operation as division by zero.*

For all numbers there are certain laws that are admitted. Of these the following are of frequent use:

(a) *If equals be added to (or subtracted from) equals, the sums (or differences) are equal.*

(b) *If equals be multiplied (or divided) by equals, the products (or quotients) are equal.*

Exercises XI

1. Express algebraically the sum of a and b diminished by the sum of c and d, and find the result for $a=7$, $b=11$, $c=6$, $d=15$.

2. Find the result of dividing the product of -3, -4, -5, -6, by the sum of -5 and -4.

3. What number must be multiplied by -7 to give $+63$, by $-a^2$ to give $-a^5$, by $-k$ to give hk, by $+a$ to give $-b$?

4. If

$$2x-3=x+7,$$

shew in succession that

$$(1) \qquad 2x=x+7+3.$$
$$(2) \qquad 2x-x=7+3.$$
$$(3) \qquad x=10.$$

5. Illustrate by measurements on a straight line the following sums:

$$(1) \qquad (+3)+(+2)+(-6)+(+5)+(-7).$$
$$(2) \qquad (-4)+(-3)+(+5)+(-2)+(+4).$$
$$(3) \qquad -3+2-7+5-9+8+6-3.$$

6. If m is any number, not zero, and it is known that

$$mx=ma,$$

shew that $x=a$.

7. If a is an implicitly negative number, shew that a^2, a^4, a^6, a^8, ... are all positive, while a^3, a^5, a^7, a^9, ... are all negative.

8. If x is any number whatever, not zero, shew that x^2 is necessarily positive.

9. It is known that

$$0.x = 0.y,$$

each being zero. Why may it not be concluded that $x = y$?

10. Shew that from any one of the statements:

$$(1)\quad x + b = y + a,$$
$$(2)\quad x - a = y - b,$$
$$(3)\quad x - y = a - b,$$

the other two may be derived, or, in other words, shew that the three statements are equivalent.

11. A man bought m acres of land at $2a$ dollars an acre, and n acres at $2b$ dollars an acre. He sold all at $(a + b)$ dollars an acre. Find his gain.

Examine if $m = 80$, $n = 30$, $a = 13$, $b = 11$.

12. If $2s = a + b + c$, find in terms of a, b, c, the value of

$$s + (s - a) + (s - b) + (s - c).$$

13. A man walks out into the country for a hours at the rate of b miles an hour and returns by stage at the rate of c miles an hour. How long was he in returning?

14. A speculator buys a property for a dollars and sells it for b dollars. By how much has he increased his capital?

Interpret if $a = 12{,}000$, $b = 10{,}350$.

15. Calculate the value of the expressions,

$$2x, \quad 3x + 5, \quad x^2,$$

for $x = -2$, $-1{\cdot}5$, -1, $-0{\cdot}5$, 0, $+0{\cdot}5$, $+1$, $+1{\cdot}5$, $+2$, shewing the results in each case in tabular form.

16. Shew that

$$(y - z) = -1.(z - y),$$

and that, of $y - z$, $z - y$, each is the *negative of the other*.

17. Without disturbing the order of the letters, write as a trinomial, in as many ways as possible,

$$l + m + n + p + q.$$

18. Shew that any trinomial whatever may be treated as a binomial.

CHAPTER IV

ADDITION AND SUBTRACTION

13. Addition. Very little explanation is here called for, as all that is essential has appeared in the preceding chapters. Numerous exercises are given in order that familiarity with algebraic expressions may be acquired.

First, suppose all the terms are *like*, *i.e.*, differ, if at all, only in sign or numerical coefficient as in the expression:

$$2b - 3b + 5b - 9b + 6b + b.$$

The result is readily seen to be $+2b$ or simply $2b$. Though the signs of both addition and subtraction occur, yet having regard to negative quantities the expression is spoken of as an *algebraic sum*. The term $+b$, the equivalent of $+1.b$, is treated as having $+1$ as numerical coefficient, and $-b$, would be said to have -1 as coefficient.

Exercises XII

Find the following algebraic sums, (1) first collecting all the the positive and all the negative terms, and then finding the result; (2) finding the sum as each new term is taken:

1. $3a - 7a + 10a - a + 5a$.
2. $5x + 3x - 7x - x + 11x$.
3. $9l - 13l + 17l - 5l$.
4. $-3p + 7p - 6p - 4p$.
5. $3hk - 4hk + 7hk - 2hk$.
6. $-2yz + 3yz - 7yz + 5yz$.
7. $3abc - 4abc - 7abc$.
8. $9x^2 - 7x^2 + 3x^2 - 4x^2$.
9. $-5m^2n^2 - 7m^2n^2 + 4m^2n^2$.
10. $3(a + b) - 7(a + b) + 6(a + b)$.
11. $-13xyz - 4xyz + 12xyz$.
12. $5(x + y)^2 - 3(x + y)^2 + (x + y)^2$.

Next, suppose that the terms are not all like, and take as example,

$$5a - 3b + 2c - 4b + 7c - 3a - a + 5b - 2c.$$

Here it is readily seen that the expression is equivalent to

$$a - 2b + 7c.$$

In this result the three terms are *unlike* and we cannot proceed further with the addition,—in other words we have to be content with this *indication of the sum*.

Exercises XIII

Find the value of:

1. $3a - 11b + 9c - 8a + 7b - 5c + 7a - 3b - 8c.$
2. $9x + 3y - 4z - 5y - 14x + 3z - 9z - 5x + 18y.$
3. $6p - 17q + 12r - 14q + 8r - 19p - 4r - 15p + 16q.$
4. $-5l + 7m - 9n - 11m + 13n + 15l - 17n - 19l - 21m.$
5. $-3x^2 + 8y^2 + 9x^2 - 11y^2 + 7x^2 - 22y^2 - 16x^4 + 28y^2.$
6. $7x^2 - 2xy + 3y^2 - 8x^2 - xy - 4y^2 + x^2 + 3xy + 5y^2.$
7. $13mn - 5nl + 6lm - 9mn - 2nl - 3lm + mn - 4nl + 2lm.$
8. $4x^3 - 5y^3 + 3z^3 - 7x^3 - 4y^3 + 13z^3 + 2x^3 + 3y^3 - 18z^3.$
9. $-5x^2 + 7x - 3 - 2x^2 - 3x + 7 + 9x^2 - 5x + 11.$
10. $8x^3 - 5x^2 + 12x - 11 + 4x^3 + 7x^2 - 19x + 9.$
11. $\frac{1}{3}x + 7 + \frac{1}{2}x + 5 - \frac{3}{4}x - 8 + \frac{2}{3}x - 5.$
12. $\frac{5}{6}h^2 - \frac{2}{3}hk + \frac{3}{4}k^2 - \frac{1}{3}h^2 - \frac{1}{4}hk - \frac{2}{3}k^2.$

It is frequently necessary to find the sum of two or more expressions enclosed in brackets, as, for example,

$$(2a - 3b),\ (5a - 7b),\ (-3a + 5b),\ (6a + 9b).$$

We have in this example to find the value of

$$(2a - 3b) + (5a - 7b) + (-3a + 5b) + (6a + 9b).$$

Recalling that for all values of a, b, c we have

$$a + (b - c) = a + b - c,$$

or

$$+a + (+b - c) = +a + b - c.$$

We have the rule that *if a bracketed expression is preceded by the sign +, the brackets may be removed, the terms retaining their original signs*, as it is readily seen that the case in which there are several terms within the brackets does not differ essentially from that in which there are two only.

Hence the sum proposed is equal to

$$2a - 3b + 5a - 7b - 3a + 5b + 6a + 9b,$$

or

$$10a + 4b.$$

Exercises XIV

Find the sum of:

1. $2x - 3y)$, $(5x - 7y)$, $(26x + 4y)$, $(-29x - 23y)$.
2. $(a + b + c)$ $(a + b - c)$, $(b + c - a)$, $(c + a - b)$.
3. $(2x - y - z)$, $(2y - z - x)$, $(2z - x - y)$.
4. $(y - z)$, $(z - x)$, $(x - y)$.
5. $(a + b - c - d)$, $(b + c - d - a)$, $(c + d - a - b)$, $(d + a - b - c)$.
6. $(3a^2 - 5b^2 + 6c^2)$, $(-5a^2 + 8b^2 + c^2)$, $(-a^2 - 6b^2 - 9c^2)$.
7. $(9yz - 11zx + 8xy)$, $(-7yz + 8zx - 19xy)$, $(5yz + zx + 3xy)$.
8. $(5u^2 - 6uv + 9v^2)$, $(-17u^2 - 3uv - 4v^2)$, $(10u^2 + 5uv + 13v^2)$.
9. $(2x^2 - 3x + 7)$, $(5x^2 + x - 9)$, $(-8x^2 - 5x + 4)$.
10. $(x^2 + x + y^2 + y)$, $(2x^2 - 3y^2 - y)$, $(5y^2 + 3y - 7x)$.
11. $(4x^3 - 5x^2 + 8x - 9)$, $(3x^3 + 2x^2 - 4x - 11)$, $(7x^3 - 13x + 21)$.
12. $(\frac{1}{2}x^2 + \frac{1}{3}x - \frac{1}{4})$, $(\frac{2}{3}x^2 - \frac{1}{2}x + \frac{1}{3})$, $(-\frac{3}{4}x^2 + \frac{2}{3}x - \frac{1}{2})$.

In extended additions it may be convenient to arrange the quantities to be added as below shewn:

$$\begin{array}{r} a + 3b - 5c \\ -2a - 4b + 6c \\ -3a + 5b - 2c \\ 9a - 8b - 3c \\ \hline 5a - 4b - 4c \end{array} \qquad \begin{array}{rrrr} 2x^3 & -\ 5x^2 & +\ 7x & -\ 2 \\ -\ x^3 & +\ x^2 & -\ 5x & +\ 4 \\ +\ 3x^3 & & -\ 8x & \\ -\ 5x^3 & -\ 7x^2 & & +\ 9 \\ \hline -\ x^3 & -\ 11x^2 & -\ 6x & +\ 11 \end{array}$$

As an exercise, the student may work the examples of Exercises XIII, in this way.

It should be noted that, in expressions involving only one letter, and that to different powers, for example,

$$5y^3 - 3y^2 + 7y - 9,$$

or

$$8 - 3z + 5z^2 - 11z^3 + z^4,$$

the practice is to write the terms in the order of descending or ascending powers.

Sometimes in an expression involving several letters certain of them are regarded as principal, the others being treated as coef[illegible]ts of the principal numbers. For example i[illegible]

$$la + ma + na + pa$$

let a be taken as the principal number. We may then express the sum thus:

$$(l + m + n + p)a.$$

So in

$$hx - ky + lx + my - px - qy$$

let x and y be regarded as principal numbers; the sum can be expressed thus:

$$(h + l - p)x + (-k + m - q)y,$$

or thus:

$$(h + l - p)x - (k - m + q)y.$$

So also the expression

$$(b - c)x + (c - a)y + (a - b)z$$

may be expanded and then combined—a, b, c being taken as principal numbers—to yield the expression

$$(z - y)a + (x - z)b + (y - x)c.$$

In these examples the literal coefficients are combined into one coefficient just as the numerical coefficients are combined in the preceding Exercises.

Exercises XV

1. Treating x as the principal number, write, as indicated above, the following sums:

(1) $ax + bx + cx + dx$.
(2) $hx - kx + lx - mx$.
(3) $a^2x - b^2x + c^2x - d^2x$.
(4) $3px - 4qx + 5rx - 6sx$.
(5) $4h^2x - 5k^2x + 9l^2x - 7m^2x$.

2. Treating x and y as the principal numbers, express, as indicated above, the following sums:

(1) $ax + by - cx - dy + ex - fy$.
(2) $mnx - lpy + nlx - mpy + lmx - npy$.
(3) $a^2x + kly + b^2x + lhy + c^2x + hky$.
(4) $px + qy + rx + py + qx + ry$.
(5) $l^2x - a^2y + m^2x - b^2y + n^2x - c^2y$.

3. Treating x, y, z as the principal numbers, express in the manner indicated the following sums:

(1) $bx + cy + az + cx + ay + bz$.
(2) $bx - cy + az - cx + ay - bz$.
(3) $ax + by + cz + bx + cy + az + cx + ay + bz$.
(4) $lmx - mny + nlz - nlx + lmy - mnz$.
(5) $klx + lhy + hkz - k^2x - l^2y - h^2z - l^2x - h^2y - k^2z$.

14. Subtraction. Such subtractions as appear in

$$5a - 3a; \quad 6x + 7x - 9x - 8x$$

have already been considered, and we pass to a subtraction such as

$$(2x - 3y) - (x - 2y).$$

It has already appeared that for all values of the involved letters

$$a - (b - c) = a - b + c,$$

or

$$a - (+b - c) = a - b + c,$$

and it follows that *if a bracketed expression is preceded by the sign –, the brackets may be removed if the signs of the terms be all changed*, since it is easy to pass from the case of a bracketed expression of two terms to one of any number of terms.

Hence

$$(2x - 3y) - (x - 2y) = 2x - 3y - x + 2y$$
$$= x - y.$$

In like manner

$$(2x - 3y + 4z) - (x + 2y - 7z) = 2x - 3y + 4z - x - 2y + 7z$$
$$= x - 5y + 11z.$$

Often the subtraction is shewn thus:

$$\begin{array}{r} 2x - 3y + \ \ 4z \\ x + 2y - \ \ 7z \\ \hline x - 5y + 11z \end{array}$$

and the rule given: *Change the signs of the terms in the subtrahend, and proceed as in addition.* It is needless to say that the change is best made mentally.

Exercises XVI

1. From $(3x - 5y + 7z)$ take $(2x + 3y - 2z)$.

2. From $(11a + 23b - 17c)$ take $(9a - 12b + 13c)$.

3. From $(9yz + 7zx - 14xy)$ take $(5yz - 3zx - 13xy)$.

4. From $(2x^2 - 7x + 3)$ take $(x^2 + 9x - 11)$.

5. Find the value of:

$$(a + b + c) - (b + c - a) - (c + a - b) - (a + b - c).$$

6. What number must be taken from $a^2 + 2ab + b^2$ to give as remainder $a^2 - 2ab + b^2$?

7. From

$$-2l^2+7m^2-8n^2+3mn-13nl-7lm$$

take

$$5l^2-\ m^2+2n^2+7mn+\ 3nl-8lm.$$

8. Find the value of:

$$2(-2a+7b-24c)-3(-5a-9b-15c).$$

9. Treating x and y as principal numbers, find the value of

$$(ax+by)-(px+qy).$$

10. Find the value of:

(1) $3(a-b)-2(a-b)+7(a-b)-5(a-b)$.

(2) $5(q-r)-7(r-p)-9(p-q)+6(p+q+r)$.

(3) $(7a^2-3b^2)-(2a^2-3ab+5b^2)-(-5a^2-ab+7b^2)$.

(4) $(2x^3-3x)-(5x^2+x-7)-(x^3-x^2+x-1)$.

11. Treating x as the principal number, find the difference:

$$(ax^2+2bx+c)-(hx^2+2kx+l).$$

12. Find the value of

$$(\tfrac{2}{3}x^3-\tfrac{3}{4}x^2+x-\tfrac{1}{2})-(\tfrac{3}{4}x^3-\tfrac{1}{3}x^2-\tfrac{2}{3}x+\tfrac{3}{4}).$$

13. From $7\frac{a}{b}-11\frac{b}{c}+18\frac{c}{a}$ take $3\frac{a}{b}+4\frac{b}{c}+19\frac{c}{a}$.

14. From what number must $4ab$ be subtracted to give the remainder $a^2-2ab+b^2$?

15. Brackets. Sometimes in an expression several pairs of brackets are employed, some being within others. Such an expression may be *reduced*, *i.e.*, written without brackets and brought to its simplest form by the rules illustrated and stated in Sections 6, 11, 14. The following examples will shew how such expressions are treated:

Ex. 1.

$$(3x-2y)-\{(5x-7y)-(3x-4y)+(7x+2y)\}$$
$$=(3x-2y)-(5x-7z)+(3x-4y)-(7x+2y)$$
$$=3x-2y-5x+7y+3x-4y-7x-2y$$
$$=-6x-3y.$$

Note that the *outer* brackets are removed first, each bracketed expression within them being still treated as one number. This is not necessary but, as a rule, it involves the least changing of signs. As an exercise, the example may be worked by removing the *inner* brackets first.

Ex. 2.

$$(2a-3b)-(a+2b)-[\{(2a+5b)-(7a-3b)\}-\{(9a-3b)-(-5a-7b)\}]$$
$$=2a-3b-a-2b-\{(2a+5b)-(7a-3b)\}+\{(9a-3b)-(-5a-7b)\}$$
$$=2a-3b-a-2b-(2a+5b)+(7a-3b)+(9a-3b)-(-5a-7b)$$
$$=2a-3b-a-2b-2a-5b+7a-3b+9a-3b+5a+7b$$
$$=20a-9b.$$

Ex. 3.

$$5[2a+3b+(7a-5b)-\{(3a-11b)-(2a+5b)\}]$$
$$=10a+15b+5(7a-5b)-5\{(3a-11b)-(2a+5b\})$$
$$=10a+15b+35a-25b-5(3a-11b)+5(2a+5b)$$
$$=10a+15b+35a-25b-15a+55b+10a+25b$$
$$=40a+70b.$$

Ex. 4.

$$(2l-3m)-\{(5l+7m-3n)-(6l-3m-\overline{7l-3m-9n})\}$$
$$=2l-3m-(5l+7m-3n)+(6l-3m-\overline{7l-3m-9n})$$
$$=2l-3m-5l-7m+3n+6l-3m-\overline{7l-3m-9n}$$
$$=2l-3m-5l-7m+3n+6l-3m-7l+3m+9n$$
$$=-4l-10m+12n.$$

Here the line above $7l-3m-9n$ serves the purpose of a pair of brackets; it is called a *vinculum*.

EXERCISES XVII

1. Reduce the following to their simplest forms:

(1) $(5p-3q)-\{(7p-11q)-(9p-23q)\}$.
(2) $5x-\{3y-(8x-3y-5z)\}$.
(3) $-(5x-3y)-[7x-\{3y-(8x-11y)\}]$.
(4) $5a-\{3a-2b-(7a-\overline{13a-5b})\}$.
(5) $-(a^2-\overline{ab-b^2})-[2ab-\{a^2-(3ab-b^2)\}]$.

2. In the following, remove the brackets at one step:

(1) $-\{(a+b-c)-(2a-3b-c)\}+\{(5a-2b)-(3a-7b+14c)\}$.
(2) $-(a+b-\overline{2a-3b})-(\overline{3a-2b}-\overline{9b-7c})$.
(3) $5u-[3u-5v-(\{7u+24v)-(16u-5v)\}]$.
(4) $(7a^2-3b^2)-\{5b^2-(3a^2-\overline{5b^2+22c^2})\}$.
(5) $a-[b-\{c-(d-\overline{e-f})\}]$.

3. In the following seek out the terms involving x, those involving y, and those involving z, and write down the value of the expressions:

(1) $(2x-3y)-\{(5y-8z)-(3z-4x)\}$.
(2) $-(9x-3y)-\{8y-(17z-\overline{3x-11y})\}$.
(3) $x-\{2y-(3z-\overline{4x-5y-5z})\}$.
(4) $(4x-7y)-3\{2y-(7z-9y)\}$.
(5) $2\{(x+y)+3(5x+\overline{2y-z})\}$.

4. Collect the powers of x in the expression

$$2x-(5+x^2)-\{(3x+8x^2)-(7+5x-13x^2)\},$$

and write the expression in its simplest form.

5. Remove the brackets from

$$3(x^3-x^2y)-3(xy^2-y^3)-2\{(x^3+y^3)-5(x^2y+xy^2)\}$$

and collect and arrange the terms.

6. Find the result of subtracting $a-\{b-(c-\overline{d-e})\}$ from $a-b-c-d-e$.

7. Find the value of

$$-a(a-\overline{b-c})-b(b-\overline{c-a})-c(c-\overline{a-b}).$$

The *inverse* process of including several terms within brackets, so as to present the aggregate as one term, has been already employed. As an illustration consider the expression

$$x - a + y - b + z - c,$$

which it is proposed to write as a binomial. It is at once seen that it can be so presented in several ways; in the forms:

$$(x + y + z) + (-a - b - c),$$

$$(x + y + z) - (a + b + c),$$

there is a familiar association of letters. Generally in such work one is attentive to the order of the letters and, as a rule, the sign before the bracket is so taken that the first term within the bracket is of positive sign.

EXERCISES XVIII

1. Write as a binomial, treating all terms in x and all terms in y as like terms,

$$ax - py + bx - qy + cx - ry.$$

2. Write as a binomial, all the terms within the brackets being of positive sign,

$$x^2 + y^2 + z^2 - yz - zx - xy.$$

3. Collect and arrange in powers of x,

$$ax^3 - bx^2 - cx + d - hx^3 - kx^2 - lx - m + px^3 - qx^2 + rx - s,$$

the first term in the bracketed coefficients in each case to be of positive sign.

4. Write as a trinomial, taking the letter x as the principal letter,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c.$$

5. Write as the product of two binomials, the one factor being the sum, the other the difference, of the same two numbers:

(1) $(a+b+c)(a+b-c)$.

(2) $(a+b-c)(a-b+c)$.

(3) $(2x+3y-4z)(-2x+3y+4z)$.

(4) $(a+b+c+d)(a+b-c-d)$.

(5) $(a-b+c-d)(d-a-b+c)$.

6. Bring to what would seem the neatest form

$$lx^2-(m+n)x+mn+mx^2-(n+l)x+nl+nx^2-(l+m)x+lm.$$

Exercises XIX

(Miscellaneous)

A

1. Give a clear statement of the proof that

$$x^7 . x^{11} = x^{7+11}.$$

2. If $a=7$ and $b=9$, find which is the greater, ab or 79.

3. Add the number of two digits of which the unit digit is n, and tens digit m, to the number given by reversing the order of the digits, and show that the sum is divisible by 11.

4. If m is any integer, write down the general even integer, and form the two even integers immediately preceding it, and the two even integers immediately succeeding it.

Add the five integers thus formed.

5. From the sum of $2a^2-3ab+11b^2$ and $5a^2-9ab+13b^2$ take $4a^2-19ab-11b^2$.

B

1. Show that $(mn)^5 = m^5n^5$.

2. There are two numbers, one of which is three times the other. If 12 is added to each, the greater result is twice the less. Find the numbers.

3. A can do a certain piece of work in m hours, and B in n hours. A works at it a hours, and then B works at it b hours. Write the expression which indicates the amount of work now done.

4. If m is any integer, including zero, write down the general odd integer, and form the two odd integers immediately preceding it and the two odd integers immediately succeeding it.

Add the five odd integers thus formed.

5. From the sum of $15x^2 - 13xy - 16y^2$ and $7x^2 - 18xy + 10y^2$ take the sum of $9x^2 + 16xy - 12y^2$ and $8x^2 - 14xy - 15y^2$.

C

1. If A denotes the measure of the area of a rectangle, and l and b the measures of its length and breadth, write the relation connecting these three numbers.

2. When asked his age, A said that in 17 years he would be twice as old as he would be in 2 years. Find his age.

3. There are two squares such that the side of one is twice as long as that of the other. Show that the area of the larger square is four times that of the smaller.

Illustrate by a figure.

4. By how much does $+3$ exceed -3? Illustrate.

5. How many hours are there in x weeks and y days?

CHAPTER V

MULTIPLICATION

16. The Product of Two or More Simple Expressions. A *simple* expression is one consisting of a single term, in contrast with a *compound* expression which consists of two or more terms. The rule for forming the product of simple expressions has been already given. Some exercises are given for practice; the answers should be written down at once, the necessary computations being made mentally.

Exercises XX

Write down the following products:

1. $2x \cdot 3y$.	2. $(+3x)(-5y)$.
3. $(7x^2)(-11y^2)$.	4. $(-3l)(+4m)(-5n)$.
5. $(-7a^2)(-2b^2)(-3c^2)$.	6. $(-yz)(-xy)$.
7. $(a^2b)(b^2c)(c^2a)$.	8. $(-x^2)(-x^3)(-x^4)$.
9. $(-la^2)(-ma^2)(-na^2)$.	10. $(-a^2)^3(-b^3)^2$.
11. $(-2a)^2(-3b)^2$.	12. $(-a^3)^5$.
13. $\frac{1}{2}yz \cdot \frac{1}{3}zx \cdot \frac{1}{4}xy$.	14. $(-yz)^2(-zx)^3(-xy)^4$.
15. $(abc)^3 \cdot (bc)^2(ca)^2(ab)^2$.	16. $\frac{3}{5}x^2 \cdot \frac{2}{3}xy \cdot (-\frac{10}{12}y^2)$.

17. Dimensions of Simple Expressions. In the monomial $5x^3$, the number x occurs as a factor three times, the expression is said to be of *the third degree in x*, or *of three dimensions in x*. So in the term $9a^2b^3$, the number a occurs as a factor 2 times, and the number b as a factor

3 times, and the term is said to be *of five dimensions in a and b.* Thus *the dimension of a monomial in one or more letters is the number of times the letter or letters appear in it as factors.*

Illustration: The expression $17ax^2y^3z^4$ is of two dimensions in x, of five in x and y, of nine in x, y, and z, and of ten in a, x, y, and z.

From the laws of indices it can readily be seen that: *The dimensions of the product of two monomials in one or more letters is the sum of the dimensions of the monomial factors.*

EXERCISES XXI

1. State the dimension in x, in x and y, and in x, y, and z of: xyz; $x^2y^2z^2$; xy^2z^3; $7x^4y^5z^6$; $(2x^3y^2z^2)^3$.

2. State the dimensions in a of

$$a^3 . a^5;\ (-a^2)^3;\ (a^2)^3 . (a^3)^2;\ la^2 . ma^3.$$

3. Write down three monomials each of four dimensions in x and y.

4. Write down four monomials of five dimensions in a and y.

5. Write down five monomials of six dimensions in l, m, n.

18. The Product of a Simple and a Compound Expression. It has been seen that

$$(a+b)c = ac + bc,$$
$$x(y+z) = xy + xz,$$
$$l(a+b+c) = la + lb + lc.$$

From these relations and the rule of signs, the product of any compound expression and a simple expression can be found. The results of the exercises proposed should be written down at once.

Exercises XXII

Write down the following products:

1. $(x+y-z)p$.
2. $l(2x-3y+4z)$.
3. $xyz(x+y+z)$.
4. $a(yz-zx+xy)$.
5. $3x(2l^2+3m^2-9n^2)$.
6. $9hk(h^2+3k^2)$.
7. $-7l^2m(mn^2-n^2l)$.
8. $7a^2bc(bc-3ca+8ab)$.
9. $pqr(a^2x+b^2y+c^2z)$.
10. $5l^2m^3(lx-my)$.

19. The Product of Two Compound Expressions. The simplest example of such a product is

$$(a+b)(x+y),$$

and this has been seen to be equal to

$$(a+b)x+(a+b)y \text{ or } ax+bx+ay+by.$$

In like manner it appears that

$$\begin{aligned}(a+b-c)(x-y) &= (a+b-c)x+(a+b-c)(-y).\\ &= ax+bx-cx-ay-by+cy.\end{aligned}$$

Thus it is seen that *the product of two compound expressions is the algebraic sum of all terms formed by multiplying each term of the one expression by each term of the other.*

Illustration 1. Find the product of $(x+2)$ and $(x+3)$.

$$\begin{aligned}(x+2)(x+3) &= (x+2)x+(x+2)3\\ &= x^2+2x+3x+6\\ &= x^2+5x+6.\end{aligned}$$

In the greater number of examples, as here, it happens that when the multiplication is completed there are like terms, and the last step in the work is that of collecting these like terms. On this account the multiplication is usually so arranged as to bring like terms together in a form convenient for collection. Thus, the example worked may be presented in the following way:

$$\begin{array}{l} x+2 \\ \underline{x+3} \\ x^2+2x \\ \underline{\quad\;\; +3x+6} \\ x^2+5x+6 \end{array}$$

Illustration 2. Find the product $(3x-4y)(5x-6y)$.

$$\begin{array}{l} 3x-4y \\ 5x-6y \\ \hline 15x^2-20xy \\ \qquad\;\; -18xy+24y^2 \\ \hline 15x^2-38xy+24y^2 \end{array}$$

Thus $(3x-4y)(5x-6y)=15x^2-38xy+24y^2$.

Illustration 3. Find the product $(x^2-2x+5)(x-3)$.

$$\begin{array}{rrrr} x^2 & -2x & +5 & \\ & x & -3 & \\ \hline x^3-2x^2 & +5x & & \\ -3x^2 & +6x & -15 & \\ \hline x^3-5x^2 & +11x & -15. & \end{array}$$

Thus $(x^3-2x+5)(x-3)=x^3-5x^2+11x-15$.

Illustration 4. Find the product of $x+a$ and $x+b$.

$$\begin{array}{l} x+a \\ x+b \\ \hline x^2+ax+bx+ab \end{array}$$

$\therefore\ (x+a)(x+b)=x^2+ax+bx+ab.$

In an example such as this, there is frequently reason for treating x as a principal number, and the answer is written as a polynomial in x, the other numbers appearing as coefficients. Thus here we would write

$$(x+a)(x+b)=x^2+(a+b)x+ab.$$

EXERCISES XXIII

Find the following products:

1. $(a-b)(x-y)$.
2. $(2x+3y)(p-q)$.
3. $(5l-7m)(3v-8w)$.
4. $(5h+3k)(11x-5y)$.
5. $(6a-5b)(4l-7m)$.
6. $(h^2+k^2)(l^2+m^2)$.
7. $(ab-cd)(pq+rs)$.
8. $(2p^2-3q^2)(5x^2-4y^2)$.
9. $(3lm-5np)(2ab-3cd)$.
10. $(al-bm)(3hx-8ky)$.

Find the following products arranged in powers of the involved letter:

11. $(x+3)(x+5)$.
12. $(x-3)(x-5)$.
13. $(x-3)(x+5)$.
14. $(x+3)(x-5)$.
15. $(2y-3)(3y+5)$.
16. $(7z-8)(3z-11)$.
17. $(5p+9)(7p-8)$.
18. $(3a+7)(9a+13)$.
19. $(3b^2-7)(4b^2-5)$.
20. $(5l^2-9)(7l^2-8)$.

Find the following products, giving attention to the order of terms:

21. $(a+b)(a+b)$.
22. $(2x+3y)^2$.
23. $(3x+2y)(5x+9y)$.
24. $(12p-5q)(8p+13q)$.
25. $(x^4-5x)(2x+3)$.
26. $(9a^2-8b^2)(4a^2-7b^2)$.
27. $(3lm-7)(8lm+11)$.
28. $(9x^3-7x)(2x^2-5)$.
29. $(7ar-15bs)(11ar-23bs)$.
30. $(8uv-9)(7uv-15)$.

Exercises XXIV

Find the following products, having regard to order of terms in the result:

1. $(1-x+x^2)(1+2x)$.
2. $(1-x+x^2)(1+x)$.
3. $(1+x+x^2)(1-x)$.
4. $(3z^2-5z+2)(2z-3)$.
5. $(5a^2-3a+7)(4a+5)$.
6. $(b^2-3b-8)(5b-4)$.
7. $(x^2+xy+y^2)(x-y)$.
8. $(x^2-xy+y^2)(x+y)$.
9. $(3p^2-2pq+4q^2)(7p-8q)$.
10. $(3r^3-5r^2+7r-4)(5r-3)$.

Multiply

11. $1+x+x^2$ by $1-x+x^2$.
12. $3x^2+2x+5$ by x^2-3x+7.
13. a^2+ax+x^2 by a^2-ax+x^2.
14. $3l^2+2lm+5m^2$ by $l^2-3lm+7m^2$.
15. $\frac{1}{2}y^2-\frac{1}{3}y+\frac{1}{4}$ by $\frac{2}{3}y^2-\frac{3}{4}y-\frac{2}{3}$.
16. $2-r^2+7r^4$ by $5+3r^2-r^4$.

Find the following continued products:

17. $(x+1)(x+2)(x+3)$.
18. $(x+a)(x+2a)(x+3a)$.
19. $(a+b)^3$.
20. $(a-b)^3$.
21. $(2x+3y)(x-5y)(4x-y)$.
22. $(x+1)(x+2)(x+3)(x+4)$.
23. $(1+x+x^2)(1-x+x^2)(1-x^2+x^4)$.
24. $(x^2+xy+y^2)(x^2-xy+y^2)(x^4-x^2y^2+y^4)$.

Multiply, giving the answers as a polynomial in x:

25. $x+h$ by $x+k$.
26. $x-a$ by $x-b$.
27. $ax+b$ by $cx+d$.
28. ax^2+bx+c by $x-h$.
29. $(ax^2-2bx+c)(px-q)$.
30. $x+a$ by $x+b$ by $x+c$.

20. Dimensions of Polynomials. In the expression

$$7x^3-3x^2+9x-5$$

the terms, in order, are of three, two, one, and zero dimensions in x. The expression itself is said to be of the same dimensions as that of its highest (dimensional) term, and therefore here to be three-dimensional. The expression

$$2x^2+3xy-5y^2-8x+11y-16$$

is of two dimensions in x and y, the first three terms being of two dimensions, the next two terms of one dimension, and the last term of zero dimensions.

The expression

$$2x^3+3y^3-8z^3+5xyz$$

is three-dimensional in x, y, z, each term being of three dimensions. When every term of an expression, as the one here taken, is of the same dimension in certain letters, the expression is said to be *homogeneous*.

Sometimes other letters than the ones considered as fixing the dimensions appear, as in

$$ax^2+2bx+c,$$

where x may be taken as the principal letter, and a, b, c regarded merely as coefficients. Here a b c may be any numbers whatever, so that $ax^2+2bx+c$ may denote *any polynomial whatever of two dimensions in x*. Thus if $a=1$, $b=0$, $c=1$ the polynomial is x^2+1, and if $a=2$, $b=-\frac{3}{2}$, $c=5$, the polynomial is $2x^2-3x+5$. However a must not be zero, for in that case the expression would not be of two dimensions. For the reasons stated, the expression $ax^2 + 2bx + c$ is called the *general polynomial of two dimensions in* x, or the *general quadratic (expression) in* x.

From what has been said in Section 17 it is readily seen that:

(a) *The dimension of the product of two polynomials in certain letters is the sum of the dimensions of the two factors.*

(b) *The product of two homogeneous expressions in certain letters is homogeneous in those letters.*

Exercises XXV

1. State the dimension in x of each of the following:

   (1) $5x^2 - 7x + 9$.

   (2) $3 + 5x - 4 + 7x^4 - 8x + 4 - 9x^3$.

   (3) $1 + x^3 + x^6$.

2. State the dimension in x of each of the products:

   (1) $(x^2 + 2x - 3)(x + 5)$.

   (2) $(x^2 + 3x - 1)(5x^3 + x + 1)$.

   (3) $(x + 1)(x + 2)(x + 3)(x + 4)$.

3. Find without going through the complete multiplication the term of two dimensions in x, in the following products:

   (1) $(x^2 + 5x + 9)(2x + 3)$.

   (2) $(2x^2 - 3x + 7)(5x - 9)$.

   (3) $(2x + 1)(2x + 3)(2x + 5)$.

4. Write down:

   (*a*) An expression of three dimensions in x.

   (*b*) An expression of two dimensions in x and y.

   (*c*) An expression of two dimensions in x, y and z.

(*d*) A homogeneous expression of four dimensions in x and y.

(*e*) A homogeneous expression of three dimensions in x, y and z.

5. Write down:

(*a*) The general expression of one dimension in x—in other words, the *general linear* expression in x.

(*b*) The general expression of three dimensions in x—in other words, the *general cubic* expression in x.

(*c*) The *general quartic* expression in x.

(*d*) The *general quintic* expression in x.

(*e*) The *general homogeneous* quadratic expression in x and y.

6. Would it be correct to say that the general quadratic expression in x is $hx^2 + 2kx + l$, or $px^2 + qx + r$, or $x^2 + mx + n$?

7. Multiply out

$$(x - y)(x + y)(x^2 + y^2).$$

What might have been predicted in regard to the dimensions of the result?

Exercises XXVI

Miscellaneous

A

1. Find the value of the expression $5x - 7$ for $x = -3, -2, -1, 0, +1, +2, +3$, arranging the results in tabular form.

2. The area of a circle is given by the relation

$$A = \pi r^2$$

where A, r denote the measures of the area and the radius, and π is a certain definite number. Taking 3·1416 as a working value of π, find the area of the circles whose radii are 1, 1·5, 2, 2·5, 3 centimetres respectively.

3. Find the product:

$$(x+3y)(x+5y)(x+7y).$$

What do this expression and the result become if y is taken equal to unity?

4. Shew that the sum of any three consecutive integers is three times the middle integer.

5. Find to what power of 2 the expression

$$2^2 \times 4^4 \times 8^8$$

is equal.

B

1. Shew that x^2 has the same value for $x=-1$ and $x=+1$; for $x=-2$ and $x=+2$; for $x=-3$ and $x=+3$; for $x=+a$ and $x=-a$.

2. A train travels for t hours at the rate of v miles an hour. If s is the number of miles "made," write the relation connecting s, v, t.

3. Find the product:

$$(x^3+x^2y+xy^2+y^3)(x-y).$$

What do this expression and the result become if x is taken equal to $+1$, and what if x is taken equal to -1?

4. An integer of two digits has its units digit greater by 2 than its tens digit. Shew that the number formed by writing the digits in the reverse order is greater by 18 than the original integer.

Ex. $42-24=18$; $75-57=18$.

5. One number exceeds another number by 7, and it is found that three times the less exceeds twice the greater by 3. Find the numbers.

CHAPTER VI

DIVISION

21. Quotient of One Monomial by Another. From the fact that $a \times b = ab$, it follows that $ab \div b = a$. So also it is readily seen that $-ab \div a = -b$, $a^2d \div (-ad) = -a$, $(-x^2y^3z^2) \div (-xy^3z) = xz$. When, as in these examples, one monomial contains as factors all the *literal* factors of another the former is said to be *algebraically divisible* by the latter. Thus $-2p^2qr$ is algebraically divisible by $3pr$, the quotient being $-\frac{2}{3}pq$; the fact that there is a fractional numerical coefficient, $-\frac{2}{3}$, in no way impairs the divisibility from the point of view of algebra.

The answers to the following exercises should be written down, and the results checked by multiplication as in Exercises XX.

EXERCISES XXVII

Write down the results of the following divisions:

1. $abc \div c$.
2. $-abcd \div cd$.
3. $-x^3yz^2 \div xyz$.
4. $-p^2q^3 \div -pq^2$.
5. $6l^2m^2n^2 \div -2lmn$.
6. $-3f^2gh^3 \div -2gh^2$.
7. $\frac{2}{3}a^2bc^2d \div -\frac{3}{4}abcd$.
8. $7u^2v^5w^7 \div 9uv^3w^5$.
9. $15abxy \div -5ax$.
10. $6(a+b)^2 \div 3(a+b)$.
11. $-9a^2b^2(x+y)^3 \div 3ab(x+y)$.
12. $15(a+b)^2(x+y)^3 \div -5(a+b)(x+y)$.

22. Quotient of a Compound by a Simple Expression. Since

$$a(x-y) = ax - ay$$

it follows that the quotient of $ax - ay$ by a is $x - y$. In $ax - ay$ the factor a is present in each term and is said to *run through* the expression. The exercises below given will recall Exercises XXII.

Exercises XXVIII

Write the results of the following divisions checking results by multiplication:

1. $(mx - nx) \div x$.
2. $(mx - nx) \div -x$.
3. $(a^2b + ab^2) \div a$.
4. $(a^2b + ab^2) \div b$.
5. $(a^2b + ab^2) \div ab$.
6. $(lmxy - npxy) \div xy$.
7. $(3x^2y + 3xy^2) \div xy$.
8. $(5a^2bc^2d - 7ab^2cd^2) \div -abcd$.
9. $(ax^2yz + bxy^2z + cxyz^2) \div xyz$.
10. $(\frac{2}{3}pq^2r^2 - \frac{5}{7}p^2qr^2 - \frac{3}{4}p^2q^2r) \div \frac{1}{2}pqr$.
11. $\{mn(x+y) - nl(x+y) + lm(x+y)\} \div (x+y)$.
12. $\{(x+y)(b-c) + (x+y)(c-a) + (x+y)(a-b)\} \div (x+y)$.
13. $\{(x+5)x + (x+5)7\} \div (x+5)$.
14. $\{(2x-7)3x - (2x-7)5\} \div (2x-7)$.

23. Quotient of One Polynomial by Another. To devise a method for such a division, it will be well first to form a product, say of $x+3$ and $x+5$.

$$\begin{array}{l} x + 3 \\ \underline{x + 5} \\ x^2 + 3x \\ \underline{\quad\;\, + 5x + 15} \\ x^2 + 8x + 15 \end{array}$$

$$\therefore\ (x+3)(x+5) = (x^2 + 8x + 15),$$

and therefore it follows that

$$(x^2 + 8x + 15) \div (x+3) = x + 5.$$

We wish now to see how this result might have been found if we had been given only $x^2 + 8x + 15$ to be divided by

$x+3$, and therefore did not know the multiplication from which $x^2+8x+15$ came.

Note first that $x^2+8x+15$ and $x+3$ are arranged in powers of x. We seek the factor that multiplied by $x+3$ will give $x^2+8x+15$. Plainly the term of highest power of x in this factor multiplied by x, the term of highest power in $x+3$, will give x^2, *i.e.*, the term of highest power in $x^2+8x+15$. Thus x is the *first term* in the factor sought. Therefore, if we were to multiply $x+3$ by the factor sought, the first multiplication would be by x and this would make up x^2+3x. There remains then $(x^2+8x+15)-(x^2+3x)$ or $5x+15$ to be made up by another multiplication. It is seen that $5x+15$ is the product of $x+3$ and 5; hence 5 is the other term in the factor sought which must therefore be $x+5$. This work can be readily shewn thus:

$$
\begin{array}{l}
x+3)x^2+8x+15(x+5 \\
\quad\;\;\underline{x^2+3x} \\
\qquad\quad +5x+15 \\
\qquad\quad \underline{+5x+15}
\end{array}
$$

It is seen that this process merely *unravels* the multiplication and shews that

$$
\begin{aligned}
x^2+8x+15 &= (x^2+3x)+(5x+15) \\
&= x(x+3)+5(x+3) \\
&= (x+3)(x+5).
\end{aligned}
$$

The following exercises may now be worked and the results checked. Further, the unravelled multiplication should be written out after the manner shewn.

Exercises XXIX

Find the following quotients:

1. $(x^2+7x+12)\div(x+4)$.
2. $(x^2-7x+12)\div(x-4)$.
3. $(x^2-x-12)\div(x-4)$.
4. $(x^2+2x-15)\div(x+5)$.
5. $(y^2-13y+40)\div(y-8)$.
6. $(2x^2+11x+12)\div(x+4)$.
7. $(3a^2+26a+35)\div(3a+5)$.
8. $(x^2+4xy+3y^2)\div(x+y)$.
9. $(x^2-8xy+15y^2)\div(x-5y)$.
10. $(6p^2+2pq-20q^2)\div(3p-5q)$.
11. $(15x^4+26x^2+8)\div(5x^2+2)$.
12. $(6a^2b^2-19ab+15)\div(2ab-3)$.
13. $(35x^2+9xy-104y^2)\div(7x+13y)$.
14. $(6-11x-35x^2)\div(2-7x)$.
15. $(4l^2m^2-3lmpq-27p^2q^2)\div(lm-3pq)$.
16. $(21-44x^2-32x^4)\div(3-8x^2)$.
17. $(15x^4-34x^2y^2+15y^4)\div(3x^2-5y^2)$.
18. $(x^2-9)\div(x-3)$.
19. $(10p^4-17p^2q^2+3q^4)\div(2p^2-3q^2)$.
20. $(x^4-y^4)\div(x^2+y^2)$.

Consider next the product $(x+3)(x^2+5x+7)$ formed in the two ways shewn:

$$\begin{array}{llll} x+3 & & & \\ x^2+5x+7 & & & \\ \hline x^3+3x^2 & & & \\ \quad +5x^2 & +15x & & \\ & +7x & +21 & \\ \hline x^4+8x^2 & +22x & +21 & \end{array} \qquad \begin{array}{llll} x^2 & +5x & +7 & \\ & x+3 & & \\ \hline x^3 & +5x^2 & +7x & \\ & +3x^2 & +15x & +21 \\ \hline x^3 & +8x^2 & +22x & +21 \end{array}$$

Then

$$(x^3+8x^2+22x+21)\div(x+3)=x^2+5x+7$$
$$(x^3+8x^2+22x+21)\div(x^2+5x+7)=x+3.$$

Let us see how these results might be found if they were not already known. Taking the first division, we see that the highest term in the factor which taken with $x+3$ will give as product $x^3+8x^2+22x+21$ is x^2, so that $x+3$ is to be multiplied by x^2, which makes up x^3+3x^2 of this product and leaves $5x^2+22x+21$ to be made up and we now ask what must $x+3$ be multiplied by to give $5x^2+22x+21$, which can be answered as in Exercises XXIX. The work may be shewn thus:

$$\begin{array}{l} x+3)x^3+8x^2+22x+21(x^2+5x+7 \\ \quad\quad \underline{x^3+3x^2} \\ \quad\quad\quad 5x^2+22x \\ \quad\quad\quad \underline{5x^2+15x} \\ \quad\quad\quad\quad 7x+21 \\ \quad\quad\quad\quad \underline{7x+21} \end{array}$$

Similar considerations lead to shewing the second division thus:

$$\begin{array}{l} x^2+5x+7)\,x^3+8x^2+22x+21\,(x+3 \\ \quad\quad\quad\quad \underline{x^3+5x^2+\ \ 7x} \\ \quad\quad\quad\quad\quad 3x^2+15x+21 \\ \quad\quad\quad\quad\quad \underline{3x^2+15x+21} \end{array}$$

The unravelling of the multiplication may be shewn thus:

$$\begin{aligned} x^3+8x^2+22x+21 &= (x^3+3x^2)+(5x^2+22x+21) \\ &= (x^3+3x^2)+(5x^2+15x)+(7x+21) \\ &= x^2(x+3)+5x(x+3)+7(x+3) \\ &= (x+3)(x^2+5x+7). \\ x^3+8x^2+22x+21 &= (x^3+5x^2+7x)+(3x^2+15x+21) \\ &= x(x^2+5x+7)+3(x^2+5x+7) \\ &= (x^2+5x+7)(x+3). \end{aligned}$$

The following exercises should be treated as were those of the preceding set.

Exercises XXX

Find the quotient in the division of:

1. $x^3 + 5x^2 + 17x + 22$ by $x + 2$.
2. $x^3 - 8x^2 + 23x - 24$ by $x - 3$.
3. $1 - 3y - 7y^2 + 6y^3$ by $1 + 2y$.
4. $x^3 - 1$ by $x - 1$.
5. $x^3 + 1$ by $x + 1$.
6. $2p^3 + 3p^2 - 32p + 15$ by $p + 5$.
7. $6a^3 - 13a^2 + 20a - 21$ by $2a - 3$.
8. $10x^3 - x^2y - 44xy^2 + 32y^3$ by $5x - 8y$.
9. $35 - 128z + 137z^2 - 36z^3$ by $5 - 9z$.
10. $6 - 25p^2 + 57p^4 - 88p^6$ by $3 - 8p^2$.
11. $x^3 - 10x^2 - 32x + 21$ by $x^2 - 13x + 7$.
12. $6x^3 - 5x^2y - 9xy^2 - 28y^3$ by $3x - 7y$.
13. $x^3 - 1$ by $x^2 + x + 1$.
14. $x^3 + 1$ by $x^2 - x + 1$.
15. $14 + 71p - 45p^2 - 66p^3$ by $7 - 3p - 6p^2$.
16. $35 + 46z^2 - 37z^4 + 6z^6$ by $5 + 8z^2 - 3z^4$.
17. $x^2 + (a + b)x + ab$ by $x + a$.
18. $x^4 - 12x^2 + 5x + 42$ by $x^2 + 5x + 6$.
19. $m^2x^2 - 1$ by $mx - 1$.
20. $x^3 - (p + q + r)x^2 + (qr + rp + pq)x - pqr$ by $x - p$.

24. Inexact Division. In algebra, as in arithmetic, an inexact division may be proposed. When one polynomial is to be divided by another the division is said to inexact when it is not possible to find a monomial or polynomial the product of which and the divisor, is the dividend. A few examples will make clear what is meant.

Ex. Divide $x^2 + 5x + 7$ by $x + 2$.

$$
\begin{array}{r}
x + 2)\,x^2 + 5x + 7\,(x + 3 \\
\underline{x^2 + 2x} \qquad\quad \\
3x + 7 \qquad \\
\underline{3x + 6} \qquad \\
1 \qquad
\end{array}
$$

Note that the division is of one *arranged* polynomial by another. A point is reached where the remainder to be divided is not of as high degree as the divisor. The division is inexact, with remainder 1. We may here write

$$\frac{x^2+5x+7}{x+2}=x+3+\frac{1}{x+2}$$

or

$$(x^2+5x+7)=(x+2)(x+3)+1.$$

Ex. 2. Divide x^3+a^3 by x^2+ax+a^2.

$$\begin{array}{r} x^2+ax+a^2)\,x^3 \qquad\qquad\qquad +a^3(x-a \\ \underline{x^3+ax^2+a^2x} \qquad\quad \\ -ax^2-a^2x+a^3 \\ \underline{-ax^2-a^2x-a^3} \\ +2a^3 \end{array}$$

Here the polynomials are arranged with respect to x, which is treated as the principal number; this requires, it is seen, arrangement with respect to a also. The division is continued until a point is reached where the remainder is of lower degree in x than the divisor. The remainder in this division is therefore $2a^3$.

Ex. 3. Divide $6z^3-19z^2+40z-37$ by $2z^2-3z+7$.

$$\begin{array}{r} 2z^2-3z+7)\,6z^3-19z^2+40z-37(3z-5 \\ \underline{6z^3-9z^2+21z} \qquad\quad \\ -10z^2+19z-37 \\ \underline{-10z^2+15z-35} \\ 4z-2 \end{array}$$

The remainder $4z-2$ is of lower degree in z than the divisor, so that the division terminates as an inexact division with this remainder.

Exercises XXXI

1. Examine whether the following are exact:

(1) $x^2+13x+17$ by $x+9$.

(2) $a^2-3ay-11y^2$ by $a-5y$.

(3) x^3-y^3 by x^2-xy+y^2.

(4) $3x^3-7x^2+11x-52$ by $x-4$.

(5) $6z^3+34z^2+25z-59$ by $3z^2+5z-7$.

2. Find what value n must have if x^2-5x+n is known to be divisible by $x-2$.

3. It is known that $x^2-*x+12$ is divisible by $x-3$, the $*$ indicating an erased numerical coefficient; restore this coefficient.

4. Divide $x^3-7x+11$ by $x+2$, by $x-3$, by $x-4$, and by $x-m$, finding the remainder in each case.

5. Divide x^3+px+q by $x-1$, by $x-2$, by $x-3$, and by $x-a$, finding the remainder in each case.

Exercises XXXII

(Miscellaneous)

A

1. If when a is divided by b the quotient is q and the remainder r, write the relation which exists among a, b, q, r without employing the sign of division.

Ex. $a=37$, $b=8$.

2. Express in ounces a pounds and b ounces.

3. Divide

$$\tfrac{1}{3}x^3-\tfrac{11}{72}x^2+\tfrac{1}{60}x-\tfrac{3}{10} \text{ by } \tfrac{2}{3}x-\tfrac{3}{4},$$

and verify by multiplication.

4. Find the sum

$$a+(a+b)+(a+2b)+(a+3b)+(a+4b),$$

and express the result in terms of the middle term.

5. Find three consecutive integers the sum of which is 57.

B

1. A district is surveyed and divided into m townships, each k miles square. How many acres are there in the district?

2. A man buys l pounds of tea at a cents a pound, m pounds at b cents a pound, and n pounds at c cents a pound; find the average cost a pound.

3. If 1, 3, 5, ... are called the 1st, 2nd, 3rd .... odd numbers, it is asserted that the nth odd number is $2n-1$. Test this for $n=2$, 5, 7.

4. Find the product

$$(x+y)(x^2+y^2)(x^3+y^3),$$

and comment on the dimensions of the result.

5. There are three consecutive odd numbers of which the sum is 51. Find the numbers.

C

1. From the sum of $15x^2-17x+23$ and $-3x^2-19x-37$ take the product of $4x-7$ and $3x+5$, expressing what is proposed algebraically, before engaging in any algebraic work.

2. The area of the surface of a sphere is given by the relation

$$A=4\pi r^2,$$

where A and r measure the area and the radius and π is a certain definite number. Shew that the area of the surface of a sphere with radius of measure 2 is four times that of a sphere with radius of measure 1.

3. There is a certain number, and it is known in regard to it that five times the number diminished by 11 is the same as three times the number increased by 27. Denoting the number by n, reason from this fact to the value of n.

4. Divide

$$x^4-14x^3+13x^2+11x-15 \text{ by } x^2-2x-3,$$

and test the accuracy of the work by dividing back by the quotient found.

5. Find the product

$$(da-bc)(db-ca)(dc-ab),$$

and arrange the result according to powers of d.

D

1. From the product of $5z-3$ and $8z+11$ take the product of $4z+7$ and $11z-9$, and multiply the result by $-3z+4$, indicating first in algebraic form what is proposed to be done.

2. The volume of a sphere is given by the relation

$$V=\tfrac{4}{3}\pi r^3,$$

where V and r measure the volume and the radius, and π is a certain definite number. Shew that the volume of a sphere of radius 2 is eight times that of a sphere of radius 1.

Taking $\frac{22}{7}$ as a working value of π, find the volume of a sphere of radius 3·5 decimeters.

3. Find the quotient of

$$6m^5+3m^4n-17m^3n^2+48m^2n^3-41mn^4+7n^5 \text{ by } 3m^2+6mn-7n^2,$$

testing the accuracy of the result in any way.

4. A could do a piece of work in l hours, B in m hours, and C in n hours. A works at it a hours, and is succeeded by B, who works for b hours, and is succeeded by C, who works for c hours. How much of the work is now done?

5. Write as a binomial employing one sign of subtraction

$$-(a-x)-(b-y)-(c-z).$$

E

1. Find the value of the expression $(x-3)(x-5)$ for $x=-1$, 0, +1, +2, +3, +4, +5, +6, +7, +8, +9, showing the results in tabular form.

2. Two circles with radii of measures a and $2a$ are described about the same centre. The circumference of the outer circle is divided into three equal parts at A, B, C, and AP, BQ, CR are straight lines drawn directly towards the common centre, meeting the inner circumference at P, Q, R. If $A=\pi r^2$, in the notation of (2) of (A), page 55, shew that the figure is divided into four parts of equal area.

3. Find the product

$$(2x+1)(x+3).$$

Repeat the work, taking x as 10, and compare with the ordinary multiplication of 21 by 13.

4. If n is an integer, find how much the cube of n exceeds the product of the three consecutive integers of which n is the middle one.

Illustrate by taking $n=7$, 23, 50.

5. A, B, C start from the same point, at the same time in the same direction along a straight roadway, at the rates of 3, 4, 5 miles an hour respectively. Shew that, at the end of t hours, B is midway between A and C.

CHAPTER VII

IMPORTANT IDENTITIES

25. Multiplication Identities. There are certain results in multiplication—many of them have appeared in preceding exercises—which it is important to keep in memory, as they will be found of frequent use. Being found by the rules of multiplication these results are valid for all values of the involved letters, just as $a^2 \times a^3 = a^5$ whatever be a, and $(a+b)x = ax + bx$ whatever be a, b, x. Statements of equality which are true for all values of the letters occurring in them are called *identities*.

(i)
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2.$$

These results should now be verified by multiplication, and illustrated by drawing lines of length $a+b$ and $a-b$, making squares on them and seeing that the parts corresponding to the terms on the right make up the square.

These identities may be given verbal statement: *The square of the sum (or the difference) of any two numbers is equal to the sum of the squares of those numbers increased (or diminished) by twice their product.*

The two identities may be combined into a single algebraic statement thus:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2.$$

Ex. 1. Deduce the second result from the first.

We have

$$(a+b)^2 = a^2 + 2ab + b^2,$$

a relation true for all values of the involved letters. We may then put $-b$ for $+b$ and the relation becomes

$$\{a + (-b)\}^2 = a^2 + 2a(-b) + (-b)^2$$

or

$$(a-b)^2 = a^2 - 2ab + b^2.$$

A verbal statement of the two relations, which brings out the fact that the two are virtually one, may be given thus, it being kept in mind that when the word *term* is used the sign is implied: *The square of a binomial is equal to the sum of the squares of the terms together with twice their product.*

Ex. 2. Find the expansion of $(2x-3y)^2$.

$$(2x-3y)^2 = (2x)^2 + 2(2x)(-3y) + (-3y)^2$$
$$= 4x^2 - 12xy + 9y^2.$$

Or thus, using the second relation directly:

$$(2x-3y)^2 = (2x)^2 - 2(2x)(3y) + (3y)^2$$
$$= 4x^2 - 12xy + 9y^2.$$

In working like exercises below, it would be well first to write them out in full as here, and then to run through them writing the final results at once.

Exercises XXXIII

1. Find the expansions of the following:

$(x+y)^2$; $(x-y)^2$; $(3p+4q)^2$; $(5a-3b)^2$;
$(x+1)^2$; $(x-9)^2$; $(3x-8)^2$; $(7x+9)^2$;
$(ab-cd)^2$; $(4lm-7np)^2$; $(\frac{2}{3}hk-\frac{3}{4}lm)^2$.

2. In $(a+b)^2 = a^2 + 2ab + b^2$, put $a = x$, and $b = -y$, and comment on the result.

3. From an examination of the three terms that give the square of $a+b$, namely

$$a^2 + 2ab + b^2,$$

(i) State the relation of the *middle term* to the other two terms.

(ii) Shew how, the first two terms being given, the third could be found.

4. Make a question, similar to question 3, referring to the square of $a-b$, and work the question.

5. Find the binomials of which the following are the squares, verifying by squaring the binomial in each case:

$$p^2+2pq+q^2;\quad m^2-2mn+n^2;\quad x^2+6x+9;$$
$$x^2-4x+4;\quad 4x^2+4xy+y^2;\quad 4x^2-12xy+9y^2;$$
$$m^2n^2+2mnpq+p^2q^2;\quad 4a^2b^2-20abcd+25c^2d^2.$$

6. Shew that

$$(10a+5)^2=100a(a+1)+25$$

and find a rule for writing down at once the square of such numbers as 25, 35, 85, 115.

7. Noting that $99^2=(100-1)^2$, make the computation.

8. Noting that $49^2=(50-1)^2$, make the computation.

9. Find the value of

$$101^2,\ 26^2,\ 74^2,\ 76^2,\ 69^2.$$

10. Give a verbal statement to the relation

$$a^2+2ab+b^2=(a+b)^2.$$

11. Expand $(-a-b)^2$ and explain why the result should be the same as the expansion of $(a+b)^2$.

12. Expand $(b-a)^2$ and explain why the result should be the same as the expansion of $(a-b)^2$.

(ii) $$(\mathbf{a}+\mathbf{b})(\mathbf{a}-\mathbf{b})=\mathbf{a}^2-\mathbf{b}^2.$$

This relation should be verified by multiplication, and it is an interesting geometrical exercise to illustrate it by a figure.

The identity admits the verbal statement: *The product of the sum and the difference of any two numbers is equal to the difference of their squares.*

EXERCISES XXXIV

1. Find, without actual multiplication, the following products:

$(p+q)(p-q)$; $(m+5)(m-5)$; $(2m+3)(2m-3)$;
$(x+1)(x-1)$; $(1+x)(1-x)$; $(3x+4y)(3x-4y)$;
$(ab+cd)(ab-cd)$; $(2mn-3pq)(2mn+3pq)$;
$(a^2+b^2)(a^2-b^2)$; $(a^3+b^3)(a^3-b^3)$; $(x^n+y^n)(x^n-y^n)$.

2. Find the following products:

$(\overline{a+b}+c)(\overline{a+b}-c)$; $(a+\overline{b+c})(a-\overline{b+c})$;
$(a+b-c)(a-b+c)$; $(2x+3y+4z)(2x+3y-4z)$;
$(3p-5q+7r)(3p+5q-7r)$; $(a^2+ab+b^2)(a^2-ab+b^2)$.

3. Give a verbal statement to the relation

$$a^2-b^2=(a+b)(a-b).$$

4. State what products yield the following results:

a^2-b^2; m^2-n^2; $4m^2-n^2$; $4m^2-9n^2$;
x^2-25; $a^2b^2-c^2d^2$; $9a^2b^2-25c^2d^2$; a^4-b^4.

5. Find the value of

$$49^2-48^2; \quad 26^2-24^2; \quad 127^2-117^2.$$

6. Shew that

$$(n+1)^2-n^2=2n+1.$$

Supposing n here to be an integer give a verbal statement to the relation.

(iii) $(x+a)(x+b)=x^2+(a+b)x+ab.$

Here x is treated as a principal number. The multiplication should be performed and the identity illustrated by a figure.

The relation may be given verbal statement: *The product of two binomials with a common term is equal to the square of the common term, increased by the product of the common term into the sum of the other two terms, together with the product of these other terms.*

EXERCISES XXXV

1. Write down, without actual multiplication, the following products:

$(x+1)(x+3)$; $(a-5)(a-7)$; $(m+4)(m+4)$;
$(2x+3)(2x+5)$; $(3y-5)(3y-11)$; $(4z-5)(4z-3)$;
$(ab+5)(ab-7)$; $(x+2y)(x+3y)$; $(m-5n)(m+2n)$;
$(2x-3y)(2x-3y)$; $(3z-8u)(3z+7u)$; $(3mn+2pq)(3mn-5pq)$.

2. Exhibit as the product of two binomials the following:

x^2+5x+6; $x^2+9x+20$; $y^2+7y+12$;
a^2-5a+6; $m^2-9m+20$; $z^2-7z+12$;
x^2-x-6; n^2-n-20; $r^2-2r-15$;
l^2+l-6; v^2+v-20; $k^2+2k-15$.

3. From the identity (iii) deduce the identities (i) and (ii).

4. Exhibit as the product of two binomials the following:

$(2x)^2+5(2x)+6$; $(5y)^2-7(5y)+12$;
$(3m)^2-2(3m)-35$; $(4r)^2+3(4r)-70$.

5. Express as the product of two binomials the following:

$x^2+5xy+6y^2$; $l^2+9lm+20m^2$;
$x^2-5xy+6y^2$; $a^2-ab-6b^2$.

6. Express as the product of two binomials the following:

$4x^2+10x+6$; $9y^2-18y+8$;
$16z^2-8z-99$; $4a^2-20ab+21b^2$.

(iv) $\quad (\mathbf{a}+\mathbf{b}+\mathbf{c})^2=\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2+2\mathbf{bc}+2\mathbf{ca}+2\mathbf{ab}.$

This relation should be verified by multiplication and illustrated by a figure. As has been suggested in an exercise, it can be derived from identity (i), for we have

$$\begin{aligned}(a+b+c)^2 &= (\overline{a+b}+c)^2\\ &= (a+b)^2+2(a+b)c+c^2\\ &= a^2+2ab+b^2+2ca+2bc+c^2\\ &= a^2+b^2+c^2+2bc+2ca+2ab.\end{aligned}$$

Exercises XXXVI

1. Write out the expansions of the following:

$(x+y+z)^2$; $(a+b-c)^2$; $(a-b-c)^2$;

$(2x+3y+5z)^2$; $(yz+zx+xy)^2$; $(ax-by+cz)^2$;

$(5l-9m+11n)^2$; $(2lx-6my-5nz)^2$; $(h^2-x^2-y^2)^2$.

2. Write out the expansion of $(\pm x \pm y \pm z)^2$ for each combination of signs.

3. Show that $(a-b+c)^2$ and $(-a+b-c)^2$ yield the same expansion, and explain why this is so.

4. Give a verbal statement to identify (iv).

5. In the expansion of a trinomial, show that if there are terms of negative sign, there are two and two only.

6. Find the expansion of

$$(a+b+c+d)^2.$$

Recalling the identities,

$$a^2 = a^2,$$
$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2bc + 2ca + 2ab,$$

we see that a square of a monomial consists of one term, the square of a binomial of three terms, the square of a trinomial of six terms, and it is plain that the squares of higher polynomials will consist of a still higher number of terms. The expressions on the right of the identities are called *perfect squares*. From what has been said, it is plain that an expression of two terms, or of four or five terms cannot be a perfect square. For example, a^2+b^2 is not a perfect square, although for certain numerical values of a and b it may be an arithmetical square; thus the values $a=3$, $b=4$, make a^2+b^2 equal to 25 or 5^2.

EXERCISES XXXVII

1. Find in each case what must be added to or subtracted from the following to yield a perfect square:

$$a^2+b^2;\quad 4x^2+y^2;\quad 4m^2+9n^2;\quad \tfrac{1}{4}h^2+\tfrac{1}{9}k^2;\quad m^2n^2+p^2q^2;\quad 25r^2+1.$$

2. Find in each case what must be added to the following to yield a perfect square:

$$a^2+2ab;\quad x^2+4xy;\quad x^2-6xy;\quad x^2+2x;\quad x^2-10x;\quad x^2-5x.$$

3. Write the different expressions in each case that may be added to the following to make a perfect square:

$$a^2+b^2+c^2;\quad a^2+b^2+c^2+2bc;\quad a^2+b^2+c^2+2bc+2ca;$$
$$x^2+y^2+1;\quad 4l^2+9m^2+16n^2;\quad p^2+q^2+2p+2q.$$

4. Exhibit $x^4+x^2y^2+y^4$ as the difference between two squares.

(v)
$$\begin{aligned}(\mathbf{a}+\mathbf{b})^3 &= \mathbf{a}^3+3\mathbf{a}^2\mathbf{b}+3\mathbf{a}\mathbf{b}^2+\mathbf{b}^3\\ &= \mathbf{a}^3+\mathbf{b}^3+3\mathbf{a}\mathbf{b}(\mathbf{a}+\mathbf{b}),\\ (\mathbf{a}-\mathbf{b})^3 &= \mathbf{a}^3-3\mathbf{a}^2\mathbf{b}+3\mathbf{a}\mathbf{b}^2-\mathbf{b}^3\\ &= \mathbf{a}^3-\mathbf{b}^3-3\mathbf{a}\mathbf{b}(\mathbf{a}-\mathbf{b}).\end{aligned}$$

These results should be verified by multiplying the square of $(a+b)$ or $(a-b)$ by $(a+b)$ or $(a-b)$. It is an interesting exercise to illustrate the identities by a figure, shewing $(a+b)^3$ as the cube described on a line of length $(a+b)$.

EXERCISES XXXVIII

1. Write down the expansion of each of the following:

$$(x+y)^3;\quad (1+y)^3;\quad (x+1)^3;\quad (x+5)^3;$$
$$(a^2+b^2)^3;\quad (pq+r)^3;\quad (mn+2)^3;\quad (lm+2np)^3;$$
$$(2x+3y)^3;\quad (2ab+3cd)^3;\quad (\tfrac{1}{2}x+\tfrac{1}{3}y)^3.$$

2. Write down the expansion of each of the following:

$$(x-y)^3;\quad (1-y)^3;\quad (x-1)^3;\quad (x-5)^3;$$
$$(a^2-b^2)^3;\quad (pq-r)^3;\quad (mn-2)^3;\quad (lm-2np)^3;$$
$$(2x-3y)^3;\quad (2ab-3cd)^3;\quad (\tfrac{1}{2}x-\tfrac{1}{3}y)^3.$$

3. Give a verbal statement to each form of each of the two identities under number (v).

4. Assuming that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, derive the expansion of $(a-b)^3$.

5. Employ the expansion for the cubes to find the value of 101^3, 99^3, 51^3, 49^3.

6. From the expansion of $(a+b)^3$ derive the expansion of $(a+b+c)^3$.

7. Find what should be added to the following, in each case, to yield a perfect cube:

$$a^3+b^3; \quad x^3-y^3; \quad 8m^3+27n^3;$$
$$a^3+3a^2b; \quad x^3-3x^2y; \quad 8m^3+12m^2n.$$

(vi) $$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (bc+ca+ab)x + abc.$$

This relation should be verified and illustrated by a figure. Here x is treated as a principal number, and it is to be noted that the product of the three binomials, linear in x, is a cubic polynomial in x.

Exercises XXXIX

1. Write out the following products:

$(x+1)(x+2)(x+3)$; $(x+4)(x+5)(x+6)$;
$(x+y)(x+2y)(x+3y)$; $(x+4y)(x+5y)(x+6y)$;
$(a-1)(a-2)(a-3)$; $(m-5)(m-1)(m-2)$;
$(p-1)(p+2)(p-3)$; $(z-2)(z-3)(z+4)$;
$(m-3n)(m+2n)(m+5n)$; $(xy-2)(xy+5)(xy-7)$;
$(2x-3)(2x-4)(2x-5)$; $(3x-1)(3x+5)(3x-8)$.

2. It is known that $x^3+9x^2+26x+24$ is the product of three factors linear in x, and that $x+3$ and $x+4$ are two of the factors. What must the third factor be?

3. It is known that $x^3-x^2-41x+105$ is the product of three factors linear in x, and that $x-5$ and $x+7$ are two of the factors. Find the third factor.

4. From identity (viii) deduce those under number (vii).

26. Identities in Division. Each of the identities of the preceding section can be read as an identity in division. For example (i) may be read

$$(a^2 + 2ab + b^2) \div (a + b) = a + b,$$

and (ii) may be read

$$(a^2 - b^2) \div (a - b) = a + b,$$

and so on. There are, in addition to these, certain results that are of frequent use.

$$\text{(i)} \qquad \frac{\mathbf{x^3 - y^3}}{\mathbf{x - y}} = \mathbf{x^2 + xy + y^2}.$$

The result should be verified by multiplication or division. The meaning of the identity is: *The quotient of the difference of the cubes of any two numbers by the difference of those numbers is the sum of the squares of the numbers increased by their product.*

EXERCISES XL

1. Write down the following quotients:

$$\frac{a^3 - b^3}{a - b}; \qquad \frac{y^3 - 1}{y - 1}; \qquad \frac{z^3 - 27}{z - 3};$$

$$\frac{8a^3 - 27b^3}{2a - 3b}; \qquad \frac{y^6 - 1}{y^2 - 1}; \qquad \frac{m^6 - n^6}{m^2 - n^2}.$$

2. Find the following quotients, by actual division where now necessary:

$$\frac{x - y}{x - y}; \quad \frac{x^2 - y^2}{x - y}; \quad \frac{x^3 - y^3}{x - y}; \quad \frac{x^4 - y^4}{x - y}; \quad \frac{x^5 - y^5}{x - y}; \quad \frac{x^6 - y^6}{x - y}; \quad \frac{x^7 - y^7}{x - y}.$$

What does this sequence of results seem to indicate?

$$\text{(ii)} \qquad \frac{\mathbf{x^3 + y^3}}{\mathbf{x + y}} = \mathbf{x^2 - xy + y^2}.$$

This result should be verified by multiplication or division.

Exercises XLI

1. Write down the following quotients:

$$\frac{a^3+b^3}{a+b};\quad \frac{y^3+1}{y+1};\quad \frac{z^3+27}{z+3};$$
$$\frac{8a^3+27b^3}{2a+3b};\quad \frac{y^6+1}{y^2+1};\quad \frac{m^6+n^6}{m^2+n^2}.$$

2. Examine the following divisions:

$$\frac{x+y}{x+y};\quad \frac{x^2+y^2}{x+y};\quad \frac{x^3+y^3}{x+y};\quad \frac{x^4+y^4}{x+y};\quad \frac{x^5+y^5}{x+y};\quad \frac{x^6+y^6}{x+y};\quad \frac{x^7+y^7}{x+y}.$$

What do the results seem to indicate?

3. Give the identity (ii) a verbal statement.

4. Deduce identity (ii) from identity (i).

Exercises XLII

(Miscellaneous)

A

1. Find the product

$$(x+1)(x+2)(x+3)(x+4),$$

first multiplying the two extreme factors and the two mean factors, and then multiplying the results.

2. If $x=3$ or 2, shew that the value of x^2-5x+6 is zero, and find the value of the expression for any other two values of x.

3. A man whose credit at the bank is good opened a current account, depositing $175; successive entries in his bank book shew that he has given a cheque for $89, has deposited $20, has given a cheque for $90, has deposited $25, has given a cheque for $110, has deposited $10, and has deposited $94. Find the amount he has in the bank after each entry.

4. From the expansion

$$(a+b+c)^2=a^2+b^2+c^2+2bc+2ca+2ab$$

deduce the expansion of $(a+b-c)^2$.

5. Find what must be added to the square of $x-y$ to yield the square of $x+y$.

B

1. Find the expansion of $(a+b)^4$,

(i) By multiplying the expansion of $(a+b)^2$ by itself.

(ii) By multiplying the expansion of $(a+b)^3$ by $a+b$.

Could the number of terms in the expansion have been foretold?

2. Shew that for $x=3$, or $x=5$ the expression $(x-3)(x-5)$ takes the value zero.

3. A vessel sails 53 degrees due south from a point of which the north latitude is 29 degrees. Find in what latitude the vessel is now.

4. Find the excess of $(a+b+c)^2$ over $a^2+b^2+c^2$.

5. Multiply $2x+3y+5z$ by $3x-7y+4z$, and then find each of the products

$$(2x+3y)(3x-7y),$$
$$(3y+5z)(-7y+4z),$$
$$(2x+5z)(3x+4z).$$

C

1. Employ the facts that

$$x^2+xy+y^2=\frac{x^3-y^3}{x-y}; \quad x^2-xy+y^2=\frac{x^3+y^3}{x+y},$$

to find the product $(x^2+xy+y^2)(x^2-xy+y^2)$.

2. Find the product

$$(ax+by)(cx+dy),$$

and shew that the product of the two terms in the coefficient of xy is equal to the product of the coefficients of x^2 and y^2.

3. Find what integer should be taken from $x^2-7x+13$ to make it divisible by $x-5$.

4. Shew that $(a+b)^3+c^3+(b+c)^3+a^3$ is divisible by $a+b+c$.

5. Find the value of

$$(b-c)^2+(c-a)^2+(a-b)^2.$$

D

1. Find the product

$$(x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2).$$

2. Find the expansion of

$$(x+y)^1, \quad (x+y)^2, \quad (x+y)^3, \quad (x+y)^4, \quad (x+y)^5$$

and seek a rule.

3. Find the value of

$$(y-z)(y+z-x) + (z-x)(z+x-y) + (x-y)(x+y-z).$$

4. The volume of a cone is given by the relation

$$V = \tfrac{1}{3}\pi h r^2$$

where V, r, h measure the volume, the radius of the base, and the height, while π denotes a certain definite number, which may be taken as 3·14. Find the volume of a cone of height 7 inches and with a base of radius 2 inches.

5. Find the polynomial which, divided by $x^2 - 3x - 5$, gives the quotient $2x^2 + 6x - 7$ and the remainder $8x - 13$.

E

1. Group the terms of

$$x^2 + y^2 + 2(xy - ab) - (a^2 + b^2)$$

so as to exhibit the expression as the difference of two squares.

2. Shew that $(c-a)^3 + (a-b)^3$ is exactly divisible by $b - c$.

3. Of what two numbers is 25 the square?

4. From the identity

$$(x+y)^2 = x^2 + 2xy + y^2$$

deduce the expansion of $(a+b+c+d)^2$.

5. Shew that $az^6 + bz^4 + cz^2 + d$ is not changed by the substitution of $-z$ for z.

CHAPTER VIII

FACTORS

27. Resolution into Factors. It has been seen, at several points in the preceding chapters, that there are polynomials which can be expressed as the product of two or more algebraic factors. The process of expressing a polynomial as the product of factors is called *resolution into factors*. We may speak also of resolving a monomial as ab or $5x^2yz^3$ into factors, but the factors are seen at once, and it would serve no purpose to offer a set of exercises.

In this chapter there will be taken up methods for dealing with such cases of resolution into factors as ordinarily occur. It will be seen that the methods are based on results in multiplication and division that can now be taken as familiar. The student should keep in mind the *types* that yield to resolution and, when new expressions present themselves, should be solicitous as to whether or not they are resolvable.

28. Resolution Based on the Identity

$$m(x+y) = mx + my = (x+y)m.$$

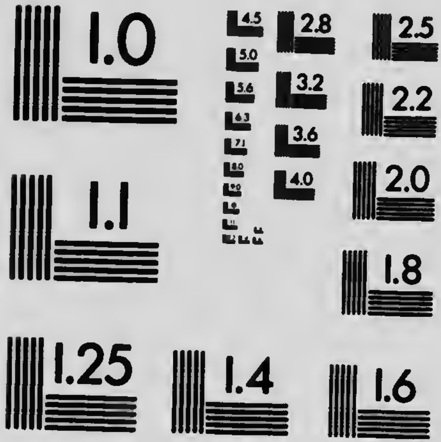
EXERCISES XLIII

Resolve into factors:

1. $mp + mq$.
2. $x^2y + xy^2$.
3. $a^2b - ab^2$.
4. $x + mx$.
5. $ab - kab$.
6. $xy - mnxy$.

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



7. $6a^2xy - 9a^2zx$.

8. $15a^2bc - 18abc^2$.

9. $m^2nx^2y - mn^2xy^2$.

10. $ax - ay + az$.

11. $(a+b)x - (a+b)y$.

12. $(a+b) - (a+b)m$.

13. $(a+b) + mn(a+b)$.

14. $xy(a+b) - uv(a+b)$.

15. $(a+b)(x+y) - (a+b)(x-y)$.

By a repeated application of this principle more complex expressions may sometimes be resolved.

Ex. 1. Resolve into factors $1 + x + y + xy$.

$$1 + x + y + xy = (1+x) + y(1+x)$$
$$= (1+x)(1+y).$$

Ex. 2. Resolve into factors $abc - bcx - ax^2 + x^3$.

$$abc - bcx - ax^2 + x^3 = bc(a-x) - x^2(a-x)$$
$$= (a-x)(bc - x^2),$$

and the expression is resolved.

If in such cases no grouping of terms suggests itself at once, it is well to seek out some one letter which occurs to one and only one power, and make all terms containing it into one group.

EXERCISES XLIV

Resolve into factors:

1. $1 - x - y + xy$.

2. $pq - qx - py + xy$.

3. $abcd + cbmy + cdlx + lmxy$.

4. $x^3 - nx^2 - m^2x + m^2n$.

5. $x^2 + ax + bx + ab$.

6. $x^3 + (h+p)x^2 + (q+hp)x + hq$.

7. $a^2c + b^2d + b^2c + a^2d$.

8. $p^2x - q^2y - p^2y + q^2x$.

29. Resolution Based on the Identities

$$(a \pm b)^2 = a^2 \pm 2ab + b^2,$$
$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2bc + 2ca + 2ab.$$

When the expression $a^2 + 2ab + b^2$ is recognized as the product of the two equal factors $a + b$ and $a + b$ the resolution is effected.

The following exercises should be worked out so as to shew that they conform to the fundamental identity, as in the example:

$$4a^2 - 12ax + 9x^2 = (2a)^2 - 2(2a)(3x) + (3x)^2$$
$$= (2a - 3x)^2.$$

They and all similar exercises should then be reworked, without detailed explanation, to develop sensitiveness to algebraic form. The verification of the results will afford exercises either in multiplication or in the methods of the preceding chapter, and will help to do away with any dependence upon the answers to the exercises.

