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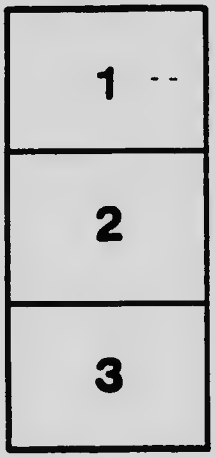
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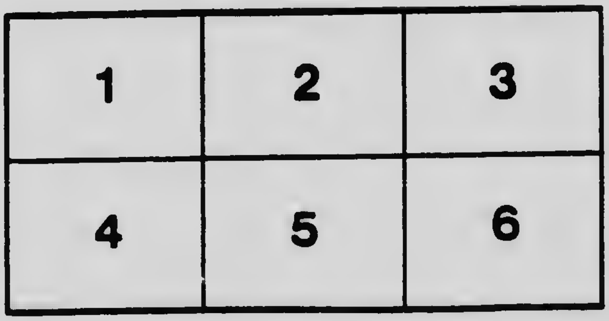
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W. L. Ellis

A BRIEF TREATISE

ON

PLANE TRIGONOMETRY

FOR

PRACTICAL SCIENCE STUDENTS

BY

N. F. DUPUIS, M.A.

QUEEN'S UNIVERSITY

KINGSTON

R. UGLOW & COMPANY

1902

W. J. Ellis

A BRIEF TREATISE ON

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MOSTLY ON THE PRACTICAL SIDE

AND INTENDED FOR PRACTICAL SCIENCE STUDENTS

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PREFACE.

This little book has been written as a help to students in practical trigonometry in the School of Mining. If it should prove to be helpful to anyone else, so much the better. It is not constructed along usual lines.

The exercises are many and varied, and are largely practical, while some of them are proofs of minor and readily obtained theorems. Exercises in transformations which may be beautiful and interesting, but are not of practical use, are not many.

The student is encouraged to work with natural functions, as in the experience and opinion of the writer these are more direct, more manageable with small angles, and fully as expeditious as logarithmic methods, and in some cases more so.

From a perusal of some modern works on the subject, a student would rise with the idea that logarithms are an essential and necessary part of trigonometry, and that nothing can be done without them. He who forms such an idea has failed to grasp the nature of the subject, and to understand the force and meaning of the trigonometric functions. In practical life men should learn to do their work with a minimum of appliances, and a small table of natural functions is all that is required in practical trigonometry.

For these reasons logarithms and logarithmic methods are relegated to the latter portions of the work, and the theory of logarithms is supposed to be learned where it should be learned, in connection with arithmetic and algebra.

CONTENTS.

A BRIEF TREATISE ON PLANE TRIGONOMETRY.

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1. Of Decimal Approximation.—In the practical measurement of lengths, angles, weights, etc., results are not usually integral or definitely fractional, but expressible by a series of decimal figures. Thus, in measuring the distance between two points with a scale divided to tenths of an inch, we might estimate the hundredths, and find the distance to be 3.14 inches say. If the measure were graduated to hundredths of an inch, we might estimate to thousandths approximately, and find the distance to be 3.141 inches. But in any case the final decimal figure in our result is only an approximation, and the expression is only an approximate value for the distance.

It is obvious that the more decimal places we include, other things being the same, the closer is the approximation.

Example.—The ratio of the circumference of a circle to its diameter, or of the semi-circumference to the radius is

$$3.1415926\ldots \qquad (1)$$

to seven decimal places. 3.14 is the approximate value to two decimals, and 3.1416 to four decimals; because, as we reject all after the 5, and 59 is nearer to 60 than to 50, we change the 5 into 6.

The majority of quantities occurring in trigonometry are of this nature, that is, they are decimal approximations carried to four, five, six, etc., decimal places.

2. Errors.—All practical measurements are affected by two sources of errors: (1) Errors of construction in the instrument employed, and (2) errors in making and recording the observation. And when the error of a decimal approximation is less than the errors due to observation, the approximation is sufficiently close for practical purposes.

Thus, it is not possible without the aid of a microscope to measure a distance to within a thousandth of an inch; hence, such a distance expressed in inches to the nearest unit in the third decimal place is sufficiently accurate as far as expression goes.

NOTE.—As this work will be largely practical, and will deal continually with four, five and six place decimals, students should learn and practise contracted methods of working with decimals, and study to become expert in their use.

EXERCISE I.

1. Find to six decimal places the length of the circumference of a circle whose radius is 1 mile or 5280 feet.

2. In regard to Ex. 1, find the error in feet resulting from taking 3.1 instead of 3.1415926. Also find the errors from taking 3.141; 3.1416.

3. In employing a six decimal approximation for 3.1415926 what number would you take, and why?

4. In expressing a length in miles, how many decimals are required to give it to the nearest foot? To the nearest inch?

5. In expressing an area in acres, how many decimals are required to give it to the nearest square foot? To the nearest square inch?

6. The area of a field is given as 18.7415 acres, which is truly expressed to the fourth decimal figure. What is the greatest possible error in the expression? Give result in square feet.

7. Multiply 1.4142136 by 0.7071068 to seven decimals in the product, using contracted multiplication.

8. Multiply 0.0471404 by 21.213204 to five decimals.

9. The sides of a rectangle are 13.3412 and 24.467 ft. Find the area to the nearest square inch.

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10. The measure 0.7312 is taken 0.085 times. Find to three decimals the length measured.

11. Divide 1 by 2.30258509 to seven decimals.

(The result of this division is the modulus of our common system of logarithms.)

12. Divide 180 by 3.1415926 to six decimals.

(The quotient gives the number of degrees in one radian.)

13. Work out to three decimals the value of

$$(7960 \times 5280 \times 3.1415926) \div 1296000.$$

(The result is the number of feet in one second of arc of latitude on the earth's surface.)

3. Measures and Units of Measure.—Every measure must be expressed in units of its own kind. Thus, lengths are expressed in units of length, such as mile, foot, inch, etc.; time in units of time, as year, day, hour, etc.

So also angles must be expressed in angular units.

Angle is generated by the rotation of a variable line about a fixed point. As the rotation may be with the hands of a clock or against them, angle may be negative or positive. Usually, but not necessarily, rotation against that of the hands of a clock is taken as positive.

A line which, starting from any given direction, makes a complete rotation, returning to its original direction, measures the simplest unit angle, the circumangle. This is subdivided thus:

$$1 \text{ circumangle} = 2 \text{ straight angles} = 4 \text{ right angles} = 360°,$$

$$\text{Then } 1 \text{ right angle} = 90°; \ 1° = 60', \ 1' = 60''.$$

Degrees, minutes, seconds of angle are marked $° \ ' \ ''$

This division of angle is very ancient, and is known as the *sexagesimal* or *degree* measure. It forms the basis of the majority of trigonometric tables.

4. Radian Measure; natural measure; circular measure of an angle.

It is proved in geometry that in the same circle the lengths of arcs are proportional to the angles which they subtend at the centre. So that if s be the length of an arc, and θ be the angle which it subtends at the centre, $s = m\theta$, where m is an arbitrary constant.

If m is taken to be the radius, then

$$s = r\theta \; .... \qquad (2)$$

and the resulting value of θ is taken to be the radian measure of the angle subtended by s.

Hence the length of an arc is *the angle which it subtends in radians at the centre* $\times$ *the radius.*

If $s = r$, then $\theta = 1$.

Therefore the unit of radian measure, or one radian, is the angle at the centre subtended by an arc equal in length to the radius.

5. Connection between the Units.—If c denotes the length of the semi-circumference of a circle, we know that

$$c = r \times 3.1415926...$$

Or, denoting 3.1415926.... by π, as is usual,

$$c = \pi r.$$

Hence π is the radian measure of two right angles.

