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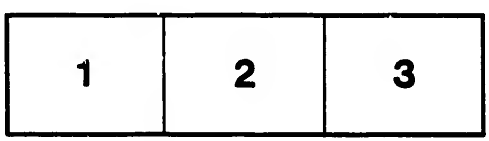
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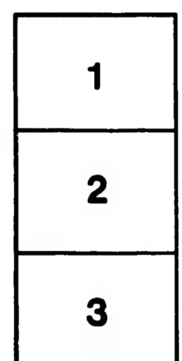
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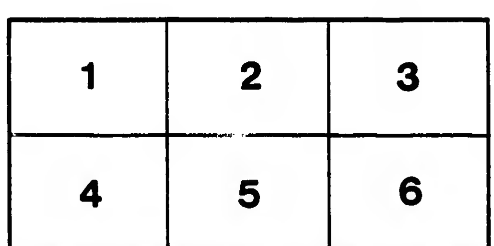
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GEOMETRICAL SOLUTIONS

OF THE LENGTHS AND DIVISION OF

CIRCULAR ARCS,

THE

QUADRATURE OF THE CIRCLE, TRISECTION OF
THE ANGLE, DUPLICATION OF THE CUBE,

AND THE

QUADRATURE OF THE HYPERBOLA.

BY
PETER FLEMING, CIVIL ENGINEER,
AUTHOR OF "A SYSTEM OF LAND SURVEYING," "A METHOD OF MEASURING A BASE LINE BY ANGULAR OBSERVATION," AND "GEOMETRICAL SOLUTIONS OF THE QUADRATURE OF THE CIRCLE."

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PREFACE.

In treating of Geometrical Solutions, of the Quadrature of the Circle, the perimeters of regular polygons of a given circle are made the chords of another given arc, having one common point on that arc—that is, the first is the perimeter of the least polygon or inscribed square (triangle) and by doubling the number of sides of this polygon, the perimeter of which becomes the second chord from the same point; and by thus successively doubling the number of sides, it is evident the perimeter of the ultimate polygon will be equal to the circumference of the given circumscribing circle. In the same manner, taking the circumscribing and corresponding polygonal perimeters, that the ultimate polygonal perimeter will also come to be equal to the same circumference. Or let p be equal to the perimeter of the inscribed square, (triangle) and be the first term of an infinite series—and p' the perimeter of the polygon double of the number of sides of the first—and p'' that of the polygon of double the number of sides of the second &c.; then $p'—p$ will be the second term, $p''—p'$ the third term, and $p'''—p''$ fourth term, &c. *ad infinitum*. In the same manner, let q be equal to the perimeter of the circumscribing square (triangle) of the same circle, be the first term of an infinite series, q' that of the polygon of double the number of sides of which q is the perimeter; then $q'—q$ will be the second term, $q''—q'$ the third term, and $q'''—q''$ the fourth term, &c. *ad infinitum*; and the first will be a converging infinite series of the form $p+(p'—p)+(p''—p')+(p'''—p'')+$ &c., and the latter will also be a converging infinite series of the form $q—(q'—q)—(q''—q')—(q'''—q'')—$ &c., and each series will be equal to the circumference of the given circle; but as the first series is an increasing one, and the other a diminishing one, it is evident that the ultimate chord of chords p, p′, p″ &c. *ad infinitum*, and the ultimate chord of q, q″, q″, &c. *ad infinitum*, must meet in a common point n on the arc, and this point joined to the common point A of meeting of all the chords p, p′, p″, and q, q′, q″; the line An, (fig. 8, Geometrical Solutions of the Quadrature of the Circle) must be the ultimate chord of both, and equal to the circumference of the given circle; by which it is resolved into a straight line. Such is the result presumed to be obtained by each of the Nine Geometrical Solutions of the Quadrature of the Circle, which are the Geometrical Summations of the infinite series, derived from the differences of polygonal perimeters equal to the circumference.

In the present work we find the lengths of circular arcs, by the same geometrical summation of the series derived from their perimeters—that is, by regular polygonal division of the arc, giving both inscribed and circumscribed perimeters, by which the length of the arc in the same manner is found; and if the arc is made a known part of the circumference, the length of the whole circumference, consequently is a multiple of the arc given in a straight line, as exemplified by propositions, 1 and 2.

The division of circular arcs, or of the whole circle, into any number of equal parts, are resolved in the same manner, by infinite series derived from the number of divisions required, as exemplified in propositions, 3, 4, and 5, and by which any equal division of circular arcs, or of the whole circumference may be obtained. Consequently the *Trisection of the Angle* comes only to be one example, of an indefinite number, of the same general application.

The *Duplication of the Cube* and the *Quadrature of the Hyperbola*, appear to be no more than examples of the same application; or the geometrical solutions of summation of infinite converging series, of which the form is alternately plus and minus in its terms—for the duplication is the summation of the Binomial Theorem $(a+x)^{\frac{1}{3}}=\sqrt[3]{2}$, and is only an example of an indefinite number, of which a power or root may be in the same manner represented by a straight line. Also, the Quadrature of the Hyperbola is only the geometrical solution of summation of a series of the same form, as shown by propositions, 6, 7, and 8.

It may now appear, by both the Geometrical Solution of the Quadrature of the Circle, and those of this work, that the whole of the hitherto unresolved problems of the ancients, namely, the Quadrature of the Circle, Trisection of the Angle, Duplication of the Cube, Quadrature of the Hyperbola, and Lengths and Division of Circular Arcs, which have vainly been attempted to be solved by finite lines or expressions sought for upon the supposition of distinct or unconnected principles for each, both by geometry and analysis, belong only to one of general application to all,—which is that of the *Geometrical summation of infinite converging series.*

It may farther appear, that the finite solutions of these ancient problems, if thus depending upon such common basis—*the geometrical summation of infinite series*—none of them could ever be found singly, or by a different construction for each; and consequently, without this common basis, or common construction being discovered, they all might still and forever remain unsolved. But even under this view of the basis known, the duplication of the cube may have remained unsolved, without the discovery of the Binomial Theorem due to Sir Isaac Newton; although in all probability, from the advancements made in analysis since his time, the discovery of this series would be an ordinary result. All which may well account for the mysterious obstacles which have so long lain in the way of all past attempts, either by geometry or analysis, to solve them; and by which these problems remained to be an open challenge to the intellectual capacity and ingenuity of past ages to remove.

The pretension of solving geometrically the Summation of Infinite Series, as given in the Geometrical Solutions of the Circle and the present work, is wholly based upon the variation or movement of the points representing the differences of the chords, or perimeters on a given arc, which differences give the terms of two infinite series, each of which evidently must vanish at a common point n; and that there must be made a point of variation of each term, or of the points of one of the series through which arcs of intersection are made, with arcs described through points of the opposite series, that those intersections continued *ad infinitum* must be on a straight line, and this line must meet the arc in that vanishing point, (" Geometrical Solutions of the Quadrature of the Circle,', Lemma 3, 4, 5, &c.) and Lemma ; by which it is evident, the chord of this point will be common to both series, and be without any approximation; therefore, this construction will become of general application, to find in a straight line the sum of any two converging series whatever, if connected by a common law, or of that in which its terms are alternately plus and minus.

PETER FLEMING.

Montreal, December, 1850.

GEOMETRICAL SOLUTIONS

OF THE LENGTHS AND DIVISION OF

CIRCULAR ARCS, &c.

LEMMA 1.

FIG. 1. With any radius CA, describe the semicircle ABDE, and make the chords AB and BD, each equal to the radius AC. Also describe from the point A, through C, the arc BC, and from B, the arc ACD, and bisect the arc CD in the point D′, and join A and D. Then from any number of points a, b, c, d, e, f, &c,. of the arc BD, make on the arc BC, the arc Ba′ equal to the arc B a,—the arc B b′ equal to the arc B b,—the arc B c′ equal to the arc B c, &c,—and from the point A, as a centre, with the distance A a, describe the arc a m; and from the point C through a′, describe the arc a′ m, intersecting the arc a m in the point m. In the same manner, from the points A and C as centres, describe the intersections n, o, p, q, r, &c., and through these intersections, which let it be granted are of indefinite number, draw the line B, m, n, o, p, q, r, &c, cutting

the arc CD;—This line shall be a curved line passing through the point D′.

For make the arc BI equal to the arc BB′; the arc CD′ is by construction, equal to the arc CB′, and BB′ is equal to CD′—hence the point D′, must be on the intersection of the arcs B′D′ and ID′—consequently the point D′, must be on the line Bmnopqr, &c.—But the radii of each intersection are unequal, and each intersection of different radii—therefore the line Bmnopqr, &c, must be a curved line, cutting the arc CD in the point D′.

LEMMA 2.

FIG. 2. From the point A as a centre, describe through B the arc BCA″, and from D′ as a centre describe through B the arc BA′A″ meeting the arc BCA″ in the point A″—and from A″ as a centre describe the arc AD′, cutting the arc BA′ in the point C′, and the arc BC in the point B, and from C′ through D′ describe the arc A″D′H; and join A′H, which by construction must pass through the point B′. Next make the arcs Ba and Ba′ equal to each other (Lemma 1)—and Bb′ equal to Bb &c, and through the points a and a′, b and b′, c and c′, &c, describe from C and A, the intersections

m,n,o,p, &c. Also make D′a′ equal to D′a, D′b′ equal to D′b &c., and through the points a and a′, describe from C′ and A″ as centres the intersection m′—In the same manner describe the intersections n′, o′, p′, &c,—It is evident by construction that a line, drawn through the points B, m, n, o, p, &c, and a line drawn through the points D′, m′, n′, o′, p′, &c. must meet in a common point G, on the straight line A′B′H ; for the arcs BH and BB′, are symmetrical with the arcs D′H and D′B′, to the straight line A′B′H. The points B, m, n, o, p, &c, and the points D′, m′, n′, o′, p′, &c., shall be on the arc of a circle described through the points BGD′.

From the points C and C′, through G, describe the arcs G e and G e′, and from the points A and A″, describe the arcs Ge″ and Ge‴,—by construction Be is equal to Be″, and D′e′ is equal to D′e‴.

Describe through the points BGD′ (constructed as fig. 2) the arc D′GBb, FIG. 3. and from G as a centre, with the distance D′B (equal AC) describe the arc b a″ r, intersecting the arc D′GBb in the points b and r,—and from b with the same radius (AC), through G describe the arc Ga, intersecting the arc b a″ in the point a′. Next on the arc b a″ r, make the arcs b c′ and r q, each equal to the arc BC′ or D′C, and with the radius AC, describe through G and c′, the arc Gc′a, meeting the arc Ga′a in the point a, and intersecting the arc BB′C in the point d, and through G and q describe the arc Gq intersecting the arc AB′D′ in the point d′. Also on the arc Ga′a, make the arc Gc equal to the arc D′C, and through the points b and c, describe the arc bc, intersecting the arc Gc′a in the point b′, and make pq′ equal to BC′, and through q′ and B describe the arc Bq′, intersecting the arc bb′c in the point b″; then from c, through a and b, describe the semi-circular arc a b e, and draw through the points a and c the straight line a c e, meeting the semi-circular arc in the point e. Also from q describe the arc Gh—from c′ describe the arc Gh′H′,—and from q′ the arc Bh″.