Exercises XLV

Resolve into factors:

1. $m^2 + 2mn + n^2$.
2. $p^2 - 2pq + q^2$.
3. $x^2 + 2x + 1$.
4. $x^2 - 2x + 1$.
5. $x^2 + 4x + 4$.
6. $x^2 - 14x + 49$.
7. $4a^2 + 12ab + 9b^2$.
8. $9a^2b^2 - 30ab + 25$.
9. $(a + b)^2 + 2(a + b)c + c^2$.
10. $(a + b)^2 - 2(a + b)(x + y) + (x + y)^2$.
11. $20^2 + 2(20)9 + 81$.
12. $1 - 22xy + 121x^2y^2$.
13. $25u^2 - 20uv + 4v^2$.
14. $\frac{4}{9}x^2 + 2x + \frac{9}{4}$.
15. $(a - x)^2 + 4x(a - x) + 4x^2$.
16. $m^4 + 2m^2n^2 + n^4$.
17. $4x^{12} - 4x^6y^6 + y^{12}$.
18. $81z^2 - 18z^2 + 1$.
19. $x^2 + y^2 + z^2 - 2yz - 2zx + 2xy$.
20. $9x^2 + 16y^2 + 25z^2 - 40yz + 30zx - 24xy$.
21. $4p^2 + 12pq + 9q^2 - 4p - 6q + 1$.
22. $a^2x^2 + b^2y^2 + c^2z^2 + 2bcyz + 2cazx + 2abxy$.
23. $y^2z^2 + z^2x^2 + x^2y^2 + 2xyz(x + y + z)$.
24. $x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 + 2x^2y^2$.
25. $\frac{1}{4}a^4 + \frac{1}{9}b^4 + \frac{1}{25}c^4 + \frac{2}{15}b^2c^2 + \frac{1}{5}c^2a^2 + \frac{1}{3}a^2b^2$.
26. $25l^2 - 10lm + m^2 - 10l + 2m + 1$.

30. Resolution Based on the Identity

$$(\mathbf{a}+\mathbf{b})(\mathbf{a}-\mathbf{b})=\mathbf{a}^2-\mathbf{b}^2.$$

Ex. 1. $9x^2-25a^2=(3x)^2-(5a)^2$

$=(3x+5a)(3x-5a).$

Ex. 2. $a^2-b^2+2bc-c^2=a^2-(b^2-2bc+c^2)$

$=a^2-(b-c)^2$

$=(a+\overline{b-c})(a-\overline{b-c})$

$=(a+b-c)(a-b+c).$

Exercises XLVI

Resolve into factors:

1. $16x^2-25y^2.$
2. $a^2-9b^2.$
3. $a^2b^2-x^2y^2.$
4. $9m^2n^2-4p^2q^2.$
5. $(a+b)^2-c^2.$
6. $a^2-2ab+b^2-c^2.$
7. $x^2-y^2-2yz-z^2.$
8. $4x^2-20xy+25y^2-9z^4.$
9. $(a+b)^2-(c+d)^2.$
10. $a^2-2ab+b^2-c^2+2cd-d^2.$
11. $1-81m^2.$
12. $1-p^2-2pq-q^2.$
13. $(a^2+b^2)^2-4a^2b^2.$
14. $(x-2y)^2-(y-2x)^2.$
15. $a^4-b^4.$
16. $1-x^6.$
17. $a^8-b^8.$
18. $y^{16}-1.$
19. $(x^2-y^2+z^2)^2-4z^2x^2.$
20. $x^2+2xy+y^2-z^2-2z-1.$
21. $a^2-b^2+2bc-c^2.$
22. $x^2-4y^2-12yz-9z^2.$
23. $(x^2+y^2)^2-(x^2-y^2)^2.$
24. $(a+b+c)^2-(a-b-c)^2.$
25. $(2x^2+3x+1)^2-(x^2-3x-2)^2.$
26. $(a-b)^2-(c-d)^2+(a-d)^2-(b-c)^2.$

31. Resolution Based on the Identity

$$(\mathbf{x}+\mathbf{a})(\mathbf{x}+\mathbf{b})=\mathbf{x}^2+(\mathbf{a}+\mathbf{b})\mathbf{x}+\mathbf{ab}.$$

It is plain that the factors of $x^2+8x+15$ can be found if two numbers can be found of which the *sum* is $+8$ and the *product* is $+15$. These numbers are seen by trial to be $+3$ and $+5$ and we can write

$$x^2+8x+15=(x+3)(x+5).$$

This trinomial may be resolved by writing it as the difference of two squares thus

$$\begin{aligned} x^2 + 8x + 15 &= x^2 + 2(4)x + 15 \\ &= x^2 + 2(4)(x) + 4^2 - 4^2 + 15 \\ &= (x + 4)^2 - 1^2 \\ &= (\overline{x + 4} + 1)(\overline{x + 4} - 1) \\ &= (x + 5)(x + 3). \end{aligned}$$

It would be well to work several of the exercises in both ways.

Exercises XLVII

Resolve into factors:

1. $x^2 + 12x + 35.$
2. $y^2 + 9x + 20.$
3. $a^2 + 10x + 21.$
4. $m^2 + 8mn + 15n^2.$
5. $p^2q^2 + 5pq + 6.$
6. $m^2n^2 + 11mnpq + 30p^2q^2.$
7. $4x^2 + 16x + 15.$
8. $9y^2 + 51y + 16y^2.$
9. $16x^2 + 12xy + 2y^2.$
10. $1 + 18xy + 65x^2y^2.$
11. $x^2 - 8x + 15.$
12. $x^2 - 2x - 15.$
13. $y^2 + 2y - 15.$
14. $x^2y^2 - 2xy - 35.$
15. $a^2b^2 + 3abcd - 70c^2d^2.$
16. $91 - 20z^2 + z^4.$
17. $1 - 20z^2 + 91z^4.$
18. $4x^2y^2 - 36xyz + 65z^2.$
19. $1 - 4z^3 - 77z^6.$
20. $x^4 - 18x^2 + 81.$
21. $(a + b)^2 - 7c(a + b) + 12c^2.$
22. $x^2y^2z^2 - 16xyz - 192.$
23. $4x^2y^2 + 10xyz^2 - 24z^4.$
24. $25x^6 + 25x^3 - 84.$
25. $(x - y)^4 - 13z^2(x - y)^2 + 36z^4.$
26. $\frac{1}{4}x^2 - \frac{4}{15}x + \frac{1}{15}.$

By an extension of this method, trinomials which are the products of two such factors as $(2x + 5y)$ and $(3x + 4y)$ can be resolved. The product is here $6x^2 + 23xy + 20y^2$, and we have first multiplying by 6 to make the first term a square and dividing by 6 to restore the original value.

$$\begin{aligned} 6x^2 + 23xy + 20y^2 &= [36x^2 + 23 \times 6xy + 120y^2] \div 6 \\ &= [(6x)^2 + 23y(6x) + 120y^2] \div 6 \\ &= [(6x + 8y)(6x + 15y)] \div 6 \\ &= (3x + 4y)(2x + 5y). \end{aligned}$$

Often, however, the factors of such an expression are readily found in the following way. Suppose that

$$6x^2 + 23xy + 20y^2$$

is to be resolved. It is known in advance that the factors to be found are of the form

$$(*x + *y)(*x + *y),$$

the star indicating a numerical coefficient not yet known. The product of the coefficients of x is 6, and of those of y is 20. We choose any such factors, say

$$\begin{aligned} &6x + 5y \\ &x + 4y. \end{aligned}$$

It is seen by cross multiplication that in the products of these two factors the coefficient of xy must be $24 + 5$ or 29, and is therefore not 23; the attempt has then failed. Try next

$$\begin{aligned} &3x + 4y \\ &2x + 5y. \end{aligned}$$

Here the coefficient of xy is 23, and we have the factors. As only a limited number of trials can be made, this can be said to be a method. A little practice will bring readiness in applying it.

The following exercises should be worked by both methods.

Exercises XLVIII

Resolve into factors:

1. $2x^2 + 13xy + 15y^2$.
2. $12a^2 + 41ab + 35b^2$.
3. $35x^2 - 71x + 24$.
4. $21x^2 + 2x - 55$.
5. $12p^2 - 7pq - 45q^2$.
6. $15m^2 + 2mn - 77n^2$.
7. $28r^2 - 103r + 39$.
8. $6z^2 + 7z - 5$.
9. $15 - 19x - 10x^2$.
10. $4a^2b^2 + 23abxy + 15x^2y^2$.
11. $7 - 60z^2 - 27z^4$.
12. $35x^2 - 23xy - 72y^2$.
13. $35p^2 - 103p + 72$.
14. $35 - 11z - 72z^2$.
15. $40x^4 - 51x^2 - 108$.
16. $6x^2 + 13xy + 6y^2$.
17. $30x^2y^2 - 61xyz^2 + 30z^4$.
18. $81 - 153x^2 + 16x^4$.
19. $52h^2 - 67hk + 21k^2$.
20. $8z^4 + 26z^2 - 99$.

32. Resolution by Completing the Square. In Section 31 an example was given in which by *completing a square*, the expression assumed a form which admitted resolution into factors. There are two or three types of expressions that can be resolved into factors by this artifice.

Ex. 1. Resolve $x^4 + x^2y^2 + y^4$ into factors.

$$x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - 2x^2y^2 + x^2y^2,$$

adding to complete the square and subtracting to correct.

$$\therefore \quad x^4 + x^2y^2 + y^4 = (x^2 + y^2)^2 - (xy)^2$$

$$= (\overline{x^2 + y^2} + xy)(\overline{x^2 + y^2} - xy)$$

$$= (x^2 + xy + y^2)(x^2 - xy + y^2)$$

$$\therefore \quad \mathbf{x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2)}$$

a result which should be kept in memory. As an exercise in the review of terms, it may be noted that the two factors are each homogeneous and of two dimensions in x and y, and the result is homogeneous and of four dimensions in x and y.

Ex. 2. Resolve $x^4 + 4y^4$ into factors.

$$\begin{aligned} x^4 + 4y^4 &= (x^4 + 4x^2y^2 + 4y^4) - 4x^2y^2 \\ &= (x^2 + 2y^2)^2 - (2xy)^2 \\ &= (\overline{x^2 + 2y^2} + 2xy)(\overline{x^2 + 2y^2} - 2xy) \\ &= (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2). \end{aligned}$$

Exercises XLIX

Resolve into factors:

1. $a^4 + a^2b^2 + b^4$.
2. $x^4 + x^2 + 1$.
3. $x^4 + 64y^4$.
4. $m^4 + 324n^4$.
5. $4p^4 + 81q^4$.
6. $x^4 + 9x^2 + 25$.
7. $x^4 - 11x^2 + 25$.
8. $m^4 - 11m^2n^2 + n^4$.
9. $4x^4 - 16x^2y^2 + 9y^4$.
10. $4x^4 + 3x^2y^2 + 9y^4$.
11. $1 + 4z^4$.
12. $1 + 4z^8$.
13. $x^8 + x^4 + 1$.
14. $x^{16} + x^8 + 1$.

82. Resolution Based on the Identities

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

The relation of the different terms in the expansion should be studied so that a perfect cube can be at once identified. Exercises XXXVIII should be re-examined.

Exercises L

Resolve into factors:

1. $a^3 + 3a^2 + 3a + 1$.
2. $a^3 - 3a^2 + 3a - 1$.
3. $x^3 + 6ax^2 + 12a^2x + 8a^3$.
4. $8p^3 - 36p^2q + 54pq^2 - 27q^3$.
5. $x^6 - 12x^4y^2 + 48x^2y^4 - 64y^6$.
6. $1 - 9x + 27x^2 - 27x^3$.
7. $27m^3 + 135m^2n + 225mn^2 + 125n^3$.
8. $(a + b)^3 + (a - b)^3 + 6a(a^2 - b^2)$.
9. $8x^6 + 60x^4y^2 + 150x^2y^4 + 125y^6$.
10. $1 - 6x + 12x^2 - 8x^3$.
11. $1 - 9z^2 + 27z^4 - 27z^6$.
12. $(a - b)^3 + 6b(a - b)^2 + 12b^2(a - b) + 8b^3$.

Sometimes, by *completing a cube*, an expression is made to assume a form which lends itself to resolution.

Ex. $x^3 + y^3 = (x^3 + 3x^2y + 3xy^2 + y^3) - (3x^2y + 3xy^2)$

$$= (x+y)^3 - 3xy\,(x+y)$$
$$= (x+y).[(x+y)^2 - 3xy]$$
$$= (x+y)(x^2 + 2xy + y^2 - 3xy)$$
$$= (x+y)(x^2 - xy + y^2),$$

in accordance with identity (ii) page 75.

Exercises LI

Resolve into factors:

1. $x^3 - y^3$.
2. $a^3 + a^2b + ab^2 + b^3$.
3. $x^3 + x^2 + x + 1$.
4. $x^3 + 2x^2 + 2x + 1$.
5. $x^6 - 4x^4 + 4x^2 - 1$.
6. $x^3 + 3x^2 - 13x - 15$.
7. $a^3 - a^2b + ab^2 - b^3$.
8. $a^3 + 2a^2b - 2ab^2 - b^3$.

33. Resolution Based on the Identities

$$\mathbf{a^3 + b^3 = (a+b)(a^2 - ab + b^2)},$$
$$\mathbf{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}.$$

These follow at once from the identities (i) and (ii) of Section 2[illegible] [illegible]y should be given verbal statement.

Ex. [illegible] into factors $8a^3 - 27b^3$.

$$[illegible]\,27b^3 = (2a)^3 - (3b)^3$$
$$= \{(2a)(3b)\}\{(2a)^2 + (2a)(3b) + (3b)^2\}$$
$$= (2a - 3b)(4a^2 + 6ab + 9b^2).$$

Ex. 2. Resolve into factors $x^6 + y^6$.

$$x^6 + y^6 = (x^2)^3 + (y^2)^3$$
$$= \{(x^2) + (y^2)\}\{(x^2)^2 - (x^2)(y^2) + (y^2)^2\}$$
$$= (x^2 + y^2)(x^4 - x^2y^2 + y^4).$$

EXERCISES LII

Resolve into factors:

1. $x^3 - 1$.
2. $y^3 + 1$.
3. $1 + a^3b^3$.
4. $1 - 343p^3q^3$.
5. $p^3 - 27$.
6. $q^3 + 64$.
7. $8a^3 + 27b^3$.
8. $27m^3 - 125n^3$.
9. $x^6 - y^6$.
10. $x^9 + y^9$.
11. $x^9 - y^9$.
12. $x^{12} - 1$.
13. $x^{12} + y^{12}$.
14. $a^{15} - b^{15}$.
15. $a^3b^3 - c^6$.
16. $(a^2 - bc)^3 - 8b^3c^3$.
17. $(x+1)^3 + (x-1)^3$.
18. $(x-2y)^3 - (2x-y)^3$.
19. $(b-c)^3 + (c-a)^3$.
20. $(b-c)^3 + (c-a)^3 + (a-b)^3$.
21. $(a+b)^3 + c^3$.
22. $(x+y)^3 - 1$.

84. Notes. If it is proposed to resolve into factors the trinomial in x

$$x^2 - 6x + 7,$$

all attempts to find two integers or fractions of which the sum is -6 and the product $+7$ fail. If the expression is treated thus:

$$x^2 - 6x + 7 = (x^2 - 6x + 9) - 9 + 7$$
$$= (x-3)^2 - 2,$$

it is seen that it does not present itself as the difference of two squares. Accordingly it would seem that the given quadratic expression cannot be resolved into two factors of the first degree. Later it will be seen that, with an extension of the meaning of number, the resolution of every quadratic in x is possible. Similar remarks apply to the resolution of the quadratic

$$x^2 - 6xy + 7y^2,$$

homogeneous in two letters x and y, into two factors,

$$(x + *y)(x + *y),$$

each homogeneous and of one dimension in x and y.

When it is found that, for example,

$$x^2 - 5x + 6 = (x-2)(x-3).$$
$$8x^2 - 22x + 15 = (2x-3)(4x-5),$$

there probably arises the question whether such expressions can be resolved in only one way. Since $2x-3=2(x-\frac{3}{2})$ and $4x-5=4(x-\frac{5}{4})$, the second set of factors may be written

$$8(x-\tfrac{3}{2})(x-\tfrac{5}{4}).$$

Here, however, there has been a mere shifting of numerical factors, and this set of factors is algebraically not different from the set $(2x-3)(4x-5)$. It will be seen later that in such cases the resolution is possible in only one way, so that, for instance, $8x^2-22x+15$ cannot be the product of factors other than $x-\frac{3}{2}$ and $x-\frac{5}{4}$, with a numerical factor. This is analogous to the arithmetical fact that integral numbers, as $35=5\times 7$, $63=3\times 3\times 7$, can be resolved into prime factors in only one way.

The factors of ab are a and b, and those of x^2-9 are $x-3$ and $x+3$. These and like statements are always to be taken in the algebraic sense, and not in the sense of factors in the arithmetic of integers. Thus if $a=\frac{3}{7}$ and $b=\frac{35}{3}$, $ab=5$, a prime number. So a^2 in algebra is a perfect square, and a is not, though if a happens to be 9 it is an arithmetical square.

35. Highest Common Factor. The two expressions

$$a^2bcp, \quad ab^2cq,$$

have several common factors, namely, a, b, c, bc, ca, ab, abc. Of these the one of ***highest degree*** in the involved letters is abc, which on this account is called the ***highest common factor***. The analogue in the theory of arithmetical integers is the ***greatest common measure;*** for example, the greatest common measure of 54 and 72 is 18. When it is said that one number is a measure of another, it is meant that it is contained in that number an integral number of times. Now, in algebra, a is a factor of ab, but the co-factor b is not necessarily an integer, and the term ***measure*** is not appropriate. Further, while abc is the ***highest*** common factor of a^2bcp and ab^2cq it may not be the ***greatest*** in the arithmetical sense, for if a, b, c, are positive proper fractions, abc is less than bc which, in turn, is less than b or c.

The highest common factor (the H.C.F.) of

$$54p^2q^3r \text{ and } 72pq^2r^3$$

is $18pq^2r$, the G.C.M. of the numerical coefficients being associated with the algebraic H.C.F.

Ex. 1. Find the H.C.F. of $x^2-17x+70$ and $x^2-10x+21$.

$$x^2-17x+70=(x-10)(x-7),$$
$$x^2-10x+21=(x-3)(x-7).$$
$$\therefore \text{ The H.C.F. is } x-7.$$

Ex. 2. Find the H.C.F. of $x^2-xy-12y^2$, $2x^2+7xy+3y^2$, and $3x^2+7xy-6y^2$.

$$x^2-xy-12y^2=(x-4y)(x+3y),$$
$$2x^2+7xy+3y^2=(2x+y)(x+3y),$$
$$3x^2+7xy-6y^2=(3x-2y)(x+3y).$$
$$\therefore \text{ the H.C.F. is } x+3y.$$

Ex. 3. Find the H.C.F. of $x^2+2x-15$ and $x^3+2x^2-9x+30$.

$$x^2+2x-15=(x-3)(x+5).$$

Not having a ready method of resolving the cubic expression, we see, by division, whether one of the expressions $x-3$, $x+5$ is a factor of it, and find that

$$x^3+2x^2-9x+30=(x+5)(x^2-3x+6).$$

Hence, since x^2-3x+6 does not contain $x-3$ as a factor, it follows that the H.C.F. is $x+5$.

Exercises LIII

Find the H.C.F. of:

1. $a^2bc,\ ab^2c,\ abc^2$.
2. $ax^2,\ bx^3,\ cx^4$.
3. $a^2bcd^2,\ ab^2c^2d$.
4. $ap^3q,\ bp^2q^2,\ cpq^2,\ dq^3$.
5. $24h^2k^2l,\ 36hkl^2$.
6. $85x^2y,\ 51xy^2$.
7. $105x^2yz,\ 91xy^2z,\ 56xyz^2$.
8. $a^2x^2y^2z^4,\ b^2x^4y^2z^2\ \ c^2x^2y^4z^2$.
9. $a^2-b^2,\ a^3-b^3,\ a^4-b^4,\ a^6-b^6$.
10. $a^4-b^4,\ a^6-b^6,\ a^8-b^8$.
11. $x^8-y^8,\ x^{12}-y^{12}$.
12. $x^2-9x+20,\ x^2-8x+15,\ x^2-7x+10$.
13. $a^2-ab-6b^2,\ a^2-7ab+12b^2,\ a^2+2ab-15b^2$.
14. $2x^2-xy-3y^2,\ 4x^2-9y^2,\ 6x^2-xy-12y^2$.
15. $x^2+(a+l)x+al,\ x^2+(a+m)x+am,\ x^2+(a+n)x+an$.
16. $(a+b)^3,\ a^3+b^3,\ a^2-b^2$.
17. $(a-b)^2,\ a^3-b^3,\ a^2-b^2$.
18. $8a^3-27b^3,\ 4a^2-12ab+9b^2,\ 4a^2-9b^2$.
19. $15+11z-12z^2,\ 6+23z+20z^2,\ 12+7z-12z^2$.
20. $a^4+a^2b^2+b^4,\ a^6-b^6,\ a^8+a^4b^4+b^8$.
21. $x^2-2x-15,\ x^3+x^2-14x-24$.
22. $a^2+3ab-18b^2,\ a^3+11a^2b+37ab^2+42b^3$.
23. $6a^2-ab-12b^2,\ 6a^3-a^2b+3ab^2+20b^3$.
24. $x^3-6x^2+11x-6,\ x^3+4x^2+x-6$.

36. Lowest Common Multiple. The lowest common multiple (L.C.M.) of

$$a^2bcp \text{ and } ab^2cq$$

is a^2b^2cpq, this being the expression of lowest degree that will admit both of the given expressions as factors. The *multiple* is to be understood in its algebraic sense, and *lowest* has reference to the *degree*, here, as in the treatment of H.C.F., the idea of degree in algebra being analogous to to that of magnitude in arithmetic.

Ex. 1. Find the L.C.M. of $2a^2bc$, $3ab^2c$, $4abc^2$. Here plainly $12a^2b^2c^2$ is the expression of lowest degree admitting each of the given expressions as a factor, and the numerical coefficient is the smallest that is a multiple of each of the coefficients.

Ex. 2. Find the L.C.M. of

$$x^2-5x+6,\ x^2-7x+12,\ x^2-6x+8.$$

$$x^2-5x+\ \ 6=(x-3)(x-2),$$
$$x^2-7x+12=(x-4)(x-3),$$
$$x^2-6x+\ \ 8=(x-2)(x-4).$$

Therefore the L.C.M. is

$$(x-2)(x-3)(x-4),$$

i.e., $$x^3-9x^2+26x-24.$$

Exercises LIV

Find the L.C.M. of:

1. $yz,\ zx,\ xy.$
2. $a^2bc,\ ab^2c,\ abc^2.$
3. $aqr,\ brp,\ cpq.$
4. $al^2,\ bm^2,\ cn^2.$
5. $2x^2y,\ 3y^2z,\ 4z^2x.$
6. $7a^2bcd^2,\ 9ab^2c^2d.$
7. $4a^2d,\ 5b^2e,\ 6c^2f.$
8. $5a^2x^3,\ 6ax^4,\ 9a^3x^2.$
9. $bcyz,\ cazx,\ abxy.$
10. $12l^2m^3n,\ 18lm^2n^3,\ 21lm^3n^2.$
11. $(x+1)(x+2),\ (x+2)(x+3),\ (x+3)(x+1).$
12. $x^2-5x+6,\ x^2-x-6,\ x^2+x-6,\ x^2+5x+6.$
13. $4x^2-9y^2,\ (2x-3y)^2,\ (2x+3y)^2.$
14. $x^2-6x+9,\ x^2+4x-21,\ x^2-8x+15.$
15. $2a^2+5ab+2b^2,\ a^2-4b^2,\ a^2+4ab+4b^2.$
16. $6x^2-x-15,\ 8x^2-2x-21,\ 10x^2+23x+12.$
17. $a+b,\ a^2+b^2,\ a-b,\ a^2-b^2.$
18. $x^2-y^2,\ x^3-y^3,\ x^4-y^4.$

19. $a^8 - b^8$, $a^{12} - b^{12}$.

20. $z^2 - 20z + 96$, $z^3 - 3z^2 - 36z - 32$.

21. $x^2 + (c+a)x + ca$, $x^2 + (a+b)x + ab$, $x^2 + (b+c)x + bc$.

22. $x^3 - 9x^2 + 26x - 24$, $x^3 - x^2 - 14x + 24$.

EXERCISES LV

(MISCELLANEOUS)

A

1. If a bullet falls, from rest, from a point at a moderate height above the surface of the earth, and if it falls through s feet in t seconds, it is known that

$$s = \tfrac{1}{2}gt^2$$

where g is a certain definite number (at any place) which, for computation, may be taken to be 32·2. Find how far the bullet would fall in 1 second, in 2 seconds, and in the second second.

2. Resolve into factors:

(i) $a^2 - 22a + 120$;

(ii) $a^2 - 22ax + 120x^2$;

(iii) $(x^2 - 5x)^2 - 22(x^2 - 5x) + 120$.

3. Find, without going through the complete multiplication, the coefficient of x^2 in each of the products:

(i) $(x^2 - 5x + 3)(2x + 11)$;

(ii) $(x^2 + 2x + 3)(x^2 + 3x + 5)$;

(iii) $(x+1)(x+2)(x+3)(x+4)$.

4. Find the product

$$(a-b)(a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8).$$

5. Divide \$190 among A, B, and C, so that B may have \$5 less than twice as much as A, and C \$5 less than twice as much as B.

B

1. A train travels p yards in h minutes; how far would it travel in s seconds at the same average rate?

2. Resolve into factors

$$a^4 - a^3b - ab^3 + b^4$$

(i) By grouping the terms in pairs;

(ii) By completing the square of which the extreme terms are parts.

3. Find, without going through the multiplication, the coefficient of x^2 in each of the products:

(i) $(2x^2 - 11x + 3)(5x - 7)$;

(ii) $(3x^2 - 5x + 8)(2x^2 + 7x - 5)$;

(iii) $(x-2)(x-3)(x-4)(x-5)$.

4. Shew that $x^2 - 7x + 12$ is equal to zero if $x = 3$ or 4, but that any other value of x than 3 or 4 will not make the expression equal to zero.

5. From the product of $x^4 - x^2y^2 + y^4$ and $x^4 + x^2y^2 + y^4$ take the square of $x^4 + y^4$.

C

1. Shew that $x - 1$ is a factor of $2x^3 - 11x^2 + 17x - 8$.

Shew also that if, in the expression, the value 1 is assigned to x the expression assumes the value zero.

2. By a fresh grouping of terms show that $b - c$ is a factor of

$$a^2(b-c) + b^2(c-a) + c^2(a-b),$$

and find all the factors.

3. If $2s = a + b + c$ find, in the terms of a, b, c, the value of

$$s + (s-a) + (s-b) + (s-c).$$

4. Shew that the difference of the squares of two consecutive odd integers is equal to four times the intermediate even number.

5. Resolve into factors

$$(x^2 + 3x)^2 - 38(x^2 + 3x) + 280.$$

D

1. Shew that $(a+b)^2+(a-b)^2=2(a^2+b^2)$.

Taking a and b as the measures of the lengths of lines, give a geometrical statement to this identity.

2. Resolve into factors

$$(a^3+b^3)+2(a+b)(a^2-b^2)+3ab(a+b).$$

3. If $s=a+b+c$, find, in terms of a, b, c, the value of

$$(s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b).$$

4. If x and y are two numbers and it is known that their sum is 57 and their difference is 21, find x and y.

5. Shew that the sum of any two odd integers is an even integer.

E

1. Find the product of $x^3+x^2y+xy^2+y^3$ and $x^3-x^2y+xy^2-y^3$.

2. Resolve into factors:

(i) $(x+y)^2-8(x+y)+7$;

(ii) $8a^2-2ab-15b^2+10a-15b$;

(iii) $p^4+q^4-r^4+2p^2q^2-2r^2-1$.

3. A merchant bought a box of tea at 30 cents a pound, and a second box of tea, holding 15 pounds more, at 45 cents a pound, paying \$63 for the two. Find the number of pounds in each box.

4. Find the H.C.F. of

$$8x^3-125y^3 \text{ and } 4x^3-2x^2y-5xy^2-75y^3.$$

5. Shew that the sum of any odd and any even integer is an odd integer.

F

1. Find the product

$$(a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

2. Resolve into factors:

(i) $m^4+4abm^2-(a^2-b^2)^2$;

(ii) $(m+n)z^2-mnz-m^3$;

(iii) $20(p+q)^2-67(p+q)r-30r^2$.

3. An integer expressed by two digits, of which the tens digit is the greater by 5, is equal to eight times the sum of the digits. Find the number.

4. Find the L.C.M. of

$$a^3+b^3, \quad a^3+a^2b-ab^2+2b^3, \quad a^3+2a^2b-ab^2-2b^3.$$

5. Shew that the product of any two odd integers is an odd integer.

G

1. If $x=a+b$, and $y=a-b$, find in terms of a and b the value of $x^3+y^3+3xy(x+y)$.

2. Resolve into factors:

(i) $a^3-b^3+ab(a-b)+b(a^2-b^2)$;

(ii) $4x^2+28xy+49y^2-25$;

(iii) x^9+y^9.

3. A starts from a place at the rate of 3 miles an hour, and at the end of one hour B starts on the same way at the rate of 4 miles an hour. At the end of what time will A overtake B?

4. Find the H.C.F. and the L.C.M. of a^2-b^2 and a^3-b^3, and shew that the product of the H.C.F. and the L.C.M. when found is equal to the product of the given expression.

5. Shew that the product of any odd and any even integer is an even integer.

CHAPTER IX

EQUATIONS

37. Problems Leading to Equations. What is to be understood by the term *equation* has already been explained. The following problems are examined, as preliminary to a study of the rules for solving equations.

Problem 1. A man has two fields, one square and the other a rectangle with sides 20 rods and 16 rods longer than the side of the square field; the area of the square field is 11 acres less than that of the other. Find the dimensions of the fields.

Let x be the measure of the side of the square field, unit 1 rod.

$\therefore$ $x+20$ and $x+16$ are the measures of the sides of the larger field.

$\therefore$ x^2 and $(x+20)(x+16)$ measure the areas of the fields, unit 1 square rod.

Therefore since 11 acres $=160\times 11$ or 1760 sq. rods,

$$(x+20)(x+16)-x^2=1760$$

$$\therefore \quad x^2+36x+320-x^2=1760$$

$$\therefore \quad 36x+320=1760.$$

It is now plain that $36x$ is less than 1760 by 320.

$$\therefore \quad 36x=1760-320$$

$$=1440.$$

It follows then that x must be the quotient of 1440 by 36, *i.e.*, the factor that with 36 gives the product 1440.

$$\therefore \quad x=\tfrac{1440}{36}$$

$$=40$$

$\therefore$ The side of the square field is 40 rods in length, and those of the other field $40+20$ and $40+16$, *i.e.*, 60 and 56 rods.

Problem 2. A has \$17 more than B, and when A has acquired an additional \$20 and B an additional \$12, it is found that A has twice as much money as B. Find how much each had at first.

Let $x =$ the number of dollars B had at first.

$\therefore\ x + 17 =$ " " A "

Therefore, after the acquisitions, A and B have

$(x + 17) + 20$, and $x + 12$ dollars respectively.

$$\therefore \qquad 2(x + 12) = (x + 17) + 20$$

$$\therefore \qquad 2x + 24 = x + 37.$$

Hence it is plain that $2x$ is 24 less than $x + 37$.

$$\therefore \qquad 2x = x + 37 - 24$$

$$= x + 13$$

$$i.e., \quad x + x = x + 13,$$

so that x must equal 13.

$\therefore$ B had at first \$13, and A had $(13 + 17)$ or \$30.

In the preceding examples, as in those of Chapter II. which may perhaps with profit be examined again, are seen the building up of the equation, which sums up what is given in the problem, and the eliciting from the equation its secret—the value of the unknown x. It is seen again that there is a real difference between an *equation*, as that of Ex. 2,

$$2x + 24 = x + 37,$$

which requires x to be 13, and is therefore true only if $x = 13$, and an *identity* as

$$(x - 2)^2 = x^2 - 4x + 4,$$

which we *know* to be true *whatever be x*.

The following problems are to be treated as have been the examples in this section.

Exercises LVI

1. One side of a rectangle is 18 yards longer than the adjacent side, and the area is 81 square yards less than that of a square of the same perimeter. Find the dimensions of the rectangle.

2. A merchant bought a box of 100 pounds of tea at 35 cents a pound. He sold part of it at 45 cents a pound, and the rest at 50 cents a pound, gaining on the whole $13·30. Find how many pounds were sold at each price.

3. Find how many pounds of tea which costs 40 cents a pound should be mixed with 80 pounds at cost of 30 cents a pound, in order that when the mixture is sold at 50 cents a pound there may be a gain of $26.

4. There are two numbers, the sum of which is 50, and the sum of three times the less and twice the greater is 71. Find the numbers.

5. A has a certain number of half-dollar pieces, and B has 7 more than twice as many quarter-dollar pieces. Between them they have $18·75. Find how much money they each have.

6. The difference of the squares of two consecutive integers is 13. Find the integers.

7. A sets out from a certain place at the rate of 3 miles an hour, and 2 hours later B starts in the same direction. At the end of what time will they be together?

8. Divide 35 into two parts, such that three times the greater exceeds 5 times the less by 9.

9. In an integer expressed by 2 digits the units digit is 5 greater than the tens digit, and the number is three times the sum of its digits. Find the number.

10. The length of a rectangular field is 18 rods greater than its width. Were it 8 rods wider and 10 rods longer its area would be 5 acres greater than it is. Find the dimensions of the field.

38. Rules for Solving Equations. The meaning of *equation* being understood, certain rules will now be adduced to facilitate their solution, for, while it is an excellent practice to solve equations without reference to rules, there is an economy in both time and thought in employing rules, supposed to be understood. In this section, as in several later sections, the equation will be studied without reference to any problem, although it is well to regard it as having arisen in some problem. In this way the student is less liable to lose sight of the fact that from the first x—or the unknown—denotes some definite number, not any number we please.

Ex. 1. Solve

$$5x-3=2x+9. \qquad \text{(i)}$$

Employing the rule: *If equals be added to equals the sums are equal,* we add 3 to each of the numbers given as equal by the equation.

$$\therefore \quad 5x-3+3=2x+9+3$$
$$\therefore \quad 5x=2x+9+3$$

and the result is as if -3 of the left member of the equation were taken to the right and its sign changed.

We have then

$$5x=2x+12. \qquad \text{(ii)}$$

Applying the axiom again we add $-2x$ to each member of the equation.

$$\therefore \quad 5x-2x=-2x+2x+12$$
$$\therefore \quad 5x-2x=12$$

and the result is as if $+2x$ of the right member of (ii) were taken to the left and its sign changed.

We have then

$$3x=12. \qquad \text{(iii)}$$

Employing the rule: *If equals be divided by the same number the quotients are equal,* we divide each member of the equation by 3. Then

$$x=\tfrac{12}{3}$$
$$i.e., \quad x=4 \qquad \text{(iv)}$$

and the equation is solved.

What has been done may now be exhibited more briefly thus:

	$5x - 3 = 2x + 9$
We know	$+3 = \quad + 3$
$\therefore$ by addition	$5x = 2x + 12$
Identically	$-2x = -2x$
$\therefore$ by addition	$3x = 12$
$\therefore$ by division by 3	$x = 4$

$\therefore$ The value of x is 4.

In the example just solved, the different equations (i), (ii), (iii), (iv) are such that from anyone the others may be derived. They are said to be equivalent.

It will be well to work a number of equations in the way just shewn, appealing to first principles, but from what has appeared the following rule for solving such equations may be stated:

Bring all the terms involving x—or whatever letter is employed for the unknown—to the left side, and take all numerical terms to the right side, changing the signs of all terms transferred, then collect terms and divide each member by the coefficient of the unknown.

Ex. 2. Solve

$$(2x - 5) + (3x + 7) = (x - 9) + (x + 23).$$

Here a preliminary step is the removal of the brackets, so that the simple like terms may be collected. The work may be shewn thus:

Given: $(2x - 5) + (3x + 7) = (x - 9) + (x + 23)$.

$\therefore \quad 2x - 5 + 3x + 7 = x - 9 + x + 23$

$\therefore \quad 2x + 3x - x - x = -9 + 23 + 5 - 7$

$\therefore \quad 3x = 12$

$\therefore \quad x = 4.$

As a rule the work is shortened by collecting all the like terms on each side before transferring. Thus we might say:

Given: $(2x - 5) + (3x + 7) = (x - 9) + (x + 23)$.

$\therefore \quad 2x - 5 + 3x + 7 = x - 9 + x + 23$

$$\therefore \quad 5x + 2 = 2x + 14$$
$$\therefore \quad 5x - 2x = 14 - 2$$
$$\therefore \quad 3x = 12$$
$$\therefore \quad x = 4.$$

Ex. 3. Solve

$$\tfrac{1}{3}(x-2) + \tfrac{1}{2}(x+1) = \tfrac{1}{6}(x+7) + \tfrac{1}{2}(x+3).$$

In general, time is saved by getting rid of numerical fractions, which may here be done by multiplying each member of the equation by 6, the L.C.M. of the denominators. The solution would then be shown thus:

Given: $\tfrac{1}{3}(x-2) + \tfrac{1}{2}(x+1) = \tfrac{1}{6}(x+7) + \tfrac{1}{2}(x+3).$

By multiplication by 6, it follows that:

$$2(x-2) + 3(x+1) = (x+7) + 3(x+3)$$
$$\therefore \quad 2x - 4 + 3x + 3 = x + 7 + 3x + 9$$
$$\therefore \quad 5x - 1 = 4x + 16$$
$$\therefore \quad x = 17.$$

Thus the value of x is 17.

Further illustrations are unnecessary, as in working examples the student will acquire artifices for shortening work.

In each case the answer found should be substituted in the proposed equation to test the accuracy of the work and to impress what is meant by an equation.

Exercises LVII

1. Solve the following equations, referring each step to first principles:

1. $4x + 11 = 3x + 19.$	2. $7x + 30 = 5x + 40.$
3. $4x + 2 = x + 7.$	4. $9x + 15 = 4x + 23.$
5. $2x - 5 = x - 2.$	6. $5x - 7 = 2x + 10.$
7. $8x + 5 = 9x + 3.$	8. $11x + 17 = 13x + 26.$
9. $3y - 8 = 5y + 7$	10. $\tfrac{1}{2}(x-7) = \tfrac{1}{5}(x+9).$
11. $7z + 11 = 12z - 9.$	12. $\tfrac{1}{3}(x-1) + \tfrac{1}{4}(x-2) = \tfrac{1}{5}(x+12).$

2. Solve the following equations:

1. $(5x-2)+(3x-7)=(15x-13)-(4x-9).$
2. $(3y+2)-(2y+3)=(7y-19)-(4y-8).$
3. $5(z-7)-3(z+1)=6(z-4)-5(z+3).$
4. $\frac{1}{3}(x+5)-\frac{1}{5}(x+8)=\frac{1}{7}(x+11)-\frac{1}{9}(x-5).$
5. $7(l+1)+5(l+2)=3(l+9)+4(l+11).$
6. $\frac{2}{3}(z+7)-\frac{3}{4}(z+11)=\frac{5}{6}(z+19)-\frac{1}{12}(z-1).$
7. $(z+1)(z+2)+(z+3)(z+4)=(z+6)(z+1)+(z-4)^2.$
8. $(x+5)^2+(x-2)^2=(x+3)^2+(x-1)^2.$
9. $(x+3)(x+5)-(x-2)(x-3)=$ $(2x+5)(x-7)-(2x-3)(x-1).$
10. $5y(7-y)+y(5y-1)=(y+1)(y-1)-(y-2)^2.$

3. Solve the following equations:

1. $(\frac{1}{3}x-\frac{4}{5})-(\frac{5}{6}x-\frac{3}{4})=(\frac{4}{5}x+\frac{2}{3})-(\frac{2}{3}x-\frac{5}{12}).$
2. $\frac{1}{3}(\frac{1}{2}x-\frac{2}{3})-\frac{1}{4}(\frac{2}{3}x-\frac{1}{5})=\frac{1}{2}(x-7)-\frac{1}{3}(x-10).$
3. $\frac{2}{3}(z+1)^2-\frac{1}{4}(z+2)^2=\frac{5}{12}(z+1)(z+2)-\frac{5}{6}(z+9).$
4. $(y-\frac{3}{4})(y-\frac{1}{2})+(y-\frac{2}{3})(y+\frac{1}{3})=2(y+1)(y-5).$
5. $\frac{1}{2}(y-1)-\frac{1}{3}(y-2)=\frac{1}{3}-(y+7)-\frac{1}{6}(y-5).$
6. $\frac{1}{3}x-\frac{2}{3}(x-5)+\frac{3}{4}+\frac{5}{6}(x-4)=1-\frac{1}{12}x.$
7. $(1+x)(3-x)-(2-x)(3+x)=\frac{1}{2}(x+1)-\frac{1}{3}(x+2).$
8. $\frac{1}{7}(x+11)-\frac{1}{11}(x+1)=(x-1)(x+1)-(x-2)(x+2)-1.$
9. $\frac{1}{2}[x+\frac{1}{3}\{(x-5)+\frac{1}{4}(x+1)\}]=2\frac{3}{4}.$
10. $(x+1)(x+2)(x+3)=(x+5)(x+7)(x-6).$

39. Literal Equations. Let it be proposed to divide a line, the measure of the length of which is a, into two parts, one of which measures k more than the other.

Let $x=$ measure of length of shorter part.

$\therefore\ x+k=$ measure of length of longer part.

$\therefore\ x+(x+k)=$ sum of the measures of the two parts.

$\therefore\ x+(x+k)=a.$

Here we have an equation in which the unknown is x; in this equation appear also a and k, which may be any two numbers whatever, k less than a, that being *given* in the problem are to be considered as known.

The equation is called a *literal* equation in contrast with those already worked, which may be called *numerical* equations.

To solve this equation means to find x in terms of the known a and k. Manifestly the rules for solving numerical equations will apply here; accordingly we have

$$x + (x + k) = a.$$
$$\therefore \quad 2x = a - k$$
$$\therefore \quad x = \tfrac{1}{2}(a - k).$$

Thus the two parts are

$$\tfrac{1}{2}(a-k) \text{ and } \tfrac{1}{2}(a-k)+k,$$
i.e., $\tfrac{1}{2}a - \tfrac{1}{2}k$ and $\tfrac{1}{2}a + \tfrac{1}{2}k$,
or $\tfrac{1}{2}(a-k)$ and $\tfrac{1}{2}(a+k)$.

In literal equations, by convention, the earlier letters are understood to indicate the known quantities.

Exercises LVIII

Solve the following equations:

1. $(a + x) + (b + x) = (c + x)$.
2. $(m + x) - (n - x) = (2p + x)$.
3. $\frac{1}{3}(a + x) + \frac{1}{4}(b + x) = \frac{1}{5}(c + x)$.
4. $hx - kx = h^2 - k^2$.
5. $a(a + x) = b(b + x)$.
6. $m^2(m + nx) + n^2(n + mx) = (m + n)^3$.
7. $a(x - a) + b(x - b) + c(x - c) = 2(bc + ca + ab)$.
8. $(x + b)(2a + 3b) = (a + 2b)(2x + b)$.
9. $(x - 2k)(5h + 6k) = (3h - 4k)(8h - 3x)$.
10. $2a^2(ax + b^2) + 2b^2(bx + a^2) = (a^2 + b^2)^2 - (a^2 - b^2)^2$.

EXERCISES LIX

(MISCELLANEOUS)

A

1. Find the coefficient of x^3 in the following products:

   (i) $(x^2-5x+7)(x^2+7x-3)$;

   (ii) $(2x^3-5x^2+7x-9)(3x^2-11x+5)$;

   (iii) $(2x^3+7x^2-13x+4)(5x^3-5x^2+12x-8)$;

   (iv) $(ax^2+bx+c)(hx^2+kx+m)$.

2. Find the result of substituting $-x$ for x in the expression

$$ax^5+bx^2+cx.$$

3. Resolve into factors

$$(3x^2+7xy+4y^2)^2-(5x^2+2xy-3y^2)^2.$$

4. Solve the equation

$$\tfrac{1}{7}(x-3)+\tfrac{1}{5}(x+5)=\tfrac{1}{8}(x+6)+\tfrac{1}{6}(x-1).$$

5. What is meant by saying that -2 is less than $+1$?

B

1. Shew that

$$(a+b)^2(a-b)^2=(a^2-b^2)^2.$$

2. Resolve into factors:

   (i) $x^2y^2-a^2x^2-a^2y^2+a^4$;

   (ii) $alx^2+amxy+blxy+bmy^2$;

   (iii) x^6-y^6.

3. For what values of x will the expression $x^2-8x+15$ assume the value zero?

4. Shew that the expression

$$(b-c)(x-b)(x-c)+(c-a)(x-c)(x-a)+(a-b)(x-a)(x-b)$$

does not really involve x.

5. Divide \$53 between A and B so that one-eighth of A's share exceeds one-seventh of B's by one dollar.

C

1. Find the L.C.M. of

$$(a+b)^2(a^3-b^3),\ (a-b)^2(a^3+b^3).$$

2. From the square of x^3-a^3 take the cube of x^2-a^2.

3. A merchant buys 100 pounds of tea at a certain price a pound. This price at once advancing 5 cents a pound, the merchant sold the tea at once at an advance of one-fifth on the new cost price and gained $12. Find the cost price of the tea.

4. Find the least value of x^2 for all positive, negative, and zero values of x.

5. Shew that

$$x^2(y-z)+y^2(z-x)+z^2(x-y),$$
$$x(z^2-y^2)+y(x^2-z^2)+z(y^2-x^2),$$
$$yz(y-z)+zx(z-x)+xy(x-y),$$

are equal to one another.

D

1. It is known that the remainder, when x^2-3x+m is divided by $x-5$, is 16. Find what m must be.

2. Find the least value of x^2-2x+1 for all positive, negative, and zero values of x.

3. Group the terms in

$$x^2(y+z)+y^2(z+x)+z^2(x+y)+2xyz$$

so as to shew that $y+z$ is a factor.

What other factors suggest themselves?

4. A has $72 and B has $33; find how much A should give B so that B may have three-fourths as much as A.

5. If $x=b+c,\ y=c+a,\ z=a+b$, find in terms of a, b, c the value of

$$x^2+y^2+z^2-yz-zx-xy.$$

CHAPTER X

FUNCTIONS

40. Meaning of Function. In many exercises the student has been asked to work such a problem as the following:

Find the value of $2x+3$ *for* $x=0, 1, 2, 3, 4, 5$.

The computation is very easy, and the results can be shewn thus:

For $x=$	0	1	2	3	4	5
The value of $2x+3=$	3	5	7	9	11	13

Here, whatever value be assigned to x, the form of the expression enables us to find its value, and the fact, that the value of the expression is thus determined for general values of x, is expressed by saying that $2x+3$ is a *function of* x. When, in an expression involving x, we think of x as capable of assuming or being assigned *a variety of values, as many values as we please*, and, as a rule, *any values we please*, x is called a *variable*, and the expression is called a *function of the variable* x.

Since, in general, any expression, in which x occurs, can be evaluated for an assigned value of x, the term *function of* x is often defined in elementary algebra as *any expression involving* x, or *any algebraic form involving* x, the question of the variability of x not being raised.

Exercises LX.

1. Calculate the values of the following functions of x for $x = -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5$:

$$x+3,\quad 2x,\quad 3x+7,\quad x^2-x+2,\quad x^2-x-2,\quad \frac{1}{x+10}.$$

2. In each of the following functions assign to x any value, at pleasure, and calculate the increase in the value of the function for successive advances of 1 on the value of x taken:

$$x+3,\quad 2x,\quad 2x+3,\quad 3x,\quad 3x-2,\quad 7x-10.$$

3. In each of the following functions of x, shew that there is one value which must not be given to x:

$$\frac{1}{x},\quad \frac{1}{x-2},\quad \frac{1}{2x+3},\quad \frac{x-3}{x+2}.$$

4. Calculate the values of the following functions of x for $x = -0{\cdot}5, -0{\cdot}4, -0{\cdot}3, -0{\cdot}2, -0{\cdot}1, 0, 0{\cdot}1, 0{\cdot}2, 0{\cdot}3, 0{\cdot}4, 0{\cdot}5$:

$$x-5,\quad 3x-7,\quad 2x+9,\quad x^2,\quad x^2-3.$$

Exhibit the values in the form of a table.

5. Shew that whatever value be given x in the function $5x-7$, an advance of 1 in the value of x compels an increase of 5 in the value of the function.

6. Shew that the expression

$$\tfrac{1}{3}(3x-5)+\tfrac{1}{5}(11-5x)$$

does not change as x is assigned different values, *i.e.*, that it is *constant*.

7. Shew that no value of x will assign to the following functions a negative value:

$$x^2,\quad (x-2)^2,\quad x^2-4x+4,\quad x^2-4x+9.$$

41. Notation. In this section the term function will be understood for the most part as referring to the form of the expression.

It is at once seen that x^2-3x-2 is a function of x. If we were examining this function for many values of x and referring to it frequently, it would be found convenient to adopt some short way of denoting it. In such a case the practice is to denote it by the suggestive symbol,

$$f(x),$$

which is read, *function of x*, or *function x*.

Hence $f(x) =$, (or means), x^2-3x-2.

Then
$$f(1) = 1^2 - 3.(1) - 2.$$
$$f(2) = 2^2 - 3.(2) - 2.$$
$$f(p) = p^2 - 3p - 2.$$

If two functions were under study, as say x^2+1 and $\dfrac{x-1}{x+1}$ we might denote the one by $f(x)$ and the other by $F(x)$, for plainly different functions or forms should be denoted by different symbols. Here

$$f(3) = 3^2 + 1.$$
$$F(3) = \frac{3-1}{3+1}.$$
$$F(p) + f(p) = \frac{p-1}{p+1} + (p^2+1).$$

As the student becomes familiar with the notation, he will see that often the suggestive letter f is not the only one so employed; thus

$$f(x),\ g(x),\ h(x)$$

may be employed to denote three different functions of x.

Exercises LXI

1. If $f(x)$ denotes x^2+3x+7, find $f(0)$, $f(1)$, $f(2)$, $f(a)$.

2. If $f(x)$ denotes $x^2+8x-11$, find $f(a)$, and shew that $f(x)-f(a)$ has $x-a$ for a factor.

3. If $f(x)=x+\frac{1}{x}$, find $f(1)$, $f(3)$, $f(a^2)$.

4. If $f(x)=x^2-2$, shew that

$$x^2+\frac{1}{x^2}=f\left(x+\frac{1}{x}\right).$$

5. If $f(x)$ denotes ax^2+bx+c, find $f(m)$, and shew that $f(x)-f(m)$ has $x-m$ for a factor.

6. If $g(x)$ denotes $ax^3+3bx^2+3cx+d$, find $g(m)$, and shew that $g(x)-g(m)$ has $x-m$ for a factor.

7. If $f(y)=2y-7$, find $f(x)$, $f(x^2)$, $f(x+2)$, and shew that

$$f(z^2)+3f(z)+8=2(z+5)(z-2).$$

8. If $f(z)=z^2$, shew that

$$f(x^2+y^2)=f(x^2)+f(y^2)+2f(x).f(y).$$

9. If $f(z)=z^2$ and $g(z)=z+1$, shew that

$$f(x)+2g(x)-1=f(x+1).$$

Exercises LXII

(Miscellaneous)

A

1. Divide the product of $x^2-9x-36$ and $x^2-4x-45$ by $x^2-21x+108$.

2. What value of x will make $2x+3$ equal to zero?

Shew that any smaller value than this of x will make $2x+3$ negative, and any greater value will make $2x+3$ positive.

3. In an orchard are m rows of m trees each, the trees being equidistant in the rows, and the whole forming a square. The orchard is enlarged by an addition of a row of trees along two adjacent sides, equidistant in their rows, so that the orchard still has a square outline. Find how many trees are added to the orchard.

4. A person when asked the day of the month on a certain day in September replied that the number of days remaining in September was one-fifth of the number of days remaining in the year. Find the day of the month.

5. If $f(x)=(x+1)^3-(x-1)^3$, shew that $f(-x)=f(x)$.

B

1. In the expression

$$(x+y+z)(xy+yz+zx)-(y+z)(z+x)(x+y)$$

find all the terms in which x does not occur.

What inference from the result?

2. For what value of x will the two functions of $2x+5$ and $3x+2$ have equal values?

Take any less value of x and find which of the functions is the greater, and any greater value of x, and find which function is the greater.

3. Resolve into factors:

(i) $x^3-y^3-x(x^2-y^2)+y(x-y)^2$;

(ii) $(2a+3b)^3+(3a+2b)^3$;

(iii) $(15x^2-xy-28y^2)^2-(21x^2+19xy-12y^2)^2$.

4. A rectangular court, the unequal sides of which are m yards and n yards in length, is surrounded on the outside by a walk a yards wide. Find the area of the walk.

5. If $f(x)=(x+1)^3+(x-1)^3$, shew that $f(-x)=-f(x)$.

C

1. Shew that

$$(y-z)^2+(z-x)^2+(x-y)^2=2(x-y)(x-z)+2(y-z)(y-x)$$
$$+2(z-x)(z-y).$$

2. Shew that

$$(b+c)^2+(c+a)^2+(a+b)^2-a^2-b^2-c^2=(a+b+c)^2.$$

3. Resolve into factors:

(i) $(ln+mp)^2-(lp+mn)^2$;

(ii) $(x+y)^2-17a(x+y)+72a^2$.

4. If $f(n)=n^2$, shew that

$$f(n)-f(n-1)=2n-1.$$

5. The volume of a sphere being given by the formula $v=\frac{4}{3}\pi r^3$, where v and r are the measures of the volume and the radius, and π is a certain definite number, find the volume of the spherical shell of thickness of measure t and of internal radius of measure x.

D

1. Find the values of x^2 for $x=0, 1, 2, 3, 4, 5$, noting anything remarkable in the changes in the value of the expression for successive advances of 1 in the value of x.

2. Shew that

$$(a+b)^2 = 4ab + (a-b)^2.$$

3. Shew that $x-1$ is a factor of

$$x^4 - 2x^3 + 7x^2 - 5x - 1.$$

Shew also that for the value 1 of x the expression assumes the value zero.

4. A merchant has two kinds of tea, which he sells at 32 cents and 40 cents a pound. How many pounds of each must he take to make a mixture of 100 pounds worth 35 cents a pound?

5. Group the terms of

$$a^3(b-c) + b^3(c-a) + c^3(a-b)$$

so as to shew that $b-c$ is a factor of the expression.

E

1. Find the value of $x^2 - 4x + 4$ for $x = -2, -1, 0, 1, 2, 3, 4, 5, 6$.

2. Shew that

$$(b+c)(c+a)(a+b) - 3abc$$

is not changed by the interchange of any two of the letters a, b, c.

3. Shew that

$$x^4 - 3x^3 - 29x^2 + 3x + 28$$

is divisible by both $x-1$ and $x+1$, shewing also that the given expression vanishes for $x=1$ or $x=-1$.

Find the other factors of the expression.

4. If $x=b+c-a$, $y=c+a-b$, $z=a+b-c$, find in terms of a, b, c the value of

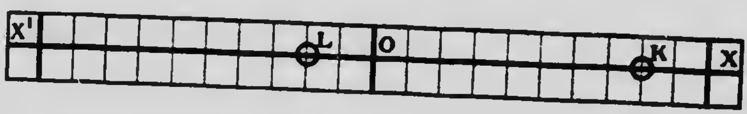
$$x^2 + y^2 + z^2 + yz + zx + xy.$$

5. If $f(x) = x^2 - 4x + 4$, shew that $f(\overline{x-2}+h) = f(\overline{x-2}-h)$.

CHAPTER XI

GRAPHS

42. Representation of Number. Many representations of positive and negative numbers have been given. We have now to consider in greater detail the representation on a straight line. Let X′OX be a straight line which may be produced indefinitely in both directions.



Let O be an *origin* for measurements and suppose positive measurements made to the right. Then the distance $(3+5)$ measured from O is given by OK, or determined by the point K. Similarly the distance $(3-5)$ measured from O is given by OL or determined by the point L. But $3-5=-2$. Therefore if positive measurements mean those taken to the right, negative measurements must be taken to the left. Thus all the numbers of algebra can be represented on the line, a point on the line corresponding to each number. It is plain that different points cannot correspond to the same number.

Similarly for a vertical line—one drawn up and down the page—it is agreed that measurements made upwards from O are to be taken as positive so that downward measurements are to be negative.

For the representations of this chapter it is almost necessary to have squared paper; every fifth or tenth line should be of heavy ruling.

Exercises LXIII

1. On a horizontal line, mark the points that indicate the following numbers:

$$7,\ 13,\ 4-3,\ 4-5,\ 3-7,\ -9.$$

2. Mark the points P and P′ corresponding to

$$3-5+7-9 \text{ and } 7-5+2-8,$$

and shew that the distance PP′ is given by

$$(3-5+7-9)-(7-5+2-8),$$

and the distance P′P by

$$(7-5+2-8)-(3-5+7-9).$$

3. On a vertical line mark the points that indicate

$$5,\ 8,\ 8-5,\ 8-11,\ 3-5+7,\ 4-9-3,\ -4.$$

43. Study of the Linear Function. In the function $2x+1$, as has been seen, x may be assigned any value, and each such value assigns to this function a value. A set of corresponding values is shewn in the table

For $x=$	0	1	2	3	4	5	6	7
$2x+1=$	1	3	5	7	9	11	13	15

Here it is plain that $2x+1$, regarded as one number, is a variable, for x may be taken so as to assign any value we please to $2x+1$. For example if $2x+1$ is to have the value 20, then since

$$\begin{aligned} 2x+1 &\text{ is to equal } 20 \\ 2x &\text{ is to equal } 19 \\ \text{and } x &\text{ is to equal } 9\tfrac{1}{2} \end{aligned}$$

so that the value $9\frac{1}{2}$ of x assigns the value 20 to $2x+1$.

Thus $2x+1$ is itself to be regarded as a variable, depending on the variable x.

Next, note that, taking any value of x as say $x=3$, each advance of 1 in the value of x, carries with it an increase of 2 in the value of $2x+1$. This was to be expected, for the variable part of $2x+1$ is $2x$ and any change in x implies twice as great a change in $2x$.

More generally, assign to x the values a, $a+h$, $a+2h$. The values of the functions are

$$2a+1,\ 2a+2h+1,\ 2a+4h+1.$$

Thus each advance of h in the value of x gives the same advance of $2h$ in the function, whatever be h and whatever be the value a of x from which the advance starts.

The x in $2x+1$ is a variable; in contrast the 2 and 1 in $2x+1$ are *constants*. So in the general linear function $ax+b$, while a and b may be any numbers whatever, a not zero, yet once their value is assigned they are not to be given other values; thus a and b while *general* are *constants*.

Exercises LXIV

1. In the function $3x-5$ assign to x any five values, and exhibit the results in the form of a table.

Shew that starting from any value—say 7—of x each advance of 1 compels an advance of 3 in the value of the function.

Give to x the values m, $m+r$, $m+2r$, $m+3r$ where m and r are general, and find the values and the growths in value of the function.

2. Treat the function $\frac{1}{2}x+3$ in the same way as the function in exercise 1.

3. What is the simplest linear function of x?

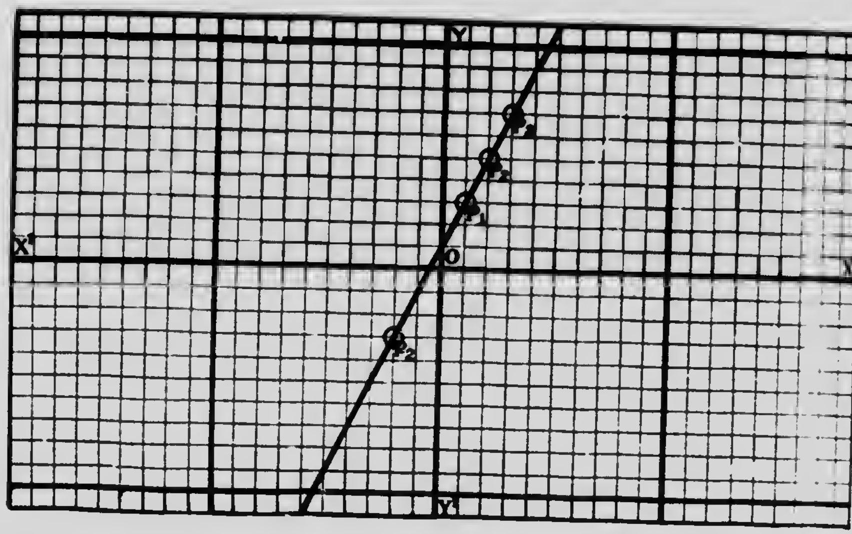
4. A train is moving at an unchanging velocity of 22 feet a second. Shew that the space s, unit 1 foot, is a linear function of the time t, unit 1 second, the measurements starting together.

5. In the function $5x+2$ give to x the values 3, 8, 4; then give to x values 1 less than these, and shew that the three changes in the value of the function are equal.

6. In the function $5-2x$ shew that each advance of 1, from 0, of the value of x implies a fall of 2 in the value of the function.

44. Graph of the Linear Function. It is proposed now to exhibit to the eye the results of the study of the linear function of x, in particular of the function $2x+1$.

Take X'OX, YOY' two straight lines at right angles to each other, and treat O as an origin for horizontal measurements.



Assign to x the value 1, and mark the corresponding point 1 on the line X'OX. The value of the function for $x=1$ is 3; then *above* 1, *i.e.*, in the direction OY for positive numbers, at a distance 3 from 1, mark the point P_1. This shews to the eye that for $x=1$ the value of the function is 3. So for $x=2$ the value of the function is 5, and the point P_2, marked as P_1, indicates this fact. Similarly for $x=3$ and 4.

For $x = -2$, the function has the value -3, and the point P_{-2} at a distance 3 *below* the place that indicates $x = -2$ shews this also. So for any value of x we please. Next for the values 1, 2, 3, of x, the values of the function are 3, 5, 7, and it is plain that each of the two successive increases of 1 in the value of x requires an increase of 2 in the value of the function. This means that the corresponding points P_1, P_2, P_3 lie in a straight line. So if to x be given *any two values whatever*

$$a, a+h,$$

the corresponding values of the function are

$$2a+1, 2a+2h+1,$$

and it is plain that the growth of h in the value of x requires a growth of $2h$ in the value of the function. The two points registering these values must then be on a straight line, so running as to rise 2 for every advance of 1 to the right. Since a and $a+h$ may be 1 and 2, this line must be the straight line through P_1 and P_2. Thus the values of the function for all values of x, registered on X′OX, are indicated by corresponding points on a certain definite straight line. This straight line is called the *graph of the function*. It shews to the eye all that we have found in the function, and may suggest additional particulars. For example, we notice that the value of the function is zero, *i.e.*, it is represented by a point neither above nor below X′OX, for what *seems* to be $x = -\frac{1}{2}$, and we see that $x = -\frac{1}{2}$ does make $2x+1$ equal to zero. The word *seems* is employed because a drawing, like all actual measurements, cannot claim absolute accuracy. So too we *see* that the value of x that makes the function equal to -9 is -5, and that the value $x = 3{\cdot}5$ gives the value 8 to the function.

To obviate the repeated use of the word function, we may denote the function by *y*, and write here

$$y = 2x + 1.$$

In this case we say, too, that the graph is the *graph of the relation* or *of the equation* $y = 2x + 1$.

The straight lines X′OX, Y′OY are called *axes*, the former the *x-axis*, the latter the *y-axis*. The point O is called the *origin*. The two numbers marking a point, as the numbers 2 (to the right) and 5 (up) mark the point P_2, are called the *co-ordinates of the point*, the *x-ordinate* being always mentioned first.

Exercises LXV

1. Construct the graph of the function

$$x + 2$$

giving reasons for saying that it is a straight line.

It being supposed known in advance that the graph is a straight line, how many computations are necessary to make the graph?

How much does the graph rise for an advance of 1 in the value of x, and what number could be taken to measure the *slant* or *slope* of the line?

Read off from the graph the values of the function for $x = 0$. 2·5, 3·5, −0·5, −3·5, and read off the values of x that make the function equal to 0, −0·5, 7, 10.

2. Construct the graphs of the functions

$$2x + 3, \quad 2x - 1, \quad 2x + 5$$

them on one sheet.

3. Construct the graphs of the equations

$$y = 3x - 1, \quad y = 3x + 1, \quad y = 3x + 2$$

shewing them on one sheet.

What do the results of this and the preceding seem to indicate?

4. Construct the graphs of the functions

$$x, \quad 2x, \quad 3x, \quad 4x,$$

shewing them on one sheet.

5. Construct the graph of the function

$$2x + 1$$

and, with respect to it, answer a series of questions similar to those asked in exercise 1.

6. Construct the graph of the equation

$$y = 1 - 2x$$

and, with respect to it, answer a series of questions as in exercise 1.

7. Shew on the same she[illegible] the graphs of the two equations

y [illegible] $+ 7$,

$$y = 2x - 1.$$

For what value of x have the two functions the same value?

8. Mark the points of which the co-ordinates are as follows:

$$(1, 1); \quad (2, 3); \quad (5, 7); \quad (0, 4); \quad (5, 0).$$

9. Mark the points of which the co-ordinates are as follows:

$$(1, -2); \quad (3, -5); \quad (0, -5); \quad (-1, -2); \quad (-3, 0); \quad (0, 0).$$

10 Mark the two points of co-ordinates (1, 2), (5, 9), and measure the distance between them, using a compass to transfer this distance to the horizontal or the vertical.

11. Mark the points of co-ordinates (0, 3), (0, −7), and measure their distance apart.

12. Mark the points of co-ordinates (3, 4), (−7, −8), and measure the distance between them.

13. Mark the triangle of which the vertices are the points given by the co-ordinates

$$(3, 5), \ (-7, -9), \ (3, -6).$$

14. Mark the quadrilateral of which the vertices are given by the co-ordinates

$$(2, 3), \ (-2, -2), \ (3, -6), \ (7, -1),$$

and test by measurement for a square.

45. Illustrations. The following examples illustrate the idea implied in the terms *variable* and *function*, and will furnish material for graphic representation.

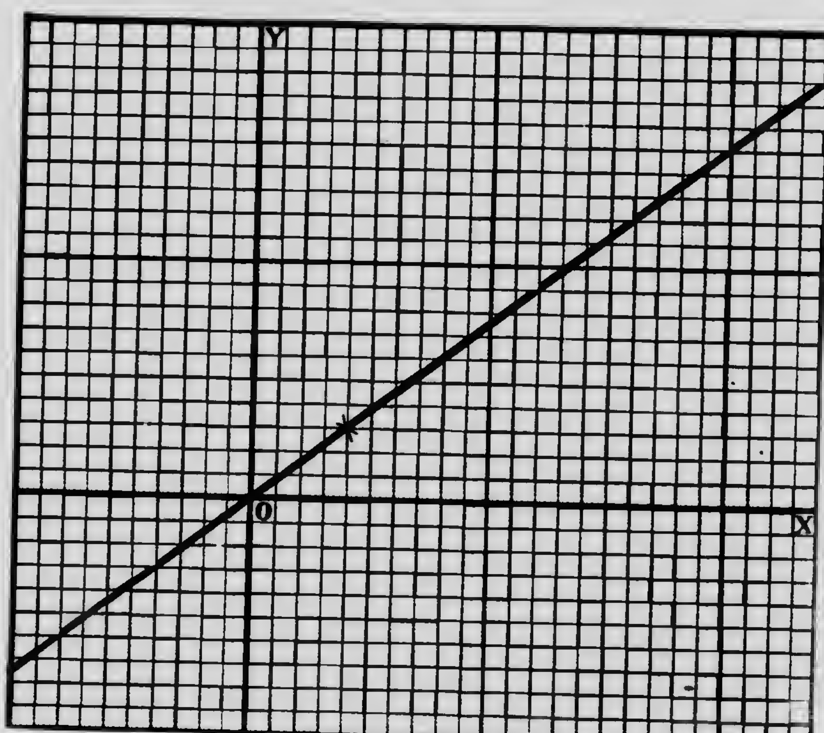
Example 1. A railway train, travelling at the rate of $\frac{3}{4}$ miles a minute, is observed to pass a certain point at a certain time. How far will it be from that point t minutes later?

Let s measure this distance, the unit being one mile. Then plainly

$$s = \tfrac{3}{4}t.$$

Here the very simple linear function $\frac{3}{4}t$ gives the distance corresponding to *any* time t, and in the passing of time there is an image of the variation of t. The number s, standing for this function, changes under the change in t.

We know in advance, from the fact that $\frac{3}{4}t$ is linear in t, that the graph is a straight line. It is necessary then to find only two points on the line. If $t=0$, $s=0$, and if $t=4$, $s=3$. The graph is then as appears below.



The graph may be employed to read off the s corresponding to any given t, or the t corresponding to any given s.

If t is negative, s is negative, and the graph takes us back to a time *before* the observed passing of the point, giving us the distance behind this point corresponding to such a time, the rate of travelling being supposed the same.

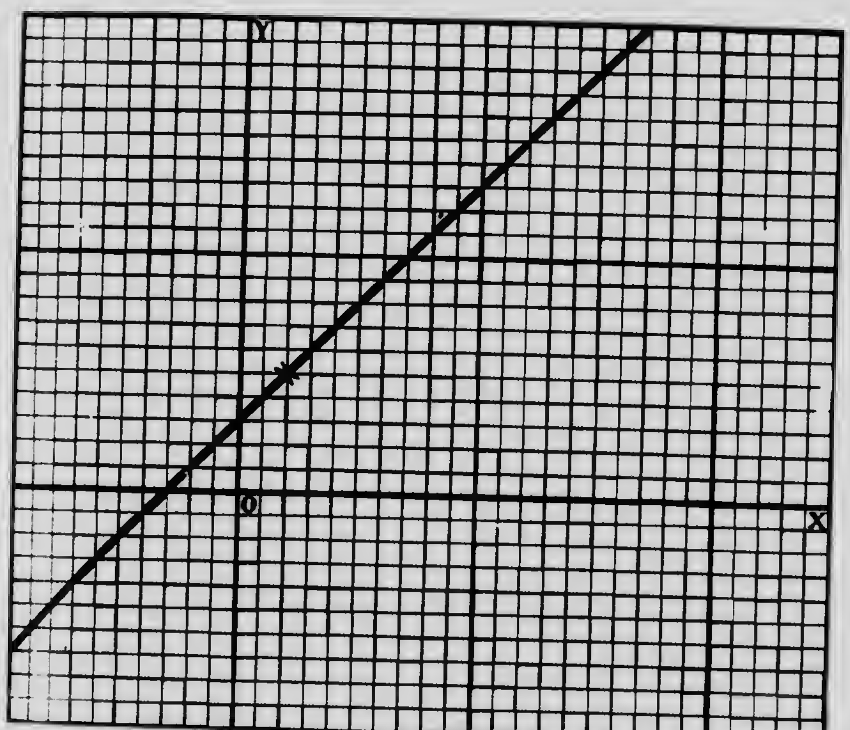
Example 2. A railway train leaves a town P and continues at the rate of a mile a minute. It passes a point 3 miles along the line at a certain time. How far from P will the train be t minutes later?

Let s measure this distance. Then plainly

$$s = t + 3.$$

Here the linear function $t+3$ gives the distance corresponding to any time t. The changing time is reflected in the variation of t, and s, standing for $t+3$, will vary under the change in t.

For $t=0$, $s=3$, and for $t=2$, $s=5$. The graph is then as appears below.



Example 3. Find the Fahrenheit reading corresponding to the Centigrade reading x degrees.

Let y degrees be the required reading. Then, since 180 degrees Fahrenheit—measuring the range from 32 degrees to 212 degrees, the freezing and boiling points of water—are equal to 100 degrees Centigrade, *i.e.* the range from 0 to 100 degrees, we have

$$100 \text{ degrees Centigrade} = 180 \text{ degrees Fahrenheit},$$
$$\therefore\ x \quad \text{"} \quad \text{"} \quad = \tfrac{9}{5}x \quad \text{"} \quad \text{"}$$

Hence the Fahrenheit reading y degrees corresponding to x degrees Centigrade is given by the relation

$$y = \tfrac{9}{5}x + 32.$$

The graph appearing on the opposite page exhibits this relation and from it can be read off the reading in either scale corresponding to a given reading in the other.

Exercises LXVI

1. A merchant sells tape at the rate of 3 yards for 5 cents. Find the cost of x yards at this rate, and construct the graph registering the cost of any number of yards.

2. Shew that the graph of the equation

$$y = mx$$

passes through the origin, m being any constant.

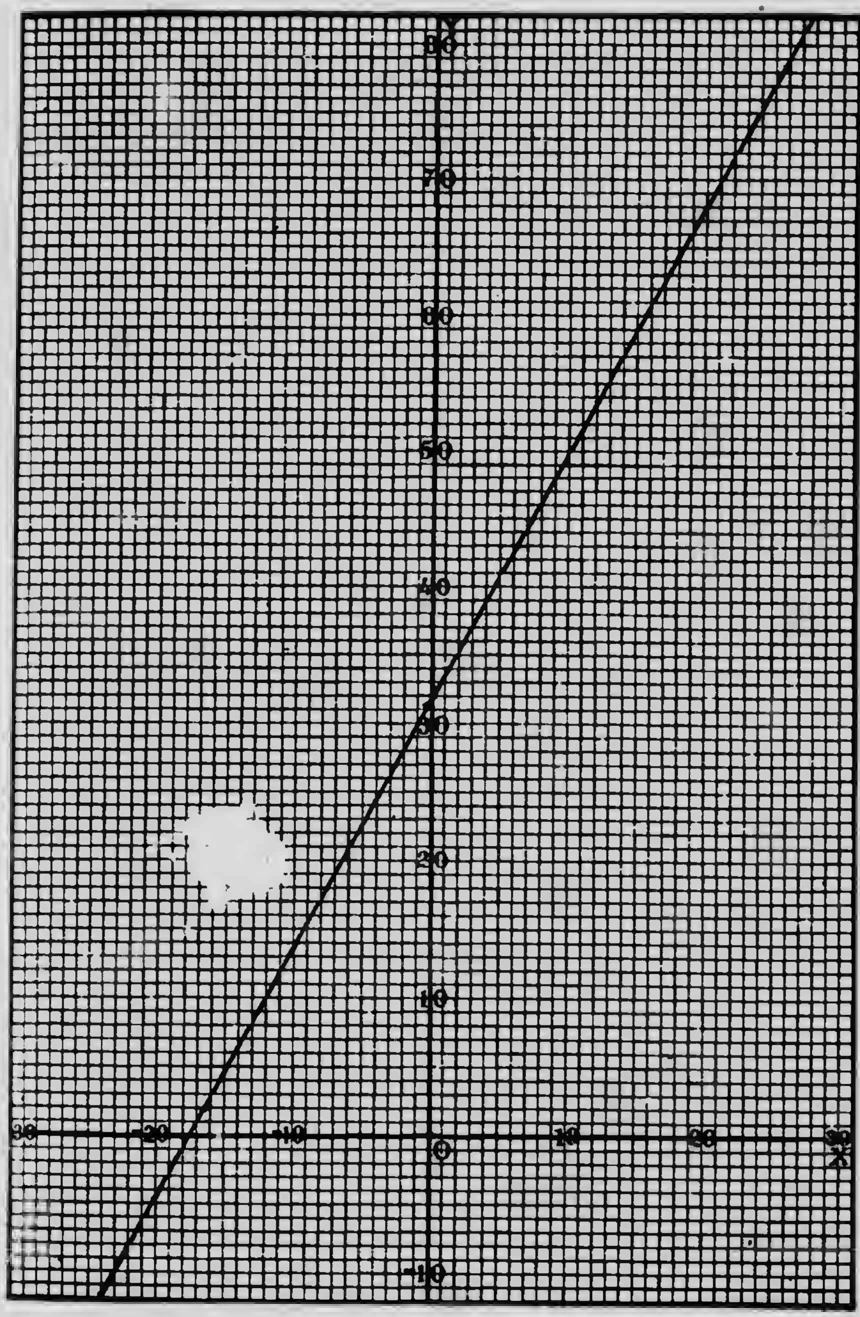
What is the significance, in the graph, of the number m?

3. A man starts from a point on a straight road, walking at the rate of 3 miles an hour. He walks for 2 hours, rests 3 hours, and then continues the journey at the rate of 4 miles an hour. Draw the graph shewing how far he is from the starting point at different times.

4. Taking 4 litres as the equivalent of 7 pints, construct a graph to exhibit the number of litres or pints in a given number of pints or litres.

5. Taking a mile as the equivalent of 1·6 kilometres, construct a graph to exhibit the number of miles corresponding to a given number of kilometres, and *vice versa*.

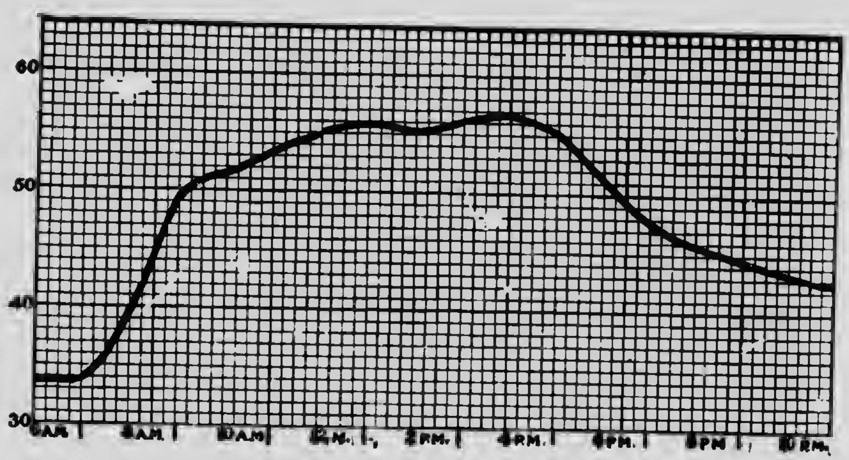
6. Taking a kilogramme as the equivalent of 2·2 pounds, construct a *conversion graph* for pounds and kilogrammes.


Y
80
70
60
50
40
30
20
10
-10
-30
-20
-10
0
10
20
30
X

46. Statistical Graphs. A series of measurements, or observations, or readings, may gain in significance by a graphic representation. In the examples given below, it will be noted that there are two variable elements involved, one depending on the other.

Illustration. The hourly readings of the thermometer, on a certain day, starting at 6 A.M., were, 33·9, 34·0, 40·0, 49·7, 51·8, 53·5, 55·1, 55·9, 55·2, 56·1, 56·8, 55·2, 51·9, 48·0, 45·9, 44·8, 43·6, 42·8. Exhibit the fluctuation of temperature graphically.

Here the variable elements are the time and the temperature. The time is measured in hours, and four graduation intervals are taken to represent one hour. The temperature is measured in degrees, and one interval taken to represent a degree. For convenience in the matter of space the horizontal axis is drawn, not through a zero-marking of temperature but through the mark 30 degrees.



A free continuous curve is drawn through the points registering the given temperatures, and this curve indicates the probable temperatures at intervening times. In contrast with the certainty of the position of each point in the graph of the linear function, is the lack of certainty in the freedom we have in filling in the curve. Yet feel also that many points on a curve determine its course in at least its bold features. Subjoined is a copy of the curve of temperature traced automatically for the same period on the same day.

Exercises LXVII

1. The population of a certain city at intervals of five years starting in 1870 was

12,000, 15,500, 19,200, 20,100, 19,400, 20,500, 27,300, 32,400.

