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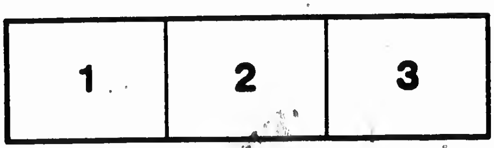

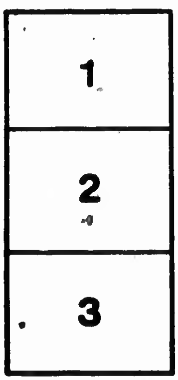

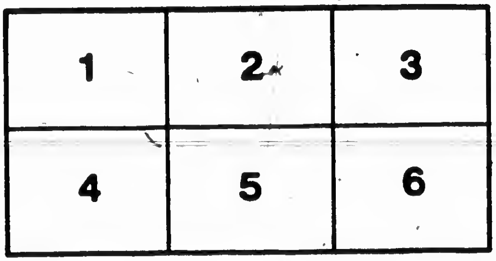

Just issued

WATSON'S

COMPOUND INTEREST AND ANNUITY,

LOAN AND VALUATION TABLES

FOR THE USE OF

Building Societies, Brokers,

AND OTHERS, REQUIRING TO BUY, SELL, OR VALUE

MORTGAGES, BONDS, DEBENTURES OR ANNUITIES.

NEW EDITION, GREATLY ENLARGED.

— BY —

WILLIAM E. WATSON,

ACCOUNTANT.

Toronto:

DUDLEY & BURNS, PRINTERS, NO. 11 COLBORNE STREET.

[illegible]

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PREFACE.

The following Tables embrace those prepared by JAMES WATSON, Manager of The People's Loan and Deposit Company, the first edition of which has been exhausted, but the stereotype plates of which have been secured by the compiler of this edition.

The former edition contained 22 rates, but the present work has been extended so as to embrace 38 rates, and the new, together with the old rates, in several of the Tables, have been extended to 45 years on the half-yearly, and 70 years on the yearly basis.

There have been also added two additional tables, showing the amount of $1, and the amount of an annuity of $1, for 50 half-years, and 50 years; and a complete list of all the nominal yearly, and half-yearly rates, showing the true rates of these, convertible yearly, half-yearly, quarterly and monthly, and the Logarithms of these true rates, to 10 places of decimals by the use of which any item in the Tables may be verified, and new problems beyond their limits solved with the greatest accuracy.

There have been added besides, complete formulæ for the calculation of nearly all questions which can arise involving compound interest and annuities, together with examples showing the practical use of Logarithms in such calculations.

The compiler hopes that the work will thus be found a useful manual for all institutions or individuals dealing in Bonds, Mortgages or Annuities—Life or otherwise.

The Tables have been prepared and revised with great care, and may be confidently relied on.

EXPLANATIONS AND EXAMPLES.

INTEREST.

The number of rates of interest included in the present edition is 38, and comprises whole and fractionary rates from 3% to 12%, proceeding by halves, and the results of these rates, compounded *yearly and half-yearly*, are shown.

The true rates of these nominal annual rates are exhibited in a classified form on page 17, which shows the true rate of interest on $1. ($t$) for one month, quarter, half-year or year of each of these 38 nominal rates. Along with these true rates is shown the Logarithm of each (Log. t) to ten places of decimals, and also the Logarithm of these rates increased by unity (Log. r, — $r = 1 + t$).

The rates compounded *half-yearly* are equivalent to one-half of these rates compounded *yearly*, for double the time as regards Tables I and VII. Thus 6 months at 3% half-yearly is equivalent to 12 months at 1½% yearly, and 1 month half-yearly to 2 months yearly, etc. This applies to all rates.

In Table II, Monthly Instalments of $1, would then represent $1 every two months.
In Table III, Quarterly " " " " " per half-year.
In Table IV, Half-yearly " " " " " per year, at half the given rate compounded yearly.

TABLES.

Table I—Gives the *present value* of a *single payment of $1*, due at the end of any month from 1 to 240 (20 years), at the various rates of interest designated at the top of each page, computed yearly, and half-yearly. It also shows the *present value* of a *single payment of $1*, due at the end of any half-year, from 40 to 90 half-years (20 to 45 years), interest half-yearly, and of a *single payment of $1*, due at the end of any year from 20 to 70 years, interest yearly, at the various rates indicated.

Table II—Gives the *present value* of a *monthly instalment of $1*, payable at the end of each month from 1 to 240 months at same rates.

Table III—Gives the *present value* of a *quarterly instalment of $1*, payable at the end of each quarter from 1 to 80 (20 years), at the various rates given, when the first instalment becomes due at the end of the 1st, 2nd or 3rd month, and the second, or corresponding instalment, at the end of the 4th, 5th or 6th month, and so on.

Table IV—Gives the *present value* of a *half-yearly instalment of $1*, payable at the end of each half-year from 1 to 40 (20 years), at same rates as above, compounded half-yearly and yearly, when the first instalment becomes due at the end of the 1st, 2nd, 3rd, 4th, 5th or 6th month, the second at the end of the 7th, 8th, 9th, 10th, 11th or 12th month respectively, and so on; and is continued showing the present value of instalments at the *end only* of each half-year, from 40 to 90 half-years, at rates compounded half-yearly.

Table V—Gives the *present value* of a *yearly instalment of $1*, payable at the end of each year from 1 to 20 at same rates compounded yearly and half-yearly, when the first instalment becomes due at the end of any month from 1 to 12, the second at the end of any month from 13 to 24 respectively, and so on; and is continued showing the present value of instalments at the *end only* of each year from 20 to 70 years at rates compounded yearly.