$\therefore$ $180° = \pi^{\wedge}$, denoting radian by $^{\wedge}$.

$$\left.\begin{aligned} &\text{Hence} \quad 1° = \frac{\pi}{180} = 0^{\wedge}.017453... \\ &\text{and} \quad 1^{\wedge} = \frac{180}{\pi} = 57°.29578.... \end{aligned}\right\} .... \qquad (3)$$

These numbers, 0.017453, the multiplier by which to change degrees into radians, and 57.29578 which changes radians to degrees, should be carefully remembered.

Thus, $64° = 64 \times 0.01745 = 1.11680...$ radians

and $0^{\wedge}.71654 = 0.71654 \times 57.29578 = 41°.045....$

For small angles, say less than 1°, the length of the chord may be taken for that of the arc in practical work without any material error, and the error reduces rapidly as the angle diminishes.

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EXERCISE II.

1. Express $36° 14' 20''$ in degrees and decimals of a degree.

2. Express $25' 15''.34$ as a decimal of a degree.

3. Express $54''.35$ as a decimal of a degree.

4. Express $3°.8472$ in degrees, minutes and seconds.

5. With radius 1 mile, find the length in inches of the arc which subtends an angle of $1''$.

6. The earth's radius being 3,980 miles, find the number of seconds in 1 foot of arc on its surface.

7. Express $1^{\wedge}.0472$ in degrees.

8. Express $50° 47' 57''$ in radians.

9. A house at the distance of a mile subtends a horizontal angle of $35' 44''$. Find the dimensions of the house.

10. A tree is known to be 76 feet high, what angle will it subtend at the distance of a mile?

11. How far from the eye must a disc 2 feet in diameter be held that it may just hide the sun, the angular diameter of the sun being $32'$?

12. How far must a man 6 feet tall go away from camp that he may subtend an angle of $50'$?

13. A man 5 feet 8 inches tall standing upon the opposite bank of a river subtends an angle of $18'$. What is the breadth of the river?

14. A wheel 12 feet radius revolves 12 times per minute. Find the rate per second at which the rim travels.

15. *A* and *B* are on the same meridian. *A's* latitude is $32° 14' 12''$ N., and *B's* is $27° 15' 40''$ N. Find the distance from *A* to *B*, the earth's radius being 3,980 miles.

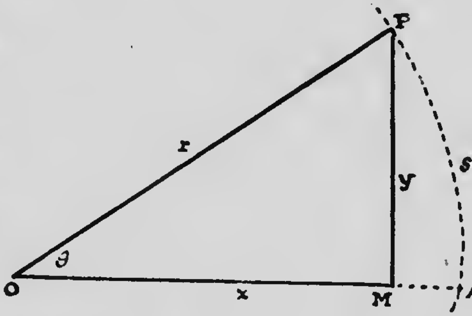
16. If the difference in latitude of *A* and *B* (Ex. 15) be $1° 6' 49''$, and their distance apart be 77.3 miles, find the earth's diameter.

17. The earth's distance from the sun is 93,000,000 miles, and it makes its annual revolution in 365.2422 days. Find its velocity in miles per second.

18. The moon's distance from the earth is 237,000 miles, a she performs her revolution about the earth in 27.32166 day Over how many degrees does she move per hour? Also ho many miles?

TRIGONOMETRIC RATIOS, OR TRIGONOMETRIC FUNCTIONS.

6. Assume any triangle, OPM, right-angled at M. Deno OM by x, MP by y, and OP by r.



Then OP may be considered as the radius of a circle passing through P and A, and the angle POA, or θ, as being generated by the radius rotating from position OA to OP. A downward rotation of OA would give a negative angle; it being remembered, however, that it is only in t case of comparison of angles that there arises any necessity f using the terms positive and negative.

From (2) the angle θ is the ratio $s : r$.

The other ratios with which we are here concerned are

(1) Three leading functions.

$\frac{y}{r}$ = sine θ, contracted to sin θ,

$\frac{x}{r}$ = cosine θ, " " cos θ,

$\frac{y}{x}$ = tangent θ, " " tan θ.

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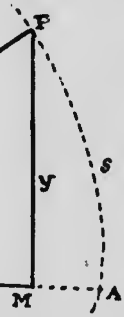

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(2) The reciprocals of (1).

$\frac{r}{y}$ = cosecant θ, contracted to cosec θ, or cox θ,

$\frac{r}{x}$ = secant θ, " " sec θ,

$\frac{x}{y}$ = cotangent θ, " " cot θ.

This assumed triangle, with its notation, and the names of the particular ratios must be carefully mastered and remembered. The following statements may help:

When r is denominator the ratio is sine, or cosine; sine when the other side is opposite the angle, cosine when adjacent.

When r is numerator, we have secant or cosecant; cosecant when the other side is opposite, secant when adjacent.

When r does not occur, we have tangent, or cotangent.

From the figure of this article, $\cos\theta = \frac{x}{r} = \sin OPM$. But OPM is the complement of θ. Therefore $\cos\theta$ = sine of complement of θ = comp. sine of θ = cosine θ, contracted to $\cos\theta$. And similarly for other co-functions.

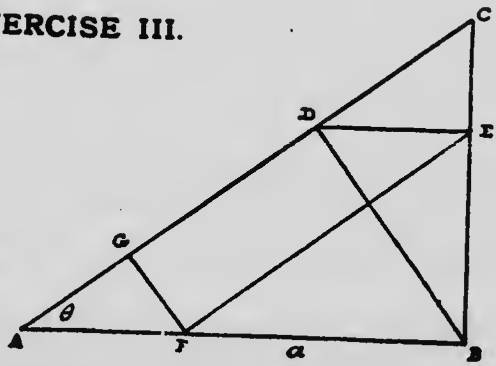
Thus, as 90° and $\frac{\pi}{2}$ both denote right angles,

$$\sin\theta = \cos(90° - \theta) \text{ or } \cos\left(\frac{\pi}{2} - \theta\right);$$

$$\cot\theta = \tan(90° - \theta);$$

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right), \text{ etc., etc.}$$

EXERCISE III.



The △ ABC is right-angled at B, and the angle BAC is θ, and $AB = a$. BD is ⊥ to AC, DE to BC, EF is parallel to AC, and FG is ⊥ to AC. Express the following in terms of a and functions of θ:

1. *BC.*	2. *AC.*	3. *BD.*
4. *BE.*	5. *DE.*	6. *EF.*
7. *FB.*	8. *FG.*	9. *AF.*
10. *AG.*	11. *GD.*	12. *EC.*

7. Tables giving the values of the foregoing ratios for ev degree and minute from 0° to 90°, or through a right an are called trigonometric tables, and in particular, tables of *natural functions*, to distinguish them from the *logarithms* these quantities, which are tabulated under the name of log ithmic functions. We shall confine our attention at present natural functions.

The tabular quantities are given to a certain number decimal places, usually not less than 4 and not more than and we thus have 4-place, 5-place, ... 7-place tables.

These tables serve, among other purposes, to give us a requir function of a given angle, and within certain limits, to give the angle when any one of its functions is known.

On account of variations in their construction, general dir tions for "working" a set of tables cannot be given. But t writer would tender this advice: Become expert at worki with 4 and 5-place decimals, and employ natural functio rather than logarithmic ones. Natural functions are more dire and simple in their applications, and present less opportunity f errors of work. Besides, to a person skilful in decimals, oper tions with natural functions are even more expeditious tha with logarithmic functions.

8. Inter-relation of the Functions.—The six function are so inter-related that when one is given the others can b found, so that all the functions are not necessary, but they ar very convenient.

The most prominent inter-relations, which are easily deduce from the definitions (Art. 6), are as follows :

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1. Relations giving unity—

(*a*) $\sin^2 \theta + \cos^2 \theta = 1.$

(*b*) $\sec^2 \theta - \tan^2 \theta = 1.$

(*c*) $\operatorname{cosec}^2 \theta - \cot^2 \theta = 1.$

(*d*) $\tan \theta \cot \theta = 1.$ (4)

2. Other relations—

(*e*) $\tan \theta = \dfrac{\sin \theta}{\cos \theta}.$

(*f*) $\sin \theta = \dfrac{\tan \theta}{\sec \theta}.$

EXERCISE IV.