Exhibit graphically the probable variation of population.

2. The quotations of a certain industrial stock at intervals of 7 days, for a certain period, were

64, 68, 66, 70, 76, 78, 75, 80, 83.

Exhibit graphically the probable fluctuation in price.

3. The record of a patient's temperature for a certain time, at intervals of a half-hour, is

97·5, 98, 98, 99, 99·5, 102, 104·5, 104, 103, 102·5, 101, 101.

Exhibit the fluctuation graphically.

4. At intervals of 100 feet horizontal measurement in a direction directly from the water's edge, the height of the land above the level of the water was found to be

15 ft., 20 ft., 30 ft., 32 ft., 28 ft., 37 ft., 45 ft., 48 ft., 50 ft., 51 ft.

Exhibit graphically the changing altitude.

Is the graph a probable profile of the surface of the land?

5. The readings of the barometer, at intervals of an hour, for a certain period were—in inches of mercury—

29·6, 29·7, 30·1. 29·9, 29·3, 29·5, 30·2, 30·5, 30·3, 29·8, 29·1, 29·8, 29·9, 29·5.

Exhibit graphically the variation.

6. A bullet is allowed to fall from rest at a height, and it is found that at the end of 1, 2, 3, 4, 5 seconds the velocity acquired by it is 9·8, 19·6, 29·4, 39·2, 49·0 metres a second. Exhibit graphically the changing velocity in its relation to the time.

7. A bullet is allowed to fall from rest at a height, and it is found that the space through which it has fallen at the end of 1, 2, 3, 4, 5 seconds is 4·9, 19·6, 44·1, 78·4, 122·5 metres. Exhibit graphically the changing space through which the bullet falls in relation to the time.

Exercises LXVIII

(Miscellaneous)

A

1. Shew that if any two of the letters a, b, c are interchanged in the expression

$$(b+c-a)(c+a-b)(a+b-c)(a+b+c)$$

the value of the expression is not changed.

2. Divide a straight line 24 inches long into three segments such that one extreme segment exceeds twice the other by 2 inches, while the mean segment is equal to the sum of the extreme segments.

3. If $f(x)=x^2+px+q$, find $f(x)-f(a)$ and factorize this last expression.

4. If $2x-3y=7$, find y in terms of x, and exhibit graphically the relation of y to x.

5. Shew that $(abc)^3=a^3b^3c^3$.

B

1. If $a=y+z-x$, $b=z+x-y$, $c=x+y-z$, shew that

$$(b+c)(c+a)(a+b)=8xyz.$$

2. Solve the equations

(1) $(x+3)(x+4)+10=(x+1)(x+11)+14.$

(2) $a(x-a)+b(x-b)+c(x-c)=2bc+2ca+2ab.$

3. Factorize

(1) $x^{21}-y^{21}.$

(2) $x^4-2abx^2-a^4-a^2b^2-b^4.$

4. If $f(n)=n^2+n+1$, shew that $f(n^2)=f(n).f(-n)$.

5. Taking a as any arbitrary number and b as any arbitrary length, represent graphically the equation

$$y=ax+b.$$

C

1. Resolve into factors $x^2-2x-24$, and find for what values of x this expression will be equal to zero.

2. Find the product

$$(x+1)(x+2)(x+3)(x-1)(x-2)(x-3).$$

3. Find two consecutive integers such that four times the greater exceeds three times the less by 10.

4. Resolve into factors

(1) $25(x+1)^2-36(x-1)^2.$

(2) $(a-b)^2-(c-d)^2.$

(3) $x^4-25x^2+144.$

5. Solve the equation,

$$\tfrac{1}{2}(x+3)+\tfrac{1}{3}(x-1)=\tfrac{1}{4}(x+1)+\tfrac{1}{5}(x+18).$$

D

1. From $7(m+n)x-11(n+l)y+8(l+m)z$ take
$$5(m+n)x-13(n+l)y+6(l+m)z.$$

2. Write down at once the following products:

(1) $(5x+7y)(5x-7y)$.

(2) $(x+7)(x+8)(x+9)$.

(3) $(x-y-z)(y-z-x)$.

3. Tea worth 35 cents a pound is taken with tea worth 50 cents a pound to form a mixture of 60 pounds worth 40 cents a pound. How many pounds of each kind are taken?

4. If $f(x)=lx^2+mx+n$, find the value of $f(2x)-f(x)$.

5. If $3x+2y-5=2x+3y-3$, find y in terms of x and exhibit the relation between y and x graphically.

E

1. For each of l days the sales of a certain article amounted to a dollars, for each of m days to b dollars, and for each of n days to c dollars. Find the daily average of the sales.

2. Resolve into factors:

(1) $x^2-21x+104$; (2) $x^2+21ax+104a^2$;

(3) $(x^2-5x-104)$; (4) $x^2+5xy-104y^2$.

3. The sum of $1000 was divided between A and B, after which A gives one-third of his share to B. Then B had as much as A had at first. Find the sums given to each in the division.

4. Multiply $a^3+3a^2b+3ab^2+b^3$ by $a+b$, and also $a^2+2ab+b^2$ by $a^2+2ab+b^2$, explaining why the results should be the same.

5. Find the L.C.M. of
$$x^2+20x+99,\ x^2+24x+143,\ x^2+22x+117.$$

CHAPTER XII

COMPLEMENTARY METHODS AND THEOREMS

47. Multiplication. A few exercises, somewhat more complicated than those given in Chapter V, will now be offered. It is advisable always to make sure that the terms of the involved expressions are arranged according to powers of one or more letters, or in an order suggested by the sequence of the letters present. The results should be given in arranged form.

The following examples will illustrate these points.

Ex. 1. Find the product

$$(a^2+b^2+c^2-bc-ca-ab)(a+b+c).$$

In writing the first factor there has been a manifest attention to order. First, there are the three squares,—terms of the same *type*—in the order of the letters a, b, c, and next, the products in pairs, the *first* one being that from which the *first* letter a is absent, and so on; even in writing the individual terms, bc, ca, ab, regard is had to order, as the letters abc are taken as a cycle ${}^{a}\underset{c}{\bigcirc}{}^{b}$ and as a comes just before b, so b comes just before c, and c just before a. In the second factor the order of the letters is observed.

The work of multiplication is as follows:

$$\begin{array}{llllllll}
a^2+b^2+c^2-bc-ca-ab & & & & & & & \\
a\ +b\ +c & & & & & & & \\
\hline
a^3+ab^2+c^2a- & abc-ca^2-a^2b & & & & & & \\
\quad -ab^2 & -\ abc & +a^2b+b^3+bc^2-b^2c & & & & & \\
\qquad\quad -c^2a- & abc+ca^2 & & -bc^2+b^2c+c^3 & & & & \\
\hline
a^3 & -3abc & +b^3 & +c^3 & & & &
\end{array}$$

Therefore the product is

$$a^3+b^3+c^3-3abc.$$

A place is opened for each new term unlike any that have appeared, and each new term like one that has presented itself is placed, in its own line, in the appropriate place. The idea in the arrangement of the result is manifest.

Ex. 2. Find the product

$$(p^2 - pq + q^2 - p - q + 1)(p + q + 1).$$

This is seen to be essentially the same as the example just worked, the earlier *degenerating* into this if c is taken as unity. Here, however, in the first factor, the arrangement is different, the terms of two dimensions being taken first and ordered with respect to p and q, then the terms of one dimension in order, and then the term of zero dimension.

The result is

$$p^3 + q^3 - 3pq + 1.$$

Ex. 3. Find the product:

$$(x^2 + px + q)(x^2 + mx + n).$$

The arrangement in each factor is that of descending powers of x. The result then is naturally given arranged in powers of x, the other letters being taken as coefficients.

$$\begin{array}{lllll}
x^2 & + px & + q & & \\
x^2 & + mx & + n & & \\
\hline
x^4 & + px^3 & + qx^2 & & \\
 & + mx^3 & + mpx^2 & + mqx & \\
 & & + nx^2 & + npx & + nq \\
\hline
\end{array}$$

Then adding the partial products the result is seen to be

$$x^4 + (p + m)x^3 + (mp + n + q)x^2 + (mq + np)x + nq.$$

Exercises LXIX

Find the following products:

1. $(4x^2 + 9y^2 + 16z^2 - 12yz - 8zx - 6xy)(2x + 3y + 4z)$.
2. $(p^2 + 9q^2 + 25r^2 + 15qr - 5rp + 3pq)(p - 3q + 5r)$.
3. $(yz + zx + xy)(x^2 + y^2 + z^2)$.
4. $(ayz + bzx + cxy)(ax + by + cz)$.

5. $(ax^3 + 3bx^2 + 3cx + d)(x - m)$.

6. $(4x^2 + 5xy - 3y^2 - 2x - 3y + 5)(x - y + 3)$.

7. $(ax + h)(bx + k)(cx + l)$.

8. $(2x + 3y - 5)(3x + 4y - 2)(x - 5y + 4)$.

9. $(3x + 4y - 5z)(2x - 3y + 4z)(x - 5y - 3z)$.

10. $(hx + k - l)(kx + l - h)(lx + h - k)$.

11. $(a + b + c)^3$.

12. $(a + b + c + d)^2$.

13. $(2a - 3b + 4c - 5d)(3a - 4b + 5c - 6d)$.

14. $(x + y + z)(ax + by + cz)(a^2x + b^2y + c^2z)$.

15. $(2x + 3y)^2(3x - 5y)^2$.

16. $(x + a)(x + b)(x + c)(x + d)$.

17. $(x - h)(x - k)(x - l)(x - m)(x - n)$.

18. $(a + b + c + d)(a^2 + b^2 + c^2 + d^2)$.

19. $(a - x)(b - x)(c - x)$.

20. $(1 - x + x^2 - x^3)(1 + x + x^2 + x^3)$.

21. $(a + 2bx + cx^2)(p + 2qx + rx^2)$.

22. $(x^2 + y^2 + z^2 + yz + zx + xy)(x + y + z)$.

23. $(x + y + z)(ax + by + cz)(bcx + cay + abz)$.

24. $(u^2 + v^2 + w^2)(lvw + mwu + nuv)$.

25. $(\frac{1}{2}x^2 + \frac{1}{3}x + \frac{1}{5})(\frac{1}{7}x^2 + \frac{1}{10}x + \frac{1}{8})$.

26. $(5x + 2y - 3z)(7x - y + 4z)(2x - 5y + 9z)$.

27. $(\frac{2}{3}x + \frac{3}{4}y)(\frac{3}{2}x + \frac{4}{5}y)(\frac{1}{5}x + \frac{1}{6}y)$.

28. $(1 + q + r - p)(1 + r + p - q)(1 + p + q - r)$.

48. Division. In the more complicated exercises in division the idea of arrangement according to powers of one or more of the involved letters is of even greater importance than in multiplication. This will appear in connection with the following examples.

Ex. 1. Find the quotient of $a^3+b^3+c^3-3abc$ by $a+b+c$.

First, we shall attempt the division as it is proposed, and see whither the work leads.

$$
\begin{array}{l}
a+b+c)\,a^3+b^3+c^3-3abc \qquad (a^2-ab-ca-bc+b^2+c^2 \\
\qquad a^3 \qquad\qquad +a^2b+ca^2 \\
\hline
\qquad b^3+c^3-3abc-a^2b-ca^2 \\
\qquad\qquad -abc-a^2b \qquad -ab^2 \\
\hline
\qquad b^3+c^3-2abc \qquad -ca^2+ab^2 \\
\qquad\qquad -abc \qquad -ca^2 \qquad -c^2a \\
\hline
\qquad b^3+c^3-abc \qquad +ab^2+c^2a \\
\qquad\qquad -abc \qquad\qquad -b^2c-bc^2 \\
\hline
\qquad b^3+c^3 \qquad +ab^2+c^2a+b^2c+bc^2 \\
\qquad b^3 \qquad\qquad +ab^2 \qquad +b^2c \\
\hline
\qquad c^3 \qquad\qquad +c^2a \qquad +bc^2 \\
\qquad c^3 \qquad\qquad +c^2a \qquad +bc^2 \\
\hline
\end{array}
$$

In seeking the second term of the quotient, we note that a, the first term of the divisor, is not contained in b^3, the first term of the remainder, and we look on to find a term of which a is a factor; the first such term is $-3abc$, but a term of higher dimension in a, namely a^2b, is seen—as also a^2c—and this we take first, wishing to work down from the higher powers of a. In like manner, we examine the remainder when seeking each new term in the quotient.

Now, contrast the division shewn above with the following, where the terms are considered with respect to a, and arranged:

$$
\begin{array}{l}
a+(b+c)\,)\,a^3 \qquad -3bc\,.\,a+(b^3+c^3)\,(a^2-(b+c)a+(b^2-bc+c^2) \\
\qquad a^3+(b+c)a^2 \\
\hline
\qquad -(b+c)a^2-3bc\,.\,a \qquad +b^3+c^3 \\
\qquad -(b+c)a^2-(b^2+2bc+c^2)\,.\,a \\
\hline
\qquad\qquad (b^2-bc+c^2)a+(b^3+c^3) \\
\qquad\qquad (b^2-bc+c^2)a+(b^3+c^3) \\
\hline
\end{array}
$$

In each case we obtain as quotient

$$a^2+b^2+c^2-bc-ca-ab.$$

Ex. 2. Divide $6x^4 - x^3 + 14x^2 - 19x + 23$ by $2x^2 + 3x + 7$.

$$
\begin{array}{l|l}
6x^4 - x^3 + 14x^2 - 19x + 23 & 2x^2 + 3x + 7 \\ \cline{2-2}
\underline{6x^4 + 9x^3 + 21x^2} & 3x^2 - 5x + 4 \\
\quad -10x^3 - 7x^2 - 19x & \\
\quad \underline{-10x^3 - 15x^2 - 35x} & \\
\qquad\qquad +8x^2 + 16x + 23 & \\
\qquad\qquad \underline{+8x^2 + 12x + 28} & \\
\qquad\qquad\qquad\quad 4x - 5 &
\end{array}
$$

Thus the quotient is $3x^2 - 5x + 4$ and the remainder $4x - 5$.

To save horizontal space, the divisor is here written to the right, and the terms of the quotient written below.

The division, in all such cases, is prosecuted until a point is reached where there is no remainder, or a remainder, the dimensions of which in the letter, or the principal letter, of arrangement is lower than that of the divisor.

Ex. 3. Divide 1 by $1 - x$.

$$
\begin{array}{l}
1 - x)\,1 \qquad\qquad (1 + x + x^2 + x^3 \\
\qquad\ \underline{1 - x} \\
\qquad\quad +x \\
\qquad\quad \underline{+x - x^2} \\
\qquad\qquad\quad +x^2 \\
\qquad\qquad\quad \underline{+x^2 - x^3} \\
\qquad\qquad\qquad\quad +x^3 \\
\qquad\qquad\qquad\quad \underline{+x^3 - x^4} \\
\qquad\qquad\qquad\qquad\quad +x^4
\end{array}
$$

The quotient, as also the divisor, is written and developed according to ascending powers of x. Plainly the division may be carried as far as we please. Stopping as shewn we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \frac{x^4}{1-x},$$

where division is indicated by a line separating dividend and divisor.

EXERCISES LXX

Divide

1. $p^3+q^3+3pq-1$ by $p+q-1$.
2. $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$ by x^2-xy+y^2.
3. $x^3+y^3-z^3+3xyz$ by $x+y-z$.
4. $6x^2-35y^2-88z^2+111yz+17zx-11xy$ by $2x-7y+11z$.
5. $30x^3+84y^3+240z^3-47x^2y-33x^2z-170y^2z-53y^2x-174z^2x-94z^2y+177xyz$ by $5x-7y-8z$.
6. $apx^4+2(aq+pb)x^3+(ar+pc+4bq)x^2+2(br+cq)x+cr$ by $ax^2+2bx+c$.
7. $(a^2-b^2)x^2-4abxy-(a^2-b^2)y^2$ by $(a-b)x-(a+b)y$.
8. $5x^6-7x^5+11x^4-19x^3-20x^2-33x+79$ by x^3-7x^2+8x-4.
9. $apx^3+(aq+pb)x^2y+(ar+bq)xy^2+bry^3$ by $ax+by$.
10. $ax^4+bx^3+cx^2+dx+e$ by $x-m$.
11. $1+x+x^2$ by $1-x$.
12. 1 by $1-2x+x^2$.
13. 1 by $1+x$.
14. 1 by $a-x$.

49. Multiplication by Detached Coefficients. The process of multiplying two polynomials, arranged according to the powers of the involved letter, can be abridged by dropping the letter and making a certain arrangement of the work. The method can be readily acquired through a study of the following examples in which the stages in the shortening of the work are shewn:

Ex. 1. Find the product of $2x^2+3x-7$ and $2x-5$.

(i)	(ii)
$2x^2+3x-7$	$2+3-7$
$2x-5$	$2-5$
$4x^3+6x^2-14x$	$4+6-14$
$-10x^2-15x+35$	$-10-15+35$
$4x^3-4x^2-29x+35$	$4-4-29+35$

$\therefore$ since $2x^2\times 2x=4x^3$, (ii) equally with (i), shews that the product is $4x^3-4x^2-29x+35$.

(iii)

$$\begin{array}{r|l}
 & 2+\ 3-\ 7 \\ \hline
2 & 4+\ 6-14 \\
-5 & \quad -10-15+35 \\ \hline
 & 4-\ 4-29+35
\end{array}$$

$\therefore$ as in (ii), the product is $4x^3-4x^2-29x+35$.

Ex. 2. Find the product of $3x^4-5x^2+7x+6$ and $2x^2-7x+9$.

$$\begin{array}{r|l}
 & 3+\ 0-\ 5+\ 7+\ 6 \\ \hline
2 & 6+\ 0-10+14+12 \\
-7 & \quad -21-\ 0+35-49-42 \\
+9 & \qquad\quad +27+\ 0-45+63+54 \\ \hline
 & 6-21+17+49-82+21+54
\end{array}$$

$\therefore$ since $3x^4\times 2x^2=6x^6$, the product is

$$6x^6-21x^5+17x^4+49x^3-82x^2+21x+54.$$

Note that, on account of the dependence on the order of terms, it is necessary to supply a zero coefficient for a corresponding absent term.

Ex. 3. Find the product of $2z^3+5z-4$ and z^2-3z+6.

$$\begin{array}{r|l}
 & 2+0+\ 5-\ 4 \\
-3 & \quad -6+\ 0-15+12 \\
+6 & \qquad\quad +12+\ 0+30-24 \\ \hline
 & 2-6+17-19+42-24
\end{array}$$

$\therefore$ since $2z^3\times z^2=2z^5$, the product is

$$2z^5-6z^4+17z^3-19z^2+42z-24.$$

Note that, since the first multiplication is by 1, it is not necessary to re-write the sequence of coefficients in the multiplicand.

In working the examples proposed it would be well to work out at least some of them in the ordinary way, and go through the work of abridging the process.

Exercises LXXI

Find the following products by the method of detached coefficients:

1. $(2x^2+5x-7)(3x^2-7x+9)$.
2. $(5x^3+7x^2-8x-13)(4x^2-5x-6)$.
3. $(7x^3-6x^2+11x-8)(x^2+3x-8)$.
4. $(2+3x-5x^2)(5-9x+7x^2)$.
5. $(x^4+x^3+x^2+x+1)(x-1)$.
6. $(1-x+x^2-x^3+x^4-x^5)(1+x)$.
7. $(x+5)(x+7)(x-11)(x-13)$.
8. $(ax^2+bx+c)(x-1)$.
9. $(px^2+2qx+r)(ax+b)$.
10. $(x^3-3mx^2+3m^2x-m^3)(x^3+3mx^2+3m^2x+m^3)$.
11. $(1+x+x^2+x^3+x^4+x^5)(1-x+x^2-x^3+x^4-x^5)$.
12. $(ax^2+2bx+c)(ax^2-2bx+c)$.

50. Division by Detached Coefficients, or by Horner's Method. As in multiplication, the work of division may be abridged. The development of the method is shewn in the following examples.

Ex. 1. Divide $10x^3-29x^2+41x+28$ by $5x-7$.

(i)

$$\begin{array}{r|l}
5x-7 & 10x^3-29x^2+41x+28\ (2x^2-3x+4 \\
& \underline{10x^3-14x^2} \\
& \quad -15x^2+41x \\
& \quad \underline{-15x^2+21x} \\
& \qquad\qquad 20x+28 \\
& \qquad\qquad \underline{20x+28}
\end{array}$$

(ii)

$$
\begin{array}{l}
5-7\,)\,10-29+41+28\,(\,2-3+4 \\
\qquad\quad -14 \\
\hline
\qquad\quad -15+41 \\
\qquad\qquad\quad +21 \\
\hline
\qquad\qquad\quad 20+28 \\
\qquad\qquad\qquad +28 \\
\hline
\end{array}
$$

$\therefore$ since $10x^3 \div 5x = 2x^2$, the quotient is

$$2x^2-3x+4.$$

(iii)

$$
\begin{array}{r|l}
5-7 & 10-29+41-28 \\
 & \quad -14+21-28 \\
\hline
 & 2-\ \ 3+\ \ 4
\end{array}
$$

(iv)

$$
\begin{array}{r|l}
 & 10-29+41-28 \\
+7 & \quad +14-21+28 \\
\hline
5 & 2-\ \ 3+\ \ 4
\end{array}
$$

In (ii), the letter x is dropped and the unnecessary 10, -15, 20, under the first term in the successive remainders do not appear. In (iii), the work is shewn in more compact form, and the first term in each remainder is found mentally, and the corresponding term in the quotient found by division by 5. In (iv), the 5 which is used only as a divisor is written in a convenient place by itself, and the sign of the -7 is changed to + so that the subtraction to find each remainder is changed to an addition.

It is to be noted that the division is completed when a term in the row below the dividend is found below the last term of the dividend.

Ex. 2. Find the quotient of $14x^5-29x^4+63x^3-56x^2+58x-30$ by $2x^2-3x+5$.

(i)

$$
\begin{array}{l}
2x^2-3x+5\,)\,14x^5-29x^4+63x^3-56x^2+58x-30\,(\,7x^3-4x^2+8x-6 \\
\qquad\qquad\quad -21x^4+35x^3 \\
\hline
\qquad\qquad\quad -\ 8x^4 \\
\qquad\qquad\qquad\quad +12x^3-20x^2 \\
\hline
\qquad\qquad\qquad\quad 16x^3 \\
\qquad\qquad\qquad\qquad -24x^2+40x \\
\hline
\qquad\qquad\qquad\qquad -12x^2 \\
\qquad\qquad\qquad\qquad\qquad +18x-30 \\
\hline
\end{array}
$$

(ii)

$$\begin{array}{r|l} & 14-29+63-56+58-30 \\ +3 & \quad\ \ +21-12+24-18 \\ -5 & \qquad\qquad -35+20-40+30 \\ \hline 2 & 7-\ \ 4+\ \ 8-\ \ 6. \end{array}$$

It will be seen that in (i), we have the ordinary work of division, except that the term that should find its place under the first term of the successive remainders does not appear, and that the subtractions in any column are not made until that column appears as giving a term in the quotient. In (ii), the first coefficient 2 is written conveniently as in the preceding example, and the signs of the other terms are changed so that the subtractions of the ordinary division are changed into additions; as each column is completed, and the sum found mentally, the division by 2 is made, and the corresponding term of the quotient written below the line in that column.

Note that the division ends when the term $+30$ is introduced into the column belonging to the last term of the dividend.

Ex. 3. Divide $2x^4+3x^3-16x^2+44x+39$ by x^2+5x+3.

$$\begin{array}{r|l} & 2+\ \ 3-16+44+39 \\ -5 & \quad -10+35-65 \\ -3 & \qquad\quad -\ \ 6+21-39 \\ \hline & 2-\ \ 7+13. \end{array}$$

$\therefore$ Since $2x^4 \div x^2 = 2x^2$ the quotient is

$$2x^2-7x+13.$$

The first coefficient being 1 is not written as when each column is added, the division by 1 gives the total as quotient.

Ex. 4. Divide x^5-1 by $x-1$

$$\begin{array}{r|l} & 1+0+0+0+0-1 \\ +1 & \quad +1+1+1+1+1 \\ \hline & 1+1+1+1+1. \end{array}$$

$\therefore$ The quotient is $x^4+x^3+x^2+x+1$.

Ex. 5. Divide $2x^4 - 7x^2 + 3x - 15$ by $x^2 + 2x - 3$.

$$\begin{array}{r|rrrrr} & 2 + 0 & -7 & +\ 3 & -15 & \\ -2 & -4 & +8 & -14 & & \\ +3 & & +6 & -12 & +21 & \\ \hline & 2 - 4 & +7; & -23 & +\ 6 & \end{array}$$

The quotient is $2x^2 - 4x + 7$, and the remainder $-23x + 6$.

Exercises LXXII

Divide by Horner's method:

1. $2x^3 - 7x^2 - 22x + 35$ by $x - 5$.
2. $3x^3 + 13x^2 - 45x + 77$ by $x + 7$.
3. $4x^4 - 31x^3 + 63x^2 - 131x + 30$ by $x - 6$.
4. $6x^3 + 23x^2 - 69x - 77$ by $2x + 11$.
5. $6a^3 + a^2b - 59ab^2 + 56b^3$ by $3a - 7b$.
6. $14 - 27z + 19z^2 - 3z^3 - 18z^4$ by $2 - 3z$.
7. $2y^3 - 7y^2 - 19y + 35$ by $y - 8$.
8. $4z^4 - 5z^3 + 16z^2 + 35z - 29$ by $z + 7$.
9. $14x^4 - 29x^3y + 4xy^3 + 29y^4$ by $2x - 3y$.
10. $35a^4 - 32a^3b + 79a^2b^2 + 110ab^3 + 29b^4$ by $7a + 2b$.
11. $x^7 - 1$ by $x - 1$.
12. $x^7 + 1$ by $x + 1$.
13. $x^{10} + 1$ by $x + 1$
14. $x^{11} + 1$ by $x + 1$.
15. $x^{11} - 1$ by $x - 1$.
16. $2x^4 + x^3y + 12x^2y^2 + xy^3 + 40y^4$ by $x^2 + 2xy + 5y^2$.
17. $20 - 47z + 51z^2 - 50z^3 - 14z^4$ by $5 - 8z - 2z^2$.
18. $7x^5 - 9x^4y + 16x^3y^2 - 21x^2y^3 - 33xy^4 + 57y^5$ by $x^2 - 3xy - 8y^2$.
19. $x^{35} - 1$ by $x^5 - 1$.
20. $a^8 + a^4b^4 + b^8$ by $a^4 + a^2b^2 + b^4$.

51. The Remainder Theorem. The division of a polynomial in x by a divisor linear in x is supposed to be carried on until there is no remainder, or a remainder in which x does not occur. The following examples suggest a very important theorem relating to the form or value of the remainder.

Ex. 1. Divide x^3-1 by $x-m$.

$$
\begin{array}{l}
x-m)\,x^3 \qquad -1\,(x^2+mx+m^2 \\
\quad\ \ \underline{x^3-mx^2} \\
\qquad\ +mx^2 \\
\qquad\ \underline{+mx^2-m^2x} \\
\qquad\qquad\quad m^2x-1 \\
\qquad\qquad\quad \underline{m^2x-m^3} \\
\qquad\qquad\qquad\quad m^3-1
\end{array}
$$

Ex. 2. Divide ax^2+bx+c by $x-m$.

$$
\begin{array}{l}
x-m)\,ax^2+bx+c\,(ax+(am+b) \\
\quad\ \ \underline{ax^2-amx} \\
\qquad\ (am+b)x+c \\
\qquad\ \underline{(am+b)-am^2-bm} \\
\qquad\qquad\qquad am^2+bm+c
\end{array}
$$

In each case the remainder is seen to be the result of substituting m for x in the dividend, or, in other words, to be the same function of m that the dividend is of x. We are thus led to propose the theorem:

Theorem. *If a polynomial in x is divided by $x-m$, the remainder is the result of substituting m for x in the polynomial.*

Proof. Let $f(x)$ denote any polynomial in x. Then by division it can be shewn that

$$f(x) = q(x)(x-m) + R.$$

Where $q(x)$ is a polynomial in x and R is the remainder in the division of $f(x)$ by $x-m$ and does not therefore involve x. This relation is an identity, *i.e.*, it is independent of the value of x, or true for all values of x. It will therefore be true if we take x equal to m. This affects the value of $f(x)$, $q(x)$, and $x-m$, but not that of R. Therefore

$$f(m) = q(m).(m-m) + R.$$
$$= q(m).0 + R.$$

i.e., $$f(m) = R.$$

so that the remainder R is the result of writing m for x in the polynomial.

The theorem may be stated more briefly thus: *If $f(x)$, a polynomial in x, is divided by $x-m$ the remainder is $f(m)$.*

The following corollaries should be noted:

COR. 1. *If $f(x)$ is a polynomial in x, and if $f(m)$ is equal to zero, then $f(x)$ is divisible by $x-m$.*

COR. 2. *If $f(x)$ is a polynomial in x which has $x-m$ as a factor, then $f(m)$ is identically zero.*

COR. 3. *If $f(x)$ is a polynomial in x, then $f(x)-f(m)$ is divisible by $x-m$.*

COR. 4. *If $f(x)$ is a polynomial in x, the value of $f(m)$ is the remainder when $f(x)$ is divided by $x-m$.*

The following examples will illustrate the application of the theorem.

Ex. 1. If n is any positive integer, x^n-1 is divisible by $x-1$.

The remainder when x^n-1 is divided by $x-1$, being found by writing 1 for x, is

$$(1)^n-1$$

which is at once seen to be zero. Therefore the division is exact.

Ex. 2. Shew that

$$2x^5-7x^4+8x^3-5x^2+13x-11$$

is divisible by $x-1$.

The remainder in the division by $x-1$, being found by writing 1 for x, is

$$2-7+8-5+13-11$$

which is zero, so that the division is exact.

It is plain that: *A polynomial in x is divisible by $x-1$ when the sum of its coefficients is zero.*

Ex. 3. Find the value of

$$3x^5-7x^4+15x^3-27x^2-15x+93$$

for $x=6$.

The direct substitution of 6 for x implies much computation. The remainder when the expression is divided by $x-6$ is the value sought, and is readily found by division by Horner's method as below given.

$$\begin{array}{r|l} & 3-\ \ 7+15-\ \ 27-\ \ \ 15+\ \ \ \ \ 93 \\ +6 & \ \ \ \ +18+66+486+2754+16414 \\ \hline & 3+11+81+459+2739;\ 16507 \end{array}$$

$\therefore$ the value sought $=16507$.

Ex. 4. Shew that

$$a^3+b^3+c^3-3abc$$

is divisible by $a+b+c$.

Treating the given expression as a polynomial in a, and writing $a+b+c$ in the form $a-(-\overline{b+c})$, we see that the remainder, being found by writing $-\overline{b+c}$ for a, is

$$(-\overline{b+c})^3+b^3+c^3-3bc(-\overline{b+c}).$$

This last
$$\begin{aligned} &= -(b+c)^3+b^3+c^3+3bc(b+c) \\ &= -(b+c)^3+(b+c)^3 \\ &= 0. \end{aligned}$$

Thus $a+b+c$ is a *factor* of the expression.

Ex. 5. Shew that x is a factor of

$$(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3.$$

It is plain that if a is a factor, the value $a=0$ will reduce the expression to zero. Putting $a=0$ we have as result

$$(b+c)^3-(b+c)^3-(c-b)^3-(b-c)^3,$$

which, since $(b-c)=-(c-b)$, is equal to zero. Thus a is a factor.

Or we may look upon a as $a-0$, and the given expression as a polynomial in a. The remainder in the division by a will then be the result of substituting 0 for a in the expression. This has been found to be zero.

Exercises LXXIII

1. Shew that

(i) If n is a positive integer, x^n-1 is divisible by $x-1$.

(ii) If n is an even positive integer, x^n-1 is divisible by $x+1$.

(iii) If n is an odd positive integer, x^n+1 is divisible by $x+1$.

(iv) If n is an odd positive integer, x^n-1 is not divisible by $x+1$.

(v) If n is an even positive integer, x^n+1 is not divisible by $x+1$.

2. Shew that

(i) $x^n - y^n$ is divisible by $x - y$ if n is any positive integer.

(ii) $x^n - y^n$ is divisible by $x + y$ if n is any even integer, but not if n is an odd integer.

(iii) $x^n + y^n$ is divisible by $x + y$ if n is an odd integer, but not if n is an even integer.

3. Shew that

$$3x^4 - 17x^3 - 19x^2 + 5x + 4$$

is divisible by $x + 1$.

4. It is known that $x^2 + px + q$ is divisible by both $x - 1$ and $x + 1$. Find the values of p and q.

5. Shew that

$$(ab + bc + ca)^2 - (b^2c^2 + c^2a^2 + a^2b^2)$$

is divisible by a, by b, by c, and by $a + b + c$.

6. Shew that $b + c$ is a factor of

$$(a + b + c)(bc + ca + ab) - abc.$$

7. Find the value of

$$3x^4 - 7x^3 + 29x^2 - 51x + 33$$

for $x = 7$.

8. Shew that

$$ax^4 + bx^3 + cx^2 + dx + e$$

is divisible by $x - 1$ if

$$a + b + c + d + e = 0.$$

Write down five polynomials of the fourth degree that are divisible by $x - 1$.

9. Shew that

$$ax^4 + bx^3 + cx^2 + dx + e$$

is divisible by $x + 1$ if

$$a + c + e = b + d.$$

Write down five polynomials of the fourth degree that are divisible by $x + 1$.

52. Symmetry. In many expressions there is a certain balance of form which may be described by saying that the involved letters, except in the matter of order, are treated alike. Thus in

$$a^2 + ab + b^2$$

where two letters a and b appear, it is seen that each letter is involved similarly to the others. So in

$$a + b + c$$
$$a^2 + b^2 + c^2 + bc + ca + ab$$
$$a^3 + b^3 + c^3 - 3abc$$

where the three letters a, b, c appear, it is seen in each that the three letters a, b, c are similarly involved. Such expressions are said to be *symmetrical*, or to be *symmetric functions of the involved letters (numbers)*. While the symmetry is in general seen at once, a definition is necessary:

An expression is symmetrical with respect to two or more involved letters when every interchange of two letters leaves the expression unchanged in algebraic value.

The terms of a symmetrical expression fall into one or more groups, in each of which the terms are of one type. Thus in $a + b + c$ the terms are of one type, and, it being understood that three letters a, b, c are concerned, the expression might be written

$$\Sigma a,$$

the Σ signifying that all such terms as a are to be taken as a sum. So in

$$a^2 + b^2 + c^2 + bc + ca + ab$$

there are two types of terms, those such as a^2 and those such as bc, and the expression may be written

$$\Sigma a^2 + \Sigma bc.$$

The following examples will illustrate the use that may be made of the fact of symmetry in an expression.

Ex. 1. Resolve into factors

$$a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc.$$

Surmising that there may be some simple factor as a, or $b-c$, or $b+c$, or $a+b+c$, we test for these in order. It is found that a is not a factor, and the fact of the symmetry of the expression at once declares that neither b nor c is a factor. It is found also that $b-c$ is not a factor, and therefore by the *principle of symmetry* as before neither $c-a$ nor $a-b$. We find that $b+c$ is a factor, for, treating the expression as a polynomial in b, the remainder on dividing by $b+c$, given by writing $-c$ for b, is

$$a^2(-c+c)+c^2(c+a)+c^2(a-c)-2ac^2,$$

which reduces to zero. Thus $b+c$ is a factor, and the fact of symmetry guarantees that $c+a$ and $a+b$ are factors. Hence $(b+c)(c+a)(a+b)$ is a factor of the expression and, since these expressions are each of three dimensions, it follows that they can differ only by a numerical factor. Accordingly we are justified in saying that identically

$$a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc=N(b+c)(c+a)(a+b),$$

where N is a numerical factor, consequently independent of a, b, c, and therefore the same for all values of a, b, c. Assigning then the values $a=0$, $b=1$, $c=1$ we have

$$0.2+1.1+1.1=N.2.1.1$$

whence $N=1$, so that

$$a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc=(b+c)(c+a)(a+b).$$

Ex. 2. Resolve into factors

$$a^3+b^3+c^3-3abc.$$

The given expression is symmetrical in a, b, c. Testing for factors of the types, a, $a+b$, $a-b$, $a+b+c$, we find that of these $a+b+c$ is a factor. For, values of a,b,c that will make $a+b+c=0$, *i.e.*, $a=-(b+c)$ will make the expression become

$$-(b+c)^3+b^3+c^3+3bc(b+c)$$

i.e., $$-(b+c)^3+(b+c)^3,$$

which is zero. Accordingly, $a+b+c$ is a factor. Now the given expression is symmetrical, and, $a+b+c$ being symmetrical and therefore not requiring any other factor of the same type, we see that, on account of the dimensions, the co-factor of $a+b+c$ must be symmetrical and everywhere of two dimensions in a,b,c. The general expression of this type is $m(a^2+b^2+c^2)+n(bc+ca+ab)$ where m and n are numerical. We may therefore say that

$$a^3+b^3+c^3-3abc=(a+b+c)\{m(a^2+b^2+c^2)+n(bc+ca+ab)\}$$

where m and n have certain definite values, whatever be a,b,c. Suppose $c=0$; then identically

$$a^3+b^3=(a+b)\{m(a^2+b^2)+nab\},$$

and by division by $a+b$

$$a^2-ab+b^2=m(a^2+b^2)+nab.$$

This last being an identical statement, we have $m=1$, $n=-1$, and accordingly,

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab).$$

In the preceding m and n might have been found simply in a variety of ways. For example, we might have taken $b=0$ and $c=0$, which would have given m, and then n might have been found. Or we might have said, still referring to the original relation involving m and n, that on the left the coefficient of a^3 is 1, and on the right m, since the only multiplication that yields a^3 is that of a into ma^2; and that on the left the coefficient of abc is -3, and on the right $+3n$, arising through the multiplication of a into nbc, b into nca, and c into nab: hence m must be $+1$ and n must be -1.

Ex. 3. Resolve into factors

$$a^3(b-c)+b^3(c-a)+c^3(a-b).$$

The given expression is not symmetrical, since an interchange of b and c gives an expression which is the negative of this. But there is a sort of symmetry, the expression being unchanged by a cyclic change of letters a into b, b into c, c into a; this is what is called *cyclo-symmetry*.

We find that $b-c$ is a factor, and consequently to complete the cyclo-symmetry, $c-a$ and $a-b$ must be factors. Noting that the given expression is of four dimensions in a,b,c, it follows that there must be a fourth *literal* factor of one dimension, which will not require, on account of symmetry, any additional factor. This can be only $a+b+c$. We have therefore

$$a^3(b-c)+b^3(c-a)+c^3(a-b)=N(b-c)(c-a)(a-b)(a+b+c),$$

where N is a definite numerical factor to be found, the relation being true whatever be a,b,c. Take a equal to zero. Then

$$b^2c - c^2b = N(-bc)(b-c)(b+c),$$

whence it is seen that $N = -1$, so that

$$a^2(b-c) + b^2(c-a) + c^2(a-b) = -(b-c)(c-a)(a-b)(a+b+c)$$
$$= (c-b)(a-c)(b-a)(a+b+c).$$

That $N = -1$ may be readily seen otherwise. Thus, on the left, the coefficient of a^2b is 1, and on the right is $-N$, so that N must be -1.

Ex. 4. Simplify

$$(x+y+z)^2 + (y+z-x)^2 + (z+x-y)^2 + (x+y-z)^2.$$

The expression is symmetrical and everywhere of two dimensions in x, y, z, so that the only terms that can appear are of the types

$$x^2, \quad yz.$$

From the expansion of the square of a trinomial, the coefficient of x^2 is seen to be 4; that of yz to be 0. Thus it is manifest that the expression reduces to

$$4(x^2 + y^2 + z^2).$$

EXERCISES LXXIV

1. Write out in full the expressions denoted by the following, it being understood that three letters appear in each:

(1) Σa.

(2) $\Sigma a^2 - \Sigma bc$.

(3) $\Sigma a^3 - \Sigma a^2b + abc$.

(4) $2\Sigma a^4 - 3\Sigma b^2c^2$.

(5) $\Sigma x^2y - 3\Sigma x^2y^2 + 5\Sigma x^4$.

2. Write in short form, employing only type terms, the following:

(1) $a^2 + b^2 + c^2 + bc + ca + ab$.

(2) $(a+b+c)(a^2+b^2+c^2)$.

(3) $3(x^3+y^3+z^3) + 11xyz$.

(4) $x^2(y+z) + y^2(z+x) + z^2(x+y) + 7xyz - x^3 - y^3 - z^3$.

(5) $x^4 + y^4 + z^4 + 3y^2z^2 + 3z^2x^2 + 3x^2y^2 + xyz(x+y+z)$.

3. Supposing that there are three involved letters, supply terms to the following to render them symmetrical:

(1) x^2+y^2+yz.
(2) $x^3+3y^2z+6xyz$.
(3) $a^2b^2+c^4$.
(4) $lmn^2+m^4+n^4$.
(5) $3a^2b+6abc+b^3+c^3$.
(6) $l^5+m^2n^3$.

4. Simplify the following expressions:

(1) $(b+c)^3+(c+a)^3+(a+b)^3$.
(2) $(b+c)a^2+(c+a)b^2+(a+b)c^2-(a^2+b^2+c^2)(a+b+c)$.
(3) $(y+z)(y-z)^2+(z+x)(z-x)^2+(x+y)(x-y)^2$.
(4) $(x+y+z)^2+(y+z-2x)^2+(z+x-2y)^2+(x+y-2z)^2$.
(5) $(x+y+z)^2+(x-y-z)^2+(y-z-x)^2+(z-x-y)^2$.

5. Resolve into factors:

(1) $x^2(y-z)+y^2(z-x)+z^2(x-y)$.
(2) $(y-z)^3+(z-x)^3+(x-y)^3$.
(3) $(b+c)(c+a)(a+b)+abc$.
(4) $mn(m-n)+nl(n-l)+lm(l-m)$.
(5) $yz(y+z)+zx(z+x)+xy(x+y)+2xyz$.
(6) $x(y-z)^3+y(z-x)^3+z(x-y)^3$.
(7) $(a+b+c)^3-a^3-b^3-c^3$.
(8) $(b-c)(b+c)^3+(c-a)(c+a)^3+(a-b)(a+b)^3$.
(9) $x^4(y-z)+y^4(z-x)+z^4(x-y)$.
(10) $(b^2-c^2)^3+(c^2-a^2)^3+(a^2-b^2)^3$.

53. Important Identities. The following identities are of outstanding importance.

(i)

$$
\begin{aligned}
(a+b+c)^3 &= a^3+b^3+c^3+3a^2(b+c)+3b^2(c+a)+3c^2(a+b)+6abc\\
&= a^3+b^3+c^3+3bc(b+c)+3ca(c+a)+3ab(a+b)+6abc\\
&= a^3+b^3+c^3+3(a+b+c)(bc+ca+ab)-3abc\\
&= a^3+b^3+c^3+3(b+c)(c+a)(a+b).
\end{aligned}
$$

We have

$$(a+b+c)^3 = (a+\overline{b+c})^3$$
$$= a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3$$
$$= a^3 + 3a^2b + 3a^2c + 3b^2a + 6abc + 3c^2a + b^3 + 3b^2c + 3bc^2 + c^3,$$

which is readily combined into the two forms first given.

Next, taking a part of the second form, note that

$$3bc(b+c) + 3ca(c+a) + 3ab(a+b) + 6abc$$
$$= \{3bc(b+c) + 3abc\} + \{3ca(c+a) + 3abc\} + \{3ab(a+b) + 3abc\} - 3abc$$
$$= 3[bc(a+b+c) + ca(a+b+c) + ab(a+b+c)] - 3abc$$
$$= 3(a+b+c)(bc+ca+ab) - 3abc,$$

and the third form is obtained.

Further, taking a part of the first form, note that

$$3a^2(b+c) + 3b^2(c+a) + 3c^2(a+b) + 6abc$$
$$= 3a^2(b+c) + 3(b^2c + bc^2) + 3b^2a + 6abc + 3c^2a$$
$$= 3a^2(b+c) + 3bc(b+c) + 3a(b+c)^2$$
$$= 3(b+c)(a^2 + bc + ab + ca)$$
$$= 3(b+c)(c+a)(a+b),$$

and the fourth form is obtained.

(ii)

$$\mathbf{a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - bc - ca - ab).}$$

For, adding $3ab(a+b)$ to complete the cube of $a+b$, and subtracting to correct, we have,

$$a^3 + b^3 + c^3 - 3abc$$
$$= a^3 + 3ab(a+b) + b^3 + c^3 - 3abc - 3ab(a+b)$$
$$= \{(a+b)^3 + c^3\} - 3ab(a+b+c)$$
$$= (\overline{a+b}+c)(\overline{a+b}^2 - \overline{a+b}\,.\,c + c^2) - 3ab(a+b+c)$$
$$= (a+b+c)(a^2 + b^2 + c^2 - bc - ca + 2ab) - 3ab(a+b+c)$$
$$= (a+b+c)(a^2 + b^2 + c^2 - bc - ca - ab).$$

The following examples are illustrative.

Ex. 1. Shew that

$$(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b).$$

This amounts to shewing that

$$(b-c)^3+(c-a)^3+(a-b)^3-3(b-c)(c-a)(a-b)=0.$$

This last expression, being of the form appearing in (ii) must equal

$$(\overline{b-c}+\overline{c-a}.+\overline{a-b})(\text{a second factor})$$

which, since $b-c+c-a+a-b=0$, must equal zero. Hence the truth of the proposed.

Ex. 2. Resolve into factors $p^3+3pq+q^3-1$.

We have

$$\begin{aligned}p^3+3pq+q^3-1 &= p^3+q^3+(-1)^3-3pq.(-1)\\ &=\{p+q+(-1)\}\{p^2+q^2+(-1)^2-q(-1)-(-1)p-pq\}\\ &=(p+q-1)(p^2-pq+q^2+p+q+1).\end{aligned}$$

EXERCISES LXXV

1. Find what identity (ii) becomes
   - (i) If b is replaced by $-b$;
   - (ii) If a is replaced by x, b by $-y$, c by $-z$;
   - (iii) If a is replaced by p, b by q, and c by -1.

2. Give the complete work of resolving each of the following into factors:

(1) $x^3+y^3+z^3-3xyz$.	(3) $x^3-y^3+z^3+3xyz$.
(2) $p^3+8q^3+27r^3-18pqr$.	(4) $a^3-b^3-c^3-3abc$.

3. Assuming the identity (ii), resolve into factors the following:
   - (1) $8x^3+27y^3+125z^3-90xyz$.
   - (2) $u^3+8v^3-27w^3+18uvw$.
   - (3) $-a^3-b^3-c^3+3abc$.

4. Expand as the cube of a trinomial

$$(\overline{b-c}+\overline{c-a}+\overline{a-b})^3$$

and in this way obtain the relation

$$(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b).$$

54. Additional Note on Resolution into Factors. By actual multiplication we find the product of the two expressions $2x+3y+5z$ and $3x-4x+7z$, linear in x, y, z.

$$\begin{array}{l}
2x+3y+5z \\
3x-4y+7z \\
\hline
6x^2+9xy+15zx \\
\quad -8xy \qquad\quad -12y^2-20yz \\
\qquad\quad +14zx \qquad\quad +21yz+35z^2 \\
\hline
6x^2+\ xy+29zx-12y^2+\quad yz+35z^2
\end{array}$$

Thus the product is

$$6x^2-12y^2+35z^2+29zx+xy+yz.$$

We now seek a method of finding the factors of this last expression. Plainly, if the terms involving z had been ignored, or blotted out, we should have had to multiply $2x+3y$ and $3x-4y$ and the product would have been $6x^2+xy-12y^2$. The last expression we know how to factor, and, knowing this, we see that the part of the expression not involving z gives the x and y parts of the factors sought.

Similarly, the part not involving x gives the y and z parts of the factors and the part not involving y gives the x and z parts of the factors. Accordingly writing

$$\begin{aligned}
6x^2+xy-12y^2 &= (2x+3y\qquad)(3x-4y\qquad) \\
-12y^2+yz+35z^2 &= (\qquad+3y+5z)(\qquad-4y+7z) \\
6x^2+29xz+35z^2 &= (2x\qquad+5z)(3x\qquad+7z).
\end{aligned}$$

We see by associating the three results that the linear factors

$$(2x+3y+5z)(3x-4y+7z)$$

are determined.

In like manner from the product

$$(2a+3b-4)(3a-4b+5).$$

The result is found to be

$$6a^2 + ab - 12b^2 - 2a + 31ab - 20.$$

Then taking in succession the part of this that is everywhere two-dimensional, the part not involving b, and the part not involving a, we have

$$\begin{aligned} 6a^2 + ab - 12b^2 &= (2a - 3b \quad)(3a - 4b \quad) \\ 6a^2 - 2a - 20 &= 2(a \quad - 2)(3a \quad + 5) \\ &= (2a \quad - 4)(3a \quad + 5) \\ -12b^2 + 31b - 20 &= (\quad 3b - 4)(\quad - 4b + 5) \end{aligned}$$

and as in the earlier example we associate these results and recover the factors

$$(2a + 3b - 4)(3a - 4b + 5).$$

When the three results are such as not to meet and yield two linear factors of three terms each, this fact is to be taken as evidence that the expression proposed will not break up into factors.

Exercises LXXVI

1. Form the following products and recover the factors in each case:

(1) $(3x - 5y + 7z)(2x + y + 3z)$. (4) $(7l - 3m + 2)(5l - 8m - 6)$.

(2) $(2a - 3b - 4c)(3a - 5b - 6c)$. (5) $(3a - 7b + 9c)(2a + 4b - 5c)$.

(3) $(4x - 5y + 7)(2x - 3y + 8)$.

2. Resolve into factors the following:

(1) $6x^2 + 6y^2 - 20z^2 + 2yz + 7zx - 13xy$.

(2) $12a^2 - 35b^2 + 16c^2 + 26bc - 38ca + 13ab$.

(3) $10x^2 - 9xy - 36y^2 - 11x + 81y - 35$.

(4) $4p^2 + 21q^2 - 18r^2 + 33qr + 6rp - 31pq$.

(5) $35x^2 - 24xy - 35y^2 + 41x + 61y - 24$.

55. Highest Common Factor. In order to find the highest common factor of two or more expressions, it has been assumed that the expressions can be resolved into factors. It may be, however, that such a resolution is not immediately possible, and in such a case a method of finding the highest common factor is sought. A method is afforded by Proposition 2 of Book VII of Euclid's Elements.

The process depends on the following lemma:

Lemma. Every common factor—arithmetical or algebraic—of two quantities is a factor of the sum or the difference of any multiples—arithmetical or algebraic—of those quantities.

For let p and q denote any two quantities, and let v be a common factor in either the arithmetical or algebraic sense. We have then

$$p = hv, \text{ and } q = kv$$

where h and k are two quantities, the co-factors of v in p and q. Take mp and nq, any two multiples—arithmetical or algebraic—of p and q. Then for the sum $mp + nq$ of these multiples, we have

$$\begin{aligned} mp + nq &= mhv + nkv \\ &= v(mh + nh), \end{aligned}$$

so that v, any common factor of p and q, is a factor of $mp + nq$.

Similarly, v is a factor of $mp - nq$, and the lemma is established.

Now let a and b be the two algebraic expressions, and suppose them arranged according to powers of some common letter. Let a be of not higher dimensions than b, and divide b by a, obtaining as quotient q and remainder r, as shewn below:

$$\begin{array}{l} a\,)\,b\,(\,q \\ \underline{\;aq\;} \\ \;\;r \end{array}$$

Then r is of lower dimensions than a in the common letter, and

$$r = b - aq, \text{ or } b = r + aq.$$

By the lemma every common factor of a and b is a factor of $b - aq$ or r, and is therefore a factor common to a and r. Also every common factor of a and r is a factor of $r + aq$ or b, and is therefore a factor common to a and b. Thus a and r have the same common factors as a and b, and it is sufficient then to find the highest common factor of a and r, a simpler problem since r is of lower degree in the common letter than a or b. In the same way as before, let a be divided by r, and let p be the quotient and s the remainder, so that s is of lower degree than r.

$$\begin{array}{l} r\,)\,a\,(\,p \\ \underline{\;rp\;} \\ \;\;s \end{array}$$

Then as before s and r have the same common factors as r and a, and therefore as a and b.

Let the process be continued and suppose that a point is reached where having to divide a remainder v into the preceding divisor u, we find the division exact.

$$\begin{array}{l} v\,)\,u\,(\,l \\ \underline{\;lv\;} \end{array}$$

It is plain that v is the highest common factor of u and v, and therefore of the two expressions in the next earlier division, and therefore ultimately of a and b, the given expressions.

If no point is reached where the division is exact the expressions are without common factor.

In practice always, if the expressions have any numerical or monomial common factor, this factor is removed.

The following examples will illustrate the theory.

Ex. 1. Find the H.C.F. of x^3+3x^2+3x+2 and x^3+4x^2+6x+4.

$$x^3+3x^2+3x+2\ )\ x^3+4x^2+6x+4\ (\ 1$$
$$\underline{x^3+3x^2+3x+2}$$
$$x^2+3x+2$$

$$x^2+3x+2\ )\ x^3+3x^2+3x+2\ (\ x$$
$$\underline{x^3+3x^2+2x}$$
$$x+2$$

$$x+2\ )\ x^2+3x+2\ (\ x+1$$
$$\underline{x^2+2x}$$
$$x+2$$
$$\underline{x+2}$$

Thus $x+2$ is the H.C.F.

It is found that

$$x^3+3x^2+3x+2=(x+2)(x^2+x+1),$$
$$x^3+4x^2+6x+4=(x+2)(x^2+2x+2).$$

It follows also—and this may be shewn independently—that x^2+x+1 and x^2+2x+2 are without common factors, and it is seen that the L.C.M. of the given expressions is

$$(x+2)(x^2+x+1)(x^2+2x+2).$$

Ex. 2. Find the H.C.F. of $x^3-4x^2-16x-35$ and $x^4+5x^3+14x^2+19x+15$.

$$x^3-4x^2-16x-35\ )\ x^4+5x^3+14x^2+19x+15\ (\ x+9$$
$$\underline{x^4-4x^3-16x^2-35x}$$
$$9x^3+30x^2+54x+15$$
$$\underline{9x^3-36x^2-144x-315}$$
$$66x^2+198x+330$$

We have now to find the H.C.F. of $66x^2+198x+330$ and $x^3-4x^2-16x-35$. We note however that 66 is a factor of the earlier of these, and we know that 66 is not a common factor of the given expressions. Therefore as we seek only common factors we drop the factor 66 and seek the H.C.F. of x^2+3x+5 and $x^3-4x^2-16x-35$.

$$
\begin{array}{r|l}
x^2+3x+5 \,) & x^3-4x^2-16x-35\ (\, x-7 \\
& x^3+3x^2+\ 5x \\ \hline
& \quad -7x^2-21x-35 \\
& \quad -7x^2-21x-35 \\ \hline
\end{array}
$$

Thus x^2+3x+5 is the H.C.F. sought.

It is found that

$$x^3-4x^2-16x-35=(x^2+3x+5)(x-7),$$
$$x^4+5x^3+14x^2+19x+15=(x^2+3x+5)(x^2+2x+3),$$

and it follows that the L.C.M. of the given quantities is

$$(x^2+3x+5)(x^2+2x+3)(x-7).$$

Ex. 3. Find the H.C.F. of $8x^3+22x^2+29x+21$ and $16x^3+18x^2-5x+6$.

$$
\begin{array}{r|l}
8x^3+22x^2+29x+21\,) & 16x^3+18x^2-\ 5x+\ 6\ (\, 2 \\
& 16x^3+44x^2+58x+42 \\ \hline
& \quad -26x^2-63x-36 \\
\end{array}
$$

We have now to find the H.C.F. of $-26x^2-63x-36$ and $8x^3+22x^2+29x+21$. The division will introduce numerical fractions, but this can be avoided if we multiply the latter by 13, since 13 has no factor common to itself and the former. Accordingly we have:

$$
\begin{array}{r|l}
-26x^2-63x-36\,) & 104x^3+286x^2+377x+273\ (\, -4x \\
& 104x^3+252x^2-144x \\ \hline
& \qquad 34x^2+233x+273 \\
\end{array}
$$

Here the division again would introduce numerical fractions, and to avoid this we multiply this remainder by 13 which will not affect the H.C.F. for the reasons given.

$$
\begin{array}{r|l}
-26x^2-63x-36\,) & 442x^2+3029x+3549\ (\, -17 \\
& 442x^2+1071x+\ 612 \\ \hline
& \qquad 1958x+2937 \\
\end{array}
$$

The remainder here may as in Example 2 be divided by the numerical factor 979.

$$\begin{array}{r|l} 2x+3\ ) & -26x^2-63x-36\ (\ -13x-12 \\ & \underline{-26x^2-39x\qquad\quad} \\ & \qquad\quad -24x-36 \\ & \qquad\quad \underline{-24x-36} \end{array}$$

Hence $2x+3$ is the H.C.F. sought.

It is found that

$$8x^3+22x^2+29x+21=(2x+3)(4x^2+5x+7),$$
$$16x^3+18x^2-5x+6=(2x+3)(8x^2-3x+2),$$

and it follows that the L.C.M. of the given expressions is

$$(2x+3)(4x^2+5x+7)(8x^2-3x+2).$$

The actual work of finding the H.C.F. of two expressions may be shortened by employing Horner's method of division, as well as by other artifices that will occur to the student in his work.

Exercises LXXVII

1. Find the H.C.F. of the following, in each case obtaining also the L.C.M.:

(1) x^3+3x^2+5x+6 and x^3+4x^2+6x+9.

(2) x^3+3x^2+5x+6 and x^3-2x^2-9.

(3) x^3+2x^2-2x+3 and x^3+x^2-3x+9.

(4) x^3-2x^2-x-6 and x^3-x^2-7x+3.

(5) x^3+3x^2+3x+2 and x^3+x^2-x+2.

(6) x^3+2x^2-2x+3 and x^3+4x^2+5x+6.

(7) $x^3+6x^2+3x-18$ and $x^3+5x^2-4x+12$.

(8) $9x^3-4x-35$ and $6x^3+19x^2+29x+21$.

(9) $8x^3-22x^2y+21xy^2-9y^3$ and $6x^3-21x^2y+32xy^2-21y^3$.

(10) $2x^3-3x^2+x+15$ and $4x^4-9x^3+18x^2-6x+35$.

2. Shew by taking the sum and the difference of the two expressions

$$x^4 + 3x^3 + 3x^2 + 5x - 12, \quad x^4 - 4x^3 - 19x^2 + 10x + 12,$$

that their H.C.F. is a common factor of

$$2x^3 - x^2 - 16x + 15 \text{ and } 7x^3 + 22x^2 - 5x - 24$$

and find the H.C.F.

3. Shew that two consecutive integers can have no common factor (integral) other than unity.

4. If m and n are two integers with g for G.C.M. and l for L.C.M. shew that

$$mn = lg.$$

5. If m and n are two algebraic expressions with g for H.C.F. and l for L.C.M. shew that

$$mn = lg.$$

EXERCISES LXXVIII

(MISCELLANEOUS)

A

1. Resolve into factors

(i) $15x^2 + 19xy - 56y^2$.

(ii) $15 \quad + 19z \quad - 56z^2$.

2. Find for what value of x the expressions $3x - 5$ and $2x - 1$ are equal.

Compare the expressions for several values of x less this value, and for several values of x greater than this value.

Draw a graph of each of the functions, both on the one sheet, *i.e.*, employing the same axes.

3. The sum of three consecutive odd integers is 57; find these integers.

4. If $f(z) = 1 + z + z^2$, find the value of

$$f(z) \, . f(-z).$$

5. Find the product

$$(x + y - 3z)(y + z - 3x)(z + x - 3y).$$

B

1. Noting that

$$6x^2 + 23xy + 20y^2 = 6x^2 + 8xy + 15xy + 20y^2$$

resolve the first-mentioned expression into factors.

The 23, the coefficient of xy, is resolved into two parts 8 and 15, the product of which is equal to the product of the coefficients 6 and 20, of x^2 and y^2.

[See C, 2 page 77.]

2. Find the result of substituting 6 for x in the expression

$$2x^3 - 15x^2 + 17x - 91$$

(i) by actual substitution;

(ii) by dividing by $x - 6$, preferably by Horner's method, to find the remainder.

3. Shew that two consecutive odd integers can have no common factor other than unity.

4. If $f(x) = 2x^3 + 7x$ shew that $f(-x) = -f(x)$.

5. Shew that the sum of five consecutive integers is five times the middle integer.

C

1. Resolve into factors, employing the suggestion in B 1, the following:

(i) $3x^2 + 17xy + 20y^2$;

(ii) $6x^2 + 7xy - 20y^2$;

(iii) $15x^2 + 19xy - 56y^2$.

2. The temperature, at intervals of 1 hour, starting at 7 o'clock, was recorded as follows:

48°, 50°, 54°, 58°, 65°, 72°, 74°, 76°, 75°, 70°, 64°, 59°.

Exhibit the variation graphically.

3. Explain the difference between the statements:

(i) $(x-1)(x-2) = x^2 - 3x + 2$;

(ii) $(x-1)(x+4) = (x+1)^2$.

4. In the product

$$(3x^2 + 5x - 8)(5x^2 - 3x + 7)$$

find without a complete multiplication, and also by means of it, the coefficient of x^2.

CHAPTER XIII

FRACTIONS

56. Meaning of Fraction. In algebra the fraction is referred back to its source, and is defined as the result of dividing one number by another, or, in other words, as an indicated quotient. A word to bring out the contrast between arithmetical and algebraic fractionality should here be said. The expressions a, $2a^2$, $\frac{9}{8}a^2+\frac{3}{4}a+5$, are all algebraically integral with respect to a, because no division by a, or by any expression involving a, occurs. However, if $a=\frac{2}{3}$ the first two expressions have as values arithmetical fractions, and the third is equal to an integer. Next, the expression $\frac{a}{b}$ is algebraically a fraction with respect to a and b, though a and b may have values that yield an integral result, as for example $a=21$, $b=7$. The expression $\frac{2a^2}{3a}$ is only in appearance a fraction, excluding the value $a=0$, since a division by zero is not to be thought of, and is equivalent to the algebraically *integral* expression $\frac{2}{3}a$.

The terms *numerator* and *denominator* are employed as in arithmetic.

Exercises LXXIX

1. Assign to the involved letters in the following values that will make each of the algebraic fractions an integer:

$$\frac{a^2}{b^2}, \quad \frac{a+b}{b}, \quad \frac{a+b}{c+d}, \quad \frac{a+7}{a+2}.$$

2. State which of the following expressions are integral, and which fractional with respect to x:

$$x+2; \quad 3x^3-9x+2; \quad \frac{2}{3}x^2-\frac{5}{7}x+\frac{1}{11};$$

$$\frac{7}{x-2}; \quad \frac{x+3}{x-5}; \quad \frac{x^3-2x+3}{x+1}; \quad \frac{x^2-2x-3}{(x+1)(x-3)}.$$

57. Fundamental Principle. As in arithmetic, the rule that *if the numerator and denominator of a fraction are multiplied by the same number the value of the fraction is not changed* is of capital importance. Stated symbolically the rule is

$$\frac{a}{b} = \frac{ma}{mb},$$

it being supposed that m is not zero.

The student will note how readily algebra lends itself to the concise expression of a general rule.

To prove the rule, we have

$$\begin{aligned}\frac{ma}{mb} &= (m \times a) \div (m \times b)\\ &= m \times a \div m \div b\\ &= m \div m \times a \div b,\end{aligned}$$

since in a set of multiplications and divisions any change in the order of operations may be made.

$$\begin{aligned}\therefore \frac{ma}{mb} &= 1 \times a \div b\\ &= a \div b\\ &= \frac{a}{b}.\end{aligned}$$

Hence $\frac{a}{b} = \frac{ma}{mb}$.

The following results should be kept in mind:

(i) $\frac{a}{b} = \frac{-1.a}{-1.b} = \frac{-a}{-b} = \frac{+a}{+b}$.

(ii) $\frac{+a}{-b} = \frac{-1.(+a)}{-1.(-b)} = \frac{-a}{+b} = -\frac{a}{b}$.

It must also be remembered that in an expression such as

$$\frac{a-b}{x-y}$$

the numerator and the denominator are each to be regarded as one number, just as if the fraction were written

$$\frac{(a-b)}{(x-y)}.$$

The line indicating the division serves then, also, as a vinculum for the two numbers it separates.

The fact that $(a-b) = -1.(b-a) = -(b-a)$ is of frequent use, as will be seen. Thus we may say

$$\frac{m}{x-a} = \frac{(-1).m}{(-1)(x-a)} = -\frac{m}{a-x}$$

$$\frac{x-a}{x-b} = -\frac{x-a}{b-x} = -\frac{a-x}{x-b} = \frac{a-x}{b-x}.$$

Exercises LXXX

1. Shew that $\frac{ma}{mb} = \frac{na}{nb}$.

2. Write the following fractions so as to exhibit them with the same denominator:

$$\frac{1}{x-y}, \quad \frac{1}{y-x}, \quad \frac{a-b}{x-y}, \quad \frac{a-b}{-x}.$$

3. Shew that the following fractions are equal:

$$\frac{ax}{by}, \quad \frac{(-ma)x}{(mb)(-y)}, \quad \frac{(-a^2)(-x^2)}{(-ab)(-xy)}, \quad \frac{(-ap)(qx)}{(-bq)(py)}.$$

4. Prove that $\frac{(-a)^7}{(-b)^7} = \frac{(-ma)^7}{(-mb)^7} = \frac{a^7}{b^7}$.

5. Shew that

$$\frac{(a-x)(b-y)}{(b-x)(a-y)} = \frac{(x-a)(y-b)}{(x-b)(y-a)} = \frac{(a-x)(y-b)}{(a-y)(x-b)}.$$

6. Write the following fractions so as to exhibit them with the same denominators:

$$\frac{m}{(b-c)(c-a)(a-b)}, \quad \frac{n}{(a-b)(a-c)(b-c)}, \quad \frac{p}{(c-b)(a-c)(b-a)}.$$

7. Shew that $\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}$.

This principle is usefully employed in reducing fractions to their *lowest terms*, by dividing out factors common to numerator and denominator, the factors divided out or cancelled being supposed not zero.

Ex.
$$\frac{x^2-5x+6}{x^3-37x+84}=\frac{(x-2)(x-3)}{(x-3)(x+7)(x-4)}$$
$$=\frac{x-2}{(x+7)(x-4)}$$
$$=\frac{x-2}{x^2+3x-28}.$$

Exercises LXXXI

Reduce the following fractions to their lowest terms:

1. $\frac{6a^2bd}{15ab^2d}$.

2. $\frac{54x^3y^5}{72x^4y^4}$.

3. $\frac{51abxy}{34abmx-85abny}$.

4. $\frac{x^2-5x+6}{x^2-7x+12}$.

5. $\frac{z^2+8z+15}{z^2+14z+45}$.

6. $\frac{x^2-x-20}{x^2+2x-35}$.

7. $\frac{14x^2-29xy+12y^2}{10x^2+xy-24y^2}$.

8. $\frac{10+43p-143p^2}{6+25p-91p^2}$.

9. $\frac{(b-c)(c-a)(a-b)}{(c-b)(a-c)(b-a)}$.

10. $\frac{x^2-4x-21}{x^3-x^2-17x-15}$.

11. $\frac{6-19z+15z^2}{18-42z+20z^2}$.

12. $\frac{x^2-n^2}{2x^3-7x^2n+11xn^2-6n^3}$.

13. $\dfrac{a^6-b^6}{(a^2+b^2)(a^2-ab+b^2)(a^2+ab+b^2)}$.

14. $\dfrac{x^3+y^3}{x^3+x^2y+xy^2+y^3}$.

58. Addition and Subtraction. The development of the theory of algebraic fractions is so much like that of arithmetical fractions that a brief treatment is all that is necessary.

(i) $$\frac{\mathbf{m}}{\mathbf{a}}+\frac{\mathbf{n}}{\mathbf{a}}=\frac{\mathbf{m+n}}{\mathbf{a}}.$$

For $$\frac{m+n}{a} \text{ means } (m+n)\div a,$$

and $$(m+n)\div a=(m\div a)+(n\div a)$$
$$=\frac{m}{a}+\frac{n}{a}.$$

$$\therefore\quad \frac{m}{a}+\frac{n}{a}=\frac{m+n}{a}.$$

Similarly $$\frac{\mathbf{m}}{\mathbf{a}}-\frac{\mathbf{n}}{\mathbf{a}}=\frac{\mathbf{m-n}}{\mathbf{a}},$$

and the rule for adding or subtracting fractions with the same denominator is evident.

(ii) $$\frac{\mathbf{m}}{\mathbf{x}}+\frac{\mathbf{n}}{\mathbf{y}}=\frac{\mathbf{my+nx}}{\mathbf{xy}}.$$

For $$\frac{m}{x}=\frac{my}{xy} \text{ and } \frac{n}{y}=\frac{nx}{xy}.$$

$$\therefore\quad \frac{m}{x}+\frac{x}{y}=\frac{my}{xy}+\frac{nx}{xy}$$
$$=\frac{my+nx}{xy}.$$

Similarly $$\frac{\mathbf{m}}{\mathbf{x}}-\frac{\mathbf{n}}{\mathbf{y}}=\frac{\mathbf{my-nx}}{\mathbf{xy}}.$$

It is easy to pass to the case of more than two fractions and to reach the rule: *To find the algebraic sum of several fractions, bring all the fractions to a common denominator—the lowest common multiple of all the denominators is the simplest—and the sum is the algebraic sum of the numerators of these fractions divided by their common denominator.*

Ex. 1. Find the value of

$$\frac{1}{x-2}+\frac{1}{x-3}-\frac{1}{x-4}.$$

The L.C.M. of denominators is $(x-2)(x-3)(x-4)$.

$$\therefore \quad \frac{1}{x-2}+\frac{1}{x-3}-\frac{1}{x-4}$$

$$=\frac{(x-3)(x-4)}{(x-2)(x-3)(x-4)}+\frac{(x-2)(x-4)}{(x-2)(x-3)(x-4)}-\frac{(x-2)(x-3)}{(x-2)(x-3)(x-4)}$$

$$=\frac{(x-3)(x-4)+(x-2)(x-4)-(x-2)(x-3)}{(x-2)(x-3)(x-4)}$$

$$=\frac{(x^2-7x+12)+(x^2-6x+8)-(x^2-5x+6)}{x^3-9x^2+26x-24}$$

$$=\frac{x^2-8x+14}{x^3-9x^2+26x-24}.$$

As a rule the simplification of the result does not call for the multiplication in the denominator, and the answer might be given thus:

$$\frac{x^2-8x+14}{(x-2)(x-3)(x-4)}.$$

Ex. 2. Find the value of

$$\frac{b+c}{(a-b)(a-c)}+\frac{c+a}{(b-c)(b-a)}+\frac{a+b}{(c-a)(c-b)}.$$

Here the L.C.M. of the denominators is $(b-c)(c-a)(a-b)$.

The sum required $= -\dfrac{b+c}{(a-b)(c-a)} - \dfrac{c+a}{(b-c)(a-b)} - \dfrac{a+b}{(c-a)(b-c)}$

$$= \frac{-(b+c)(b-c)-(c+a)(c-a)-(a+b)(a-b)}{(b-c)(c-a)(a-b)}$$

$$= \frac{-b^2+c^2-c^2+a^2-a^2+b^2}{(b-c)(c-a)(a-b)}$$

$$= \frac{0}{(b-c)(c-a)(a-b)} = 0.$$

In such an example it is well to bring the letters in the different factors into some standard order, so that the relation of such factors as $(b-c)$ and $(c-b)$ may not be overlooked.

EXERCISES LXXXII

Find the following algebraic sums:

1. $\dfrac{a}{xy} + \dfrac{b}{xy}$.
2. $\dfrac{1}{m} + \dfrac{1}{n}$.
3. $\dfrac{1}{p} - \dfrac{1}{q}$.
4. $\dfrac{a}{x} - \dfrac{b}{y}$.
5. $\dfrac{x}{y} - \dfrac{y}{x}$.
6. $\dfrac{a^2}{b} + \dfrac{b^2}{a}$.
7. $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$.
8. $\dfrac{a}{x} + \dfrac{b}{y} + \dfrac{c}{z}$.
9. $\dfrac{1}{yz} + \dfrac{1}{zx} - \dfrac{1}{xy}$.
10. $\dfrac{p}{yz} + \dfrac{q}{zx} + \dfrac{r}{xy}$.
11. $\dfrac{a}{px} + \dfrac{b}{qx} + \dfrac{c}{rx}$.
12. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$.
13. $\dfrac{1}{x-2} + \dfrac{1}{x-3}$.
14. $\dfrac{4}{x+5} - \dfrac{3}{x+7}$.
15. $\dfrac{11}{x-5} + \dfrac{7}{5-x}$.
16. $\dfrac{13}{x+1} - \dfrac{9}{x-1}$.
17. $\dfrac{x-1}{x+1} + \dfrac{x+1}{x-1}$.
18. $\dfrac{x-9}{x+7} + 3$.

19. $\frac{1}{x-2}+\frac{1}{x-3}+\frac{1}{x-4}$.

20. $\frac{1}{x-5}-\frac{2}{x-7}+\frac{1}{x-9}$.

21. $\frac{x+1}{x+8}-\frac{2x+7}{x+10}+\frac{x+11}{x+12}$.

22. $\frac{1}{x-a}-\frac{2}{x}+\frac{1}{x+a}$.

23. $\frac{a^2}{x^2-a^2}+\frac{2a}{a-x}+\frac{a}{x+a}$.

24. $\frac{1}{x^2+x+1}-\frac{3}{x-1}+\frac{4}{x^3-1}$.

25. $\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}$.

26. $\frac{c}{a+b}+\frac{b}{c+a}+\frac{a}{b+c}$.

27. $\frac{x+9}{x^2-3x+2}-\frac{2x+8}{x^2-4x+3}+\frac{x+11}{x^2-5x+6}$.

28. $\frac{x+5}{(x-3)(7-x)}+\frac{x+7}{(x-5)(3-x)}+\frac{x+3}{(x-7)(5-x)}$.

29. $\frac{3z}{15(1-z)(z-5)}-\frac{9z+8}{12(z-5)(2z-7)}-\frac{3z+5}{18(7-2z)(z-1)}$.

30. $\frac{2y+3}{y-1}+\frac{5y}{3-y}-\frac{17y}{1-y}-\frac{4y+19}{y-3}-\frac{5y}{y-1}$.

31. $\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}$.

32. $\frac{y-z}{(x-y)(x-z)}+\frac{z-x}{(y-z)(y-x)}+\frac{x-y}{(z-x)(z-y)}$.

33. $\frac{y+z}{(x-y)(x-z)}+\frac{z+x}{(y-z)(y-x)}+\frac{x+y}{(z-x)(z-y)}$.

34. $\frac{a-b}{a^2-ab+b^2}+\frac{a+b}{a^2+ab+b^2}+\frac{1}{a-b}+\frac{1}{a+b}$.

35. $\frac{1+x+x^2}{1-x+x^2}-\frac{1-x+x^2}{1+x+x^2}+\frac{1-x^3}{1+x^3}-\frac{1+x^3}{1-x^3}$.

59. Multiplication. The product of two fractions is reached as follows:

$$\frac{a}{b} \times \frac{c}{d} = (a \div b) \times (c \div d)$$
$$= a \div b \times c \div d.$$

Therefore, since a set of multiplications and divisions may be taken in any order,

$$\frac{a}{b} \times \frac{c}{d} = a \times c \div b \div d$$
$$= (a \times c) \div (b \times d).$$
$$\therefore \quad \mathbf{\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}},$$

which agrees with the rule of arithmetic however obtained. It is easy to pass to the product of any number of fractions. The result is usually presented in its lowest terms, and it is well to cancel factors common to numerator and denominator, before making the implied multiplications.

Ex. 1. $$\frac{x^2 - 5x + 6}{x^2 - 8x + 15} \times \frac{x^2 + x - 30}{x^2 + 2x - 8} = \frac{(x-3)(x-2)}{(x-5)(x-3)} \times \frac{(x-5)(x+6)}{(x+4)(x-2)}$$
$$= \frac{x+6}{x+4}.$$

Ex. 2. $$\frac{a^3 - b^3}{a^3 + b^3} \times \frac{(a-b)^2 + ab}{(a+b)^2 - ab} \times \frac{a^2 + b^2}{a^2 - b^2}$$
$$= \frac{(a-b)(a^2 + ab + b^2)}{(a+b)(a^2 - ab + b^2)} \times \frac{a^2 - ab + b^2}{a^2 + ab + b^2} \times \frac{a^2 + b^2}{(a+b)(a-b)}$$
$$= \frac{a^2 + b^2}{(a+b)^2} = \frac{a^2 + b^2}{a^2 + 2ab + b^2}.$$

Exercises LXXXIII

Express in simplified form:

1. $\dfrac{a^2b}{ab^2} \times \dfrac{apx}{bqy} \times \dfrac{b^2}{a^2} \times \dfrac{y^2}{x^2}$.

2. $\left(-\dfrac{a}{b}\right)^3 \times \left(-\dfrac{a^2}{b^3}\right) \times \dfrac{b^5x}{a^5y}$.

3. $\dfrac{12}{15}\dfrac{ax^3}{by^2} \times \dfrac{17}{21}\dfrac{y^3}{x^2} \times \dfrac{35}{51}\dfrac{a}{b}$.

4. $\dfrac{2}{3}\dfrac{(a+b)^2}{c^2} \times \dfrac{3}{4}\dfrac{ab}{a+b} \times \dfrac{5}{7}\dfrac{c}{a+b}$.

5. $\dfrac{x^2-3x-28}{x^2+8x+16} \times \dfrac{x^2-14x+45}{x^2-12x+35}$.

6. $\dfrac{35x^2+17x-132}{24x^2-2x-117} \times \dfrac{28x^2-111x+108}{15x^2-7x-88}$.

7. $\dfrac{x^2+(a+b)x+ab}{x^2-c^2} \times \dfrac{x+c}{x+a}$.

8. $\dfrac{(x-a)(b-x)}{(x-c)(d-x)} \times \dfrac{(x+a)(x-d)}{(x+c)(x-b)}$.

9. $\dfrac{a^2-(b-c)^2}{a^2-(b+c)^2} \times \dfrac{b^2-(c+a)^2}{b^2-(c-a)^2} \times \dfrac{c^2-(a-b)^2}{c^2-(a+b)^2}$.

10. $\dfrac{(a-b)^2-(c-d)^2}{(a-d)^2-(b-c)^2} \times \dfrac{(a+b)^2-(c+d)^2}{(a+d)^2-(b+c)^2}$.

11. $\left(\dfrac{1}{x-y} + \dfrac{1}{x+y}\right) \times \dfrac{x^4-y^4}{xy^2+x^2y}$.

12. $\dfrac{a^2-b^2}{a^3+b^3} \times \dfrac{a^2-ab+b^2}{a^2-3ab+2b^2} \times \dfrac{a^2}{b} \times \dfrac{ab-2b^2}{a^2+b^2}$.

13. $\dfrac{1-x^3}{1+x^2} \times \dfrac{9-x^2}{25-x^2} \times \dfrac{5+4x-x^2}{3-2x-x^2}$.

14. $\dfrac{a^2+x^2+3ax(a+x)}{a^2-x^2} \times \dfrac{(a-x)^2}{a^3-x^3-3ax(a-x)}$.

15. $\dfrac{6x^2-xy-15y^2}{4x^2-9xy+5y^2} \times \dfrac{12x^2+xy-20y^2}{6x^2+19xy+15y^2} \times \dfrac{3x^2+2xy-5y^2}{9x^2-3xy-20y^2}$.