Table VI—Gives the *instalment* required to repay a loan of $1000, when said instalment is payable *monthly, quarterly, half-yearly or yearly*, or the *monthly, quarterly, half-yearly or yearly annuities* which $1000 will purchase, at the various rates enumerated, compounded yearly and half-yearly, from 1 to 20 years.

Table VII—Gives the *amount of $1* increased by its interest: 1st, at the end of any half-year from 1 to 50 half-years, interest compounded half-yearly; and 2nd, at the end of any year from 1 to 50 years, interest compounded yearly, at the rates designated.

Table VIII—Gives the *amount of an annuity of $1* for same period and rates as Table VII.

NOTE.—*In table VI the instalments are given exact to the next cent above the true value when the fraction exceeds one-tenth of a cent. In all the other tables the values are made true to the nearest decimal.*

USES OF THE TABLES.

1. To find the present value of an ordinary Mortgage, or any sum to be paid at the end of a period of years, or years and some months.

By Table I, find the present value of $1 for the term required, which will be on the same line as the required number of months in the column headed by the required rate of interest, and multiply this factor by the sum of the Mortgage, or other payment, of which the present value is desired.

EXAMPLE.—Required the present value of $1000 due 14 years and 7 months hence, interest at 10% compounded half-yearly. By Table I, factor for 14 years 7 months is .24098 and .24098 × 1000=$240.98, the present value.

2. To find the present value of a Mortgage or Debenture when the principal becomes due after a number of years, or a broken period of years and some months, and bearing any rate of interest, compounded half-yearly or yearly.

By Table I, as above, find the present value of $1 due at the end of the period required, and multiply the principal by this factor. Then, by Table IV or V, as the case may be that the interest or coupons are payable half-yearly or yearly, find the present value of a half-yearly or yearly instalment of $1 for the same time; multiply this factor by the half yearly or yearly sum of interest, which will give the present value of the interest or coupons. Then add together the present values of the principal and the interest, which will give the present value of the whole.

EXAMPLE 1.—Required the present value of a Mortgage bearing interest at 7% payable half-yearly, principal $4000, due 11 years and 2 months hence; to pay 9%, compounded half-yearly.

By Table I, the present value of $1 due 134 months hence at 9%=.37417 × 4000=	$1496.68
And by Table IV p. v. of half-yearly instalments of $1 for same time and rate=14.569. The half-yearly interest on $4000 at 7%=$140, and 14.569 × 140=	2039 66
The present value of the Mortgage is	$3536.34

EXAMPLE 2.—A Debenture of $100, having 19 years and 2 months to run, and bearing interest at 6%, compounded half-yearly, is offered at a price to pay the purchaser 5% half-yearly. Required its present value.

By Table I, present value of $1, due 230 months hence, at 5%=.38808 × 100=	$38.808
And by Table IV, present value of half-yearly instalments of $1 for same time=25.141 × 3 (value of half-yearly coupon)=	75.423
The present value of the Debenture is	$114.231

Thus the present value of any Mortgage or Debenture may be calculated to pay the purchaser any rate on his purchase money, notwithstanding the rate of the security being different. It will be seen that if the purchaser gets a higher rate than that borne by the security, he will pay less than its face value, and, if he realizes a lower rate he will pay more than the face value. If the rate borne by the Mortgage or Debenture be the same as that which the purchaser obtains, he will pay for it the par value. This may be shown by the tables as in the above cases.

EXAMPLE 3.—What is the present value of a Debenture of $100 due 18 years hence, having coupons of $8 each payable yearly, to pay 8% yearly?

By Table I, present value of $1 due 216 months hence at 8%=.25025 × 100=	$25.025
And by Table V, p. v. of yearly instalments of $1 for same time=9.3719 × 8=	74.975
The present value of the Debenture is	$100.000

3. To find the amount to which any sum would accumulate after a given number of months.

Divide the sum by the present value of $1 due at the end of the term, and at the rate required. The quotient will be the amount.

EXAMPLE.—To what sum will $100, now invested at 10% interest compounded half-yearly, amount in 18 years and 7 months?

By Table I, the present value of $1, due 223 months hence = .16310 ∴ $\frac{100}{.16310}$ = $613.12, answer.

For any even number of half-years or years up to 50 respectively, Table VII gives the amount of $1, and this factor multiplied by the given sum will give its amount for same period.

4. To find the present value of any Instalment or Annuity, payable yearly, half-yearly, quarterly or monthly, during a given number of years, or a broken period of years and some months, at any rate of interest given.

Find the present value of an Instalment of $1 for the proper time and rate in the Table corresponding to the periodic payment, and multiply this factor by the given Instalment.

EXAMPLE 1.—A Mortgage payable by monthly instalments of $20 each, has 8 years and 4 months to run. What is its present value, interest 10%, convertible half-yearly?
By Table II, the present value of an instalment of $1 for 100 months at 10%=$68.164 × 20=$1,363.28, answer.

EXAMPLE 2.—A Mortgage, payable by quarterly instalments of $25 each, has 8 years and 1 month to run. What is its present value, interest 9%, and convertible half-yearly.
By Table III, the present value of a quarterly instalment for 8 years and 1 month (*i.e.* 33 instalments, first due one month hence), at 9%=$23.545 × 25=$588.62, answer.

EXAMPLE 3.—A Lease, payable by half-yearly rents of $60 each, has 7 years and 3 months to run (*i.e.* 15 rents unpaid, first due three months hence). What is its present value, interest at 10%, convertible yearly?
By Table IV, the present value of a half-yearly instalment of $1 for 7 years and 3 months at 10% yearly=$10.716 × 60=$642.96, answer.

EXAMPLE 4.—A Mortgage, payable by yearly annuities of $210, has 15 years and 3 months to run, when last instalment matures. What is its present value, interest 9½%, annually?
By Table V, the present value of 16 yearly instalments of $1, last instalment due 183 months hence, at 9½%, annually=$8 6301 × 210=$1,812.321, answer.