1. From the equilateral triangle prove that

(*a*) $\sin 60° = \cos 30° = \frac{1}{2}\sqrt{3}.$

(*b*) $\sin 30° = \cos 60° = \frac{1}{2}.$

2. From the square prove that $\sin 45° = \cos 45° = \frac{1}{2}\sqrt{2}.$

3. Given $\sin \theta = \frac{1}{3}$, find $\cos \theta$ and $\tan \theta$.

4. Given $\tan \theta = 5$, find $\sin \theta$ and $\cos \theta$.

5. Given $\tan \theta = \dfrac{b}{a}$, show that

$$\sin \theta = \frac{b}{\sqrt{a^2 + b^2}}, \quad \cos \theta = \frac{a}{\sqrt{a^2 + b^2}}.$$

6. Given $\sin \theta = \dfrac{m}{n}$, show that

$$\tan \theta = \frac{m}{\sqrt{n^2 - m^2}}, \quad \sec \theta = \frac{n}{\sqrt{n^2 - m^2}}.$$

(Results of 5 and 6 should be remembered.)

7. If $\tan \theta = \frac{3}{4}$, find $\sin \theta$, $\cos \theta$, $\sec \theta$.

8. Express $(\tan \phi + \cos \phi) \cot \phi \, / \sec \phi$ in terms of x, where $x = \sin \phi$.

EXERCISE V.

(In these Exercises natural functions are to be used, and the results completely worked out.)

1. A plank 12 feet long on a level floor has one end raised until it describes an arc 3 feet 4 inches in length. Find (1) the

angle that the plank makes with the floor; (2) the perpendi-lar height of the raised end.

2. One end of the plank of Ex. 1 is placed on a box 22 inc high. What angle does the plank make with the floor?

3. A saw-cut through a board 20 inches wide is 23.45 inc long. Find the angle which the cut makes with the edge of t board.

4. Across a board 12 inches wide a line is to be drawn fr edge to edge so as to be 16.97 inches long. What angle mu the line make with the edge?

5. A line is drawn from one vertex of a square to the mi point of one of the sides. Find the angle which this line mak with either diagonal.

6. A hill has an elevation of 36° 12′. In going 120 feet the hill how far does one go (*a*) upon the level? (*b*) upon t vertical?

7. A post 6 feet high stands vertically in level ground. Fi the length of its shadow when the elevation of the sun 18° 40′.

8. A vertical post 3 feet high in level ground casts a shado 5 feet $8\frac{1}{4}$ inches. Find the sun's elevation.

9. In the triangle ABC the angle $A = 64° 18'$, $B = 44° 13'$, an the altitude to side AB is 24. Find the remaining parts of th triangle.

10. The legs of a pair of compasses are each 6 inches lon How far apart must the points be that the lines from points t centre may make an angle of 24° with each other?

11. Given two sides of a triangle 44 and 37, and the altitud to side 44, 32, to find the angles of the triangle.

12. Given the base-angles of a triangle and the altitude to th base, find the sides.

13. In the triangle ABC, $A = 58° 42'$, $B = 36° 20'$, and th side $AB = 20$. Find the other parts.

(Draw $CD \perp$ to AB. Put $AD = x$, $CD = y$. Then $y = x$ $\tan A = (c - x) \tan B$; whence x is known.)

14. Given two sides of a triangle and the altitude to the third side, to find the remaining parts.

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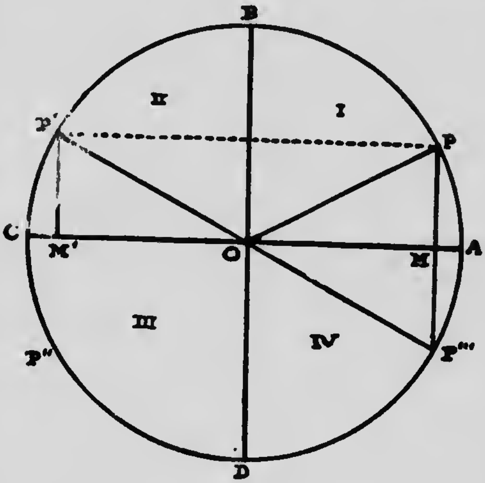
9. Complement and Supplement. Negative Angle.

Divide the circle into four parts by two orthogonal diameters, AC and BD. The section AOB is the first quadrant, and any angle lying in this, as AOP, is > 0 and $< 90°$. BOC is the second quadrant, and any angle lying in this, as AOP', is $> 90°$ and $< 180°$. Similarly, an angle in the third quadrant is $> 180°$ and $< 270°$; and in the fourth quadrant, it is $> 270°$ and $< 360°$.



The angle POB is the complement of AOP, and *vice versa*, since together they make up the right angle AOB.

But $$\sin POB = \frac{PN}{OP} = \frac{OM}{OP} = \cos AOP.$$

Hence the cosine of an angle is the sine of its complement, and the sine of an angle is the cosine of its complement.

(See Art. 6.)

Again, $\angle\ AOP + \angle\ POC$ make up two right angles, or a straight angle, and are therefore supplementary to one another.

But if $AOP = P'OC$, then $AOP' = POC$, and therefore AOP and AOP' are supplementary; and $P'M' = PM$.

Then, $$\sin AOP' = \frac{M'P'}{OP'} = \frac{MP}{OP} = \sin AOP;$$

(*a*) Or, *the sine of an angle is the same as the sine of its supplement.*

Again, we are taught in geometry that if segments be measured along the same line in opposite directions, they have contrary signs. Therefore, if OM is positive, OM' is negative, and *vice versa.*

Now, $$\cos AOP' = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos AOP.$$

(*b*) Hence, *the cosine of an angle and the cosine of its sup*[p]*ment are equal in numerical value, but have opposite algebr*[aic] *signs.*

Again, an angle of a triangle cannot be greater than 180°, a[nd] must always, therefore, be confined to the first two quadran[ts]. Angles in the third and fourth quadrants, as those determin[ed] by P'' and P''', may be considered as taken negatively, in re[la]tion to the angles of a triangle. Thus, AOP''' is got by rotati[ng] OA backwards in relation to AOP, and AOP''' is according[ly] $-AOP$,

But the $$\sin AOP''' = \frac{MP'''}{OP'''} = \frac{-MP}{OP} = -\sin AOP.$$

And $$\cos AOP''' = \frac{OM}{OP'''} = \frac{OM}{OP} = \cos AOP.$$

(*c*) Therefore, *the sine of a negative angle is minus the sine* [of] *the equal positive angle;* and *the cosine of a negative angle is t*[he] *same as the cosine of the equal positive angle.*

Or, when an angle changes sign, the sine of the angle chang[es] sign, and the cosine remains unchanged.

EXERCISE VI.

1. Given sin. 21° 35′ = 0.36785, to find the sine and tangen[t] of 68° 25′.

2. Given tan. 75° = 3.732, to find the sine and cosine of 15°.

3. Trigonometric tables extend only to 90°. Show how, fro[m] the tables, to find (*a*) sin. 123°; (*b*) cos. 165° 44′; (*c*) tan. 105°

4. Prove that the tangent of an angle and the tangent of it[s] supplement are equal in magnitude and opposite in sign.

5. Show that the limits of magnitude for the sine of an angl[e] are -1 and $+1$; that the cosine has the same limits; and tha[t] the limits of the tangent are from $-\infty$ to $+\infty$.

6. Make a table of the variations of the sine, cosine, tangent and secant in each quadrant.

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7. Taking a horizontal line-segment $= 6.28$ in length to represent the circumference of the circle with radius 1, construct the graphs of the sine, of the cosine, of the tangent, and of the secant.

8. Show from the graphs, or from the table of Ex. 6, that a magnitude changes sign when it passes through zero or infinity.

NOTE.—The graph of the sine is called the *sinusoid*, and is a curve of some importance.

THE TRIANGLE.