60. Division. The rule for division may be found by employing the rule for multiplication.

For
$$\frac{a}{b} \div \frac{c}{d} = \left(\frac{a}{b} \times \frac{d}{c} \times \frac{c}{d}\right) \div \frac{c}{d}$$
$$= \frac{a}{b} \times \frac{d}{c} \times \left(\frac{c}{d} \div \frac{c}{d}\right)$$
$$= \frac{a}{b} \times \frac{d}{c}.$$
$$\therefore \quad \frac{\mathbf{a}}{\mathbf{b}} \div \frac{\mathbf{c}}{\mathbf{d}} = \frac{\mathbf{ad}}{\mathbf{bc}}$$
$$= \frac{\mathbf{a}}{\mathbf{b}} \times \frac{\mathbf{d}}{\mathbf{c}}.$$

Thus it is seen, as might have been independently shewn, that
$$(\mathbf{a} \div \mathbf{b}) \div (\mathbf{c} \div \mathbf{d}) = \mathbf{a} \div \mathbf{b} \div \mathbf{c} \times \mathbf{d}.$$

Exercises LXXXIV

Express in simplified form:

1. $\frac{a^2b}{c^2d} \times \frac{a^2d}{c^2b} \div \frac{a^2x}{b^4y}.$

2. $\frac{3x^2z}{5z^2y} \div \left(\frac{4x^2yz}{5xyz^2} \times \frac{10xyz}{12x^2y^2z}\right).$

3. $\frac{1-x^2}{(1-x)^2} \times \frac{1+x^2}{(1+x)^2} \div \frac{1+2x^2+x^4}{1-2x^2+x^4}.$

4. $\left(a+b-\frac{4ab}{a+b}\right) \div \left(a-b+\frac{4ab}{a-b}\right).$

5. $\left(\frac{a^4-b^4}{a^2-b^2} \div \frac{a+b}{a^2-ab}\right) \div \left(\frac{a^2+b^2}{a+b} \div \frac{a-b}{ab-b^2}\right).$

6. $\left(\frac{x+1}{x+2} - \frac{x-1}{x-2}\right) \div \left(\frac{x+2}{x+1} - \frac{x-2}{x-1}\right).$

7. $\frac{a}{a-x} \div \left(\frac{a^2}{a^2-x^2} \div \frac{a^3}{a^3-x^3}\right).$

8. $\frac{a}{a-x} \div \frac{a^2}{a^2-x^2} \div \frac{a^3}{a^3-x^3}.$

61. Ratios and the Equality of Ratios. In algebra, as in arithmetic, the ratio of two numbers is expressed by the fraction given by dividing the one by the other. Thus the ratio of a to b is denoted by the fraction $\frac{a}{b}$.

When two fractions are equal the ratios denoted by them are equal. For example, if $\frac{a}{b}=\frac{c}{d}$, then *a is to b as c is to d*, or in symbols $a:b::c:d$, and the numbers a, b, c, d, taken in order, are said to be ***proportionals*** or to be ***in proportion***. Such an equality is not different in character from that of, say, $\frac{2}{3}$ and $\frac{4}{6}$, but in algebra it is frequently of an importance not suggested by the arithmetical illustration. As an example consider the following:

Ex. AB is a rod 9 feet high standing on a level court at a distance of 135 feet from a tower XY. A point P on the ground, 15 feet from B, is seen to be in line with A and X. Find the height of the tower.

Let x measure the height of the tower, unit 1 foot. Then from the similarity in shape of the triangles PBA, PYX, it follows that

XY is to PY as AB is to PB,

or in symbols

XY : PY : : AB : PB.

Therefore introducing the measures

$$\frac{x}{15+135}=\frac{9}{15},$$

$$\therefore \quad \frac{x}{150}=\frac{9}{15}.$$

Multiply these equal numbers each by 150.

$$\therefore \quad x=\frac{9}{15}\times 150$$
$$=90,$$

and the height of the tower is 90 feet.

Here it is seen that the idea of ratio has led to the formation of an equation, the proportion being that equation and determining the value of an involved unknown.

Certain theorems in regard to two equal ratios will now be proved.

Let

$$\frac{a}{b} = \frac{c}{d}.$$

Multiply each of the equals by bd: then

$$\frac{a}{b} \times bd = \frac{c}{d} \times cd,$$

i.e., $$ad = bc. \qquad \text{(i)}$$

Thus: *If two fractions are equal the results of cross multiplication are equal*, or in the language of ratios: *In a proportion the product of the extremes is equal to the product of the means.*

Now divide each of the equals in (i) by cd; then

$$\frac{ad}{cd} = \frac{bc}{cd},$$

i.e., $$\frac{a}{c} = \frac{b}{d}. \qquad \text{(ii)}$$

Thus: *If two fractions are equal, the ratio of their numerators is equal to that of their denominators*, which is the converse of the fundamental principle; or otherwise: *If the means of a proportion are interchanged, the rearranged numbers are in proportion.*

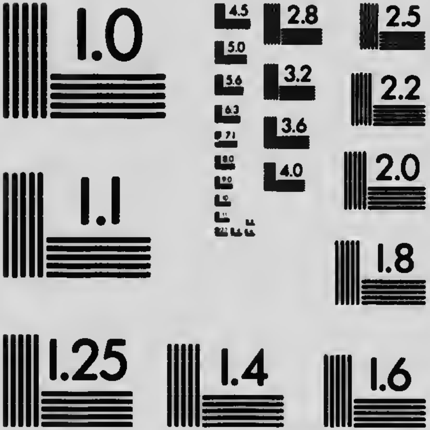
Also divide 1 by each of the given equals; then

$$1 \div \frac{a}{b} = 1 \div \frac{c}{d},$$

i.e., $$\frac{b}{a} = \frac{d}{c}. \qquad \text{(iii)}$$

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



Next, to each of the given equals add 1; then

$$\frac{a}{b}+1=\frac{c}{d}+1,$$

i.e., $$\frac{\mathbf{a+b}}{\mathbf{b}}=\frac{\mathbf{c+d}}{\mathbf{d}}. \qquad \textbf{(iv)}$$

Also from each of the given equals subtract 1; then

$$\frac{a}{b}-1=\frac{c}{d}-1,$$

i.e., $$\frac{\mathbf{a-b}}{\mathbf{b}}=\frac{\mathbf{c-d}}{\mathbf{d}}. \qquad \textbf{(v)}$$

Then from (iv) and (v), by division

$$\frac{a+b}{b}\div\frac{a-b}{b}=\frac{c+d}{d}\div\frac{c-d}{d},$$

and therefore $$\frac{\mathbf{a+b}}{\mathbf{a-b}}=\frac{\mathbf{c+d}}{\mathbf{c-d}}. \qquad \textbf{(vi)}$$

Now multiply each of the given equals by $\frac{m}{n}$; then

$$\frac{ma}{nb}=\frac{mc}{nd},$$

whence by an application of (vi) and (ii)

$$\frac{\mathbf{ma+nb}}{\mathbf{ma-nb}}=\frac{\mathbf{mc+nd}}{\mathbf{mc-nd}}. \qquad \textbf{(vii)}$$

$$\frac{\mathbf{ma+nb}}{\mathbf{mc+nd}}=\frac{\mathbf{ma-nb}}{\mathbf{mc-nd}}. \qquad \textbf{(viii)}$$

Further, multiply each of the equals of (ii) by m and add n; then

$$\frac{ma}{c}+n=\frac{mb}{d}+n,$$

i.e., $$\frac{ma+nc}{c}=\frac{mb+nd}{d}.$$

Whence by (ii)

$$\frac{c}{d}=\frac{ma+nc}{mb+nd}=\frac{a}{b}. \qquad \text{(ix)}$$

Hence: *If two fractions are equal, each of them is equal to the quotient of the sum of any multiples of the numerators, by the sum of the same multiples of corresponding denominators.*

The theorem can be extended so as to apply to any number of equal fractions, as is seen in Ex. 1, below.

All of the results of this section, as well as the methods employed in treating them, should be known so as to be readily available.

It is understood always that, when a fraction is referred to, the denominator is not zero.

Ex. 1. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then each is equal to

$$\frac{la+mc+ne}{lb+md+nf}.$$

A method, not yet illustrated, for treating problems in ratios will be employed.

Let the common value of the three given fractions be denoted by k. Then

$$a=bk, \quad c=dk, \quad e=fk.$$

Therefore

$$\frac{la+mc+ne}{lb+md+nf}=\frac{lbk+mdk+nfk}{lb+md+nf}$$

$$=\frac{k.(lb+md+nf)}{lb+md+nf}$$

$$=k, \text{ the value of } \frac{a}{b}, \text{ or } \frac{c}{d}, \text{ or } \frac{e}{f}.$$

Hence each of the given fractions is equal to the fraction

$$\frac{la+mc+ne}{lb+md+nf}.$$

Ex. 2. If $\frac{a}{x}=\frac{b}{y}=\frac{c}{z}$ and $x+y+z=0$, then $a+b+c=0$.

Denote each of the given equal fractions by k,

$$\therefore \quad a=kx, \quad b=ky, \quad c=kz,$$

$$\therefore \quad a+b+c=kx+ky+kz$$
$$=k(x+y+z).$$

But $x+y+z=0$, so that $k(x+y+z)=0$. Therefore also

$$a+b+c=0.$$

It thus appears that: *If two or more fractions are equal, and the sum of their denominators is zero, then the sum of their numerators is also zero.*

This important principle is of frequent use in solving problems in ratios, but for a time perhaps it will be best not to assume it, establishing it when necessary in the manner shewn.

Exercises LXXXV

1. If $\frac{a}{b}=\frac{c}{d}$, shew that

$$\frac{a^2}{b^2}=\frac{ac}{bd}=\frac{c^2}{d^2}; \quad \frac{c}{a}=\frac{d}{b}; \quad \frac{a+3b}{b}=\frac{c+3d}{d}.$$

2. If $\frac{m}{x}=\frac{n}{y}$, shew that $\frac{2m+3n}{2x+3y}=\frac{7m-5n}{7x-5y}$.

3. If $\frac{x-5}{x+5}=\frac{3}{7}$, shew that $\frac{x}{5}=\frac{10}{4}$.

4. If $\frac{m+n}{m-n}=\frac{p+q}{p-q}$, shew that $\frac{m}{n}=\frac{p}{q}$.

5. If $\frac{x}{a}=\frac{y}{b}$, then $\frac{x^3}{a^3}=\frac{y^3}{b^3}=\frac{x^3+y^3}{a^3+b^3}=\frac{(x+y)^3}{(a+b)^3}$.

6. If $\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}$, shew that

(i) $x+y+z=0$

(ii) $ax+by+cz=0$.

7. If $\dfrac{l}{b+c-a}=\dfrac{m}{c+a-b}=\dfrac{n}{a+b-c}$, shew that

$$(b-c)l+(c-a)m+(a-b)n=0.$$

8. If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$, then each equals $\dfrac{x+y+z}{a+b+c}$ and

$$\frac{x^3}{a^3}=\frac{y^3}{b^3}=\frac{z^3}{c^3}=\frac{(x+y+z)^3}{(a+b+c)^3}=\frac{xyz}{abc}.$$

9. If $\dfrac{a}{p}=\dfrac{b}{q}=\dfrac{c}{r}$, then $\dfrac{a^2+b^2+c^2}{p^2+q^2+r^2}=\dfrac{bc+ca+ab}{qr+rp+pq}$.

10. Solve the equation: $\dfrac{x+7}{x-7}=\dfrac{11}{5}$.

11. If $\dfrac{2y-3z}{b-c}=\dfrac{4z-7x}{c-a}=\dfrac{11x-5y}{a-b}$, then $4x-3y+z=0$.

12. If $\dfrac{2y+3z}{5(b-c)}=\dfrac{5z+4x}{6(c-a)}=\dfrac{7x+3y}{4(a-b)}$, then

$$145x+69y+86z=0.$$

13. If $\dfrac{ax^2+bx+c}{px^2+qx+r}=\dfrac{ax+b}{px+q}$, then each is equal to $\dfrac{c}{r}$.

14. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, shew that

$$\frac{a^3-7a^2c+11c^2e}{b^3-7b^2d+11d^2f}=\frac{(c+e)(e+a)(a+c)}{(d+f)(f+b)(b+d)}.$$

15. If $\dfrac{a}{l}=\dfrac{b}{m}=\dfrac{c}{n}$ write down three expressions that are equal to $\dfrac{a^2+b^2+c^2}{l^2+m^2+n^2}$.

62. Complex Fractions When in the numerator or denominator of a fraction there appear one or more fractions, the fraction is said to be complex. The rules of ordinary fractions apply to them.

Ex. 1. Simplify $\dfrac{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}{\dfrac{a}{x}+\dfrac{x}{a}}$.

The given fraction $=\dfrac{\dfrac{(a+x)^2+(a-x)^2}{a^2-x^2}}{\dfrac{a^2+x^2}{ax}}$

$$=\frac{2(a^2+x^2)}{a^2-x^2}\times\frac{ax}{a^2+x^2}=\frac{2ax}{a^2-x^2}.$$

Ex. 2. Simplify $\dfrac{\dfrac{a}{b}+\dfrac{c}{d}}{1-\dfrac{a}{b}\cdot\dfrac{c}{d}}$.

Here multiply numerator and denominator by bd, as plainly this will clear them of fractions.

Then, given fraction $=\dfrac{ad+bc}{bd-ac}$.

Ex. 3. Simplify $\dfrac{\left(\dfrac{x+y}{x-y}\right)^2-2+\left(\dfrac{x-y}{x+y}\right)^2}{\left(\dfrac{x+y}{x-y}\right)^2+2+\left(\dfrac{x-y}{x+y}\right)^2}$.

Noted that $\left(\dfrac{x+y}{x-y}\right)\left(\dfrac{x-y}{x+y}\right)=1$, it is seen that the numerator and denominator are perfect squares.

Then given fraction

$$=\frac{\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right)^2}{\left(\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}\right)^2}=\frac{\left\{\dfrac{(x+y)^2-(x-y)^2}{x^2-y^2}\right\}^2}{\left\{\dfrac{(x+y)^2+(x-y)^2}{x^2-y^2}\right\}^2}$$

$$= \frac{\dfrac{(4xy)^2}{(x^2-y^2)^2}}{\dfrac{\{2(x^2+y^2)\}^2}{(x^2-y^2)^2}} = \frac{(4xy)^2}{\{2(x^2+y^2)\}^2},$$

multiplying numerator and denominator by $(x^2-y^2)^2$, or, which amounts to the same thing, cancelling the $(x^2-y^2)^2$ in the denominators of the numerator and denominator.

$$\therefore \text{ The given fraction } = \frac{16x^2y^2}{4(x^2+y^2)^2}$$

$$= \frac{4x^2y^2}{x^4+2x^2y^2+y^4}.$$

The example should be worked by multiplying numerator and denominator by $(x-y)^2(x+y)^2$ at the outset to clear of fractions above and below.

Exercises LXXXVI

Simplify the following:

1. $\dfrac{\dfrac{x^2}{y}+\dfrac{y^2}{x}}{\dfrac{x}{y^2}+\dfrac{y}{x^2}}$.

2. $\dfrac{\dfrac{x^2}{y}-\dfrac{y^2}{x}}{\dfrac{x}{y}-\dfrac{y}{x}}$.

3. $\dfrac{\dfrac{x^2}{y^2}+2+\dfrac{y^2}{x^2}}{\dfrac{x^2}{y^2}-\dfrac{y^2}{x^2}}$.

4. $\dfrac{1+\left(\dfrac{x-y}{x+y}\right)^2}{1-\left(\dfrac{x-y}{x+y}\right)^2}$.

5. $\dfrac{\left(\dfrac{x+y}{x-y}\right)^2-1}{\left(\dfrac{x+y}{x-y}\right)^2+1}$.

6. $\dfrac{\dfrac{a^2}{b^2}-2+\dfrac{b^2}{a^2}}{\dfrac{a^2}{b^2}-\dfrac{b^2}{a^2}}$.

7. $\dfrac{\dfrac{1}{1-x}+\dfrac{1}{1+x}}{\dfrac{1}{x-1}-\dfrac{1}{x+1}}$.

8. $\dfrac{\dfrac{1}{a^3}-\dfrac{1}{b^3}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}$.

9. $\dfrac{\dfrac{a+b}{2a}-\dfrac{2b}{a+b}}{\dfrac{a-b}{2b}+\dfrac{2a}{a-b}}$.

10. $\dfrac{\dfrac{2}{x-3}+\dfrac{3}{x-2}}{\dfrac{x+4}{x-3}-\dfrac{x+5}{x-2}}$.

11. $\dfrac{a^3\left(1-\dfrac{x^3}{a^3}\right)}{x^3\left(1-\dfrac{a^3}{x^3}\right)}$.

12. $\dfrac{\left(1+\dfrac{a}{x}+\dfrac{a^2}{x^2}\right)\left(1+\dfrac{a^3}{x^3}\right)}{\left(1-\dfrac{x}{a}+\dfrac{x^2}{a^2}\right)\left(1-\dfrac{x^3}{a^3}\right)}$.

13. $\dfrac{1-\dfrac{x^2+y^2-z^2}{2xy}}{1+\dfrac{y^2+z^2-x^2}{2yz}}$.

14. $\dfrac{a+\dfrac{x^2}{a}+\dfrac{x^4}{a^3}}{\left(a+x+\dfrac{x^2}{a}\right)\left(a-x+\dfrac{x^2}{a}\right)}$.

15. $\dfrac{\dfrac{1}{x^2-5x+6}+\dfrac{1}{x^2-7x+12}}{\dfrac{1}{x^2-6x+8}+\dfrac{1}{x^2-4x+4}}$.

16. $\dfrac{x^2+\left(\dfrac{a}{b}+\dfrac{b}{a}\right)x+1}{\left(1+\dfrac{a}{bx}\right)\left(1+\dfrac{b}{ax}\right)}$.

17. $\dfrac{1-\dfrac{bc}{yz}-\dfrac{ca}{zx}-\dfrac{ab}{xy}}{\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}-\dfrac{abc}{xyz}}$.

18. $\dfrac{\dfrac{ax}{x-a}-\dfrac{bx}{x-b}}{\dfrac{x-b}{(x-a)^2}-\dfrac{x-a}{(x-b)^2}}$.

19. $\dfrac{a}{x+\dfrac{a}{x+\dfrac{a}{x}}}$.

20. $\dfrac{a}{x-\dfrac{a}{x-\dfrac{a}{x}}}$.

21. $a+\dfrac{1}{b+\dfrac{1}{c+\dfrac{1}{d}}}$.

22. $a-\dfrac{1}{b-\dfrac{1}{c-\dfrac{1}{d}}}$.

Exercises LXXXVII

(Miscellaneous)

A

1. Divide:

(i) $6x^2+23x+21$ by $2x+3$;

(ii) $6.10^2+23.10+21$ by $2.10+3$;

(iii) 851 by 23;

and compare the work and results.

2. Express a ft. b in. as a fraction of a yard.

3. If the rate of interest is r per cent., write down the fractions (of the principal) that give

(i) The interest for one year;

(ii) The amount for one year;

(iii) The amount for five years;

(iv) The interest for five years.

4. Simplify

$$\left(m+\frac{mn}{m-n}\right)\left(m-\frac{mn}{m+n}\right)\div\frac{m^2+n^2}{m^2-n^2}.$$

Shew that

$$(m^2-n^2)^2+(2mn)^2=(m^2+n^2)^2.$$

Give to m and n different sets of values and obtain sets of three integers such that the sum of the squares of two of them is equal to the square of the third.

B

1. Shew that

$$[(a+b)x+(a-b)y][(a-b)x-(a+b)y]=(a^2-b^2)(x^2-y^2)-4abxy.$$

2. Express xm. $y$dm. $z$cm. as a fraction of a kilometre.

3. Write down, without going through the complete multiplication, the products:

(i) $(x-5)(x-7)(x-9)$;

(ii) $(x-5y)(x-7y)(x-9y)$;

(iii) $(1-5z)(1-7z)(1-9z)$.

4. If $b+c=c+a=a+b$, shew that $a=b=c$

5. Divide \$25 between A and B in such a way as to ensure that A after giving \$1 to B will still have \$1 more than B.

C

1. Find the product

$$(2x-3y+5z)(3x+4y-6z)(5x-7y-8z).$$

2. If a and b are two numbers, and it is known that neither of them is zero, what can be asserted of their product?

Could the same be asserted of their sum, their difference, their quotient?

3. Shew that the product of any three consecutive integers increased by the middle integer is equal to the cube of the middle integer.

4. If $2s=a+b+c$, shew that

$$(s-a)+(s-b)+(s-c)=s.$$

Illustrate by taking $a=123{\cdot}7$, $b=135{\cdot}4$, $c=151{\cdot}5$.

5. If 1 metre is taken as equal to 39·37 inches, find how many metres there are in x yd. y ft. z in.

D

1. Shew that

$$\frac{1}{(a-b)^2(a-c)^2}+\frac{1}{(b-c)^2(b-a)^2}+\frac{-1}{(c-a)^2(c-b)^2}$$

$$=\frac{-2}{(b-c)(c-a)(a-b)}\cdot\left[\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}\right].$$

2. What factor of

$$x^3 - 4x^2 - 15x + 18$$

is suggested by an examination of the coefficients?

Find all the factors and find for what values of x the expression will vanish.

3. Find the value of

$$3x^4 - 7x^3 + 19x^2 - 23x + 53$$

for $x = 6$.

4. Construct the graphs, referred to the same axes, of the two equations,

$$3x - 4y = 12;$$
$$4x - 3y = 23.$$

5. If

$$\frac{x}{11} = \frac{y}{13} = \frac{z}{24},$$

shew that $x + y = z$.

E

1. Find by actual division the remainder when $ax^2 + bx + c$ is divided by $x - 1$, and by $x + 1$.

2. Find the expansion of $(x+y)^5$ and from the result derive that of $(x-y)^5$.

3. S[illegible] that

$$\left(\frac{\;}{b}\right)^3 = \frac{a^3}{b^3}; \quad \left(\frac{mn}{xy}\right)^5 = \frac{m^5n^5}{x^5y^5}; \quad \left(\frac{1}{pq}\right)^4 = \frac{1}{p^4q^4}.$$

4. Find the product

$$(x^3 - y^3)(x^3 + y^3)(x^6 + y^6)(x^{12} + y^{12}).$$

5 If it is known that the product xy is zero, what can be said in regard to the numbers x and y?

CHAPTER XIV

FRACTIONAL EQUATIONS

63. Fractional Equations. If an equation involves fractional numbers, whether arithmetical or algebraic, the rules for solution adduced in Chapter IX may be applied, for those rules were derived from axioms that apply equally to all numbers. It is not necessary then to establish them anew. A few examples, however, are given to illustrate working methods.

Ex. 1. Solve

$$\frac{5}{7}(x-13)+\frac{3}{5}(x-18)=\frac{2}{3}(x-1).$$

We may here add the fractions on the left and find that

$$\frac{25(x-13)+21(x-18)}{35}=\frac{2(x-1)}{3}$$

or

$$\frac{46x-703}{35}=\frac{2x-2}{3}.$$

Then by the rule of cross multiplication, which means merely multiplying the two equal numbers by 3×35, we find that

$$3(46x-703)=35(2x-2)$$

$$\therefore \quad 138x-2109=70x-70$$

$$\therefore \quad 138x-70x=2109-70$$

$$\therefore \quad 68x=2039$$

$$\therefore \quad x=29\tfrac{67}{68}.$$

Or we may—as in general is the readiest way—at once multiply the numbers, given as equal, by $7 \times 5 \times 3$, the L.C.M. of the denominators, which at one step clears the equation of fractions. We have then

$$15 \times 5(x-13)+21 \times 3(x-18)=35 \times 2(x-1)$$

$$\therefore \quad 75x - [illegible] + 63x-1134=70x-70$$

$$\therefore \quad 75x+63x-70x=975+1134-70$$

$$\therefore \quad 68x=2039$$

$$\therefore \quad x=29\tfrac{67}{68}.$$

Ex. 2. Solve

$$\frac{3}{3x-1}-\frac{4}{4x-3}=\frac{20}{4x+7}-\frac{15}{3x+5}.$$

Here it is best first to effect the subtractions indicated on each side. Hence we have,

$$\frac{3(4x-3)-4(3x-1)}{(3x-1)(4a-3)}=\frac{20(3x+5)-15(4x+7)}{(4x+7)(3x+5)}$$

$$\therefore \quad \frac{-5}{(3x-1)(4x-3)}=\frac{-5}{(4x+7)(3x+5)}.$$

Then dividing each of the equals by -5, and cross multiplying we have

$$(4x+7)(3x+5)=(3x-1)(4x-3).$$

$$\therefore \quad 12x^2+47x+21=12x^2-13x+3$$

$$\therefore \quad 47x+13x=-21+3$$

$$\therefore \quad 60x=-18$$

$$\therefore \quad x=\tfrac{3}{10}.$$

It would be well to work the example by multiplying by the L.C.M. of the denominators, with a view to finding reasons for preferring the procedure here given.

Ex. 3. Solve

$$\frac{1}{x+a}-\frac{1}{x+b}=\frac{1}{a-x}-\frac{1}{b-x}.$$

Here we have to do with a literal equation, and we have to keep in mind that a and b are supposed to be given, and therefore known numbers. We are required to find the value of x that will satisfy the given relation, and this value may involve, or be given in terms of, a and b. Proceeding as in this example just treated we have

$$\frac{(x+b)-(x+a)}{(x+a)(x+b)}=\frac{(b-x)-(a-x)}{(a-x)(b-x)}.$$

$$\therefore \quad \frac{b-a}{(x+a)(x+b)}=\frac{b-a}{(a-x)(b-x)}.$$

We shall suppose that $b-a$ is not zero, *i.e.*, that b is not equal to a. We should, however, examine what the equation would be were b equal to a; in this case it is seen that each side is zero whatever be x, so that the equation would be satisfied for every value of x. Hence, supposing a not equal to b we have, after division by $b-a$,

$$(a-x)(b-x)=(x+a)(x+b)$$

$$\therefore \quad ab-(a+b)x+x^2=x^2+(a+b)x+ab$$

$$\therefore \quad 2(a+b)x=0.$$

Therefore, if $a+b$ is not zero,

$$x=0.$$

Should it be that $a+b$ is equal to zero, or $a=-b$, it is readily seen that the given equation holds whatever be x.

In solving the equations given below, the student should work to a concise form of presentation which indicates, with the least possible verbal explanation, the course of the reasoning up to the result sought. It would be well also to test the results by substituting them in the original equations; practice in algebraic work is thus afforded, and the real meaning of the equation is impressed.

Exercises LXXXVIII.

1. Solve the following numerical equations:

(1) $\frac{2}{3}(x+9)+\frac{5}{6}(x+7)=\frac{3}{4}(x+2)+\frac{1}{3}(x+23).$

(2) $\frac{7}{11}(x-5)-\frac{5}{8}(x-13)=\frac{5}{12}(x+3)-\frac{3}{11}(x+7).$

(3) $\frac{3}{x-5}-\frac{5}{x-7}=\frac{8}{7-x}-\frac{1}{5-x}.$

(4) $\frac{2x+7}{3}-\frac{8x+19}{12}=\frac{5x+11}{7x+9}.$

(5) $\frac{5x+11}{8}-\frac{9x+5}{13x+1}=\frac{15x+13}{24}.$

(6) $\frac{7}{x-11}-\frac{13}{x-15}=\frac{7}{x+2}-\frac{13}{x-8}.$

(7) $\frac{3}{5x-7}=\frac{7}{11x+5}$.

(8) $\frac{x-1}{x-2}+\frac{x-5}{x-3}=2$.

(9) $\frac{2x}{x-7}-2=\frac{13}{x-8}$.

(10) $\frac{1}{x-3}-\frac{5}{2x+3}=\frac{1}{x+2}-\frac{5}{2x+7}$.

2. Solve the following equations:

(1) $\frac{5}{2x+3a}=\frac{7}{8x+4a}$.

(2) $\frac{2a}{3x+7a}=\frac{3a}{7x-5a}$.

(3) $\frac{x+2a}{x+b}=\frac{3x+7a}{3x+2b}$.

(4) $\frac{a}{b}\cdot x+\frac{b}{a}\cdot x=a+b$.

(5) $\frac{x}{a+b}+\frac{x}{a-b}=\frac{a}{a+b}+\frac{b}{a-b}$.

(6) $\frac{1}{x-2a}+\frac{1}{x-2b}=\frac{2}{x+a+b}$.

(7) $\frac{x+a}{x-a}=\frac{c+d}{c-d}$.

(8) $\frac{a(x-b)}{c(x-d)}=\frac{ax+cd}{cx+ab}$.

(9) $\frac{a}{b+cx}-\frac{b}{a+cx}=\frac{a-b}{c+cx}$.

(10) $\frac{x+4a+b}{x+a+b}+\frac{4x+a+2b}{x+a-b}=5$.

3. Divide 27 into two parts such that one-fourth of the one may be equal to one-fifth of the other.

4. Divide 43 into two parts such that one-third of the one may exceed one-fifth of the other by unity.

5. *A*'s money exceeds *B*'s by \$12. *A* increases his by one-seventh and *B* increases his by one-fifth and then *A*'s money is to *B*'s as 4 to 3. Find how much each had before the increase.

6. The denominator of a certain fraction exceeds its numerator by 4, and it is known that if each term of the fraction is increased by 9 the resulting fraction is equal to $\frac{3}{4}$. Find the fraction.

7. *A*'s age is three-fourths that of *B*, and in 5 years it will be four-fifths that of *B*. Find their ages.

8. *B*'s age exceeds *A*'s age by 30 years and in 15 years *A*'s age will be one-half *B*'s age. Find the age of each.

9. Divide \$75 between A and B giving A five dollars more than three-fourths of the sum given B.

10. When \$75 are divided between A and B, it is found that two-thirds of A's share increased by \$12 is equal to five-eighths of B's share. Find the share of each.

11. A line a inches long is divided into two parts and it is noticed that two-thirds of one part and one-fourth of the other make up one-half the line. Find the parts.

12. On a straight line, the distances of the points A and B from O are 7 and 11 inches. A point P between A and B divides AB in the ratio of $3:5$. Find the distance of P from O.

13. Divide a straight line a inches long into two parts such that the one increased in length c inches is to the other diminished by c inches as $2:3$.

14. If O, A, B, are three points on a straight line, and OA, OB are a and b inches in length, find the distance from O of the point P, between A and B, if it is known that $AP : PB :: m : n$.

15. Find three consecutive integers such that the sum of the first two is five-sixths of twice the third.

16. A train makes the first 90 miles of a run of 150 miles at a rate 5 miles an hour faster than its usual rate; owing to an accident the remainder of the journey has to be made at a rate 5 miles an hour slower than its usual rate, and it is found that the time of the run is the same as if made at the usual rate. Find the usual rate,

17. The daily wage of a boy is 5 cents more than one-half the daily wage of a man. In one day the wages of 10 men and 7 boys is to the wages of 7 men and 10 boys as 233 is to 209. Find the daily wage of a man and of a boy.

18. A and B walk round a block 6 miles square. A's rate is one mile an hour greater than B's, and it is found that the time required by B is to that required by A as 4 is to 3. Find the rate of each, and shew that for a block m miles square the ratio of their times would be also as 4 to 3.

19. Find three consecutive odd numbers such that the sum of 6 times the first and 4 times the last is to 10 times the middle number as 53 is to 55.

EXERCISES LXXXIX
(MISCELLANEOUS)

A

1. A rectangular field is a yd. long and b yd. wide. A boy runs round the field 7 times; find how far he runs.

Find also how far he travels in running round the field n times.

2. It is known that $3x+2a$ exceeds $2x+3b$ by c, where a, b, c are supposed known. Find the value of x and verify that the value found satisfies the condition stated.

3. Resolve into factors:

(i) $ab+bx+ay+xy$.

(ii) $35x^2-37xy-88y^2$.

4. Represent graphically the relation between x and y given by the equation

$$4y=3x+12.$$

5. It is proposed to find three consecutive integers such that the sum of the first and the third is twice the second. Construct the equation and shew that it is satisfied for every value of the unknown.

B

1. A rectangular block a in. by b in. by c in. is plated at h cents a sq. in. Find the cost.

2. It is known that $x+a$ exceeds $x+b$ by c. Shew that this fact tells nothing about x but gives merely a relation among a, b, c,—these numbers being supposed known.

If a is taken as 5 and b as 3, find what c must be.

3. Resolve into factors

(i) $ab-bx-ay+xy$.

(ii) $35x^2+37xy-88y^2$.

4. $X'OX$ and $Y'OY$ are two intersecting roads, OX running east and OY north. At H, a point 1 mile east of O, a third road crosses $X'OX$ obliquely, bearing 3 miles north for 4 east. A person starting from H walks until he reaches a point x miles east of the road $Y'OY$; find how far he is from the road $X'OX$.

5. Find the product

$$(a+b)(a^2+b^2)(a^4+b^4).$$

C

1. The circumference of a circle is given in terms of its radius by the formula

$$c = 2\pi r$$

where π is a certain definite number. By a careful measurement the circumference of a circle of radius of measure ·5 is found to measure 3·14. What information does this give as to the value of π?

2. Resolve into factors:

(i) $(x^2+5x)^2+10(x^2+5x)+24$.

(ii) p^6+q^6.

3. A certain fraction has a denominator 2 greater than its numerator. When 7 is added to each term, the resulting fraction differs as much from unity as it does from the earlier fraction. Find the fraction.

4. Exhibit graphically the values of $\frac{4}{5}x+\frac{3}{5}$ for varying values of x.

5. Explain what is meant by saying that $(x-2)(x-3)$ and x^2-5x+6 are *identically equal*.

D

1. The area a of a circle is given in terms of the radius r by the formula

$$a = \pi r^2$$

where π is a certain definite number, while a and r refer to any circle. Shew that this implies that the areas of two circles are to each other as the squares of their radii.

2. Resolve into factors:

(i) $(x^2-3x)^2-8(x^2-3x)-20$.

(ii) $m^2p^2+n^2q^2+m^2q^2+n^2p^2$.

3. Find the greatest common measure of 44117 and 56363 explaining the reason for each step in the work.

4. Solve

$$\frac{1}{x-1}-\frac{1}{x-7}=\frac{1}{x+1}-\frac{1}{x-5}.$$

5. Shew that the sum of the two numbers obtained by writing two digits in their two different orders is divisible by 11, and that their difference is divisible by 9.

CHAPTER XV

SIMULTANEOUS EQUATIONS

65. Introductory. Consider the following problem:

Problem. The cost of 2 yd. of one kind of cloth and 3 yd. of another was $1.20. Had the prices been interchanged the cost would have been $1.30. Find the prices.

Let x and y measure the prices, (unit 1 cent).

Then from what is given

$$\left.\begin{aligned} 2x + 3y &= 120, \\ 3x + 2y &= 130. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad \text{(i)}$$

Thus the problem leads to *two equations in two unknowns, i.e.* to two equations which are to be satisfied by a certain value of x and a certain value of y. We expect then to elicit from the equations the value of x and the value of y which will satisfy the first equation and *at the same time* the second equation.

To make the coefficients of y the same in the two equations, multiply the members of the first equation by 2 and those of the second by 3 to find

$$\left.\begin{aligned} 4x + 6y &= 240, \\ 9x + 6y &= 390, \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad \text{(ii)}$$

which two equations mean no more and no less than the two from which they were derived.

Keeping in mind that x—as also y—denotes the same number in each equation, we have by subtraction,

$$9x - 4x = 390 - 240,$$

or $$5x = 150,$$

whence $$x = 30.$$

To find y, we may obtain from the original equations two equations, as (ii), in which the coefficients of x are the same, 3 and 2 being taken as multipliers. Or we may substitute the value of x, just found, in *either* of the original equations,—say the first. Then

$$30 \times 2 + 3y = 130,$$

an equation, involving the one unknown y, which we know how to solve. We have then

$$\begin{aligned} 3y &= 120 - 30 \times 2 \\ &= 60, \\ \therefore \quad y &= 20. \end{aligned}$$

The prices are therefore 30 cents and 20 cents a yard.

The values

$$\left.\begin{aligned} x &= 30, \\ y &= 20. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(iii)}$$

constitute the *solution* of the equations or are the *roots*. The three *systems* (i), (ii), (iii) are *equivalent*, each being derivable from any other.

The solution might have been effected otherwise. For from the equations we have

$$\left.\begin{aligned} y &= \tfrac{1}{3}(120 - 2x), \\ y &= \tfrac{1}{2}(120 - 3x). \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad \text{(iv)}$$

But y denotes the same number in each, so that we must have

$$\tfrac{1}{3}(120 - 2x) = \tfrac{1}{2}(130 - 3x),$$

an equation involving only one unknown, or, *multiplying through* by 6,

$$\begin{aligned} 2(120 - 2x) &= 3(130 - 3x) \\ \therefore \quad 240 - 4x &= 390 - 9x \\ \therefore \quad 9x - 4x &= 390 - 240 \\ 5x &= 150 \\ \therefore \quad x &= 30. \end{aligned}$$

Then y, which is equal to either of the equal numbers on the right in (iv), is given by

$$\begin{aligned} y &= \tfrac{1}{3}(120 - 2 \times 30) \\ &= 20. \end{aligned}$$

The difference between the two methods is not great, the latter emphasizing, perhaps, the fact that each equation gives the same value of y, when the value of x, already found, is substituted in it.

Indeed, a third way of regarding the equations may be given. From the first equation we have

$$y = \tfrac{1}{3}(120 - 2x).$$

Therefore, since y, as also x, denotes the same number in each equation, we have, by substituting in the second equation the value of y just found in terms of x from the first,

$$3x + 2 \times \tfrac{1}{3}(120 - 2x) = 130,$$

an equation involving only one unknown or, by multiplying through by 3,

$$9x + 2(120 - 2x) = 390.$$

$$\therefore \quad 9x - 4x = 390 - 240$$

$$\therefore \quad 5x = 150$$

$$\therefore \quad x = 30,$$

and the value of y may be found as before.

Two equations, involving two unknown quantities, as those in (i), are said to be *simultaneous*, or to be a *simultaneous system, in two unknowns*.

In the first method of solution, the central idea is the *elimination* of y from the two equations, *i.e.* the finding from them of an equation in which y does not occur; in the second method the central idea is the *identification* of the two values of y, found, in terms of x, from the two equations; and in the third, the *substitution* in the one equation, of the value of y found, in terms of x, from the other equation, the result being an equation which involves x only.

The equations in the following exercises may be regarded as having come from certain problems. They should be worked by each of the three methods given, and the results found should be verified by substitution in the original equations.

Exercises XC

1. Solve the following sets of simultaneous equations:

(1) $\left.\begin{aligned} x+y&=11,\\ x-y&=1. \end{aligned}\right\}$ (2) $\left.\begin{aligned} 2x+3y&=18,\\ 3x+5y&=29. \end{aligned}\right\}$

(3) $\left.\begin{aligned} 4x+3y-41&=0,\\ 3x+5y-50&=0. \end{aligned}\right\}$ (4) $\left.\begin{aligned} 11x-5y&=43,\\ 7x-3y&=29. \end{aligned}\right\}$

(5) $\left.\begin{aligned} 5y-12z&=5,\\ 11y-13z&=78. \end{aligned}\right\}$ (6) $\left.\begin{aligned} 8u+11v&=101,\\ 11u-4v&=5. \end{aligned}\right\}$

(7) $\left.\begin{aligned} 3l+5m&=2,\\ 4l+7m&=1. \end{aligned}\right\}$ (8) $\left.\begin{aligned} 15z-11w&=145,\\ 8z+9w&=3. \end{aligned}\right\}$

(9) $\left.\begin{aligned} 3x+11y+103&=0,\\ 2x-3y-14&=0. \end{aligned}\right\}$ (10) $\left.\begin{aligned} 7y-3z&=3y+2z+2,\\ 3y-4z&=3z-3y. \end{aligned}\right\}$

2. Solve the following sets:

(1) $\left.\begin{aligned} 2x-3y-7&=x-4y+11,\\ 3x-11y-4&=4x-5y+9. \end{aligned}\right\}$ (2) $\left.\begin{aligned} 3x-5y&=5x-8y,\\ 3x-11&=7y-23. \end{aligned}\right\}$

(3) $\left.\begin{aligned} 2y-5z+19&=7y-8z+53,\\ 2y-7z+11&=5y-6z+29. \end{aligned}\right\}$ (4) $\left.\begin{aligned} 5z-23x&=54-13z,\\ 11x-15z&=37-5x \end{aligned}\right\}$

(5) $\left.\begin{aligned} \tfrac{1}{3}(3u-29)&=\tfrac{1}{4}(7v-31),\\ \tfrac{1}{5}(4u-5v)&=\tfrac{1}{6}(7u-10v). \end{aligned}\right\}$ (6) $\left.\begin{aligned} \tfrac{2}{3}x+\tfrac{5}{6}y&=\tfrac{7}{8},\\ \tfrac{5}{8}x-\tfrac{7}{12}y&=\tfrac{5}{9}. \end{aligned}\right\}$

(7) $\left.\begin{aligned} (x+2y+3):(2x+3y+4)&::7:12,\\ 5x:7y&::8:15. \end{aligned}\right\}$ (8) $\left.\begin{aligned} \tfrac{1}{5}x+\tfrac{3}{8}y&=\tfrac{3}{8}x+\tfrac{1}{5}y,\\ \tfrac{1}{4}x+\tfrac{1}{5}y&=\tfrac{11}{12}. \end{aligned}\right\}$

(9) $\left.\begin{aligned} \tfrac{2}{3}(x-1)+\tfrac{3}{4}(y-3)&=\tfrac{1}{2}(x+y),\\ 2\tfrac{1}{2}x+3\tfrac{1}{3}y&=25. \end{aligned}\right\}$

(10) $\left.\begin{aligned} 7z-11x+3&=5z-8x+20,\\ &=11z+13x+47. \end{aligned}\right\}$

3. Solve the following sets, the earlier letters of the alphabet signifying known numbers:

(1) $\left.\begin{aligned} 2x+3y&=11a,\\ 5x-y&=7a. \end{aligned}\right\}$ (2) $\left.\begin{aligned} x+y&=a+b,\\ x-y&=a-b. \end{aligned}\right\}$

(3) $\left.\begin{aligned} 7y-12z&=4c,\\ 8y-9z&=5d. \end{aligned}\right\}$ (4) $\left.\begin{aligned} ax+by&=2ab,\\ x+y&=a+b. \end{aligned}\right\}$

(5) $\left.\begin{aligned}\frac{x}{a}+\frac{y}{b}&=1,\\ \frac{x}{b}-\frac{y}{a}&=1.\end{aligned}\right\}$

(6) $\left.\begin{aligned}\frac{3x}{a}+\frac{5y}{b}&=5,\\ \frac{5x}{b}-\frac{7y}{a}&=1.\end{aligned}\right\}$

(7) $\left.\begin{aligned}ax+by&=l,\\ cx+dy&=m.\end{aligned}\right\}$

(8) $\left.\begin{aligned}x:y&::m:n,\\ x+3n&=y+2m.\end{aligned}\right\}$

(9) $\left.\begin{aligned}a(u-b)+b(v-a)&=5ab,\\ 3b(u-b)+2a(v-a)&=6(a^2+b^2).\end{aligned}\right\}$

(10) $\left.\begin{aligned}lx+my&=2lm,\\ l^2x+m^2y&=lm(x+y).\end{aligned}\right\}$

4. A man bought a certain number of yards of cloth at 75 cents a yard, and a certain number of yards at $1.25 a yard, the total cost being $11.25. If the cloth had been bought at a uniform price of $1.00 a yard the cost would have been $11.00. How many yards of each kind were bought?

5. In one day 5 men and 8 boys earn $13.40 while 9 men and 13 boys earn $23.00. Find the daily wage of a man and of a boy.

6. Five years hence *A*'s age will be four-fifths of *B*'s age, and ten years hence *A*'s age will be five-sixths of *B*'s age. Find the present age of each.

7. An integer expressed by two digits is 36 less than the integer formed by reversing the order of the digits, and the sum of these integers is 110. Find the integer first taken.

8. Two sums of money are lent, one at 4 per cent., the other 5 per cent. per annum, and yield a yearly income of $82. When the rate on each is changed to $4\frac{1}{2}$ per cent. the yearly income is reduced $1. Find the sums.

9. If *A* were to give $5 to *B*, then *A* would have only one-half as much as *B*, but if *B* were to give $10 to *A*, then *A* would have four-fifths as much as *B*. Find how much each has.

10. Two boats start at the same time from *A* to *B*, a sail of 120 miles. At the end of 4 hours the faster is as far from *B* as the slower is from *A*, and at the end of five hours, the faster is midway between the slower and *B*. Find the rates of sailing.

66. Significance of a System of Two Equations in Two Unknowns. While the methods of solving equations of the kind now being studied can be supposed known, a further examination of them with a view to grasping the full *meaning* or *force* or *office* of the equation is necessary. Take, for example, the two equations

$$2x + 3y = 12,$$
$$3x + 2y = 13,$$

or, which is the same thing,

$$2x + 3y - 12 = 0,$$
$$3x + 2y - 13 = 0.$$

Turning from the *equations*, consider first the *expressions* or *functions*

$$2x + 3y - 12, \text{ and } 3x + 2y - 13.$$

Here we look upon x and y as variables, each of them being free to take or to be given any value we please. As we give x and y different sets of values, the expressions will be expected to take different values. Thus for $(x = 3, y = 4)$, for $(x = -2, y = +1)$, for $(x = 0, y = 5)$, the values of the former are 6, -13, 3, and of the latter are 4, -17, -3.

Limiting the attention for the moment to the one expression $2x + 3y - 12$, consider now by itself the one equation involving it,

$$2x + 3y - 12 = 0.$$

Plainly, if this equation is to be satisfied, *i.e.* if x and y are to make the expression $2x + 3y - 12$ take the value zero, x and y may not have any values we please. But plainly, too, *one* of the numbers x and y, say x, may be given any value, and then the equation determines the value of the other, y. Thus if x is taken as 1,

$$2 \times 1 + 3y - 12 = 0$$

whence

$$y = 3\tfrac{1}{3}$$

and ($x = 1$, $y = 3\tfrac{1}{3}$) is a solution. Accordingly there is no limit to the number of solutions of this one equation. The following table gives a number of solutions:

$x =$	-1	0	$\tfrac{1}{2}$	1	2	3	4	..
$y =$	$4\tfrac{2}{3}$	4	$3\tfrac{2}{3}$	$3\tfrac{1}{3}$	$2\tfrac{2}{3}$	2	$1\tfrac{1}{3}$	..

Next, if we consider only the equation

$$3x + 2y - 13 = 0,$$

it is seen as before that not every value of x and y will satisfy the equation, but that one of these may be taken as we please, and then the other is determined by the equation. The following table gives a number of solutions:

$x =$	-1	0	$\tfrac{1}{2}$	1	2	3	4	..
$y =$	8	$6\tfrac{1}{2}$	$5\tfrac{3}{4}$	5	$3\tfrac{1}{2}$	2	$\tfrac{1}{2}$	..

If now we ask for a value of x and a value of y that will satisfy the *two* equations considered, we see that in the table there occurs one such set, namely ($x = 2$, $y = 3$). This is the set of values that can be found by solving the equations, as simultaneously true. For, from

$$\left.\begin{aligned} 2x + 3y - 12 = 0 \\ 3x + 2y - 13 = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad \text{(i)}$$

we find the equivalent set

$$4x + 6y - 24 = 0$$
$$9x + 6y - 39 = 0$$

whence, by subtraction

$$5x = 15,$$

so that x *must* equal 3, and consequently, as we have seen, y *must* equal 2.

Thus, noting that the equations are of one dimension in the two numbers x and y, we see that

(*a*) *One linear equation in x and y, has solutions unlimited in number: the equation while not allowing x and y to have any values we please, does not determine x and y.*

(*b*) *Of two linear equations in x and y, each has solutions unlimited in number: among the solutions of each, there is one and only one solution that belongs to both, so that the two linear equations determine x and y.*

Exercises XCI

1. In the expression $5x - 3y + 2$, substitute the following values of (x, y):

$$(2, 3),\quad (3, 4),\quad (0, 0),\quad (-1, +2),\quad (-\tfrac{2}{3}, -\tfrac{3}{4}),\quad (-3, -5).$$

2. Find five sets of values of (x, y) that will make $5x - 3y + 2$ equal to zero.

3. Find five sets of values of (x, y) that will make $5x - 3y + 2$ equal to 23.

4. Find five sets of values of (x, y) that will make $3x - 4y - 3$ equal to zero, and substitute those values in the expression $x + y - 8$.

5. Shew that there is only one set of values of (x, y) that will make $3x - 4y - 3$, and $x + y - 8$ both equal to zero.

6. Shew that if x and y are two numbers that satisfy the equation

$$5x - 3y = 16$$

then x and y cannot satisfy

$$15x - 9y = 50.$$

7. Point out the difficulty that presents itself when it is proposed to solve the set of equations

$$3x + 7y = 27,$$
$$12x + 28y = 100.$$

(NOTE: The equations of such a set are said to be *inconsistent.*)

8. Point out the difficulty that presents itself when it is proposed to solve the set of equations

$$3x + 7y = 27,$$
$$12x + 28y = 108.$$

Can values of x and y be found to satisfy these two equations?

9. Find the two numbers which satisfy the conditions that twice the first added to three times the second gives the sum 49 while three times the first exceeds twice the second by 2.

10. There are two numbers such that the sum of twice the first and three times the second is 49, while the sum of twice a number 2 greater than the first, and three times a number 3 greater than the second is 62. Point out the difficulty that presents itself, and shew that such numbers exist.

11. A man when asked the dimensions of his garden replied that if it were 5 yd. longer and 5 yd. wider it would be 375 sq. yd. larger, and that if it were 5 yd. less in each dimension it would be 375 sq. yd. smaller. He was told that this could not be the case, and, correcting himself, substituted 325 sq. yd. for the latter 375 sq. yd. Examine the problem.

67. Graphic Representation. A single equation of the first degree in x and y, as

$$2x + 3y = 12,$$

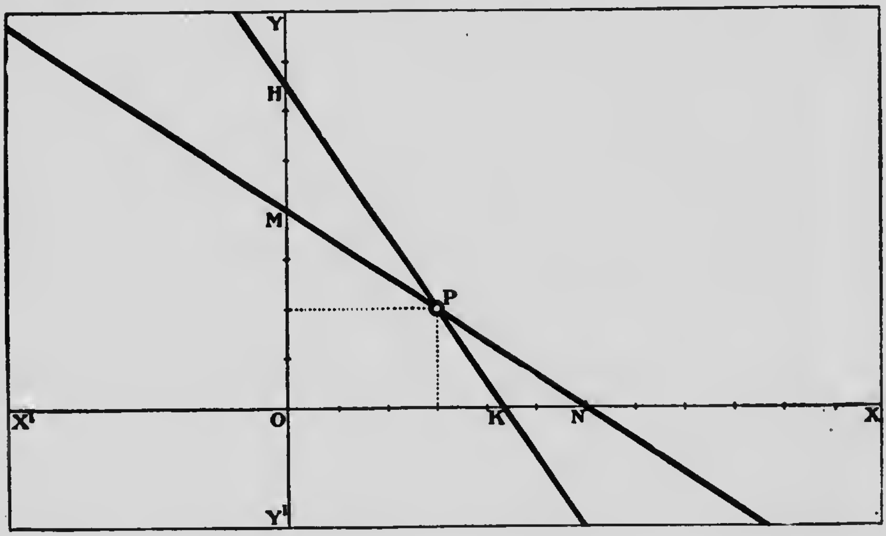
may be written so as to shew directly that it gives y in terms of x, or as a function of x. Here we have

$$3y = -2x + 12,$$

whence

$$y = -\tfrac{2}{3}x + 4.$$

On the right, in this last equation, we think of x as being ***capable of taking any value whatever***, ***i.e.*** as being a ***variable:*** when x is given any value, the equation gives the value of y corresponding to it. This is only re-stating what has been said in the preceding article. Now $-\frac{2}{3}x+4$, which from the equation means the value of y, is a function of x of the first degree, and, as has been seen in Chapter XI, can be represented graphically by a straight line. This straight line MPN (see figure) is such



that on it lies each point whose x and y is one of the unlimited many solutions of the equation

$$2x + 3y = 12.$$

It is spoken of as *the line whose equation is*

$$2x + 3y = 12$$

and this means neither more nor less than what has been said.

Consider next the equation

$$3x+2y=13$$

which may be written in the forms

$$2y=-3x+13,$$

$$y=-\tfrac{3}{2}x+6\tfrac{1}{2}.$$

This equation is represented graphically by the line *HPK*, and the sets of values of x and y, in number unlimited, that satisfy this equation, are represented by the x and y of points on this line.

The two straight lines intersect at the point *P*, and it follows then that the x and y of the point *P* satisfy each of the equations, and *P* is the only point of which this may be said. Thus the fact, *already known*, that there is only one set of values of x and y that satisfy two equations of the kind being studied, is reflected, in the graphical representation, by the fact that two straight lines intersect in only one point.

Exercises XCII

1. Construct the line that represents graphically the equation

$$2x-3y=5.$$

From the figure find the value of y corresponding to the values -1, $-\frac{1}{2}$, 0, 3, 4, of x, and the values of x corresponding to the values of -1, 0, 3, 4, of y, in each case testing by substituting in the equation.

2. Construct the line that represents graphically the equation

$$3x-4y=12.$$

From the figure find three sets of values of x and y that satisfy the equation.

Mark the point $(x=5,\ y=2)$ and substitute $(x=5,\ y=2)$ in the equation.

Mark the point $(x=8,\ y=3)$ and substitute $(x=8,\ y=3)$ in the equation.

3. Construct the line whose equation is

$$2x - 3y = -6.$$

Noting that the equation is the equivalent of

$$y = \frac{2}{3}x + 2$$

find what is denoted in the figure by $\frac{2}{3}$ and 2, the coefficients on the right.

4. Construct the line whose equation is

$$3x + 4y = 24$$

and examine as in the preceding example.

5. Construct the lines represented by the equations

$$3x + 4y = 24,$$
$$4x - 3y = 7$$

and from the figure find the set of values of x and y that will satisfy *simultaneously* these equations.

6. Construct the lines that represent the equations

$$4x + 5y = 40,$$
$$x + y = 1,$$

and from the figure find three sets of values of x and y that will satisfy the second equation without satisfying the first.

7. Construct the lines that represent the equations

$$3x - 4y = 3,$$
$$15x - 20y = 15.$$

8. Construct the lines whose equations are

$$3x - 4y = 3,$$
$$3x - 4y = 5.$$

What fact in the figure corresponds to the fact that there is no set of values of x and y that can satisfy these two equations?

9. Shew that the equations

$$2x - 3y = 3,$$
$$2x - 3y = 4$$

are represented graphically by lines of the same *slope*.

10. The line that represents the equation

$$2x-3y=0$$

passes through the origin.

11. Construct the lines whose equations are

$$x-\ \ y=0,$$
$$x-2y=0,$$
$$x-3y=0.$$

68. The General Simultaneous Set of Equations. Let us now consider a set of equations, in two unknowns, in which the coefficients are algebraic or general rather than numerical or particular, say

$$ax+by=c,$$
$$a'x+b'y=c'.$$

Here a', b', c'—read *a prime*, *b prime*, etc.; or *a dash*, *b dash*, etc.—denote known numbers in no way related to a, b, c; this notation is employed to save the use of a great many different letter forms, and to gain any advantage that may come through having two or more expressions of the same form written with similar letters in corresponding parts. Here we have the a's as coefficients of x, the b's as coefficients of y, and the c's as the absolute terms.

To solve, multiply through with a view to eliminating y. Then

$$ab'x+bb'y=b'c,$$
$$a'bx+bb'y=bc'.$$

Therefore by subtraction

$$(ab'-a'b)x=b'c-bc$$

whence $$x=\frac{b'c-bc'}{ab'-a'b}.$$

Similarly it is found that

$$y = \frac{ac' - a'c}{ab' - a'b},$$

and the solution of the equation is

$$x = \frac{cb' - c'b}{ab' - a'b}, \quad y = \frac{ac' - a'c}{ab' - a'b}.$$

By regarding the association of letters

$$\begin{matrix} x, & y, & 1 \\ a, & b, & c \\ a', & b', & c' \end{matrix}$$

it is not difficult to frame a rule for writing down the solution without going through the work of solving.

The student may treat similarly the set

$$ax + by + c = 0,$$
$$a'x + b'y + c' = 0.$$

Exercises XCIII

Write down the solutions of the following sets, and test in each case :

1. $3x + 4y = 5,$
   $4x + 5y = 7.$

2. $3x - 4y = 2,$
   $4x - 5y = 3.$

3. $5x + 7y - 12 = 0,$
   $8x + 5y - 13 = 0.$

4. $7y - 10z + 13 = 0,$
   $2y + 3z - 15 = 0.$

5. $\frac{1}{3}x - \frac{1}{2}y = \frac{1}{6},$
   $\frac{3}{4}x + \frac{2}{3}y = 5\frac{7}{12}.$

6. $\frac{3}{7}x + \frac{5}{8}y = \frac{2}{3},$
   $\frac{3}{8}x + \frac{2}{7}y = \frac{1}{2}.$

7. $ax + by = c,$
   $lx + my = n.$

8. $ax - by = c,$
   $a'x - b'y = c'.$

9. $a_1x + a_2y = a_3,$
   $b_1x + b_2y = b_3.$

10. $\frac{x}{a} + \frac{y}{b} = 1,$
    $\frac{x}{a'} + \frac{y}{b'} = 1.$

69. Simultaneous Equations in Three Unknowns. In the *expression*

$$x + 2y + 3z - 10$$

we think of x, y, z as being variables, the value of the expression depending upon the values assumed by or assigned to x, y, z. Thus for $x = 1$, $y = 2$, $z = 3$ the value of the expression is 2. If, however, we have the *equation*

$$x + 2y + 3z - 10 = 0,$$

a restriction is put upon the values that x, y, z may have. Yet plainly two of x, y, z may be given any values we please, and then the equation determines the value of the third. Thus for $z = 2$, and $y = 1$, we must have

$$x + 2 \times 1 + 3 \times 2 - 10 = 0$$

i.e. $$x = 2,$$

so that $(x = 2,\ y = 1,\ z = 2)$, or more briefly $(2, 1, 2)$ is a solution of the equation. Manifestly the number of solutions that may be found is unlimited.

Now suppose that x, y, z have to satisfy two equations, the one already taken and one other,

$$x + 2y + 3z - 10 = 0,$$
$$2x + y + 5z - 13 = 0.$$

Here we see that a further restriction is put upon the values that x, y, z may have, for while one of the three x, y, z, say z, may be given any value we please, the values of both of the others are then determined by the two equations. Thus if we give to z the value 2, then must

$$x + 2y + 3 \times 2 - 10 = 0,$$
$$2x + y + 5 \times 2 - 13 = 0.$$

Or, in simpler form,

$$x + 2y = 4,$$
$$2x + y = 3,$$

whence, as in the earlier paragraphs,

$$x = \tfrac{2}{3},\ y = 1\tfrac{2}{3},$$

so that $(\frac{2}{3}, 1\frac{2}{3}, 2)$ is a solution of the two equations. Plainly we can find as many solutions as we please.

Finally, suppose that x, y, z have to satisfy three equations, those already taken and one other,

$$\left.\begin{aligned} x + 2y + 3z - 10 &= 0, \\ 2x + y + 5z - 13 &= 0, \\ 3x + 5y + 4z - 23 &= 0. \end{aligned}\right\} \cdots\cdots\cdots \text{(i)}$$

We see, or shall see, that the values that x, y, z may now have are still further restricted. Multiplying the first of these through by 5 and the second by 3 we have

$$5x + 10y + 15z - 50 = 0,$$
$$6x + 3y + 15z - 39 = 0.$$

Then by subtraction, since x, y, z stand for the same numbers in each equation,

$$x - 7y + 11 = 0 \cdots\cdots\cdots \text{(ii)}$$

an equation that does not involve z. Similarly *eliminating* z from the first and the third, we find

$$8x + 4y + 20z - 52 = 0,$$
$$15x + 25y + 20z - 115 = 0,$$

whence by subtraction

$$7x + 21y - 63 = 0,$$

or, after division through by 7,

$$x + 3y - 9 = 0 \cdots\cdots\cdots\cdots \text{(iii)}$$

Thus we see that the set, compiled from (i), (ii), (iii),

$$\left.\begin{aligned} x + 2y + 3z - 10 &= 0, \\ x - 7y \qquad + 11 &= 0, \\ x + 3y \qquad - 9 &= 0, \end{aligned}\right\} \cdots\cdots\cdots (iv)$$

is the equivalent of the given set.

Now the last two of these equations determine the values of x and y, which are found to be $x = 3$, $y = 2$. Then from the first equation it follows that

$$3 + 2 \times 2 + 3z - 10 = 0,$$

whence $$z = 1.$$

It follows then that in order that the three equations may be satisfied, x, y, z must equal 3, 2, 1 respectively. Therefore the three equations determine the values of x, y, z, and the one and only solution is (3, 2, 1).

The method of solving appears in what precedes. While the student may not have frequent occasion to treat simultaneous equations in more than two unknowns, a few exercises are added. The results should in every case be tested.

EXERCISES XCIV

1. Find the value of

$$3x + 4y + 5z - 25$$

for $(x = 1,\ y = 2,\ z = 3)$, for $(x = 0,\ y = -1,\ z = +6)$, for $(x = -1,\ y = +5,\ z = +3)$, for $(x = +5,\ y = -2,\ z = 0)$.

Find also a set of values for x, y, z that will make the expression equal to zero.

2. Find two sets of values of x, y, z, that will make $x + y + z + 1$ equal to 10.

3. Find three solutions of the equation

$$2x - 3y + 4z = 15.$$

Take any values of x, y, z *at random*, and see if they satisfy the equation.

4. Find three solutions of the simultaneous set

$$\begin{aligned} x + y + z &= 12, \\ 2x + 3y + 4z &= 38 \end{aligned}$$

5. Solve

$$\begin{aligned} x + y + z &= 12, \\ 2x + 3y + 4z &= 38, \\ 3x + 5y + 7z &= 64. \end{aligned}$$

6. Examine the set

$$\begin{aligned} x + y + z &= 12, \\ 2x + 3y + 4z &= 38, \\ 4x + 5y + 6z &= 62. \end{aligned}$$

7. Solve:

(1) $\left.\begin{aligned} x - y + 2z &= 11, \\ 3x - 4y + 5z &= 22, \\ 5x + 6y - 7z &= 8. \end{aligned}\right\}$

(2) $\left.\begin{aligned} 5u + 6v + 8w &= 17, \\ 2u - 3v + 7w &= 9, \\ 4u + 11v + 2w &= 11. \end{aligned}\right\}$

(3) $\left.\begin{aligned} l + m + n &= 8, \\ l - m + n &= 6, \\ l + m - n &= -2. \end{aligned}\right\}$

(4) $\left.\begin{aligned} k + l &= 18, \\ l + h &= 15, \\ h + k &= 11. \end{aligned}\right\}$

(5) $\left.\begin{aligned} 7y + 8z &= 68, \\ 5z + 4x &= 37, \\ 11x + 9y &= 69. \end{aligned}\right\}$

(6) $\left.\begin{aligned} 7x - 3y + 8z &= 44, \\ 5x + 13y - 6z &= 10, \\ -19x + 5y + 7z &= 69. \end{aligned}\right\}$

(7) $\left.\begin{aligned} x + y + z &= 32, \\ x + 3y + 5z &= 124, \\ x + 5y + 9z &= 216. \end{aligned}\right\}$

(8) $\left.\begin{aligned} 2u - v - w + 5 &= 0, \\ 2v - w - u + 3 &= 0, \\ 2w - u - v - 7 &= 0. \end{aligned}\right\}$

(9) $\left.\begin{aligned} 2x - 3y + 4z &= 11, \\ 5x + 4y - 8z &= 23, \\ 9x + 11y + 5z &= 93. \end{aligned}\right\}$

(10) $\left.\begin{aligned} \tfrac{1}{2}x + \tfrac{1}{3}y + \tfrac{1}{4}z &= \tfrac{1}{12}, \\ \tfrac{1}{3}x + \tfrac{1}{4}y + \tfrac{1}{5}z &= \tfrac{1}{15}, \\ \tfrac{1}{5}x + \tfrac{1}{6}y + \tfrac{1}{7}z &= \tfrac{1}{21}. \end{aligned}\right\}$

70. Problems. The following problems will supply further exercises in simultaneous equations and will help to impress the meaning of a simultaneous set. The results should in every case be verified. It would be well also for the student to attempt an arithmetical solution of each problem.

Exercises XCV

1. A person bought a certain number of pounds of tea at 40 cents a pound, and a certain number at 30 cents a pound paying therefor $7·30. The price of each having increased 5 cents, he paid $8·40 for a similar purchase. Find how many pounds of each were bought.

2. A pound of tea and 3 pounds of sugar cost 90 cents, but if sugar were to rise 50 per cent. in price, and tea 10 per cent. the cost would be $1·05. Find the prices of tea and sugar.

3. The two digits of a number change places when 9 is added to the number, and the sum of the two numbers concerned is 33. Find the numbers.

4. The sum of two numbers is 83 and their difference is 27; find the numbers.

5. Find the fraction which becomes equal to $\frac{2}{3}$ when 3 is added to the numerator and 4 to the denominator, or when 1 is subtracted from the numerator and 2 from the denominator.

6. P and Q are two places 35 miles apart. A starts from P for Q, and B starts from Q for P at the same time, and they meet at the end of 5 hours. If A has travelled 5 miles more than B, find their rates.

7. If the length of a field were increased by 20 rods and its breadth diminished by 10 rods its area would be unchanged. If the breadth were increased by 20 rods and the length diminished by 10 rods the area would be increased 900 square rods. Find the dimensions of the field.

8. If A were to give \$2 to B, then B's money would be twice A's, but if B were to give \$3 to A, then B's money would be only one-fourth greater than A's. Find how much each has.

9. A merchant mixes 3 pounds of one kind of tea with 2 pounds of another to make a mixture worth 36 cents a pound. With the same kinds of tea he makes a second mixture worth 37 cents a pound by taking 7 pounds of the former with 3 pounds of the latter. Find what each kind of tea is worth.

10. A walks from P to Q in a certain time. Had his rate of walking been $\frac{1}{2}$ mi. an hr. greater he would have made the journey in $\frac{1}{2}$ hr. less time, and had his rate been 1 mi. an hr. less he would have taken $1\frac{1}{2}$ hr. longer for the journey. Find the rate and the distance from P to Q.

11. A sum of money is divided among A, B, and C. If B and C together receive \$24, C and A together \$29, and A and B together \$18, find the shares of each.

12. A fraction becomes equal to $\frac{3}{4}$ when 1 is added to each of its terms, and becomes equal to $\frac{2}{3}$ when 1 is subtracted from each of its terms. Find the fraction.

13. A man walks from P to Q in 7 hours. If his rate of walking had been one-half a mile an hour less it would have taken him one hour longer. Find his rate and the distance from P to Q.

14. A vessel contains a mixture of wine and water. Two gallons of water are added and the strength of the mixture is found to be four-fifths: then 10 gallons of wine are added and it is found that the mixture has resumed its original strength. Find how many gallons of wine and water were contained in the original mixture.

15. Two trains 76 yards and 100 yards long, meeting on parallel tracks, pass each other in 6 seconds. Had they been going in the same direction, the former would have taken 36 seconds to pass the latter. Find the rates of the trains.

16. Two sums of \$1,200 and \$1,000 were lent at certain rates, and yielded a yearly interest of \$98. Had the rates been interchanged the yearly interest would have been \$2 greater. Find the rates.

EXERCISES XCVI

(MISCELLANEOUS)

A

1. Find by actual division the remainder when px^2+qx+r is divided by $x-a$.

2. How many odd integers are there between (i) 0 and 72, (ii) 24 and 52, (iii) $2m$ and $2n$, where m and n are certain positive integers, n greater than m?

3. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, shew that $\frac{a}{d}=\frac{a^3}{b^3}$.

4. If a and b are positive and a less than b, find the difference

$$\frac{a+1}{b+1}-\frac{a}{b}$$

shewing that it is positive.

Illustrate by assigning to a and b specific values.

5. Construct the two lines whose equations are

$$2x-3y=6; \quad 3x+2y=35.$$

B

1. Shew by actual division that

$$x^4-3x^3-33x^2+46x+61=(x-7)(x^3+4x^2-5x+11)+138,$$

and hence that the value of $x^4-3x^3-33x^2+46x+61$ for $x=7$ is 138.

2. Find the L.C.M. of x^5-xy^4, $x^9+x^7y^2$, $x^6+y^6+x^2y^2(x^2+y^2)$.

3. If

$$\frac{m+n}{m-n}=\frac{x+y}{x-y}$$

shew that $m:n::x:y$, and conversely.

4. If a and b are positive and a greater than b, find the difference

$$\frac{a+1}{b+1}-\frac{a}{b}$$

shewing that it is negative.

Illustrate by assigning to a and b specific values.

C

1. Shew by repeated division by 7 that

$$5478 = 2.7^4 + 1.7^3 + 6.7^2 + 5.7 + 4$$
$$= 21654 \textit{ in the scale of seven.}$$

2. Write down immediately the results of the following multiplications:

(i) $(x+3)(x-5)$; (ii) $(2x-7)(2x+5)$;
(iii) $(x+5)(x+7)(x+9)$; (iv) $(x-7)(x+6)(x-4)$.

3. Find and tabulate the values of x^2+2x+2, or which is the same thing $(x+1)^2+1$, for $x = -5, -4, -3, -2, -1, 0, +1, +2, +3$.

4. If a, b, h are all positive and a less than b, find the difference

$$\frac{a+h}{b+h} - \frac{a}{b}$$

shewing that the result is positive.

5. Construct the lines whose equations are

$$2x + 3y - 13 = 0$$
$$4x - 3y + 1 = 0$$
$$3(2x + 3y - 13) = 5(4x - 3y + 1)$$

and shew that the solution of the first two equations satisfies also the third.

D

1. Shew by repeated division that

$$x^4 - 5x^3 + 12x^2 - 23x + 37$$
$$= (x-2)^4 + 3(x-2)^3 + 6(x-2)^2 - 3(x-2) + 15.$$

2. Write down immediately the results of the following multiplications:

(i) $(2x+3)(2x+5)(2x+7)$; (ii) $(x+3y)(x-2y)(x+y)$;
(iii) $(a+b-c)(a-b+c)$; (iv) $(x^2-xy+y^2)(x^2+xy+y^2)$.

3. In $x^2 - 4xy + 5y^2$ substitute (i) $x=4$, $y=1$; (ii) $x=7$, $y=3$; (iii) $x = -4$, $y = +5$; (iv) $x = -2$, $y = -3$.
Is it accidental that in each case the result is positive?

4. If a, b, h are all positive and a greater than b, find the difference

$$\frac{a+h}{b+h} - \frac{a}{b}$$

shewing that the result is negative.

CHAPTER XVI

QUADRATIC EQUATIONS

71. Introductory. The following problems will lead to equations which differ somewhat from those previously met.

Problem 1. Divide a straight line 8 inches long into two parts such that the area of their rectangle may be fifteen-sixteenths of that of the square on half the line.

The unit of length be 1 inch, let x measure the length of one part, and therefore $8-x$ the length of the other part. Then the measures of the rectangle under the parts, and the square on half the line are

$$x(8-x), \text{ and } 4^2 \text{ or } 16$$

$$\therefore \quad x(8-x)=\frac{15}{16} \text{ of } 16$$

$$\therefore \quad 8x-x^2=15.$$

Then, bringing 15 to the left side, we have

$$-x^2+8x-15=0$$

or, multiplying the equals by -1,

$$x^2-8x+15=0.$$

This then is the equation, in a sort of standard form, to which the problem leads, and from it we expect to find the value of x. In the equations of earlier chapters the unknown x occurred to only one power, and that the first, but here there are two different powers, the first and the second. An equation that involves the second power, but no higher power, of the unknown is called a *quadratic equation*, and the equations of Chapters II and IX are called *simple* or *linear equations*.

To solve this equation, we note that $x^2-8x+15$ and $(x-5)(x-3)$ are two different forms of the one quantity.

$$\therefore \quad (x-5)(x-3)=0.$$

Now in order that the product on the left may be equal to zero it is necessary that one factor should be equal to zero, and the vanishing of either factor will ensure that the equation is satisfied. Thus the equation will be satisfied if

either $$x-5=0 \text{ or } x-3=0,$$

in other words, if $$x=5 \text{ or } x=3.$$

Thus the equation has *two* roots, and recalling what x has been taken to measure we see that one root gives the parts 5 inches and 3 inches, and the other gives them as 3 inches and 5 inches; hence while the equation has *two* solutions, there is effectively *one* division of the line.

The equation of this problem may be otherwise written, as, for example,

$$x^2-8x=-15, \text{ or } x^2+15=8x,$$

but for most purposes it is best considered under the form

$$x^2-8x+15=0.$$

This equation has been shown to have two roots, and plainly no number other than 5 or 3 will make $(x-5)(x-3)$ equal to zero. The roots of the *equation* correspond to the factors $x-5$ and $x-3$ of the quadratic expression.

Problem 2. The length of a rectangle exceeds its breadth by 20 rods, and its area is 40 acres. Find its dimensions.

Taking 1 rod as the unit of length, and therefore 1 sq. rd. as the unit of area, let x measure the breadth and therefore $x+20$ the length.

$$\therefore \quad x(x+20) \text{ measures the area.}$$

But 40×160 measures the area, since 1 acre equals 160 square rods.

$$\therefore \quad x(x+20)=40\times 160$$

$$\therefore \quad x^2+20x-4800=0.$$

This last, then, is the quadratic equation to which the problem leads. From it we find at once

$$(x-40)(x+60)=0.$$

Hence, in order that the equation may be satisfied, we must have either $x-40=0$, or $x+60=0$, *i.e.*, either $x=40$ or $x=-60$; either of these values of x, and no other value of x, will satisfy the equation. It remains to relate these values to the problem.

If $x=40$, the breadth and the length of the field are

40 rods and 60 rods.

If $x = -60$, the breadth and the length of the field are

$$-60 \text{ rods and } -40 \text{ rods.}$$

Plainly the two results point to the same field. If + and − signify opposite directions in the same straight line, this is not of moment in determining the dimensions of the field. It is to be noted that the breadth is taken as the shorter dimension, so that when -60 appears as giving the breadth, the length is given by $-60+20$ or -40, since, from the point of view of algebra, -60 is 20 less than -40.

The problems just considered have been seen to lead to *quadratic equations*, the solutions of which were obtained by resolving the implied *quadratic expressions*. While it is well always to think of an equation as having a source in some such problem, a number of examples of equations, divorced from their problems, are given in order to afford ready practice. In working them the student may assume the axioms relating to equations and the rules derived from them

Ex. 1. Solve

$$3x^2-5x+11=2x^2-x+8.$$

We have at once

$$x^2-4x+3=0$$
$$\therefore\quad (x-3)(x-1)=0$$
$$\therefore\quad x-3=0 \textit{ or } x-1=0$$
$$\therefore\quad x=3 \text{ or } 1, \text{ the solutions required.}$$

Ex. 2. Solve

$$12x^2-100x+75=0.$$

After multiplication through by 3, it is seen that

$$(6x)^2-50(6x)+225=0.$$
$$\therefore\quad (6x-5)(6x-45)=0.$$

Therefore either $6x-5=0$ or $6x-45=0$, and the solutions are $\frac{5}{6}$ and $7\frac{1}{2}$.

Exercises XCVII

1. Solve the following equations, in each case substituting in the original equation the roots found, substituting also in the quadratic expression on the right of the sign of equality when the equation is brought to the form,

Quadratic expression $=0$,

(i) a value of x, intermediate to the roots, (ii) a value of x smaller than the smaller root, (iii) a value of x larger than the larger root:

(1) $x^2-10x+21=0$; (2) $x^2-12x+35=0$;
(3) $x^2-14x+48=0$; (4) $x^2-28x+192=0$;
(5) $2x^2-3x+11=x^2+5x-4$; (6) $x^2-7x=3x-21$;
(7) $15x-x^2=2x+36$; (8) $2x^2+11=x^2+12x$;
(9) $(x+1)(x+2)=6x$; (10) $(2x-3)(x-5)=(x-3)^2$.

2. Solve, treating each equation as in No. 1:

(1) $x^2-x-20=0$; (2) $x^2+3x-28=0$;
(3) $x^2+9x+20=0$; (4) $x^2+15x+44=0$;
(5) $x^2+2x-48=0$; (6) $x^2-2x-48=0$;
(7) $x^2-9=0$; (8) $x^2-289=0$;
(9) $x^2=81$; (10) $5x^2=80$.

3. Solve, treating each equation as in the preceding:

(1) $(2x-3)(x-5)=0$; (2) $(2x-3)(3x-4)=0$;
(3) $(3x+5)(x-7)=0$; (4) $(5x+11)(7x-8)=0$;
(5) $2x^2+7x-30=0$; (6) $21x^2+41x-40=0$;
(7) $6y^2-13y+6=0$; (8) $30z^2-43z+15=0$;
(9) $(2x-1)^2-(x-3)^2=0$; (10) $(3x-2)^2=(2x+3)^2$.

4. Construct the equation, expressing it in the form

Quadratic expression $=0$,

whose roots are

(i) 2, 3; (ii) 5, 8; (iii) $-2, -3$; (iv) $-5, +8$; (v) a, b.

5. As equations, is there any difference between

$$6x^2-17x+12=0 \text{ and } x^2-\tfrac{17}{6}x+2=0?$$

6. Solve the equation

$$12x^2-31x+20=0$$

and construct the equation whose roots are the reciprocals of those found.

72. The Quadratic Function Resolvable into Factors in only One Way. Take as example the expression x^2-5x+6, where we think of x as a variable to which may be assigned any value we please. We have identically, *i.e.*, whatever be x,

$$x^2-5x+6=(x-2)(x-3) \quad \ldots\ldots\ldots(\text{i})$$

The question arises whether x^2-5x+6 can admit a different resolution into factors. If possible, let $x-m$ be a factor of x^2-5x+6, where m is neither 2 nor 3. There must then be a second factor $x-n$, which may be found by division, so that identically

$$x^2-5x+6=(x-m)(x-n) \quad \ldots\ldots\ldots(\text{ii})$$

Therefore from (i) and (ii), we have identically

$$(x-2)(x-3)=(x-m)(x-n).$$

In this latter relation, true for all values of x, we may put $x=m$. Hence

$$\begin{aligned}(m-2)(m-3)&=(m-m)(m-n)\\&=0(m-n)\\&=0.\end{aligned}$$

But m is different from 2 and 3, so that neither $m-2$ nor $m-3$, nor therefore $(m-2)(m-3)$, can be zero. Here the supposition that $x-m$, where m is neither 2 nor 3, is a factor of x^2-5x+6 is wrong. Thus x^2-5x+6 can be resolved into factors in only one way.

Next consider the expression $8x^2-22x+15$. We find

$$\begin{aligned}8x^2-22x+15&=(2x-3)(4x-5),\\&=(x-\tfrac{3}{2})(8x-10),\\&=(8x-12)(x-\tfrac{5}{4}),\\&=8(x-\tfrac{3}{2})(x-\tfrac{5}{4}).\end{aligned}$$

Here the apparently different factors of the expression are effectively the same, for they differ only through the moving of a merely numerical factor. The last may be regarded as a sort of standard form, and, though in it there is a third factor which is merely numerical, we say that the quadratic expression in x is the product of two factors linear in x. With this explanation it is readily seen here as in the earlier example that the resolution is unique.