EXAMPLE 5.—What is the present value of a Lease of $100 per annum, for 62 years, at 5%, compounded yearly?
By Table V, the present value of $1 per annum, for 62 years, at 5%=$19.0288 × 100=$1,902.88, answer.

By Tables III, IV and V the present value of a quarterly, half-yearly or yearly payment of Rent, or Interest on Mortgages or Debentures, can be determined to pay any of the rates given, and in the case of Mortgages or Debentures the present value of the principal may be found by Table I, and added to that of the interest as in the examples of § 2.

5. Assuming that a Mortgagor has arranged with the Mortgagee to prepay his Mortgage, or a portion of same, in addition to his usual annuity. To find how such a payment would affect equitably the subsequent annuities, as to amount, or as to time.

EXAMPLE 1.—A Mortgage, payable by monthly instalments of $20 each, yields 10½% interest, convertible half-yearly, and has 7 years and 5 months to run. The borrower wishes to pay down $600, and to find how long his instalments of the same amount must continue to pay off the debt.

By Table II, the present value of a monthly instalment of $1 for 89 months =62.101 × 20 .. .. $1242.02

Deduct	600.00
Balance	$642.02

Dividing t is by the amount of the instalment, viz. $20, will give the value of an instalment of $1 for the necessary time-642.02 ÷ 20= 32.10

And p. v. of a monthly instalment of $1 for 37 months= .. .. 31.59 (nearest amt. below)

Difference on $1 instalment= 0.51 × 20=$10.20

The time therefore would be 37 months, and $10.20 additional cash to be paid now; or if postponed till 38 months would be (3) $10.20 ÷ .72320=$14.10, to be paid as a last instalment.

EXAMPLE 2.—A Mortgage, payable by quarterly instalments of $60 each, and yielding 10% half-yearly on investment, has 5 years and 5 months to run before maturity of last instalment. The Borrower wishes to pay $300 on account, and to know how much his instalments are to be reduced for balance of period.

First method.—By Table III, the present value of a quarterly instalment of $1 for 5 years and 5 months=$16.955 × 60= $1017.30

Deduct	300.00
Present value of balance	= $717.30

and this amount, divided by the present value of quarterly instalments of $1 for 5 years and 5 months, viz. $\frac{717.30}{16.955}$=$42.31; or

Second method.—Divide the amount paid down by the present value of an instalment of $1 for the period to run, and deduct the quotient from the former instalment for the new instalment. Thus, in the above example, amount paid down = $300, present value of instalments of $1 for 5 years and 5 months=$16.955; then 300 ÷ 16.955=$17.69; and 60 − 17.69=$42.31.

6. The addition of a percentage to the amount loaned for the whole term, and that amount divided by the number of the instalments to be made during this period, yields a variable rate of interest, according to the time for which the loan is made, and the number of instalments,—monthly yielding a better rate than quarterly, and quarterly than half-yearly.

EXAMPLE 1.—A Borrower receives $1000 cash, at 6% for 10 years, to be repaid by monthly instalments. To the $1000 there is added interest at 6% per annum for 10 years=$600 + 1000 =1600, and this amount is divided by the number of payments, 1600 ÷ 120=13.34. It is required to determine the rate of interest half-yearly which this investment yields.

By Table VI an instalment of $13.11 will repay $1000 in 10 years at 10% half-yearly, while 10½% would require an instalment of $13.37 The rate would therefore be between 10% and 10½%.

EXAMPLE 2.—A Loan of $4000 on same terms is made for 5 years. Required the rate this investment produces. To 4000 add 5 years' interest at 6%=$1200 + 4000=5200 ÷ 60=$86.67 monthly, or per $1000=$\frac{1000}{4000}$ × 86.67=$21.67.

By Table VI, instalment to repay $1000 in 5 years, at 11%, *half yearly*=$21.63, and at 11½%, *yearly*=$21.71, which are therefore approximate rates for such a loan.

Thus for any loan, when the number of years and the amount of instalment to repay it are given, by reducing the instalment to the basis of a loan of $1000, the rate of interest, if between 3% and 12%, may be found approximately, in Table VI. If beyond these rates it must be found by independent calculation.

7. To find the rate per cent. yielded by the return of a certain sum, at the end of a definite number of years or half-years, for the use of a smaller sum, when between 3% and 12%, and less than 50 years or half-years.

Divide the amount returned by the amount loaned, which will give what was repaid for the use of $1. Then find in Table VII the nearest amount for a corresponding time, and the column will show the rate per cent.

EXAMPLE.—A Borrower returns $5000 for the use of $1335 during 30 years, find the rate per cent. yearly. $5000 ÷ 1335 = $3.7453. By Table VII, the amount of $1 for 30 years at 4½%, yearly = $3.7453. The rate is thus 4½%, yearly.

8. To determine the time required for any definite sum to accumulate to a larger given sum, at a given rate.

Reduce to the basis of $1 by dividing the larger sum by the smaller. In the column of the given rate in Table VII, find the amount which corresponds to the result of this division, or the nearest given, and the line will give the exact, or approximate, time required.

EXAMPLE.—How long will it take $4000, bearing interest at the rate of 6%, convertible yearly, to amount to $12,828.40? 12,828.40 ÷ 4,000 = $3 2071, and by Table VII, the amount of $1 at 6%, yearly, for 20 years = $3.2070. Required time = 20 years.

In case questions may arise beyond the limits of these tables, or a more accurate result be required by additional decimals, the following formulæ are added, and the true rates for one year, half year, quarter and month of all the nominal rates of interest, compounded half-yearly and yearly, are given on page 17, together with their logarithms, to 10 places of decimals.