By construction the curvilinear angle d′Gg′, is equal to the curvilinear angle d′D′g. The curvilinear angle dBg, is equal to the curvilinear angle dGg, and the curvilinear angle b″ b g″ is equal to the curvilinear angle b″Bg′ —also by the same the angle g″ b h″ is equal to the angle g″Bh″— gBh′ to gGh′—g′Gh to g′D′h. Hence it is evident that the straight line Lh, must pass through the intersection d′, and bisect the arc D′G, in the point g′, and Lh′ bisect the arc GB, in the point g, and Lh″ bisect the arc bB in the point g″ (the point L, the centre of the arc bBGD′.)

Also by construction, the arc BGD′, is related to the semi-circle ABDE, the same as the arc bBG is related to the semi-circle abH′e; and the corresponding centres of construction in each, are in position symmetrical and equal—that is, the point a, corresponds to the point A—the centre c, to the centre C—and c′ to C′—b′ to B′ &c. Also the curvilinear triangle bb′G, is similar and equal to the triangle BB′D′ — the triangles bb″B — BdG — Gd′D′, are similar and equal to each other, and each similar to the triangle bb′G or to BB′D′.

Again, from the centre C, through the point g, describe the arc gi′, and from A describe the arc g i; also from c′ describe the arc gf′ and from q the arc gf. Now, by construction the quadrilateral curvilinear figure BdGh′, is similar to the quadrilateral curvilinear figure BB′D′H of which D′e′ is equal to De‴, and Be″ is equal to Be—(fig. 2,) Therefore, Gf′ must be equal to Gf, and Bi′ equal to Bi. Also Gf must be equal to D′i, and Gf′ equal to Di′—Bi equal to Gl, and Bi equal to Gl′—and the same of bm equal to bm′, Bn equal to Bn′, and the whole of the corresponding arcs of the quadrilateral figures bh″Bb″—Bh′Gd — and GhD′d′ must be equal to each other; but the point g′ by construction is a point of the curve D′G (fig 2.) and the point g is a point of the curve BG. Also G is a point of the curve Bgg′D (fig. 2.) but by construction the line BgGg′D′ is the arc of a circle described from the centre L—hence the points D′, g′, G, g, B, must be on a circular arc.

LEMMA 3.

FIG. 4. Let any arc An be equal to the sum of any converging series—that is the arcs A1 + 12 + 23 + &c. the first, second, third, &c. terms. Also the arc An, equal to, or the sum of another series, and the arcs A1′—1′2′—2′3′—&c. the first, second, third, &c. terms; but the one series varying as the other varies or dependant upon each other by the same law, and consequently the terms of each equally vanishing at the point n.*

* Instance such series may be formed, by the chord A 1 made equal to the perimeter of the inscribed

From any point H with the radius AC, describe the arc FCA′, and bisect the arc CF in the point D″, and with the radius AC through H and D″, describe the arc HD″, and bisect the arc HD″ in the point G′, and from the middle of the arc of the two last terms, as 33′, 44′, 55′, &c. with the distance CG′, cut the arc nE in the point D, and from D with the distance AC or radius, cut the arc nA in the point H′, and from H′, with the same radius describe the arc DCA″. Next bisect the arc CD in the point D′, and with the radius AC, describe the arc HD′, and bisect H′D′ in the point G.

Again from the point A′, describe through the points H′ 1, 2, 3, &c. the arcs H′C, 1t, 2s, 3r, &c. Next make the arcs H′1″, H′2″, H′3″, &c, equal to H′1, H′2, H′3, &c, and from the centre C des cribe through the points 1″, 2″, 3″, &c., the arcs 1″t′, 2″s′, 3″r′, &c., making the intersections d,e,f, &c., and through H′, d,e,f, &c., describe the circular arc H′G D′. The points of intersections, d,e,f, &c., shall all be only on the circular arc

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square or polygon of four sides of a given circle—the chord A2, equal to that of eight sides, &c.—and the chord A1′ equal to the perimeter of the circumscribing polygon of four sides, and the chord A2′, equal to that of eight sides, &c. (See Geom. Sol. of the Circle.)

(Lemma 2) H′G, described from the centre L or on half of the arc H′D′.

For from the centre C, with the distance CG, describe the arc Gg, meeting the arc H′C in the point g, and let H′h equal Hn, and because (always supposed) 1′n is greater than 1n, the arc 3′n is greater than 3n—hence Hg must be greater than Hn′—therefore all the intersections d, e, f...n must be on the circular arc H′G (Lemma 2, fig. 2) or between the points H′ and G.

LEMMA 4.

FIG. 4. Through the intersections d,e,f, &c., describe the circular arc H′GD′ from the centre L (Lemma 2.) and from the point D′ through the points of intersection e″ f″ &c., the arcs e″e′, f″f′ &c. And from thè point A′, describe through the points e f′ g′, &c., the arcs e′x, f′y, g′z, &c., meeting the arc Hn, in the points x,y,z, &c.

The arc 1x shall be the variation of the point 1 towards the point 2, and the arc 2y shall be the variation of the point 2 towards the point 3, &c. (Lemma 9, Solutions Quadrature of the Circle). Then from the point D, through the point x, y, z, &c, and from the point H′, through the points 2′, 3′, 4′, &c. describe the intersections a, b, c, &c. The points a, b, c, shall be on one straight line a b c...n.

For from the point H′ through the points 1′, 2′, 3′, &c. and the point D through the points 1, 2, 3, &c. describe the intersections *l*, *a*, *b*, *c*, &c. and draw the curve line *l a b c*...n. Also, continue 2′ a to meet *l* 1 in the point *a′*— and 3′ *b* to meet 2 *a* in the point *b′* &c., and draw the curve line *a b c*,...n.— Now by construction the intersection *a′* has moved to be the point *a*—the intersection *b′* has moved to be the point *b*, and c′ has moved to be the point c, &c. on the arcs 2′*a′*, 3′*b′*, 4′*c′*, &c. and the ultimate intersection of each arc *a′b′c′*, *a,b,c*, a,b,c, must be in the point n, (Lemma 9, Quad.) But if the points x, y, z, &c., be not the points of variation by which the intersections a, b, c, &c. shall be on one straight line with the point n—let *p*, *q*, *r*, &c. be the true points—that is 1 *p* be the variation of 1 towards 2—2 *q* that of 2 towards 3, &c., and on the arc H′C, make the arc H′*p′* equal to the arc H′*p*–H′*q′* equal to H′*q*— H′r′ equal to H′r, &c., and through the points *p* and *p′*, from the points A and C, describe the arcs *p m′* and *p′ m′*, which by construction must meet in the point m' on the circular arc H′ G D′ (Lemma 2.) In the same manner describe the arcs $q\,n$ and $q'\,n'$—r o and r' o′ &c.—and from the point D′ describe through the points m', n', o', &c—the arcs $m'\,m''$, $n'\,n''$, $o'\,o''$, &c*—and from C through the points m'', n'', o'', &c. describe through the point m'' &c.* the arcs p″ m″ u &c.—hence 1″ p' should be the variation of 1″ towards 2′—and in the same manner 2″ q' towards 3″ &c. But the arc 1 p is made the variation of 1 towards 2, &c., but now 1″p is the variation p″ to 1″ and not the same ratio of 1 p to 12, &c., which by construction should be equal; for H′p'' should be equal to H 2, —for H 2″ is equal to H 2—therefore 1 x is the true variation of 1 towards 2, &c., and through which the intersections a,b,c, &c., must be in one straight line with the ultimate intersection n on the arc B D (Lemma 9. Sol. Quad.)

Corrollary,—It is evident if the points 1, 2, 3, 4, are laid off from the point A by the chords A1, A2, A3, &c. that the chord An will be equal to the sum of infinite terms, the differences of the chords—and if supposed on the arc A H′ n, the length of the arc An will

* The point n″, o″, &c. are not shown on the figure on account of the smallness of the scale.

be equal to the sum of the infinite series, or arcs A 1 + 1 2 + 2 3 + 3 4, &c.

SOLUTIONS.

CASE FIRST.—*To find the length of any circular arc in a straight line.*

PROPOSITION 1.

PROBLEM.

Let Almnl be a Quadrantal arc. Draw the chord A 1, and divide the arc Almnl by continued bisections into regular inscribed and circumscribing sides—first into two sides A m, m 1, and c d, d e—second into four sides A l, l m, m n, n 1 and f g, g h, h i, i k &c. which will be the inscribed and circumscribed perimeters of the arc A l m n 1. Then from the point A make A 1 the inscribed perimeter of one side, and the chord A 2 equal to the inscribed perimeter of two sides—A 3 equal to the inscribed perimeter of four sides, &c. Also A 1′ the diameter of the semi-circle equal to ab the circumscribing perimeter of one side—and A 2′ equal to the circumscribing perimeter of two sides, &c. FIG. 5.

Then from the point I (Lemma 3) through the points B, 1,2,3, &c. describe the arcs BC, 1t,2s,3r, &c.—and on the arc B C, make the arc B1″ equal to the arc B1,—B2″ equal to B2—B3″ equal to

B3, &c.—and from the point C, describe through the points 1″, 2″, 3″, &c. the arcs 1″t′, 2″s′, 3″r′, &c. making the intersections d,e,f, &c. Also through the intersections d,e,f, &c. from the centre L (Fig. 2) describe the arc B d e f...G D′, bisecting the arc B C in the point D′, (Lemma 1 and 2.) Again from the point D′, through the intersections e″, f″, &c., describe the arcs e″e′, f″f′, &c.—and from the point I, through the points e′, f′, &c. describe the arcs e′x, f′y, &c. meeting the arc BD, in the points x, and y, &c.

From the point B, through the points 2′,3′, &c., and from the point D, through the points x, y, &c. describe the intersections a,b,c, &c.—and through a,b,c, &c. draw the straight line a b c &c. meeting the arc B D in the point n, and join A n. For that a b c...n is a straight line (Lemma 4)—let A1 = x,—A2 = x′, —A3 = x″ &c—then A n, must be equal to the sum of the series $x + (x' - x) + (x'' - x') + (x''' - x'') +$ &c. Therefore the distance A n must be equal to the perimeter of infinity of sides on the arc A l m n 1, or equal to the Quadrantal arc.

Corrollary,—Hence the Quadrature of the Circle; for four times A n, must be equal to the whole circumference of the circle of the diameter A1′.

NOTE.—In the case of the given arc being greater than the Quadrantal arc, or greater than the semi-circular arc—then make the first perimeter, be from bisection of the arc into two sides, and the second from bisection of the first, and the third from bisection of the second, &c.—and make the diameter of the semi-circle or A1′, equal to the first or greatest circumscribing perimeter, and construct by this proposition, Fig. 5, and (Lemma 3.)

PROPOSITION 2.

PROBLEM.

Of the length of the circumference or Quadrature of the Circle.

Let the circle A B C be given, and inscribe and circumscribe regular polygons of four, eight, sixteen, &c., sides, and with half of the perimeter of the circumscribing square, from the centre C, describe the semi-circle A...1′ —then from the point A, make the chord A1, equal to the inscribed perimeter of four sides, A2 equal to the perimeter of eight sides, A3 equal to the perimeter of sixteen sides, &c. Also make the chord A2′ equal to the circumscribing

FIG. 6.