Turn now to the equation

$$x^2 - 5x + 6 = 0$$

where x is no longer a variable, but one or other of the constants which are the roots of the equation. We have at once

$$(x-2)(x-3) = 0$$

so that 2 and 3 are the roots. There can be no root different from 2 or 3. For if possible let m be such a root. Then

$$m^2 - 5m + 6 = 0$$

$$\therefore \quad (m-2)(m-3) = 0.$$

But m being neither 2 nor 3, neither $(m-2)$ nor $(m-3)$ can be zero, so that the last relation is impossible. Hence m cannot be a root.

So in the case of the equation

$$8x^2 - 22x + 15 = 0$$

we have

$$(2x-3)(4x-5) = 0$$

or

$$8(x-\tfrac{3}{2})(x-\tfrac{5}{4}) = 0;$$

accordingly the roots are $\frac{3}{2}$ and $\frac{5}{4}$, and, as before, it is readily seen that there can be no root different from these.

Hence it appears that the quadratic equation has two and only two roots. The general proof of this theorem will be given later.

73. Extension of the Number System. The quadratic expressions, hitherto met, having been seen to have two and only two linear factors, we are led to ask if this is true of all quadratic expressions. The resolution, when the factors are not at once recognised, is effected by exhibiting the expression as the difference of two squares, one of which contains all the terms involving the variable, the case in which the expression is a perfect square being disregarded for the present. Thus taking $x^2-28x+187$, we have, adding the square of $\frac{28}{2}$ to complete the square whose first two terms are x^2 and $-28x$, and subtracting to correct,

$$\begin{aligned} x^2-28x+187 &= x^2-28x+14^2-14^2+187 \\ &= (x-14)^2-9 \\ &= (x-14)^2-3^2 \\ &= (\overline{x-14}+3)(\overline{x-14}-3) \\ &= (x-11)(x-17), \end{aligned}$$

and in like manner

$$\begin{aligned} 2x^2-11x+15 &= 2\{x^2-\tfrac{11}{2}x+\tfrac{15}{2}\} \\ &= 2\{x^2-\tfrac{11}{2}x+(\tfrac{11}{4})^2-(\tfrac{11}{4})^2+\tfrac{15}{2}\} \\ &= 2\{(x-\tfrac{11}{4})^2-(\tfrac{1}{4})^2\} \\ &= 2(\overline{x-\tfrac{11}{4}}+\tfrac{1}{4})(\overline{x-\tfrac{11}{4}}-\tfrac{1}{4}) \\ &= 2(x-\tfrac{5}{2})(x-3) \\ &= (2x-5)(x-3). \end{aligned}$$

Conversely, the product of two factors, linear in x, may be exhibited as the difference of two squares. Thus

$$\begin{aligned} (x+4)(x+6) &= (\overline{x+5}-1)(\overline{x+5}+1) \\ &= (x+5)^2-1^2, \end{aligned}$$

and

$$(2x-3)(4x-5)=8(x-\tfrac{3}{2})(x-\tfrac{5}{4})$$
$$=8(\overline{x-\tfrac{11}{8}}-\tfrac{1}{8})(\overline{x-\tfrac{11}{8}}+\tfrac{1}{8})$$
$$=8\{(x-\tfrac{11}{8})^2-(\tfrac{1}{8})^2\},$$

which, except for the numerical factor, is the difference of two squares.

Let it now be proposed to resolve x^2-6x+7. We have

$$x^2-6x+7=x^2-6x+9-9+7$$
$$=x^2-6x+9-2$$
$$=(x-3)^2-2.$$

Here, as 2 is not the square of any number that we could have met up to this point, progress is arrested, as the expression does not assume the form of the difference of two squares. Numbers can be found, the squares of which are as near to 2 as we please. Thus

$$1,\ 1\cdot4,\ 1\cdot41,\ 1\cdot414,\ 1\cdot4142,\ \ldots$$

are numbers, increasing in value, the squares of which are

$$1,\ 1\cdot96,\ 1\cdot9881,\ 1\cdot999396,\ 1\cdot99996164,\ \ldots$$

and accordingly afford approximations,—the later ones *close approximations*,—to what we may designate as a *desired number*. To meet the difficulty we shall introduce this desired number and, not being able to represent it as an integer or a fraction,* shall denote it by a new symbol $\sqrt{2}$ or $\sqrt[2]{2}$. This and similar numbers as $\sqrt{5}$, $\sqrt{7}$, will

*That there is no integer whose square is 2 is at once evident, and that there is no fraction whose square is 2 may easily be shewn. For if possible let $\frac{h}{k}$ be such a fraction. h and k being integers prime to each other. Then we should have $\left(\frac{h}{k}\right)^2=2$. Therefore

$$\frac{h\times h}{k\times k}=2.$$

That this might be so it would be necessary that k divide the numerator exactly, which is impossible, since k is prime to h. Thus there is no fraction whose square is 2.

be associated in addition, subtraction, multiplication, and division, with themselves, and with integers and fractions, according to the fundamental rules of algebra; and through these rules the meaning of multiplication, division, etc., for such numbers, will appear. Thus, by definition,

$$\sqrt{2} \times \sqrt{2} = 2.$$

Further

$$\sqrt{2} + \sqrt{2} = \sqrt{2}(1+1) = \sqrt{2} \times 2 = 2\sqrt{2};$$

$$(\sqrt{3} \times \sqrt{7}) \times (\sqrt{3} \times \sqrt{7}) = \sqrt{3} \times \sqrt{7} \times \sqrt{3} \times \sqrt{7}$$

$$= \sqrt{3} \times \sqrt{3} \times \sqrt{7} \times \sqrt{7}$$

$$= 3 \times 7$$

$$\therefore \quad \sqrt{3} \times \sqrt{7} = \sqrt{3 \times 7};$$

and

$$\frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{5} \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{\sqrt{15}}{3}.$$

From the approximations to $\sqrt{3}$ and $\sqrt{7}$, we can find approximations to $\sqrt{3} + \sqrt{7}$, and these approximations can be regarded as determining $\sqrt{3} + \sqrt{7}$.

Returning now to the question in hand we may write

$$x^2 - 4x + 2 = (x-2)^2 - (\sqrt{2})^2$$

$$= (\overline{x-2} + \sqrt{2})(x - 2 - \sqrt{2})$$

$$= (x - \overline{2 - \sqrt{2}})(x - \overline{2 + \sqrt{2}}).$$

Thus through the introduction of a new kind of number we are able to resolve the expression, and to say of the equation

$$x^2 - 4x + 2 = 0$$

that it can be given the form

$$(x - \overline{2 - \sqrt{2}})(x - \overline{2 + 2\sqrt{2}}) = 0$$

and that it admits the two roots

$$2 - \sqrt{2},\ 2 + 2\sqrt{2}.$$

Just as 2 does not admit an integral or fractional square root, so 5, 6, 11, for example, do not admit a cube root, and to meet such cases we may introduce new numbers, calling them cube roots, and denoting them by the symbols $\sqrt[3]{5}$, $\sqrt[3]{6}$, $\sqrt[3]{11}$. Similarly too for higher roots. All such numbers are spoken of as ***irrational numbers***, and the numbers hitherto dealt with, namely, integers and fractions, are called ***rational numbers***.

Thus through the introduction of irrational numbers we are enabled to say that certain quadratic equations, which otherwise would not admit of solutions, have two roots. It may be shewn, as before, that such equations have only two roots.

Exercises XCVIII

1. Solve the following equations, in each case finding the best approximations to the first decimal place, and substituting the approximations in the equation:

(1) $x^2-2x-5=0$; (2) $x^2+4x-7=0$;

(3) $x^2-3x+1=0$; (4) $x^2-6x-3=0$;

(5) $x^2+7x-11=0$; (6) $2x^2-3x-4=0$;

(7) $3x^2-18x+5=0$; (8) $5x^2-12x+6=0$;

(9) $7x^2-5x-9=0$; (10) $4x^2-11x+5=0$.

2. Find the following in simple form, and an approximation, the best not involving more than two places of decimals, to each result:

(1) $(\sqrt{2}+3)^2$; (2) $(\sqrt{3}+5)(2-\sqrt{3})$;

(3) $(\sqrt{5}+7)(\sqrt{6}+8)$; (4) $(\sqrt{3}+\sqrt{5})(5\sqrt{3}-2\sqrt{5})$;

(5) $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$; (6) $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$;

(7) $\dfrac{\sqrt{3}+\sqrt{5}}{2\sqrt{3}-\sqrt{5}}$; (8) $\dfrac{\sqrt{7}+\sqrt{5}}{\sqrt{2}+\sqrt{3}}$.

3. Explain on what grounds is based each of the following statements:

(1) $\sqrt{20} = 2\sqrt{5}$; (2) $(\sqrt{7})^3 = 7\sqrt{7}$.

4. A certain unit length being assumed, construct geometrically the line to which is ascribed the *length** $\sqrt{2}$; the length $\sqrt{3}$; the length $\sqrt{15}$.

5. A certain unit length being assumed, construct a rectangle to which the area $\sqrt{15}$ is ascribed.

6. Solve the following equations and test the result by substituting the roots found in the equation concerned:

(1) $x^2 - 12x + 2 = 0$; (2) $3x^2 - 15x + 10 = 0$;
(3) $7x^2 - 15x + 6 = 0$; (4) $8x^2 - 7x - 3 = 0$;
(5) $3x^2 - 16x + 12 = 0$; (6) $3x^2 + 7x + 3 = 0$;
(7) $5x^2 - 11x + 3 = 0$; (8) $2\frac{1}{3}x^2 - 4\frac{1}{2}x - 5 = 0$.

74. Further Extension of the Number System. Let it now be proposed to resolve into factors the quadratic expression $x^2 - 6x + 11$. Proceeding as in the earlier sections we have

$$\begin{aligned} x^2 - 6x + 11 &= x^2 - 6x + 9 - 9 + 11 \\ &= (x-3)^2 + 2 \\ &= (x-3)^2 - (-2). \end{aligned}$$

In the attempt to express the given function as the *difference* of two squares, we are brought face to face with the need of a number whose square is -2. Now the squares of all the numbers that we have had to deal with—whether these numbers are positive or negative—are positive, and

* It must be remembered that in algebra we have to do with numbers, and therefore with the *measures of lengths*, for example, rather than *lengths*. Yet for brevity of statement *length*, *weight*, &c., are frequently employed to signify *measure of length*, *measure of weight*, &c.

we are forced to say that there is no number whose square is -2. To meet the difficulty, we introduce a number of a new kind, one, namely, whose square is the negative number -2; this number we denote by the symbol $\sqrt{-2}$, and for it we have the following relation—a fact of multiplication—

$$(\sqrt{-2})^2 = \sqrt{-2} \times \sqrt{-2} = -2.$$

We may now say

$$\begin{aligned} x^2 - 6x + 11 &= (x-3)^2 - (\sqrt{-2})^2 \\ &= (\overline{x-3} + \sqrt{-2})(\overline{x-3} - \sqrt{-2}) \\ &= (x - \overline{3 - \sqrt{-2}})(x - \overline{3 + \sqrt{-2}}). \end{aligned}$$

We have here supposed—less we could not do—that the $\sqrt{-2}$ is to be associated with other numbers in accordance with the fundamental rules of algebra. In like manner we may arrive at other such numbers as $\sqrt{-1}$, $\sqrt{-\frac{3}{4}}$, $\sqrt{-4}$. . .

This new kind of number is designated *imaginary*, whereas the numbers hitherto considered are called *real*. For such numbers addition, subtraction, multiplication and division are a natural outcome of the rules of algebra assumed to hold for them. A number as $3 + \sqrt{-2}$, in which there is a real part and an imaginary part, is spoken of as a *complex* number.

If now we have the equation

$$x^2 - 6x + 11 = 0$$

we may say

$$\begin{aligned} (x^2 - 6x + 9) - 9 + 11 &= 0 \\ \therefore \quad (x-3)^2 - (-2) &= 0 \\ \therefore \quad (x-3)^2 - (\sqrt{-2})^2 &= 0 \\ \therefore \quad (x - \overline{3 - \sqrt{-2}})(x - \overline{3 + \sqrt{-2}}) &= 0 \end{aligned}$$

Accordingly, in order that the equation be satisfied, we must have either

$$x = 3 - \sqrt{-2} \text{ or } x = 3 - \sqrt{-2},$$

and here, as in like cases before, there are two, and only two, roots.

Since imaginary numbers are to be combined according to the fundamental rules for real numbers, we have

$$\begin{aligned}(2\sqrt{-1})^2 \text{ or } (\sqrt{-1} \times 2)^2 &= \sqrt{-1} \times 2 \times \sqrt{-1} \times 2\\ &= \sqrt{-1} \times \sqrt{-1} \times 2 \times 2\\ &= -1 \times 2^2\end{aligned}$$

$$\therefore\ \sqrt{-1 \times 2^2},\ i.e.,\ \sqrt{-4} = 2\sqrt{-1}.$$

Similarly

$$\sqrt{-n} = \sqrt{n}.\sqrt{-1}.$$

Thus all imaginary numbers can be expressed as the product of a real number and the imaginary number $\sqrt{-1}$, so that $\sqrt{-1}$ has the same relation to the system of imaginary numbers that 1 has to the system of real numbers. The letter i is very commonly employed to denote $\sqrt{-1}$, so that we may write

$$i^2 = -1, \quad i^2 + 1 = 0$$
$$\sqrt{-5} = \sqrt{5}.i = i\sqrt{5}.$$

The student will now see that, with the extended system of numbers, it is possible to express every quadratic expression as the product of two linear factors, and therefore to find for every quadratic equation two and only two roots.

One special case, to which allusion has already been made, should be considered; this is the case in which the quadratic expression is a perfect square. Take f... ...xample x^2-6x+9. Here

$$x^2-6+9=(x-3)(x-3)$$

and the two linear factors are the same. If w... take the equation

$$x^2-6x+9=0$$

we have

$$(x-3)(x-3)=0$$

and there is only one value of x, namely 3, which ... the equation. Yet on account of the two facto... that there are *two equal roots*.

EXERCISES XCIX

1. Give reasons for saying :

(i) $\sqrt{-7}\times\sqrt{-5}=-\sqrt{35}$;

(ii) $\sqrt{-7}+\sqrt{-5}=(\sqrt{7}+\sqrt{5})\sqrt{-1}$;

(iii) $(\sqrt{-1})^4=1$.

2. Resolve into factors :

(1) x^2-2x+2; (2) x^2+2x+2;
(3) x^2-3x+3; (4) x^2+3x+3;
(5) $x^2-8x+20$; (6) $x^2+10x+30$;
(7) $2x^2+3x+2$; (8) $3x^2-5x+4$;
(9) x^2+x+1; (10) x^2+ax+a^2.

3. Solve the equations :

(1) $x^2-4x+5=0$; (2) $x^2+4x+5=0$;
(3) $x^2-5x+8=0$; (4) $x^2+7x+15=0$;
(5) $y^2-6y+12=0$; (6) $z^2-4z+7=0$;
(7) $2x^2-7x+7=0$; (8) $5x^2-12x+9=0$,
(9) $x^2-x+1=0$; (10) $y^2-ay+a^2=0$.

4. S the following equations, in each case finding the sum and the p. duct of the roots :

(1) $x^2-8x+15=0$; (2) $x^2+12x+35=0$;

(3) $x^2-5x+3=0$; (4) $x^2-10x+27=0$;

(5) $2x^2-9x+5=0$; (6) $2x^2-3x+5=0$;

75. ne Solution of the Quadratic Equation Otherwise V. wed. The following way of treating the quadratic eq ion is not uncommon. Let

$$x^2-8x+15=0$$

be the e ation pr d for solution. Then

$$x^2-8x=-15$$

ach of the equals add 16, which completes the square of the expression to the left. Then

$$x^2-8x+16=16-15$$

or $$(x-4)^2=1.$$

Then $x-4$ must be the number whose square is 1. We know two such numbers namely $+1$ and -1. That these are the only such numbers can be readily seen, for no positive number greater than $+1$ or less than $+1$ can have 1 as its square.

$$\therefore \quad x-4=+1 \text{ or } -1$$

or, in a more condensed statement,

$$x-4=\pm 1$$

$$\therefore \quad x=4+1 \text{ or } 4-1$$

i.e., $x=5$ or 3.

Similarly to solve the equation $3x^2 - 7x + 2 = 0$ we have

$$
\begin{aligned}
& 3x^2 - 7x + 2 = 0 \\
\therefore\quad & 3x^2 - 7x = -2 \\
\therefore\quad & x^2 - \tfrac{7}{3}x = -\tfrac{2}{3} \\
\therefore\quad & x^2 - \tfrac{7}{3}x + \tfrac{49}{36} = \tfrac{49}{36} - \tfrac{2}{3} \\
\therefore\quad & (x - \tfrac{7}{6})^2 = \tfrac{25}{36} \\
\therefore\quad & x - \tfrac{7}{6} = \pm \tfrac{5}{6} \\
\therefore\quad & x = +\tfrac{7}{6} + \tfrac{5}{6} \text{ or } +\tfrac{7}{6} - \tfrac{5}{6} \\
& \quad = +2 \text{ or } +\tfrac{1}{3}.
\end{aligned}
$$

A quadratic equation which when brought to the standard form

Quadratic expression = 0

has no term of one dimension in the unknown as

$$x^2 - 7 = 0,$$

is called a *pure* quadratic equation. If a term of one dimension is present the equation is an *adfected* quadratic.

Exercises C

1. Solve, after the manner of this section:

(1) $x^2 - 6x + 8 = 0$; (2) $x^2 - x = 12$;

(3) $x^2 - 5x = 3$; (4) $x^2 - 6x + 11 = 0$;

(5) $2x^2 - 3x - 5 = 0$; (6) $3x^2 - 5x + 7 = 0$;

(7) $2x^2 - 5x - 7 = 0$; (8) $\frac{3}{4}x^2 - \frac{2}{3}x - \frac{5}{6} = 0$.

2. Explain on what grounds it is asserted that 25 has two, and only two, square roots.

3. Shew that the equation

$$\frac{1}{x-9}-\frac{1}{x-8}=\frac{1}{12}$$

when simplified assumes the form

$$x^2-17x+60=0$$

and find the roots, testing for accuracy by substituting in the original equation.

4. Solve the equation

$$(x+1)(2x+3)-(x+2)^2=55.$$

5. Solve the equation

$$\frac{1}{x-3}-\frac{1}{x-2}=\frac{1}{x+6}.$$

6. Shew that in order that x^2 should be equal to y^2 it is not necessary that $x=y$.

7. A man when asked how many dollars he had, stated that the square of the number was 21 less than 10 times the number. Can it be definitely known from this how many dollars he had?

76. Problems. The quadratic equation often offers itself as the key to the solution of a problem. The problems of this section are of this kind. The student should study carefully each problem, denote by some letter the measure of the unknown, which, if *known*, would satisfy all the conditions given, and then, employing this unknown, construct with care, from the given conditions, the equation proper to the problem. It will be found that of the two values of the unknown, which are given by the equation, each may furnish a solution of the problem, or one only may be applicable. It is interesting and instructive, in the latter case, to seek to account for the presence of the alien value.

Ex. 1. Find a number the square of which is 35 less than 12 times the number.

Let x be the number sought.

Then, at once, from the condition stated

$$12x - x^2 = 35$$

$$\therefore\quad -x^2 + 12x - 35 = 0$$

$$\therefore\quad x^2 - 12x + 35 = 0$$

$$\therefore\quad (x-5)(x-7) = 0$$

$$\therefore\quad x = 5 \text{ or } x = 7.$$

The two values are seen to apply, and we can therefore say that the number is either 5 or 7.

Ex. 2. The perimeter of a field is 152 rods and its area is 9 acres. Find the dimensions of the field.

Take 1 rod as the unit of length and therefore 1 square rod as the unit of area. Then area of the field measures 160×9 or 1440, since 1 acre equals 160 square rods; also the semi-perimeter measures $152 \div 2$ or 76.

Let x = measure of breadth of field.

$\therefore$ $76 - x$ = measure of length of field.

$\therefore$ $x(76 - x)$ = measure of area of field.

$$\therefore\quad x(76 - x) = 1440$$

$$\therefore\quad 76x - x^2 - 1440 = 0$$

$$\therefore\quad x^2 - 76x + 1440 = 0$$

$$\therefore\quad (x-36)(x-40) = 0$$

$$\therefore\quad x = 36 \text{ or } 40.$$

Accordingly the width of the field is 36 rods or 40 rods and the length of the field corresponding to these values is $(76-36)$ rods or $(76-40)$ rods, *i.e.*, 40 rods and 36 rods. Thus both roots have a significance in relation to the problem.

EXERCISES CI

1. The denominator of a fraction exceeds its numerator by 5. If 3 is added to each term of the fraction the resulting fraction exceeds the original fraction by $\frac{1}{12}$. Find the original fraction.

2. The product of the first and last of three consecutive odd integers increased by the square of the intermediate one is 46. Find the integers.

3. Find the number whose square diminished by 44 is 7 times the number.

4. The length of a rectangular plot of grass exceeds its breadth by 10 yards. The plot is surrounded by a walk 3 yards wide. If the area of the walk is to that of the plot as 19 to 50, find the dimensions of the plot.

5. Find the price of tea a pound from the fact that if the price were increased 5 cents a pound, a buyer would receive 10 pounds less for $21.

6. Two trains make the run from P to Q, a distance of 150 miles, the one at a rate 5 miles an hour greater than the other. The slower train requires one hour more for the run. Find the rates.

7. Divide 20 into two parts such that the sum of their squares may be 36 more than twice their product.

8. A train starts on a run from P to Q, a distance of 150 miles. At the end of 3 hours it is delayed half an hour, and then, increasing its speed by 5 miles an hour, reaches Q on schedule time. Find the normal rate of the train.

9. The sum of a number and its reciprocal is 2·9; find the number.

10. The length of a rectangle is 10 yards greater than its breadth. A second rectangle whose breadth and length exceed those of the first by 6 yards and 10 yards has an area $1\frac{1}{2}$ times that of the first. Find the dimensions of the fields.

11. Of the two sides of a right-angled triangle one is 7 inches longer than the other, and the hypothenuse is 13 inches in length. Find the lengths of the sides.

12. A number of men are first formed into a solid square, and afterwards into a hollow square 3 deep; the front in the latter formation has 75 men more than the front in the solid square. Determine the number of men.

13. One man can do a piece of work in 5 days less than another man, and if they work together they can do it in 6 days. Find the time in which each can do the work.

14. Find the price of eggs a dozen when two less in a shilling's worth raises the price one penny a dozen.

15. The men in a regiment can be arranged in a column twice as long as it is wide. If their number were 224 less they could be arranged in a hollow square 4 deep, having in each outer side of the square as many men as there were in the length of the column. Find the number of men.

16. The fore wheel of a carriage is 3 feet less in circumference than the hind wheel, and makes 180 more revolutions in going one mile. Find the circumference of each wheel.

17. Divide a straight line of *length* 1 into two parts such that the rectangle contained by the whole and one part may be equal to the square on the other part.

Give a geometrical construction for this division and compare the solutions.

18. Divide a straight line of length a into two parts such that the rectangle contained by the whole and one part may be equal to the square on the other part.

EXERCISES CII

(MISCELLANEOUS)

A

1. Noting that x^2-2x-3 is identically equal to $(x-3)(x+1)$ compute and tabulate the value of x^2-2x-3 for

$$x=-5,-4,-3,-2,-1,0,+1,+2,+3,+4,+5,+6,+7.$$

2. Resolve into factors:

$$\text{(i)} \quad x^2 - 4x - 6;$$
$$\text{(ii)} \quad x^2 - 4xy - 6y^2.$$

3. If interest is at the rate of r per cent., shew that the amount of \$1 at the end of 1, 2, 3 years is

$$\$\left(1+\frac{r}{100}\right), \quad \$\left(1+\frac{r}{100}\right)^2, \quad \$\left(1+\frac{r}{100}\right)^3.$$

4. If $x=b+c-2a$, $y=c+a-2b$, $z=a+b-2c$, find the value of

$$x^2+y^2+z^2+2yz+2zx+2xy.$$

5. If $f(x)=x^2+x+1$, shew that

$$f(z) \,.\, f(-z)=f(z^2).$$

B

1. Find and tabulate the values of x^2 for

$$x=-3, -2\tfrac{1}{2}, -2, -1\tfrac{1}{2}, -1, -\tfrac{1}{2}, 0, +\tfrac{1}{2}, +1, +1\tfrac{1}{2}, +2, +2\tfrac{1}{2}, +3.$$

2. Find the roots of the following quadratic equations:

$$\text{(i)} \quad x^2-x=0;$$
$$\text{(ii)} \quad x^2=0.$$

3. If interest is at the rate of r on the unit, shew that the amount of \$1 at the end of 1, 2, 3 years is

$$\$(1+r), \quad \$(1+r)^2, \quad \$(1+r)^3.$$

What would be the amount at the end of n years, n a positive integer?

4. If $x=b-c$, $y=c-a$, $z=a-b$, shew that

$$x^3+y^3+z^3=3xyz.$$

5. If

$$\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}$$

shew that

$$(b-c)x+(c-a)y+(a-b)z=0.$$

C

1. Find and tabulate the values of x^2+3 for
$$x=-2,-1\cdot5,-1,-0\cdot5,0,+0\cdot5,+1,+1\cdot5,+2.$$

2. Construct the quadratic equation whose roots are

(i) 0,1;
(ii) 0,0.

3. If interest is at the rate of r per cent., the present value of \$1 due at the end of 1, 2, 3 years is
$$\$\left(\frac{100}{100+r}\right), \quad \$\left(\frac{100}{100+r}\right)^2, \quad \$\left(\frac{100}{100+r}\right)^3.$$

4. If
$$ax=ay,$$
does it follow that $x=y$?

5. Shew that the equation
$$(x-1)(x-2)(x-3)=0$$
has three and only three roots.

D

1. Find and tabulate the values of x^2-3 for
$$x=-2,-1,-0\cdot8,-0\cdot6,-0\cdot4,-0\cdot2,0,+0\cdot2,+0\cdot4,+0\cdot6,+0\cdot8,+1,+2.$$

2. Shew that x^2+3x+5 and $2x^2-x+8$, in general unequal, are equal for certain values of x.

3. Construct the equation whose roots are 3,4,5.

4. If interest is at the rate of r on the unit, shew that the present value of \$1 due at the end of 1,2,3 years is
$$\$\left(\frac{1}{1+r}\right), \quad \$\left(\frac{1}{1+r}\right)^2, \quad \$\left(\frac{1}{1+r}\right)^3.$$

5. If a heavy particle is projected vertically upwards with a velocity v (ft. a sec.), its distance s (ft.) from the earth at the end of time t (sec.) is given by the formula
$$s=vt-16t^2.$$

Assuming this, find when a particle projected thus with a velocity 72 ft. a sec. is at a height 77 ft. above the earth.

CHAPTER XVII

GRAPH OF THE QUADRATIC EXPRESSION

76. Introductory Note. In a quadratic function as x^2, x^2+3, x^2-4x+3, the x is to be regarded as a variable. Our enlarged number system now includes real numbers, as $2-1, 0, \sqrt{3} \ldots$ and imaginary or complex numbers as $\sqrt{-1}, \sqrt{-7}, 3+\sqrt{-5}, \ldots$ In this chapter it will be supposed that the variable x is to have or to be assigned only real values, and, in general, any real value we please, positive, negative, or zero. Attention may be called to the fact that *the square of a real number is positive, except in the case of the number zero, the square of which has the value zero.*

Further, it may be recalled (see *Illustration* 3, *p.* 24), that, in accordance with general usage, we say, for example, that -3 is *less* than -2, although *numerically* or in *absolute value* this is not so. Indeed, the fact that 4 is less than 7 may be expressed algebraically by stating that $7-4$ is *positive*; and, carrying this mode of statement over to negative numbers, we say that, as $(-2)-(-3)$, or $-2+3$, is positive, -3 is less than -2, or in symbols

$$-3 < -2.$$

We say also that -2 is greater than -3, or in symbols

$$-2 > -3.$$

77. Study of the Function x^2.

(1) *Variation of the Function* x^2. As x is to have only real values, it is readily seen that

(i) For every positive value of x and every negative value of x the value of x^2 is positive, *i.e.*, greater than zero, while for the value zero of x the value of x^2 is zero. Thus, while x may have any value we choose to assign it, ***the value of*** x^2 ***cannot be less than zero.***

(ii) Any negative value of x as, say -5, gives to x^2 the same value as the corresponding positive value of x, here $+5$.

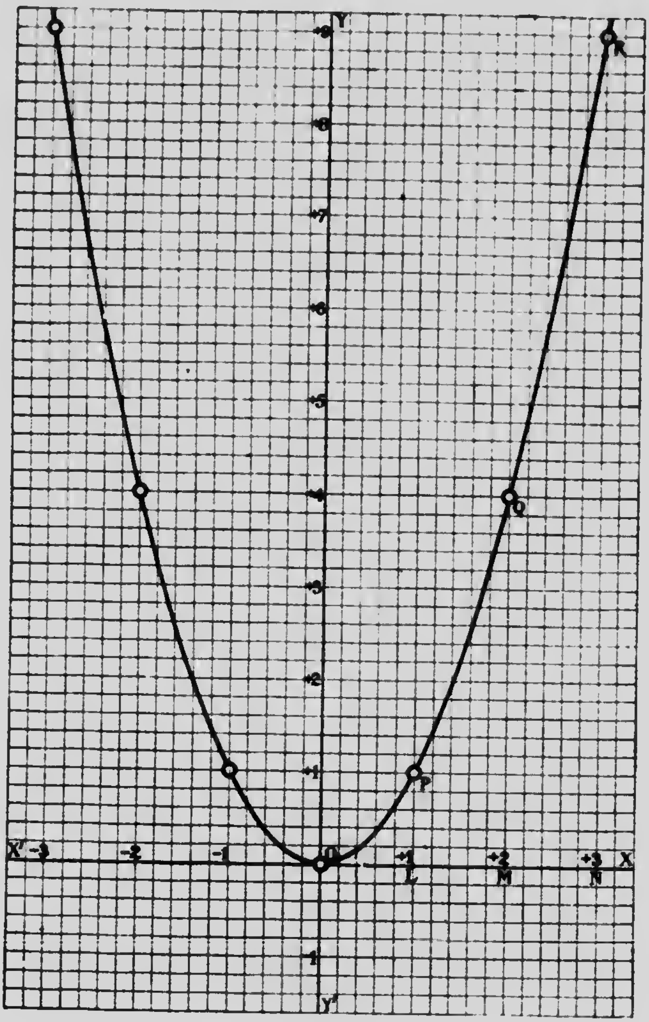
(iii) The *farther from zero*, or the *greater numerically*, x is taken, the greater the value of x^2.

The facts stated in (i), (ii), (iii) are all illustrated in the following tabular statement:

For $x =$	-3	-2	-1	0	$+1$	$+2$	$+3$
$x^2 =$	$+9$	$+4$	[illegible]	0	$+1$	$+4$	$+9$

(2) *The Graph of the Function* x^2. As previously explained, let X′OX, YOY′ be a set of axes. A value of x, as $+2$, is marked by the point M distant 2 from the origin O, and to the right of O since $+2$ is positive; the corresponding value of x^2, namely $+4$, is marked by the point Q, distant 4 from M in the direction Y′OY and in the *sense* OY since $+4$ is positive. Thus the point Q, in virtue of its position, registers the fact that for $x = +2$ the value of x^2 is $+4$. Let all the values of x and x^2 in the table be similarly marked by points. Additional values of x, as many as we please, may be taken, and these with the corresponding values of x^2 marked by additional points.

These points tend to fill out a line the points of which carry the corresponding values of x and x^2. That this line is not a straight line, as is the case for the linear function, is at once evident. From (i), (ii), (iii), it follows that the line must be as given in the figure.



Only a small part of the curve can be drawn, since as x becomes large the value of x^2 becomes too large to be marked on a small sheet.

If we denote the function x^2 by y, we may say that the graph is the graph of the *function* x^2 or of the *equation* $y = x^2$.

The student should study this example with the greatest care, satisfying himself of the symmetry of the curve with respect to the axis YOY′, and seeing that as x grows from zero, either through positive or through negative values, the function x^2 grows from zero through positive and *always increasing* values, the curve registering those increasing values; or, otherwise stated, that, as x passes through negative or through positive values to zero, the curve registers the values, *always declining*, of x^2, until at zero the *minimum value* of x^2, or briefly the *minimum* of x^2, is reached.

Exercises CIII

1. Compute the value of x^2 for values of x at intervals of 0·2 between 0 and 3, and employ these values to construct on a sheet ruled to tenths the graph of $y = x^2$ for values of x between -3 and $+3$.

Employ the graph to read off approximate values of

(1) $(1{\cdot}7)^2$, $(2{\cdot}3)^2$, $(2{\cdot}9)^2$.

(2) $\sqrt{2}$, $\sqrt{3}$, $\sqrt{3{\cdot}4}$, $\sqrt{4{\cdot}5}$, $\sqrt{6{\cdot}2}$, $\sqrt{8{\cdot}5}$.

2. Construct the graph of the function $2x^2$ for values of x between $-2{\cdot}5$ and $+2{\cdot}5$.

3. Construct the graph of the equation

$$2y = 3x^2$$

for values of x between -2 and $+2$.

4. Construct on the sheet employed in the first of these exercises, and with the same axes, the graph of the function $-x^2$.

What is the *maximum* value of $-x^2$?

78. Study of the Function $x^2 - 4x + 3$.

(1) *Variation of the Function* $x^2 - 4x + 3$. For the values $-1, 0, +1, +2, +3, +4, +5$ the values of the given expression as given in the following table:

For $x =$	-1	0	$+1$	$+2$	$+3$	$+4$	$+5$
$x^2 - 4x + 3 =$	$+8$	$+3$	0	-1	0	$+3$	$+8$

A glance at the table reveals the fact that as x advances by steps of 1 from the value $x = 2$ the function has the same values as when x recedes by steps of 1 from the value $x = 2$. The reason for this will appear. We have identically

$$x^2 - 4x + 3 = (x - 2)^2 - 1$$

and it is evident that the latter form is the more convenient one for computing the value of the function. Thus for $x = 7$ we see at once that the value of the function is $(7 - 2)^2 - 1$ or 24. But the form is otherwise of great value. First, since $(x - 2)^2$ is the square of a real number, its lowest value is zero, and it takes this value for $x = 2$; hence the lowest value, the *minimum*, of $(x - 2)^2 - 1$ is $0 - 1$ or -1, and it takes this value for $x = 2$. Next, give to x any two values, the one as much greater than 2 as the other is less than 2. Let $2 + h$ and $2 - h$ be two such values where h is any real number. Then for $x = 2 + h$ and for $x = 2 - h$ the values of the expression $x^2 - 4x + 3$ or $(x - 2)^2 - 1$ are

(i) $(2 + h - 2)^2 - 1$ or $h^2 - 1$,

(ii) $(2 - h - 2)^2 - 1$ or $h^2 - 1$,

i.e., are the same, and this is true whatever h may be. Thus the symmetry in the values of the expression as x advances and recedes from the value $x = 2$ is established.

Next note that identically

$$x^2-4x+3=(x-1)\ (x-3)$$

a form useful for the computation of the value of the function for assigned values of x. We see that the *roots* of the *equation*

$$x^2-4x+3=0$$

are 1 and 3.

For any value of x between the *roots* of the *equation* the value of the *expression is negative*. For example, for $x=1\frac{1}{2}$ which is greater than 1 and less than 3, we have, as value of the expression,

$$(1\tfrac{1}{2}-1)\ (1\tfrac{1}{2}-3)$$

which, since one factor is negative and the other positive, must be negative. Similarly for any other value of x between 1 and 3.

For any value of x less than the smaller root 1, the value of the expression is positive. For example, for $x=\frac{1}{2}$, which being less than 1 is also less than 3, the value of the expression is

$$(\tfrac{1}{2}-1)(\tfrac{1}{2}-3)$$

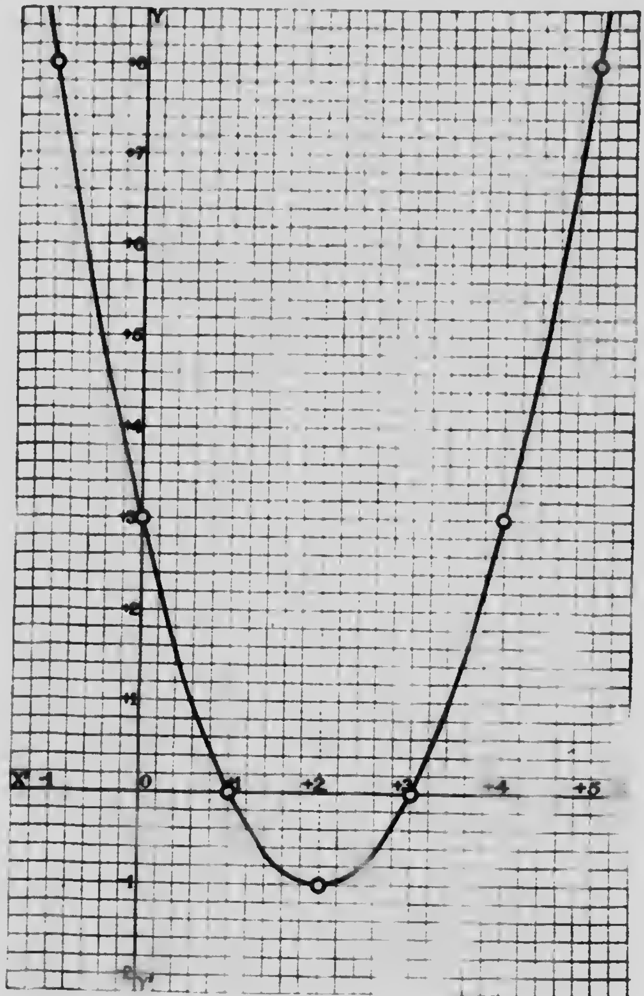
which is positive, both factors being negative.

Similarly for any value of x greater than the greater root 3, the value of the expression is positive, since for such a value the two factors of the expression are positive.

(2) *Graph of the Function* x^2-5x+6. As in the earlier section, take a set of axis, mark the values given in the table by points and draw through the points the curve suggested by them. The reason for the symmetry of the curve about the dotted line has already appeared.

The graph presents to the eye all that has been developed in the study of the function. Thus the function has a minimum value -1, and this for $x = 2$. The *function* has a zero value, *i.e.*, the *equation*

$$x^2 - 4x + 3 = 0$$



is satisfied for the values 1 or 3 of x; *i.e.*, the curve crosses the x-axis at points which give the roots of the equation. The curve lies below the x-axis, *i.e.*, the function is negative, for values of x between the roots of the equation. The curve lies above the x-axis, *i.e.*, the function is positive, for value, of x *outside* of the roots of the equation. The decline in value of the function as x passes through values up to 2, the minimum at $x=2$, the rise in value as x passes beyond 2, are all reflected in the graph.

Exercises CIV

1. Noting that $-x^2+4x-3$ is equal to $-(x^2-4x+3)$, study the variation in value of $-x^2+4x-3$, finding for what values of x it vanishes, for what values of x it is negative, for what values of x it is positive, and for what value of x it is a *maximum*.

Also construct the graph of the function comparing it with the graph of x^2-4x+3.

2. Study the variation and construct the graph of each of the following:

(1) x^2-x-2; (2) $2+x-x^2$;
(3) x^2+x-2; (4) $2-x-x^2$;
(5) x^2-4; (6) $4-x^2$;
(7) x^2+x-6; (8) $6-x-x^2$;
(9) $x^2-8x+15$; (10) $15-2x-x^2$.

3. Noting that $2x^2+x-3$ is equal to $2(x^2+\frac{1}{2}x-\frac{3}{2})$, or to $2(x+\frac{3}{2})(x-1)$, study the variation of $2x^2+x-3$, finding its minimum value and the values of x for which it is zero, positive, negative.

Construct also the graph of the function.

4. Study the variation of and represent graphically each of the following functions:

(1) $5x^2+3x-14$; (2) $14-3x-5x^2$;
(3) $6x^2+5x-6$; (4) $6-5x-6x^2$;
(5) $10x^2+x-21$; (6) $21-x-10x^2$;
(7) $10x^2-x+21$; (8) $21+x-10x^2$;
(9) $6x^2-19x+10$; (10) $-10+19x-6x^2$.

5. Study the variation of the function x^2-4x+2, finding its minimum.

Represent the function graphically on a liberal scale, and from the figure find an approximate value to each root.

Find also the roots of the equation, and by taking a good arithmetical approximation to their values, test the estimate made from the graph.

6. Treat as in the exercise just proposed each of the following:

(1) x^2-6x+7; (2) x^2-4x-4;

(3) $6x^2-x-10$; (4) $2x^2+3x-4$;

(5) x^2-3x+1; (6) $2x^2-3x-3$.

7. Construct on the same sheet and with the same axes the graphs of the three functions

$$x^2-4x+3,\quad x^2-4x+4,\quad x^2-4x+5.$$

8. Noting that x^2-2x is equal to $(x-1)^2-1$, find the minimum value of x^2-2x.

9. Noting that $2x-x^2$ is equal to $1-(1-x)^2$, find the maximum value of $2x-x^2$.

10. Divide a straight line of length 2 into two parts such that the rectangle contained by the two parts may be the greatest possible.

11. Construct the graph of the function

$$x^2-5x+3.$$

Also solve the equation

$$x^2-5x+3=0$$

and state how the graph indicates the character of the roots.

12. Construct on the same sheet and with the same axes the graphs of the three functions

$$x^2-2x,\quad x^2-2x+1,\quad x^2-2x+2.$$

Also solve the corresponding equations and state how the nature of the roots is indicated in the graphs.

79. Minimum of the Sum of Two or More Squares. It has been seen that the expression x^2, where x is supposed to be real, assumes its smallest value for the value zero of x. For the same reason the expression

$$x^2 + y^2$$

when x and y are supposed to be real variables, assumes its minimum when $x = 0$ and $y = 0$. For if x or y or both x and y are not zero then $x^2 + y^2$ is positive, whereas if x and y are both zero the value of $x^2 + y^2$ is zero. Thus $x^2 + y^2$ has zero for its minimum.

Further, if x and y are supposed real and it is given that

$$x^2 + y^2 = 0,$$

then it must be that $x = 0$ and $y = 0$, since any other hypothesis as to the values of x and y would give to $x^2 + y^2$ a positive value, *i.e.*, a value different from zero. Similarly if

$$(x-1)^2 + (y-2)^2 = 0$$

and x and y are real it follows that $x = 1$ and $y = 2$.

Like considerations apply when more than two squares are thus involved.

Exercises CV

1. If the variables that occur are supposed real, find the minimum value of each of the following:

(1) $v^2 + w^2$; (2) $v^2 + w^2 + 1$;

(3) $u^2 + v^2 + w^2$; (4) $u^2 + v^2 + w^2 + 5$;

(5) $2x^2 + 3y^2$; (6) $3x^2 + 5y^2 - 7$;

(7) $(x-3)^2 + (y-5)^2$; (8) $x^2 - 4x + y^2 - 6x$.

2. If x and y are real, find the maximum value of

$$5 - x^2 - y^2.$$

3. If x, y, z are real and

$$x^2 + y^2 + z^2 = 0,$$

shew that x, y, z are each zero.

4. If l and m are real and

$$(l-3)^2 + (m+4)^2 = 0,$$

find the values of l and m.

5. If u, v, w are real and a, b, c are given real constants and

$$(u-a)^2 + (v-b)^2 + (w-c)^2 = 0,$$

then $x = a$, $y = b$, $z = c$.

6. Express by a simple equation

(1) That $x = 0$ and $y = 0$;

(2) That $x = 2$ and $y = 3$;

(3) That $x = 1$, $y = 2$, $z = 3$.

7. If x, y, z are all real and

$$(y-z)^2 + (z-x)^2 + (x-y)^2 = 0,$$

then $x = y = z$.

8. If u and v are real and

$$u^2 + v^2 = 2uv,$$

then must $u = v$.

9. Shew that in general

$$u^2 + v^2 > 2uv,$$

u and v being real numbers.

10. If u, v, w are all real and

$$u^2 + v^2 + w^2 = vw + wu + uv,$$

then must $u = v = w$.

[Note that

$$u^2 + v^2 + w^2 - vw - wu - uv = \tfrac{1}{2}\{(v-w)^2 + (w-u)^2 + (u-v)^2\}].$$

11. Find the maximum value of

$$7 - 4x - 2y - x^2 - y^2,$$

if x and y are real variables.

Exercises CVI

(Miscellaneous)

A

1. Find the value of the indicated product

$$(1+x+x^2+x^3+x^4)(1-x+x^2-x^3+x^4)$$

(i) by ordinary multiplication, (ii) by multiplication by Horner's method, (iii) by writing the product as that of the sum and the difference of two quantities, (iv) by employing the fact that $(1-x^5)\div(1-x)$ is the equivalent of the first factor with an analogous equivalent of the second factor.

2. If $yz=20$, $zx=15$, $xy=12$, shew that xyz must have one of two values, and find the value of x, of y, and of z corresponding to each.

3. Find the quotient of x^5, (i) by x^4; (ii) by x^5; (iii) by x^6.

4. Find the value of $(-1+\sqrt{3})^3$, and of $(-1+\sqrt{-3})^3$, obtaining an approximation to the former to the second place of decimals.

5. If x is a real number shew that the value of x^2+x+1 is necessarily positive.

B

1. Find the only numbers which are equal to their squares.

2. If $3s=a+b+c+d$ shew that

$$\frac{s-a}{s}+\frac{s-b}{s}+\frac{s-c}{s}+\frac{s-d}{s}=1.$$

3. Write the expansions of

$$(a-b-c)^3,\ (b+c-a)^3$$

and explain why it is that the one expansion is the negative of the other.

4. Shew that 2 is the only real number whose cube is 8.

5. By a geometrical construction find a line whose *length* is $\sqrt{13}$.

C

1. A sum of money is divided among A, B, and C. The combined shares of B and C being \$80, of C and A being \$72, and of A and B being \$66, find the sum and the share of each.

2. Simplify

$$\frac{x^2+xy+y^2}{x^2-xy+y^2}\times\frac{x^3+y^3}{x^3-y^3}\div\frac{(x+y)^2}{(x-y)^2}.$$

3. If

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{w}{d}$$

then

$$\frac{x^4+y^4+z^4+w^4}{a^4+b^4+c^4+d^4}=\frac{xyzw}{abcd}.$$

4. If $x^2+x=a^2+a$, where a is supposed known, find the value of x.

5. If $x-3$ is a factor of $x^2-mx+12$, find what the value of m must be.

D

1. The *lengths* of the edges of a rectangular solid are a, b, c; find the length of a diagonal.

2. Solve the two equations

$$x^2-5x+3=0,\ x^2-11x+27=0$$

and shew that the roots of the latter exceed those of the former by 3.

3. Resolve into factors

$$x^3-(a-b)x^2-(ab+2b^2)x+2ab^2.$$

4. If x, y, z, l, m, n are all real, express in a single statement that $x=l$, $y=m$, $z=n$.

5. Divide 35 into three parts such that one-fourth of the first is 2 less than one-third of the second, and one less than one-fifth of the third.

CHAPTER XVIII

THE GENERAL QUADRATIC EQUATION

80. Solution of the General Equation. In the quadratic equations considered up to the present the coefficients of the powers of the unknown and the absolute term have been *numerical*, and on this account these equations are said to be *particular* equations. We consider now the equation

$$ax^2 + bx + c = 0$$

where a, b, c are *definite* numbers, *known* numbers, yet *any integers or fractions we please*, except that a is supposed not to be zero in order that the equation may be veritably of the second degree. Every quadratic equation can be brought to this standard form, and this equation is, in its generality, *every* quadratic equation, the attention being limited to equations with rational coefficients. Thus if we take $a = 2$, $b = -7$, $c = -3$, it is the equation

$$2x^2 - 7x - 3 = 0.$$

To solve the general equation we proceed as in the case of any of the particular equations solved, and resolve the quadratic expression involved into factors linear in x. Turning then to the quadratic expression, we have *identically, i.e., for all values of x*,

$$ax^2 + bx + c$$

$$= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

$$= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2\right]$$

$$= a\left[\left\{\left(x + \frac{b}{2a}\right) - \frac{\sqrt{b^2 - 4ac}}{2a}\right\}\left\{\left(x + \frac{b}{2a}\right) + \frac{\sqrt{b^2 - 4ac}}{2a}\right\}\right]$$

$$= a\left[\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\right]$$

and the resolution into linear factors is effected.

If now we have the *equation* $ax^2 + bx + c = 0$, it follows from what has just been shewn that

$$a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = 0.$$

In order that this may be true, one of the three factors on the left must be zero. Now a is not zero, and consequently it is necessary that one of the two other factors equal zero, while the vanishing of either will suffice. In order then that the equation be satisfied it must be that

$$x - \frac{-b + \sqrt{b^2 - 4ac}}{2a} = 0, \text{ or } x - \frac{-b - \sqrt{b^2 - 4ac}}{2a} = 0,$$

i.e., it must be that

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

which may be stated more briefly thus:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

and it follows that the general quadratic equation has two and only two roots.

The result (i) should be retained in memory, as by means of it the roots of any proposed equation may be written down without going through the process of solution. Thus the roots of the equation

$$2x^2-7x-3=0$$

where $a=2$, $b=-7$, $c=-3$ are

$$\frac{-(-7)\pm\sqrt{(-7)^2-4(2)(-3)}}{2\times 2}$$

i.e.,
$$\frac{+7\pm\sqrt{73}}{4}.$$

From what precedes it is plain that the solving of a quadratic equation and the resolving of a quadratic expression into linear factors are virtually one problem. In other words, if the roots of a quadratic equation are known, then the factors of the corresponding quadratic expression may be written down, and *vice versa*.

Further, the resolution of a quadratic expression into linear factors is unique. For, denoting by m and n the somewhat complex expressions found for the roots of the equation

$$ax^2+bx+c=0,$$

we have identically

$$ax^2+bx+c=a(x-m)(x-n) \ldots\ldots \text{(ii)}$$

Then ax^2+bx+c cannot admit a different resolution. For if so, let $x-h$ be a factor where h is different from m and n. Then by division we find another factor $x-k$, so that identically

$$ax^2+bx+c=a(x-h)(x-k) \ldots\ldots \text{(iii)}$$

Hence from (ii) and (iii) we have, for all values of x,

$$a(x-m)(x-n)=a(x-h)(x-k).$$

Therefore, since we may assign to x the value h,

$$a\,(h-m)(h-n)=a(h-h)(h-k)$$
$$=0.$$

Now a is not zero, and neither $h-m$ nor $h-n$ is zero, so that the last statement is impossible. Therefore the hypothesis of a different resolution into factors is untenable.

It follows then that no other treatment of the quadratic equation can lead to roots different from those found.

To solve the general quadratic equation $ax^2+bx+c=0$, attention was withdrawn from the *equation* and directed to the *expression* ax^2+bx+c. While this is important as ensuring a complete view of the problem, it is the practice to proceed as follows:

$$ax^2+bx+c=0$$

$$\therefore\quad a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]=0$$

$$\therefore\quad a\left[x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right]=0,\text{ adding }\left(\frac{b}{2a}\right)^2$$

within [ ] to complete the square of the part involving the unknown x, and subtracting to correct.

$$\therefore\quad a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]=0$$

$$\therefore\quad a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2\right]=0$$

$$\therefore\quad a\left[\left\{\left(x+\frac{b}{2a}\right)-\frac{\sqrt{b^2-4ac}}{2a}\right\}\left\{x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right\}\right]=0$$

$$\therefore\quad a\left[\left(x-\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x-\frac{-b-\sqrt{b^2-4ac}}{2a}\right)\right]=0.$$

Hence since a is not zero, one or other of the remaining factors must be zero, and the vanishing of either will ensure that the equation be satisfied.

$$\therefore \quad x - \frac{-b + \sqrt{b^2 - 4ac}}{2a} = 0 \text{ or } x - \frac{-b - \sqrt{b^2 - 4ac}}{2a} = 0$$

or

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

By a slight modification of the process, algebraic fractions may be avoided until the very end.

$$ax^2 + bx + c = 0.$$

Then by multiplication through by $4a$,

$$4a^2x^2 + 4abx + 4ac = 0$$

$$\therefore \quad (2ax)^2 + 2b(2ax) + 4ac = 0$$

$$\therefore \quad \{(2ax)^2 + 2b(2ax) + b^2\} - \{b^2 + 4ac\} = 0$$

$$\therefore \quad (2ax + b)^2 - (\sqrt{b^2 - 4ac})^2 = 0$$

$$\therefore \quad (2ax + b - \sqrt{-b^2 4ac})(2ax + b + \sqrt{b^2 - 4ac}) = 0.$$

From this, as before,

$$2ax = -b + \sqrt{b^2 - 4ac} \text{ or } -b - \sqrt{b^2 - 4ac},$$

and since a is not zero,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Exercises CVII

1. Employing the formula, write down and bring to their simplest form the roots of each of the following:

(1) $2x^2 + 3x - 7 = 0$; (2) $3x^2 - 13x + 11 = 0$;
(3) $x^2 - 7x + 11 = 0$; (4) $5x^2 - 8x + 14 = 0$.

2. Give the complete work of solving the equation

$$x^2 + px + q = 0.$$

Also write down the roots from the formula.

3. Give the complete work of solving the equation

$$ax^2 + 2bx + c = 0.$$

Also write down the roots from the formula, and reduce them to the form found.

Does this equation differ in generality from the equation

$$ax^2 + bx + c = 0?$$

4. Write down the roots of each of the following equations:

(1) $7x^2 - 12x + 16 = 0$; (2) $3x^2 - 7x - 10 = 0$;

(3) $hx^2 + 2kx + l = 0$; (4) $hx^2 - 2kx + l = 0$;

(5) $hx^2 + 2kx - l = 0$; (6) $hx^2 - 2kx - l = 0$;

5. Solve the equation

$$x^2 - 31x + 40 = 0$$

and for the particular expression $x^2 - 31x + 40$, shew that the resolution into linear factors is unique.

6. Bring the following equations to their simplest form shewing that they are of the second degree and writing down their roots:

(1) $\dfrac{3}{x+1} + \dfrac{4}{x-5} = \dfrac{6}{x-2}$; (2) $\dfrac{6x+5}{2x-7} + \dfrac{4x-1}{x-2} = \dfrac{7x+1}{x-3}$;

(3) $\dfrac{x+4}{x-4} + \dfrac{x+9}{x-9} = \dfrac{x-4}{x+4} + \dfrac{x-9}{x+9}$; (4) $\dfrac{2x+1}{x+2} - \dfrac{x+2}{4x+4} = \dfrac{7x+8}{4x+13}$;

(5) $\dfrac{1}{x^2+11x-8} + \dfrac{1}{x^2+2x-8} + \dfrac{1}{x^2-13x-8} = 0$;

(6) $\dfrac{2x+3}{2(2x-1)} - \dfrac{7-x}{2(x+1)} = \dfrac{7-3x}{4-3x}$;

7. Find what the roots of the general equation become

(i) If $b = 0$;

(ii) If $c = 0$;

(iii) If $b = c = 0$.

81. Simple Relations Involving the Roots. If we are given the equation

$$ax^2 + bx + c = 0$$

the x is to be regarded as an *unknown* which through the solution of the equation becomes *known*, and is one of two perfectly definite numbers, viz.,

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a} \ldots\ldots\ldots\ldots \text{(i)}$$

For brevity denote these by m, n, so that m and n signify the two roots of the equation. Then

$$m + n = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2b}{2a}$$

$$= -\frac{b}{a}.$$

$$mn = \frac{\{(-b) + \sqrt{b^2 - 4ac}\}\{(-b) - \sqrt{b^2 - 4ac}\}}{2a.2a}$$

$$= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2}$$

$$= \frac{+b^2 - b^2 + 4ac}{4a^2}$$

$$= \frac{4ac}{4a^2}$$

$$= \frac{c}{a}.$$

These two results

$$\left.\begin{aligned} m + n &= -\frac{b}{a} \\ mn &= \frac{c}{a} \end{aligned}\right\} \ldots\ldots\ldots\ldots \text{(ii)}$$

interpreted mean:

The sum of the two roots of a quadratic equation is equal to the quotient, with its sign changed, of the second coefficient by the first; and the product of the two roots is equal to the quotient of the absolute term by the first coefficient.

The expressions (i) for the roots are algebraically irrational, while the relations (ii) are rational. It frequently happens that facts involving the roots of the equation are more easily established by employing these simple relations than by making use of the actual values of the roots.

Ex. 1. If m and n are the roots of the equation

$$2x^2-7x-9=0,$$

find the value of m^3+n^3.

Noting that

$$m+n=\tfrac{7}{2}\;,\quad mn=-\tfrac{9}{2}$$

we have

$$\begin{aligned} m^3+n^3 &= (m+n)(m^2-mn+n^2) \\ &= (m+n)[(m^2+2mn+n^2)-3mn] \\ &= (m+n)[(m+n)^2-3mn]. \end{aligned}$$

Thus m^3+n^3 is expressed in terms of the known quantities $m+n$ and mn, and we have

$$\begin{aligned} m^3+n^3 &= \tfrac{7}{2}\times[(\tfrac{7}{2})^2+3\times\tfrac{9}{2}] \\ &= -\tfrac{721}{8}. \end{aligned}$$

The dependence of m^3+n^3 upon $m+n$ and mn may be otherwise established thus:

$$\begin{aligned} m^3+n^3 &= m^3+3mn(m+n)+n^3-3mn(m+n) \\ &= (m+n)^3-3mn(m+n). \end{aligned}$$

The student should find the value of m^3+n^3 by finding the two roots and computing the sum of their cubes.

The expression m^3+n^3 is *symmetrical* in m and n, and every symmetrical expression can, as m^3+n^3, be expressed in terms of $m+n$ and mn.

Ex. 2. If m and n are the roots of the equation

$$3x^2-11x+7=0,$$

find the equation whose roots are $m+3$ and $n+3$.

Plainly the equation whose roots are the proposed numbers is

$$(x-\overline{m+3})(x-\overline{n+3})=0.$$

This however is not what is wanted, as the expression on the left involves m and n which are *as yet unknowns*. We have then to find m and n and substitute their values in this last equation. But we may proceed thus. The equation is

$$x^2-(m+n+6)x+(m+3)(n+3)=0$$

or

$$x^2-(m+n+6)x+mn+3(m+n)+9=0.$$

Now $m+n=\frac{11}{3}$ and $mn=\frac{7}{3}$, so that the equation becomes

$$x^2-(\tfrac{11}{3}+6)x+\tfrac{7}{3}+3\times\tfrac{11}{3}+9=0$$

or in simplest form

$$3x^2-29x+67=0.$$

The student should solve the two equations and verify that the roots of this last exceed those of the proposed by 3.

Ex. 3. If m and n are the roots of the equation

$$ax^2+bx+c=0,$$

shew that the expression ax^2+bx+c is identically equal to

$$a(x-m)(x-n).$$

This fact has already appeared in the actual solution of the equation, so that now it is merely an exercise in verification. We have, identically,

$$ax^2+bx+c=a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]$$
$$=a\left[x^2-(m+n)x+mn\right],$$

where m and n are the roots of the equation $ax^2+bx+c=0$.

Resolving into factors the expression on the right, we have identically

$$ax^2+bx+c=a(x-m)(x-n).$$

EXERCISES CVIII

1. If m and n are the roots of the equation

$$7x^2 - 13x + 5 = 0,$$

find the value of each of the following:

$$m^2 + n^2,\ m^3 + n^3,\ m^4 + n^4,\ m^3 + m^2n + mn^2 + n^3,\ m^3n + mn^3,$$

$$\frac{1}{m} + \frac{1}{n},\ \frac{1}{m^2} + \frac{1}{n^2},\ \frac{m^2}{n} + \frac{n^2}{m},\ \frac{m^2}{n^2} + \frac{n^2}{m^2},\ \left(\frac{1}{m} - \frac{1}{n}\right)^2.$$

Verify each result by substituting for m and n their values found by solving the equation.

2. If m and n are the roots of the equation

$$4x^2 - 7x - 11 = 0,$$

find the equation whose roots are

(i) $m + 2,\ n + 2$; (ii) $m^2,\ n^2$; (iii) $\frac{1}{m},\ \frac{1}{n}$; (iv) $\frac{m}{n},\ \frac{n}{m}$.

Verify each result by solving the equation found and comparing the roots with those of the given equation.

3. If m and n are the roots of the equation

$$ax^2 + bx + c = 0,$$

find in terms of the known numbers a, b, c, the value of each of the following:

$$m^2 + n^2,\ m^3 + n^3,\ \frac{1}{m} + \frac{1}{n},\ \frac{1}{m^2} + \frac{1}{n^2},\ \frac{m^2}{n} + \frac{n^2}{m}.$$

4. If m and n are the roots of the equation

$$ax^2 + bx + c = 0,$$

find the equation whose roots are

(i) $m + 4,\ n + 4$; (ii) $m + h,\ n + h$; (iii) $\frac{1}{m},\ \frac{1}{n}$; (iv) $hm,\ hn$;

(v) $m - n,\ n - m$.

5. If m and n are the roots of the equation

$$x^2 + px + q = 0,$$

find the value of $m - n$, and explain why it should be not determined absolutely but as one of two perfectly definite numbers.

82. Nature of the Roots of the Quadratic. In solving the general quadratic equation a point is reached where, in order to resolve the quadratic expression into factors, we say that b^2-4ac is equal to $(\sqrt{b^2-4ac})^2$, and the algebraic irrationality thus introduced appears in the roots

$$\frac{-b+\sqrt{b^2-4ac}}{2a},\quad \frac{-b-\sqrt{b^2-4ac}}{2a}$$

of the equation. Now a,b,c are any integers or fractions whatever, a not zero, and are often such as to make b^2-4ac the square of some integer or fraction. Thus if $a=12$, $b=23$, $c=10$, so that the implied equation is

$$12x^2+23x+10=0,$$

the roots yielded are

$$\frac{-23+\sqrt{49}}{24},\quad \frac{-23-\sqrt{49}}{24},$$

and, since 49 is the square of 7, we find the roots to be

$$-\frac{2}{3},\quad -\frac{5}{4}.$$

But if $a=4$, $b=-11$, $c=5$, so that the equation is

$$4x^2-11x+5=0,$$

the roots yielded are

$$\frac{11+\sqrt{41}}{8},\quad \frac{11-\sqrt{41}}{8},$$

and, as $\sqrt{41}$ is no integer or fraction, the importance of the irrational numbers, introduced in Section 73, appears. Similarly for $a=2$, $b=3$, $c=5$, *i.e.*, for the equation

$$2x^2+3x+5=0,$$

the roots yielded are

$$\frac{-3+\sqrt{-31}}{4}, \quad \frac{-3-\sqrt{-31}}{4},$$

and, as $\sqrt{-31}$ is no real number, the importance of the imaginary and complex numbers, introduced in Section 74, appears.

Suppose now that, in our work with particular or, as we may say, *numerical* equations, we had not come upon any such as actually did lead to the bringing in of irrational and imaginary numbers. In our treatment of the general equation we should come upon the expression b^2-4ac, and this we would like to write as $(\sqrt{b^2-4ac})^2$. There would then arise the question whether $\sqrt{b^2-4ac}$, which does not admit algebraic simplification, has always an arithmetical meaning. We should then see that b^2-4ac may be, as in the examples just considered, a number as 41 or -31, so that $\sqrt{b^2-4ac}$, being $\sqrt{41}$ or $\sqrt{-31}$, *i.e.*, being no integer or fraction, is by supposition without meaning. In face of this difficulty we should probably be led to say: *$\sqrt{b^2-4ac}$ sometimes has a meaning, and sometimes not; in dealing with the general equation it will be necessary to be always on guard, making sure that a,b,c are such as to give $\sqrt{b^2-4ac}$ a meaning. To make the equation truly general, and to simplify matters, let us treat $\sqrt{b^2-4ac}$ as if it always had a meaning, and accept the consequences.* Thus, in a way, not really different from that actually followed in Sections 73 and 74, we should be led to introduce as *consequences* the irrational number and the imaginary number.

We might now suppose that, in the equation $ax^2+bx+c=0$, the coefficients are not merely any integers or fractions we please, but any real numbers we please. This, however, would demand a treatment of irrational numbers that would be out of place in an elementary work. We suppose then, unless it may be in a few cases that will be noted, that in the general equation $ax^2+bx+c=0$ the coefficients a,b,c are any integers or fractions, a not zero.

A reference to the two roots will now shew that:

(i) If b^2-4ac is positive, *i.e.*, if b^2 is greater than $4ac$, the roots are real and unequal, and arithmetically either rational or irrational;

(ii) If b^2-4ac is zero, *i.e.*, if $b^2=4ac$, the two roots are equal, being $\frac{-b}{2a}$ and $\frac{-b}{2a}$, which are real and rational;

(iii) If b^2-4ac is negative, *i.e.*, if b^2 is less than $4ac$, the two roots are complex.

Or more briefly:

The roots of a quadratic equation are real and different, real and equal, or complex according as $b^2>4ac$, $b^2=4ac$, $b^2<4ac$.

It is to be noted that when one root is complex, so also is the other. Thus the roots of the equation

$$4x^2-9x+6=0$$

are

$$\frac{9+\sqrt{-15}}{8},\quad \frac{9-\sqrt{-15}}{8}$$

or

$$\frac{9}{8}+i.\frac{\sqrt{15}}{8},\quad \frac{9}{8}-i.\frac{\sqrt{15}}{8}.$$

Two numbers, as $h+i.k$, $h-i.k$, where h and k are real and i denotes $\sqrt{-1}$, are said to be *conjugate*.

EXERCISES CIX

1. Pronounce directly on the character of the roots of each of the following equations:

(1) $2x^2-13x+12=0$; (2) $3x^2-8x+7=0$;

(3) $5x^2+11x+9=0$; (4) $7x^2+12x+4=0$;

(5) $3x^2-5x-7=0$; (6) $11x^2-3x-5=0$.

2. In the equation

$$ax^2+bx-c=0$$

where a,b,c are any real numbers, and a and c are either both positive or both negative, *i.e.*, are *of the same sign*, the roots of the equation are real.

3. If the roots of the equation

$$ax^2+bx+c=0$$

are real, then also are those of the equation

$$ax^2+2bx+c=0$$

Examine the converse of this.

4. The roots of the two equations,

$$ax^2+bx+c=0, \quad ax^2-bx+c=0$$

are of the same character.

5. Find in its simplest form the quadratic equation of which one root is $5+\sqrt{-7}$.

6. Represent graphically the three functions

$$x^2-6x+8, \quad x^2-6x+9, \quad x^2-6x+10$$

and point out how these graphs indicate the character of the roots of the corresponding equations.

7. Discuss the problem: To divide a straight line of *length* 4 into two parts such that the rectangle under the parts may be of *area* 5.

EXERCISES CX

(MISCELLANEOUS)

1. The roots of the equation

$$ax^2 + c = 0$$

are equal in numerical value and each the negative of the other. When will the roots of this equation be real?

From the relations of Section 81, find when the roots of an equation are each the negative of the other.

2. Taking the side of a given equilateral triangle as of length 2, construct a line of length $\sqrt{7}$.

3. If

$$3x + 4y = 25,$$

explain what is to be understood by x and y.

4. Continue the series of numbers

$$1 + 3 + 5 + 7 + \cdots$$

to 13 terms, then write the numbers in reverse order under those already written, then add vertically and thus find the sum of the 13 numbers first written.

5. Find the result of substituting in the expression

$$x^2 + y^2 + z^2 - yz - zx - xy$$

the values

$$x = b - c, \quad y = c - a, \quad z = a - b$$

6. The roots of the equation

$$ax^2 + bx + a = 0$$

are each the reciprocal of the other.

From the relations of Section 81 find when the roots are so related.

7. Solve the equation

$$x^2 - 6x + 11 = 0$$

and find the minimum value of the expression

$$x^2 - 6x + 11$$

and comment on the results.

8. If the side of an equilateral triangle is of *length* a, shew by means of a figure that the triangle is of *area* $\frac{a^2\sqrt{3}}{4}$.

CHAPTER XIX

SIMULTANEOUS EQUATIONS OF THE SECOND DEGREE

83. Illustrative Problem. The following problem will lead to a set of equations, in two unknowns, unlike those studied in Chapter XV in that the unknowns occur to a degree higher than the first.

PROBLEM. Find the fraction which becomes equal to $\frac{3}{4}$ when 7 is added to its numerator and 8 to its denominator, and which becomes equal to its reciprocal when 11 is added to its numerator and 2 to its denominator.

Let x and y be the numerator and the denominator of the fraction sought.

Then, at once, from the given conditions we have

$$\left.\begin{aligned}\frac{x+7}{y+8}&=\frac{3}{4}\\ \frac{x+11}{y+2}&=\frac{y}{x}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

These then are two equations which we expect to determine the values of the unknowns, x and y. We say nothing of their degree until they are brought to a form free from algebraic fractions. Simplifying, we find the equations to be the equivalent of the set

$$\left.\begin{aligned}4x-3y+4&=0\\ x^2-y^2+11x-2y\quad&=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

The first equation has in it terms of the first degree, but no terms of degree higher than the first, in either or both of x,y; it is then a *simple* or *linear equation in two unknowns*. The second equation has in it terms of the second degree, but no terms of degree higher than the second, in either or both of x,y; it is then a *quadratic equation in two unknowns.*

As in Section 66, it is seen that sets of values of x,y—as many as we please—can be found to satisfy the first equation;

thus if we take $x=2$ the equation requires that $y=3$, and $x=2$, $y=3$ is then a solution. A similar remark applies to the second equation. But the x and y sought are to satisfy *both* equations, and it will be seen that their values can be found. From the simple equation express one of the unknowns, say y, in terms of the other. We have then

$$y=\frac{4x+4}{3} \quad \ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

In the quadratic equation of (ii) substitute for y its value in terms of x, given in (iii), which is permissible since y in each equation stands for the same number. Then

$$x^2-\left(\frac{4x+4}{3}\right)^2+11x-2\left(\frac{4x+4}{3}\right)=0,$$

an equation which might have been foreseen to be of the *second degree in one unknown* x, which therefore may be solved. Simplifying we find

$$x^2-\frac{16x^2+32x+16}{9}+11x-\frac{8x+8}{3}=0$$

$$\therefore \quad 9x^2-16x^2-32x-16+99x-24x-24=0$$

$$\therefore \quad -7x^2+43x-40=0$$

$$\therefore \quad 7x^2-43x+40=0$$

$$\therefore \quad (x-5)(7x-8)=0$$

and x must have one of two values, 5 or $\frac{8}{7}$. If $x=5$ we have, since x and y must satisfy the simple equation (iii),

$$y=\frac{4\times 5+4}{3}=8$$

and if $x=\frac{8}{7}$,

$$y=\frac{4\times\frac{8}{7}+4}{3}=\frac{20}{7}.$$

Thus there are two solutions, or two sets of values of x and y, namely

$$\left.\begin{matrix}x=5\\ y=8\end{matrix}\right\},\qquad \left.\begin{matrix}x=\frac{8}{7}\\ y=\frac{20}{7}\end{matrix}\right\}$$

or, more briefly,

$$(5,\ 8),\qquad (\tfrac{8}{7},\ \tfrac{20}{7}).$$

Examine now whether the problem admits both solutions. It is seen that the fraction $\frac{5}{8}$ satisfies the conditions, but it is also seen that the fraction $\frac{\frac{8}{7}}{\frac{20}{7}}$ satisfies the conditions. Thus if we admit complex fractions the problem admits the two answers

$$\frac{5}{8}, \qquad \frac{\frac{8}{7}}{\frac{20}{7}}.$$

84. Illustrative Examples. In the following examples the student may regard the unknowns as referring to some problem, and as being required to satisfy the equations proposed for solution.

Ex. 1. Solve

$$\left.\begin{aligned} x+y&=9 \\ xy&=20 \end{aligned}\right\}$$

Here the first equation is linear but the second, while simple in either x or y, is of two dimensions in these two unknowns. We may solve the set then as in the preceding section. From the simple equation we have

$$y=9-x.$$

Substituting for y, in terms of x, in the other equation we have

$$x(9-x)=20$$

$$\therefore \quad 9x-x^2-20=0$$

$$\therefore \quad x^2-9x+20=0$$

$$\therefore \quad (x-4)(x-5)=0,$$

and x must equal 4 or 5.

If $x=4$, we have *from the simple equation* $y=9-4$, or 5; and if $x=5$ we have $y=9-5$, or 4.

Thus the solutions are

$$(4, 5), (5, 4).$$

From the symmetry of the two given equations it might have been seen that if there were a solution as (4, 5) there would also be a solution (5, 4).

The method of solving two equations, the one simple, the other of two dimensions, in two unknowns is now manifest. Sometimes, however, as here, the set may be solved without a substitution, by a manipulation appealing more perhaps to algebraic taste or skill. From the first equation we have, by *squaring the equals*,

$$x^2 + 2xy + y^2 = 81,$$

and from the second by multiplying

$$4xy \qquad = 80.$$

Then by subtraction

$$x^2 - 2xy + y^2 = 1,$$

whence

$$x - y = +1 \text{ or } x - y = -1.$$

Associating the former with the equation

$$x + y = 9,$$

which must be satisfied, we find

$$x = 5, \ y = 4,$$

and associating the latter with the same equation we find

$$x = 4, \ y = 5.$$

It is thus seen that the given equations are equivalent to the two sets of simple equations

$$\left.\begin{array}{r} x + y = 9 \\ x - y = 1 \end{array}\right\} ; \quad \left.\begin{array}{r} x + y = 9 \\ x - y = -1 \end{array}\right\}$$

Ex. 2. Solve the set, in three unknowns,

$$\left.\begin{array}{r} x - y + z - 4 = 0 \\ 2x - 3y + 4z - 9 = 0 \\ 2x^2 - 5yz + 5x - 7y - 24 = 0 \end{array}\right\} \text{......................(i)}$$

Of these equations the first two are simple and the third is quadratic. The first two equations will not determine x, y, z, but from them we can determine any two of the unknowns, say y and z, in terms of the third. Multiplying the first through by 4, and employing the second in a subtraction, we find

$$2x - y - 7 = 0,$$

an equation in two unknowns, whence

$$y = 2x - 7.$$

Similarly, eliminating y from the first two equations, we find

$$x - z - 3 = 0$$

or

$$z = x + 3.$$

Thus the given set is equivalent to the set

$$\left.\begin{aligned} y &= 2x - 7 \\ z &= x - 3 \\ 2x^2 - 5y^2z + 5x - 7y - 24 &= 0 \end{aligned}\right\} \text{.....................(ii)}$$

Substituting from the first two, in the third, we find

$$2x^2 - 5(x - 3)(2x - 7) + 5x - 7(2x - 7) - 24 = 0,$$

an equation of the second degree in the one unknown x, as might have been foreseen.

$$\therefore \quad 2x^2 - 10x^2 + 65x - 105 + 5x - 14x + 49 - 24 = 0$$

$$\therefore \quad -8x^2 + 56x - 80 = 0$$

whence, by division through by -8,

$$x^2 - 7x + 10 = 0,$$

and from this

$$x = 5 \text{ or } x = 2.$$

If $x = 5$ we have from the simple equations of (ii)

$$y = 2 \times 5 - 7 = 3$$

$$z = \quad 5 - 3 = 2.$$

If $x = 2$ we have similarly

$$y = 2 \times 2 - 7 = -3$$

$$z = 2 - 3 \quad = -1.$$

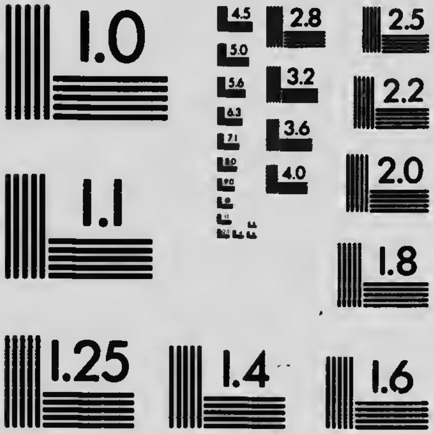
Thus the given set is equivalent to the two sets of simple equations

$$\left.\begin{aligned} x &= 5 \\ y &= 3 \\ z &= 2 \end{aligned}\right\}; \quad \left.\begin{aligned} x &= 2 \\ y &= -3 \\ z &= -1 \end{aligned}\right\}$$

which are the solutions sought.

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



EXERCISES CXI

1. Solve each of the following simultaneous sets:

(1) $\left.\begin{aligned} x-y&=1,\\ xy&=20.\end{aligned}\right\}$ (2) $\left.\begin{aligned} x+y&=12,\\ x^2+y^2&=74.\end{aligned}\right\}$

(3) $\left.\begin{aligned} 2x+3y-12&=0,\\ 2x^2+3y^2-30&=0.\end{aligned}\right\}$ (4) $\left.\begin{aligned} 5x-3y&=23,\\ x^2+xy+y^2&=93.\end{aligned}\right\}$

(5) $\left.\begin{aligned} 7x-4y&=16,\\ 2x^2-3xy+2y^2-5x+7y&=15.\end{aligned}\right\}$ (6) $\left.\begin{aligned} x-y&=\tfrac{2}{3}(x+y),\\ x^2+xy&=270.\end{aligned}\right\}$

(7) $\left.\begin{aligned} \frac{6x+5y}{7}&=\frac{11x+8y}{12},\\ (4x+5y-1)(x+y-3)&=(3x-y+3)(2x+3y+1).\end{aligned}\right\}$

2. If m, n denote the unknown roots of the equation

$$15x^2-47x+28=0,$$

it is known that

$$m+n=\frac{47}{15},$$

$$mn=\frac{28}{15}.$$

From these relations find the roots of the equation.

3. Shew that the simultaneous set

$$\left.\begin{aligned} x^3-y^3&=98,\\ x-y&=2,\end{aligned}\right\}$$

is the equivalent of the set

$$\left.\begin{aligned} x^2+xy+y^2&=49,\\ x-y&=2,\end{aligned}\right\}$$

and solve the set.

4. Shew that the simultaneous set

$$\left.\begin{aligned} xy&=35,\\ xy+15x+10y&=190,\end{aligned}\right\}$$

is the equivalent of the set

$$\left.\begin{aligned} 15x+10y&=155,\\ xy&=35,\end{aligned}\right\}$$

and may therefore be solved as in the exercises of this chapter.

5. The sum of two numbers is 13, and the sum of their squares is 89; find the numbers.

Solve by employing two unknowns and also by employing only one, and compare the solutions.

6. The sides BC, CA, AB of a triangle are of *lengths* 14, 15, 13. A perpendicular AL is drawn to BC. If AL, BL are of *lengths* x, y, find two relations connecting x and y and from them determine the *lengths* AL, BL, CL.

7. The area of a rectangle is 180 square yards. If its length and its breadth were 5 yards and 3 yards greater, the area would be 300 square yards. Find the dimensions of the rectangle.

8. The sum of two numbers is $\frac{17}{12}$, and the sum of their reciprocals is $\frac{17}{6}$. Find the numbers.

Solve by employing two unknowns and also by employing only one.

9. Shew that the set of equations

$$\left.\begin{aligned} x-y &= 2, \\ x^2-y^2 &= 16, \end{aligned}\right\}$$

is the equivalent of a set each of the first degree.

10. The difference of two numbers is 5 and the difference of their squares is 105. Find the numbers.

11. The perimeter of a rectangle is 34 feet, and its diagonal is of length 13 feet. Find the dimensions of the rectangle.