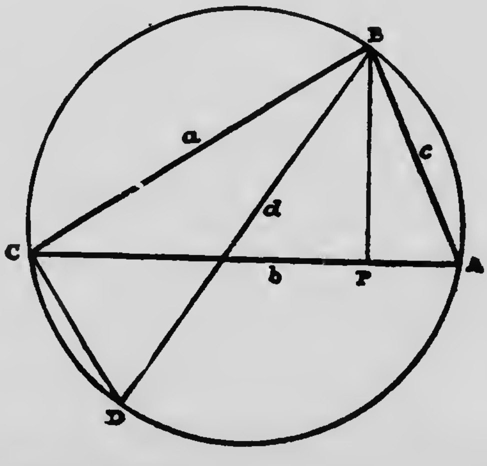
10. It is shown in geometry that a triangle is given or known when any three of its parts are given, except the three angles, and two sides and the angle opposite the shorter side. When the three angles alone are given, the triangle is given in *form*, but not in magnitude; and when two sides and the angle opposite the shorter side are given, there are two triangles, in general, satisfying the conditions, and the triangle is said to be *ambiguous.*

The trigonometric solution of triangles consists in finding the remaining parts of a triangle, when three parts, sufficient for its determination, are given. And the practical problems belonging to trigonometry come largely under this head.

With the use of natural functions the general solution of triangles is effected mostly by two direct formulas, which we proceed to develop.

(*a*) The sine formula.

ABC is a triangle, and *BP* is the altitude to side *CA*.



Then $BP = a \sin C = c \sin A$,

$$\therefore \frac{a}{\sin A} = \frac{c}{\sin C} = \frac{b}{\sin B}$$

by symmetry.

And this is the sine formula.

Its statement is: *In any triangle the sides are proportional the sines of the opposite angles.*

(*b*) The circumcircle.

Let BD be the diameter of the circumcircle of ABC.

Then $\angle CDB = \angle CAB = \angle A$, since they stand on the sar arc CB.

But $\sin CDB = \frac{CB}{BD} = \frac{a}{d} = \sin A$, or, $\frac{a}{\sin A} = d$, and hence

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = d \ldots . \qquad ($$

This is, in a way, the completion of the sine formula.

EXERCISE VII.

1. In a triangle $a = 20$, $b = 26$, $A = 35° 22'$, to find all th other parts.

2. In a triangle $a = 35$, $b = 48$, $A = 62° 40'$, to find the oth parts. Explain the difficulty here.

3. In a triangle $A = 51° 20'$, $B = 16° 35'$, $c = 17.45$, to find th other parts.

4. In a triangle $a = 24.60$, $c = 45.33$, $B = 67° 15'$, to find th other parts.

(Draw CD perpendicular to side c. Then $BD = a \cos$ and $CD = a \sin B$, and BD and CD are known. Then $CD = A$ $\tan A$; whence $\angle A$ is known. Therefore, etc.)

5. A post 18 feet long leans to the north at an angle of 20 with the vertical. Find the length of its shadow on leve ground, when the sun is south and at an elevation of $47° 50'$.

6. Solve Ex. 5 on condition that the post leans to the eas at the same angle.

7. Solve Ex. 5 on the condition that the post leans to th south at the same angle.

8. A triangle right-angled at B has $a = 4$, $c = 10$, and the lin BD meets AC in D, and makes the angle $CBD = 75°$. Find th length of BD.

9. If in Ex. 8, $a = 6$, $c = 13$, at what angle with AB must B be drawn, meeting AC in D, so that BD may be 10?

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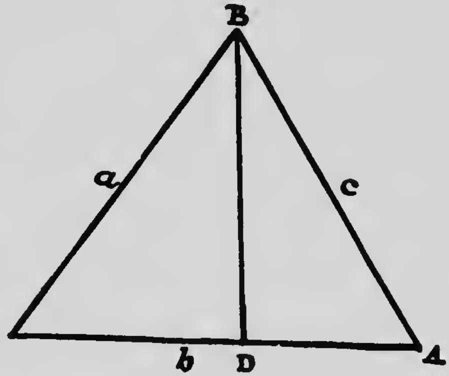
10. The sides of a triangle are 13, 14, 15, and the diameter of its circumcircle is 16.25. Prove that this is correct, by showing that the sum of its three angles is 180°.

11.—The Cosine Formula.—This may be developed in several different ways, but the following is one of the simplest.

ABC is a triangle, and *BD* is the altitude to *AC*.



Then $AD = c \cos A$;

$BD = c \sin A$.

But $BC^2 = BD^2 + DC^2 = (c \sin A)^2 + (b - c \cos A)^2$.

Or, squaring and collecting,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

And we have the following sets of forms, which are practically all the same, being obtained by the principles of symmetry.

$$(a)\begin{cases} a^2 = b^2 + c^2 - 2bc \cos A \\ b^2 = c^2 + a^2 - 2ca \cos B \\ c^2 = a^2 + b^2 - 2ab \cos C \end{cases}$$

$$(b)\begin{cases} \cos A = \dfrac{b^2 + c^2 - a^2}{2bc} \\ \cos B = \dfrac{c^2 + a^2 - b^2}{2ca} \\ \cos C = \dfrac{a^2 + b^2 - c^2}{2ab} \end{cases} \quad \ldots \quad (6)$$

Cor.—If any of the angles concerned are greater than 90° in (*a*), we must remember that its cosine is negative, Art. 9 (*b*), and treat it accordingly.

In (*b*), if the value of the fraction is negative, *i.e.*, if the numerator be negative, this indicates that the angle is greater than 90°.

EXERCISE VIII.

1. If $a=26$, $b=28$, $c=30$, find the angles.

2. When $a=24.3$, $b=17.75$, $A=27^\circ\ 15'$, $B=19^\circ\ 16'$, find t side c.

3. Given $a=15.71$, $b=18.37$, $A=14^\circ\ 47'$, $B=162^\circ\ 38'$, to fi the side c.

4. Given $a=42.3$, $b=56.1$, and $C=37^\circ\ 44'$, to find the remai ing parts.

5. Starting from A, I measure off 320 rods in a certain dire tion to B. I then change my direction through $42^\circ\ 50'$ an measure off 480 rods to C. How far is it from A to C in straight line?

6. The road from A to B goes by way of C. From A to C i 23 miles direct north, and from C to B is 42 miles 27° east c north. How much will the road from A to B be shortened b making it direct, and what will be its direction?

7. Starting from A, I wish to measure a 10-mile straight line Arriving at B, 4 miles from A, I find a large swamp. I turn t the right 50°, and go 2.5 miles. I then go to the left 97° from my previous course. How far must I go to strike my first lin at C, and what distance intervenes between B and C?

8. In any triangle show that $\sin A=\sin(B+C)$.

9. In any triangle show that $a=b\cos C+c\cos B$, with two symmetrical expressions.

10. From Ex. 9 eliminate a, b and c, by means of the sine formula, and show that $\sin(A+C)=\sin A\cos C+\cos A\sin C$.

11. The sides of a parallelogram are a and b, and the angle between them is θ; show that the two diagonals are

$$\sqrt{(a^2+b^2-2ab\cos\theta)} \text{ and } \sqrt{(a^2+b^2+2ab\cos\theta)}.$$

12. From Ex. 9 prove the cosine formula by transposing $b\cos C$, and squaring.

13. If one diagonal of a parallelogram is double the other, then $\cos\theta=\frac{3}{10}\left(\frac{a}{b}+\frac{b}{a}\right)$.

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14. If one diagonal of a parallelogram is a mean proportional between the sides, the other diagonal is $\sqrt{2(a+b)^2 - 5ab}$.

15. The sides of a quadrilateral inscribed in a circle are 4, 7, 5, 8. Find the angles of the figure.

12. The Ambiguous Case.—When an angle of a triangle is determined by its sine alone, it may be either a certain angle or its supplement, since these have the same sine.

Thus, if $\sin A = 0.5$, then A is either 30° or 150°, and the angle is thus ambiguous. And if there is nothing in the nature of the triangle which excludes one of these values, the triangle is ambiguous, or may have either one of two different forms.

An example will make this plain.