FIG. 7

perimeter of eight sides—A′3′ equal to the perimeter of sixteen sides, &c., and describe the arcs F C F′ and D C I, and and from I describe through B the arc B C, (Lemma 3.) Next from I describe the arcs 1t, 2s, 3r, &c., and from the center C, describe the arcs 1″t′, 2″s′, 3″r′, &c. making the intersections d,e,f, &c., and describe the points of variation x, y, z, &c. (Lemma 3 and 4) on the arc B D. Then from the points B and D, through the points 2′, 3′, &c., and through the points x, y, z, &c. describe the intersections a, b, c, &c. and draw the straight line a b c.. &c. meeting the arc B D, in the point n, and join A and n. The distance A n shall be the determinate length of the circumference of the given circle A B C (Fig. 6.) The demonstration the same as Prop. 1.

Corrollary,—It is evident that the distance A n may be equally derived from the perimeters by division or bisection of the radius of the given circle, only the semi-circle A...1′, would be of diameter equal to the circumscribing triangle of the given circle A B C.

Case First.—*The Division of Circular Arcs and the Circle into equal Arcs.*

LEMMA A

Let the arc H F be divided into n equal arcs, and the arc H n be the unity of division, consequently the arc F n will be equal to n—1 parts. Now let the arc H n, be equal to the Geometrical series $x + \frac{x}{n-1} + \frac{x}{(n-1)2} + \frac{x}{(n-1)3} + \frac{x}{(n-1)4}$ &c.; then the arc F n, will be equal to $(x + \frac{x}{n-1} + \frac{x}{(n-1)2} + \frac{x}{(n-1)3} + \frac{x}{[n-1]4} +$ &c.) n—1; for the whole arc F n must be equal to H n [n—1.] Then let the arc H 1 be equal to x, the arc 1 2, equal to $\frac{x}{n-1}$, the arc 2 3, equal to $\frac{x}{[n-1]2}$, the arc 3 4, equal to $\frac{x}{(n-1)3}$, the arc 4 5 equal to $\frac{x}{[n-1]4}$ and &c. Also the arc F 1′ must be equal to x [n—1]—the arc 1′ 2′ equal to $\frac{x(n-1)}{n-1} = x$,—the arc 2′ 3′ equal to $\frac{x(n-1)}{(n-1)2} = \frac{x}{n-1}$, the arc 3′ 4 equal to $\frac{x(n-1)}{x(n-1)3} = \frac{x}{(n-1)2}$, the arc 4′ 5′ equal to $\frac{x(n-1)}{(n-1)4} = \frac{x}{(n-1)3}$, &c., hence the arc F n, will be equal to the series

$$x\,[n-1] + x + \frac{x}{n-1} + \frac{x}{(n-1)2} + \frac{x}{(n-1)3} \text{ \&c.}$$

FIG. 8.

From the points B and D, [Lemma 3.] as centers, through the points 1′ 2′ 3′ &c., and through the points 1.2-3. &c. describe the intersections *a.b.c*, &c. and draw the line *a b c* &c., and also through the intersections *a′,b′,c′*, &c. draw the line *a′b′c′* &c, The lines *a b c*, and *a′b′ c′* &c. shall meet in the point n, on the arc H F.

For by construction the arc H n is equal to the series $x + \frac{x}{n-1} + \frac{x}{(n-1)2} + \frac{x}{(n-1)3} + \frac{x}{(n-1)4} +$ &c.—and F n is equal to the series $x\,[n-1] + x + \frac{x}{n-1} + \frac{x}{(n-1)2} + \frac{x}{(n-1)3} + \frac{x}{(n-1)4} +$ &c.— Hence by leaving out the first term x [n—1] of the latter series, equal to the arc F 1′, we have the arc H n equal to the arc 1′n, consequently the term or arc 1′2′ is equal to H 1, the arc 2′3′ equal to the arcs 1 2, the arc 3′4′ equal to the arc 2 3, the arc 4′5′, equal to the arc 3 4, &c., therefore the series of H n is the same as that of 1′n, and each must approach by equal terms or arcs to the common vanishing point in n; therefore the curve lines *a b c* &c. and *a′b′c′* must meet the arc H F, in the point n.

Corrollary,—Let the series $x' + \frac{x'}{n-1} + \frac{x'}{(n-1)2} + \frac{x'}{(n-1)3} + \frac{x'}{(n-1)4}$ + &c. be greater than the arc H n, then the series $x'[n-1] + x' + \frac{x'}{n-1} + \frac{x'}{(n-1)2} + \frac{x'}{(n-1)3}$ + &c. will be greater than F n. It is evident that there cannot be an ultimate intersection in the arc or point n, for n cannot be the vanishing point of either series nor can there be a common vanishing point; but let the series $x'' + \frac{x''}{n-1} + \frac{x''}{[n-1]2} + \frac{x''}{[n-1]3} + \frac{x''}{[n-1]4}$ + &c. be less than the arc H n or equal to H t, and consequently $x''[n-1] + x'' + \frac{x''}{n-1} + \frac{x''}{[n-1]2} + \frac{x''}{[n-1]3} + \frac{x''}{[n-1]4}$ + &c. less than F n, or equal to F t′. It is also evident that the ultimate intersection described from B and D, through the points t and t′ must be without the arc H F, and be in the point t″.

LEMMA B.

Let the arc H F, to be divided into n equal parts. Divide the length of the FIG. 8.

arc in a straight line into n equal parts [Case 1.] Then through any point R of the same arc, draw the tangent P Q and make R P and P Q each equal to the half of the nth part of the straight line which is equal to the arc H F, and join PC and QC, intersecting the arc in the points U and V, and join U and V. The distance U V, shall be less than the chord of the nth part of the arc H F.

For the arc U V is less than P Q, and the chord U V is less than the arc U V, therefore the chord U V must be less than the chord of the nth part of the arc H F.

PROPOSITION 3.

Problem.

Division of an Arc into three equal parts, or the Trisection of an Angle.

FIG. 8. Let H F be the given arc to be trisected. Describe the distance P Q [Lemma B.] and make the chords H1 and 1t, each equal to the half of U V—and let it be supposed H n is equal to the third part of the arc H F—then H n, will be equal to the series $x + \frac{x}{n-1} + \frac{x}{[n-1]\,2}$ &c. and the arc Fn, will be equal to $x\,[n-1] + x + \frac{x}{n-1} + \frac{x}{[n-1]\,2} +$

&c. [Lemma A.] Now on both arcs make $x = 1$, and $n = 3$, we have for the value of $H n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c.$, and $F n = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c.$, but as the value of x is unknown in relation to H n or F n, make Ht equal to the infinite series $1' + \frac{1'}{2} + \frac{1'}{4} + \frac{1'}{8} + \&c. = 2$*, and bisect the arc Ht, in the point 1, and make the arc 1 2 equal to half of the arc $H1 = \frac{1'}{2}$, make the arc 2 3 equal to half of the arc $1\ 2 = \frac{1}{4}'$, &c. In the same manner F1′ equal to twice the arc $H1 = 2'$,—the arc 1′ 2′ equal to $H1 = 1'$, —the arc 2′ 3′ equal $\frac{1'}{2}$,—the arc 3′ 4′ equal $\frac{1'}{4}$, &c. Hence the terms, or arcs on Ft′ are double the corresponding arcs on Ht, and the remaining arc t′n must be double the arc tn.

From the point B and D, as centers [Lemma 3] describe through the points 2′ 3′ 4′...t′, and through the points x, y, z...t, the intersections a, b, c...t″, and draw through these intersections the straight line a b c...t″ meeting the arc H F in the point n. The arc H n shall be the third part of the arc H F [Lemma 4.]

For Ht is equal to the series $1' + \frac{1'}{2} + \frac{1'}{4} + \frac{1'}{8} + \&c.$, the points of variation x y z &c. become evanescent in the

* Bonnycastle's Algebra. Vol. 1, page 329.

point t; and t is the ultimate point of variation,--consequently the intersection t'' is the ultimate intersection of the series Ht and Ft', and therefore must be on the line a b c &c. (Lemma 4.) But the points 1. 2. 3...t and the points 1', 2', 3',...t' are in position proportional to the arcs Hn and Fn. Therefore the straight line a b c...t'', must cut the arc H F in the same proportion in the point n, and the arc Hn must be equal to the third part of the arc H F, or the arc Fn be equal to twice the arc Hn.

PROPOSITION 4.

Problem.

Division of the Semi-circle into three equal parts.

FIG. 9. Let A B D E be the semi-circle—with the distance At [Lemma B] from A cut the semi-circle in the point t, and make Et' equal to twice At. Next bisect At in the point o, and make E o' equal to twice A o—the arc o 1 equal to the half of A o, and o' 1' equal to twice o1, &c. Then from the points B and D [Lemma 3] describe through the point 3', 4', 5'...t', and through the points of variation x, y, z...t [Lemma 4] the intersections a, b, c...t'', and draw the straight line a b c...t'', meeting the semi-

circular arc in the point n. The arc An shall be equal to the third part of the semi-circular arc A E, [Prop. 3 Prob.] or the chord An shall be equal to the radius A C. The demonstration the same as Prop. 3.

PROPOSITION 5.

Problem.

Division of the circumference of the Circle into seven equal parts.

Let A B C be the given circle. FIG. 10. From any point A of the circumference make At equal to less than the seventh part of the circumference A B C [Lemma B.]—and At equal to the series $1' + \frac{1'}{n-1} + \frac{1'}{[n-1]2} + \frac{1'}{[n-1)3} + \frac{1'}{[n-1]4}$ +&c., then n will be equal to 7 and n--1 = 6; hence At is equal to $1' + \frac{1'}{6} + \frac{1'}{6^2} + \frac{1'}{6^3} + \frac{1'}{6_4}$ [Lemma A.] The sum of which series, or At is equal to $\frac{1'}{1'-\frac{1'}{6}}$* $= \frac{6}{5} = 1' + \frac{1'}{5}$. Therefore make the arc A1 equal to $\frac{5}{6}$ of the arc At, [Prop 3];

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* Wood's Algebra, page 113. Bonnycastle's Algebra, vol. 1, page 92.

nextmake thc arc 1 2, equal to the second term of the series or to $\frac{1}{6}$ of A1, the arc 2 3,equal to the third term $= \frac{1}{6^2} = \frac{1}{36}$ &c. Also make the arc A C 1′ equal to six times of A1—the arc 1′ 2′ equal to six times the arc 1 2,—the arc 2′ 3′ equal to six times the arc 2 3, &c. Then from the point B and D, [Lemma 3] through the points 2′ 3′ 4′...t′, and through the points of variation x,y,z.,.t, describe the intersections a,b,c...t″, and draw the straight line a b c...t″, meeting the circumference in the point n—the arc An shall be the seventh part of the circle A B C.

NOTE.—In the same manner may be described the nth part of any circle whatever,† observing as n increases the scale of the figure will require to be enlarged accordingly, if it is desired to have more intersections upon the straight line a.b...t″n than a and t″.