12. The problem of resolving into factors the expression

$$x^2-6x-91$$

reduces to the finding of two numbers whose sum is -6 and product -91. Denoting the two numbers by m and n, find them from these conditions.

13. Solve the following sets of simultaneous equations:

(1) $$\left.\begin{aligned} x+y+z &= 12, \\ z+x &= 2y, \\ z^2 &= x^2+y^2. \end{aligned}\right\}$$

(2) $$\left.\begin{aligned} x+2y-3z &= 4, \\ 2x-y+z &= 5, \\ x^2-3y^2+8z^2 &= yz+3x-6. \end{aligned}\right\}$$

(3) $$\left.\begin{aligned} x+y+z &= 13, \\ x-y-1 &= 0, \\ xy-2z &= 0. \end{aligned}\right\}$$

(4) $$\left.\begin{aligned} 4x-5y+3z &= 14, \\ 5x+7y-8z &= 9, \\ (2x-y)^2 &= (z-8)^2. \end{aligned}\right\}$$

Exercises CXII

(Miscellaneous)

1. If $2y = 3x + 1$ and $5z = 3y + 2$ find z in terms of x.

2. If the series of terms

$$1 + 3 + 5 + 7 + \cdots\cdots$$

be continued find what would be the 13th term, the 37th term, the nth term, n being a positive integer.

3. Shew in any way that $x + y + z$ is a factor of

$$x^3 + y^3 + z^3 + 2(y^2z + yz^2 + z^2x + zx^2 + x^2y + xy^2) + 3xyz,$$

and that the other factor is $x^2 + y^2 + z^2 + yz + zx + xy$.

4. Give the complete work of the solution of the equation

$$x^2 - 5x + 5 = 0,$$

and find an approximation to each root *correct to tenths.*

Construct the graph of the function $x^2 - 5x + 5$ on a sheet ruled to tenths and thus obtain an approximation to each root of the corresponding equation.

5. Two tanks are of the same shape and size. The escape pipe of one will empty it in 15 min., and that of the other will empty it in 12 min. Both tanks being full, the escape pipes are opened; at the end of what time will the depth of the water in the one be twice the depth of the water in the other?

6. Shew that there are two, and only two, numbers which are equal to their reciprocals.

7. If m and n are two integers with p as their G.C.M., shew that p is a common factor of $m + n$ and $m - n$. Shew also that the G.C.M. of $m + n$ and $m - n$ is either p or $2p$.

8. An agent receives goods to be sold on a commission of a per cent., the proceeds to be invested in other goods after the deduction of a commission for investing of b per cent. Shew that the two commissions make up

$$\frac{a + b}{100 + b}$$

of the value of the goods consigned.

CHAPTER XX

EQUATIONS REDUCIBLE TO QUADRATICS

85. General Note. The general equation of the third degree, the general equation of the fourth degree, and many special types of equations of higher degree, in one unknown, may be solved. Such equations will not be considered. There are, however, many special equations, or special classes of equations, of the third, fourth, and even higher degrees, the solution of which depend so immediately upon the quadratic equation as to claim attention. So also there are certain simultaneous equations, of higher degree as a set than those treated in the preceding chapter, the solution of which presents no new difficulty.

86. Simpler Types. In the following examples it will be seen that in each case the equation becomes a quadratic if the unknown is taken to be, not the unknown of the equation, but some expression involving that unknown.

Ex. 1. Solve the equation

$$x^4-9x^2+20=0.$$

Here the equation is of the fourth degree in x, but, regarding x^2 as the unknown for the time being, we see that the equation is a quadratic, as it may be written

$$(x^2)^2-9(x^2)+20=0$$

$$\therefore \quad x^2=4 \text{ or } 5$$

$$\therefore \quad x=+2,\ -2,\ +\sqrt{5},\ \text{or } -\sqrt{5}.$$

Note that the equation is of the fourth degree in x and admits four roots, and also that the equation may be exhibited thus:

$$(x-2)(x+2)(x-\sqrt{5})(x+\sqrt{5})=0.$$

Ex. 2. Solve

$$(x+1)(x+2)(x+3)(x+4)=840.$$

It is seen that if the equation be simplified it is of the fourth degree. When simplified it does not assume a form that suggests a quadratic. If, however, on the left of the sign of equality we associate the first and last factors, and the two other factors, we have

$$(x^2+5x+4)(x^2+5x+6)=840.$$

Now take x^2+5x as the unknown for the time being and denote it by y so that the equation becomes

$$(y+4)(y+6)=840.$$

Then

$$y^2+10y+24=840$$

$$\therefore \quad y^2+10y-816=0$$

$$\therefore \quad (y-24)(y+34)=0$$

$$\therefore \quad y=24 \text{ or } -34.$$

Then, returning to the original unknown, we see that the given equation will be satisfied if

$$x^2+5x=24 \text{ or } x^2+5x=-34,$$

which two equations are together the equivalent of the proposed.

From the first of these we have

$$x=3 \text{ or } -8,$$

and from the second

$$x=\frac{-5+\sqrt{-111}}{2} \text{ or } \frac{-5-\sqrt{-111}}{2}.$$

Thus the given equation, which is of degree four, admits four solutions:

$$3,\ -8,\ \frac{-5-\sqrt{-111}}{2},\ \frac{-5+\sqrt{-111}}{2}.$$

It is to be noted that the equation

$$(x+1)(x+2)(x+3)(x+4)-840=0$$

is no other than

$$(x-3)(x+8)\left(x-\frac{-5-\sqrt{-111}}{2}\right)\left(x-\frac{-5+\sqrt{-111}}{2}\right)=0.$$

The student may solve the equation, denoting x^2+5x+4 by y.

EXERCISES CXIII

1. Solve the following equations:

(1) $x^4 - 10x^2 + 21 = 0.$

(2) $3x^4 - 17x^2 + 12 = 0.$

(3) $(x^2 + 4x)^2 - 2(x^2 + 4x) - 15 = 0.$

(4) $(x-3)(x-1)(x+1)(x+3) = 105.$

(5) $(x+3)(x+4)(x+5)(x+6) = 1680.$

(6) $(2x-1)(2x+1)(2x+3)(2x+5) = 384.$

(7) $\dfrac{2x^2+3}{5x} + \dfrac{5x}{2x^2+3} = \dfrac{221}{110}.$

(8) $\dfrac{x^2+15}{x} - 4 \cdot \dfrac{x}{x^2+15} = 7\frac{1}{2}.$

(9) $\dfrac{x^2+1}{x^2-1} + \dfrac{x^2-1}{x^2+1} = \dfrac{41}{20}.$

(10) $\left(x+\dfrac{1}{x}\right)^2 + 12\left(x+\dfrac{1}{x}\right) = 30\frac{25}{36}.$

2. Construct the equation whose roots are

(1) $+1, -1, +3, -3$; (2) $+3, -3, +\sqrt{5}, -\sqrt{5}$;

(3) $4, 5, 6, 7$; (4) $-2, +3, +4, +9$;

(5) $3, 4, \dfrac{7+\sqrt{13}}{2}, \dfrac{7-\sqrt{13}}{2}$,

shewing in each case that the equation admits no other roots than those given.

87. Reciprocal Equations. We now consider equations which, when brought to the form

$$\text{Polynomial} = 0,$$

have the coefficients, equidistant from the beginning and the end of the arranged polynomial, equal.

Ex. 1. Solve the equation

$$12x^4+4x^3-41x^2+4x+12=0 \text{.......................(i)}$$

This equation is of the fourth degree. It is, however, of special form, the coefficients of the polynomial of the equation which are equidistant from the beginning and the end being equal. We think of x as being the number, or some one of several numbers, that will satisfy the equation. Now x cannot be zero, for $x=0$ does not satisfy the equation. We may then divide through by x^2, obtaining

$$12x^2+4x-41+4\cdot\frac{1}{x}+12\cdot\frac{1}{x^2}=0 \text{...........(ii)}$$

Then, associating terms with equal coefficients, we have

$$12\left(x^2+\frac{1}{x^2}\right)+4\left(x+\frac{1}{x}\right)-41=0 \text{...........(iii)}$$

Now $x^2+\frac{1}{x^2}$ suggests $\left(x+\frac{1}{x}\right)^2$ and it is seen that $\left(x+\frac{1}{x}\right)^2=\left(x^2+\frac{1}{x^2}\right)+2$. Then, adding 2 within the first brackets of (iii), which on account of the coefficient 12 means the addition of 24, and subtracting to correct we have

$$12\left(x^2+2+\frac{1}{x^2}\right)+4\left(x+\frac{1}{x}\right)-24-41=0 \text{...........(iv)}$$

$$\therefore\quad 12\left(x+\frac{1}{x}\right)^2+4\left(x+\frac{1}{x}\right)-65=0.$$

Then, treating, for the time being, $x+\frac{1}{x}$ as the unkn[illegible]t-ing it by y if any advantage comes therefrom, we h[illegible]d-ratic in $\left(x+\frac{1}{x}\right)$. Solving it we find that

$$x+\frac{1}{x}=\frac{13}{6} \text{ or } -\frac{5}{2}.$$

Thus the original equation is the equivalent of the two

$$x+\frac{1}{x}=\frac{13}{6}, \qquad x+\frac{1}{x}=-\frac{5}{2},$$

or of the two

$$x^2-\frac{13}{6}x+1=0, \qquad x^2+\frac{5}{2}x+1=0.$$

From the relation $mn = \frac{c}{a}$ of Section 81, it is plain that for each of these equations the product of the roots is unity, so that for each equation each root is the reciprocal of the other. Clearing of arithmetical fractions we have

$$6x^2 - 13x + 6 = 0, \qquad 2x^2 + 5x + 2 = 0,$$

whence $x = \frac{2}{3}, \ \frac{3}{2}, \ -2, \ -\frac{1}{2}$, the roots occurring in reciprocal pairs as foreseen.

Note that the equation is of the fourth degree, has four roots, and may be exhibited in the form

$$(x - \tfrac{2}{3})(x - \tfrac{3}{2})(x + 2)(x + \tfrac{1}{2}) = 0.$$

EXERCISES CXIV

1. Solve the following equations, in each case verifying that the roots occur in reciprocal pairs:

(1) $72x^4 - 306x^3 + 469x^2 - 306x + 72 = 0;$

(2) $72x^4 + 306x^3 + 469x^2 + 306x + 72 = 0;$

(3) $72x^4 - 6x^3 - 181x^2 - 6x + 72 = 0;$

(4) $30x^4 - 31x^3 - x^2 - 31x + 30 = 0;$

(5) $60x^4 + 17x^3 - 167x^2 + 17x + 60 = 0.$

2. Solve as a reciprocal equation, and also as a quadratic in the unknown x^2

$$x^4 + x^2 + 1 = 0.$$

3. Construct the equation of which the roots are:

(1) $\frac{3}{4}, \ \frac{4}{3}, \ \frac{4}{5}, \ \frac{5}{4}$; (2) $-\frac{5}{7}, \ -\frac{7}{5}, \ \frac{2 - \sqrt{-5}}{3}, \ \frac{3}{2 - \sqrt{-5}}$.

4. If m is a root of the equation

$$3x^4 - 17x^3 + 29x^2 - 17x + 3 = 0,$$

then also is $\frac{1}{m}$ a root.

88. Equations Solvable by a Direct Resolution into Factors. The quadratic equation was solved by the resolution into factors of the involved quadratic expression. There are many equations of higher degree in which the involved expression admits resolution into factors, and through the resolution the roots of the equation are found. The following examples will afford sufficient illustrations.

Ex. 1. Solve

$$x^3 - 1 = 0.$$

Here it is seen that $x^3 - 1 = (x-1)(x^2+x+1)$, and the equivalent equation is

$$(x-1)(x^2+x+1) = 0,$$

which will be satisfied by values of x that will satisfy either

$$x - 1 = 0, \text{ or } x^2 + x + 1 = 0,$$

and by no other values, so that this set is equivalent to the given equation. The former equation gives $x = 1$, and the latter, a quadratic, gives $x = \frac{-1+\sqrt{-3}}{2}$, or $x = \frac{-1-\sqrt{-3}}{2}$.

The given equation, which is of the third degree, has the three roots

$$1, \quad \frac{-1+\sqrt{-3}}{2}, \quad \frac{-1-\sqrt{-3}}{2}.$$

This important equation may be written in the form

$$x^3 = 1,$$

whence it is seen that x denotes what one is led to call the *cube root of unity*. The solution of the equation reveals the fact that there are *three distinct cube roots of unity*. Two of these cube roots are imaginary, as might have been foreseen; for 1 is the cube of no negative number, of no positive number less than 1, and of no positive number greater than 1. The two imaginary numbers are the roots of the equation

$$x^2 + x + 1 = 0.$$

Denote these roots by m and n. Then

$$\left.\begin{aligned} m+n&=-1 \\ mn&=+1 \end{aligned}\right\} \text{..................(i)}$$

This last relation shews that each of the imaginary cube roots of unity is the reciprocal of the other. Further, since m and n are roots of the given equation, we have

$$m^3=1\ ; \quad n^3=1 \text{.............(ii)}$$

Next, by multiplication by m^2, we have from the second equation of (i)

$$m^3n=m^2.$$

But $m^3=1$. Therefore

$$n=m^2.$$

Similarly

$$m=n^2,$$

and each of the imaginary cube roots of unity is the square of the other. Thus if w denotes an imaginary cube root of unity, the three cube roots of unity are

$$1, \quad w, \quad w^2.$$

Ex. 2. Solve

$$6x^3-7x^2-11x+12=0.$$

A method of solving any cubic equation not being available, we seek a factor of the expression on the right of the sign of equality. The fact that the sum of the coefficients is zero shews that $x-1$ is a factor, and by division the other factor of the second degree is found. The equation is thus seen to be the equivalent of

$$(x-1)(6x^2-x-12)=0.$$

Here the equation will be satisfied by the values of x that will satisfy the separate equations

$$x-1=0, \quad 6x^2-x-12=0,$$

and by those values alone. The roots of the given cubic are thus found to be

$$1, \quad \tfrac{3}{2}, \quad -\tfrac{4}{3}.$$

Ex. 3. Solve

$$2x^4 - 7x^3 + x^2 + 7x - 3 = 0.$$

Here the involved expression is seen to have both $x-1$ and $x+1$ as factors, and the given equation is the equivalent of

$$(x-1)(x+1)(2x^2 - 7x + 3) = 0.$$

The roots are then found to be

$$1, \quad -1, \quad \tfrac{1}{2}, \quad 3.$$

Ex. 4. Solve

$$x^3 - a^3 = ax^2 - a^2x.$$

It is seen that $x-a$ is a factor of the two expressions given as equal. The equation will therefore be satisfied if $x-a=0$ or if $x=a$. If, however, $x-a$ is not zero, we may divide through by $x-a$, and the equation will be satisfied if

$$x^2 + ax + a^2 = ax.$$

Thus the given equation is the equivalent of

$$x - a = 0 \text{ and } x^2 + ax + a^2 = ax.$$

From this latter we find

$$x^2 + a^2 = 0,$$

or $$x^2 = -a^2,$$

or $$x = +a\sqrt{-1}, \text{ or } -a\sqrt{-1},$$

and the roots of the equation are

$$a, \quad +a\sqrt{-1}, \quad -a\sqrt{-1}.$$

It is to be noted then that when the "both sides" of an equation have a common factor involving the unknown, the equating to zero of that factor leads to a root of the given equation. In other words, if P, Q, R are three expressions involving x, the equation

$$PR = QR$$

is the equivalent of the two equations

$$R = 0, \; P = Q.$$

The student should solve the equation proposed by bringing all the terms to the left of the sign of equality.

EXERCISES CXV

Solve the following equations:

1. $x^4-1=0$.
2. $x^3-6x^2+11x-6=0$.
3. $x^3+6x^2+11x+6=0$.
4. $x^3-a^3=0$.
5. $x^4+x^2+1=0$.
6. $2x^3-9x^2+2x+5=0$.
7. $(x^2-9)(2x+7)=(x-3)(x^2+5x+11)$.
8. $(x-5)(2x^2-\ \ -15)=(x-3)(3x^2-19x+20)$.
9. $x^6-35x^3+216=0$.
10. $(x-2)(2x^3-7x^2+2x+3)=(x-3)(3x^3-11x^2+8x+2)$.
11. $x(x^3+x^2-17x+15)=(x-1)(x^3+3x^2-28x)$.
12. $x^5-1=0$.
13. $x^6-1=0$.

89. Special Type Involving Two Unknowns. As a rule two equations of the second degree in two unknowns are not solvable by available methods. The following example is illustrative of a type that may be solved through a knowledge of the quadratic equation in one unknown.

Ex. 1. Solve the simultaneous set

$$\left.\begin{aligned} x^2+2xy+3y^2&=43 \\ 2x^2-3xy+4y^2&=26 \end{aligned}\right\} \text{..............(i)}$$

In these equations the absolute terms, *i.e.*, the terms not involving the unknowns, are to the right of the sign of equality and the terms involving the unknowns to the left. It is seen that the expressions on the left of the sign of equality are homogeneous, every term being of two dimensions in the unknowns. On this account such a set is sometimes called a *homogeneous system*.

The method of solution is as follows: Put

$$y=mx \text{..............(ii)}$$

where m is a third unknown, and substitute for y in each of the given equations.

Then

$$\left.\begin{aligned} x^2 + 2mx^2 + 3m^2x^2 &= 43, \\ 2x^2 - 3mx^2 + 4m^2x^2 &= 26. \end{aligned}\right\} \ldots\ldots\ldots\ldots\text{(iii)}$$

or, which is seen to be a result of the homogeneity pointed out,

$$\left.\begin{aligned} x^2(1 + 2m + 3m^2) &= 43, \\ x^2(2 - 3m + 4m^2) &= 26. \end{aligned}\right\} \ldots\ldots\ldots\ldots\text{(iv)}$$

Now it is readily seen by appeal to the given equations that zero is not an admissible value. Then by division of the equals and cancellation

$$\frac{1 + 2m + 3m^2}{2 - 3m + 4m^2} = \frac{43}{26}. \quad \ldots\ldots\ldots\ldots\text{(v)}$$

$$\therefore \quad 78m^2 + 52m + 26 = 172m^2 - 129m + 86$$

whence $\quad 94m^2 - 181m + 60 = 0,$

a quadratic in the unknown m, from which we find

$$m = \tfrac{3}{2} \text{ or } \tfrac{20}{47}.$$

Take first $m = \frac{3}{2}$ and substitute this value in either, say the first, of (iv). Then

$$x^2(1 + 3 + \tfrac{27}{4}) = 43$$

$$\therefore \quad x^2 = 4$$

$$\therefore \quad x = +2 \text{ or } -2$$

and since $y = mx$, and $m = \frac{3}{2}$, we have for $x = 2$, $y = 3$, and for $x = -2$, $y = -3$. Next take $m = \frac{20}{47}$, and, as before,

$$x^2(1 + \tfrac{80}{47} + \tfrac{4800}{2209}) = 43$$

$$\therefore \quad x^2 = \frac{2209 \times 43}{10769}$$

$$\therefore \quad x = \pm 47\sqrt{\tfrac{43}{10769}}.$$

whence, since $y = mx$ and $m = \frac{40}{47}$,

$$y = \pm 40\sqrt{\tfrac{43}{10769}}.$$

Thus the equations admit four solutions

$$(+2,\ +3), \quad (-2,\ -3), \quad (+47\sqrt{\tfrac{43}{10769}},\ +40\sqrt{\tfrac{43}{10769}}),$$

$$(-47\sqrt{\tfrac{43}{10769}},\ -40\sqrt{\tfrac{43}{10769}}).$$

Exercises CXVI

Solve the following sets of equations:

1. $x^2+y^2=34;\ xy=15.$
2. $2x^2+xy-y^2=5;\ 7xy-4y^2=6.$
3. $3x^2-xy+5y^2=21;\ x^2+xy+y^2=7.$
4. $4x^2-y^2=20;\ 5xy-3y^2=12.$
5. $3x^2-xy+7y^2=11;\ x^2+y^2=2.$
6. $3x^2-2xy=5;\ 5xy-2y^2=77.$
7. $10x^2+7xy-12y^2=0;\ 2x^2+3xy+4y^2=192.$
8. $x^2+xy+y^2=148;\ x^2-xy+y^2=52.$
9. $3x^2-y^2=26;\ xy=77.$
10. $10x^2-31xy+24y^2=0;\ 2x^2-3xy+4y^2=108.$
11. $2x^2+3xy+y^2=70;\ 6x^2+xy-y^2=50.$
12. $x^2+xy-6y^2=24;\ x^2+3xy-10y^2=32.$
13. $6x^2-17xy+12y^2=0;\ x^2+3xy-5y^2=17.$

90. Special Type Involving Three Unknowns. The type is illustrated in the following example:

Example. Solve the simultaneous set,

$$\left.\begin{aligned} x-2y+z&=0,\\ 5x+6y-7z&=0,\\ 2x^2+5y^2-3z^2&=7x-11y+9z-14. \end{aligned}\right\}$$

The first two equations are linear, and completely homogeneous, every actual term appearing in the equations being of the first degree in the unknowns. The third equation is of the second degree, and of a quite general form. It will be found that the first two equations, while not sufficient to yield the values of x,y,z, determine the ratios $x:y:z$.

Multiplying the first equation through by 7 and combining the result, in addition, with the second we find

$$12x-8y=0$$

or

$$12x=8y$$

or

$$\frac{x}{8}=\frac{y}{12}\text{.........(i)}$$

Similarly multiplying the first through by 3 and combining, in addition, with the second we find

$$8x-4z=0$$

whence $$\frac{x}{4}=\frac{z}{8}\text{..........(ii)}$$

Then multiplying through by 4 in (i) and by 2 in (ii), to have x associated with the same denominator in each, we find

$$\frac{x}{2}=\frac{y}{3}=\frac{z}{4}\text{.........(iii)}$$

or $$x : y : z :: 2 : 3 : 4\text{........(iv)}$$

and the ratios, though not the values, of x,y,z are found.

Denote the common value of each fraction in (iii) by m. Then

$$x=2m,\quad y=3m,\quad z=4m\text{..........(v)}$$

Substitute these values for x,y,z in the third given equation. Then

$$8m^2+45m^2-48m^2=14m-33m+36m-14,$$

whence

$$5m^2-17m+14=0,$$

a quadratic which gives for m the two values

$$2,\quad \tfrac{7}{5}.$$

If $m=2$ we have from (v)

$$x=4,\quad y=6,\quad z=8,$$

and if $m=\frac{7}{5}$ we have

$$x=2\tfrac{4}{5},\quad y=4\tfrac{1}{5},\quad z=5\tfrac{3}{5}.$$

Thus the equation admits the two solutions

$$(4, 6, 8),\quad (2\tfrac{4}{5}, 4\tfrac{1}{5}, 5\tfrac{3}{5}).$$

EXERCISES CXVII

1. Solve the following sets of simultaneous equations:

(1) $$\left.\begin{aligned} x+y+z&=0,\\ 2x-3y+z&=0,\\ x^2+y^2+z^2&=42.\end{aligned}\right\}$$

(2) $$\left.\begin{aligned} 3x-2y-z&=0,\\ x-y+2z&=0,\\ x^2+y^2+z^2&=75.\end{aligned}\right\}$$

(3) $$\left.\begin{aligned} x+y-z&=0,\\ 3x-4y+z&=0,\\ x^2+y^2+z^2&=6x+3y+z. \end{aligned}\right\}$$

(4) $$\left.\begin{aligned} 3x-4y+z&=0,\\ 2x+y-z&=0,\\ x^2-3y^2+2yz&=5x+4y+10z-101. \end{aligned}\right\}$$

(5) $$\left.\begin{aligned} 2x-3y+z&=0,\\ x+2y-2z&=0,\\ x^2+2y^2-z^2&=x+4y-z. \end{aligned}\right\}$$

(6) $$\left.\begin{aligned} \frac{x+y}{3}=\frac{y+z}{4}&=\frac{z+x}{5},\\ 6\,(yz+zx+xy)&=11(x+y+z). \end{aligned}\right\}$$

2. If x, y, z satisfy the two homogeneous linear equations

$$ax+by+cz=0,$$
$$a'x+b'y+c'z=0,$$

shew that

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b}.$$

From the association of the three triads of letters

$$\begin{matrix} x, & y, & z\\ a, & b, & c\\ a', & b', & c' \end{matrix}$$

devise a rule for writing down at once the ratios $x : y : z$.

Apply the rule to the following sets of equations:

(1) $$\left.\begin{aligned} 4x-5y+7z&=0,\\ 6x+9y-11z&=0. \end{aligned}\right\}$$

(2) $$\left.\begin{aligned} 4x-5y+7z&=0,\\ 6x-y+3z&=0. \end{aligned}\right\}$$

(3) $$\left.\begin{aligned} 3x-4y+z&=0,\\ 5x+y-2z&=0. \end{aligned}\right\}$$

(4) $$\left.\begin{aligned} 7x+10y+3z&=0,\\ 5x-y+11z&=0. \end{aligned}\right\}$$

3. If

$$3x-5y+8z=0,$$
$$7x+3y-9z=0,$$

shew that

$$33x-11y+5z=0.$$

4. Solve

$$2x+5y-8z=0$$
$$5x-3y+7z=0$$
$$6x+5y-7z=24.$$

91. General Note. There are many equations the solution of which is effected by quite special artifices. These artifices are acquired only through considerable practice, and are in the main the outcome of a familiarity with algebraic forms and relations. The following exercises are designed for review in the solution of familiar types and for the introduction of a few new types not too difficult. The student should be extremely careful to find all the solutions as well as to examine whether all the solutions found are applicable. The object should be less a facility in obtaining results, important as this is, and more the obtaining of a complete view of the meaning of the equation or system.

Exercises CXVIII

A

Solve the following equations or systems of equations:

1. $$\frac{7x+4}{5}+\frac{3x+1}{x-1}=\frac{11x+7}{4}.$$

2. $$\left.\begin{array}{r} x^2+5xy=14, \\ y^2+6xy=13. \end{array}\right\}$$

3. $$21x^4-37x^3-16x^2-37x+21=0.$$

4. $$\left.\begin{array}{r} xy(x+y)=5\div 36, \\ \frac{1}{x}+\frac{1}{y}=5. \end{array}\right\}$$

5. $$\left.\begin{array}{r} x+y=9, \\ x^3+y^3=189. \end{array}\right\}$$

B

Solve the following equations or systems of equations:

1. $$\frac{x-2}{x+2}+2\frac{x+2}{x-2}=3.$$

2. $\left.\begin{aligned} x^2 + 2xy &= 16, \\ xy + y^2 &= 15. \end{aligned}\right\}$

3. $\dfrac{a}{x^2+1} + \dfrac{x^2+1}{x} = \dfrac{109}{30}.$

4. $\left.\begin{aligned} xy + x + y &= 11, \\ xy(x+y) &= 30. \end{aligned}\right\}$

5. $\begin{aligned} x - y &= 3, \\ x^3 - y^3 &= 279. \end{aligned}$

C

Solve the following equations or systems of equations:

1. $\dfrac{2x-3}{2x+1} + \dfrac{3x-7}{3x+5} = 2.$

2. $\left.\begin{aligned} x^2 + xy - 2y^2 &= 7, \\ x^2 + 3xy + 2y^2 &= 35. \end{aligned}\right\}$

3. $\left.\begin{aligned} 6x + 5y + z &= -1, \\ x - 2y + 3z &= 14, \\ 3x - 4y - 2z &= 5. \end{aligned}\right\}$

4. $\left.\begin{aligned} x^3 + y^3 &= 65, \\ x^2y + xy^2 &= 20. \end{aligned}\right\}$

5. $yz = 63; \quad zx = 45; \quad xy = 35.$

D

Solve the following equations or systems of equations:

1. $\dfrac{3}{x-5} + \dfrac{4}{x-6} = \dfrac{2}{x-4} + \dfrac{5}{x-7}.$

2. $\left.\begin{aligned} x^2 - 2y^2 &= 4y, \\ 3x^2 + xy - 2y^2 &= 16y. \end{aligned}\right\}$

3. $\left.\begin{aligned} (x+y)^2 - 7(x+y) + 12 &= 0, \\ x^2y^2 - 6xy + 8 &= 0. \end{aligned}\right\}$

4. $$30x^4 - 91x^3 - 278x^2 - 91x + 30 = 0.$$

5. $$\left.\begin{aligned} zx + xy &= 80, \\ xy + yz &= 98, \\ yz + zx &= 108. \end{aligned}\right\}$$

E

Solve the following equations or systems of equations:

1. $$\left.\begin{aligned} (z + x)(x + y) &= 56, \\ (x + y)(y + z) &= 63, \\ (y + z)(z + x) &= 72. \end{aligned}\right\}$$

2. $$\left.\begin{aligned} x^2 + y^2 &= 74, \\ xy + (x + y) &= 47. \end{aligned}\right\}$$

3. $$x^3 + 3x^2 - 6x - 8 = 0.$$

4. $$\left.\begin{aligned} \frac{2}{x} + \frac{3}{y} &= \frac{17}{12}, \\ \frac{5}{x} - \frac{7}{y} &= -\frac{1}{12}. \end{aligned}\right\}$$

5. $$\frac{4x^2 + 5x + 9}{3x^2 + 7x + 10} = \frac{8x + 10}{6x + 14}.$$

F

Solve the following equations or systems of equations:

1. $$\left.\begin{aligned} (y + 3)(z + 3) &= 110, \\ (z + 3)(x + 3) &= 99, \\ (x + 3)(y + 3) &= 90, \end{aligned}\right\}$$

2. $$\left.\begin{aligned} x^2 + y^2 - x - y &= 78, \\ xy + x + y &= 39, \end{aligned}\right\}$$

3. $$(x^2 - 8x + 11)^2 + (x - 4)^2 = 25.$$

4. $$\left.\begin{aligned} 21x + 35y &= 10xy, \\ 70x - 30y &= 4xy. \end{aligned}\right\}$$

5. $$3x^5 - 10x^4 + 10x - 3 = 0.$$

EXERCISES CXIX

(MISCELLANEOUS)

A

1. Find the H.C.F. of

$$1+5z+z^2-13z^3+6z^4 \text{ and } 1+6z+10z^2-2z^3-15z^4.$$

2. Find the least value of $x^2-4x+11$ if x assumes only real values.

3. Shew that the system

$$x^2+xy+y^2=7, \quad 2x^2+3xy+4y^2=24,$$

is equivalent to the system

$$xy+2y^2=10, \quad x^2-y^2=-3,$$

and find the solutions.

4. If

$$\frac{a}{3(y-z)}=\frac{b}{4(z-x)}=\frac{c}{5(x-y)}$$

shew that

$$20a+15b+12c=0.$$

5. An express train runs from Toronto to Hamilton at a uniform rate. If the speed of the train were increased 10 miles an hour the time occupied in the journey would be 8 minutes less, but if the speed were decreased 10 miles an hour the time would be 12 minutes greater. Find the distance between Hamilton and Toronto.

B

1. Shew that

$$x^2+2xy-3y^2+4y-1 \text{ and } x^2+xy-6y^2-x+2y$$

have one factor in common.

2. In the product of

$$1+2x+3x^2+4x^3+5x^4+6x^5+7x^6 \text{ and } x-2x^2+3x^3-4x^4+5x^5-6x^6$$

find the coefficients of x^7 and x^{11}.

3. Prove that

$$(a+b+c)^2+a^2+b^2+c^2=(b+c)^2+(c+a)^2+(a+b)^2.$$

4. Verify directly that

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^2=\frac{-1-\sqrt{-3}}{2}; \quad \left(\frac{-1-\sqrt{-3}}{2}\right)^2=\frac{-1+\sqrt{-3}}{2}.$$

5. Solve the system:

$$x^2-3xy+y^2=-5; \quad x+y-xy=-1.$$

C

1. Resolve into factors:

(i) $x^5+x^3-x^2-1$;

(ii) $2x^2-y^2-2z^2+3yz-xy$;

(iii) $(x+y)^2-3(x+y)z+2z^2$.

2. Shew that the minimum value of $x^2-8x+17$ is 1, and explain how it is that there may be proposed the solution of the equation

$$x^2-8x+17=0.$$

3. Form the equation whose roots are m and n where

$$m^2+n^2=28, \quad m+n=4.$$

4. If $a+3b=2c$, prove that

$$a^3+27b^3=8c^3-18abc.$$

5. Divide $1+2x$ by $1-x+x^2$ to shew that

$$\frac{1+2x}{1-x+x^2}=1+3x+2x^2-\frac{x^3+2x^4}{1-x+x^2}.$$

6. There are three consecutive integers such that the square of the middle integer exceeds four times the greatest or six times the least by 1. Determine the integers.

7. If

$$(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax+by+cz)^2,$$

then

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

CHAPTER XXI

SQUARE AND CUBE ROOT

92. Introductory. So many exercises involving the squares and the cubes of algebraic expressions have been proposed that the student may be assumed to be quick to recognize at least the simpler perfect squares and perfect cubes. In this chapter the object is to devise a formal process for obtaining the square root or the cube root, as the case may be, of a given expression, and this principally with a view to an arithmetical application.

93. Square Root. It is well to begin by constructing the squares of, say, $x+a$, $x+7$, 37.

$$\begin{array}{rcr}
x+a & \qquad & x+7 \\
\underline{x+a} & & \underline{x+7} \\
x^2+\ ax & & 7x+49 \\
\underline{+\ ax+a^2} & & \underline{x^2+\ 7x\quad} \\
x^2+2ax+a^2 & & x^2+14x+49
\end{array}$$

$$\begin{array}{rcr}
3\,.\,10+7 & = & 37 \\
\underline{3\,.\,10+7} & = & \underline{37} \\
3\times 7\,.\,10+49 & = & 259 \\
\underline{3^2\,.\,10^2+3\times 7\,.\,10\quad} & = & \underline{111\ } \\
3^2\,.\,10^2+2(3\times 7)\,.\,10+49 & = & 1369
\end{array}$$

In the squares of $x+a$ and $x+7$ the terms do not lose their individuality, whereas in the square of 37, on account of the nature of our arithmetical notation, the record of the origin of the different parts is in the main lost. If, however, a method of recovering the square root in the former cases is devised, this method will be of service in the latter. We seek then how to elicit from $x^2+2ax+a^2$ its square root, known to be $x+a$.

On examination we see that the term in $x^2+2ax+a^2$ of ***highest dimension*** in the ***first*** term x of the root is the square of this term. Thus the first term x^2 determines the first term of the root, and it is plain that the second term of the root must be determined by the remaining terms of the expression. We have then

$$\begin{array}{l} x^2+2ax+a^2(x \\ \underline{x^2 \qquad\qquad\quad} \\ \quad +2ax+a^2 \end{array}$$

How, then, is the second term a of the root to be found from this remainder? Plainly, if the first term $2ax$ be divided by *twice the part x already found*, we reach the term a, and the question is closed by noting that the square of a, the part last found, is the remaining term. The whole work may be exhibited thus:

$$\begin{array}{rl} & x^2+2ax+a^2(x+a \\ & \underline{x^2 \qquad\qquad\quad} \\ 2x+a & \quad +2ax+a^2 \\ & \quad \underline{+2ax+a^2} \end{array}$$

So for $x^2+14x+49$ we have:

$$\begin{array}{rl} & x^2+14x+49(x+7 \\ & \underline{x^2 \qquad\qquad\quad} \\ 2x+7 & \quad +14x+49 \\ & \quad \underline{+14x+49} \end{array}$$

Turn now to the arithmetical number 1369, whose square we know to be 37. The first term or part of the root is 3, which we know to signify 3 . 10. From an examination of the first two figures of 1369 we see that the root lies between 30, whose square is 900, and 40, whose square is 1600. We proceed, then, as follows:

$$\begin{array}{r} 1369(30 \\ 900 \\ \hline 469 \end{array}$$

We now multiply 30 by 2, and seek by division into 469 an indication of the second part. This indication in arithmetic is not always just, for the reasons stated, but this will appear in the verification. The indication here is 7, and we write:

$$\begin{array}{rl} & 1369(30+7 \\ & 900 \\ \cline{2-2} 60+7 & 469 \\ & 469 \\ \cline{2-2} \end{array}$$

and 7 stands the test. We are thus satisfied that the square root is 37.

This work may be presented in an equivalent though more condensed form, thus:

$$\begin{array}{rl} & 13'69(37 \\ & 9 \\ \cline{2-2} 67 & 469 \\ & 469 \\ \cline{2-2} \end{array}$$

The further theory, which is treated in detail in all works on arithmetic, need not be developed here beyond what will appear in the exercises.

EXERCISES CXX

1. Find, by the process of this section, the square roots of the following:

$$a^2+4ab+4b^2;\quad 9a^2-12ab+4b^2;\quad 49x^2-70ax+25a^2;\quad 5329;$$
$$3249;\quad 2401;\quad 9025;\quad 53{\cdot}29;\quad 2{\cdot}89.$$

2. Find the square of x^2+3x+5, and from the result recover, term by term, its square root.

3. Find the square of $a+b+c$, and from the result recover, term by term, the square root.

4. The square of 356 is 126736. Shew that the first figure of the square root of 126736 is to be found from 12, the first two figures. Shew also that the first two figures of the roots are to be found from 1267, the first four figures.

5. The square of 234 is 54756. Shew that the first figure of the root is to be found from 5, the first figure, and that the first two figures of the root are to be found from 547, the first three figures.

6. Find, by an extension of the process of this section, the square roots of the following:

$$x^4+2x^3+3x^2+2x+1;\quad y^4+4y^3+10y^2+12y+9;$$
$$9x^4+12ax^3+34a^2x^2+20a^3x+25a^4;\quad 4z^4-12z^3+25z^2-24z+16;$$
$$15129;\quad 151{\cdot}29;\quad 289444;\quad 28{\cdot}9444;\quad 622521.$$

7. The process of extracting the square roots yields $x+3$ as the square root of x^2+6x+9. Will the process yield also $-x-3$, which is known to be a square root of x^2+6x+9?

8. It is known that 7 has not a rational square root; find an approximation, correct to the second decimal place, to the irrational root.

9. Find the square roots of

$$1+2x+x^2;\quad 1+2x+3x^2+2x^3+x^4;\quad 4-12x+29x^2-30x^3+25x^4.$$

10. Shew that $1+x+x^2$ is not the square of any polynomial, and find an *approximation* to its square root.

94. Cube Root. In view of what has been said in the treatment of square roots, the matter of the extraction of the cube root may be dealt with more concisely. We know that

$$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3; \quad 37^3 = 50653,$$

and seek a formal process for recovering the cube root of $x^3 + 3ax^2 + 3a^2x + a^3$ and 50653. For reasons given in the preceding section we attack the algebraic problem. We see that the first term x of the cube root of $x^3 + 3ax^2 + 3a^2x + a^3$ is suggested or given by x^3, the term of highest dimension in x, and that the second term of the root must therefore be determined by the remaining terms of the expression. Accordingly we have

$$\begin{array}{l} x^3 + 3ax^2 + 3a^2x + a^3(x \\ x^3 \\ \hline \quad + 3ax^2 + 3a^2x + a^3 \end{array}$$

We now ask how a is to be found from this remainder, and see that if the term of highest dimension in x in the remainder be divided by *three times the square of x, the part found*, we reach the second term of the root. It is necessary, however, that a, the part thus found, be such that the product of three times its square and x, the part first found, together with the cube of a, make up the rest of the remainder. This can be exhibited thus:

$$\begin{array}{ll} & x^3 + 3ax^2 + 3a^2x + a^3(x + a \\ & x^3 \\ \hline 3x^2 + 3a.x + a^2 & \quad + 3ax^2 + 3a^2x + {}^3 \\ & \quad + 3ax^2 + 3a^2x + a^3 \\ \hline \end{array}$$

Turn now to the problem of finding the cube root of 50653, known to be 37. We see that the first part 3, which signifies 30, is found from the first two figures 50 through the fact that 30^3, which equals 27,000, is less, and 40^3, which equals 64,000, is greater than 50,653. We write then:

$$\begin{array}{l} 50{,}653(30 \\ \underline{27{,}000} \\ 23{,}653 \end{array}$$

and see that the second part of the root, 7, must be found from this remainder. As in the algebraic example, we take three times the square of 30 and divide it into 23,653 to get an indication of the second part of the root. Here the indication might be taken to be 8, but this would be found to be too great; 7 however stands the test. We have then

$$\begin{array}{cccl} & & & 50{,}653(30+7 \\ 3.30^2 & 3.30.7 & 7^2 & \underline{27{,}000} \\ 2700 + & 630 + & 49 & 23{,}653 \\ & 3379 & & \underline{23{,}653} \end{array}$$

which may be more concisely presented thus:

$$\begin{array}{rl} & 50{,}653(37 \\ & \underline{27} \\ 2700 & 23{,}653 \\ 630 & \\ \underline{49} & \\ 3379 & \underline{23{,}653} \end{array}$$

Thus the process revealed by a study of an algebraic perfect cube serves for the finding of the cube root of an arithmetical cube.

Exercises CXXI

1. Find, by the process of this section, the cube roots of

$x^3 - 3ax^2 + 3a^2x - a^3$; $8x^3 + 36x^2y + 54xy^2 + 27y^3$;
$27x^3 - 108x^2 + 144x - 64$; $64 + 240x + 300x^2 + 125x^3$;
157,464; 157·464; 12,167; 658,503.

2. Find the cube of $x^2 + 2x + 3$, and from the result recover, term by term, its cube root.

3. The cube of 234 is 12,812,904. Shew that the first figure of the square root is to be found from 12, the first two figures, and that the first two figures of the root are to be found from 12,812, the first five figures.

4. Find, by an extension of the process of this section, the cube roots of:

$x^6 + 9x^5 + 42x^4 + 117x^3 + 210x^2 + 225x + 125$;
$8 - 36x + 102x^2 - 171x^3 + 204x^4 - 144x^5 + 64x^6$;
1,860,867; 189,119,224; 189·119224.

5. It is known that 7 has not a rational cube root; find an approximation, correct to the second decimal place, to the irrational cube root.

6. Find the fourth root of

$1 + 12x + 54x^2 + 108x^3 + 81x^4$;
$16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$;
$81x^4 - 432x^3 + 864x^2 - 768x + 256$;
1,336,336; 10,556,001.

7. Find the sixth root of

$x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$;
$64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$;
148,035,889; 887,503,681.

8. Find an approximation, correct to the second decimal place, to the square root and to the cube root of $\frac{5}{7}$.

Exercises CXXII

(Miscellaneous)

A

1. Solve

$$(x-b)(x-c)=(a-b)(a-c).$$

2. Prove that

$$(1+x)^2(1+y^2)-(1+x^2)(1+y)^2=2(x-y)(1-xy).$$

3. Resolve into factors:

(i) $6x^2-23xy+21y^2$;

(ii) $a^2+2ab-3b^2-ca+5bc-2c^2$;

(iii) $(a+b+c)^3+(a+b-c)^3$.

4. A and B start in a race of 100 yards. If A can run 100 yards in 10 seconds and B in 12 seconds, at the end of what time is A midway between B and the end of the course?

5. If

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$$

then

$$\frac{(y-z)(z-x)(x-y)}{(b-c)(c-a)(a-b)}=\frac{xyz}{abc}$$

B

1. Resolve into factors:

(i) y^6+z^6;

(ii) $64y^6+729z^6$;

(iii) $(a-b)^2-7(a^2-b^2)+12(a+b)^2$.

2. The sides AB, BC, CA, of a triangle measure 13, 14, 15. From the formula

$$\triangle=\sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, c, measure the sides, s half the sum of the sides, and $\triangle$ the area of a triangle, compute the area.

3. A number x is known to satisfy the two equations

$$10x^2-31x+24=0,\quad 8x^2-2x-15=0.$$

Shew that the number is determined uniquely.

4. Solve

$$13x-8y+21z+19=0,$$
$$19x+6y+14z+7=0,$$
$$x+24y+35z+13=0.$$

5. If

$$x=a+d,\quad y=b+d,\quad z=c+d$$

shew that

$$x^2+y^2+z^2-yz-zx-xy=a^2+b^2+c^2-bc-ca-ab.$$

C

1. Find the integer whose square is less than the square of the next higher integer by 37.

2. Shew that the system of equations

$$x-y=5;\quad (x^2+y^2)(x^3-y^3)=7955,$$

is the equivalent of the system

$$x-y=5;\quad (x^2+y^2)(x^2+xy+y^2)=1591.$$

which is the equivalent of the system

$$x-y=5;\quad (25+2xy)(25+3xy)=1591.$$

Hence solve the given system.

3. If the difference of the roots of $x^2+px+q=0$ is the same as the difference of the roots of $x^2+qx+p=0$, then either $p=q$ or $p+q=-4$.

4. The sides AB, BC, CA of a triangle measure 13, 14, 15. From A a perpendicular AL is drawn to BC. If BL measures x, state what is the measure of LC and express in two ways the square on AL. In this way find first x, then AL and then the area.

5. State all the linear factors that are to be considered as factors of $x^3+x^2-14x-24$, and, testing for them, find the factors of this expression.

CHAPTER XXII

INDICES AND SURDS

95. Restatement and Proofs of Laws. The fundamental definition in the theory of indices may be stated thus:

If a is any number and m any positive integer, a^m means the product of m factors a.

The following laws may now be proved, the exponents in every case being positive integers, as required by the definition.

$$\text{I.} \quad \mathbf{a^m \times a^n = a^{m+n}}.$$

For by definition

$$a^m \times a^n = (a \times a \times a \ldots\text{to } m \text{ factors}) \times (a \times a \times a \ldots \text{ to } n \text{ factors})$$
$$= a \times a \times a \ldots\text{to } (m+n) \text{ factors}$$
$$= a^{m+n}, \text{ by definition.}$$

It follows readily that

$$a^m \times a^n \times a^r \times a^s \times \ldots = a^{m+n+r+s\ldots}.$$

$$\text{II.} \quad (1) \quad \frac{\mathbf{a}^m}{\mathbf{a}^n} = \mathbf{a}^{m-n} \text{ if m} > \text{n}.$$

For by definition

$$\frac{a^m}{a^n} = (a \times a \times a \ldots \text{ to } m \text{ factors}) \div (a \times a \times \ldots \text{ to } n \text{ factors})$$
$$= \{(a \div a) \times (a \div a) \times \ldots\text{to } n \text{ factors}\} \times \{a \times a \times \ldots\text{to}(m-n) \text{ factors}\}, \text{ by rearranging and combining multiplications and divisions}$$
$$= a \times a \times a \ldots\text{to}(m-n) \text{ factors, since } a \div a = 1$$
$$= a^{m-n}, \text{ by definition.}$$

(2) $$\frac{a^m}{a^n} = \frac{1}{a^{n-m}} \text{ if } n>m.$$

The proof is left as an exercise, as it follows by a slight modification of that of II. (1).

III. $$(a^m)^n = a^{mn}.$$

For

$$(a^m)^n = (a^m) \times (a^m) \times (a^m) \ \dots \text{ to } n \text{ factors}$$
$$= a^{m+m+m+ \ \dots \text{ to } n \text{ terms}}, \text{ by I}$$
$$= a^{mn}.$$

IV. $$(ab)^m = a^m b^m.$$

For

$$(ab)^m = (ab) \times (ab) \times (ab) \ \dots \text{ to } m \text{ factors}$$
$$= (a \times a \ \dots \text{ to } m \text{ factors}) \times (b \times b \times \ \dots \text{ to } m \text{ factors})$$
$$= a^m \times b^m$$
$$= a^m b^m.$$

These laws, though they are not completely independent of one another, are usually spoken of as the fundamental laws of indices.

Exercises CXXIII

1. Assuming that $a^m \times a^n = a^{m+n}$, shew that
   (i) $a^m . a^n . a^r = a^{m+n+r}$; (ii) $a^m . a^n . a^r . a^s = a^{m+n+r+s}$.

2. Shew that $\{(a^m)^n\}^r = a^{mnr}$.

3. Assuming that $(ab)^m = a^m b^m$, shew that $(abc)^m = a^m b^m c^m$.

4. Prove that $\left(\frac{a}{b}\right) = \frac{a^m}{b^m}$.

5. If a is a positive number greater than unity, how do higher and higher powers of a compare with one another in magnitude?
As an illustration take a equal to 3, and find a^2, a^3, a^6, a^{12}.

6. If a is a positive number less than unity; how do higher and higher powers of a compare with one another in magnitude?
As an illustration take a equal to $\frac{1}{2}$, and find a^2, a^4, a^8, a^{16}, a^{32}.

96. Extension of the Index System. In a^m, as we have just seen, the definition *requires* that m be a positive integer, so that a symbol as a^0, $a^{\frac{1}{3}}$, a^{-2} is wholly without meaning. However, it is in harmony with several steps already taken in arithmetic and algebra to remove this restriction on the value of m in a^m, and allow m to have any positive or negative integral or fractional value whatever. This we shall do, and for the new fractional and negative *powers* and *indices* thus introduced, as yet without meaning, we shall *assume* or *require* that they obey the laws proved for positive integral powers and indices. It will now be seen that the application of these laws—it must be kept in mind that they are assumed or postulated—assigns a meaning to such symbols as a^0, $a^{\frac{1}{3}}$, a^{-2}.

(*a*) *Meaning determined for a^0 by the assumed laws.*

By the rule for the addition of indices,

$$a^m \times a^0 = a^{m+0} = a^m.$$

Also identically $\quad a^m \times 1 \quad = a^m,$

$$\therefore \quad a^m \times a^0 = a^m \times 1, \text{ for general values of } a$$

$$\therefore \quad \mathbf{a^0} = 1, \text{ for general values of } a.$$

Thus hereafter

$$\mathbf{a^0 = 1.}$$

(*b*) *Meaning determined by the assumed laws for $a^{\frac{1}{m}}$, where m is a positive integer.*

Consider $a^{\frac{1}{3}}$. Then by the rule for the addition of indices

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$$
$$= a^1 \text{ or } a.$$

Hence $a^{\frac{1}{3}}$ is the number which, taken as a factor three times yields the result a; it is therefore the cube root of a and we write

$$a^{\frac{1}{3}} = \sqrt[3]{a}.$$

More generally, m being a positive integer.

$$\mathbf{a}^{\frac{1}{m}} = \sqrt[m]{\mathbf{a}}.$$

(*c*) *Meaning determined for $a^{\frac{p}{q}}$, where p and q are positive integers, by the assumed laws.*

Consider the particular case $a^{\frac{3}{5}}$. By the law of multiplication of indices

$$(a^{\frac{3}{5}})^5 = a^{\frac{3}{5}\times 5} = a^3.$$

Hence $a^{\frac{3}{5}}$ is a number which, taken as a factor five times, gives the result a^3; it is therefore the fifth root of a^3, and we can write

$$a^{\frac{3}{5}} = \sqrt[5]{a^3}.$$

Very slight modification in the reasoning will shew that

$$\mathbf{a}^{\frac{p}{q}} = \sqrt[q]{\mathbf{a}^p}.$$

(*d*) *Meaning determined by the assumed laws for a^{-m}, where m is any positive integer or fraction.*

By the law of addition of indices

$$a^{-m} \times a^m = a^{-m+m} = a^0.$$
$$= 1.$$

Also by the theory of fractions

$$\frac{1}{a^m} \times a^m = 1$$

$$\therefore \quad a^{-m} \times a^m = \frac{1}{a^m} \times a^m \text{ for general values of } a.$$

Therefore, for general values of a,

$$\mathbf{a}^{-m} = \frac{1}{\mathbf{a}^m}.$$

97. Important Convention. In the preceding, in connection with fractional indices, mention was made of *the* cube root, *the* fifth root, *the* mth root. Now it is known that a number has *two* square roots, *three* cube roots, ..., *six* sixth roots (see Exercises CXV), and the general fact, of which these are illustrations, may be assumed. It is plain then that error may arise through employing, for example, either $a^{\frac{1}{3}}$ or $\sqrt[3]{a}$ as if it denoted a single definite number when, as yet, it is equally any one of three definite numbers. On this account a limitation will be made.

Consider $a^{\frac{1}{2}}$ where a is a real positive number. Then, in virtue of what has appeared in Sections 73 and 75, there is one and only one positive real number whose square is a. There is also a negative real number, of equal numerical value, whose square is a.

Similarly if a is positive, and m a positive integer, there is among the m values of $a^{\frac{1}{m}}$ one and only one which is real and positive.

Next in $a^{\frac{1}{m}}$, suppose a negative, m being a positive integer. To fix attention, take -5 as the value of a. If $m=2$, then $(-5)^{\frac{1}{2}}$ means $\sqrt{-5}$, which is not real. If $m=3$, then $(-5)^{\frac{1}{3}}$ means $\sqrt[3]{-5}$, which if real must be negative. Now we know that $+5$ has one positive real cube root which denote by x, so that

$$x^3 = +5.$$

Then it follows that

$$(-x)^3 = -5,$$

so that -5 has one and only one real root, and that negative. Thus the real cube root of -5 is obtained from that of

+5 by a mere change of sign. A little consideration will shew that if in $a^{\frac{1}{m}}$ we suppose a negative, there is no real root when m is an even integer, and there is one real root and that negative when m is an odd integer, which real root is determined by the real positive root of the corresponding positive number.

In this chapter the attention will be restricted to real numbers, and from what has been said it will be seen to be sufficient to consider the case in which the a in $a^{\frac{1}{m}}$ is a positive real number. We now agree that:

By $a^{\frac{1}{m}}$, m a positive integer, is to be understood the *one positive real number*, whose mth power is the *positive number a*.

Consequently $a^{\frac{p}{q}}$ will denote the *one positive real number* whose qth power is a^p, it being supposed that p and q are integers and q positive.

In view of this consideration we can say for example that

$$9 = 9^1 = 9^{\frac{2}{2}} = 9^{\frac{3}{3}} = 9^{\frac{4}{4}} = \text{etc.}$$

when otherwise $9^{\frac{2}{2}}$ or $81^{\frac{1}{2}}$ might mean $+9$ or -9, and therefore not $+9$ exclusively.

The consideration of irrational exponents will not be undertaken.

Exercises CXXIV

1. Accepting the meaning assigned to fractional indices, shew that

$$2^{\frac{1}{2}} = 2^{\frac{2}{4}};\ 2^{\frac{1}{3}} = 4^{\frac{1}{6}};\ 5^{\frac{1}{2}} \times 5^{\frac{1}{3}} = 5^{\frac{1}{2}+\frac{1}{3}};\ 5^{\frac{1}{2}} \div 5^{\frac{1}{3}} = 5^{\frac{1}{2}-\frac{1}{3}}.$$

2. Find the following products:

$$(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x^{\frac{1}{2}}-y^{\frac{1}{2}});\ (x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}})(x^{\frac{1}{3}}-y^{\frac{1}{3}});$$
$$(x^{\frac{3}{4}}+x^{\frac{2}{4}}y^{\frac{1}{4}}+x^{\frac{1}{4}}y^{\frac{2}{4}}+y^{\frac{3}{4}})(x^{\frac{1}{4}}-y^{\frac{1}{4}}).$$

3. Express $2^{\frac{1}{3}}$ and $3^{\frac{1}{5}}$ each as a fifteenth root and thereby determine which of these numbers is the greater.

4. Express, as the root of an integer, each of the following:

$$5^{\frac{3}{7}};\ 3^{\frac{1}{4}}.3^{\frac{1}{5}};\ 2^{\frac{1}{3}}.3^{\frac{1}{4}};\ 3^{\frac{2}{3}}\div 9^{\frac{1}{4}};\ 2^{\frac{1}{2}}\div 4^{\frac{1}{3}}\times 8^{\frac{1}{4}}.$$

5. Shew that, in finding the meaning assigned to a^0 by the laws of indices, the value $a=0$ is excluded.

6. Shew that

$$x^{m-n}.x^{n-l}.x^{l-m}$$

is independent of the value of x, except that x may not be zero.

7. Shew that

$$12^{\frac{1}{2}}=2.3^{\frac{1}{2}};\ 36^{\frac{1}{3}}=2^{\frac{2}{3}}.3^{\frac{2}{3}};\ 108^{\frac{1}{3}}=3.2^{\frac{2}{3}};\ 80^{\frac{1}{4}}=2.5^{\frac{1}{4}}.$$

8. Shew that if n is a positive integer

$$(-a)^{-2n}=\frac{1}{a^{2n}};\quad (-a)^{-2n+1}=-\frac{1}{a^{2n-1}};\quad (-a)^{n-1}.(-a)^{-n}=-\frac{1}{a}$$

9. Find the following quotients:

$$(x+y)\div(x^{\frac{1}{3}}+y^{\frac{1}{3}});\ (x-y)\div(x^{\frac{1}{4}}-y^{\frac{1}{4}});\ (x+y)\div(x^{\frac{1}{5}}+y^{\frac{1}{5}}).$$

10. Shew that

$$(x^1+x^{-1})^2=x^2+x^{-2}+2;\ (x^1-x^{-1})^2=x^2+x^{-2}-2.$$

11. Shew that

$$(x^n+x^{-n})(x^1+x^{-1})=(x^{n+1}+x^{-(n+1)})+(x^{n-1}+x^{-(n-1)}).$$

12. Shew that

$$(\tfrac{1}{2})^{\frac{2}{3}}.(\tfrac{2}{3})^{\frac{3}{4}}.(\tfrac{3}{4})^{\frac{4}{5}}=\frac{3^{\frac{1}{20}}}{2.2^{\frac{31}{60}}}=\frac{3^{\frac{1}{20}}.2^{\frac{29}{60}}}{4}.$$

98. Surds. In Section 73 was introduced the *irrational* number as $\sqrt{2}$, $\sqrt[3]{7}$. An expression as $\sqrt{a}$ is spoken of as an *algebraic irrationality*, though a may have a value which would make $\sqrt{a}$ *arithmetically rational*. So $\sqrt{(x+7)}$, $\sqrt[3]{(3x^2+5)}$,....are algebraically irrational. In this section we shall have to do with arithmetically irrational numbers.

An irrational root of a rational number is called a *surd*. Thus $\sqrt{5}$, $\sqrt{\frac{2}{3}}$, $\sqrt{11}$ are surds, while $\sqrt{9}$, though of surd form, is not a surd. A number as $3\sqrt{5}$, which equals $3.5^{\frac{1}{2}}$ or $9^{\frac{1}{2}}.5^{\frac{1}{2}}$ or $45^{\frac{1}{2}}$ or $\sqrt{45}$, is consequently a surd. Numbers as $3\sqrt{5}$, $7\sqrt{5}$, $\frac{2}{3}\sqrt{5}$, or as $7\sqrt[3]{3}$, $13\sqrt[3]{3}$, $\frac{5}{7}\sqrt[3]{3}$, which are different rational multiples of the same surd, are called *similar*. By an extension of language, a combination of surds, or of surds and rational numbers, as $\sqrt{3}+\sqrt{11}$, or $(3+5\sqrt{7})\div(4+\sqrt{5})$, unless it chances to reduce to a rational number, is called a surd.

A number as $\sqrt{7}$, in which the root to be extracted is the square root, is called a *quadratic surd*, and a surd as $\sqrt{7}+\sqrt{11}$, where the two terms are not similar, is called a *binomial quadratic surd*.

As we agreed that a number symbol as $7^{\frac{1}{2}}$ should denote the real positive root, so in this chapter it will be understood that by $\sqrt{7}$ is meant the real positive root, so that the symbol has an absolutely definite meaning.

We have now the following facts:

I. *A surd cannot equal a rational number.*

This follows at once from the definition.

II. *A quadratic surd cannot equal the sum of a rational number and a quadratic surd.*

For if possible let

$$\sqrt{m} = a + b\sqrt{n},$$

where $\sqrt{m}$, $\sqrt{n}$ denote actual surds, and a, b, m, n are rational. Then squaring the two expressions assumed equal, we have

$$m = a^2 + b^2n + 2ab\sqrt{n}$$

whence

$$\sqrt{n} = \frac{m - a^2 - b^2n}{2ab}$$

so that $\sqrt{n}$ is rational, which shews that the assumption is wrong.

III. If

$$m + \sqrt{n} = p + \sqrt{q}$$

where m, n, p, q *are rational and* $\sqrt{n}$, $\sqrt{q}$ are *surds*, *then*

$$m = p \text{ and } n = q.$$

For it follows that

$$\sqrt{n} = (p - m) + \sqrt{q}$$

which (by II) is impossible, unless it be that $p = m$, which would then require that $n = q$.

99. Applications. The following examples will illustrate the simpler applications.

Ex. 1. Find to the second decimal place the value of

$$\frac{5 + \sqrt{3}}{\sqrt{5} - \sqrt{2}}.$$

It would be tedious to find the approximations to the values of the numerator and denominator and then effect the division, but we note that

$$\frac{5 + \sqrt{3}}{\sqrt{5} - \sqrt{2}} = \frac{(5 + \sqrt{3})(\sqrt{5} + \sqrt{2})}{(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2})}$$

$$= \frac{5\sqrt{5} + \sqrt{15} + 5\sqrt{2} + \sqrt{6}}{5 - 2}.$$

The values of the surds in the numerator may now be found approximately and combined, the result being then divided by the rational number $5-2$ or 3 to give as the value sought 8·19.

In this process we *rationalized the denominator*, and $\sqrt{5}+\sqrt{2}$ is called a *rationalizing factor* of $\sqrt{5}-\sqrt{2}$.

Ex. 2. Find the value, correct to two places of decimals, of $\sqrt{(5+\sqrt{24})}$.

This raises the question of finding the square root of an expression of the form $m+\sqrt{n}$. We may have noted that $(\sqrt{h}+\sqrt{k})^2=h+k+2\sqrt{hk}$, which is of the form $m+\sqrt{n}$. We are thus l to ask if there is an expression $\sqrt{x}+\sqrt{y}$ such that

$$(\sqrt{x}+\sqrt{y})^2=5+\sqrt{24}.$$

If so, effecting the square, we have

$$x+y+2\sqrt{xy}=5+\sqrt{24},$$

whence, from a theorem established,

$$x+y=5, \text{ and } 2\sqrt{xy}=\sqrt{24},$$

which is seen to be equivalent to the system

$$x+y=5, \quad xy=6,$$

which yields the solutions $(x=2,\ y=3)$, $(x=3,\ y=2)$. These lead to the same result, and we have

$$(\sqrt{2}+\sqrt{3})^2=5+\sqrt{24}$$

$$\therefore \quad \sqrt{(5+\sqrt{24})}=\sqrt{2}+\sqrt{3},$$

from which we find

$$\sqrt{(5+\sqrt{24})}=1{\cdot}4142\ldots+1{\cdot}7321\ldots$$

$$=3{\cdot}15, \text{ correct to the second decimal place.}$$

The example should be worked by finding $\sqrt{24}$ to a sufficient degree of accuracy, adding this result to 5 and then extracting the root.

The process given for investigating the square root of an expression of the form $p+\sqrt{q}$ will not always yield rational values of x and y, as in this example, though it will always yield a result. This result may be more involved than the original root proposed, or it may be imaginary.

Ex. 3. Solve

$$\sqrt{x+4}+\sqrt{x+11}=7.$$

It is necessary to rationalize the equation. With a view to simplicity of result when we square the involved expressions, we put the equation in the form

$$\sqrt{x+4}=7-\sqrt{x+11}.$$

Then squaring the equals, we have

$$x+4=49+(x+11)-14\sqrt{x+11},$$

an equation, involving only one irrationality, which reduces to

$$14\sqrt{x+11}=56.$$

$$\therefore \quad \sqrt{x+11}=4$$

$$\therefore \quad x+11=16$$

$$\therefore \quad x=5.$$

Thus the root is 5.

The student may solve by squaring the equals as they stand in the given equation.

Ex. 4. Solve

$$\sqrt{2x+3}+\sqrt{3x+7}=7.$$

Proceeding to rationalize as in the earlier example, we find

$$\sqrt{3x+7}=7-\sqrt{2x+3}$$

$$\therefore \quad 3x+7=49+2x+3-14\sqrt{2x+3}$$

$$\therefore \quad 14\sqrt{2x+3}=-x+45$$

$$\therefore \quad 392x+588=x^2-90x+2025$$

$$\therefore \quad x^2-482x+1437=0$$

$$\therefore \quad (x-3)(x-479)=0$$

$$\therefore \quad x=3 \text{ or } 479.$$

It is found that 3 satisfies the equation while, in order that 479 should satisfy it, we should have

$$\sqrt{961}+\sqrt{1444}=7$$

i.e. $$\sqrt{31^2}+\sqrt{38^2}=7.$$

If, though this is not in accordance with our convention, we take $\sqrt{31^2}$ as -31 and $\sqrt{38^2}$ as $+38$ the test would be met, for

$$-31+38=7.$$

In such equations it is often the case that there present themselves roots that do not apply to the equation as it stands, but do apply if certain of the irrationalities be regarded as of different sign. The reason for this is that, in the process of rationalization. the result would have been the same whichever sign had been associated with the irrationality. Thus here the student should work the examples in the manner given except that $\sqrt{2x+3}$ be taken as the negative root and denoted by $-\sqrt{2x+3}$. The student should also work the example by squaring without transference of either irrationality and by transferring $\sqrt{3x+7}$ and then squaring.

Exercises CXXV

1. Bring to a form more suitable for computation the following:

$$\frac{5+\sqrt{7}}{3+\sqrt{5}};\quad \frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}-\sqrt{5}};\quad \frac{5+\sqrt{11}}{\sqrt{5}+\sqrt{7}}.$$

2. If $x=2+\sqrt{3}$, $y=2-\sqrt{3}$ find to two places of decimals

$$\frac{x-y}{x+y}+\frac{x+y}{x-y}.$$

3. Express by means of a single root sign

$$\sqrt{2}\,.\,\sqrt[3]{3};\quad \sqrt[3]{5}\,.\,\sqrt[4]{3}.$$

4. Find the product

$$(\sqrt{3}+\sqrt{5}+\sqrt{7})(-\sqrt{3}+\sqrt{5}+\sqrt{7})(\sqrt{3}-\sqrt{5}+\sqrt{7})$$
$$(\sqrt{3}+\sqrt{5}-\sqrt{7}).$$

5. Shew that

$$(3+\sqrt{5})^2=14+6\sqrt{5}$$

and determine by inspection the square roots of

$$4+2\sqrt{3};\quad 23+8\sqrt{7};\quad 36-10\sqrt{11}.$$

6. Shew that

$$(\sqrt{3}+\sqrt{5})^2=8+2\sqrt{15}$$

and determine by inspection the square roots of

$$12+2\sqrt{35};\quad 17+2\sqrt{66};\quad 18-2\sqrt{65};$$
$$7-2\sqrt{10};\quad 10-2\sqrt{21};\quad 14+\sqrt{180}.$$

7. Determine the square roots of

$$14+8\sqrt{3};\quad 21-6\sqrt{10};\quad 30+12\sqrt{6};$$
$$13-4\sqrt{10};\quad 16+4\sqrt{15};\quad 51-36\sqrt{2}.$$

8. Shew that

$$(a^{\frac{1}{3}}-b^{\frac{1}{3}})(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})=a-b$$

so that each of the factors in this product is a rationalizing factor of the other.

9. If

$$z=\sqrt[3]{p}+\sqrt[3]{q}$$

shew that

$$z^3=p+q+3\sqrt[3]{pq}.z.$$

10. Simplify

$$\frac{\sqrt{12+6\sqrt{3}}}{\sqrt{3}+1}.$$

11. Solve the following equations:

(1) $\sqrt{2x+5}+\sqrt{2x-4}=9.$

(2) $\sqrt{3x-5}+\sqrt{2x+11}=9.$

(3) $\sqrt{x+3}+\sqrt{x+10}=\sqrt{x+43}.$

(4) $\sqrt{3x^2-2x+9}+\sqrt{3x^2-2x-4}=13.$

(5) $\sqrt{x+a}+\sqrt{x+b}=\sqrt{a-b}.$

(6) $\dfrac{\sqrt{3x+1}+\sqrt{3x}}{\sqrt{3x+1}-\sqrt{3x}}=4.$

(7) $b+\sqrt{(a^2+bx)}=\sqrt{(b^2+ax)}+a.$

(8) $a+\sqrt{(a^2+bx)}=\sqrt{(b^2+ax)}+b.$

(9) $\sqrt[3]{x+24} - \sqrt[3]{x+5} = 1.$

(10) $\sqrt[3]{1+x} + \sqrt[3]{1-x} = 4.14^{-\frac{1}{3}}.$

(11) $x^2 + \sqrt{x^2-5} = 11.$

(12) $x^2 + 3x + \sqrt{2x^2+6x+1} = 49.$

(13) $x^2 - x + 3\sqrt{2x^2-3x+2} = \frac{x}{2} + 7.$

(14) $3x^2 - 2x - \sqrt{3x^2-4x-6} = 18 + 2x.$

(15) $p + x + \sqrt{2px+x^2} = q.$

12. Shew that

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})(-\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})$$
$$= 2bc + 2ca + 2ab - a^2 - b^2 - c^2$$

13. Find a rationalizing factor of $\sqrt{5} + \sqrt{6} + \sqrt{7}$.

14. Divide $a^{\frac{2}{3}} + a^{-\frac{2}{3}} + 1$ by $a^{\frac{1}{3}} + a^{-\frac{1}{3}} + 1$.

15. Form the equation with rational coefficients one of whose roots is $5 - \sqrt{7}$.

16. If

$$\sqrt{(a+\sqrt{b})} = \sqrt{x} + \sqrt{y},$$

shew that

$$\sqrt{(a-\sqrt{b})} = \sqrt{x} - \sqrt{y}.$$

17. Find the square of

$$\sqrt{x} + \sqrt{y} + \sqrt{z}$$

and investigate whether a square root in simple form can be found for

$$24 + 8\sqrt{5} + 4\sqrt{15} + 8\sqrt{3}.$$

Exercises CXXVI

(Miscellaneous)

A

1. Resolve into factors,

(i) $3x^3-14x^2-24x$;

(ii) $2x^2-6y^2-15z^2+19yz-zx+xy$;

(iii) $4(ad+bc)^2-(a^2-b^2-c^2+d^2)^2$.

2. One root of

$$x^2-5x+d=0$$

is 8 find the other root and the value of d.

3. Divide

$$x^{\frac{5}{2}}-x^{-\frac{5}{2}} \text{ by } x^{\frac{1}{2}}-x^{-\frac{1}{2}}.$$

4. Solve

$$\left(x+\frac{1}{x}\right)+\left(x+\frac{1}{x}\right)^{-1}=2\frac{9}{10}.$$

5. A square plot of ground is surrounded by a gravel walk of uniform width which covers 891 square feet of ground. Outside this walk at a uniform distance of half the length of the plot is a second walk, of the same width as the first, which covers 1903 square feet. Find the size of the plot.

B

1. Shew that the system of equations

$$x^4+y^4=7x^2y^2,\ x+y=5$$

is the equivalent of the two systems

$$\left.\begin{array}{r}x^2+y^2=3xy\\ x+y=5\end{array}\right\};\quad \left.\begin{array}{r}x^2+y^2=-3xy\\ x+y=5\end{array}\right\}$$

and find all the solutions.

2. If one root of

$$2x^2-14x+a=0$$

is 3, find the other root and the value of a.

3. Construct the two lines whose equations are

$$2(x-1)=3(y-2);\quad 3(x-1)=4(y-2).$$

Find the solution of the simultaneous set.

4. Shew that there are two numbers which are their own reciprocals.

5. Shew that when the product

$$(3x^3-7x^2+13x-9)(5x^3+8x^2-11x-43)$$

is formed the sum of the coefficients is zero.

C

1. One of the roots of the equation

$$24x^3-46x^2+29x=6$$

is $\frac{3}{4}$. Find the other roots.

2. Resolve into factors

$$x^4-2a^2x^2-2b^2x^2+a^4+b^4-2a^2b^2.$$

3. Solve

$$\frac{1}{x-2}+\frac{1}{x-6}=\frac{1}{x-7}+\frac{1}{x-1}.$$

4. For what value of c, will

$$3x^2+5x=6c$$

have equal roots ?

5. If $2x^2+5x+7$ and ax^2+bx+c are equal whatever be the value of x, shew that

$$a=2,\ b=5,\ c=7.$$

6. Resolve into factors

(i) $(m^2-n^2)x^2+2(m^2+n^2)x+(m^2-n^2)$;

(ii) $(x^2-9x)^2+4(x^2-9x)-140$.

7. The first of two numbers added to four times the reciprocal of the second gives 4; the second diminished by three times the reciprocal of the first gives 3. Find the numbers.

CHAPTER XXIII.

ELIMINATION

100. Explanatory. If two simple equations in x are written down at random we feel that, while it is possible that they have the same root, it is quite unlikely. Thus if we write

$$7x-11=0, \quad 8x-17=0,$$

we see that the roots are not the same. Suppose now that we are given that the two equations

$$ax+b=0, \quad px+q=0$$

are satisfied by the same value of x. In view of what has been said we feel that some special relation must exist between the equations. From the first we have $x=-\frac{b}{a}$ and from the second $x=-\frac{p}{q}$. But they are satisfied by the same value,

$$\therefore -\frac{b}{a}=-\frac{p}{q},$$

or, in a form free from fractions,

$$ap-bq=0.$$

Thus in order that both equations may be satisfied by the same value of x a certain relation must exist among the coefficients of the two equations. This relation is called the *eliminant* of the system of equations.

Next consider the system

$$ax + by = 0, \quad px + qy = 0$$

and suppose the equation, which is satisfied by zero values of x and y, to be satisfied also by other values. Then the system may be written

$$a\left(\frac{x}{y}\right) + b = 0, \quad p\left(\frac{x}{y}\right) + q = 0$$

and treating the *ratio* $\left(\frac{x}{y}\right)$ as the unknown we see, as before, that the given *homogeneous linear system in* x, y, has for eliminant,

$$ap - bq = 0.$$

Examine now the system

$$ax + b = 0, \quad px^2 + 2qx + r = 0,$$

supposed to be satisfied by the same value of x. The one value of x that will satisfy the first equation is

$$x = -\frac{b}{a}.$$

This then must satisfy the second equation, so that

$$p\left(-\frac{b}{a}\right)^2 + 2q\left(-\frac{b}{a}\right) + r = 0,$$

or, in a form free from fractions,

$$pb^2 - 2qab + ra^2 = 0,$$

which is the relation that the coefficients in the given system must satisfy if the equations have a common root, the eliminant therefore of the system.

Similarly the eliminant of the homogeneous system

$$ax + by = 0, \; px^2 + 2qxy + ry^2 = 0,$$

supposed satisfied by the same values of x, y—other than zero—is

$$pb^2 - 2qab + ra^2.$$

Consider next the system

$$ax + by + c = 0,$$
$$hx + ky + l = 0,$$
$$px + qy + r = 0,$$

supposed satisfied by the same values of x and y. The last two equations determine x and y, and we find

$$x = \frac{kr - lq}{hq - kp}, \quad y = \frac{lp - hr}{hq - kp}.$$

These values must also satisfy the first equation, so that

$$a\left(\frac{kr - lq}{hq - kp}\right) + b\left(\frac{lp - hr}{hq - kp}\right) + c = 0,$$

or, in a form free from fractions,

$$a(kr - lq) + b(lp - hr) + c(hq - kp) = 0.$$

This is, then, the eliminant of the system. In the same way it is seen that the eliminant of the homogeneous system

$$ax + by + cz = 0,$$
$$hx + ky + lz = 0,$$
$$px + qy + rz = 0,$$

supposed satisfied by other than zero values of all of x, y, z, is found to be the same as that of the earlier system of three equations in x and y.

Thus generally, when a system of equations, one more in number than is sufficient to determine the unknowns, is satisfied by the same values of the unknowns, a relation must exist among the coefficients of all the equations. This relation is the eliminant of the system, and should be given in a form free from fractions or irrationalities. If the equations are all homogeneous and in number as many as there are unknowns, and therefore sufficient to determine the *ratios* of the unknowns, there will in like manner be an eliminant.

101. Illustrative Exercises. The following exercises will be found instructive, and will exhibit certain applications.

Ex. 1. One root of the equation

$$x^2 + px + q = 0$$

is double of the other, find what relation exists between p and q.

Let m and n be the two roots of the equation. Then from the theory of the quadratic equation we have

$$m + n = -p \qquad \text{(i)}$$
$$mn = q \qquad \text{(ii)}$$

and from the given condition

$$m = 2n \qquad \text{(iii)}$$

Thus m and n have to satisfy *three* equations, one equation more than sufficient to determine them. Substituting, from (iii), in (i) and (ii) we have

$$3n = -p \qquad \text{(iv)}$$
$$2n^2 = q \qquad \text{(v)}$$

whence

$$2\left(-\frac{p}{3}\right)^2 = q$$

or

$$2p^2 = 9q$$

which is therefore the relation sought.

The student should obtain this result by finding the two roots of the given equation and directly equating one to twice the other.

Ex. 2. Find the condition that $x^2 + px + q$ and $x^2 + rx + s$ may have a common factor, linear in x.

Let $x - m$ be the common factor, where m is, it goes without saying, unknown. Then m is a root of the corresponding equations, and it follows that

$$m^2 + pm + q = 0$$
$$m^2 + rm + s = 0.$$

These are *two* equations that m must therefore satisfy, one more than is sufficient to determine m. We have then to eliminate m from these equations.

By subtraction

$$m(p-r)+(q-s)=0$$

whence

$$m=-\frac{q-s}{p-r}$$

Then substituting in either of the equations, say the first, we have

$$\left(-\frac{q-s}{p-r}\right)^2+p\left(-\frac{q-s}{p-r}\right)+q=0,$$

or, in a form free from fractions,

$$(q-s)^2-p(q-s)(p-r)+q(p-r)^2=0$$

which is the eliminant of the equations and therefore the relation sought.

The student should also study the following solution:

If the two expressions have a common factor it is a factor of their difference, which is

$$x(p-r)+(q-s).$$

But this is linear and, from the point of view of algebraic factors, is not different from

$$x+\frac{q-s}{p-r}.$$

This last must then be the common factor, so that $-\frac{q-s}{p-r}$ is a root of each of the equations

$$x^2+px+q=0$$
$$x^2+rx+s=0.$$

Substituting this root in, say, the former, we find as before

$$(q-s)^2-p(q-s)(p-r)+q(p-r)^2=0.$$

Ex. 3. Find the condition that x^2+px+q and x^2+rx+s may have the same linear factors.

Let $x-m$, $x-n$ be those factors. Then, since the factors of the expressions give the roots of the corresponding equations, m and n are the roots of each of the equations

$$x^2+px+q=0$$
$$x^2+rx+s=0.$$

Hence

$$m+n=-p \qquad \text{(i)}$$
$$mn=q \qquad \text{(ii)}$$
$$m+n=-r \qquad \text{(iii)}$$
$$mn=s \qquad \text{(iv)}$$

four equations that must be satisfied by m and n, *two* more than sufficient to determine m and n, so that we may expect, in a way, *two eliminants.* We see at once that it must be that

$$p=r$$
$$q=s$$

which two relations constitute the condition sought.

Ex. 4. If

$$\frac{x}{y+z}=a, \quad \frac{y}{z+x}=b, \quad \frac{z}{x+y}=c,$$

find the relation among a,b,c.

This means that x,y,z are to be eliminated from the equations. If the equations are cleared of fractions it is seen that they are of a kind to admit this elimination. But here, as in many other examples, the elimination is more readily effected by means of an artifice suggested by the special form of the equations.

From the first we have

$$\frac{y+z}{x}=\frac{1}{a}$$

$$\therefore \quad 1+\frac{y+z}{x}=\frac{1}{a}+1$$

or $$\frac{x+y+z}{x}=\frac{1+a}{a}$$

$$\therefore \quad \frac{x}{x+y+z}=\frac{a}{1+a}.$$

Similarly

$$\frac{y}{x+y+z}=\frac{b}{1+b}, \quad \frac{z}{x+y+z}=\frac{c}{1+c}$$

$$\therefore \quad \frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=1,$$

the relation sought. This relation may easily be given a form free from fractions.