Ex. 1.—Given $a = 50$, $b = 25.87$, and $A = 30°$, to find B.

Here $\quad \sin A = 0.5$ and $\sin B = \frac{25.87}{50} \times 0.5 = 0.2587$,

and $\quad B = 15°$ or $165°$.

But as the side a opposite the given angle A is greater than the side b opposite B; $\therefore \angle A$ is $> \angle B$, and hence 165° must be rejected, and the triangle is determinate.

Ex. 2.—Given $a = 14.14$, $b = 20$, $A = 30°$, to find B.

$$\sin B = \frac{20}{14.14}, \quad \sin A = \frac{20}{14.14} \times 0.5 = 0.707,$$

$$\therefore B = 45° \text{ or } 135°.$$

And as a is $< b$, either of these angles may belong to the triangle, or it is ambiguous, having either one of two forms.

13. Area of the Triangle, in terms of two sides and the included angle.

It is shown in geometry that the area of a triangle is one-half the product of the base and altitude.

Now, fig. of Art. 11, $BD = c \sin A$, and the base is b.

$$\therefore \Delta = \frac{1}{2} bc \sin A \dots . \qquad (7)$$

with two symmetrical expressions.

Dividing a by each number of (7) gives

$$\frac{a}{\Delta}=\frac{2}{bc}\cdot\frac{a}{\sin A}=\frac{2}{bc}\cdot d,$$

whence $$d=\frac{abc}{2\Delta};$$

and $$R=\frac{abc}{4\Delta}. \quad \Bigg\}\cdots\cdot \quad (8$$

And (8) gives the circum-diameter or circum-radius of the tr angle in terms of its sides and area.

Again, $$\sin^2 A=1-\cos^2 A=1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2, \ (6)b$$

and factoring the right side as the difference of two squares, w get

$$\sin^2 A=\frac{(b+c)^2-a^2}{2bc}\cdot\frac{a^2-(b-c)^2}{2bc}$$

$$=\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}.$$

And denoting $a+b+c$ by $2s$, and therefore $a+b-c$ by $2(s-c)$ etc., reduces the expression to

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}. \quad (9)$$

But, (7) $$\Delta=\frac{1}{2}bc\sin A.$$

$$\therefore\ \Delta=\sqrt{s(s-a)(s-b)(s-c)}. \quad (10)$$

And this gives the area of the triangle in terms of its three sides.

EXERCISE IX.

1. Show that the area of a triangle is $\frac{1}{4}\sqrt{2\Sigma a^2b^2-\Sigma a^4}$; (expand the expression for $\sin^2 A$ without factoring).

2. Find the sines of the angles of the triangle whose sides are 13, 14, 15.

3. Find the area of the triangular field in which two sides are 14 and 23 rods, and the included angle is 76° 17′.

4. Find the area of the triangle whose sides are 52, 56 and 60.

5. Two sides of a triangle are respectively 14 feet and 23 feet,

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and the area is 125 square feet. Find the angle between the sides. Point out any ambiguity and explain it.

6. In Ex. 5 the area is 172. Explain the difficulty which arises.

7. Given two sides of a triangle, what must be the angle between them that the area may be greatest?

8. A triangle is to be inscribed in a circle of 20 feet radius, and two sides are to be 16 feet and 12 feet. Find the other parts of the triangle.

9. The diameter of a circle is 125 feet. Find the side of the equilateral triangle inscribed in it.

10. The sides of a triangle are 78, 91, 100. Find the distance from a vertex of a point equidistant from the three vertices.

11. The sides of a triangle being a, b, c, find the length of the median m to the side b.

(If M be the foot of the median, and ϕ be the angle which it makes with the base, apply the cosine formula to the triangles BAM and BCM.)

12. In Ex. 11, to find the angle ϕ. Find its sine, its cosine, and its tangent.

13. If ϕ be the angle which the long diagonal of the parallelogram a, b, θ, makes with side b, show that $\tan\phi = \dfrac{a \sin\theta}{b + a\cos\theta}$.

ANGLES AS AUXILIARIES.

14. Angles and their functions are often conveniently employed in obtaining solutions of problems in two different ways: (1) When neither the angle nor any of its functions appears in the final result. These angles may be called auxiliaries by elimination.

And (2) when the solution is effected principally through the inter-relation of functions, and some of these functions remain in the final result.

As an example of (1), let $\sin\theta = a$, and $\tan\theta = b$, to find the relation between a and b.

Since $$\sin\theta = \frac{\tan\theta}{\sec\theta} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

$$\therefore a = \frac{b}{\sqrt{1+b^2}},$$

which is the required relation.

As an example of (2), let it be required to find the value of $\sqrt{a^2+b^2}$, where a and b are numbers.

Dividing by a, we have $\sqrt{a^2+b^2} = a\sqrt{1+\frac{b^2}{a^2}}$.

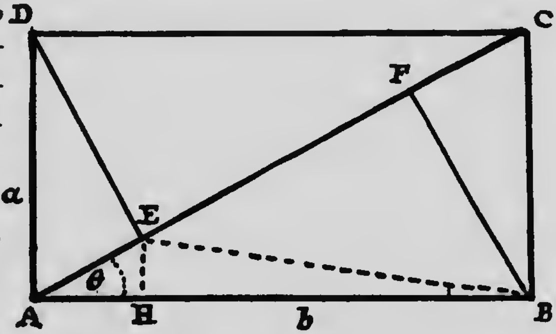
Now, if $\frac{b}{a} = \tan\theta$, then $1+\frac{b^2}{a^2} = 1+\tan^2\theta = \sec^2\theta$,

$$\therefore \sqrt{a^2+b^2} = a\sec\theta.$$

Hence we find θ where $\tan\theta = \frac{b}{a}$, and then multiply $\sec\theta$ by a for the solution.

The angle θ may be called an auxiliary by inter-relation of functions.

EXERCISE X.



ABCD is a rectangle, having *BF* and *DE* perpendiculars on the diagonal *AC*, and *EH* parallel to *AD*.

Let the sides *AB* and *AD* be denoted by b and a, and the angle *CAB* be denoted by θ.

1. To find the ratio of $a : b$ when $AC = m.BF$.

($AC = b\sec\theta$, and $BF = a\cos\theta$. $\therefore b\sec\theta = ma\cos\theta$, whence by eliminating θ, $a : b = \frac{1}{2}(m \pm \sqrt{m^2-4})$).

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2. Given the sides a and b, to find BE.

(From cosine form. $BE^2 = CE^2 + CB^2 - 2CE.CB \cos ACB$, or, $BE^2 = (b \cos \theta)^2 + a^2 - 2a(b \cos \theta) \sin \theta$.

But $\tan \theta = \frac{a}{b}$; and eliminating θ between these two equations, and reducing, $BE = \sqrt{\left\{\frac{a^4 - a^2b^2 + b^4}{a^2 + b^2}\right\}} = \frac{\sqrt{a^6 + b^6}}{a^2 + b^2}$.)

3. Find $a : b$ when $DE = m.EF$.

4. Find $a : b$ when $EF = m.AC$.

5. Prove that $BE = \sqrt{(AC^2 - 3BF^2)}$.

6. Prove the following areas :

(i.) $\triangle\ BEC = \frac{1}{2} \cdot \frac{ab^3}{a^2 + b^2}$.

(ii.) $\triangle\ BEH = \frac{1}{2} \cdot \frac{a^3b^3}{(a^2 + b^2)^2}$.

(iii.) $\triangle\ HEA = \frac{1}{2} \cdot \frac{a^5b}{(a^2 + b^2)^2}$.

(iv.) $\square\ BEDF = ab \cdot \frac{b^2 - a^2}{b^2 + a^2}$.

7. Prove that $\cos EBF = ab / \sqrt{a^4 - a^2b^2 + b^4}$.

8. Show that $\tan ABE = \tan^3 BAE$.

9. Show that $\sin ABE = \frac{a^3}{\sqrt{a^6 + b^6}}$.

10. Obtain a solution of $\sqrt{a^2 - b^2}$ by inter-relation of functions, and apply the method to find the value of

$$\sqrt{(3.146)^2 - (1.432)^2}.$$

If the figure $ABCD$ is a parallelogram, with the angle at $A = \omega$, prove the following:

11. $EC = \frac{b}{d}(b + a \cos \omega)$ where d is the diagonal AC.