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† In Legendre's Geometry, page 112, he gives the following note:—" We have believed a long time that these were the only polygons which we were able to inscribe by the procedure of Elementary Geometry, or that which is the same, by the resolution of equations of the first and second degree; but M. Gauss has proven, in a work entitled *Disquisitiones Arithmeticæ, Lepsiæ*, 1801, that we may inscribe by the same means the regular polygon of *seventeen sides*, and in general those of $2^n + 1$ sides, provided that $2^n + 1$, shall be a prime number. M.

Case Third.—*Of the Geometrical Summation of the Infinite Converging Series, of which the terms are alternately plus and minus.*

PROPOSITION 6.

Problem.

The Duplication of the Cube.

Let the side of the cube be required, of which the solid content is 2, or the double of that of which the side is 1. Then the cube root of 2, will be the length of the side of the cube, double of that whose side is 1.

By the Binomial Theorem we have $[a + x]^{\frac{m}{n}} = a^{\frac{m}{n}} (1 + \frac{m}{n}. [\frac{x}{a}] + \frac{m}{n}.\frac{m-n}{2n}.(\frac{x}{a})^2 + \frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}.[\frac{x}{a}]^3 +$ &c. Now make a $=1$, and $x = 1$, and $\frac{m}{n} = \frac{1}{3}$, we shall have the numerical series $1 + \frac{1}{3} - \frac{2}{18} + \frac{10}{162} - \frac{20}{486} + \frac{22}{729} - \frac{41}{17496} + \frac{71}{39366} -$ &c, $= \sqrt[3]{2}$, and by reducing the first and second

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Legendre adds, page 419—We shall terminate these applications of Trigonometry by giving, after the excellent work of *Gauss* cited page 112, the manner of describing the regular polygon of 17 sides by the simple resolution of equations of the second degree."

terms, third and fourth terms, &c., we shall have the series $\frac{4}{3} - \frac{12}{243} - \frac{72}{6541} - \frac{26345038}{12244097} -$ &c. $= \sqrt[3]{2}$. Also by reducing the second and third term, the fourth and fifth terms, the sixth and seventh terms &c. we have the series $1 + \frac{2}{9} + \frac{45}{2187} + \frac{128302}{19131113} +$ &c. $= \sqrt[3]{2}$.

FIG. 11. Draw the straight line a b and make ac = 1, then make a 1 equal to $\frac{4}{3}$ of a c,—12 equal $\frac{12}{243}$, —23 equal to $\frac{72}{6561}$, and 3 4 equal to $\frac{26345038}{122414097}$ &c. Also make A1 =ac= 1, 1′2′ equal to $\frac{2}{9}$ of ac, —2′3′ equal to $\frac{45}{2187}$, 3′4′equal to $\frac{128302}{19131113}$, &c. Next describe the semi-circle A B D E, with a radius, not less than the distance a c, and make the chord A1 equal to a1, the chord A2 equal to a2, the chord A3 eqnal to a3, and A4 equal to a4 &c. Also make the chord A1′ equal to a1′ = 1, A2′ equal to a2′, A3′ equal to a3′, and A4 equal to a4′, &c., and describe

the arcs 1t, 2s, 3r, &c. [Lemma 3]—and the arcs $1''t'$, $2''s'$, $3''r'$, &c., and also describe the points of variation x, y, z &c. [Lemma 4.] Then from the points B and D [Lemma 3.] describe through the points $2'$, $3'$, $4'$, &c., and through the points x, y, z, &c., the intersections a, b, c, &c., and draw through a, b, c, &c., the straight line a b c...n, meeting the semi-circular arc A B D E in the point n—and join A and n. The distance An shall be equal to the side of the cube, whose solid content is 2, or equal to the sum of the infinite series 1

$$+\frac{1}{3}-\frac{2}{18}+\frac{10}{162}-\frac{20}{486}+\&c.=\sqrt{2}.$$

NOTE.—To obtain a number of intersections a, b, c,...&c. more than two or three, would require that the distance a c = 1, to be of a much greater scale than that of this figure; for it is evident that the parts of this line expressed by the terms of the series become rapidly to decrease.

PROPOSITION 7.

PROBLEM.

The Quadrature of the Hyperbola.

The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+$ &c. found by Lord Bounker*

* Bonnycastle's Algebra, vol. 1, page 334. Note a.

equal to the Appollonian Hyperbolic space, or area between its asymtotes, the sum of which according to Lorgna is .6931471.†

This series by reduction [Prop 6] is found to be $\frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\frac{1}{56}+\frac{1}{90}$ + &c. = .6931471 and $1-\frac{1}{6}-\frac{1}{20}-\frac{1}{42}-\frac{1}{72}$ &c. = .6931471.

FIG. 12. On the line a b, make a 1′=1, and 1′2′ equal to $\frac{1}{6}$ of a1′,—2′3′ equal to $\frac{1}{20}$, —3′4′ equal to $\frac{1}{42}$, and 4′5′ equal to $\frac{1}{72}$, &c. Also make a1 equal to ½ of a1′, and 1 2 equal to $\frac{1}{12}$, — 2 3 equal to $\frac{1}{30}$, 3′4′ equal to $\frac{1}{56}$, and 4′5′ equal to $\frac{1}{90}$, &c. Then in the same manner done in the preceding [Prop. 5, 6.] describe the right line a b c ..n, meeting the arc B D in the point n, and join An,—the distance An shall be to 1 or a1′—as the area of the Hyperbola is to 1^2 or .6931471 :1^2.

†See a Desertation on the Summation of infinite converging series by A. M. Lorgna, Professor of Mathematics in the Military College of Verona. Translated from the Latin by H. Clarke. Table page 122.

Corrollary,—It will appear the following eleven series to the above in Lorgna's Table, that the summation or areas of each may be found in the same manner, for they are all the same form of this.

PROPOSITION 8.

Problem.

The Quadrature of the Circle.

The series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}$ — &c. $= \frac{1}{8}$ of the circumference of the circle whose radius is 1—first given by Liebuitz*.

By reduction [Prop. 6] this series is resolved into the two following:—

$\frac{2}{8} + \frac{2}{35} + \frac{2}{99} + \frac{2}{195} + \frac{2}{328} +$ &c. $=$ $\frac{1}{8}$ Circumference.

$1 - \frac{2}{15} - \frac{2}{63} - \frac{2}{143} - \frac{2}{155} -$ &c. $=$ $\frac{1}{8}$ Circumference.

Upon a straight line a b, make a 1′ $=1$, — 1′2 equal to $\frac{2}{15}$ of a1′,—2′3′ equal to $\frac{2}{63}$, —3′4′ equal to $\frac{2}{143}$, &c. Also make a1 equal to $\frac{2}{3}$ of a1′,—1 2 equal to FIG. 13.

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* Introduction a L'analyse Infinitesimale par Leonard Euler. Tom. 1, page 104.

$\frac{2}{35}$, 2 3 equal to $\frac{2}{99}$, 3 4 eqnal to $\frac{2}{195}$, 4 5 equal to $\frac{2}{328}$, &c. Then from the point A on the semi-circle A B D E, make the chord A1′ equal to a1′, A2′ equal to a2′, A3′ equal to a3′, &c., and describe the points of variations x, y, z, &c. [Lemma 4]---and from the points B and D [Lemma 3] describe the intersections a, b, c, &c., and draw the straight line a b c ..n, meeting the arc B D in the point n, and join An. The straight line An, shall be equal to $\frac{1}{8}$ of the circumference of the circle whose radius is equal to a1′, or the sum of the parts represented by $\frac{2}{3}+\frac{2}{35}+\frac{2}{99}+$ &c. of the circumference, or $1-\frac{2}{15}-\frac{2}{63}$ —&c.= $\frac{1}{8}$ of the circumference—and consequently eight times An, must be equal to the whole circumference of the circle whose radius is equal to a 1′ =1

* Newton has remarked with regard to this series, that to exhibit its value to twenty places of decimals, would require the computation of no less than five thousand million of its terms, the performance of which would occupy more than a thonsand years.—See Newton's second letter to Oldenburgh on the Commercium Epistolicum, page 159." (Bonnycastle's Algebra, vol. 1, page 335.)

CASE FOURTH.—*The perimeter given of any Polygon—to find the inscribed or circumscribed circle of the Polygon.*

PROBLEM.

Let the diameter A E of the semicircle A B E, be equal or greater than the perimeter of the circumscribing triangle of a given circle, and the chord An equal to its circumference—the chord A*t* equal to the perimeter of the inscribed triangle—As equal to that of the inscribed square—A*o* equal to that of the inscribed Octagon, &c. Also At′ equal to the perimeter of the circumscribing triangle — As′ equal to the perimeter of the circumscribing square,—Ao′ equal to that of the circumscribing Octagon, &c. [Prop. 1, &c., Geo. Sol. Quad. of Circle] and [Prop. 2.]. Now let A*a*, be the perimeter of any polygon of which is required the circumscribing circle—which in this case let the polygon be the inscribed square, and its perimeter to be equal to A*a*—then with the distance A*a*, from A intersect the chord A*s* in *b*, and through *b* draw *b b′* parallel to *s* C, and from *b′* with the distance *b′b* describe the semicircle A *b* E′ intersecting the chord A*t′* in the point *t″*. Next divide A*t″* into three equal parts, from which construct FIG. 14.

the equilateral triangle A B C [Fig 15.] and inscribe the circle *a b c*—the circle *a b c* shall be the circumscribing circle of the square *d e f g*, whose perimeter shall be equal A*a* or A*b*.

For by intersection the triangle A*bb*′ is similar to the triangle A *s* C, and the segment A *h t*″ is similar to the segment A B *t*′. Therefore A*s* is to A*b*, so is A*t*′ to A*t*″, equal to the perimeter of the circumscribing triangle of a circle *a b c*, and *d e f g* the inscribed square of the inscribed circle *a b c* in the equilateral. triangle A B C [Fig. 15.] the side of which is equal to the third part of A*t*′. Therefore the circle *a b c* must be the circumscribing circle of a square whose perimeter *defg* is equal to A*b* or A*a*.

Corrollary 1,—It is evident that A*n*′ must be equal to the circumference of the circle *a b c*—A*p* must be equal to the perimeter of the inscribedtriangle—A*q* must be equal to the perimeter of the inscribed octagon, &c. Also A*h* must be equal to the perimeter of the circumscribing square, and A*o*′ equal to the perimeter of the circumscribing octagon of the same circle.

Corrollary 2,—In the same manner from the given perimeter of any polygon the inscribed circle is found—for let Ae' be the given perimeter of a circumscribing square—then Af' must be the perimeter of the circumscribing triangle, the inscribed circle of which must be the circle required, — and the perimeter of the circumscribing square must be equal to Ae'', &c.