Exercises CXXVII

1. If $\frac{y}{z}=l$, $\frac{z}{x}=m$, $\frac{x}{y}=n$, find the relation among l,m,n.

2. If

$$x+\frac{1}{x}=a; \quad x^2+\frac{1}{x^2}=b$$

shew that

$$a^2-b=2.$$

3. Eliminate x from the two equations

$$x+\frac{1}{x}=a, \quad x^3+\frac{1}{x^3}=c.$$

4. Eliminate x from the two equations

$$x+\frac{1}{x}=a, \quad x^4+\frac{1}{x^4}=d.$$

5. If

$$ax^2+2bx+c=0, \quad px^2+2qx+r=0$$

have a common root find the relation among the coefficients.

6. Eliminate x and y from the equations

$$x+y=a, \; x^2+y^2=b^2, \; x^3+y^3=c^3.$$

7. If

$$ax^2+2bx+c=0, \; px^2+2qx+r=0$$

have the same roots, find what relations must exist among the coefficients.

8. One root of

$$ax^2+2bx+c=0$$

is the reciprocal of the other. Find the necessary relation among the coefficients.

9. It is known that

$$y = mx + n, \qquad y^2 = 4ax$$

have two solutions. Find the condition that the two solutions be the same.

10. Find the condition that the two solutions of

$$\frac{x}{l} + \frac{y}{m} = 1, \qquad x^2 + y^2 = a^2$$

be the same.

11. Find the condition that the two solutions of

$$lx + my = 1, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

be the same.

12. Find the condition that the roots of the equation

$$ax^2 + 2bx + c = 0$$

may be the reciprocals of those of the equation

$$px^2 + 2qx + r = 0.$$

13. If

$$\frac{w}{x+y+z} = a; \quad \frac{x}{y+z+w} = b; \quad \frac{y}{z+x+w} = c; \quad \frac{z}{x+y+w} = d;$$

find the relation among a, b, c, d.

14. If

$$\frac{y}{z} + \frac{z}{y} = a; \quad \frac{z}{x} + \frac{x}{z} = b; \quad \frac{x}{y} + \frac{y}{x} = c;$$

prove that

$$abc = a^2 + b^2 + c^2 - 4.$$

EXERCISES CXXVIII

(MISCELLANEOUS)

A

1. Construct the equation whose roots are the reciprocals of the roots of the equation

$$17x^2+53x-97=0.$$

2. It is said that 4 is the cube root of 64: as it is only one of the cube roots, find the others.

3. If

$$a=\frac{2}{2-b},\quad b=\frac{2}{2-c},\quad c=\frac{2}{2-d},\quad d=\frac{2}{2-x},$$

shew that $x=a$.

4. Find the square root of

$$x^6+4x^5-2x^4-10x^3+13x^2-6x+1.$$

5. Solve the simultaneous set

$$x^2+2yz=13,\ y^2+2zx=10,\ z^2+2xy=13.$$

B

1. For what values of x is the expression $65x^2-241x+83$ zero, for what values negative, and for what values positive?

2. If

$$x+y+z-xyz=2,$$

shew that

$$(1-x)^2=(1-xy)(1-xz).$$

3. If

$$\frac{yr-zq}{ya-zb}=\frac{zp-xr}{za-xc},$$

then each of these fractions is equal to

$$\frac{xq-yp}{xb-ya}.$$

4. If p be the difference between x and $\frac{1}{x}$, and q the difference between x^2 and $\frac{1}{x^2}$, shew that

$$p^2(p^2+4)=q^2.$$

5. The left-hand digit of a certain number, of two digits, exceeds the right digit by 5, and when the number is divided by the sum of the digits the quotient is 8. Find the number.

C

1. Resolve into factors,

$$x^2(x^2-a^2)-y^2(y^2-a^2)+2xy(x^2-y^2).$$

2. Simplify

$$\frac{(y+z-x)^2}{(y+z)^2-x^2}+\frac{(z+x-y)^2}{(z+x)^2-y^2}+\frac{(x+y-z)^2}{(x+y)^2-z^2}.$$

3. Solve the equation

$$\frac{(x-3)(2x-5)}{(x-2)(3x-7)}=\frac{(x-3)(4x-9)}{(x-2)(5x-12)}$$

shewing that $x=3$ must be a solution, and examining whether $x=2$ is a solution.

4. Shew that the product of any four consecutive integers increased by unity is a perfect square.

5. A train after a run of one hour is detained 15 minutes. It then goes on at three-fourths its former speed and arrives 24 minutes late. Had the detention occurred 5 miles further on, the train would have been only 21 minutes late. Find the original speed and the whole distance.

D

1. Shew that

$$x^2+4x+2=(x+2+\sqrt{2})(x+2-\sqrt{2})$$

and resolve similarly

(i) x^2+6x+7; (ii) x^2-5x+3.

2. Noting that the equation

$$x^2+y^2=25$$

may be put in the form

$$y=\pm\sqrt{25-x^2}$$

represent y, as a function of x, graphically.

What geometrical figure is the graph? Does this appear in the original equation?

3. Shew that it is not possible to find values of x that will satisfy the two equations

$$12x-15y=37,$$
$$20x-25y=45.$$

4. If x^2+6x+b and $x^2+12x+3b$ have a common linear factor, what numerical values can b have, and what is the common factor?

5. The volume of the frustum of a cone is given by the formula

$$v=\tfrac{1}{3}\pi h(a^2+ab+b^2)$$

where v,h,a,b measure the volume, the vertical height, and the radii of the base and the top, and π is a certain definite number which may be taken to be $\frac{22}{7}$. If $v=154$, $h=3$, $b=3$, find a.

E

1. If the product of two rational numbers, not both integers, is an integer, then the sum of the two numbers is an arithmetical fraction.

2. A can do a piece of work in l days, B in m days, C in n days. They work together at the work. Find (i) the part of the work done in one day, (ii) the time required to do all the work.

Simplify this latter result and state a rule for finding the time for all such cases of three men working together.

3. Solve

$$2x^2-3xy+11y^2=13,$$
$$3x^2-5xy+5y^2=7.$$

4. Resolve into factors

(i) $2x^2-15y^2+15z^2-16yz-11zx+7xy$

(ii) x^3+px^2+px+1.

5. Prove that

$$\frac{y-z}{1+yz}+\frac{z-x}{1+zx}+\frac{x-y}{1+xy}=-\frac{(y-z)(z-x)(x-y)}{(1+yz)(1+zx)(1+xy)}.$$

F

1. A and B are travelling at uniform rates. In an hour A goes 2 miles farther than B. It takes B 90 seconds longer than A to go a mile. Find each person's rate of travelling.

2. If

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c},$$

shew that

$$\frac{x^2+a^2}{x+a}+\frac{y^2+b^2}{y+b}+\frac{z^2+c^2}{z+c}=\frac{(x+y+z)^2+(a+b+c)^2}{(x+y+z)+(a+b+c)}.$$

3. Solve

$$a(b-c)x^2+b(c-a)x+c(a-b)=0.$$

4. Write down, without the work of actual multiplication, the products in the following multiplications:

(i) $(x+1)(x+3)(x+5)$;

(ii) $(x-y+a-b)(x+y-a-b)$.

5. If

$$3x^2-10x-7=ax^2+bx+c$$

for $x=1$, $x=2$, $x=3$, shew that $a=3$, $b=-10$, $c=-7$.

G

1. Represent graphically, with reference to the same axes, the relation between x and y given separately by the equations

$$x^2+y^2=25;$$
$$x+y=1.$$

Determine from the figure the values of x and y that will satisfy *both* equations, testing by substituting in the equations.

2. If

$$3x^3-5x^2+7x-11=ax^3+bx^2+cx+d$$

for $x=1$, $x=2$, $x=3$, $x=4$, shew that $a=3$, $b=-5$, $c=7$, $d=-11$.

3. Solve

$$\frac{x-1}{x-4}+\frac{x-5}{x-6}=4.$$

4. Resolve into factors

(i) $10x^2+11xy-6y^2$,

(ii) $3x^2-4y^2-3z^2+8yz-8zx+4xy$.

5. Shew that

$$3x^3-7x^2+12x-13=3(x-2)^3+11(x-2)^2+20(x-2)+7,$$

and exhibit

$$5x^3-8x^2+17x-23$$

as a sum of powers of $x-3$.

H

1. Represent graphically the function $6-x-x^2$, finding its maximum value.

2. Shew that the sum of a positive fraction and its reciprocal is greater than 2.

Illustrate and find the simple fraction of which this is not true.

3. Shew that

$$\sqrt{7+4\sqrt{3}}+\sqrt{7-4\sqrt{3}}$$

is rational.

4. Solve

$$\frac{a}{x}+\frac{b}{y}=5,$$

$$\frac{b}{2x}-\frac{a}{3y}=\frac{b^2-a^2}{ab}.$$

5. Prove without expanding that

$$(y+z-2x)^3+(z+x-2y)^3+(x+y-2z)^3= 3(y+z-2x)(z+x-2y)(x+y-2z).$$

I

1. Shew by division that

$$\frac{a^n-b^n}{a-b}=a^{n-1}+b\cdot\frac{a^{n-1}-b^{n-1}}{a-b}$$

or that

$$a^n-b^n=a^{n-1}(a-b)+b(a^{n-1}-b^{n-1}).$$

Hence prove that a^n-b^n is divisible by $a-b$ if $a^{n-1}-b^{n-1}$ is divisible by $a-b$, and infer the divisibility of a^n-b^n by $a-b$.

2. Shew that it is not possible that

$$\sqrt{p}=\sqrt{q}+\sqrt{r}$$

if $\sqrt{q}$ and $\sqrt{r}$ are dissimilar surds.

3. A pound of tea and three pounds of sugar cost $1·20, but if sugar were to rise 50 per cent. and tea 10 per cent. the cost would be $1·40. Find the prices of tea and sugar.

4. If the equation

$$x^2-15-m(2x-8)=0$$

has equal roots, find the values of m, and the roots.

5. If $2s=a+b+c$, shew that

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{abc}{s(s-a)(s-b)(s-c)}.$$

J

1. Solve

(i) $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c)$;

(ii) $3x^{\frac{3}{2}}-4x^{\frac{3}{4}}=7$.

2. There is a number expressed by two digits. The number is equal to three times the sum of the numbers denoted by its digits and if 45 be added to the number the digits interchange their places. Find the number.

3. If

$$x = 2a - b - c, \ y = 2b - c - a, \ z = 2c - a - b.$$

shew that

$$x^3 + y^3 + z^3 = 3xyz.$$

4. Resolve into factors

(i) $acx^3 + (ad - bc)x^2 - (ac + bd)x + bc.$

(ii) $2x^4 - 3x^3 - 21x^2 - 2x + 24.$

5. Find the minimum value of $8x^2 - 18x + 9$ and shew that the value of x which gives the expression its minimum value is one-half the sum of the roots of the corresponding equation.

K

1. Shew that

$$a^n + b^n = (a^{n-1} - a^{n-2}b)(a - b) + b^2(a^{n-2} + b^{n-2})$$

and hence shew that $a^n + b^n$ is divisible by $a + b$ for every odd value of n.

2. Solve

$$x^2 + xy + y^2 = 109; \ x^4 + x^2y^2 + y^4 = 4251.$$

3. If

$$xy = 2$$

find the value of y for $x = 4, 3, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$, and represent the relation of y to x graphically.

4. Examine whether the system

$$6x + 5y = 8xy, \ 8x + 11y = 15xy$$

is the equivalent of the system

$$\frac{6}{y} + \frac{5}{x} = 8, \ \frac{8}{y} + \frac{11}{x} = 15.$$

5. Express $x^5 - 1$ as a sum of powers of $x - 1$.

EXERCISES

(SUPPLEMENTARY TO THOSE IN CHAPTERS I—XII)

A (CHAPTER I)

1. Verify that

$$x(y+z)=xy+xz$$

if $x=7$, $y=3$, $z=8$.

2. Find how many pounds there are in m tons n cwt. p lb.

3. Prove that $7x-x^2=12$ if $x=3$ or 4, but not if $x=2$, or if $x=5$.

4. The volume of the frustum of a cone is given by the formula

$$v=\frac{1}{3}\pi h(a^2+ab+b^2),$$

where a and b are the measures of the radii of the base and the top, h the measure of the vertical height, v the measure of the volume and π a certain definite number which may be taken to be $\frac{22}{7}$. Find v if $a=12$, $b=9$, $h=4$.

5. Find the value of

$$\frac{a^2b}{7}+\sqrt{7ab(2c^2-ab)}-(2a-3b)^2$$

when $a=3$, $b=2\frac{1}{3}$, $c=2$.

6. A can walk at the rate of a miles an hour and B at the rate of b miles an hour. They start together on a walk of x miles. If a is greater than b, find the interval between the arrivals of A and B.

7. Taking one metre as 39·37 inches, and being given that y metres are equivalent to x yards, find y in terms of x.

8. There is a certain number and it is known that three times the number increased by 17 exceeds 5 times the number by 1. Express this algebraically, and find the number.

9. Shew that the statement

$$5x-7=3x+11$$

cannot be true unless x has a certain value.

10. If the pound sterling is quoted as equal to $4·87, find the value in dollars of x pounds, and of y pounds z shillings.

11. The sum of $120 is divided among A, B, C, the share of B being $10 more than that of A, and the share of C being $8 less than the combined shares of B and C. Taking $3x$ as A's share, write down the shares of B and C and the relation that the corresponding numbers must satisfy.

12. Test for accuracy the statement

$$(a+b)=a^2+2ab+b^2$$

by substituting for a and b the sets of values

(i) $a=3,\ b=0$; (ii) $a=0,\ b=4$; (iii) $a=5,\ b=7$.

13. Find the value of

$$\frac{x^3+y^3}{x^2-xy+y^2}-\frac{x^3-y^3}{x^2+xy+y^2}$$

for $x=\frac{1}{3}$, $x=\frac{1}{4}$.

14. Find how many days there are between the mth of July and the nth of August, one of these dates being included.

15. The area of a circle being given by the formula $A=\pi r^2$, where A and r measure the area and the radius while π is a certain definite number, the same for all circles, and it being given of three circles whose radii measure l, m, n that the area of the first is equal to the combined areas of the other two, shew that

$$l^2=m^2+n^2.$$

16. Test the equality

$$(a-b)-(c-d)=a-b-c+d$$

by taking $a=11$, $b=2$, $c=5$, $d=3$.

17. If $a=2$ and $b=4$ shew that $a^b=b^a$

18. If $a=2$ and $b=3$ shew that a^b is less than b^a.

19. A town site is laid out into equal blocks by m streets running east and west and n streets running north and south. How many blocks are there?

20. A rectangular piece of ground a ft. by b ft. is surrounded on the outside by a path x feet wide. Find the area of the walk.

21. A train travels m yards in h minutes; find its rate in miles an hour.

22. The shorter side of a rectangle is b inches long, and is x inches shorter than the longer side. Find the perimeter of the rectangle, and the side of a square of the same perimeter.

23. Find the joint earnings in a day of l men at a dollars a day, m women at b dollars a day, and n boys at c dollars a day.

24. x and y are two numbers such that m times the first exceeds n times the second by p. Express this in algebraic language.

25. If m yards of cloth are worth p dollars find the value of x yards.

26. If n men can do a certain piece of work in h days, find how long it would take a men to do the work.

27. If a, b, c measure the lengths of the sides of a right-angled triangle, it is known that $a^2+b^2=c^2$, c belonging to the hypothenuse. Find the length of the hypothenuse of a right-angled triangle whose sides are 5 ft. and 12 ft. in length.

B (Chapter II)

1. One number exceeds another number by 5, while twice that number is equal to three times the other. Find the numbers.

2. A can run 8 yards and B 7 yards in a second. In a race A gives B a start of 6 yards; at the end of what time will A overtake B?

3. A's age is three times B's age, but 9 years hence will be only twice B's age. Find the ages of each.

4. A has $27 more than B. A doubles his money and B converts his into a sum three times as great, and then B has $1 more than A. Find the sums at first held by them.

5. The units digit of a number expressed by two digits is 1 greater than the tens digit, and the number is 5 times the sum of the digits. Find the number.

6. A merchant bought 100 yards of cloth at 75 cents a yard. A certain number of yards were rendered valueless, yet by selling the remainder at $1·00 a yard he gained $5 on the transaction. Find how many yards were ruined.

7. A has $[illegible] and B has $40. How many dollars should B give A in order that A may have twice as much as B?

8. Find two consecutive integers such that the sum of 5 times the first and 6 times the second is 83.

9. Divide 23 into two parts such that twice the greater exceeds three times the less by unity.

10. A man bought a certain number of yards of one kind of cloth at $1·50 a yard, and 10 yards more of another kind at $1·80 a yard, paying in all $150. How many yards of each kind were bought?

C (Chapter III)

1. A man has property worth \$8000 and his debts amount to \$1000. What number (of dollars) expresses his net worth and what his net indebtedness?

2. A cistern has two pipes, one which can supply 10 gallons a minute and one which can empty 12 gallons a minute. If the cistern has some water in it and both pipes are set in action, state at what rate the cistern is being filled and at what rate being emptied.

3. Test the statement

$$a(x+y-z)=ax+ay-az$$

for the following values of the involved letters:

(i) $a=3,\ x=7,\ y=5,\ z=9$; (ii) $a=-3,\ x=4,\ y=5,\ z=12$.

4. Prove, ability to make the multiplications not being assumed, that

$$(p-q)(y-z)-(q-p)(z-y)=0.$$

5. Find the average of the numbers in each of the following sets:

(i) 3 numbers 7, 5 numbers 12, 7 numbers 15:

(ii) l numbers x, m numbers y, n numbers z;

(iii) 5 numbers -7, 3 numbers 23, 4 numbers -6.

6. The question is raised whether a^2 is greater than a. Examine the matter for

$$a=3,\ a=2,\ a=1,\ a=\tfrac{3}{4},\ a=\tfrac{1}{2},\ a=0,\ a=-\tfrac{1}{2},\ a=-1.$$

7. Regarding north latitude and east longitude as positive, express, referring to a map, the latitude and longitude of the following cities:

Toronto, Rome, Pekin, Cape Town,
Rio Janeiro, Quito, Vancouver.

8. Test for accuracy the statement

$$m(x-y)=mx-my,$$

taking $m=+5$, $x=+3$, $y=-2$.

9. Shew that

$$-7+5-3-4+2=7(-1)+5(+1)+3(-1)+4(-1)+2(+1).$$

10. Shew that

$$(-4)^2=(+4)^2;\ (-3)^5=(-1)^5.3^5=-3^5;\ (-3)^5+(+3)^5=0.$$

D (Chapter IV)

1 Find the value of each of the following:—

(1) $(2l+3m+7n)+(5l-8m+2n)+(-4l+2m+n)$;

(2) $(3x-5y+8z)-(2x+5y+4z)-(-5x-2y+7z)$;

(3) $(5p^2-7q^2+4r^2)+(-2p^2-5q^2+7r^2)-(-3p^2+4q^2-10r^2)$;

(4) $(-3bc+4ca-7ab)-(-4bc-4ca+5ab)+(4bc+10ca-9ab)$;

(5) $(2x+3y-4z)+(\frac{3}{5}x-\frac{2}{3}y+z)+(-\frac{3}{4}x+2y-\frac{5}{6}z)$.

2. Express in simplest form

(1) $3x+2y+5z+8x-y-3z-6x+3y-z$;

(2) $3bc+5ca+8ab-2bc+3ca-4ab+bc-6ca-3ab$;

(3) $5(v+w)-7(w+u)+11(u+v)+9(w+u)-4(v+w)-9(u+v)$;

(4) $\frac{2}{3}x^2+\frac{3}{4}x-\frac{5}{6}+\frac{1}{2}x^2-\frac{3}{8}x-\frac{2}{3}+\frac{5}{6}x^2-\frac{7}{12}x+\frac{1}{4}$;

(5) $\frac{1}{4}al-\frac{2}{3}bm+\frac{5}{8}cn-\frac{1}{12}al-\frac{1}{4}bm-\frac{3}{4}cn+\frac{1}{3}al+\frac{1}{4}bm+\frac{5}{8}cn$.

3. From the sum of $3x^2+5xy+7y^2$ and $2x^2-9xy+y^2$ take $4x^2-3xy-4y^2$.

4. What number must be added to $-b$ to yield a sum $+b$?

5. Find what number must be added to the sum of $17z^2-13z+7$ and $12z^2+5z-13$ to yield the sum $33z^2-17z-23$.

6. Remove the brackets and reduce to simplest form the following:

(1) $(2a-3b)-\{3c-(5a-4b)+(6a-7c)\}$;

(2) $x-\{y-(z-\overline{x-y-z})\}$;

(3) $\{3a-(4b-7c)\}-\{-5a+(5b-\overline{2c-3a})\}$;

(4) $5[3(l-m)-4\{3n-2(7l-\overline{3m-4n})\}]$;

(5) $2yz-\overline{3zx-7xy}-4\{3(yz-2xy)-(2yz-3xy)\}$.

7. Express as a sum of powers of x

(1) $ax^3-3bx^2+3cx-d+hx^3+3kx^2+3lx+m$;

(2) $(b+c)x^2+bcx+abc+(c+a)x^2+cax+abc+(a+b)x^2+abx+abc$;

(3) $2ax^2-3bx^2+(2a+b)x-(a-b)+5bx^2-(2a-3b)x^2+(7a-5b)x+(5a+b)$;

(4) $a^2x^2+3abx+b^2+(ab-b^2)x^2-(5a^2-2b^2)x-(3ab-b^2)$;

(5) $(b-c)x^2+(ab-ca)x+(b^2-c^2)+(c-a)x^2+(bc-ab)x+(c^2-a^2)+(a-b)x^2+(ca-bc)x+(a^2-b^2)$.

8. Point out the arithmetical difficulty in saying

$$7-3+5-4=-3-4+7+5.$$

9. What is meant by saying that -2 is greater than -3? Is $(-2)^2$ greater than $(-3)^2$?

10. Explain what is meant when it is said that, in algebra, subtraction is addition.

E (CHAPTER V)

1. Find the results of the following multiplications:

(1) $(x-11)(x-13)$; (2) $(x^2-11)(x^2-13)$;

(3) $(y+8)(y+7)$; (4) $(ab+8)(ab-7)$;

(5) $(z-9)(z-17)$; (6) $(p-9q)(p-17q)$;

(7) $(3x-5)(7x-11)$; (8) $(3yz-5)(7yz-11)$;

(9) $(5z+9)(8z-15)$; (10) $(5\overline{x+y}+9)(8\overline{x+y}-15)$;

2. Find the following products arranging the results according to the powers of x:

(1) $(x+p)(x+q)$; (2) $(x-p)(x-q)$;
(3) $(x+p)(x-q)$; (4) $(x-p)(x+q)$;
(5) $(ax+b)(hx+k)$; (6) $(ax-b)(hx-k)$;
(7) $(x^2+px+q)(x+a)$; (8) $(x+a)(x^2+a^2)$;
(9) $(lx+m)(mx+l)$; (10) $(x-a)(x-5a)(x-9a)$.

3. Find the product

$$(2x-3)(4x-5)(3x-4)$$

varying in as many ways as possible the order in which the factors are taken in multiplication.

4. State of what dimension in x is each of the following:

(1) x^2-3x+5; (2) $3x^4-x+11$;
(3) $(x+1)(x+2)(x+3)$; (4) $(x^2+5)(x^2+7)(x^2+9)$;
(5) $(x+1)(x^2+1)(x^3+1)$; (6) $(x+1)^2(x+2)^3$.

5. On what grounds is it said that the product

$$(x+y)(x^2+y^2)(x^3+y^3)$$

is homogeneous and of six dimensions in x and y?

6. Shew that the product

$$(x-1)(x+1)(x-2)(x+2)$$

will not be changed if for x there be substituted $-x$.

7. In the product

$$(3x^2-5x+11)(3x-4)$$

find the coefficient of x, avoiding all unnecessary work.

8. From an examination of the factors state what can be said of the dimensions of the product

$$(x+y)(x^2+xy+y^2)(x^4+x^2y^2+y^4).$$

F (CHAPTER VI)

1. Find the quotient in the case of each of the following divisions:

(1) $(x^2-8x+15)\div(x-5)$; (2) $(x^2+8x+15)\div(x+5)$;

(3) $(6a^2-31a+35)\div(3a-5)$; (4) $(6p^2-31pq+35q^2)\div(3p-5q)$;

(5) $(35z^2-z-88)\div(7z+11)$; (6) $(65x^2-33xy-68y^2)\div(5x+4y)$;

(7) $(56+13z-30z^2)\div(8-5z)$;

(8) $299x^2+98xy-285y^2)\div(23x-19y)$;

(9) $(\frac{15}{28}a^2+\frac{251}{672}a-\frac{11}{32})\div(\frac{5}{7}a-\frac{3}{8})$;

(10) $(\frac{9}{20}x^2+\frac{49}{60}xy+\frac{8}{15}y^2)\div(\frac{3}{5}x+\frac{2}{3}y)$

2. Find the following quotients, testing in each case either by multiplication or by division back:

(1) $(x^3+10x^2+26x+15)\div(x+3)$;

(2) $(x^3-10x^2+26x-15)\div(x-3)$;

(3) $(6x^3+5x^2-6x+35)\div(3x+7)$;

(4) $(6x^3-5x^2y-6xy^2-35y^3)\div(3x-7y)$;

(5) $(21z^3-47z^2+57z-55)\div(3z^2-2z+5)$;

(6) $(20+l-38l^2+45l^3)\div(4-7l+5l^2)$;

(7) $(7-30pq-8p^2q^2+45p^3q^3)\div(1-3pq-5p^2q^2)$;

(8) $(33v^3+39v^2-77v-91)\div(11v+13)$;

(9) $(\frac{1}{4}+\frac{13}{40}x+\frac{11}{60}x^2+\frac{1}{15}x^3)\div(\frac{1}{4}+\frac{1}{5}x)$;

(10) $(\frac{5}{9}z^3-\frac{29}{24}z^2+\frac{127}{96}z-\frac{7}{10})\div(\frac{2}{3}z^2-\frac{3}{4}z+\frac{4}{5})$.

3. Shew that $x+4$ and $x+5$ are each factors of $x^3+3x^2-34x-120$, and find the remaining factor explaining why a remaining factor is known to exist.

4. Shew that

$$(2x^2-5x+7)=(x-3)(2x+1)+10.$$

Hence shew that the result of substituting 3 for x in $2x^2-5x+7$ is 10.

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5. Shew that

$$(2x^3-5x^2+7x-29)=(x-7)(2x^2+9x+70)+461.$$

Hence shew that the result of substituting 7 for x in $2x^3-5x^2+7x-29$ is 461.

6. Find the quotient and the remainder when

(1) $7x^2-13x+19$ is divided by $x-a$;

(2) $3x^2-7x-11$ is divided by $x-p$;

(3) $5x^2-12x-17$ is divided by $x+a$.

7. Find the quotient and the remainder when x^2+px+q is divided by $x-h$, and also when divided by $x+h$.

8. Perform the following divisions:

(1) $(x^2-lx+mx-lm)\div(x-l)$;

(2) $(x^2-(l+m)x+lm)\div(x-m)$;

(3) $acx^2-(bc-ad)xy-bdy^2\div(cx+dy)$;

(4) $\{al+(bl-a)x+(cl-b)x^2-cx^3\}\div(l-x)$;

(5) $(1-6ax+12a^2x^2-8a^3x^3)\div(1-2ax)$;

(6) $\{1-7(a+x)+12(a+x)^2\}\div\{1-3(a+x)\}$.

9. Employ division to express

(i) 45783 in the form $2.7^5+5.7^4+3.7^2+2.7+3$;

(ii) $4x^3-7x^2+11x-5$ in the form

$$4(x+2)^3-31(x+2)+87(x+2)--87.$$

10. Form the following products and test the accuracy of the work by division:

(1) $(x^2-3x+11)(5x^2+3x-8)$;

(2) $(7-5y+8y^2)(2+3y+5y^2)$;

(3) $(2p^2-3pq-q^2)(5p^2+7pq-8q^2)$;

(4) $(2x-5)(2x-9)(2x-13)$.

G (CHAPTER VII)

1. Write down, without actual multiplication, the expansions of the following:

$(3x+5y)^2$; $(2+3z)^2$; $(5a-3b)^2$; $(7-2p)^2$; $(2a+b+3c)^2$;
$(1+2x+3x^2)^2$; $(2a-3b+4c)^2$; $(\overline{a+b}+\overline{x+y})^2$;
$(bc+ca+ab)^2$; $(yz-zx+xy)^2$; $(a^2-b^2+c^2)^2$.

2. Without formal multiplication find the following products:

$(3x+4y)(3x-4y)$; $(5ab-3cd)(5ab+3cd)$;
$(5ax-7by)(5ax+7by)$; $(a+b+x)(a+b-x)$;
$(l-m+n)(l-m-n)$; $(l+m-n)(l-m+n)$;
$(a+b+c+d)(a+b-c-d)$; $(l-m+n-p)(l+m-n-p)$;
$(2x-3y+5z)(2x+3y-5z)$;
$(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

3. Write down the results of the following multiplications:

$(x+4)(x+7)$; $(x-5)(x+11)$;
$(2x-5)(2x-9)$; $(7x+3y)(7x+5y)$;
$(yz+11)(yz-10)$; $(\overline{a+b}+5)(\overline{a+b}+9)$;
$(2x+3y+5z)(2x+3y+7z)$; $(x+1)(x+4)(x+7)$;
$(x-5)(x-6)(x-7)$; $(z-1)(z+2)(z-3)$;
$(2x+3)(2x+5)(2x+7)$; $(1+2x)(1+3x)(1+5x)$;
$(x+y)(x+2y)(x+3y)$.

4. Write the expansions of the following:

$(u+v)^3$; $(p-q)^3$; $(3x+4)^3$; $(3x+4y)^3$;
$(2x-3y)^3$; $(1+2z)^3$; $(2p-1)^3$; $(xy+z^2)^3$;
$(2-3x)^3$; $(4z-5)^3$; $(x^2+y^2)^3$.

5. Shew that

$$\frac{m^2}{2}+\frac{n^2}{2}=\left(\frac{m-n}{2}\right)^2+\left(\frac{m+n}{2}\right)$$

and shew that this is equivalent to the relation

$$2m^2+2n^2=(m-n)^2+(m+n)^2$$

Taking m and n as measuring the lengths of two lines, illustrate this relation geometrically.

6. Prove that

$$(bn-cm)^2+(cl-an)^2+(am-bl)^2+(al+bm+cn)^2=(a^2+b^2+c^2)(l^2+m^2+n^2).$$

7. Find in simple form

$$(x^2+xy+y^2)^2-(x^2-xy+y^2)^2.$$

H (Chapter VIII)

1. Express as a product of factors each of the following:

$9x^2+42xy+49y^2$ $x^2-16x+63;$ $1-15z+56z^2;$
$49z^2-81a^2;$ $(a+b)^2-(a-b)^2;$ $4x^2-36x+77;$
$3x^2+8ax+5a^2;$ $6x^2-23xy+21y^2;$ $ap-bq+bp-aq;$
$a^3b-b^2p+a^2p-ab^3;$ $x^2+px+x+p;$ $2p^2+qr+2pq+pr.$

2. Resolve into factors:

$x^3+a^3;$ $x^6+a^6;$ $8x^3+125a^3;$ $1+27z^3;$
$x^3-a^3;$ $x^6-a^6;$ $8x^3-125x^3;$ $1-27z^3;$
$(a+b)^3+(a-b)^3;$ $(a+b)^3-(a-b)^3;$ $a^3+2a^2b+2ab^2+b^3$

3. Resolve into factors:

$1+4z^4;$ $1+z^2+z^4;$ $9a^4+5a^2+1;$
$4x^4+3x^2y^2+9y^4;$ $4x^4-37x^2y^2+9y^4;$ $(x+y)^4+(x^2-y^2)^2+(x-y)^4.$

4. Find the H.C.F. and the L.C.M. of

(1) $x^4-13x^2+36,\ x^3+2x^2-5x-6,\ x^3-6x^2+11x-6.$

(2) $a-b,\ a^2-b^2,\ a^3-b^3,\ a^4-b^4.$

5. The question of resolving

$$x^3-2x^2-8x-35$$

into factors arises. Name all the linear factors that should be tested for, and find the factors of the expression.

6. Express each of the following as the difference of two squares, and thus arrive at the factors:

(i) $x^2-24x+135;$ (ii) $x^2+6x-315;$
(iii) $a^2-5ab-84b^2;$ (iv) $9a^2+39ab+40b^2.$

7. Shew that after multiplication by a numerical factor each of the following may be expressed as the difference of two squares and thus arrive at the factors:

(i) $12a^2+41a+35$; (ii) $18x^2-41xy+21y^2$; (iii) $6p^2+pq-15q^2$.

8. Shew that the following may be resolved into linear factors if such numbers as $\sqrt{2}$, $\sqrt{3}$, etc., are employed as terms in the factors:

(i) $x^2-8x+11$; (ii) x^2-6x+7; (iii) $x^2+10x+23$.

9. Express

$$a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2.$$

as the difference of two squares and resolve the expression into factors.

10. Resolve into factors

(1) $x^4-(m+n)x^3+(m+n)mnx-m^2n^2$;

(2) $(1+y)^2-2x^2(1+y^2)+x^4(1-y)^2$;

(3) $(a^2-4b^2)x^2+2(a^3+2b^3)x+(a^4-b^4)$.

11. Examine into the reason for the following method of resolving into factors an expression as

$$20x^2+71xy+63y^2.$$

Seek two numbers whose product is $+20\times+63$ *and sum is* $+71$. *These numbers being found to be* $+36$ *and* $+35$, *write the expression thus:*

$$20x^2+36xy+35xy+63y^2$$

This is seen to be

$$4x(5x+9y)+7y(5x+9y),$$

or

$$(5x+9y)(4x+7y).$$

In this way resolve

$20x^2-71xy+63y^2$; $20x^2-xy+63y^2$;

$20+z-63z^2$; $6a^2-29ab+35b^2$;

$24y^2+26yz-63z^2$; $36x^2+60ax+25a^2$.

I (Chapter IX)

1. If

$$mx - pq = rs$$

outline the argument by which it is shewn that

$$x = \frac{rs + pq}{m}.$$

2. Give the complete argument employed in solving the equation.

$$3(5x - 3) = 4(3x + 5).$$

3. Contrast the two statements:

(i) $(a + b)(a^2 + b^2) = a^3 + a^2b + ab^2 + b^3$;

(ii) $(x - 3)(x + 7) = (x - 1)(x + 1)$.

4. Find the number which when subtracted from each term of the fraction $\frac{a}{b}$ will yield a fraction equal to the square of the given fraction.

Illustrate the general result by applying it to the fractions

$$\tfrac{2}{3},\ \tfrac{5}{8},\ \tfrac{4}{7},$$

verifying in each case.

5. Solve the following equations:

(1) $3(x - 7) + 5(x + 3) = 6(x + 2) + 2$.

(2) $\dfrac{x + 1}{4} + \dfrac{x + 4}{5} = \dfrac{3x + 2}{7} + \dfrac{x - 4}{8}$.

(3) $a(x - l) + b(x - m) + c(x - n) = 0$.

(4) $(a - b)(x + c) = (a + b)(x - c)$.

(5) $\frac{2}{3}(x + \frac{1}{2}) + \frac{5}{6}(x - \frac{3}{4}) = \frac{3}{8}(5x + 9)$.

6. Shew that $3x + 7$ and $2x + 13$ are ordinarily unequal, but that there is a value of x that will make them equal.

7. Find for what value of x the value of $5x + 8$ exceeds that of $4x + 11$ by 29.

8. Divide 37 into three parts, such that the second may exceed the first by 3 and the third may be 5 less than the sum of the first and the second.

J (Chapter X)

1. Find the results of substituting in the expression $3x-5$ the following values of x:

$$-4,\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3,\ +4.$$

2. If $f(x)$ denotes x^2-3x+7, find the results of dividing $f(x)$, by $x-1$, $x+1$, $x-3$, $x+3$, $x-9$, $x+9$, and shew that these results are

$$f(1),\ f(-1),\ f(3),\ f(-3),\ f(a),\ f(-a.)$$

3. If $f(y)$ denotes y^3+y^2+y+1, shew that $y^3 f\left(\frac{1}{y}\right)=f(y)$.

4. If $f(z)$ denotes z^3-z^2-z-1, shew that $-z^3 f\left(\frac{1}{z}\right)=f(z)$ and that $f(z)+f(-z)=-2(z^2+1)$.

5. If $f(x)=2x+3$, find the growth in $f(x)$ for each successive advance of 1 in the value of x, starting from $x=0$.

6. If $f(a)=a+1$ and $F(a)=a^2+a+1$, shew that

$$F(a)-f(a)+1=f(a^2).$$

7. If $f(x)=ax+b$, find the growth in $f(x)$ for an advance of h in the value of x from the value k of x.

8. If $F(x)=x^2-8x+15$, shew that $F(3)=0$ and that $F(5)=0$.

9. Write down any polynomial of say the third degree; denoting it by $f(x)$ write down $f(a)$ and then shew that $f(x)-fa$ is divisible by $x-a$.

10. If $f(x)=x+1$ and $F(x)=x^2+x+1$, shew that

$$f(x)+F(x)=F(\overline{x+1}^2).$$

11. If $f(x)=x^2-4x+7$, shew that

$$f(2-h)=f(2+h)$$

whatever be h.

12. If $F(x)=x^2-6x+7$, shew that

$$F(x)=F(6-x).$$

K (Chapter XI)

1. Construct the graphs of the following functions:

$$3x-5,\ 2x-3,\ \tfrac{2}{3}x+\tfrac{3}{4},\ 3x-4.$$

2. Represent graphically the relation between the variables x and y given by each of the following equations:

$$y=x+5;\quad 2y=3x-7;\quad 3y=4x-6$$

3. Represent graphically, on a single sheet and referred to the same axes, the three equations

$$5y=4x-5,\ 5y=4x-8,\ 5y=4x-11.$$

4. Bring the equation

$$2x-3y-5=5x-4y+2$$

to a form exhibiting y as a function of x, and represent this function graphically.

5. From the graph of

$$3x+4y=12$$

estimate the value of y for $x=1{\cdot}2,\ 1{\cdot}3,\ 1{\cdot}4,\ 1{\cdot}5,\ 1{\cdot}6$ and the value of x for $y=1{\cdot}6,\ 1{\cdot}8,\ 2{\cdot}1,\ 2{\cdot}2$, and check results by reference to the equation.

6. From the graphs, referred to the same axes, of

$$y=\tfrac{2}{3}x-\tfrac{1}{3}$$
$$y=\tfrac{3}{4}x-\tfrac{3}{4}$$

find the value of x that makes the two functions equal.

7. The length of one degree on a parallel of latitude is given for certain latitudes in the table

Lat.	0°	[illegible]	20°	30°	40°	50°	60°	70°	80°	90°
Length in Miles	69·2	[illegible]	65·0	60·0	53·1	44·6	34·7	23·7	12·1	0·0

Plot these values, draw a curve freely through them, and employ it to estimate the length of a degree in latitudes 15°, 23°, 37°, 44°, 56°, 73°.

L (CHAPTER XII)

1. Find the product of

$$(a-b)^3 \text{ and } (a+b)^3,$$

employing any artifice to reduce the amount of work.

2. Find the following products:

(1) $(x^2+y^2+z^2-yz+zx+xy)(x-y-z)$;

(2) $(2x+y-3z)(3x-y+2z)(x-2y+3z)$;

(3) $(2+x-3x^2)(3-x+2x^2)(1-2x+3x^2)$;

(4) $(1-x+x^2-x^3+x^4)(1+x+x^2+x^3+x^4)$.

3. Find by a contracted method the following products:

(1) $(3x^3-5x^2+7x-4)(x-3)$;

(2) $(4z^3-7z^2+8z-11)(2z-5)$;

(3) $(2-3y+5y^2-7y^3)(1-4y+6y^2)$.

4. Find the following quotients:

(1) $(x^6-2x^3+1)\div(x^2-2x+1)$;

(2) $(x^5-2x^4-4x^3+13x^2-11x-7)\div(x^2-3x+7)$;

(3) $\{(x+y)^2-3(x+y)z+2z^2\}\div(z-x-y)$;

(4) $(a+c)^3+(b+d)^3+3(a+c)^2(b+d)+3(a+c)(b+d)^2\div(a+b+c+d)$;

(5) $(1-11y^2+27y^3-62y^4+63y^5)\div(1+3y-7y^2)$.

5. Divide

$$2+3x \text{ by } 1+x+x^2,$$

carrying the quotient to the fifth power of x.

6. Divide, by Horner's method,

$$x^8-7x^7+16x^6-3x^5-9x^4+13x^3+4x^2-7x-1800 \text{ by } x-3,$$

and shew that the remainder is the equivalent of the dividend for the value 3 of x.

7. Divide

$$x^6-3x^5+2x^4-x^3-3x^2+2x-2 \text{ by } x^2-x+1,$$

and shew that the remainder is the equivalent of the dividend if $x^2=x-1$.

8. Find the value for $x=13$ of

$$2x^6-17x^5-127x^4+1298x^2+30x+54.$$

9. Divide, by Horner's method,

(1) $x^8+5x^6+11x^4+19x^2-36$ by $x^4-2x^3+2x^2+2x-3$;

(2) $x^6-x^2+10x-10$ by x^3-3x^2+4x-2;

(3) $6x^4-23x^3+22x-16$ by $2x^2-5x-8$.

10. Find the H.C.F. of

(1) $1+5z+z^2-13z^3+6z^4$ and $1+6z+10z^2-2z^3-15z^4$;

(2) $3x^3+14x^2+22x+21$ and $6x^4+10x^3+2x^2-20x-28$;

(3) $4x^4+2x^3-18x^2+3x-5$ and $6x^5-4x^4-11x^3-3x^2-3x-1$;

(4) $2a^5-4a^4+8a^3-12a^2+6a$ and $3a^5-3a^4-6a^3+9a^2-3a$.

11. Resolve into factors:

(1) $\{(1-a)x+by\}^3+\{ax+(1-b)y\}^3$;

(2) $(x+y)(1+x)(1+y)+xy$;

(3) $(x+yz)(y+zx)(z+xy)+(x^2-1)(y^2-1)(z^2-1)$;

(4) $(x^2+x)^2-14(x^2+x)+24$;

(5) $x^3y+x^2yz+x^3z-xy^3-zy^3-zxy^2$.

12. If $2s=a+b+c$, where a, b, c measure the lengths of the sides of a triangle, it is known that the measure of the area. A, is given by the formula

$$A=\sqrt{s(s-a)(s-b)(s-c)}.$$

Shew that if $a^2=b^2+c^2$, the formula becomes

$$A=\tfrac{1}{2}bc,$$

and point out the geometrical significance of this.

13. Find the H.C.F. of

(1) $20x^4+x^2-1$, $25x^4+5x^3-x-1$, $25x^4-10x^2+1$;

(2) x^3-x^2-2x+2 and $x^4-3x^3+2x^2+x-1$;

(3) x^3-3x^2+3x-2 and x^3+x^2-5x-2;

(4) $x^3-7x^2+14x-8$ and $x^3-6x^2+11x-6$.

14. If $x^3+y^3+z^3+mxyz$ is divisible by $x+y+z$, find the value that m must have.

15. Find the products in ascending powers of x
$(1-x+x^2)^2(1+x+x^2)^2$; $(1-x+x^2)^3(1+x+x^2)^3$.

16. Resolve into factors
$$8b^2c^2+2c^2a^2+8a^2b^2-a^4-16b^4-c^4.$$

17. Find to the sixth power of x the product of
$$\frac{x}{1}-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\dots\dots$$
and
$$1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6}+\dots\dots$$
and shew that it corresponds to one-half of the result of substituting $2x$ in the earlier expression.

18. Find for what value of m the expression $x^2-mx-108$ is divisible by $x+6$.

19. Resolve into factors

(1) $15x^2-32y^2-30z^2+68yz-7zx+28xy$;
(2) $21+53p-89q+30p^2-103pq+88q^2$;
(3) $28x^2+43xy-45y^2-59x+137y-104$.

20. Divide
$(x+y+z)^3-x^3-y^3-z^3$ by $x(y^2+z^2)+y(z^2+x^2)+z(x^2+y^2)+2xyz$.

21. Resolve into factors

(1) $a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc$.
(2) $(a+b)^3+(b+c)^3+(c+a)^3+3(c+2a+b)(a+2b+c)(b+2c+a)$.

22. Find the quotient of
$$a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$$
by
$$a^2(b-c)+b^2(c-a)+c^2(a-b).$$

23. Shew that in the product
$$(3x^4-5x^3-2x^2+7x+1)(4x^3+6x^2-7x+5)$$
the sum of the coefficients of the odd powers of x must be equal to the sum of the coefficients of the even powers, the term without x being regarded as belonging to the even powers.

(MISCELLANEOUS).

1. Add to each of the following a term to make the expression a prefect square:

$x^2 - 2ax$; $x^2 + a^2$; $x^2 - 6x$; $x^2 + 4x$;
$x^2 + 5x$; $4y^2 + 6y$; $9z^2 - 12z$; $1 - 8z$.

2. Find the value of

$$(x + y - z - u)^2 - (u + z - x - y)^2.$$

3. In the product

$$(1 + 3x + 5x^2 + 7x^3 + 9x^4)(1 - 5x + 9x^2 - 13x^3 + 17x^4)$$

find the coefficient of x^5.

4. Divide 1 by $1 - 2x + x^2$ out to the term of the quotient involving x^4.

5. Write down the results of the following multiplications:

(1) $(x-5)(x-9)$; (2) $(x+7)(x+9)(x+11)$;
(3) $(3x-7)(3x-10)$; (4) $(2x-3)(2x-5)(2x-7)$.

6. Verify that

$$2x^2 - 5x + 9 \text{ and } x^2 + 3x - 6$$

are equal if $x = 3$ or if $x = 5$ but not if $x = 4$.

7. Shew that $a^4 + b^4$ is divisible by neither $a - b$ nor $a + b$, but that, admitting $\sqrt{2}$, where $(\sqrt{2})^2 = 2$, as a number that may be employed as a coefficient,

$$a^4 + b^4 = (a^2 + \sqrt{2}\,.\,ab + b^2)(a^2 - \sqrt{2}\,.\,ab + b^2).$$

8. Find the product $(b+c)(c+a)(a+b)$ and group the terms of the result so as to shew that $b + c$ is a factor of it, and thus arrive at all the factors.

9. Find the product of

$$x^2 - (3p - 4q)x - 12pq \text{ and } x^2 - (3q - 4p)x - 12pq.$$

10. Find the quotient of $a^2x^8 + (2ac - b^2)x^4 + c^2$ by $ax^4 + bx^2 + c$.

11. Divide the product of $x^2 - 7x + 12$ and $x^2 - 11x + 30$ by $x^2 - 9x + 18$.

12. It is known that

$$x^4 - x^3 - 4x^2 + mx + n$$

is divisible by

$$x^2 + 2x - 3.$$

Find the values of m and n.

13. If A, B, C, D are four points in a straight line, shew that

$$BC.AD + CA.BD + AB.CD = 0.$$

where BC, . . . denote the measures of the lengths of the segments of the line, and measurements in one direction are to be taken as positive, and, therefore, measurements in the opposite direction as negative.

14. If, as in the preceding, A, B, C, D are four points on a straight line, shew that

$$BC.AD^2 + CA.BD^2 + AB.CD^2 = -BC.CA.AB.$$

15. A, who can travel at the rate of m miles an hour, sets out in pursuit of B who can travel at the rate of n miles an hour, and has had h hours start. In what time will A overtake B?

16. Is it possible for ax and bx to be equal if a and b are not equal?

17. Is it possible for $a + x$ and $b + x$ to be equal if a and b are unequal?

18. In the identity

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$$

substitute the values

(i) $x, \ -y, \ -z$ for a,b,c;
(ii) $-x, \ +y, \ +z$ for a,b,c.

19. If $x = 0{\cdot}1$ find the error in taking $1 + 3x$ for $(1 + x)^3$.

20. Shew that

$$(y-z)^3 + (z-x)^3 + (x-y)^3 = 2(x-y)(x-z) + 2(y-z)(y-x) + 2(z-x)(z-y).$$

21. Represent graphically the equation

$$3y = 4 - 2x.$$

22. Find the remainder when

(i) $x^{100} + m$ is divided by $x - 1$;
(ii) $x^{2n} - 1$ is divided by $x - 1$;
(iii) $x^{2n} - 1$ is divided by $x + 1$;
(iv) $x^{3n} - 1$ is divided by $x^2 - 1$.

ANSWERS

EXERCISES I. Pages 6—7.

1. (1) $\$nr$; (2) lb cents; (3) amn cents; (4) $\$h^2k^2$. **2.** (1) $100x$; (2) $100h+k$; (3) $36x+12y+z$; (4) $100l+10m+n$. **3.** $2\frac{5}{8}$, $1\frac{1}{9}$, $\frac{5}{42}$. **4.** (i) $8\frac{2}{3}$; (ii) $14a$; (iii) $\frac{a+b+c}{3}$; (iv.) $\frac{2a+3b+4c}{3}$. **5.** $a+b+c$; $\frac{a+b+c}{2}$. **6.** $\frac{ma+nb}{m+n}$ cents. **7.** $12ab$. **8.** $mnab$; mn times as great. **9.** 6 ways; 6 ways. **10.** (i) $m+n$; (ii) $m-n$; (iii) $\frac{a}{m}+\frac{b}{n}$; or similar expressions in any letters. **11.** $2n$; $2n+1$; $n=43$; $n=26$. **12.** $\frac{1}{m}$th; $\frac{1}{n}$th; $(\frac{1}{m}+\frac{1}{n})$th. **13.** $10m+n$. **14.** (i) $5a$; (2\ $13x+5y$; (3) a^3b^3; (4) $30a^3b^3$. **15.** 0, 2, 4, 6, 8, 10; 7, 10, 13, 16, 19, 22; 0, 1, 4, 9, 16, 25; 5, 10, 17, 26. 37, 50. **16.** $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{9}{10}$, $\frac{11}{12}$; 1, $\frac{5}{3}$, $\frac{17}{5}$, $\frac{37}{7}$, $\frac{65}{9}$, $\frac{101}{11}$; 1, $\frac{7}{3}$, $\frac{21}{13}$, $\frac{43}{31}$, $\frac{73}{57}$, $\frac{111}{91}$.

EXERCISES II. Page 9.

1. x^7, y^{16}, l^{12}. **2.** x^3, h^5, ab^2. **3.** a^{21}, a^{12}, a^6b^{11}, x^2y^5, $4m^3n^3$, $4x^3y^5$. **4.** a^3b^3, 1, $\frac{1}{y^2}$.

EXERCISES III. Pages 10—13.

1. $(40-x)$ years; $(40+m)$ years. **3.** $(m-n)$ years; $(m+l)$ years; in m years. **5.** $\frac{a}{l}$. **7.** (1) $(m+n)(m-n)$; (2) $(a+b)-(p-q)$. **8.** (1) $n(n+1)(n+2)$, or $(n-1)(n)(n+1)$; (2) $2n(2n+2)(2n+4)$;

(3) $(2n+1)(2n+3)(2n+5.)$ **9.** (1) 5 mi. east of P; (2) 5 mi. west of P; (3) $a-b$) mi. east of P. **10.** A, $m-x+y$; B, $n-y+z$. **11.** The sum of any two consecutive odd integers is twice the intermediate even integer. **12.** $100l+10m+n$. **16.** $10m+n$; $10n+m$; the sum $=11(m+n)$ and is divisible by 11 whatever be the digits m, n. **17.** (i) 24 ways; (ii) 24 ways; (iii) 4 ways; (iv) 4 ways.

EXERCISES IV. Page 15.

2. $a-b-c-d$; $a+b-c-d$; $x+y-z-w$. **3.** $(a+b)+(c+d)$, or otherwise; $(x-z)+(y-u)$, or otherwise; $(z-y)+x$, or otherwise; $(h-k)+(l-m)$, or otherwise. **4.** $a-(b+c)$; $(x+y)-z+u$; $(x-y)-z$; $(h-m)-(k+l)$; each may be given otherwise. **5.** $ax+bx-ay-by$. **6.** $ax+bx+cx$; $ax+bx+cx+ay+by+cy$; $ax+bx+cx+ay+by+cy+az+bz+cz$. **7.** $x(m+n)$; $x(m-n)$; $x(a+b+c)$. **8.** $a+(b+c)$ or $(a+b)+c$; three ways; four ways; **9.** $(a+b)^2=a^2+2ab+b^2$.

EXERCISES V. Page 18.

1. $75, $160, $330. **2.** $90, $30. **3.** 45 yr., 15 yr. **4.** 24, 13. **5.** 36. **6.** 150 lb., 300 lb. **7.** 120 by 40 rd., 80 by 80 rd. **8.** 4 and 5 per cent. **9.** (1) is true whatever be x; (2) is true only if $x=12$.

EXERCISES VI. Pages 25—26.

1. (13 − 15) mi. or = 2 mi.; 28 mi. **2.** (17 − 23 + 9 − 5) mi. north, or −2 mi. north, or 2 mi. south; 54 mi. **4.** At the end 480 ac. more than at the beginning. **5.** −1120 ac.; 1120 ac. **6.** −10. **7.** $-5°$. **8.** $-5°$. **9.** −30 dol.; −20 per cent. **10.** −25 %. **11.** (i) −2; (ii) −3; (iii) 0: (iv) $-a$; (v.) $a-5b$. **13.** $(a-b)$ mi.; $(a+b)$ mi.; **14.** $(a-b+c)$ degrees.

EXERCISES VII. Page 27.

1. −300 dol. **2.** (i) −5; (ii) $3a-2b$; (iii) $x+y-z$. **4.** (i) 5; (ii) −1; (iii) −6.

EXERCISES VIII. Page 28.

1. (1) 11; (2) 5; (3) 1; (4) -3; (5) 10. **4.** $(+75)+(-30)+(+40)$; $75-30+40$.

EXERCISES IX. Page 31.

1. $+21$; -65; -84; $+99$. **2.** $+hk$; $-6yz$; $+15mn$; $40pz$. **3.** $+25$, -27, $+256$, -32, $+729$. **4.** $+a^2$, $-p^7$, $-x^{13}$, $+y^{20}$ $-z^{99}$. **5.** $+30$; $-xyz$; $+30xyz$. **6.** x^{2n}, $-x^{2n+1}$. **7.** $+x^3y^4z^5$; $-288h^3k^2l$.

EXERCISES X. Page 32.

1. $+5$; $+3$; -7; -5. **2.** $+\frac{2}{3}$; $+\frac{3}{5}$; $-\frac{3}{4}$; $-\frac{2}{3}$. **3.** $-\frac{3}{2}$; $\frac{5}{11}$. **4.** $-\frac{bc}{a^2}$; $+\frac{ap^2x^3}{bq^2y^3}$.

EXERCISES XI. Pages 34—35.

1. $(a+b)-(c+d)$; -3. **2.** $+18$. **3.** -9, $+a^3$, $-h$, $-\frac{b}{a}$. **11.** $(am+bn-an-bm)$ dol. **12.** $a+b+c$. **13.** $\frac{ab}{c}$ hours. **14.** $(b-a)$ dol. **18.** $a+b+c=(a+b)+c$.

EXERCISES XII. Page 36.

1. $10a$. **2.** $11x$. **3.** $8l$. **4.** $-6p$. **5.** $4hk$. **6.** $-yz$. **7.** $-8abc$. **8.** $-x^2$. **9.** $-8m^2n^2$. **10.** $2(a+b)$. **11.** $-5xyz$. **12.** $3(x+y)^2$.

EXERCISES XIII. Page 37.

1. $2a-7b-4c$. **2.** $-10x+16y-10z$. **3.** $-28p-15q+16r$. **4.** $-9l-25m-13n$. **5.** $-3x^2+3y^2$. **6.** $4y^2$. **7.** $5mn-11nl+5lm$. **8.** $-x^3-6y^3-2z^3$. **9.** $2x^2-x+15$. **10.** $12x^3+2x^2-7x-2$. **11.** $\frac{3}{4}x-1$. **12.** $\frac{1}{2}h^2-\frac{11}{12}hk+\frac{1}{12}k^2$.

EXERCISES XIV. Page 38.

1. $4x-29y$. **2.** $2a+2b+2c$. **3.** 0. **4.** 0. **5.** 0. **6.** $-3a^2-3b^2-2c^2$. **7.** $7yz-2zx-8xy$. **8.** $-2u^2-4uv+18v^2$. **9.** $-x^2-7x+2$. **10.** $3x^2+3y^2-6x+3y$. **11.** $14x^3-3x^2-9x+1$. **12.** $\frac{5}{12}x^2+\frac{1}{2}x-\frac{5}{12}$.

EXERCISES XV. Page 40.

1. (1) $(a+b-c+d)x$; (2) $(h-k+l-m)x$; (3) $(a^2-b^2+c^2-d^2)x$; (4) $(3p-4q+5r-6s)x$. (5) $(4h^2-5k^2+9l^2-7m^2)x$. **2.** (1) $(a-c+e)x+(b-d-f)y$; (2) $(mn+nl+lm)x-(lp+mp+np)y$; (3) $(a^2+b^2+c^2)x+(kl+lh+hk)y$; (4) $(p+q+r)x+(p+q+r)y$; (5) $(l^2+m^2+n^2)x-(a^2+b^2+c^2)y$. **3.** (1) $(b+c)x(c+a)y+(a+b)z$; (2) $(b-c)x-(c-a)y+(a-b)z$; (3) $(a+b+c)x+(a+b+c)y+(a+b+c)z$; (4) $(lm-nl)x-(mn-lm)y+(nl-mn)z$; (5) $(kl-k^2-l^2)x+(lh-l^2-h^2)y+(hk-h^2-k^2)z$.

EXERCISES XVI. Pages 41—42.

1. $x-8y+9z$. **2.** $2a+35b-30c$. **3.** $4yz+10zx-xy$. **4.** $x^2-16x+14$. **5.** 0. **6.** $4ab$. **7.** $-7l^2+8m^2-10n^2-4mn-16nl+lm$. **8.** $11a+41b-3c$. **9.** $(a-p)x+(b-q)y$. **10.** (1) $3a-3b$; (2) $4p+20q-6r$; (3) $10a^2+4ab-15b^2$; (4) x^3-4x^2-5x+8. **11.** $(a-h)x^2+2(b-k)x+(c-l)$. **12.** $-\frac{1}{12}x^3-\frac{5}{12}x^2+\frac{5}{3}x+\frac{5}{4}$. **13.** $4\frac{a}{b}-15\frac{b}{c}-\frac{c}{a}$. **14.** $a^2+2ab+b^2$.

EXERCISES XVII. Page 44.

1. (1) $7p-15q$; (2) $13x-6y-5z$; (3) $-20x+17y$; (4) $-4a+7b$; (5) $-4ab$. **3.** (1) $-2x-8y+11z$; (2) $-12x+6y+17z$; (3) $-3x+3y+8z$; (4) $4x-40y+21z$; (5) $32x+14y-6z$. **4.** $-22x^2+4x+2$. **5.** $x^3+7x^2y+7xy^2+y^3$. **6.** $-2c-2e$. **7.** $-a^2-b^2-c^2$.

EXERCISES XVIII. Pages 45—46.

1. $(a+b+c)x-(p+q+r)y$. **2.** $x^2+y^2+z^2-(yz+zx+xy)$. **3.** $(a-h+p)x^3-(b+k+q)x^2-(c+l-r)x+(d-m-s)$. **4.** $ax^2+2(hy+g)x+(by^2+2fy+c)$. **5.** (1) $\{(a+b)+c\}\{(a+b)-c\}$; (2) $(a+\overline{b-c})(a-\overline{b-c})$; (3) $\{3y-(4z-2x)\}\{3y+(4z-2x)\}$; (4) $(\overline{a+b}+\overline{c+d})(\overline{a+b}-\overline{c+d})$; (5) $\{(c-b)+(a-d)\}\{(c-b)-(a-d)\}$. **6.** $(l+m+n)x^2-2(l+m+n)x+(mn+nl+lm)$.

EXERCISES XIX. Pages 46–47.

A 2. 79, by 16. **4.** $2n$: $2n-2$, $2n-4$; $2n+2$, $2n+4$; $10n$. **5.** $3a^2+7ab+35b^2$.

B 2. 36 and 12. **3.** $\left(\frac{a}{m}+\frac{b}{n}\right)$ of work. **4.** $2n+1$: $2n-1$, $2n-3$; $2n+3$, $2n+5$: $10n+5$. **5.** $5x^2-33xy+21y^2$.

C 1. $A=lb$. **2.** 13 yr. **4.** 6. **5.** $168x+24y$.

EXERCISES XX. Page 48.

1. $6xy$. **2.** $-15xy$. **3.** $-77x^2y^2$. **4.** $60lmn$. **5.** $-42a^2b^2c^2$. **6.** xy^2z. **7.** $a^3b^3c^3$. **8.** $-x^9$. **9.** $-lmna^6$. **10.** $-a^6b^6$. **11.** $36a^2b^4$. **12.** $-a^{15}$. **13.** $2\frac{1}{4}x^2y^2z^2$. **14.** $-x^7y^6z^5$. **15.** $a^7b^7c^7$. **16.** $-\frac{1}{3}x^3y^3$.

EXERCISES XXI. Page 49.

1. One, two, three; two, four, six; four, nine, fifteen; six, twelve, eighteen. **2.** Eight; six; twelve; five.

EXERCISES XXII. Page 50.

1. $px+py-pz$. **2.** $2lx-3ly+4lz$. **3.** $x^2yz+xy^2z+xyz^2$. **4.** $ayz-azx+axy$. **5.** $6l^2x+9m^2x-27n^2x$. **6.** $9h^3k+27hk^3$. **7.** $-7l^2m^2n^2+7l^3mn^2$. **8.** $7a^2b^2c^2-21a^3bc^2+56a^3b^2c$. **9.** $a^2pqrx+b^2pqry+c^2pqrz$. **10.** $5l^3m^3x-5l^2m^4y$.

EXERCISES XXIII. Pages 51–52.

1. $ax-bx-ay+by$. **2.** $2px+3py-2qx-3qy$. **3.** $15lv-21mv-40lw+56mw$. **4.** $55hx+33kx-25hy-15ky$. **5.** $24al-20bl-42am+35bm$. **6.** $h^2l^2+k^2l^2+h^2m^2+k^2m^2$. **7.** $abpq-cdpq+abrs-cdrs$. **8.** $10p^2x^2-15q^2x^2-8p^2y^2+12q^2y^2$. **9.** $6ablm-10abnp-9cdlm+15cdnp$. **10.** $3ahlx-3bhmx-8akly+8bkmy$. **11.** $x^2+8x+15$. **12.** $x^2-8x+15$. **13.** $x^2+2x-15$. **14.** $x^2-2x-15$. **15.** $6y^2+y-15$. **16.** $21z^2-101z+88$. **17.** $35p^2+23p-72$. **18.** $27a^2+$

$102a + 91$. **19.** $12h^4 - 43h^2 + 35$. **20.** $35l^4 - 103l^2 + 72$. **21.** $a^2 + 2ab + b^2$. **22.** $4x^2 + 12xy + 9y^2$. **23.** $15x^2 + 37xy + 18y^2$. **24.** $96p^2 + 116pq - 65q^2$. **25.** $2x^3 - 7x^2 - 15x$. **26.** $36a^4 - 95a^2b^2 + 56b^4$. **27.** $24l^2m^2 - 23lm - 77$. **28.** $18x^5 - 59x^3 + 35x$. **29.** $77a^2r^2 - 326abrs + 395b^2s^2$ **30.** $56u^2v^2 - 183uv + 135$.

EXERCISES XXIV. Pages 52—53.

1. $1 + x - x^2 + 2x^3$. **2.** $1 + x^3$. **3.** $1 - x^3$. **4.** $6z^3 - 19z^2 + 19z - 6$. **5.** $20a^3 + 13a^2 + 13a + 35$. **6.** $5b^3 - 19b^2 - 28b + 32$. **7.** $x^3 - y^3$. **8.** $x^3 + y^3$. **9.** $21p^3 - 42p^2q + 44pq^2 - 32q^3$. **10.** $15r^4 - 34r^3 + 50r^2 - 41r + 12$. **11.** $1 + x^2 + x^4$. **12.** $3x^4 - 7x^3 + 20x^2 - x + 35$. **13.** $a^4 + a^2x^2 + x^4$. **14.** $3l^4 - 7l^3m + 20l^2m^2 - lm^3 + 35m^4$. **15.** $\frac{1}{8}y^4 - \frac{43}{72}y^3 - \frac{1}{12}y^2 + \frac{5}{144}y - \frac{1}{6}$. **16.** $10 - r^2 + 30r^4 + 22r^6 - 7r^8$. **17.** $x^3 + 6x^2 + 11x + 6$. **18.** $x^3 + 6ax^2 + 11a^2x + 6a^3$. **19.** $a^3 + 3a^2b + 3ab^2 + b^3$. **20.** $a^3 - 3a^2b + 3ab^2 - b^3$. **21.** $8x^3 - 30x^2y - 53xy^2 + 15y^3$. **22.** $x^4 + 10x^3 + 35x^2 + 50x + 24$. **23.** $1 + x^4 + x^8$. **24.** $x^8 + x^4y^4 + y^8$. **25.** $x^2 + (h + k)x + hk$. **26.** $x^2 - (a + b)x + ab$. **27.** $acx^2 + (ad + bc)x + bd$. **28.** $ax^3 + (b - ah)x^2 + (a - bh)x - ch$. **29.** $apx^3 - (2bp + a)x^2 + (4cp + 2bq)x - qc$. **30.** $x^3 + (a + b + c)x^2 + (bc + ca + ab)x + abc$.

EXERCISES XXV. Pages 54—55.

1. (1) Two; (2) three; (3) six. **2.** (1) Three; (2) five; (3) four. **3.** (1) 13; (2) -33; (3) 36. **6.** It would. **7.** $x^4 - y^4$; four.

EXERCISES XXVI. Pages 55—56.

A 1. $-22, -17, -12, -7, -2, +3, +8$. **2.** 3·1416, 4·7124, 6·2832, 7·8540, 9·4248. **3.** $x^3 + 15x^2y + 71xy^2 + 105x^3$. **5.** The thirty-fourth.

B 2. $s = tv$. **3.** $x^4 - y^4$; $(1 + y + y^2 + y^3)(1 - y) = 1 - y^4$; $(-1 + y - y^2 + y^3)(-1 - y) = 1 - y^4$. **5.** 24 and 17.

EXERCISES XXVII. Page 57.

1. ab. **2.** $-ab$. **3.** $-x^2z$. **4.** pq. **5.** $-3lmn$. **6.** $\frac{3}{2}f^2h$. **7.** $-\frac{8}{9}ac$. **8.** $\frac{7}{9}uvw^3$ **9.** $3by$. **10.** $2(a + b)$. **11.** $-3ab(x + y)^3$. **12.** $-3(a + b)(x + y)^2$.

EXERCISES XXVIII. Page 58.

1. $(m-n)$. **2.** $-m+n$ or $n-m$. **3.** $ab+b^2$. **4.** a^2+ab. **5.** $a+b$. **6.** $lm-np$. **7.** $3x+3y$. **8.** $-5ac+7bd$. **9.** $ax+by+cz$. **10.** $\frac{4}{3}qr-\frac{10}{7}rp-\frac{3}{2}pq$. **11.** $mn-nl+lm$. **12.** $(b-c)+(c-a)+(a-b)$ which equals zero. **13.** $x+7$. **14.** $3x-5$.

EXERCISES XXIX. Page 60.

1. $x+3$. **2.** $x-3$. **3.** $x+3$. **4.** $x-3$. **5.** $y+5$. **6.** $2x+3$. **7.** $a+7$. **8.** $x+3y$. **9.** $x-3y$. **10.** $2p+4q$. **11.** $3x^2+4$. **12.** $3ab-5$. **13.** $5x-8y$. **14.** $3+5x$. **15.** $4lm+9pq$. **16.** $7+4x^2$. **17.** $5x^2-3y^2$. **18.** $x+3$. **19.** $5p^2-q^2$. **20.** x^2-y^2.

EXERCISES XXX. Page 62.

1. $x^2+3x+11$. **2.** x^2-5x+8. **3.** $1-5y+3y^2$. **4.** x^2+x+1. **5.** x^2-x+1. **6.** $2p^2-7p+3$. **7.** $3a^2-2a+7$. **8.** $2x^2+3xy-4y^2$. **9.** $7-13z+4z^2$. **10.** $2-3p^2+11p^4$. **11.** $x+3$. **12.** $2x^2+3xy+4y^2$. **13.** $x-1$. **14.** $x+1$. **15.** $2+11p$. **16.** $7-2z^2$. **17.** $x+b$. **18.** x^2-5x+7. **19.** m^2x^2+mx+1. **20.** $x^2-(q+r)x+qr$.

EXERCISES XXXI. Pages 63—64.

1. Quotients and remainders: (1) $x+4$; -19. (2) $a+2y$; $-y^2$. (3) $x+y$; $-2y^3$. (4) $3x^2+5x+31$; 72. (5) $2z+8$; $-z-3$. **2.** 6. **3.** 7.

EXERCISES XXXII. Pages 64—66.

A 1. $a=bq+r$. **2.** $16a+b$. **3.** $\frac{1}{2}x^2+\frac{1}{3}x+\frac{2}{5}$. **4.** $5a+10b=5(a+2b)$. **5.** 18, 19, 20.

B 1. mk^2. **2.** $\dfrac{la+mb+nc}{l+m+n}$. **4.** $x^6+x^5y+x^4y^2+2x^3y^3+x^2y^4+xy^5+y^6$. **5.** 15, 17, 19.

C 1. $-35x+21$. **3.** 19. **4.** x^3+2x^2-7x+5. **5.** $abcd^3-(b^2c^2+c^2a^2+a^2b^2)d^2+abc(a+b+c)d-a^2b^2c^2$.

D 1. $12z^3+200z^2-378z+120$. **2.** 179·666......c.dm. **3.** $2m^3-3m^2n+5mn^2-n^3$. **4.** $\left(\frac{a}{l}+\frac{b}{m}+\frac{c}{n}\right)$ of work. **5.** $(x+y+z)-(a+b+c)$.

E 4. By n.

EXERCISES XXXIII. Pages 68—69.

1. $x^2+2xy+y^2$; $x^2-2xy+y^2$; $9p^2+24pq+16q^2$; $25a^2-30ab+9b^2$; x^2+2x+1; $x^2-18x+81$; $9x^2-48x+64$; $49x^2+126x+81$; $a^2b^2-2abcd+c^2d^2$; $16l^2m^2-56lmnp+49n^2p^2$; $\frac{4}{9}h^2k^2-hklm+\frac{9}{16}l^2m^2$. **5.** $p+q$; $m-n$; $x+3$; $x-2$; $2x+y$; $2x-3y$; $mn+pq$; $2ab-5cd$.

EXERCISES XXXIV. Page 70.

1. p^2-q^2; m^2-25; $4m^2-9$; x^2-1; $1-x^2$; $9x^2-16y^2$; $a^2b^2-c^2d^2$; $4m^2n^2-9p^2q^2$; a^4-b^4; a^6-b^6; $x^{2n}-y^{2n}$. **2.** $a^2+2ab+b^2-c^2$; $a^2-b^2-2bc-c^2$; $a^2-b^2+2bc-c^2$; $4x^2-12xy+9y^2-16z^2$; $9p^2-25q^2+70qr-49r^2$; $a^4+a^2b^2+b^4$. **4.** $(a+b)(a-b)$; $(m+n)(m-n)$; $(2m+n)(2m-n)$; $(2m+3n)(2m-3n)$; $(x-5)(x+5)$; $(ab+cd)(ab-cd)$; $(3ab+5cd)(3ab-5cd)$; $(a^2+b^2)(a+b)(a-b)$.

EXERCISES XXXV. Page 71.

1. x^2+4x+3; $a^2-12a+35$; $m^2+8m+16$; $4x^2+16x+15$; $9y^2-48y+55$; $16z^2-32z+15$; $a^2b^2-2ab-35$; $x^2+5xy+6y^2$; $m^2-3mn-10n^2$; $4x^2-12xy+9y^2$; $9u^2-3z-56u^2$; $9m^2n^2-9mnpq-10p^2q^2$. **2.** $(x+2)(x+3)$; $(x+5)(x+4)$; $(y+3)(y+4)$; $(a-3)(a-2)$; $(m-5)(m-4)$; $(z-4)(z-3)$; $(x-3)(x+2)$; $(l+3)(l-2)$; $(v+5)(v-4)$; $(n-5)(n+4)$; $(r-5)(r+3)$; $(k+5)(k-3)$. **4.** $(2x+3)(2x+2)$; $(5y-4)(5y-3)$; $(3m-7)(3m+5)$; $(4r+10)(4r-7)$. **5.** $(x+3y)(x+2y)$; $(l+5m)(l+4m)$; $(x-3y)(x-2y)$; $(a-3b)(a+2b)$. **6.** $(2x+3)(2x+2)$; $(3y-4)(3y-2)$; $(4z-11)(4z+9)$; $(2a-7b)(2a-3b)$.

EXERCISES XXXVI. Page 72.

1. $x^2+y^2+z^2+22x+2yz+2xy$; $a^2+b^2+c^2-2bc-2ca+2ab$; $a^2+b^2+c^2+2bc-2ca-2ab$; $4x^2+9y^2+25z^2+30yz+20zx+12xy$; $y^2z^2+z^2x^2+x^2y^2+2x^2yz+2y^2zx+2z^2xy$; $a^2x^2+b^2y^2+c^2z^2-2bcyz+2cazx-2abxy$; $25l^2+81m^2+121n^2-198mn+110nl-90lm$; $4l^2x^2+36m^2y^2+25n^2z^2+60mnyz-20nlzx-24lmxy$; $h^4+x^4+y^4+2x^2y^2-2h^2x^2-2h^2y^2$.

EXERCISES XXXVII. Page 73.

1. $2ab$; $4xy$; $12mn$; $\frac{1}{3}hk$; $2mnpq$; $10r$. **2.** b^2, $4y^2$, $9y^2$, 1, 25, $\frac{25}{4}$. **3.** $2bc+2ca+2ab$, $2bc-2ca-2ab$, $-2bc+2ca-2ab$, $-2bc-2ca+2ab$; $2ca+2ab$, $-2ca-2ab$; $2ab$; $2xy+2x+2y$, $2xy-2x-2y$, $-2xy+2x-2y$, $-2xy-2x+2y$; $24mn+16nl+12lm$, etc.; $2pq+1$, $-2pq+1$. **4.** $(x^2+y^2)^2-(xy)^2$.

EXERCISES XXXVIII. Pages 73—74.

1. $x^3+3x^2y+3xy^2+y^3$; $1+3y+3y^2+y^3$; x^3+3x^2+3x+1; $x^3+15x^2+75x+125$; etc. **2.** $x^3-3x^2y+3xy^2+y^3$; $1-3y-3y^2-y^2$; x^3-3x^2+3x-1; $x^3-15x^2+75x-125$; etc. **6.** $a^3+b^3+c^3+3b^2c+3bc^2+3c^2a+3ca^2+3a^2b+3ab^2+6abc$. **7.** $3ab(a+b)$; $-3ab(a-b)$; $36m^2n+54mn^2$; $3ab^2+b^3$; $3xy^2-y^3$; $6mn^2+n^3$.

EXERCISES XXXIX. Page 74.

1. $x^3+6x^2+11x+6$; ——; $x^3+6x^2y+11xy^2+6y^3$; ——; $a^3-6a^2+11a-6$; ——; p^3-2p^2-5p+6; ——; ——: $x^3y^3-4x^2y^2-31xy+70$; $8x^3-48x^2+94x-60$; ——. **2.** $x+2$. **3.** $x-3$.

EXERCISES XL. Page 75.

1. a^2+ab+b^2; y^2+y+1; z^2+3z+9; $4a^2+6ab+9b^2$; y^4+y^2+1; $m^4+m^2n^2+n^4$.

EXERCISES XLI. Page 76.

1. a^2-ab+b^2; y^2-y+1; etc.

EXERCISES XLII. Pages 76—78.

A 1. $x^4+10x^3+35x^2+50x+24$. **3.** +175 dol.; +86 dol.; +106 dol.; +16 dol.; +41 dol.; −69 dol.; −59 dol.; +35 dol. **5.** $4xy$.

B 1. $a^4+4a^3b+6a^2b^2+4ab^3+b^4$. **3.** 24° south lat. or −24°. **4.** $2(bc+ca+ab)$. **5.** $6x^2-21y^2+20z^2-23yz+23zx-5xy$.

C 1. $x^4+x^2y^2+y^4$. **2.** $acx^2+(ad+bc)xy+bdy^2$. **3.** 3. **5.** $2(a^2+b^2+c^2-bc-ca-ab)$.

D **1.** x^4+4y^4. **3.** 0. **4.** 29·31. **5.** $2x^4-35x^2-x+22$.

E **1.** $(x^2+2xy+y^2)-(a^2+2ab+b^2)$. **3.** $+25$ and -25.

EXERCISES XLIII. Pages 79—80.

1. $m(p+q)$. **2.** $xy(x+y)$. **3.** $ab(a-b)$. **4.** $x(1+m)$. **5.** $ab(1-k)$. **6.** $xy(1-mn)$. **7.** $3a^2x(2y-3z)$. **8.** $3abc\,(5a-6c)$. **9.** $mnxy(mx-ny)$. **10.** $a(x-y+z)$. **11.** $(a+b)(x-y)$. **12.** $(a+b)(1-m)$. **13.** $(a+b)(1+mn)$. **14.** $(a+b)(xy-uv)$. **15.** $2(a+b)y$.

EXERCISES XLIV. Page 80.

1. $(1-x)(1-y)$. **2.** $(p-x)(q-y)$. **3.** $(ab+lx)(cd+my)$. **4.** $(x^2-m^2)(x-n)$. **5.** $(x+a)(x+b)$. **6.** $(x+h)(x^2+px+q)$. **7.** $(a^2+b^2)(c+d)$. **8.** $(p^2+q^2)(x-y)$.

EXERCISES XLV. Page 81.

1. $(m+n)^2$. **2.** $(p-q)^2$. **3.** $(x+1)^2$. **4.** $(x-1)^2$. **5.** $(x+2)^2$. **6.** $(x-7)^2$. **7.** $(2a+3b)^2$. **8.** $(3ab-5)^2$. **9.** $(a+b+c)^2$. **10.** $(a+b-x-y)^2$. **11.** $(20+9)^2=29^2$. **12.** $(1-11xy)^2$. **13.** $(5u-2v)^2$. **14.** $(\frac{2}{3}x+\frac{3}{2})^2$. **15.** $(\overline{a-x}+2x)^2=(a+x)^2$. **16.** $(m^2+n^2)^2$. **17.** $(2x^6-y^6)^2$ **18.** $(9z^2-1)^2$. **19.** $(x+y-z)^2$. **20.** $(3x-4y+5z)^2$. **21.** $(2p+3q-1)^2$. **22.** $(ax+by+cz)^2$. **23.** $(yz+zx+xy)^2$. **24.** $(x^2+y^2-z^2)^2$. **25.** $(\frac{1}{2}a^2+\frac{1}{3}b^2+\frac{1}{5}c^2)^2$. **26.** $(5l-m-1)^2$.

EXERCISES XLVI. Page 82.