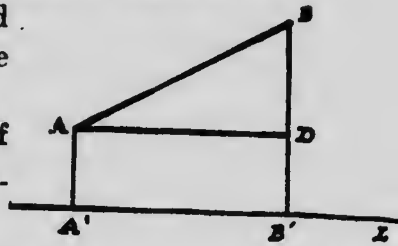
12. $AE = \frac{a}{d}(a + b \cos \omega)$.

13. The tangent of the angle between the diagonals is $2ab \sin \omega / (b^2 - a^2)$.

PRINCIPLE OF ORTHOGONAL PROJECTION.

15. AB is a line-segment, and L is any line. AA' and BB' are perpendicular to L.

Then $A'B'$ is the projection of AB on L; and $B'A'$ is the projection of BA on L.



Let AD be parallel to L; then $AD = A'B'$, and the $\angle\ BAD$ is the angle between AB and L.

Now $AD = A'B' = AB \cos. BAD$.

Therefore, *the projection of a given line-segment on any line is the length of the segment multiplied by the cosine of the angle which the segment makes with the line.*

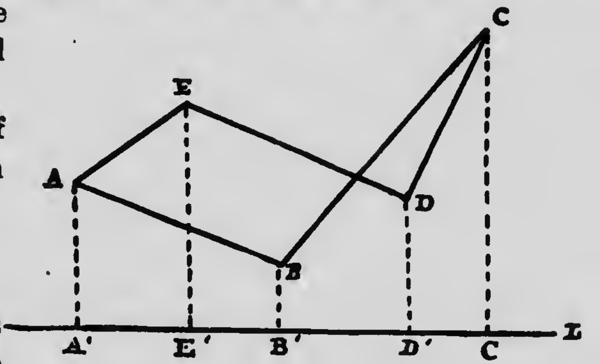
16. Theorem.—The sum of the projections, upon any line, of the sides of a closed polygon taken in order is zero.

$ABCDEA$ is the closed polygon, and L is any line.

The projections of the sides taken in order are

$A'B' + B'C' + C'D' + D'E' + E'A'$.

But this sum is evidently zero, since we start from A' and end at A'.



In taking projections analytically we must pay particular attention to the quadrant in which the segment to be projected lies.

Thus, AB is in the 4th $Q.$, and as cosines of angles in the 4th $Q.$ are positive, the projection of AB is positive. BC lies in the 1st $Q.$, and its projection is $+$; CD lies in the 3rd $Q.$, and its projection is $-$; DE, in the 2nd $Q.$ has its projection $-$; and EA in the 3rd $Q.$ has its projection $-$.

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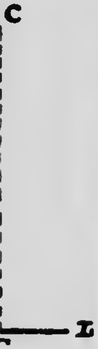

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EXERCISE XI.

1. Prove by projection that in any triangle $b = a \cos C + c \cos A$.

2. In any parallelogram the sum of the projections of two adjacent sides upon the conterminous diagonal is the diagonal.

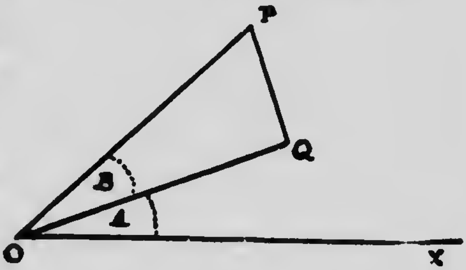
3. $ABCD$ is a quadrangle having A and C right angles, and the angle ADC is θ:

Prove (1) $AB \sin \theta = DC - AD \cos \theta$.

(2) $BC \sin \theta = AD - DC \cos \theta$.

(3) $DB \sin \theta = \sqrt{(AD^2 + DC^2 - 2AD.DC \cos \theta)}$.

4. OQ makes angle A with OX, and OP makes angle $A+B$ with OX, and PQO is a right angle.



Project the triangle OPQ on OX. The sum of the projections of the sides in order is zero.

$$\therefore OP \cos (A+B) + PQ \sin A - QO \cos A = 0.$$

Divide by OP, and

$$\cos (A+B) + \frac{PQ}{OP} \sin A - \frac{QO}{OP} \cos A = 0.$$

But $$\frac{PQ}{OP} = \sin B, \text{ and } \frac{QO}{OP} = \cos B.$$

$$\therefore \cos (A+B) = \cos A \cos B - \sin A \sin B.$$

APPLICATIONS TO FORCES AND VELOCITIES.

17. The following fundamental principles are established in the subjects of statics and dynamics.

(*a*) A force may be completely represented by a line-segment, the length of the segment representing the magnitude of the force, and the direction and position of the segment representin the direction and position of the force.

(b) The action of a force along any line, or the part of a forc acting in a direction parallel to that line, is given by the projec tion upon that line of the segment representing the force.

(c) If two forces are represented by adjacent sides of a paral lelogram, they are together exactly equivalent to the single forc represented by the conterminous diagonal of the parallelogram.

This single force is called the ***Resultant*** of the two.

It follows from (b) that a force exerts no effect in a directior perpendicular to its own direction, but that it exerts an effec in every other direction.

The word "force" may be replaced by "velocity" in the preceding.

EXERCISE XII.

1. Two forces of 6 and 15 pounds act at right angles to one another. Find the resultant in pounds, and its direction relatively to the greater force.

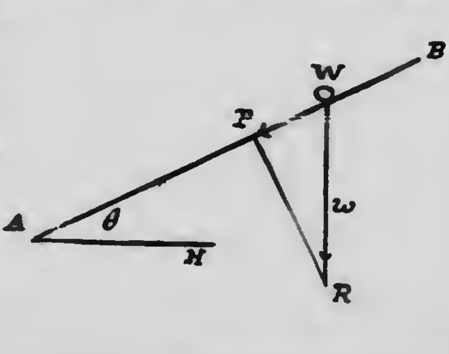
2. The forces of Ex. 1 act at an angle of 120°.

3. Forces of 30 and 40 grams act at an angle of 55°. Find the force which will exactly annul their effect.

4. Three forces acting at a point are such that their representative line-segments when taken in length and direction can form a triangle. Prove that any one of the forces, reversed in direction, is the resultant of the other two.

5. A ball of weight, W, lies on a plane inclined at angle θ to the horizontal. Find (a) the force with which the ball tends to roll down the plane; (b) the pressure of the ball upon the plane.

AB is the plane, making angle θ with AH, the horizontal, and W is the ball on the plane. The force in the question is the weight, w, of the ball, and acts vertically downwards. Hence, draw WR perpendicular to AH, to represent w.



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Project WR on AB by the perpendicular RP. Then, Art. 17 (*b*), WP represents the force down the plane, which is evidently $w \sin \theta$.

And evidently PR, which is $w \cos \theta$, represents the pressure on the plane.

Cor.—If $\theta = 0$, the force along the plane is zero, and the pressure on the plane is w. And if $\theta = \frac{\pi}{2}$, the force along the plane is w, and the pressure on the plane is zero.

6. In Ex. 5, the ball weighs 60 pounds, and the plane is inclined 60° to the horizon.

7. Interpret the result of 5, when θ is greater than $\frac{\pi}{2}$.

8. A car runs up a hill inclined 18° at the rate of 4 miles an hour. Find its velocity on the level, and also in a vertical direction.

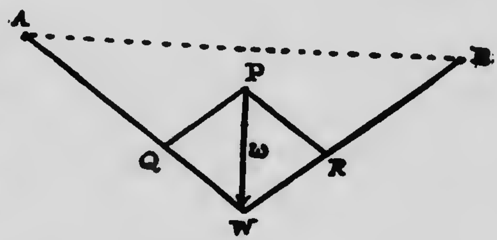
9. A ball of 100 pounds hangs by a rope. A string fastened to the ball pulls it 5° out of the vertical. Find the tension on the rope and on the string:

(*a*) When the pull is horizontal.