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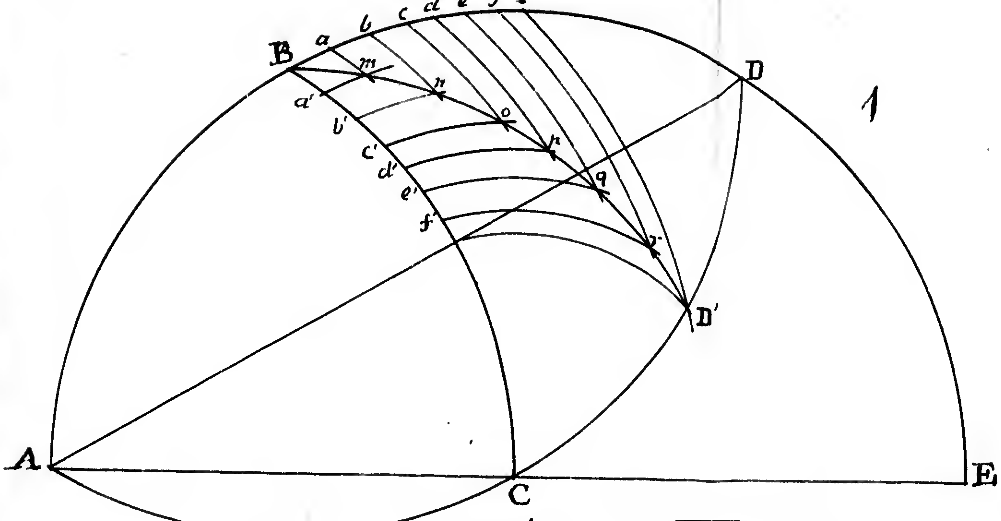
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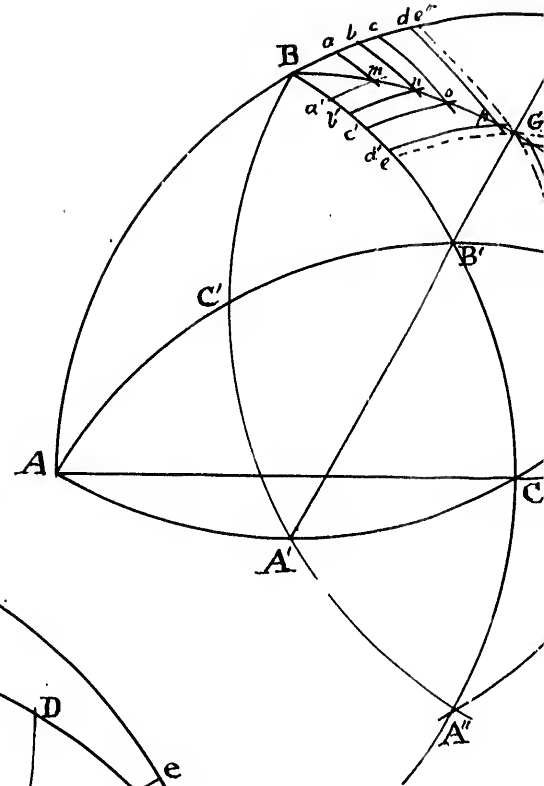
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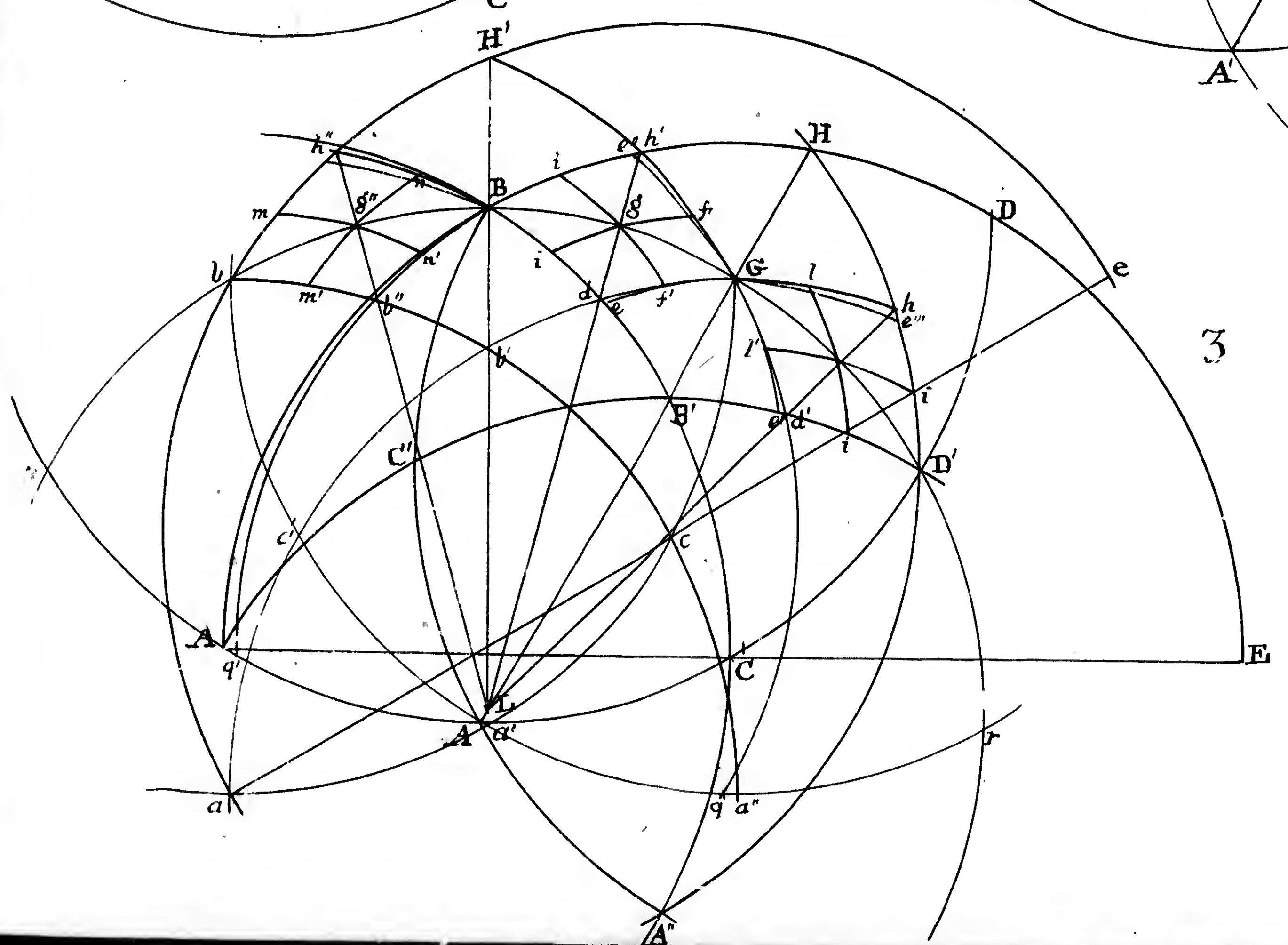
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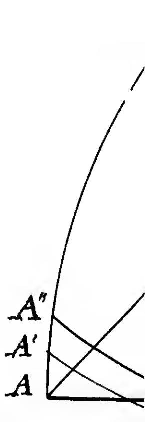
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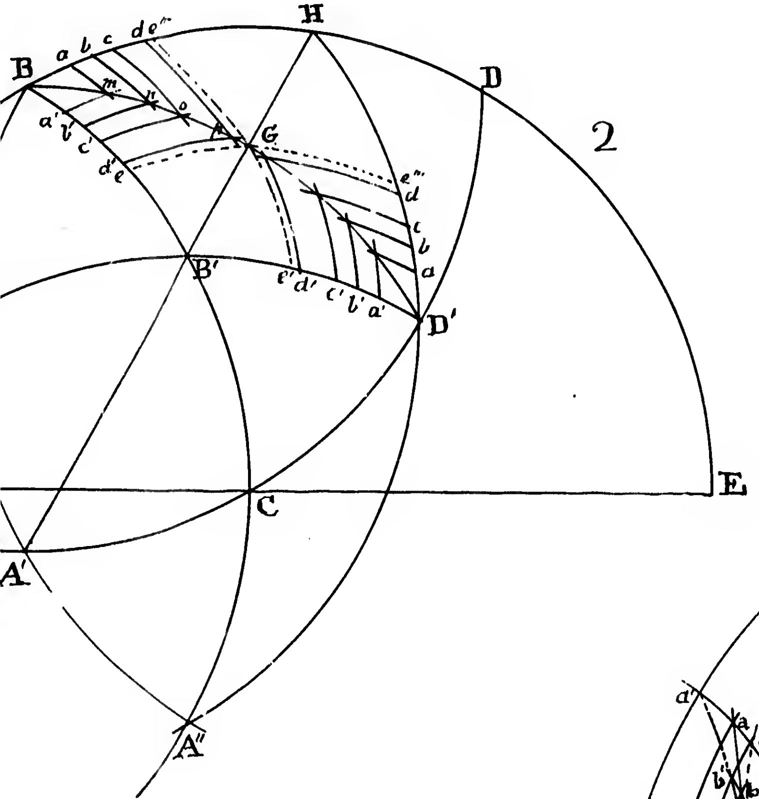

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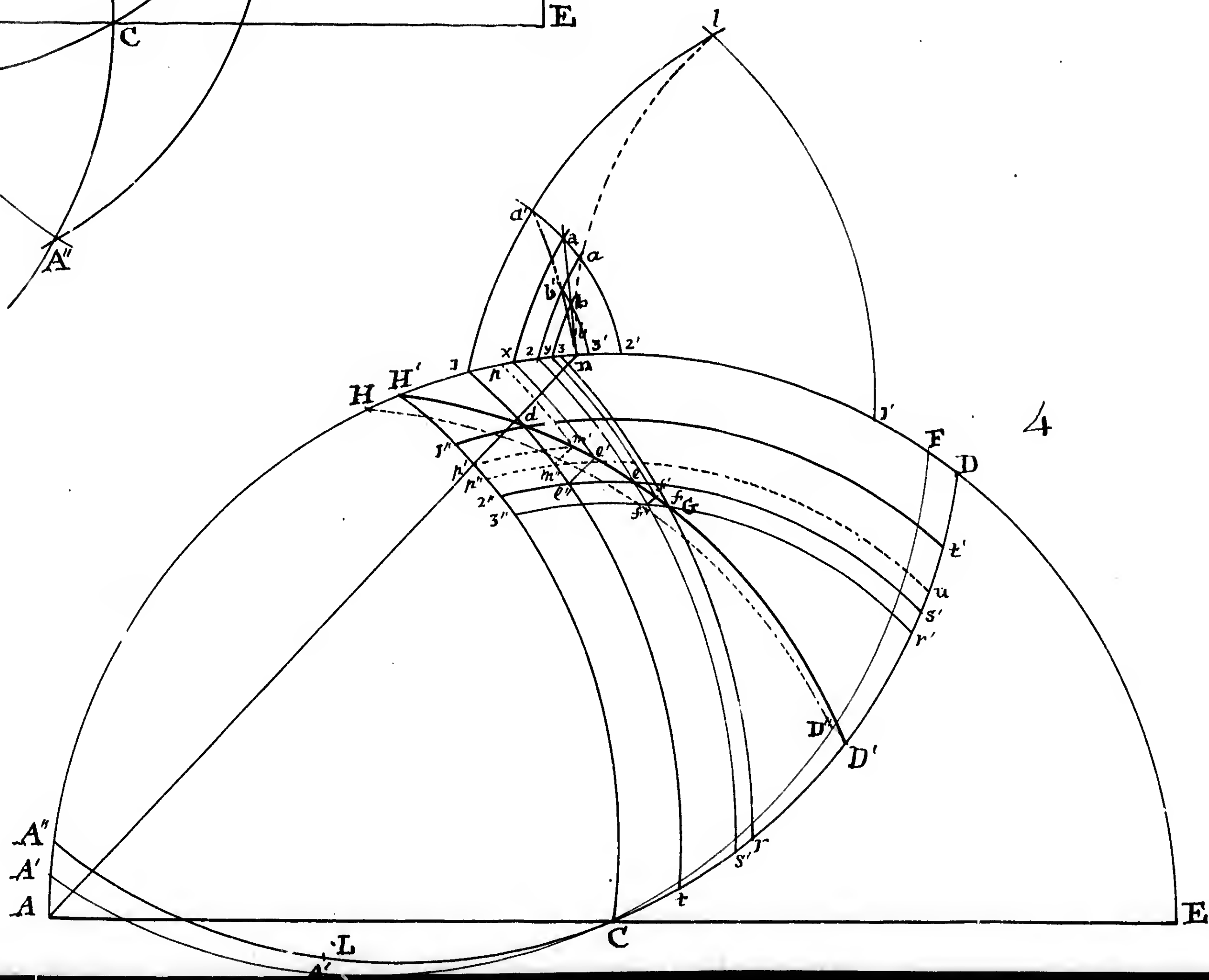