9. Formulæ of Compound Interest, with or without the use of logarithms.

I.—FOR A SINGLE SUM OF MONEY

Let P = the *principal*; or *present value* of an amount M.

M = the *amount* of this principal at the end of a given time; or the *sum* due at the end of a given time, of which we wish to find the present value.

i = the *interest* on $1, £1, or other unity for one year or other given period.

$r = (1 + i)$ the *sum* of $1, £1, or other unity for one year or period.

n = the whole number of years or periods. Then

For the *amount*

(1) $M = Pr^n$ or $\log. M = \log. P + n \times \log. r$.

For the *present value*

(2) $P = \dfrac{M}{r^n}$ or $\log. P = \log. M - n \times \log. r$.

For the *rate of interest*

(3) $r = \sqrt[n]{\dfrac{M}{P}}$ or $\log. r = \dfrac{\log. M - \log. P}{n}$. Then $r - 1$ gives i or the rate of interest for one year or period.

For the *number of years or periods*

(4) $r^n = \dfrac{M}{P}$ or $n = \dfrac{\log. M - \log. P}{\log. r}$

II.—AS RELATING TO ANNUITIES.

Let t = the *true interest* on 1 for single period of the annuity.
r = $(1 + t)$ the *amount* of 1 for one period.
n = the *number* of periods or instalments of the annuity.
A = the *amount* of the whole of the annuities at the end of n periods.
V = the *principal*; or the *present value* of the annuity of $\$a$ or £a for n periods (each annuity being assumed to be payable at the *end* of its own period)
a = the periodic *payment* or *annuity*.
S = the *sinking fund*, or the excess of the annuity (a) over the interest on the principal for one period (Vt). $S = a - Vt$.
D = the *present value* of a *deferred annuity*, first payment being made at the end of $d + 1$ periods.
d = the *number* of periods during which the annuity is *deferred*. Then

To find the *present value* of any annuity.

(5) $V = \frac{a}{t}\left(1 - \frac{1}{r^n}\right)$ or $\log. V = \log.\left(1 - \frac{1}{r^n}\right) + \log. a - \log. t$; $\log. \frac{1}{r^n} = 0 - n \times \log. r$.

or (6) $V = \frac{a\,(r^n - 1)}{tr^n}$ or $\log. V = \log.(r^n - 1) + \log. a - \log. t - n \times \log. r$.

or (7) $V = \frac{a}{t} - \frac{a}{tr^n}$ or $\log. \frac{a}{t} = \log. a - \log. t$; and

$\log. \frac{a}{tr^n} = \log. a - \log. t - n \times \log. r$.

To find the *amount* of any annuity.

(8) $A = \frac{ar^n}{t}\left(1 - \frac{1}{r^n}\right)$ or $\log. A = \log.\left(1 - \frac{1}{r^n}\right) + \log. a + n \times \log. r - \log t$.

or (9) $A = \frac{a}{t}(r^n - 1)$ or $\log. A = \log.(r^n - 1) + \log. a - \log. t$.

or (10) $A = \frac{ar^n}{t} - \frac{a}{t}$ or $\log. \frac{ar^n}{t} = \log. a + n \times \log. r - \log. t$; and

$\log. \frac{a}{t} = \log. a - \log. t$.

To find the *annuity* which a given sum V will purchase.

(11) $a = \frac{Vt}{1 - \frac{1}{r^n}}$ or $\log. a = \log. V + \log. t - \log.\left(1 - \frac{1}{r^n}\right)$

or (12) $a = \frac{Vtr^n}{r^n - 1}$ or $\log. a = \log. V + \log. t + n \times \log. r - \log.(r^n - 1)$.

The period and rate being given, to find what *annuity* it would take *to amount* to a given sum (A) at the end of a certain number of periods.

(13) $a = \frac{At}{r^n\left(1 - \frac{1}{r^n}\right)}$ or $\log. a = \log. A + \log. t - n \times \log. r - \log.\left(1 - \frac{1}{r^n}\right)$

or (14) $a = \frac{At}{r^n - 1}$ or $\log. a = \log. A + \log. t - \log.(r^n - 1)$

To find the *number* of annuities.

(15) $r^n = \frac{a}{a - Vt}$ then $n = \frac{\log. a - \log. (a - Vt)}{\log. r}$ or, as $S = a - Vt$,

$$n = \frac{\log. a - \log. S}{\log. r}$$

or (16) $r^n = 1 + \frac{At}{a}$ then $n = \frac{\log. \left(1 + \frac{At}{a}\right)}{\log. r}$

To find *present value* of a *deferred* annuity.

(17) $D = \frac{V}{r^d}$ or $\log. D = \log. V - d \times \log. r$ (first find log. V by form 5, 6 or 7).

General formulæ applied to interest compounded, or convertible into principal, half-yearly, quarterly, &c.

10. The following Tables are based upon interest convertible *yearly* and *half-yearly*, and although payments are also made quarterly and monthly, these are based on interest convertible yearly and half-yearly. Should it be required to find the values of same rates *convertible half-yearly*, *quarterly* or *monthly*, &c., the following formulæ will apply. (t being the nominal rate for one year on $1 and n the number of years).

(18) For interest convertible half yearly.

$$M = P.\left(1 + \frac{t}{2}\right)^{2n} \text{ or } \log. M = \log. P + 2n \times \log. \left(1 + \frac{t}{2}\right)$$

(19) For interest convertible quarterly.

$$M = P.\left(1 + \frac{t}{4}\right)^{4n} \text{ or } \log. M = \log. P + 4n \times \log. \left(1 + \frac{t}{4}\right)$$

(20) For interest convertible monthly.

$$M = P.\left(1 + \frac{t}{12}\right)^{12n} \text{ or } \log. M = \log. P + 12n \times \log. \left(1 + \frac{t}{12}\right)$$

To find the true rate of interest for any given period of time.