1. $(4x+5y)(4x-5y)$. **2.** $(a-3b)(a+3b)$. **3.** $(ab+xy)(ab-xy)$. **4.** $(3mn-2pq)(3mn+2pq)$. **5.** $(a+b+c)(a+b-c)$. **6.** $(a-b+c)(a-b-c)$. **7.** $(x+y+z)(x-y-z)$. **8.** $(2x-5y-3z)(2x-5y+3z)$. **9.** $(a+b+c+d)(a+b-c-d)$. **10.** $(a-b+c-d)(a-b-c+d)$. **11.** $(1-9m)(1+9m)$. **12.** $(1-p-q)(1+p+q)$. **13.** $(a^2-2ab+b^2)(a^2+2ab+b^2)=(a-b)^2(a+b)^2$. **14.** $-3(x+y)(x-y)$. **15.** $(a-b)(a+b)(a^2+b^2)$. **16.** $(1-x^3)(1+x^3)$. **17.** $(a-b)(a+b)(a^2+$

$b^2)(a^4+b^4)$. **18.** $(y-1)(y+1)(y^2+1)(y^4+1)(y^8+1)$. **19.** $(x^2+2zx+z^2-y^2)(x^2-2zx+z^2-y^2)=(x+y+z)(x-y+z)(x-y-z)(x+y-z)$. **20.** $(x+y+z+1)(x+y-z-1)$. **21.** $(a+b-c)(a-b+c)$. **22.** $(x+2y+3z)(x-2y-3z)$. **23.** $4x^2y^2$. **24.** $4a(b+c)$. **25.** $(3x^2-1)(x^2+6x+3)$. **26.** $2(a-c)(a-b+c-d)$.

EXERCISES XLVII. Page 83.

1. $(x+7)(x+5)$. **2.** $(y+5)(y+4)$. **3.** $(a+7)(a+3)$. **4.** $(m+5n)(m+3n)$. **5.** $(pq+3)(pq+2)$. **6.** $(mn+5pq)(mn+6pq)$. **7.** $(2x+5)(2x+3)$. **8.** $(3y+16)(3y+1)$. **9.** $(4x+y)(4x+2y)$. **10.** $(1+5xy)(1+13xy)$. **11.** $(x-5)(x-3)$. **12.** $(x-5)(x+3)$. **13.** $(y+5)(y-3)$. **14.** $(xy-7)(xy+5)$. **15.** $(ab+10cd)(ab-7cd)$. **16.** $(7-z^2)(13-z^2)$. **17.** $(1-7z^2)(1-13z^2)$. **18.** $(2xy-5z)(2xy-13z)$. **19.** $(1-11z^3)(1+7z^3)$. **20.** $(x^2-9)(x^2-9)=(x-3)^2(x+3)^2$. **21.** $(a+b-4c)(a+b-3c)$. **22.** $(xyz-24)(xyz+8)$. **23.** $(2xy+8z^3)(2xy-3z^3)$. **24.** $(5x^3+12)(5x^3-7)$. **25.** $\{(x-y)^2-9z^2\}\{(x-y)^2-4z^2\}=(x-y+3z)(x-y-3z)(x-y-2z)(x-y+2z)$. **26.** $(\frac{1}{2}x-\frac{1}{5})(\frac{1}{2}x-\frac{1}{3})$.

EXERCISES XLVIII. Page 85.

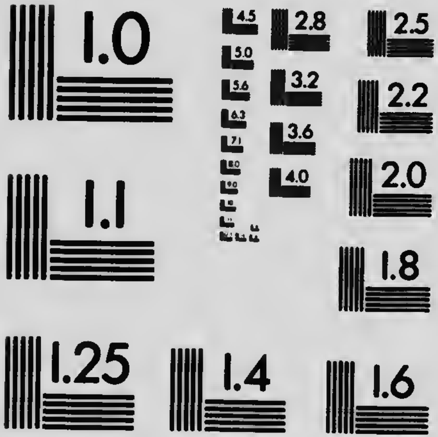
1. $(2x+3y)(x+5y)$. **2.** $(4a+7b)(3a+5b)$. **3.** $(5x-8)(7x-3)$. **4.** $(7x-11)(3x+5)$. **5.** $(4p-9q)(3p+5q)$. **6.** $(5m-11n)(3m+7n)$. **7.** $(4r-13)(7r-3)$. **8.** $(3z+5)(2z-1)$. **9.** $(5+2x)(3-5x)$. **10.** $(4ab+3xy)(ab+5xy)$. **11.** $(1-3z)(1+3z)(7+3z^2)$. **12.** $(5x-9y)(7x+8y)$. **13.** $(5p-9)(7p-8)$. **14.** $(5-8z)(7+9z)$. **15.** $(5x^2-12)(8x^2+9)$. **16.** $(3x+2y)(2x+3y)$. **17.** $(5xy-6z^2)(6xy-5z^2)$. **18.** $(3-4x)(3+4x)(3-x)(3+x)$. **19.** $(4h-3k)(13h-7k)$. **20.** $(2z-3)(2z+3)(2z^2+11)$.

EXERCISES XLIX. Page 86.

1. $(a^2+ab+b^2)(a^2-ab+b^2)$. **2.** $(x^2+x+1)(x^2-x+1)$. **3.** $(x^2+4xy+8y^2)(x^2-4xy+8y^2)$. **4.** $(m^2+6mn+18n^2)(m^2-6mn+18n^2)$. **5.** $(2p^2+6pq+9q^2)(2p^2-6pq+9q^2)$. **6.** $(x^2+x+5)(x^2-x-5)$. **7.** $(x^2+x-5)(x^2-x-5)$. **8.** $(m^2-3mn-n^2)(m^2+3mn-n^2)$.

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



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9. $(2x^2-2xy-3y^2)(2x^2+2xy-3y^2)$. **10.** $(2x^2-3xy+3y^2)(2x^2+3xy+3y^2)$. **11.** $(1+2z+2z^2)(1-2z+2z^2)$. **12.** $(1+2z^2+2z^4)(1-2z^2+2z^4)$. **13.** $(x^2+x+1)(x^2-x+1)(x^4-x^2+1)$. **14.** $(x^2+x+1)(x^2-x+1)(x^4+x^2+1)(x^8-x^4+1)$.

EXERCISES L. Page 86.

1. $(a+1)^3$. **2.** $(a-1)^3$. **3.** $(x+2a)^3$. **4.** $(2p-3q)^3$. **5.** $(x^2-4y^2)^3$. **6.** $(1-3x)^3$. **7.** $(3m+5n)^3$. **8.** $8a^3$. **9.** $(2x^2+5y^2)^3$. **10.** $(1-2x)^3$. **11.** $(1-3z^2)^3$. **12.** $(a+b)^3$.

EXERCISES LI. Page 87.

1. $(x-y)(x^2+xy+y^2)$. **2.** $(a+b)(a^2+b^2)$. **3.** $(x+1)(x^2+1)$. **4.** $(x+1)(x^2+x+1)$. **5.** $(x-1)(x+1)(x^4-3x^2+1)$. **6.** $(x+1)(x-5)(x+3)$. **7.** $(a-b)(a^2+b^2)$ **8.** $(a-b)(a^2+3ab+b^2)$.

EXERCISES LII. Page 88.

1. $(x-1)(x^2+x+1)$. **2.** $(y+1)(y^2-y+1)$. **3.** $(1+ab)(1-ab+a^2b^2)$ **4.** $(1-7pq)(1+7pq+49p^2q^2)$. **5.** $(p-3)(p^2+3p+9)$. **6.** $(q+4)(q^2-4q+16)$. **7.** $(2a+3b)(4a^2-6ab+9b^2)$. **8.** $(3m-5n)(9m^2+15mn+25n^2)$. **9.** $(x-y)(x+y)(x^2-xy+y^2)(x^2+xy+y^2)$. **10.** $(x^3+y^3)(x^6-x^3y^3+y^6)$. **11.** $(x^3-y^3)(x^6+x^3y^3+y^6)$. **12.** $(x-1)(x+1)(x^2-x+1)(x^2+x+1)(x^2+1)(x^4-x^2+1)$. **13.** $(x^4+y^4)(x^8-x^4y^4+y^8)$. **14.** $(a^5-b^5)(a^{10}+a^5b^5+b^{10})$. **15.** $(ab-c^2)(a^2b^2+abc^2+c^4)$. **16.** $(a^2-3bc)(a^4+3b^2c^2)$. **17.** $2x(x^2+3)$. **18.** $-(x+y)(7x^2-13xy+7y^2)$. **19.** $(b-a)(a^2+b^2+3c^2-3bc-3ca+ab)$. **20.** $3(b-c)(c-a)(a-b)$. **21.** $(a+b+c)(a^2+b^2+c^2-bc-ca+2ab)$. **22.** $(x+y-1)(x^2+2xy+y^2+x+y+1)$.

EXERCISES LIII. Page 91.

1. abc. **2.** x^2. **3.** $abcd$. **4.** q. **5.** $12hkl$. **6.** $17xy$. **7.** $7xyz$. **8.** $x^2y^2z^2$. **9.** $a-b$. **10.** a^2-b^2. **11.** x^4-y^4. **12.** $x-5$. **13.** $a-3b$. **14.** $2x-3y$. **15.** $x+a$. **16.** $a+b$. **17.** $a-b$. **18.** $2a-3b$. **19.** $3+4z$. **20.** $a^4+a^2b^2+b^4$. **21.** $x+3$. **22.** $a+6b$. **23.** $3a+4b$ **24.** $x-1$.

EXERCISES LIV. Pages 92—93.

1. xyz. **2.** $a^2b^2c^2$. **3.** $abcpqr$. **4.** $abcl^2m^2n^2$. **5.** $12x^2y^2z^2$. **6.** $63a^2b^2c^2d^2$. **7.** $60a^2b^2c^2def$. **8.** $90a^3x^4$. **9.** $abcxyz$. **10.** $252l^2m^3n^3$. **11.** $(x+1)(x+2)(x+3)$. **12.** $(x+2)(x-2)(x+3)(x-3)$. **13.** $(2x-3y)^2(2x+3y)^2$. **14.** $(x-3)^2(x-5)(x+7)$. **15.** $(a+2b)^2(a-2b)(2a+b)$. **16.** $(3x-5)(2x+3)(4x-7)(5x+4)$. **17.** a^4-b^4. **18.** $(x^4-y^4)(x^2+xy+y^2)$. **19.** $(a^8-b^8)(a^8+a^4b^4+b^8)$. **20.** $(z-8)(z-12)(z+4)(z+1)$. **21.** $(x+a)(x+b)(x+c)$. **22.** $(x-2)(x-3)(x-4)(x+4)$.

EXERCISES LV. Pages 93—96.

A 1. 16·1 ft., 64·4 ft., 48·3 ft. **2.** (i) $(a-10)(a-12)$; (ii) $(a-10x)(a-12x)$; (iii) $(x^2-5x-10)(x^2-5x-12)$. **3.** (i) 1; (ii) 14; (iii) 35. **4.** $a^{16}-b^{16}$. **5.** $30, $55, $105.

B 1. $sp \div 60h$ yards. **2.** $(a-b)^2(a^2+ab-b^2)$. **3.** (i) -69; (ii) -34; (iii) 71. **5.** $-x^4y^4$.

C 2. $-(b-c)(c-a)(a-b)$. **3.** $a+b+c$. **5.** $(x+5)(x+7)(x-4)(x-2)$.

D 2. $(a+b)^2(3a-b)$. **3.** $\frac{1}{2}(bc+ca+ab)-\frac{1}{4}(a^2+b^2+c^2)$. **4.** 39 and 18.

E 1. $x^6+x^4y^2-x^2y^4-y^6$. **2.** (i) $(x+y-7)(x+y-1)$; (ii) $(2a-3b)(4a+5b+5)$; (iii) $(p^2+q^2+r^2+1)(p^2+q^2-r^2-1)$. **3.** 75 lb. and 90lb. **4.** $4x^2+10xy+25y^2$.

F 1. $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$. **2.** (i) $(m+a-b)(m-a+b)(m^2+a^2+2ab+b^2)$; (ii) $(z-m)(mz+nz+m^2)$; (iii) $(4p+4q-15r)(5p+5q+2r)$. **3.** 72. **4.** $(a+b)(a-b)(a+2b)(a^2-ab+b^2)$

G 1. $8a^3$. **2.** (i) $(a-b)(a+b)(a+2b)$; (ii) $(2x+7y+5)(2x+7y-5)$; (iii) $(x+y)(x^2-xy+y^2)(x^6-x^3y^3+y^6)$. **3.** 3 hours. **4.** $a-b$; $(a-b)(a+b)(a^2+ab+b^2)$.

EXERCISES LVI. Page 99.

1. 30 yards and 48 yards. **2.** 34 pounds and 66 pounds. **3.** 100 pounds. **4.** 17 and 33. **5.** 17 and 41. **6.** 6 and 7. **7.** 6 hours. **8.** 23 and 12. **9.** 27. **10.** 96 rods and 114 rods.

EXERCISES LVII. Pages 102—103.

1. 1. 8. 2. 5. 3. $1\frac{2}{3}$. 4. $1\frac{3}{5}$. 5. 3. 6. $5\frac{2}{3}$. 7. 2. 8. $-4\frac{1}{2}$. 9. $7\frac{1}{2}$. 10. $17\frac{2}{3}$. 11. 4. 12. $8\frac{10}{23}$. **2.** 1. $-1\frac{2}{3}$. 2. 5. 3. -1. 4. $20\frac{9}{32}$. 5. $10\frac{4}{5}$. 6. $-23\frac{2}{5}$. 7. $\frac{8}{11}$. 8. $-9\frac{1}{2}$. 9. -2. 10. $-\frac{1}{6}$. **3.** 1. $-1\frac{15}{19}$. 2. $\frac{1}{2}$. 3. 76. 4. $-\frac{731}{462}$. 5. $-4\frac{1}{2}$. 6. $\frac{3}{7}$. 7. 1. 8. 10. 9. 5. 10. $-4\frac{28}{47}$.

EXERCISES LVIII. Page 104.

1. $c-a-b$. **2.** $2p-m+n$. **3.** $\frac{1}{23}(12c-20a-15b)$. **4.** $h+k$. **5.** $-(a+b)$. **6.** 3. **7.** $a+b+c$. **8.** $a+b$. **9.** $3h-6k$. **10.** 0.

EXERCISES LIX. Pages 105—106.

A 1. (i) $+2$, (ii) $+86$, (iii) $+153$, (iv) $ak+bh$. **2.** $-(ax^5+bx^3+cx)$. **3.** $-(2x-7y)(8x+y)(x+y)^2$. **4.** 10.

B 2. (i) $(x-a)(x+a)(y-a)(y+a)$; (iii) $(ax+by)(lx+my)$; (iii) $(x-y)(x^2+xy+y^2)(x^6+x^3y^3+y^6)$. **3.** $x=-5$ and $x=3$. **5.** $32 and $21.

C 1. $(a^6-b^6)(a^2-b^2)$. **2.** $3x^4a^2-2x^3a^3-3x^2a^4+a^6$. **3.** 30 cents a pound. **4.** 0.

D 1. 6 **2.** 0. **4.** $12. **5.** $a^2+b^2+c^2-bc-ca-ab$.

EXERCISES LX. Page 108.

3. The value of x must not be, 0, 2, $-1\frac{1}{2}$, -2.

EXERCISES LXI. Pages 109—110.

1. 7, 11, 17, a^2+3a+7. **2.** $a^2+8a-11$. **3.** 2, $3\frac{1}{3}$, $a^2+\frac{1}{a^2}$. **7.** $2x-7$, $2x^2-7$, $2x-3$.

EXERCISES LXII. Pages 110—112.

A 1: $x^2+8x+15$. **2.** $-1\frac{1}{2}$. **3.** $2m+1$. **4.** 7th of Sept.

B 1. No **2.** 3. **3.** (i) $xy(x-y)$; (ii) $5(a+b)(7a^2+11ab+7b^2)$; (iii $(3x+4y)^2(6x-5y)(-x-2y)$. **4.** $2(m+n)a+4a^2$.

C 3. (i) $(l+m)(l-m)(n+p)(n-p)$; (ii) $(x+y-9a)(x+y-8a)$. **5.** $\frac{4}{3}\pi(3tx^2+zt^2x+t^3)$.

D 4. $62\frac{1}{2}$ and $37\frac{1}{2}$.

E 3. $(x-7)(x+4)$. **4.** $2(a^2+b^2+c^2)$.

EXERCISES LXIV. Pages 115—116.

3. x itself. **4.** $s=22t$.

EXERCISES LXVIII. Pages 126—128.

A 2. 5in., 12in., 7in. **3.** $(x-a)(x+a+p)$. **4.** $y=\frac{2}{3}x-\frac{7}{3}$.

B 2. (1) $-\frac{3}{5}$. (2) $(a+b+c)$. **3.** (1) $(x^7-y^7)(x^{14}+x^7y^7+y^{14})$. (2) $(x^2+a^2-ab+b^2)(x^2-a^2-ab-b^2)$.

C 1. 6 and -4. **2.** $x^6-14x^4+49x^2-36$. **3.** 6 and 7. **4.** (1) $(11-x)(11x-1)$. (2) $(a-b+c-d)(a-b-c+d)$. (3) $(x-3)(x+3)(x-4)(x+4)$. **5.** 7.

D 1. $2(m+n)x+2(n+l)y+2(l+m)z$. **3.** 40 lb. and 20 lb. **4.** $3lx^2+mx$. **5.** $y=x-2$.

E 1. $(la+mb+nc)\div(l+m+n)$. **2.** (1) $(x-8)(x-13)$; (2) $(x-8a)(x-13a)$; (3) $(x-13)(x+8)$; (4) $(x+13y)(x-8y)$. **3.** \$600, \$400. **5.** $(x+9)(x+11)(x+13)$.

EXERCISES LXIX. Pages 130—131.

1. $8x^3+27y^3+64z^3-72xyz$. **2.** $p^3-27q^3+125r^3+45pqr$. **3.** $x^3y+x^3z+y^3z+y^3x+z^3x+z^3y+x^2yz+y^2zx+z^2xy$. **4.** $(a^2+b^2+c^2)xyz+cax^2y+abx^2z+aby^2z+bcy^2x+bcz^2x+caz^2y$. **5.** $ax^4+(3b-ma)x^3+3(c-mb)x^2+(d-3mc)x-md$. **6.** $4x^3+x^2y-8xy^2+3y^4+10x^2+14xy-6y^2-x-14y+15$. **7.** $abcx^3+(bch+cak+abl)x^2+(akl+blh+chk)x+hkl$. **8.** $6x^3-13x^2y-73xy^2-60y^3-14x^2+137xy+178y^2-66x-154y+40$. **9.** $6x^3+60y^3-80z^3-31x^2y+26x^2z-203y^2z-7y^2x-12z^2x+134z^2y$. **10.** $hklx^3+\{kl(k-l)+lh(l-h)+hk(h-k)\}x^2+\{h(l-h)(h-k)+k(h-k)(k-l)+l(k-l)(l-h)\}x+(k-l)(l-h)(h-k)$. **11.** $a^3+b^3+c^3+3a^2b+3a^2c+3b^2c+3b^2a+3c^2a+3c^2b+6abc$. **12.** $a^2+b^2+c^2+d^2+2bc+2ca+2ab+2da+2db+2dc$. **13.** $6a^2+12b^2+20c^2+30d^2-31bc+22ca-17ab-27da+38db-49dc$. **14.** a^3x^3+

$b^3y^2-c^3z^2+a(a^2+ab+b^2)x^2y+a(a^2+ac+c^2)x^2z+b(b^2+bc+c^2)y^2z+b(b^2+ba+a^2)y^2x+c(c^2+ca+a^2)z^2x+c(c^2+cb+b^2)z^2y\{a^2(b+c)+b^2(c+a)+c^2(a+b)\}xyz$. **15.** $36x^4-12x^3y-179x^2y^2+30xy^3+225y^4$. **16.** $x^4+(a+b+c+d)x^3+(bc+ca+ab+da+db+dc)x^2+(abc+dbc+dca+dab)x+abcd$. **17.** $x^5-(h+k+l+m+n)x^4+(\Sigma hk)x^3+(\Sigma hkl)x^2+(\Sigma hklm)x+hklmn$. **18.** $a^3+b^3+c^3+d^3+a^2b+a^2c+a^2d+b^2c+b^2d+b^2a+c^2d+c^2a+c^2b+d^2a+d^2b+d^2c$. **19.** $abc-(bc+ca+ab)x+(a+b+c)x^2-x^3$. **20.** $1+x^2+x^4+x^6$. **21.** $ap+2(aq+bp)x+(ar+cp+4bq)x^2+2(br+cq)x^3+crx^4$. **22.** $x^3+y^3+z^3+2(y^2z+yz^2+z^2x+zx^2+x^2y+xy^2)+3xyz$. **23.** $abc(x^3+y^3+z^3)+a(y^2z+yz^2)(b^2+bc+c^2)+\cdot\cdot+\cdot\cdot+(b^2c+bc^2+c^2a+ca^2+a^2b+ab^2)xyz$. **24.** $l(v^3w+vw^3)+\cdot\cdot+\cdot\cdot+lu^2vw+mv^2wu+nw^2uv$. **25.** $\frac{1}{14}x^4+\frac{41}{420}x^3+\frac{209}{1680}x^2+\frac{37}{600}x+\frac{1}{40}$. **26.** $70x^3+10y^3-108z^3-73y^2z+159yz^2-33z^2x+313zx^2-157x^2y-49xy^2+108xyz$. **27.** $\frac{1}{5}x^3+\frac{41}{72}x^2y+\frac{577}{2160}xy^2+\frac{1}{8}y^3$. **28.** $1-p-q-r-p^2-\cdot\cdot-\cdot\cdot+2qr+\cdot\cdot+\cdot\cdot+q^2r+\cdot\cdot+\cdot\cdot+\cdot\cdot+\cdot\cdot+\cdot\cdot-p^3-\cdot\cdot-\cdot\cdot-2pqr$.

EXERCISES LXX. Page 134.

1. $p^2-pq+q^2+p+q+1$. **2.** $x^3+2x^2y+2xy^2+y^3$. **3.** $x^2+y^2+z^2+yz+zx-xy$. **4.** $3x+5y-8z$. **5.** $6x^2-12y^2-30z^2+38yz+3zx-xy$. **6.** $px^2+2qx+r$. **7.** $(a+b)x+(a-b)y$. **8.** $Q=5x^3+28x^2+167x+946$; $R=5378x^2-6933x+3863$. **9.** $px^2+qxy+ry^2$. **10.** $Q=ax^3+(am+b)x^2+(am^2+bm+c)x+am^3+bm^2+cm+d$; $R=am^4+bm^3+cm^2+dm+e$. **11.** $1+2x+3x^2+3x^3+3x^4$ with, at this stage $R=3x^5$. **12.** $1+2x+3x^2+4x^3+5x^4$ with, at this stage, $R=6x^5-5x^6$. **13.** $1-x+x^2-x^3+x^4$ with, at this stage, $R=-x^5$. **14.** $\frac{1}{a}+\frac{x}{a^2}+\frac{x^2}{a^3}+\frac{x^3}{a^4}+\frac{x^4}{a^5}$ with, at this stage, $R=\frac{x^5}{a^5}$.

EXERCISES LXXI. Page 136.

1. $6x^4+x^3-38x^2+94x-63$. **2.** $20x^5+3x^4-97x^3-54x^2+113x+78$. **3.** $7x^5+15x^4-63x^3+73x^2-112x+64$. **4.** $10-3x-38x^2+66x^3-35x^4$. **5.** x^5-1. **6.** $1-x^6$. **7.** $x^4-12x^3-110x^2+876x+5005$. **8.** $ax^3-(a-b)x^2-(b-c)x-c$. **9.** $apx^3+(2aq+bp)x^2+(ar+2bq)x+br$. **10.** $x^6-3m^2x^4+3m^4x^2-m^6$. **11.** $1+x^2+x^4-x^6-x^8-x^{10}$. **12.** $a^2x^4+(2ac-4b^2)x^2+c^2$.

EXERCISES LXXII. Page 139.

1. $2x^2+3x-7$. **2.** $3x^2-8x+11$. **3.** $4x^3-7x^2+21x-5$. **4.** $3x^2-5x-7$. **5.** $2a^2+5ab-8b^2$. **6.** $7-3z+5z^2+6z^3$. **7.** $2y^2+9y+53$: $R=459$. **8.** $4z^3-33z^2+247z-1694$: $R=11829$. **9.** $7x^3-4x^2y-6xy^2-7y^3$: $R=8y^4$. **10.** $5a^3-6a^2b+13ab^2+12b^3$, $R=5b^4$. **11.** $x^6+x^5+x^4+x^3+x^2+x+1$. **12.** $x^6-x^5+x^4-x^3+x^2-x+1$. **13.** $x^9-x^8+x^7-x^6+x^5-x^4+x^3-x^2+x-1$: $R=2$. **14.** $x^{10}-x^9+x^8-x^7+x^6-x^5+x^4-x^3+x^2-x+1$. **15.** $x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1$. **16.** $2x^2-3xy+8y^2$. **17.** $4-3z+7z^2$. **18.** $7x^3+12x^2y+108xy^2+399y^3$: $R=2028y+3249$. **19.** $x^{30}+x^{25}+x^{20}+x^{15}+x^{10}+x^5+1$. **20.** $a^4-a^2b^2+b^4$.

EXERCISES LXXIII. Pages 143—144.

4. $p=0$, $q=-1$. **7.** 5899.

EXERCISES LXXIV. Pages 148—149.

4. (1) $2\Sigma a^3+3\Sigma b^2c$. (2) $-\Sigma a^3$. (3) $2\Sigma x^3-\Sigma y^2z$. (4) $7\Sigma x^2-4\Sigma yz$. (5) $4\Sigma x^2$. **5.** (1) $-(y-z)(z-x)(x-y)$ or $(z-y)(x-z)(y-x)$. (2) $3(y-z)(z-x)(x-y)$. (3) $(a+b+c)(bc+ca+ab)$. (4) $-(m-n)(n-l)(l-m)$. (5) $(y+z)(z+x)(x+y)$. (6) $(y-z)(z-x)(x-y)(x+y+z)$. (7) $3(b+c)(c+a)(a+b)$. (8) $(b-c)(c-a)(a-b)(a+b+c)$. (9) $-(y-z)(z-x)(x-y)(x^2+y^2+z^2+yz+zx+xy)$. (10) $3(b^2-c^2)(c^2-a^2)(a^2-b^2)$.

EXERCISES LXXV. Page 151.

1. (i) $a^3-b^3+c^3+3abc=(a-b+c)(a^2+b^2+c^2+bc-ca+ab)$; (ii) $(x^3-y^3-z^3-3xyz)=(x-y-z)(x^2+y^2+z^2-yz+zx+xy)$; (iii) $p^3+q^3+3pq-1=(p+q-1)(p^2-pq+q^2+p+q+1)$. **3.** (1) $(2x+3y+5z)(4x^2+9y^2+25z^2-15yz-10zx-6xy)$; (2) $(u+2v-3w)(u^2+4v^2+9w^2+6vw+3wu-2uv)$; (3) $-(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$.

EXERCISES LXXVI. Page 153.

2. (1) $(2x-3y+5z)(3x-2y-4z)$. (2) $(3a+7b-8c)(4a-5b-2c)$. (3) $(5x-12y+7)(2x+3y-5)$. (4) $(p-7q+3r)(4p-3q-6r)$. (5) $(5x-7y+8)(7x+5y-3)$.

EXERCISES LXXVII. Pages 158—159.

1. (1) x^2+x+3; $(x+2)(x+3)(x^2+x+3)$. (2) x^2+x+3; $(x+2)(x-3)(x^2+x+3)$ (3) $x+3$; $(x+3)(x^2-x+1)(x^2-2x+3)$. (4) $x-3$; $(x-3)(x^2+x+2)(x^2+2x-1)$. (5) $x+2$; $(x+2)(x^2+x+1)(x^2-x+1)$. (6) $x+3$; $(x+3)(x^2-x+1)(x^2+x+2)$. (7) $x+6$; $(x+6)(x^2+x-3)(x^2-x+2)$. (8) $3x^2+5x+7$; $(3x^2+5x+7)(2x+3)(3x-5)$. (9) $2x-3y$; $(2x-3y)(4x^2-5xy+3y^2)(3x^2-6xy+7y^2)$. (10) x^2-3x+5; $(x^2-3x+5)(2x+3)(4x^2+3x+7)$. **2.** x^2-2x-3.

EXERCISES LXXVIII. Pages 159—160.

A 1. (i) $(5x-7y)(3x+8y)$. (ii) $(5-7z)(3+8z)$. **2.** $x=4$. **3.** 17, 19, 21. **4.** $f(z^2)$. **5.** $-3x^2-3y^2-3z^2+7y^2z+7yz^2+7z^2x+7zx^2+7x^2y+7xy^2-34xyz$.

B 1. $(2x+5y)(3x+4y)$. **2.** -97.

C 1. (i) $(3x+5y)(x+4y)$. (ii) $(3-4y)(2x+5y)$. (iii) $(5x-7y)(3x+8y)$. **3.** (i) is an *identity, i.e.*, it is true whatever be x; (ii) is an *equation*, and is true only if $x=5$. **4.** -34.

EXERCISES LXXIX. Page 161.

1. $a=4$, $b=2$; $a=8$, $b=4$; $a=7$, $b=5$, $c=4$, $d=2$, $a=3$. **2.** The first three integral and the last virtually so; the others fractional.

EXERCISES LXXX. Pages 163—164.

2. $\frac{1}{x-y}$, $\frac{-1}{x-y}\left(=-\frac{1}{x-y}\right)$, $\frac{a-b}{x-y}$, $\frac{b-a}{x-y}\left(=-\frac{a-b}{x-y}\right)$.

6. Take as den'r $(b-c)(c-a)(a-b)$; the num'rs in order are m, $-n$, $-p$.

EXERCISES LXXXI. Pages 164—165.

1. $\frac{2a}{5b}$. **2.** $\frac{3y}{4x}$. **3.** $\frac{3xy}{2mx-5ny}$. **4.** $\frac{x-2}{x-4}$. **5.** $\frac{z+3}{z+9}$.

6. $\frac{x+4}{x+7}$. **7.** $\frac{7x-4y}{5x+8y}$. **8.** $\frac{5-11p}{3-7p}$. **9.** -1. **10.** $\frac{x-7}{x^2-4x-5}$.

11. $\frac{2-3z}{6-4z}$. **12.** $\frac{x+n}{2x^2-5nx+6n^2}$. **13.** $\frac{a^2-b^2}{a^2+b^2}$. **14.** $\frac{x^2-xy+y^2}{x^2+y^2}$.

EXERCISES LXXXII. Pages 167—168.

1. $\frac{a+b}{x+y}$. **2.** $\frac{m+n}{mn}$. **3.** $\frac{q-p}{qp}$. **4.** $\frac{ay-bx}{xy}$. **5.** $\frac{x^2-y^2}{xy}$.

6. $\frac{a^3+b^3}{ab}$. **7.** $\frac{yz+zx+xy}{xyz}$. **8.** $\frac{ayz+bzx+cxy}{xyz}$. **9.** $\frac{x+y-z}{xyz}$.

10. $\frac{px+qy+rz}{xyz}$. **11.** $\frac{aqr+brp+cpq}{pqrx}$. **12.** $\frac{b^2c^2x^2+c^2a^2y^2+a^2b^2z^2}{a^2b^2c^2}$.

13. $\frac{2x-5}{(x-2)(x-3)}$. **14.** $\frac{x+13}{(x+5)(x+17)}$. **15.** $\frac{4}{x-5}$. **16.** $\frac{4x-22}{x^2-1}$. **17.** $\frac{2x^2+2}{x^2-1}$. **18.** $\frac{4x+12}{x+7}$. **19.** $\frac{3x^2-18x+26}{(x-2)(x-3)(x-4)}$.

20. $\frac{8}{(x-5)(x-7)(x-9)}$. **21.** $\frac{5x^2+88x+328}{(x+8)(x+10)(x+12)}$.

22. $\frac{2a^2}{x(x^2-a^2)}$. **23.** $\frac{-ax-2a^2}{x^2-a^2}$. **24.** $\frac{-3x^2-2x}{x^3-1}$.

25. $\frac{6abc-b^2c-bc^2-c^2a-ca^2-a^2b-ab^2}{(b-c)(c-a)(a-b)}$.

$\frac{a^3+b^3+c^3+b^2c+bc^2+c^2a+ca^2+a^2b+ab^2+3abc}{(b+c)(c+a)(a+b)}$.

27. $\frac{12x-22}{(x-1)(x-2)(x-3)}$. **28.** $-\frac{3x^2-83}{(x-3)(x-5)(x-7)}$.

29. $-\frac{177z^2-167z+130}{180(z-1)(z-5)(2z-7)}$. **30.** $\frac{5y^2-49y+10}{(y-1)(y-3)}$. **31.** 0.

32. $\frac{2(yz+zx+xy-x^2-y^2-z^2)}{(y-z)(z-x)(x-y)}$. **33.** 0. **34.** $\frac{4a^5+2ab^4}{a - b^6}$.

35. $\frac{4a^5+2ab^4}{a^6-b^6}$.

EXERCISES LXXXIII. Page 170.

1. $\frac{py}{7x}$. **2.** $\frac{x}{y}$. **3.** $\frac{4}{9} \cdot \frac{a^2x}{b^2y}$. **4.** $\frac{5}{14} \cdot \frac{ab}{c}$. **5.** $\frac{x-9}{x+4}$.

6. $\frac{49x^2-168x+144}{18x^2-9x-104}$. **7.** $\frac{x+b}{x-c}$. **8.** $\frac{x^2-a^2}{x^2-c^2}$.

9. $\frac{(b-c-a)^3}{(c+a+b)(c-a-b)(c-a+b)}$. **10.** 1. **11.** $\frac{2}{y}$. **12.** $\frac{a^2}{a^2+b^2}$.

13. $\frac{3+2x+2x^2-x^3}{5-4x+4x^2+x^3}$. **14.** $\frac{(a+x)^2}{(a-x)^2}$. **15.** 1.

EXERCISES LXXXIV. Page 171.

1. $\frac{a^2by^4}{c^4x}$. **2.** $\frac{9}{10}x^2$. **3.** $\frac{1-x^2}{1+x^2}$. **4.** $\left(\frac{a-b}{a+b}\right)^3$. **5.** $\frac{a(a-b)}{b}$.

6. $-\frac{x^2-1}{x^2-4}$. **7.** $\frac{a^2(a+x)}{a^3-x^3}$. **8.** $\frac{(a+x)(a^3-x^3)}{a^4}$.

EXERCISES LXXXVI. Pages 179 – 180.

1. xy. **2.** $\frac{x^2+xy+y^2}{x+y}$. **3.** $\frac{x^2+y^2}{x^2-y^2}$. **4.** $\frac{x^2+y^2}{2xy}$. **5.** $\frac{2xy}{x^2+y^2}$.

6. $\frac{a^2-b^2}{a^2+b^2}$. **7.** -1. **8.** $\frac{a^2+ab+b^2}{ab(a+b)}$. **9.** $\frac{b(a-b)^3}{a(a+b)^3}$. **10.** $\frac{5x-13}{7}$. **11.** -1. **12.** $\frac{a^5(a+x)}{x^5(a-x)}$. **13.** $\frac{z(x-y+z)}{x(x+y+z)}$. **14.** $\frac{1}{a}$.

15. $\frac{x-2}{x-3}$. **16.** x^2. **17.** $\frac{xyz-bcx-cay-abz}{ayz+bzx+cxy-abc}$.

18. $\frac{x^2(x-a)(x-b)}{3x^2-3ax-3bx+a^2+ab+b^2}$. **19.** $\frac{ax^2+a^2}{x^3+2ax}$. **20.** $\frac{ax^2-a^2}{x^3-2ax}$.

21. $\frac{abcd+ad+dc+ab+1}{bcd+b+d}$. **22.** $\frac{abcd-ad-dc-ab+1}{bcd-b-d}$.

EXERCISES LXXXVII. Pages 181—183.

A 1. (i) $3x+7$; (ii) $3.10+7$; (iii) 37. **2.** $\frac{12a+b}{36}$ of a yard.
3. (i) $\frac{r}{100}$; (ii) $\left(\frac{100+r}{100}\right)$; (iii) $\left(\frac{100+r}{100}\right)^5$; (iv) $\left(\frac{100+r}{100}\right)^5-1$. **4.** $\frac{m^4}{m^2+n^2}$ **5.** 3, 4, 5; 5, 12, 13; 20, 21, 29; etc.

B 2. $\frac{100x+10y+z}{100000}$ of a *km*. **3.** (i) $x^3-21x^2+143x-315$; (ii) $x^3-21x^2y+143xy^2-315y^3$; (iii) $1-21z+143z^2-315z^3$. **5.** \$14, \$11.

C 1. $30x^3+84y^3+240z^3-170y^2z-94yz^2-174z^2x-33zx^2-47x^2y-53xy^2+177xyz$. **2.** Product not zero; no; no; yes. **5.** $(36x+12y+z)\div 39\cdot 37$.

D 2. $x-1$; $x-6$, $x+3$; -3, 1, 6. **3.** 2975.

E 1. $a+b+c$, $a-b+c$. **4.** $x^{24}-y^{24}$. **5.** One or other is zero, or both are zero.

EXERCISES LXXXVIII. Pages 186—188.

1. (1) $6\frac{2}{5}$. (2) $42\frac{9}{35}$. (3) $5\frac{4}{5}$. (4) 17. (5) $2\frac{3}{11}$. (6) $10\frac{1}{7}$. (7) 32. (8) 1. (9) 21. (10) $-1\frac{7}{8}$. **2.** (1) a. (2) $6\frac{1}{3}a$. (3) $-\frac{3ab}{a+b}$. (4) $\frac{ab(a+b)}{a^2+b^2}$. (5) $\frac{a^2+b^2}{2a}$. (6) $\frac{a^2+6ab+b^2}{2(a+b)}$. (7) $\frac{ac}{d}$. (8) $\frac{ab-cd}{a-c}$. (9) $\frac{ab-ac-bc}{c^2}$. (10) $-b$. **3.** 12, 15. **4.** 18, 25. **5.** \$42, \$30. **6.** $\frac{3}{7}$. **7.** 15 and 20 yrs. **8.** 15 and 45 yrs. **9.** \$35 and \$40. **10.** \$2[illegible] and \$48. **11.** $\frac{3}{5}a$, $\frac{2}{5}a$. **12.** $8\frac{1}{2}$ in. **13.** $\frac{2}{5}a-c$, $\frac{3}{5}a+c$. **14.** $\frac{na+mb}{m+n}$ ir **15.** 7, 8, 9. **16.** 25 mi. an hour. **17.** \$1·70, \$·90. **18.** 4 mi. and 3 mi. an hr. **19.** 9, 11, 13.

EXERCISES LXXXIX. Pages 189—190.

A 1. $7(a+b)$ yd.; $n(a+b)$ yd. **2.** $3b-2a$. **3.** (i) $(a+x)(b+y)$. (ii) $(7x+8y)(5x-11y)$. **5.** $(x-1)+(x+1)=2x$.

B 1. $2h(bc+ca+ab)$ cents. **2.** $a-b=c$; 2. **3.** (i) $(a-x)(b-y)$. (ii) $(7x-8y)(5x+11y)$. **4.** $\frac{3}{4}(x-1)$. **5.** $a^7+a^6b+\ldots\ldots+b^7$.

C 1. $\pi=3{\cdot}14$, the accuracy in the value depending on the accuracy of the given measurements, which cannot be absolute. **2.** (i) $(x+1)(x+2)(x+3)(x+4)$; (ii) $(p^2+q^2)(p^4-p^2q^2+q^4)$. **3.** $\frac{5}{7}$.

D 2. $(x-1)(x-2)(x-5)(x+2)$; (ii) $(m^2+n^2)(p^2+q^2)$. **3.** 157. **4.** 3.

EXERCISES XC. Page 194—195.

1. (1) 6, 5; (2) 3, 4; (3) 5, 7; (4) 8, 9; (5) 13, 5; (6) 3, 7; (7) $+9, -5$; (8) $+6, -5$; (9) $-5, -8$; (10) $-7, -6$. **2.** (1) $+15\frac{4}{5}$, $-4\frac{4}{5}$; (2) $7\frac{1}{5}$, $4\frac{4}{5}$; (3) $-6\frac{2}{7}$, $+\frac{6}{7}$; (4) $-25\frac{17}{19}$, $-30\frac{5}{57}$; (5) $51\frac{1}{5}$, $28\frac{1}{5}$; (6) $-\frac{41}{114}$, $+\frac{305}{228}$; (7) $-3\frac{109}{113}$, $-5\frac{85}{113}$; (8) $2\frac{1}{27}$, $2\frac{1}{27}$; (9) -50, $+40$; (10) $-2\frac{19}{30}$, $+4\frac{11}{20}$. **3.** (1) $1\frac{15}{17}a$, $2\frac{7}{17}a$; (2) a, b; (3) $\frac{1}{11}(20d-12c)$, $\frac{1}{33}(35d-32c)$; (4) b, a; (5) $ab(a+b)\div(a^2+b^2)$, $ab(a-b)\div(a^2+b^2)$; (6) $\dfrac{ab(5a+35b)}{25a^2+21b^2}$, $\dfrac{ab(25a-3b)}{25a^2+21b^2}$; (7) $\dfrac{ld-mb}{ad-bc}$, $\dfrac{ma-lc}{ad-bc}$; (8) $\dfrac{2m^2-3mn}{m-n}$, $\dfrac{2mn-3n^2}{m-n}$; (9) $3b$, $4a$; (10) m, l; **4.** 5 yd. and 6 yd. $1·60, $0·80. **6.** 15 yr., 20 yr. **7.** 37. **8.** $800, $1000. **9.** $50, $85. **10.** 18 mi. and 12 mi. an hr.

EXERCISES XCI. Pages 198—199.

5. $x=5$, $y=3$. **9.** 8, 11.

EXERCISES XCIII. Page 204.

1. 3, −1. **2.** 2, 1. **3.** 1, 1. **4.** $\frac{111}{41}$, $\frac{181}{41}$. **5.** $\frac{209}{43}$, $\frac{125}{43}$.

6. $\frac{1148}{1053}$, $\frac{112}{351}$. **7.** $\frac{cm-bn}{am-bc}$, $\frac{an-cl}{am-bl}$. **8.** $\frac{b'c-bc'}{ab'-a'b}$, $\frac{ca'-c'a}{ab'-a'b}$.

9. $\frac{a_3b_2-a_2b_3}{a_1b_2-a_2b_1}$, $\frac{a_1b_3-a_3b_1}{a_1b_2-a_2b_1}$. **10.** $\frac{aa'(b'-b)}{ab'-a'b}$, $\frac{bb'(a-a')}{ab'-a'b}$.

EXERCISES XCIV. Pages 207—208.

5. Not sufficient. **6.** Not sufficient. **7.** (1) 4, 5, 6. (2) −[illegible], +1, 2. (3) 2, 1, 5. (4) 4, 7, 11. (5) 3, 4, 5. (6) 0, 4, [illegible] (7) Not sufficient. (8) Inconsistent. (9) $\frac{3289}{553}$, $\frac{1683}{553}$, $\frac{1221}{553}$. (10) $x=0$, $y=0$, $z=$ [illegible].

EXERCISES XCV. Pages 209—210.

1. 7 lb. and 15 lb. **2.** 75c., 5c. **3.** 12 and 21, **4.** 55, 28. **5.** $\frac{5}{8}$. **6.** 4 mi. and 3 mi. an hr. **7.** 80 rods by 50 rods. **8.** \$17 and \$28. **9.** 40c. and 30c. a lb. **10.** 4 mi. an hour; 18 mi. (Note: take *rate* and *time* as the two unknowns). **11.** \$7, \$11, \$13. **12.** $\frac{5}{7}$. **13.** 4 mi. an hr.; 28 mi. **14.** 15 gal. and 3 gal. **15.** 35 mi. and 25 mi. an hr. **16.** 4 per cent. and 5 per cent. per annum.

EXERCISES XCVI. Pages 211—2[illegible]

A 1. pa^2+qa+r. **2.** (i) 36, (ii) 14, (iii) $n-m$.

B 2. $x^7(x^8-y^8)$.

EXERCISES XCVII. Page 216.

1. (1) 3, 7; (2) 5, 7; (3) 6, 8; (4) 12, 16; (5) 3, 5; (6) 3, 7; (7) 4, 9; (8) 1, 11; (9) 1, 2; (10) 1, 6. **2.** (1) +5, −4; (2) +4, −7; (3) −4, −5; (4) −4, −11; (5) +6, −8; (6) −6, +8; (7) +3, −3; (8) +17, −17; (9) +9, −9; (10) +4, −4. **3.** (1) $1\frac{1}{2}$, 5; (2) $1\frac{1}{2}$, $1\frac{1}{3}$; (3) +7, $-1\frac{2}{3}$; (4) $+1\frac{1}{7}$, $-2\frac{2}{5}$; (5) $+1\frac{1}{2}$, −6; (6) $+\frac{5}{7}$, $-2\frac{2}{3}$; (7) $\frac{2}{3}$, $\frac{3}{2}$; (8) $\frac{3}{5}$, $\frac{5}{6}$; (9) $+1\frac{1}{3}$, −2; (10) +5, $-\frac{1}{5}$. **4.** (i) $x^2-5x+6=0$; (ii) $x^2-13x+40=0$; (iii) $x^2+5x+6=0$; (iv) $x^2-3x-40=0$; (v) $x^2-(a+b)x+ab$. **5.** There is not. **6.** $1\frac{1}{5}$, $1\frac{1}{4}$; $20x^2-31x+12=0$.

EXERCISES XCVIII. Page 222.

1. (1) $1\pm\sqrt{6}$; (2) $-2\pm\sqrt{11}$; (3) $(3\pm\sqrt{5})\div 2$; (4) $3\pm\sqrt{12}$; (5) $(-7\pm\sqrt{93})\div 2$; (6) $(3\pm\sqrt{41})\div 4$; (7) $(9\pm\sqrt{66})\div 3$; (8) $(6\pm\sqrt{6})\div 5$; (9) $(5\pm\sqrt{277})\div 14$; (10) $(11\pm\sqrt{41})\div 8$. **2.** (1) $11+6\sqrt{2}$, 19·49; (2) $7-3\sqrt{3}$, 1·80; (3) $56+8\sqrt{5}+7\sqrt{6}+\sqrt{30}$, 96·51; (4) $5+3\sqrt{15}$, 16·62; (5) $4+\sqrt{15}$, 7·87; (6) $7+4\sqrt{3}$, 13·93; (7) $(11+3\sqrt{15})\div 7$, 3·23; (8) $\sqrt{21}+\sqrt{15}-\sqrt{14}-\sqrt{10}$, 1·55. **6.** (1) $6\pm\sqrt{34}$; (2) $(15\pm\sqrt{105})\div 6$; (3) $(15\pm\sqrt{57})\div 14$; (4) $(7\pm\sqrt{145})\div 16$; (5) $(8\pm\sqrt{28})\div 3$; (6) $(-7\pm\sqrt{13})\div 6$; (7) $(11\pm\sqrt{61})\div 10$; (8) $(27\pm\sqrt{2409})\div 28$.

EXERCISES XCIX. Page 226.

2. (1) $(x-1-\sqrt{-1})(x-1+\sqrt{-1})$; (2) $(x+1+\sqrt{-1})(x+1+\sqrt{-1})$; (3) $(x-\frac{3}{2}-\sqrt{-\frac{3}{4}})(x-\frac{3}{2}+\sqrt{-\frac{3}{4}})$; (4) $(x+\frac{3}{2}+\sqrt{-\frac{3}{4}})(x+\frac{3}{2}+\sqrt{-\frac{3}{4}})$; (5) $(x-4-2\sqrt{-1})(x-4+2\sqrt{-1})$; (6) $(x+5-\sqrt{-5})\ (x+5-\sqrt{-5})$; (7) $2(x+\frac{3}{4}+\sqrt{-\frac{7}{16}})(x+\frac{3}{4}-\sqrt{-\frac{7}{16}})$; (8) $3(x-\frac{5}{6}-\sqrt{-\frac{23}{36}})(x-\frac{5}{6}+\sqrt{-\frac{23}{36}})$; (9) $(x+\frac{1}{2}+\sqrt{-\frac{3}{4}})\ (x+\frac{1}{2}-\sqrt{-\frac{3}{4}})$; (10) $\{x+a(\frac{1}{2}+\sqrt{-\frac{3}{4}})\}\{x+a(\frac{1}{2}-\sqrt{-\frac{3}{4}})\}$. **3.** (1) $2\pm\sqrt{-1}$; (2) $-2\pm\sqrt{-1}$; (3) $(5\pm\sqrt{-7})\div 2$; (4) $(-7\pm\sqrt{-11})\div 2$; (5) $3\pm\sqrt{-3}$; (6) $2\pm\sqrt{-3}$; (7) $(7\pm\sqrt{-7})\div 4$; (8) $(6\pm 3\sqrt{-1})\div 5$; (9) $(1\pm\sqrt{-3})\div 2$; (10) $a(1\pm\sqrt{-3})\div 2$. **4.** (1) $4\pm\sqrt{-1}$; 8; 15; (2) $-6\pm\sqrt{-1}$; -12; 35; (3) $(5\pm\sqrt{-13})\div 2$; 5; 3; (4) $5\pm\sqrt{-2}$; 10; 27; (5) $(9\pm\sqrt{-41})\div 4$; $\frac{9}{2}$; $\frac{5}{2}$; (6) $(3\pm\sqrt{-31})\div 4$; $\frac{3}{2}$; $\frac{5}{2}$.

EXERCISES C. Page 228.

1. (1) 2, 4; (2) $+4$, -3; (3) $(5\pm\sqrt{37})\div 2$; (4) $3\pm\sqrt{-2}$; (5) $+2\frac{1}{2}$, -1; (6) $(5\pm\sqrt{-59})\div 6$; (7) $(5\pm\sqrt{81})\div 4$; (8) $(4\pm\sqrt{106})\div 9$. **3.** 5, 12. **4.** $+8$, -7. **5.** 0, 6. **6.** x may equal $-y$. **7.** It cannot for the sum may be \$3 or \$7.

EXERCISES CI. Pages 231—233.

1. $\frac{7}{12}$, though $\frac{-20}{-17}$ satisfies conditions. **2.** 3, 5, 7, though -3, -5, -7 satisfy conditions. **3.** 11, or -4. **4.** 30 yd. by 40 yd. **5.** 30 cents. **6.** 25 mi. and 30 mi. an hr. **7.** 13 and 7. **8.** 25 mi. an hr. **9.** $\frac{5}{2}$ or $\frac{2}{5}$. **10.** 30 yd. by 40 yd. and 36 yd. by 50 yd. **11.** 5 in., 12 in. **12.** 1296 men. **13.** 10 da. and 15 da. **14.** 8*d*. **15.** 800 men. **16.** 8 ft., 11 ft.
17. $(\sqrt{5}-1)\div 2$, $(3-\sqrt{5})\div 2$. **18.** $a(\sqrt{5}-1)\div 2$, etc.

EXERCISES CII. Pages 232—234.

A 2. (i) $(x-2-\sqrt{10})(x-2+\sqrt{10})$; (ii) $(x-\overline{2+\sqrt{10}}.y)(x-\overline{2-\sqrt{10}}.y)$. **4.** 0.

B 2. (i) 0, 1; (ii) 0, 0. **3.** $\$(1+r)^n$.

C 2. (i) $x^2-x=0$; (ii) x^2-0. **4.** It does if a is not zero, but does not if a is zero.

D 2. Equal for $x=3$ or $x=1$. **3.** $x^3-12x^2+47x-60=0$. **5.** At end of $1\frac{3}{4}$ sec., also at end of $2\frac{3}{4}$ sec.

EXERCISES CIV. Pages 242—243.

1. Vanishes for $x=1$ or $x=3$; is positive for $1<x<3$; is negative for $x<1$ and for $x>3$; is a maximum for $x=2$. **3.** Vanishes for $x=-1\frac{1}{2}$ or $x=1$; is negative for $-1\frac{1}{2}<x<1$; is positive for $x<-1\frac{1}{2}$ and for $x>1$; has a minimum value $-3\frac{1}{8}$ and this for $x=-\frac{1}{4}$. **5.** Minimum -2. **8.** -1. **9.** 1. **10.** 1, 1.

EXERCISES CV. Pages 244—245.

1. (1) 0; (2) 1; (3) 0; (4) 5; (5) 0; (6) -7; (7) 0; (8) -13. **2.** 5. **4.** $l=3$, $m=-4$ **6.** (1) $x^2+y^2=0$; (2) $(x-2)^2+(y-3)^2=0$; (3) $(x-1)^2+(y-2)^2+(z-3)^2=0$. **11.** 12.

EXERCISES CVI. Pages 246—247.

A 1. $1+x^2+x^4+x^6+x^8$. **2.** $+60$ or -60; $x=3$, $y=4$, $z=5$ and $x=-3$, $y=-4$, $z=-5$. **3.** x; 1; $\frac{1}{x}$. **4.** $-10+6\sqrt{3}$; 8; 0·39.

B 1. 0, 1. **3.** $a^3-b^3-c^3-3b^2c-3bc^2+3c^2a-3ca^2-3a^2b+3ab^2+6abc$; $-a^3+b^3+c^3+3b^2c+3bc^2-3c^2a+3ca^2+3a^2b-3ab^2-6abc$.

C 1. \$109; \$29, \$37, \$43. **2.** $\frac{x-y}{x+y}$. **4.** a or $-(a+1)$. **5.** 7.

D 1. $\sqrt{a^2+b^2+c^2}$. **2.** $(5\pm\sqrt{13})\div 2$; $(11\pm\sqrt{13})\div 2$. **3.** $(x-a)(x-b)(x+2b)$. **4.** $(x-l)^2+(y-m)^2+(z-n)^2=0$. **5.** 8, 12, 15.

EXERCISES CVII. Pages 252—253.

1. (1) $(-3\pm\sqrt{65})\div 4$; (2) $(13\pm\sqrt{37})\div 6$; (3) $(7\pm\sqrt{5})\div 2$; (4) $(4\pm 3\sqrt{-6})\div 5$. **4.** (1) $(6\pm\sqrt{-76})\div 7$; (2) -1, $+\frac{10}{3}$; (3) $(-k\pm\sqrt{k^2-hl})\div h$; (4) $(+k\pm\sqrt{k^2-hl})\div h$; (5) $(-k\pm\sqrt{k^2+hl})\div h$; 6. $(+k\pm\sqrt{k^2+hl})\div h$. **5.** $(31\pm\sqrt{801})\div 2$. **6.** (1) $(1\pm\sqrt{209})\div 2$; (2) $(21\pm\sqrt{401})\div 4$; (3) 0, +6, −6; (4) 8, $-1\frac{1}{7}$; (5) ± 1, ± 8; (6) 3, $\frac{3}{4}$. **7.** (i) $\pm\sqrt{\left(-\frac{c}{a}\right)}$; (ii) 0, $-\frac{b}{a}$; (iii) 0, 0.

EXERCISES CVIII. Page 257.

1. $\frac{99}{49}$, $\frac{832}{343}$, $\frac{7351}{2401}$, $\frac{1287}{343}$, $\frac{495}{343}$, $\frac{13}{5}$, $\frac{99}{25}$, $\frac{832}{245}$, $\frac{7351}{1225}$, $\frac{29}{25}$. **2.** (i) $4x^2-23x+19=0$; (ii) $16x^2-137x+121=0$; (iii) $11x^2+7x-4=0$; (iv) $44x^2+137x+44=0$. **3.** $\frac{b^2-2ac}{a^2}$, $\frac{3abc-b^3}{a^3}$, $-\frac{b}{c}$, $\frac{b^2-2ac}{c^2}$, $\frac{3abc-b^3}{a^2c}$. **4.** (i) $ax^2+(b-8a)x+(c-4b+16a)=0$; (ii) $ax^2+(b-2ah)x+c(-bh+ah^2)=0$; (iii) $cx^2+bx+a=0$; (iv) $ax^2+bhx+ch^2=0$; (v) $a^2x^2-(b^2-4ac)=0$. **5.** $\pm\sqrt{b^2-4ac}\div a$.

EXERCISES CIX. Page 261.

1. (1) Real and unequal; (2) imaginary; (3) imaginary; (4) real and unequal; (5) real and unequal; (6) real and unequal. **3.** The converse is not necessarily true. **5.** $x^2-10x+32=0$.

EXERCISES CX. Page 262.

1. When c and a are of opposite sign. When $b=0$. **2.** Draw AL perpendicular to BC: produce BC to D where LD = BC. Then AD is of length $\sqrt{7}$. **3.** All sets of values which are such as to satisfy the equation. **4.** Sum = 169. **5.** $3(a^2+b^2+c^2-bc-ca-ab)$. **6.** When $c=a$. **7.** $3\pm\sqrt{-2}$; 2. For real values of x, the expression $x^2-6x+11$ cannot be less than zero: therefore, if the equation $x^2-6x+11=0$ is to be satisfied, it cannot be by real values of x.

EXERCISES CXI. Pages 268—269.

1. (1) $(5, 4), (-4, -5)$. (2) $(5, 7), (7, 5)$. (3) $(3, 2), (1\frac{4}{5}, 2\frac{4}{5})$. (4) $(7, 4), (-\frac{44}{49}, -9\frac{9}{49})$. (5) $(4, 3), (-\frac{261}{46}, -\frac{22}{23})$. (6) $(+15, +3), (-15, -3)$. (7) $(4, 5), (0, 0)$. **2.** $\frac{4}{5}, 2\frac{1}{5}$. **3.** $(5, 3), (-3, -5)$. **4.** $(7, 5), (3\frac{1}{3}, 10\frac{1}{2})$. **5.** 8, 5. **6.** 12, 5, 9. **7.** Either 15 yd. by 12 yd. or 20 yd. by 9 yd. **8.** $\frac{2}{3}$ and $\frac{3}{4}$. **9.** $(5, 3)$. **10.** 8 and 13. **11.** 5 ft. by 12 ft. **12.** $-13, +7$. **13.** (1) $(3, 4, 5)$. (2) $(3, 2, 1), (\frac{75}{19}, \frac{164}{19}, \frac{109}{19})$. (3) $(4, 3, 6), (-7, -6, +27)$. (4) $(7, 10, 12), (\frac{39}{11}, \frac{16}{11}, \frac{26}{11})$.

EXERCISES CXII. Page 270.

1. $z=(9x+7)\div10$. **2.** 25, 73, $2x-1$. **3.** Expression $=(x+y+z)^3-(xyz+y^2z+yz^2)-\cdots-\cdots$, it may be interesting to note. **4.** 3·6, 1·4. **5.** 10 min. **6.** $1, -1$.

EXERCISES CXIII. Page 273.

1. (1) $\pm\sqrt{3}, \pm\sqrt{7}$. (2) $\pm\sqrt{(17\pm\sqrt{145})\div2}$. (3) $-5, -3, -1, +1$. (4) $+4, -4, \pm\sqrt{-6}$. (5) $2, -11, -9\pm\sqrt{-159}$. (6) $3\frac{1}{2}, -1\frac{1}{2}, (-2\pm\sqrt{-15})\div2$. (7) $2, \frac{3}{4}, (25\pm\sqrt{-101})\div22$. (8) $3, 5, (-1\pm\sqrt{-239})\div4$. (9) $\pm3, \pm3\sqrt{-1}$. (10) $\frac{2}{3}, \frac{3}{2}, (-85\pm\sqrt{7081})\div12$. **2.** (1) $x^4-10x^2+9=0$; (2) $x^4-14x^2+45=0$; (3) $x^4-22x^3+179x^2-638x+840=0$; (4) $x^4-14x^3+43x^2+42x-216=0$; (5) $x^4-14x^3+70x^2-147x+108=0$.

EXERCISES CXIV. Page 275.

1. (1) $\frac{2}{3}$, $\frac{3}{2}$, $\frac{3}{4}$, $\frac{4}{3}$; (2) $-\frac{2}{3}$, $-\frac{3}{2}$, $-\frac{3}{4}$, $-\frac{4}{3}$; (3) $\frac{2}{3}$, $\frac{3}{2}$, $-\frac{3}{4}$, $-\frac{4}{3}$; (4) $\frac{5}{6}$, $\frac{6}{5}$, $\frac{-1\pm\sqrt{-3}}{2}$; (5) $\frac{4}{5}$, $\frac{5}{4}$, $\frac{-7\pm\sqrt{13}}{6}$. **2.** $\frac{-1\pm\sqrt{-3}}{2}$, $\frac{+1\pm\sqrt{-3}}{2}$. **3.** (1) $240x^4-992x^3+1505x^2-992x+240=0$; (2) $105x^4+82x^3-86x^2+82x+105=0$.

EXERCISES CXV. Page 279.

1. ± 1, $\pm\sqrt{-1}$. **2.** 1, 2, 3. **3.** -1, -2, -3. **4.** a, aw, aw^2. **5.** $(+1\pm\sqrt{-3})\div 2$, $(-1\pm\sqrt{-3})\div 2$. **6.** 1, $(7\pm\sqrt{89})\div 4$. **7.** 3, $-4\pm\sqrt{6}$. **8.** 3, 5, 9. **9.** 2, $2w$, $2w^2$, 3, $3w$, $3w^2$. **10.** 2, 3, $2\pm\sqrt{5}$. **11.** 0, 1, 13. **12.** $(m\pm\sqrt{m^2-4})\div 2$ where $m=(-1\pm\sqrt{5})\div 2$. **13.** $+1$, -1, $(+1\pm\sqrt{-3})\div 2$, $(-1\pm\sqrt{-3})\div 2$.

EXERCISES CXVI. Page 281.

1. $(+3, +5)$, $(-\cdot\cdot, -\cdot\cdot)$, $(+5, +3)$, $(-\cdot\cdot, -\cdot\cdot)$. **2.** $(+2, +3)$, $(-\cdot\cdot, -\cdot\cdot)$, $(+7\div\sqrt{22}, +4\div\sqrt{22})$, $(-\cdot\cdot, -\cdot\cdot)$. **3.** $(+1, +2)$, $(-\cdot\cdot, -\cdot\cdot)$, $(+\sqrt{7}, 0)$, $(-\cdot\cdot, 0)$. **4.** $(+3, +4)$, $(-\cdot\cdot, -\cdot\cdot)$, $(+8\div\sqrt{11}, +6\div\sqrt{11})$, $(-\cdot\cdot, -\cdot\cdot)$. **5.** $(+1, -1)$, $(-\cdot\cdot, +\cdot\cdot)$, $(+3\div\sqrt{17}, +5\div\sqrt{17})$, $(-\cdot\cdot, -\cdot\cdot)$. **6.** $(+5, +7)$, $(-\cdot\cdot, -\cdot\cdot)$, $(+\sqrt{-6}\div 6, +11\sqrt{-6}\div 4)$, $(-\cdot\cdot, -\cdot\cdot)$. **7.** $(+4, +5)$, $(-\cdot\cdot, -\cdot\cdot)$, $(+6\sqrt{3}, -4\sqrt{3})$, $(-\cdot\cdot, +\cdot\cdot)$. **8.** $(+6, +8)$, $(-\cdot\cdot, -\cdot\cdot)$, $(+8, +6)$, $(-\cdot\cdot, -\cdot\cdot)$. **9.** $(+7, +11)$, $(-\cdot\cdot, -\cdot\cdot)$, $(+11\div\sqrt{-3}, -21\div\sqrt{-3})$, $(-\cdot\cdot, +\cdot\cdot)$. **10.** $(+5, +8)$, $(-\cdot\cdot, -\cdot\cdot)$, $(+9\sqrt{3}\div 2, +3\sqrt{3}\div 2)$, $(-\cdot\cdot, -\cdot\cdot)$. **11.** $(+3, +4)$, $(-\cdot\cdot, -\cdot\cdot)$; $m=-2$ does not yield a root. **12.** $(+6, +2)$, $(-\cdot\cdot, -\cdot\cdot)$; $2m=1$ does not yield a root. **13.** $(+3, +2)$, $(-\cdot\cdot, -\cdot\cdot)$, $(+4\sqrt{17}\div 7, +3\sqrt{17}\div 7)$, $(-\cdot\cdot, -\cdot\cdot)$.

EXERCISES CXVII. Pages 282—283.

1. (1) $(+4, +1, -5)$, $(-\cdot\cdot, -\cdot\cdot, +\cdot\cdot)$. (2) $(+5, +7, +1, -\cdot\cdot, -\cdot\cdot, -\cdot\cdot)$. (3) $(\frac{3}{2}, \frac{4}{2}, \frac{7}{2})$, $(0, 0, 0)$. (4) $(3, 5, 11)$, $(303\div 44, 505\div\cdot\cdot, 1111\div\cdot\cdot)$. (5) $(4, 5, 7)$, $(0, 0, 0)$. (6) $(2, 1, 3)$, $(0, 0, 0)$. **2.** (1) $-4 : 43 : 33$. (2) $-4 : 15 : 13$. (3) $7 : 11 : 23$. (4) $113 : -62 : -57$. **4.** $(22, -108, -62)$.

EXERCISES CXVIII. Pages 284—286.

A 1. $3, -1\frac{3}{27}$. **2.** $(1, 2), (-\cdot\cdot, -\cdot\cdot), (7\div\sqrt{-29}, -13\div\sqrt{-29}), (-\cdot\cdot, +\cdot\cdot)$. **3.** $\frac{3}{7}, \frac{7}{3}, -\frac{1\pm\sqrt{-1}}{2}$. **4.** $(\frac{1}{2}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{2}), (\frac{1}{6}, -1), (-1, \frac{1}{6})$. **5.** $(4, 5), (5, 4)$.

B 1. -6. **2.** $(+2, +3), (-\cdot\cdot, -\cdot\cdot), (+8i, -5i), (-\cdot\cdot, +\cdot\cdot)$. **3.** $3, \frac{1}{5}, (3\pm\sqrt{-391})\div 20$. **4.** $(2, 3), (3, 2), (5, 1), (1, 5)$. **5.** $(7, 4), (-4, -7)$.

C 1. $-\frac{8}{5}$. **2.** $(+3, +2), (-\cdot\cdot, -\cdot\cdot)$; $m=-\frac{1}{2}$ does not lead to a root. **3.** $(1, -2, 3)$. **4.** $(1, 4), (4, 1), (w, 4w), (w^2, 4w^2), (4w, w), (4w^2, w^2)$. **5.** $(5, 7, 9), (-5, -7, -9)$.

D 1. $4\frac{1}{2}, 5$. **2.** $(\frac{12}{7}, \frac{4}{7}), (0, 0), (-4, +2), (0, 0)$.
3. $(2, 2), (2, 2), (2+\sqrt{2}, 2-\sqrt{2}), (2-\sqrt{2}, 2+\sqrt{2})$, $\left(\frac{3+\sqrt{-7}}{2}, \frac{3-\sqrt{-7}}{2}\right), \left(\frac{3-\sqrt{-7}}{2}, \frac{3+\sqrt{-7}}{2}\right)$, $(2, 1), (1, 2)$. **4.** $-\frac{2}{3}, -\frac{5}{2}, 5, \frac{1}{5}$. **5.** $(5, 7, 9), (-5, -7, -9)$.

E 1. $(3, 4, 5), (-3, -4, -5)$. **2.** $(5, 7), (7, 5)$, $(-7+\sqrt{-12}, -7-\sqrt{-12}), (-7-\sqrt{-12}, -7+\sqrt{-12})$.
3. $-4, -1, +2$. **4.** $(3, 4)$. **5.** 1.

F 1. $(6, 7, 8), (-12, -13, -14)$. **2.** $(3, 9), (9, 3)$, $\left(\frac{-13+\sqrt{-143}}{2}, \frac{-13-\sqrt{-143}}{2}\right), \left(\frac{-13-\sqrt{-143}}{2}, \frac{-13+\sqrt{-143}}{2}\right)$. **3.** 1, 7, 4, 4. **4.** $(0, 0), (5, 7)$. **5.** 1, $(7\pm\sqrt{13})\div 6$.

EXERCISES CXIX. Pages 287—288.

A 1. $1+2z-3z^2$. **2.** 7. **3.** $(1, 2), (-1, -2)$, $(4\div\sqrt{3}, -5\div\sqrt{3}), (-4\div\sqrt{3}, +5\div\sqrt{3})$. **5.** 40 mi.

B 1. $z+3y-1$. **2.** $-3, -1$. **5.** $(2, 3), (3, 2), (i, -i), (-i, +i)$.

C 1. (i) $(x^2+1)(x-1)(x^2+x+1)$; (ii) $(x-y+z)(2x+y-2z)$; (iii) $(x+y-z)(x+y-2z)$. **3.** $x^2-4x+3=0$.

EXERCISES CXX. Page 292.

1. $a+2b$; $3a-2b$; $7x-5a$; 73; 57; 49; 95; 7·3; 1·7. **6.** x^2+x+1; y^2+2y+3; $3x^2+2ax+5a^2$; $2z^2-3z+4$; 123; 12·3; 538; 5·38; 789. **8.** 2·65. **9.** $1+x$; $1+x+x^2$; $2-3x+5x^2$. **10.** $1+\frac{x}{2}+\frac{3x^2}{8}$.

EXERCISE CXXI. Page 295.

1. $x-a$; $2x+3y$; $3x-4$; $4+5x$; 54; 5·4; 23; 87. **4.** x^2+3x+5; $2-3x+4x^2$; 123; 574. **5.** 1·91. **6.** $1+3x$, $2x+3y$, $3x-4$, 34, 57. **7.** $x+y$, $2x-3$, 23, 31. **8.** 0·35, 0·89.

EXERCISES CXXII. Pages 296—297.

A 1. a, $b+c-a$. **3.** (i) $(2x-3y)(3x-7y)$; (ii) $(a+3b-2c)(a-b+c)$; (iii) $2(a+b)(a^2+b^2+3c^2+2ab)$. **4.** $8\frac{4}{7}$ sec.

B 1. (i) $(y^2+z^2)(y^4-y^2z^2+z^4)$; (ii) $(4y^2+9z^2)(16y^4-36y^2z^2+81z^4)$; (iii) $2(a+2b)(3a+5b)$. **2.** 84. **3.** $\frac{3}{2}$. **4.** $(0, \frac{1}{2}, -\frac{5}{7})$.

C 1. 18. **2.** $(6, 1), (-1, -6)$, $\left(\frac{5+\sqrt{-82\frac{1}{3}}}{2}, \frac{-5+\sqrt{-82\frac{1}{3}}}{2}\right)$, $\left(\frac{5-\sqrt{-82\frac{1}{3}}}{2}, \frac{-5-\sqrt{-82\frac{1}{3}}}{2}\right)$. **4.** $x=5$; 84. **5.** $x\pm1$, $x\pm2$, $x\pm3$, $x\pm4$, $x\pm6$ $x\pm8$, $x\pm12$, $x\pm24$ are all possible linear factors. *The* factors are $(x+2)(x+3)(x-4)$.

EXERCISES CXXIV. Pages 303 – 304.

2. $x-y$; $x-y$; $x-y$. **3.** $32^{\frac{1}{15}}$, $27^{\frac{1}{15}}$. **4.** $\sqrt[6]{125}$; $\sqrt[20]{19683}$; $\sqrt[12]{432}$; $\sqrt[6]{3}$; $\sqrt[12]{128}$. **6.** 1. **9.** $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$; $x^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{1}{4}}+x^{\frac{1}{4}}y^{\frac{2}{4}}+y^{\frac{3}{4}}$; $x^{\frac{4}{5}}-x^{\frac{3}{5}}y^{\frac{1}{5}}+$ etc

EXERCISES CXXV. Pages 309—311.

1. $(15+3\sqrt{7}-5\sqrt{5}-\sqrt{35})\div 4$; $-(8+2\sqrt{15})\div 2$; $(\sqrt{77}-\sqrt{55}+5\sqrt{7}-5\sqrt{5})\div 2$. **2.** 2·02. **3.** $\sqrt[6]{72}$; $\sqrt[12]{16875}$. **4.** 59. **5.** $1+\sqrt{3}$; $4+\sqrt{7}$; $5-\sqrt{11}$. **6.** $\sqrt{5}+\sqrt{7}$; $\sqrt{11}+\sqrt{6}$; $\sqrt{13}-\sqrt{5}$; $\sqrt{5}-\sqrt{2}$; $\sqrt{7}-\sqrt{3}$; $3+\sqrt{5}$. **7.** $\sqrt{6}+\sqrt{8}$; $\sqrt{15}-\sqrt{6}$; $\sqrt{12}+\sqrt{18}$, $\sqrt{8}-\sqrt{5}$; $\sqrt{10}+\sqrt{6}$; $\sqrt{27}-24$. **10.** $\sqrt{3}$. **11.** (1) 10. (2) 7. (3) 6. (4) 4 or $-\frac{10}{3}$. (5) $-b$. (6) $\frac{3}{16}$. (7) 0. (8) $4(a+b)$. (9) 3 or -32. (10) $\pm\frac{13}{14}$. (11) ± 3. (12) 5 or -8. (13) 2 or $-\frac{1}{2}$. (14) $(4\pm\sqrt{280})\div 6$. (15) $(p-q)^2\div 2q$. **13.** $(\sqrt{5}+\sqrt{6}-\sqrt{7})(\sqrt{5}-\sqrt{6}+\sqrt{7})(\sqrt{5}-\sqrt{6}-\sqrt{7})$. **14.** $a^{\frac{1}{3}}-1+a^{-\frac{1}{3}}$. **15.** $x^2-10x+18=0$. **17.** $\sqrt{6}+\sqrt{8}+\sqrt{10}$.

EXERCISES CXXVI. Pages 312—318.

A 1. (i) $x(x+6)(3x-4)$; (ii) $(2x-3y+5z)(x+2y-3z)$; (iii) $(a+b-c+d)(a-b+c+d)(b+c-a+d)(b+c+a-d)$. **2.** -3; -24. **3.** $x^{\frac{4}{2}}+x^{\frac{3}{2}}y^{\frac{1}{2}}+x^{\frac{2}{2}}y^{\frac{2}{2}}+x^{\frac{1}{2}}y^{\frac{3}{2}}+y^{\frac{4}{2}}$. **4.** 2, $\frac{1}{2}$, $(1\pm 2\sqrt{-6})\div 5$. **5.** $50\frac{5}{8}$ ft. by $50\frac{5}{8}$ ft.

B 1. $\left(\dfrac{5\pm\sqrt{5}}{2}, \dfrac{5\mp\sqrt{5}}{2}\right)$, $\left(\dfrac{5\pm\sqrt{125}}{2}, \dfrac{5\mp\sqrt{125}}{2}\right)$. **2.** 4; 24. **4.** 1, -1.

C 1. 2, $\frac{1}{3}$. **2.** $(x+a+b)(x+a-b)(x-a+b)(x-a-b)$. **3.** $x=4$. **4.** $-\frac{25}{72}$. **6.** (i) $(\overline{m+nx}+\overline{m-n})(\overline{m-nx}+\overline{m+n})$. (ii) $(x+1)(x-2)(x-7)(x-10)$. **7.** (3, 4), $(-\frac{4}{3}, \frac{3}{4})$.

EXERCISES CXXVII. Pages 320—321.

1. lmn. **3.** $a^3=c+3a$. **4.** $a^4=d+4a^2-2$. **5.** $(ar-cp)^2-4(br-cq)(aq-bp)=0$. **6.** $a^3+2c^3=3ab^2$. **7.** $\dfrac{a}{p}=\dfrac{b}{q}=\dfrac{c}{r}$. **8.** $c=a$. **9.** $mn-a=0$, or $n=\dfrac{m}{a}$. **10.** $l^2m^2=a^2(l^2+m^2)$, or $1=\dfrac{a^2}{l^2}+\dfrac{a^2}{m^2}$. **11.** $a^2l^2+b^2m^2=1$. **12.** $\dfrac{a}{r}=\dfrac{b}{q}=\dfrac{c}{p}$. **13.** $\dfrac{1}{a+1}+\dfrac{1}{b+1}+\dfrac{1}{c+1}+\dfrac{1}{d+1}=3$.

EXERCISES CXXVIII. Pages 322—328.

A 1. $97x^2 - 53x - 17 = 0$. **2.** $2(-1 \pm \sqrt{-3})$. **4.** $x^3 + 2x^2 - 3x + 1$. **5.** (1, 2, 3), (3, 2, 1), $\{(6 \pm \sqrt{-3}) \div 3,\ (6 \mp 2\sqrt{-3}) \div 3,\ (6 \pm \sqrt{-3}) \div 3\}$, $(-1, -2, -3)$, $(-3, -2, -1)$, $\{(-6 \pm \sqrt{-3}) \div 3,\ (-6 \mp 2\sqrt{-3}) \div 3,\ (-6 \pm \sqrt{-3}) \div 3\}$.

B 1. For $x = \frac{2}{5}$ and $3\frac{4}{13}$; for $\frac{2}{5} < x < 3\frac{4}{13}$; for $x < \frac{2}{5}$ and for $x > 3\frac{4}{13}$. **3.** Each = sum of x and y multiples of numerators divided by the x and y multiples of denominators, etc. **5.** 72.

C 1. $(x-y)(x+y)(x+y-a)(x+y+a)$. **2.** 1. **3.** 3, $(3 \pm \sqrt{3}) \div 2$. **5.** $33\frac{1}{3}$ mi. an hour; $48\frac{1}{3}$ mi.

D 1. (i) $(x+3+\sqrt{2})(x+3-\sqrt{2})$; (ii) $(x - \frac{5}{2} + \sqrt{\frac{13}{4}})(x - \frac{5}{2} - \sqrt{\frac{13}{4}})$. **2.** A circle of radius 5. **4.** b is zero, and then the common factor is x, or b is 9 and then the common factor is $x + 3$. **5.** 5.

E 2. (i) $\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)$ of work; (ii) $\frac{lmn}{mn + nl + lm}$ da. **3.** (2, 1), $(-2, -1)$, $(\pm 6 \div \sqrt{569},\ \mp 25 \div \sqrt{569})$. **4.** (i) $(2x - 3y - 5z)(x + 5y - 3z)$; (ii) $(x+1)(x^2 + \overline{p-1}x + 1)$.

F 1. 8 mi. and 10 mi. an hr. **3.** 1, $c(a-b) \div a(b-c)$. **4.** (i) $x^3 + 9x^2 + 23x + 15$; (ii) $x^2 - 2bx + b^2 - y^2 + 2ay - a^2$.

G 1. $(4, -3)$, $(-3, 4)$. **3.** 5, 7. **4.** (i) $(5x - 2y)(2x + 3y)$; (ii) $(3x - 2y + z)(x + 2y - 3z)$. **5.** $5(x-3)^3 + 37(x-3)^2 + 148(x-3) + 91$.

H 1. $6\frac{1}{4}$. **2.** Not true of $\frac{1}{4}$. **3.** Equals 4. **4.** $\left(\frac{a}{2}, \frac{b}{3}\right)$.

I 3. \$1.00 and $6\frac{2}{3}$c. a lb. **4.** $m = 3$ and the two roots are 3 and 3, or $m = 5$ and the two roots are 5 and 5.

J 1. (i) $(a + b + c) \div 3$; (ii) $\sqrt[3]{\frac{2401}{81}}$. **2.** 27. **4.** (i) $(ax - b)(cx^2 - dx + c)$; (ii) $(x-1)(x+2)(x-4)(2x+3)$. **5.** $-\frac{9}{64}$; minimum for $x = \frac{9}{8}$.

K 2. $(+5, +7)$, $(+7, +5)$, $(-5, -7)$, $(-7, -5)$. **3.** The latter excludes the solution $(x = 0, y = 0)$ of the former. **5.** $(x-1)^5 + 5(x-1)^4 + 10(x-1)^3 + 10(x-1)^2 + 5(x-1)$.