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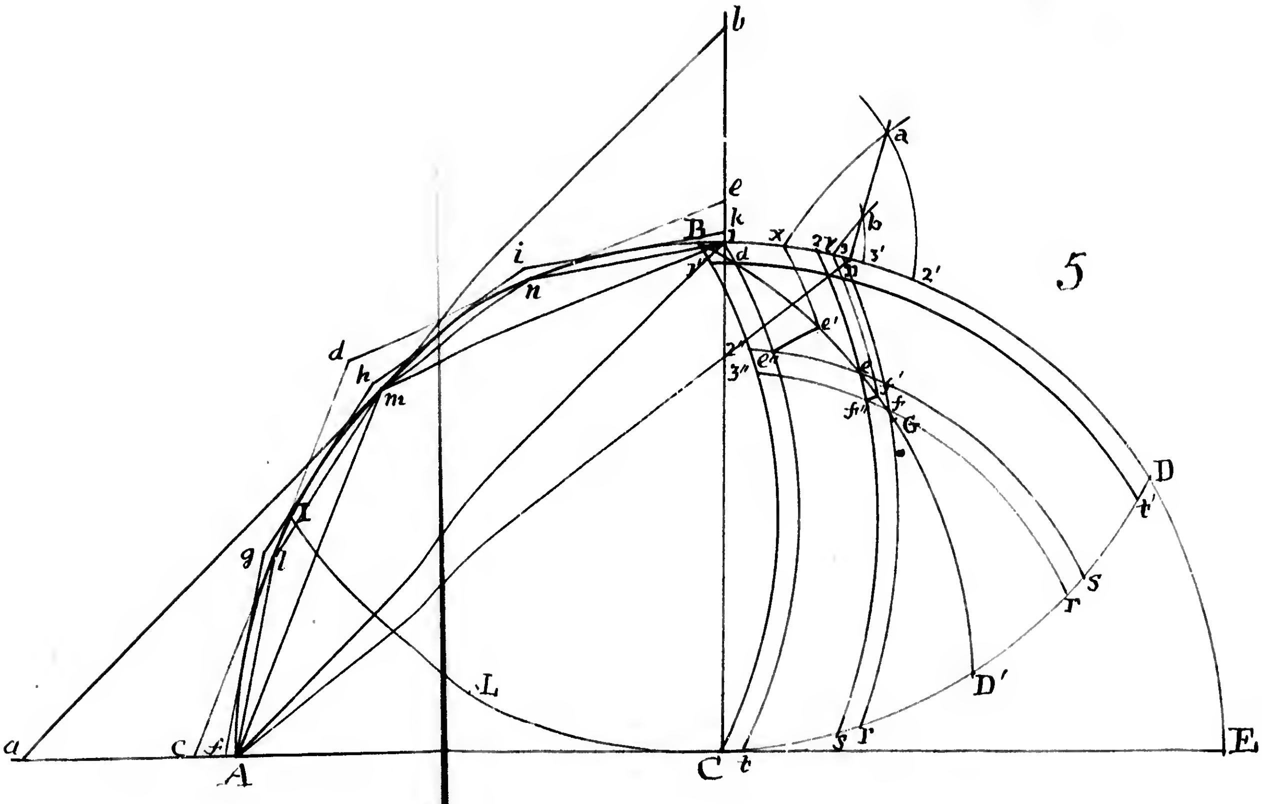

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Pl. 2.

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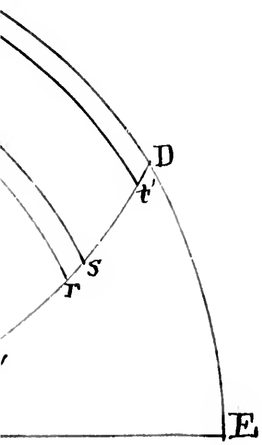