Let $r = 1 +$ its true interest for one year. By considering p as 1 and n as 1, M will then represent r. Then raise this formula to the power represented by the number by which 1 year would have to be multiplied in order to produce the given time for which we wish to find the rate. The result will show 1 + its true interest for the given time, and this result minus 1 will leave the rate of interest.

Ex 1.—What is the rate of interest per 2 years at 10%, compounded half-yearly? Here 1 year has to be multiplied by 2 for 2 years ∴ we raise the formula to the power of 2. Then $r^2 = \left\{\left(1 + \frac{.10}{2}\right)^2\right\}^2$ or $r^2 = 1.05^4 = 1.2155$ rate = ·2155 per 2 years on 1, or 21.55%.

Ex. 2.—What is the rate per month at 10% convertible half-yearly? Here 1 year is multiplied by the fraction $\frac{1}{12}$ to make 1 month ∴ we raise the formula to the power of $\frac{1}{12}$. Then $r^{\frac{1}{12}} = \left\{\left(1 + \frac{.10}{2}\right)^2\right\}^{\frac{1}{12}} = 1.05^{2 \times \frac{1}{12}} = 1.05^{\frac{1}{6}}$. We then raise 1.05 to the power of 1 = 1.05, and extract the 6th root of the result $\sqrt[6]{1.05} = 1.008164$ . . Rate per month = ·008164 . . or 0.8164 . . % (see tabulated true rates).

Ex. 3 —What is the true rate for 7 days of 10% convertible half-yearly? Here 1 year has to be multiplied by $\frac{7}{365}$ to make 7 days $\therefore r^{\frac{7}{365}} = \left\{\left(1 + \frac{.10}{2}\right)^2\right\}^{\frac{7}{365}}$ $= 1.05^{2 \times \frac{7}{365}} = 1.05^{\frac{14}{365}}$. Here we have to raise 1.05 to the 14th power and then extract the 365th root of the result, or by logarithms, multiply log. r by 14 and divide the result by 365, and then, taking the corresponding number, $r^{\frac{7}{365}} = 1.00187$ .. Rate per 7 days = 0.187%.

Single payment Examples.

Ex. 1 (Amount, Form 1).—What is the amount of $527.75 put out at compound interest for 34 years at 4½% yearly? $P = 527.75$, $r = 1.045$, log. r = ·01911,62904, $n = 34$.

$$\log. 527.75 = 2{\cdot}7224282$$
$$34 \times \log. r = {\cdot}6499539$$
$$\log. M = 3{\cdot}3723821 = \$2357.12, \text{ answer.}$$

By Table VII, 4.4664 × 527.75 = $2357.14.

Ex. 2 (Present value, Form 2).—What is the present value and the discount of $3600 due after 7 years, the interest being 6% convertible half-yearly? $M = 3600$, $r = 1.03$, log. r = ·01283,72247, $n = 14$ half-years.

$$\log. 3600 = 3{\cdot}5563025$$
$$14 \times \log. r = {\cdot}1797211$$
$$\log. P = 3{\cdot}3765814;\ P = \$2380.02.$$
$$\text{Discount} = 3600 - 2380.02 = \$1219.98.$$

By Table I, .66112 × 3600 = $2380.03.

Ex. 3 (Rate, Form 3).—A Borrower returns $5000 for the use of $1335 during 30 years. What is the rate per cent. yearly? $M = 5000$, $P = 1335$, $n = 30$.

$$\log. 5000 = 3{\cdot}6989700$$
$$\log. 1335 = 3{\cdot}1254813$$
$$\log. r^{30} = {\cdot}5734887 \text{ and} \div 30,$$
$$\log. r = {\cdot}0191163,\ r = 1.045,\ \text{rate} = 4\tfrac{1}{2}\%.$$

By Tables, rate = 4½%. See Example § 7.

Ex. 4 (Time, Form 4).—How many years must $3000 be put out at 4% interest, compounded yearly, in order to amount to $102,358. Here $M = 102{,}358$, $P = 3000$, $r = 1.04$, log. r = ·0170334.

$$\log. M = 5{\cdot}0101218$$
$$\log. P = 3{\cdot}4771213$$
$$\log. M - \log. P = 1{\cdot}5330005 \text{ and} \div {\cdot}0170334 = 90 \text{ years.}$$

This example is beyond the limits of the Tables.

Ex. 5 (Amount, Forms 18, 19 and 20).—What will $1 amount to at the end of 50 years when put out at compound interest at 8%. the interest being convertible half-yearly, quarterly and monthly? Here $P = 1.00$, $t = .08$, $n = 50$, $\left(1 + \frac{t}{2}\right)^{2n} = 1.04^{100}$, $\left(1 + \frac{t}{4}\right)^{4n} = 1.02^{200}$, $\left(1 + \frac{t}{12}\right)^{12n} = 1.006^{600}$.

8% half-yearly.	8% quarterly.	8% monthly.
(18) log. P = 0·	(19) log. P = 0·	(20) log. P = 0·
100 × log. 1.04 = 1·7033339	200 × log. 1.02 = 1·7200344	600 × log. $1.0\frac{2}{3}$ = 1·7314129
log. M = 1·7033339	log. M = 1·7200344	log. M = 1·7314129
M = $50.5049	M = $52.4849	M = $53.8782

Annuity Examples.

Ex. 1 (Present value, Forms 5, 6 and 7).—What is the present value of 20 half-yearly instalments of \$1 at 10% interest convertible half-yearly? Here $a = 1$, $r = 1.05$, $t = .05$, log. $r = \cdot 02118,92991$, log. $t = \bar{2}\cdot 6989700$, $n = 20$ periods.