(*b*) When the pull is perpendicular to the rope.

10. A rope is fastened to two horizontal supports, and has a weight of w pounds suspended from the middle. To find the tension of the rope, and the vertical and horizontal forces acting on the supports.

A and B are the supports in a horizontal line, and W is the suspended weight. Then evidently each part, AW and BW, of the rope has the same tension and the same inclination to AB.



Denote the tension by t, and the angle BAW by α. Draw PW vertically to represent w, and complete the parallelogram $PQWR$. $PQ = QW =$ the tension on the rope $= t = \dfrac{w}{2 \sin \alpha}$.

The horizontal force on the support is found by projecting QW on $AB = QW \cos \alpha = \dfrac{\cos \alpha}{2 \sin \alpha} \cdot w$, etc.

11. The rope of Ex. 10 is 20 feet long and the supports are 16 feet apart.

12. In Ex. 10 the weight is not at the middle, and the inclinations of the two parts of the rope are α and β respectively.

13. A beam 20 feet long, weighing 5 pounds per foot, stands against an upright wall, and the foot of the beam is 8 feet from the wall. Find the pressure that the beam exerts on the wall.

14. A beam (whose weight may be neglected) has one end on the ground, and the other end is held by a stay rope, so that the beam makes angle α with the horizon and the rope angle β. A weight, w, is suspended from the upper end of the beam. Find the tension of the stay rope, and the end-thrust on the beam.

ADDITION THEOREMS.

18. A theorem which gives a function of the sum or difference of two angles in terms of functions of the separate angles is called an *addition theorem.*

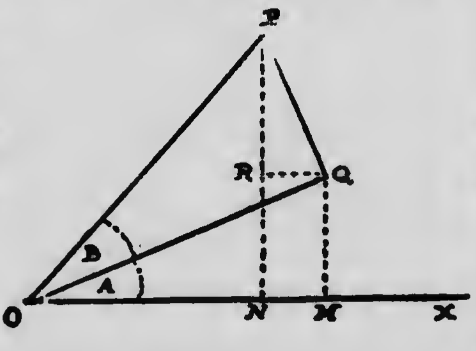
The principal addition theorems are those for the sine, the cosine and the tangent.

These theorems may be developed separately and independently, but, like the functions in general, they can all be derived from any one of them.

The addition theorem for $\cos (A+B)$ is developed in Ex. 4, of Exercise XI.

We proceed to develop $\sin (A+B)$ by another means.

The $\angle\, QOX = A$ and $POQ = B$. PQ is $\perp$ to OQ, and PN, QM are $\perp$ to OX, and QR to PN.



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Then $\sin(A+B)=\frac{PN}{OP}$

$$=\frac{PR}{OP}+\frac{QM}{OP}$$

$$=\frac{PR}{PQ}\cdot\frac{PQ}{OP}+\frac{QM}{OQ}\cdot\frac{OQ}{OP}$$

$$=\cos A\,.\,\sin B+\sin A\,.\,\cos B.$$

$\therefore\quad \sin(A+B)=\sin A\cos B+\cos A\sin B\ldots\ldots(a)\ldots\ldots$

Now write $-B$ for B, and

$$\sin(A-B)=\sin A\cos(-B)+\cos A\sin(-B)$$
$$=\sin A\cos B-\cos A\sin B,\ (c),\qquad \text{Art. 9 } (b).$$

Write $\frac{\pi}{2}-A$ for A in $\sin(A-B)$, and

$$\sin\left(\frac{\pi}{2}-\overline{A+B}\right)=\sin\left(\frac{\pi}{2}-A\right)\cos B-\cos\left(\frac{\pi}{2}-A\right)\sin B;$$

or $\quad \cos(A+B)=\cos A\cos B-\sin A\sin B\ldots\ldots\qquad (c)$

and finally writing $-B$ for B in the last,

$$\cos(A-B)=\cos A\cos B+\sin A\sin B.$$

Collecting the four theorems, we have:

$$\left.\begin{aligned}\sin(A+B)&=\sin A\cos B+\cos A\sin B &&\text{(i.)}\\ \sin(A-B)&=\sin A\cos B-\cos A\sin B &&\text{(ii.)}\\ \cos(A+B)&=\cos A\cos B-\sin A\sin B &&\text{(iii.)}\\ \cos(A-B)&=\cos A\cos B+\sin A\sin B &&\text{(iv.)}\end{aligned}\right\}\quad(11)$$

EXERCISE XIII.

1. From the addition theorems of (11) prove the following:

(a) $\sin 2\theta=2\sin\theta\cos\theta.\qquad (12)$

(b) $\left.\begin{aligned}\cos 2\theta&=\cos^2\theta-\sin^2\theta. &&\text{(i.)}\\ &=2\cos^2\theta-1. &&\text{(ii.)}\\ &=1-2\sin^2\theta. &&\text{(iii.)}\end{aligned}\right\}\quad(13)$

(c) $\sin 3\theta=3\sin\theta-4\sin^3\theta.\qquad (14)$

(d) $\cos 3\theta=4\cos^3\theta-3\cos\theta.\qquad (15)$

2. Prove that $\sin n\theta=2\sin(n-1)\theta\cos\theta-\sin(n-2)\theta$.

(Put $\phi=(n-1)\theta$ in $\sin(\phi+\theta)$, and in $\sin(\phi-\theta)$, and add the results.)

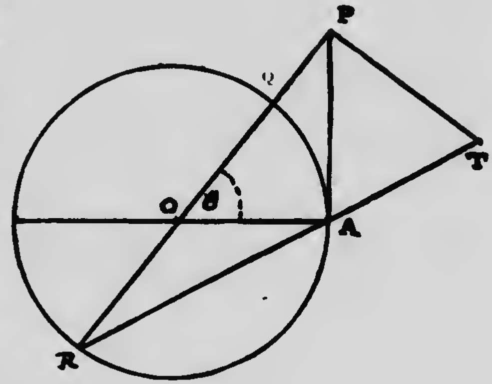
3. Show that $\sin(A+60°)+\sin(A-60°)=\sin A$.

4. Show that $\cos(A+60°)+\cos(A-60°)=\cos A$.

5. Find the sine and cosine of 15° $(15=45-30)$.

6. A flag pole stands on the top of a tower; at a feet from the bottom of the tower the top of the tower has an elevation α, and the top of the pole an elevation β. Prove that the length of the pole is $a \cdot \dfrac{\sin(\beta-\alpha)}{\cos\alpha\ \cos\beta}$.

7. AP, in the figure, is tangent at A, and PT is $\perp$ to the centre line PR, and meets RA in T. The radius of the circle is r, and the angle $POA=\theta$.



(a) Find θ when $AT = AQ$.

(b) Find the area of $\triangle\ APT$ in terms of r and θ.

(c) Prove that $AP=PT$ for all values of θ.

(d) Find the value of θ that APT may be equilateral.

(e) Find θ when $AR=AP$.

(f) Show that $AR:RP=\cos\theta:\cos\frac{\theta}{2}$.

(g) When $\theta=60°$, show that $RP^2=3AR^2$.

(h) When $\theta=60°$, show that $\triangle\ RAQ=\frac{1}{3}\triangle RPT$.

(i) Find θ when $AQ=\frac{1}{2}AR$.

19. Addition Theorem for Tangent.—

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}.$$

Divide numerator and denominator by $\cos A\cos B$, and reduce to tangents, and we get

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$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}, \qquad (16)$$

and by writing $-B$ for B, and putting $\tan(-B) = -\tan B$,

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}, \qquad (17)$$

Thence we easily obtain

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}. \qquad (18)$$

EXERCISE XIV.