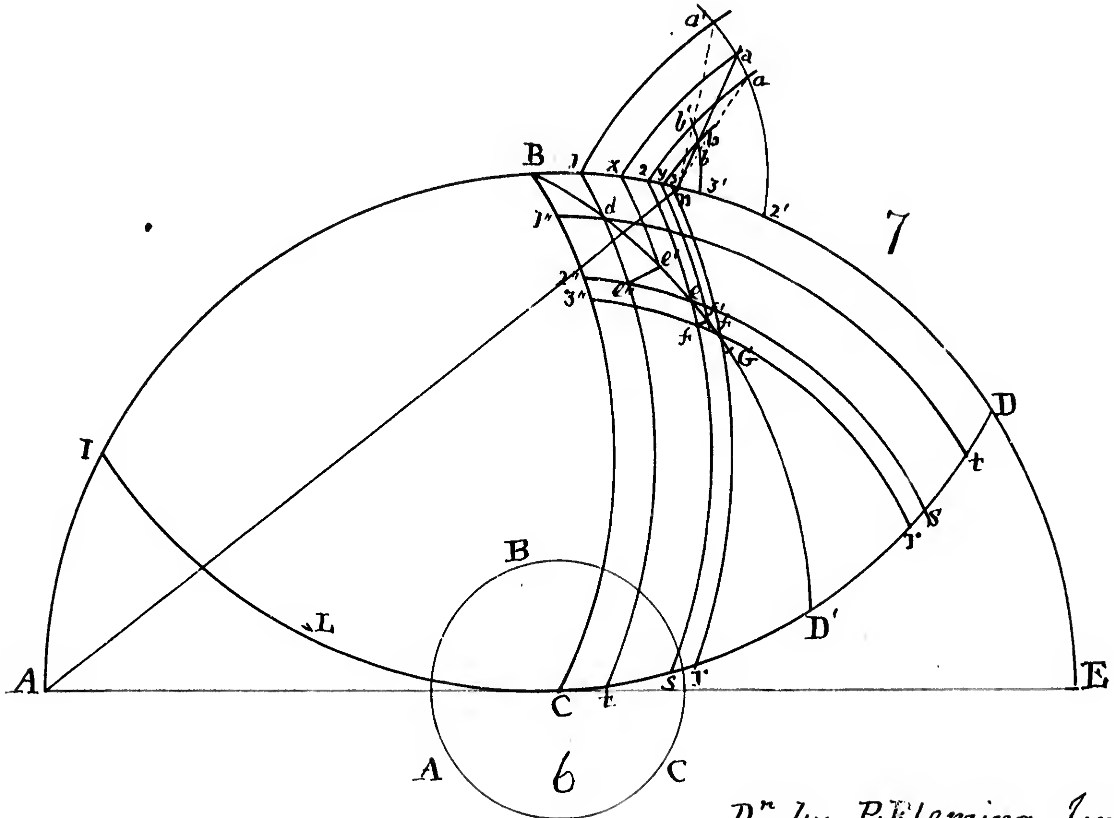

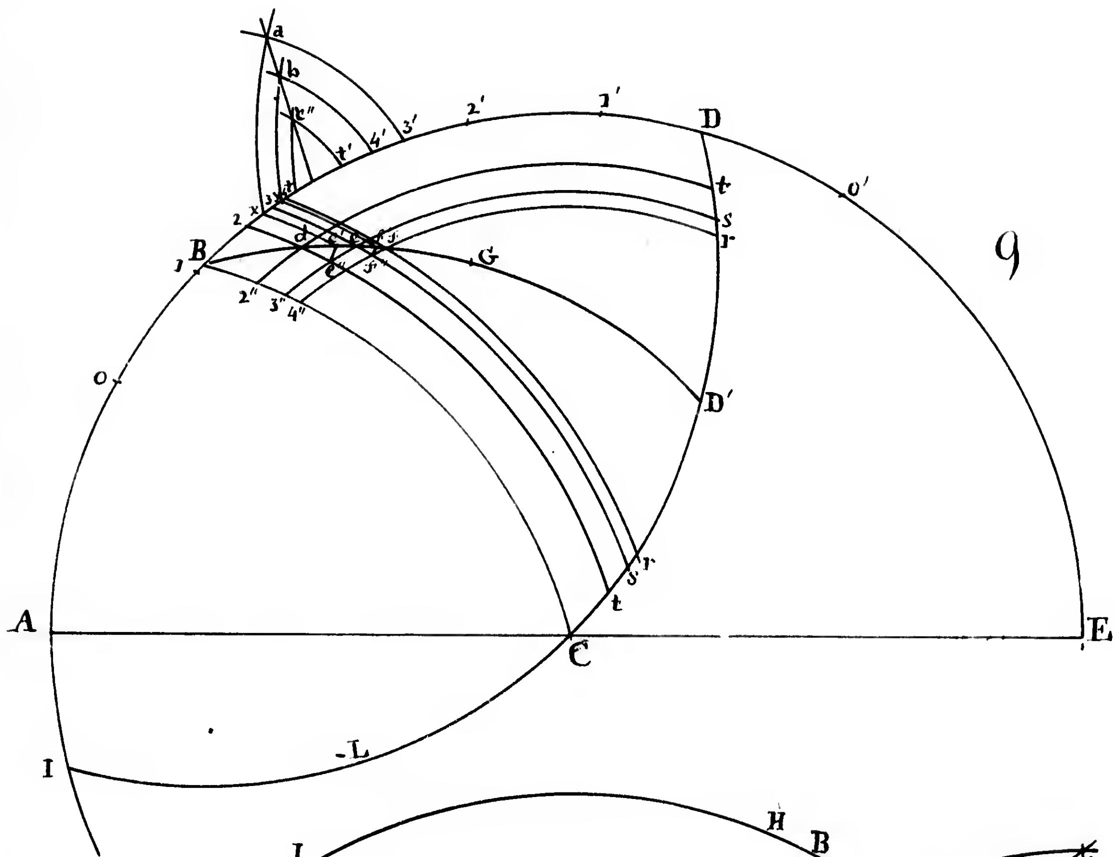

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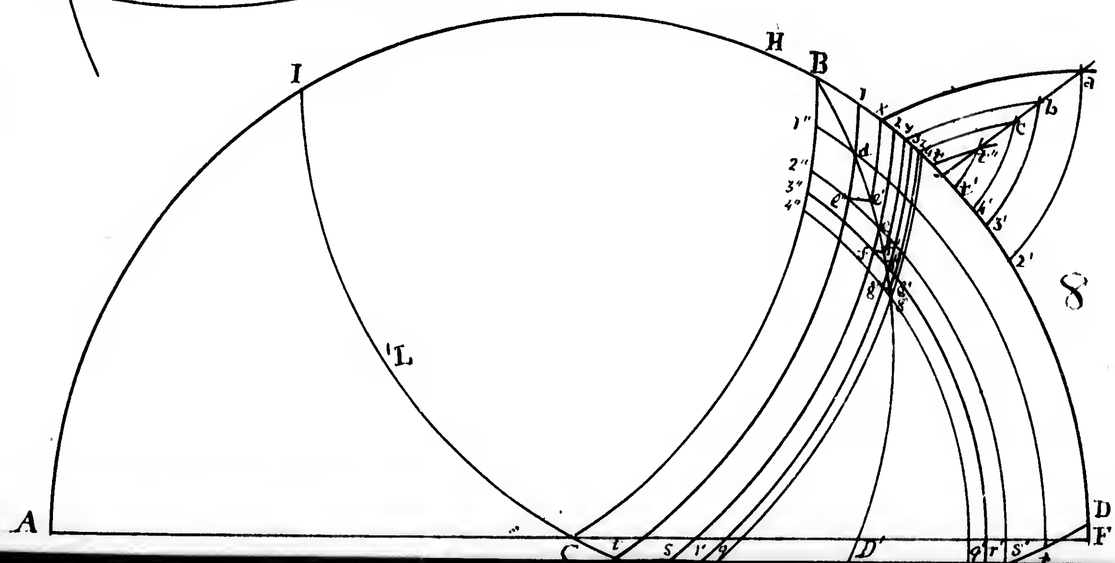

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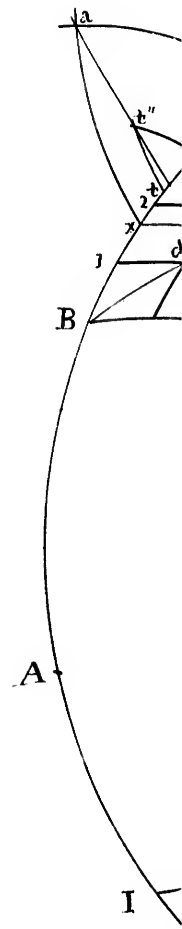
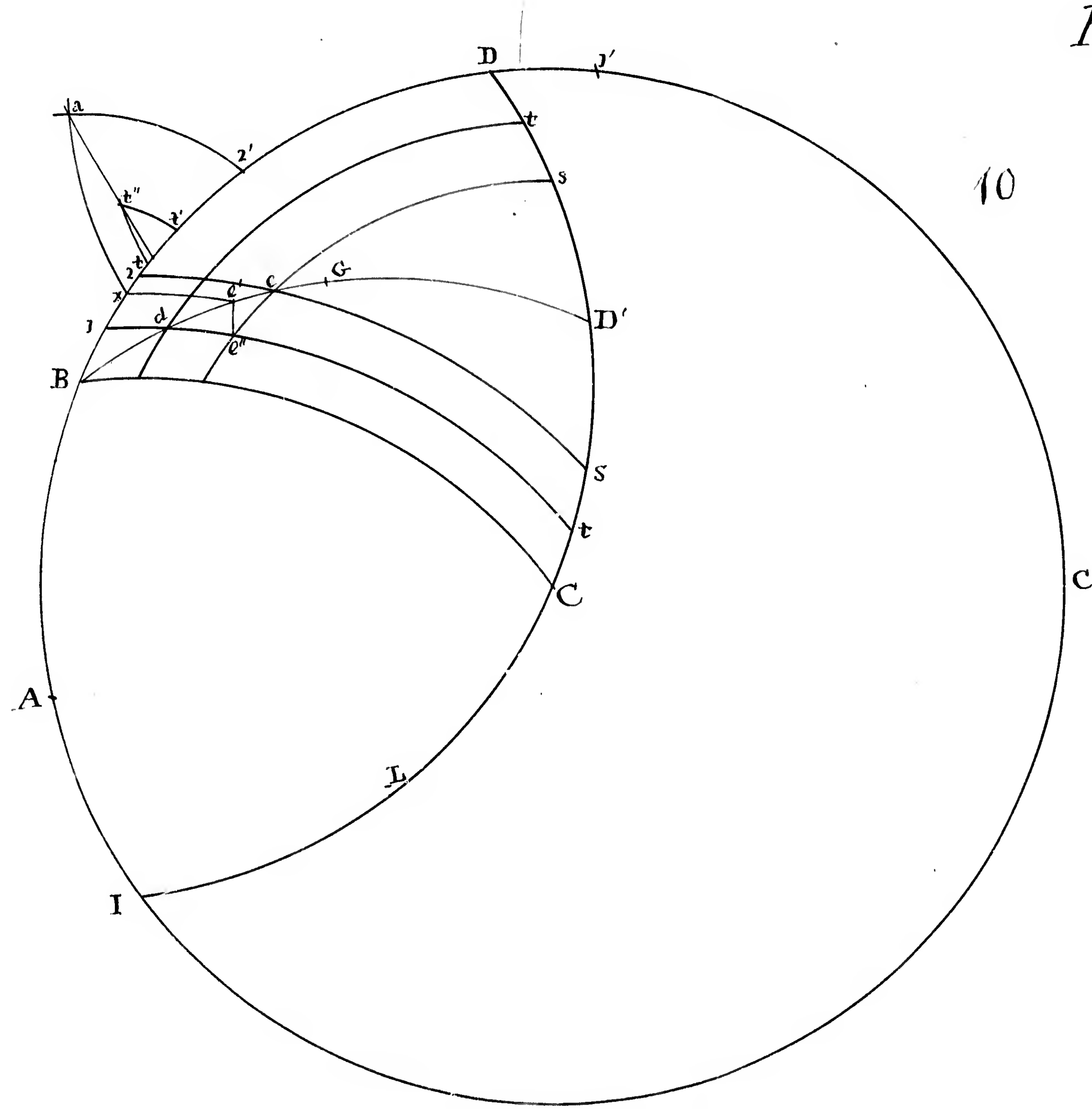


Pl. 3.



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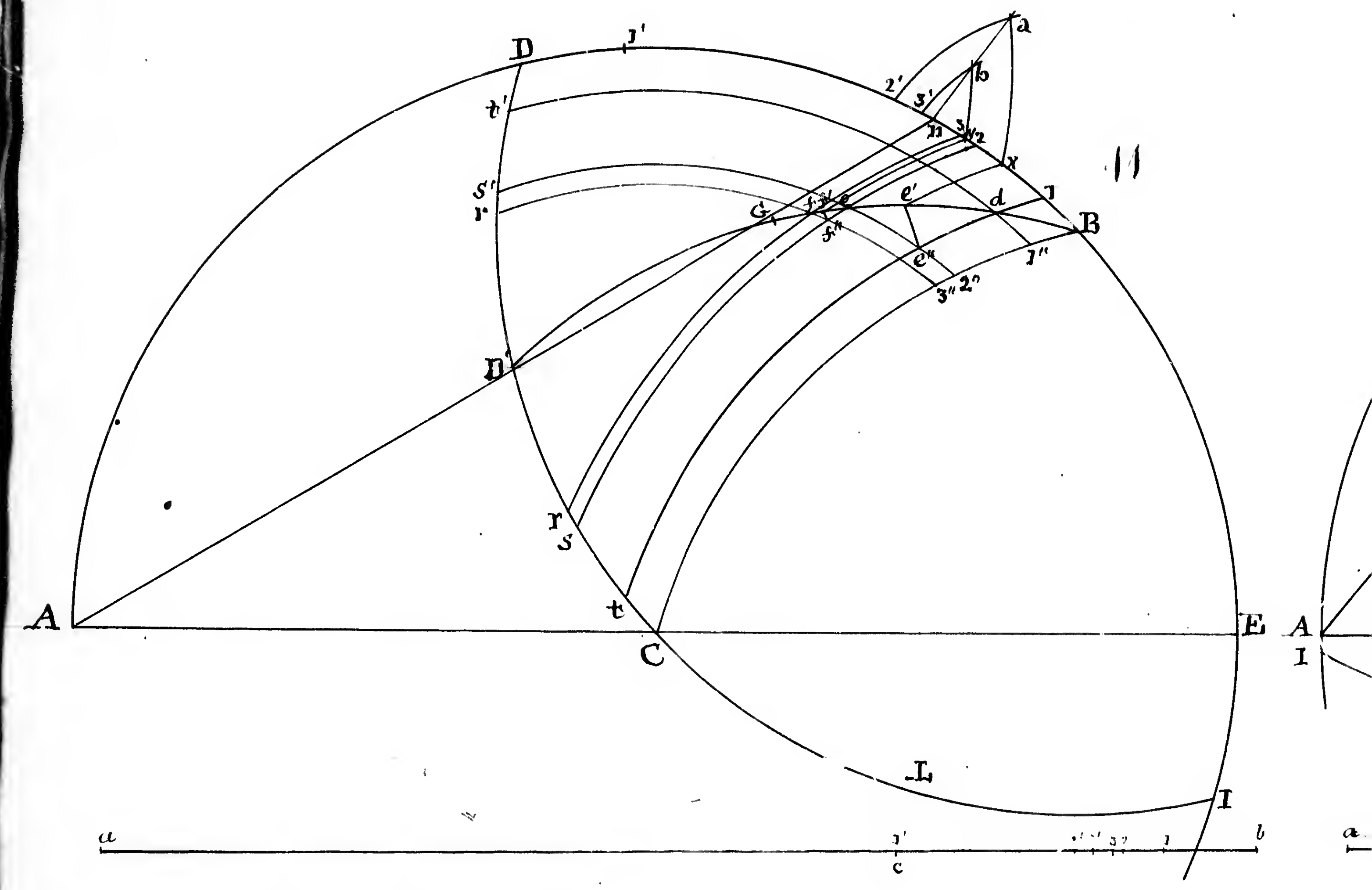
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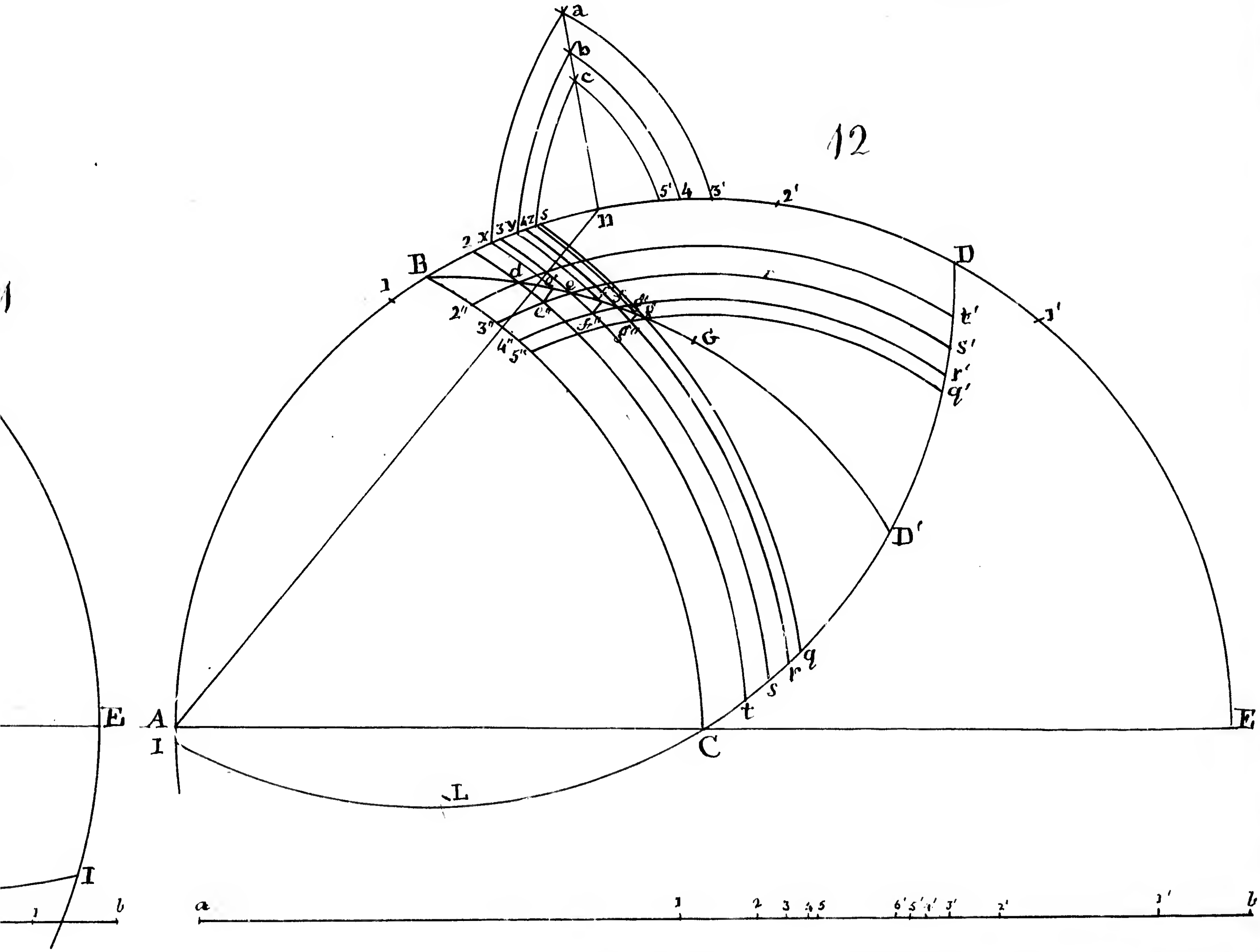
Pl. 4.