(5)

$$\log. 1 = 0\cdot$$
$$20 \times \log. r = \cdot 4237860$$
$$\log. \frac{1}{r^{20}} = \bar{1}\cdot 5762140$$
$$\frac{1}{r^{20}} = .3768895$$
$$1 - \frac{1}{r^{20}} = .6231105$$
$$\log. \left(1 - \frac{1}{r^{20}}\right) = \bar{1}\cdot 7945650$$
$$\log. a = 0\cdot$$
$$= \bar{1}\cdot 7945650$$
$$\log. t = \bar{2}\cdot 6989700$$
$$\log. V = 1\cdot 0955950$$
$$V = \$12.4622$$

(6)

$$20 \times \log. r = \cdot 4237860$$
$$r^{20} = 2.65330$$
$$r^{20} - 1 = 1.65330$$
$$\log. (r^{20} - 1) = \cdot 2183510$$
$$\log. a = 0\cdot$$
$${}^{*}C^{t} \log. r^{20} = \bar{1}\cdot 5762140$$
$$C^{t} \log. t = 1\cdot 3010300$$
$$\log. V = 1\cdot 0955950$$
$$V = \$12.4622$$

(7)

$$\log. r^{20} = \cdot 4237860$$
$$\log. t = \bar{2}\cdot 6989700$$
$$\log. tr^{20} = \bar{1}\cdot 1227560$$
$$\log. a = 0. - \log. tr^{20} = \cdot 8772440$$
$$\frac{a}{tr^{n}} = \$7\cdot 53779$$

Perpetuity or

$$\frac{a}{t} = \frac{1}{.05} = \$20.$$
$$V \quad \$12.46221$$

(By Table IV, 12.462 = p. v. 20 instalments).

11. From Form 5 it will be seen that the *present value* of an annuity of \$1 may be found by dividing the discount of a single payment of \$1 due at the end of the required number of periods by the interest on \$1 for a single period of such annuity, and the present value of any similar annuity can then be found by multiplying the result by the annuity.

Ex. 2 —Required the present value of an annuity of \$1, payable yearly, half-yearly, quarterly and monthly, for 20 years at 6% per annum, convertible half-yearly?

By Table I the present value of \$1 due 20 years hence = .30656 ∴ the discount = .69344. As shown in tabulated nominal and true rates the interest for 1 year = .0609; half-year = .03; quarter = .014889..; month = .0049386.. Then

$$\frac{.69344}{.0609} = \$11.3866 = \text{p. v. 20 yearly instalments of \$1 (see Table V).}$$

$$\frac{.69344}{.03} = \$23.115 = \text{p. v. 40 half-yearly instalments of \$1 (see Table IV).}$$

$$\frac{.69344}{.014889} = \$46.574 = \text{p. v. 80 quarterly instalments of \$1 (see Table III).}$$

$$\frac{.69344}{.0049386} = \$140.412 = \text{p. v. 240 monthly instalments of \$1 (see Table II).}$$

As shown by Form 9 the *amount* of an annuity may also be found in a similar manner by dividing the interest on \$1 to the end of the required number of periods (or the amount of \$1 − 1) by the interest on \$1 for a single period of the annuity.

* C^{t} is a contraction for arithmetical complement. It is the algebraic remainder after subtracting the number from zero or 0. By using the C^{t} we save subtractions, the addition of it to a number having the same effect as subtracting the original number, thus $z - x = z + C^{t}\,x$.

Ex. 3.—By Table VII. the amount of $1 at 3% yearly for 10 years = $1.3439 and − 1 = .3439 its interest. To find the amount of 10 yearly annuities of $1 at 3% yearly, divide .3439 by the interest on $1 for 1 period = 1 year or .03; $\frac{.3439}{.03} = \$11.463$. (See Table VIII).

Ex. 4.—(*Amount*, Forms 8, 9 and 10).

What is the amount of a yearly annuity of $10 for 20 years, at 3% interest, convertible yearly?

$a = 10$; $n = 20$; $r = 1.03$; $t = .03$; $\log. r = .01283,72247$; $\log. t = \bar{2}.4771213$

(8)

$$\begin{aligned}
\log. 1 &= 0 \\
20 \times \log. r &= .2567445 \\
\log. \frac{1}{r^{20}} &= \bar{1}.7432555 \\
\frac{1}{r^{20}} &= .553676 \\
1 - \frac{1}{r^{20}} &= .446324 \\
\log. \left(1 - \frac{1}{r^{20}}\right) &= \bar{1}.6496505 \\
\log. a &= 1 \\
C^t \log. t &= 1.5228787 \\
\log. r^{20} &= .2567445 \\
\log. A &= 2.4292737 \\
A &= \$268.704
\end{aligned}$$

(9)

$$\begin{aligned}
20 \times \log. r &= .2567445 \\
r^{20} &= 1.80611 \\
r^{20} - 1 &= .80611 \\
\log. (r^{20} - 1) &= \bar{1}.9063950 \\
\log. a &= 1 \\
C^t \log. t &= 1.5228787 \\
\log. A &= 2.4292737 \\
A &= \$268.704
\end{aligned}$$

(10)

$$\begin{aligned}
\log. a &= 1 \\
20 \times \log. r &= .2567445 \\
C^t \log. t &= 1.5228787 \\
\log. \frac{a\,r^{20}}{t} &= 2.7796232 \\
\frac{a\,r^{n}}{t} &= \$602.037
\end{aligned}$$

Perpetuity or

$$\frac{a}{t} = \frac{10}{.03} = 333.333$$

$$A = \$268.704$$

(By Table VIII, $26.8704 × 10 = $268.704)

Ex. 5.—(*Annuity*, Forms 11 and 12).

What quarterly annuity for 15 years will $2,038.75 purchase at 5½%, compounded half-yearly?

$V = 2,038.75$; $r = 1.013657$; $t = .013657$; $n = 60$; $\log r = .00589,09158$; $\log. t = \bar{2}.1353472$

(11)

$$\begin{aligned}
\log. 1 &= 0 \\
60 \times \log. r &= .3534549 \\
\log. \frac{1}{r^{60}} &= \bar{1}.6465451 \\
\frac{1}{r^{60}} &= .443144 \\
1 - \frac{1}{r^{60}} &= .556856 \\
\log. \left(1 - \frac{1}{r^{60}}\right) &= \bar{1}.7457412 \\
C^t \log. \quad " &= 0.2542588 \\
\log. V &= 3.3093640 \\
\log. t &= \bar{2}.1353472 \\
\log. a &= 1.6989700 \\
a &= \$50.00
\end{aligned}$$

(12)

$$\begin{aligned}
60 \times \log. r &= .3534549 \\
r^{60} &= 2.25660 \\
r^{60} - 1 &= 1.25660 \\
\log. (r^{60} - 1) &= .0991961 \\
C^t \log. \quad " &= \bar{1}.9008039 \\
\log. V &= 3.3093640 \\
\log. t &= \bar{2}.1353472 \\
60 \times \log. r &= .3534549 \\
\log. a &= 1.6989700 \\
a &= \$50.00
\end{aligned}$$

(By Table III $\frac{2038.75}{40.775} = \$50.00$)

Ex. 6 —(*Number*, Form 15).

How many monthly annuities of \$1 will it require to repay a loan of \$128 at 6%, convertible half-yearly?

$V = 128$; $a = 1$; r (Table A) $= 1.004938..$; $t = .004938..$ log. $r = \cdot 0021395$; $Vt = .632143$

$$\log. a = 0\cdot$$
$$\log. (a - Vt) \text{ or } S = \bar{1}\cdot 5656791$$
$$\log. \frac{a}{S} = \cdot 4343209$$

$\log. \frac{a}{S} = \cdot 4343209 \div \cdot 0021395$ (log. r) $= 203$ monthly instalments (very nearly).

(By Table II, 203 monthly instalments of \$1 = \$128.001).

Ex. 7.—(*Deferred Annuity*, Form 17).

A has a term of 9 years in an estate worth \$100 per annum; B has a term of 18 years in the same estate in reversion after the term of 9 years; C has a further term of 27 years in reversion after the 27 years, and D has the reversion in perpetuity after the 54 years. What is the present value of the interest of each in the estate, at 4%, compounded yearly?

$a = 100$; $r = 1.04$; $t = .04$; log. $r = \cdot 01703,33393$; log. $t = \bar{2}\cdot 6020600$; for A, $n = 9$; for B, $n = 18$ and $d = 9$; for C, $n = 27$ and $d = 27$; for D, $V = \frac{a}{t}$ or p. v. of perpetuity and $d = 54$

A

$$(6)\ 9 \times \log. r = \cdot 1533001$$
$$r^9 = 1.42331$$
$$r^9 - 1 = .42331$$
$$\log. r^9 - 1 = \bar{1}\cdot 6266604$$
$$\log. a = 2\cdot$$
$$C^t \log. t = 1\cdot 3979400$$
$$C^t \log. r^9 = \bar{1}\cdot 8466999$$
$$\log. V = 2\cdot 8713003$$
$$V = \$743.533$$

B

$$(6)\ 18 \times \log. r = \cdot 3066001$$
$$r^{18} = 2.02582$$
$$r^{18} - 1 = 1.02582$$
$$\log. r^{18} - 1 = \cdot 0110697$$
$$\log. a = 2\cdot$$
$$C^t \log. t = 1\cdot 3979400$$
$$C^t \log. r^{18} = \bar{1}\cdot 6933999$$
$$\log. V = 3\cdot 1024096$$
$$(17)\ 9 \times \log. r = \cdot 1533001$$
$$\log. D = 2\cdot 9491095$$
$$D = \$889.425$$

C

$$(6)\ 27 \times \log. r = \cdot 4599002$$
$$r^{27} = 2.88337$$
$$r^{27} - 1 = 1.88337$$
$$\log. r^{27} - 1 = \cdot 2749354$$
$$\log. a = 2\cdot$$
$$C^t \log. t = 1\cdot 3979400$$
$$C^t \log. r^{27} = \bar{1}\cdot 5400998$$
$$\log. V = 3\cdot 2129752$$
$$(17)\ 27 \times \log. r = \cdot 4599002$$
$$\log. D = 2\cdot 7530750$$
$$D = \$566.337$$

(By Table V).

p. v. of \$1 for 9 years = 7·4353 × 100 = 743·53 = A's share
" 27 " 16.3296
− 9 " 7·4353
8·8943 × 100 = 889·43 = B's "
" 54 " = 21.9930
− 27 " = 16·3296
5·6634 × 100 = 566·34 = C's "
2199·30

D = perpetuity $\frac{a}{t}$ = 2500.00
− (A + B + C) = 2199·30
D's share = \$300·70

12. Terminable annuities may be considered as being composed of two distinct parts—the Interest and Sinking Fund. The annuity, in order to pay off the principal, must exceed the interest. The difference between the annuity and the interest is termed the Sinking Fund. If S denotes the yearly sum appropriated to the Sinking Fund, the amount of which will produce the principal, we have $a = Vt + S$ and from Form 5 we deduce directly $a = Vt + \frac{a}{r^n}$, so that $S = \frac{a}{r^n}$. But a denotes the annuity and $\frac{a}{r^n}$ the present value of the n^{th} or last annuity, therefore the Sinking Fund is equal to the present value of the last annuity.