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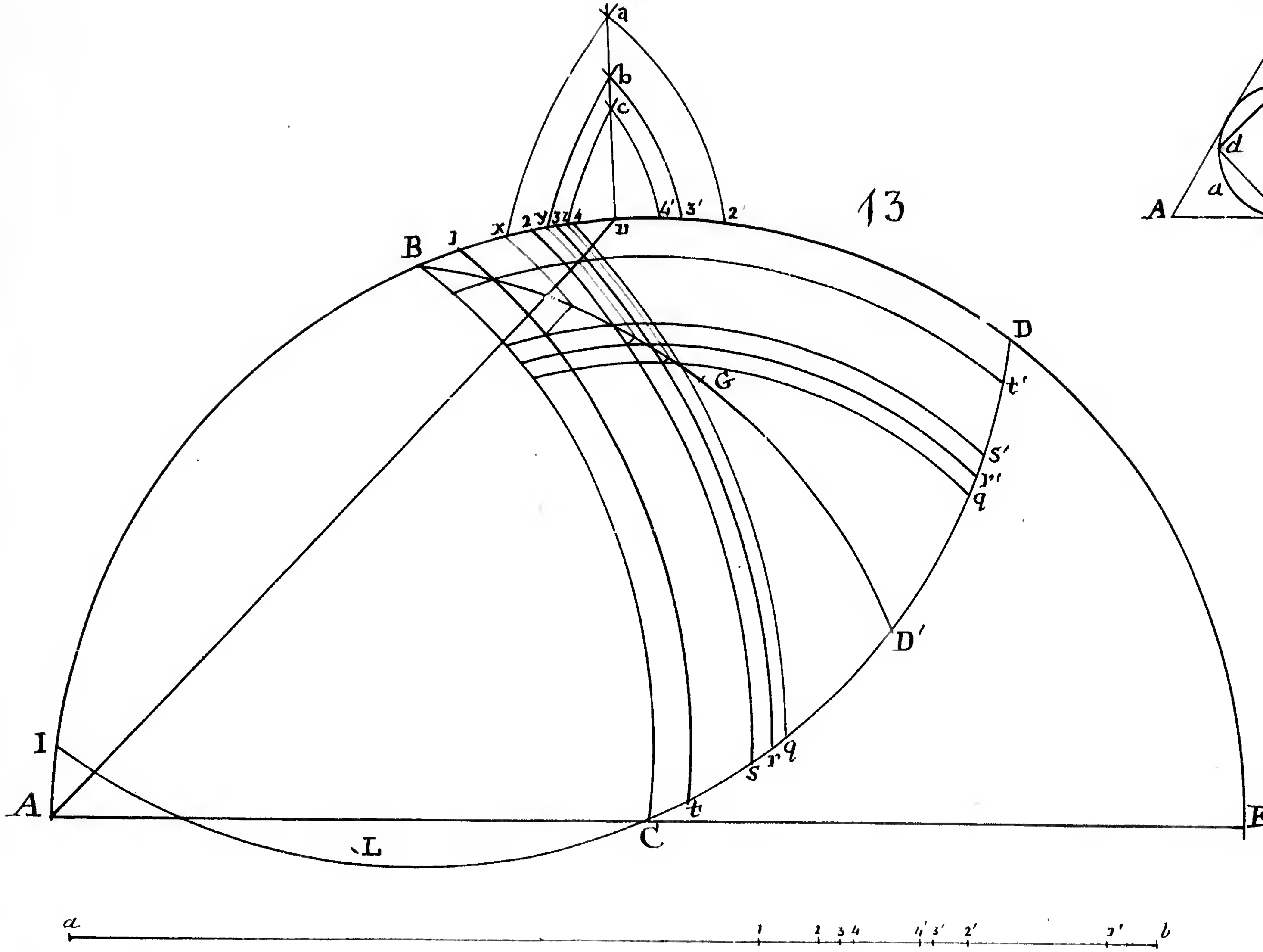

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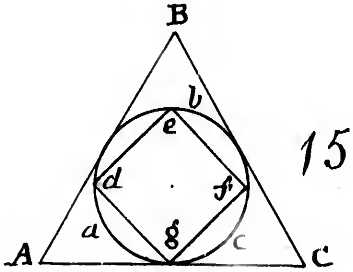
Pl. 5.



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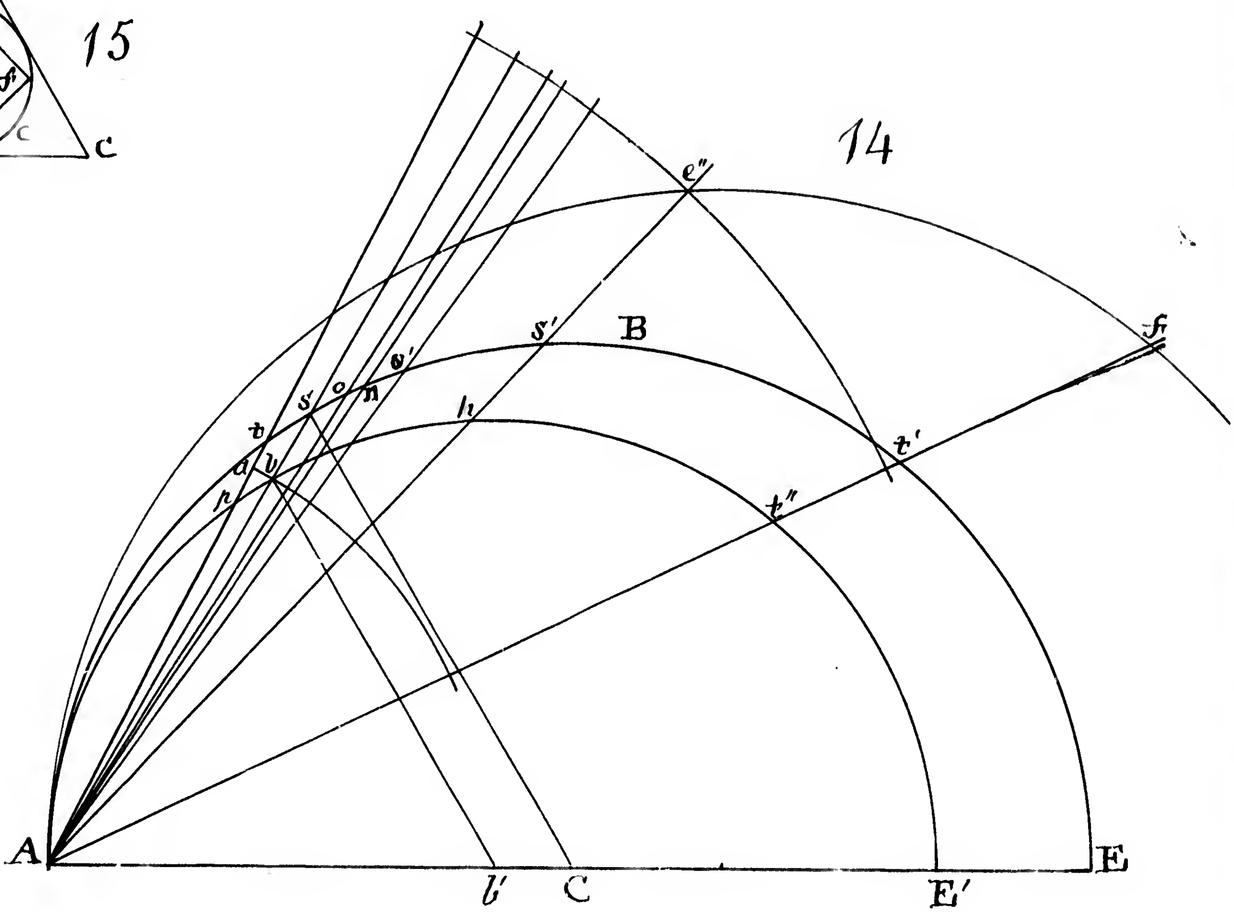

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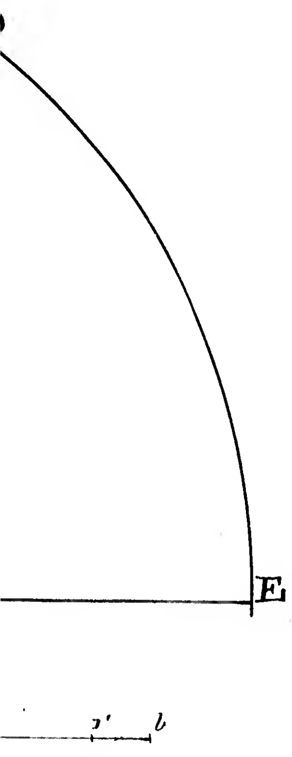

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