Ex. 8.—A loan of $500,000 is contracted by a corporation at 5%, payable half-yearly, to be reimbursed by means of 90 half-yearly annuities. What sum is to be paid as a Sinking Fund, besides the interest, for the purpose of redeeming this loan?

Here $V = 500{,}000$; $r = 1.025$; $t = .025$; $n = 90$; log. $r = \cdot 01072{,}38654$; Vt (interest) $= 12{,}500$; then

(12) $90 \times \log. r = 0\cdot9651479$	$\log. r^{90} = \cdot9651479$	a (annuity) $= \$14{,}019.04$
$r^{90} = 9.22886$	$\log. Vt = 4\cdot0969100$	Vt (interest) $= 12{,}500.00$
$r^{90} - 1 = 8.22886$	$C^t \log. (r^{90} - 1) = \bar{1}\cdot0846603$	S (sinking fund) $= \$\ 1{,}519.04$
$\log. (r^{90} - 1) = \cdot9153397$	$\log. a = 4\cdot1467182$	To prove $S =$ p. v. of last annuity.
	$a = \$14{,}019.04$	$\log. a = 4\cdot1467182$
		$\log. r^{90} = \cdot9651479$
		$\log. S = 3\cdot1815703$
		$S = \$1{,}519.04$

(By Table *IV*; $\frac{500{,}000}{35.666} = \$14{,}018.97$)

The sinking fund is a little over ½ of 1% per annum, consequently an addition annual rate of about 6 mills on the $ would extinguish the debt in the period of 45 years..

Ex. 9.—How many yearly annuities would be required to repay a loan of $200,000, at 5½% interest, payable yearly, 1% additional being added to form a sinking fund?

$V = 200{,}000$; $r = 1.055$; $t = .055$; log. $r = \cdot 02325{,}24596$; interest (Vt) $= 11{,}000$; Sinking Fund (S) $= 2{,}000$; a or ($Vt + S$) $= 13{,}000$

(14) $\log. a = 4\cdot1139434$

$\log. S = 3\cdot3010300$

$\log. (a \div S) = \cdot8129134$ and $\div \cdot 02325$ ($\log. r$) $= 34.96$ or 35 annuities (nearly).

($\frac{200{,}000}{13{,}000} = 15.3846$. In Table V, 35 years $= 15.3905$ and 34 years $= 15.2370$)

Ex. 10.—A Corporation issues 2,000 Debentures of $500 each, bearing 5% interest, and to be liquidated by means of 30 annuities. What sum is to be paid annually, and what number of Debentures will be redeemed in each of the first 3 years, and also the 20th year?

$V = 1{,}000{,}000$; $r = 1.05$; $t = .05$; $n = 30$; log. $r = \cdot0211893$; log. $t = \bar{2}\cdot6989700$; $Vt = 50{,}000$

(12) $30 \times \log. r = \cdot 6356790$ log. $V = 6 \cdot$ annuity = \$65,051.44

$r^{30} = 4.32194$ log. $t = \cdot 6989700$ interest = 50.000

$r^{30} - 1 = 3.32194$ log. $r^{30} = \cdot 6356790$ sinking fund = 15,051.44

log. $(r^{30} - 1) = \cdot 5213918$ $C^{t}\log.(r^{30} - 1) = \bar{1}\cdot 4766082$ 30 debentures, as a whole

log. $a = 4 \cdot 8132572$ number must be used.

a (mean annuity) = \$65 051.44

Therefore, the Corporation will have paid for the 1st and 2nd half-years' interest .... = \$50 000

and for redemption of 30 Debentures 15 000

Total, ... \$65 000

At the end of the 2nd year, mean annuity .... = \$65.051 44

Interest on 1,970 Debentures = \$985,000 .... = 49,250

Left for Sinking Fund \$15 801 44

This would redeem 31 more Debentures, which brings their number down to 1,939, and the Corporation would have paid for the 2nd year, interest on 1.970 Debentures ... . = \$49.250

and for redemption of 31 more .... = 15.500

\$64,750

at the end of the 3rd year, mean annuity .... = \$65,051.44

interest on 1,939 Debentures = \$969,500 .... = 48,475

Left for Sinking Fund \$16,576 44

which would redeem 33 Debentures. And so on for the balance of the time. For the number of Debentures redeemed in the 20th year, take the present value of the last instalment at that date and divide by 500.

(2) log. $a = 4 \cdot 8132572$

log. $r^{10} = \ \ 2118930$

log $S = 4 \cdot 6013642$; $S = \$39,935.96$ or nearly 80 Debentures.

13 It may be mentioned that the sums paid for the Sinking Fund are in a geometrical progression, similar to the progression of \$1 principal invested at the same rate of interest.

It often happens that Corporations or Companies, when they make all their bonds payable at one given date, have to lose at least 1% on the reinvestment of their Sinking Fund. It would, therefore, save them the work and risk of re-investment by making their bonds in the form of annuities, and discounting these so as to pay the buyer the given rate of interest; or otherwise, when the bonds are made to bear a given rate of interest, and all payable at the same time, they might bear an agreement that the Sinking Fund, or excess over interest, of the annual assessment for redemption of the whole issue should be used to redeem bonds to the amount of the sum which remains after paying the interest for each year on the balances remaining unpaid. This mode of payment has been adopted by various Companies, and the bonds to be redeemed are usually chosen by lot, and, as compensation for terminating the investment, some distribute prizes, and others an additional percentage, to those whose bonds are called in during the first few years. As the amounts applicable to redemption are relatively very small during the first years' of bonds running a long period, the cost of an additional ½% or 1% would not amount to one-tenth of the loss usually incurred in reinvesting the Sinking Fund.