1. Prove that $\tan(45^\circ + \theta) = \dfrac{1 + \tan\theta}{1 - \tan\theta}$.

2. Find tan 15° and tan 75°.

3. Find an expression for $\tan 3\theta$ in terms of $\tan\theta$.

4. Show that $\tan\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1 + \cos\theta}$.

5. Show that $\tan^2\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{1 + \cos\theta}$.

6. Show that $\sin(\theta+\phi)\sin(\theta-\phi) = (\sin\theta + \sin\phi)(\sin\theta - \sin\phi) = \sin^2\theta - \sin^2\phi$.

7. Show that $\cos(\theta+\phi)\cos(\theta-\phi) = \cos^2\theta - \sin^2\phi$.
$= \cos^2\phi - \sin^2\theta$.

8. Find the sine of 18°.

(We have $2 \times 18^\circ =$ the complement of $3 \times 18^\circ$. and hence $\sin 2 \times 18^\circ = \cos 3 \times 18^\circ$; or $2 \sin 18^\circ \cos 18^\circ = 4\cos^3 18^\circ - 3\cos 18^\circ$. Divide out cos 18°, and reduce the quotient.)

9. Find the sine and cosine of 3°. $(3^\circ = 18^\circ - 15^\circ.)$

(This is the smallest whole number of degrees of which we can find the functions in terms of surd expressions. Thence, they can be so found for every three degrees throughout the quadrant.

20. Formulas for changing Sums and Differences of Functions to Products.

Add (i.) and (ii.) of (11), Art. 18, and we get

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B.$$

Now put $A=\frac{1}{2}(\theta+\phi)$, $B=\frac{1}{2}(\theta-\phi)$, and, therefore,

$A+B=\theta$, and $A-B=\phi$, and we have

$$\sin\theta+\sin\phi=2\sin\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi).$$

Similarly, by subtracting (ii.) from (i.), by adding (iii.) to (iv.), and by subtracting (iv.) from (iii.), we get the four forms:

$$\left.\begin{aligned}
\sin\theta+\sin\phi&=2\sin\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi) &&\text{(i.)}\\
\sin\theta-\sin\phi&=2\cos\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi) &&\text{(ii.)}\\
\cos\theta+\cos\phi&=2\cos\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi) &&\text{(iii.)}\\
\cos\theta-\cos\phi&=-2\sin\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi) &&\text{(iv.)}
\end{aligned}\right\}\quad(19)$$

Ex. 1.—$\sin 4\theta+\sin 2\theta=2\sin 3\theta\cos\theta$,

$\sin 4\theta-\sin 2\theta=2\cos 3\theta\sin\theta$.

Ex. 2.—To express $\cos\theta\sin^3\theta$ in terms of multiples of θ,

$$\cos\theta\sin^3\theta=\frac{1}{4}(4\cos\theta\sin^3\theta)=\frac{1}{4}(3\sin\theta\cos\theta-\sin 3\theta\cos\theta)$$

$$=\frac{1}{4}\left(\frac{3}{2}\sin 2\theta-\frac{1}{2}[\sin 4\theta+\sin 2\theta]\right).$$

$$=\frac{1}{8}(2\sin 2\theta-\sin 4\theta).$$

EXERCISE XV.

1. Express $\cos\theta\sin^2\theta$ in terms of multiples of θ.
2. Express $\tan A+\tan B$ as a product.
3. If $A+B+C=\pi$, they are angles of a triangle, and possess certain peculiar relations.

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(*a*) Prove that

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

(*b*) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1.$

(*c*) $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$

$$\left(\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}, \text{ and}\right.$$

$$\sin C = 2 \sin \frac{C}{2} \cos \frac{C}{2} = 2 \sin \left(\frac{\pi}{2} - \frac{A+B}{2}\right)$$

$$\cos \left(\frac{\pi}{2} - \frac{A+B}{2}\right) = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}.$$

$$\therefore \sin A + \sin B + \sin C = 2 \cos \frac{C}{2}\left(\cos \frac{A-B}{2} + \cos \frac{A+B}{2}\right)$$

$$\left. = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.\right)$$

LOGARITHMIC FORMULAS.

21. Logarithms are employed to simplify and extend arithmetical operations, and their proper relation is with algebra and arithmetic. They are introduced into practical trigonometry not as a matter of necessity, but as one of convenience, and because a large part of the work in that subject consists of arithmetical operations.

The trigonometrical functions, being ratios, are numbers, and the logarithms of these numbers are tabulated under the head of log-sines, log-tangents, etc. The tabulating of them in this way is a great convenience, but all these logarithms can, of course, be got from a table of logarithms of numbers.

The log-sines, etc., offer some peculiarities, for the natural quantities being fractional, their logs have a negative characteristic; and to get over this inconvenience 10 is added to the characteristic.

Rules for working these tables are generally found in conjunction with the tables, and there is no advantage in giving them here, as facility in the use of the tables is to be acquired only by practice and experience. We shall, therefore, assume that the reader is acquainted with the general properties of logarithms, and that he has some knowledge of the tables. For convenience we here state the three most important working properties of logarithms:

(a) $\log a + \log b = \log ab$.

(b) $\log a - \log b = \log \dfrac{a}{b}$.

(c) $n \log a = \log a^n$ for all values of n.

Thus, as the addition of logarithms corresponds to the multiplication of numbers, and the subtraction of logarithms to the division of numbers, there is no operation with logarithms corresponding to the addition or subtraction of numbers. And being given $\log. a + \log. b$, there is no direct logarithmic means of finding $\log. (a+b)$, except by repeated operations.

Hence, formulas involving additions or subtractions are not adapted to logarithms, and when additions or subtractions are necessary, they must be effected before the application of logarithms.

Thus the sine formula is adapted to logarithms directly, since it involves only multiplications and divisions. But the cosine formula is not adapted to logarithms, as it involves additions and subtractions of the squares of quantities; and if these arithmetical operations are to be carried out first, the subsequent application of logarithms would be more laborious than helpful.

Hence the necessity of transforming our formulas, so as to adapt them to logarithmic computation.

22. Transformation of Cosine Formula.

(a) $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$.

$$\therefore\ 2\cos^2\frac{A}{2} = 1 + \cos A = 1 + \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{(b+c)^2 - a^2}{2bc}.$$

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$$\therefore \cos^2\frac{A}{2} = \frac{(a+b+c)(b+c-a)}{4bc} = \frac{s(s-a)}{bc}.$$

$$\therefore \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}. \qquad (20)$$

This is adapted to logarithms, since the only additions and subtractions are between simple numbers, and not between squares of numbers or trigonometric functions.

Its logarithmic form is

$$l.\cos\frac{A}{2} = \frac{1}{2}\{l.s + l.(s-a) - l.b - l.c\}, \qquad (20l.)$$

and this serves to find an angle when the three sides are given.

(*b*) $$2\sin^2\frac{A}{2} = 1 - \cos A = 1 - \frac{b^2+c^2-a^2}{2bc},$$

$$= \frac{a^2-(b-c)^2}{2bc}.$$

$$\therefore \sin^2\frac{A}{2} = \frac{(a+b-c)(a-b+c)}{4bc} = \frac{(s-b)(s-c)}{bc},$$

and $$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}. \qquad (21)$$

The logarithmic form is

$$l.\sin\frac{A}{2} = \frac{1}{2}\{l.(s-b) + l.(s-c) - l.b - l.c\}, \qquad (21l.)$$

and this serves the same purpose as (*a*).

(*c*) Another method is by an auxiliary angle.

As in (*a*) $$2\cos^2\frac{A}{2} = \frac{(b+c)^2-a^2}{2bc},$$

$$\therefore \cos^2\frac{A}{2} = \frac{(b+c)^2}{4bc}\left(1 - \left(\frac{a}{b+c}\right)^2\right),$$

and $$\cos\frac{A}{2} = \frac{b+c}{2\sqrt{bc}}\sqrt{1-\left(\frac{a}{b+c}\right)^2}.$$

Now, as a, b, c are sides of a triangle, $b+c > a$, therefore $\frac{a}{b+c}$ being less than unity, is the sine of some angle, θ say.

Then, $$\cos\frac{A}{2}=\frac{b+c}{2\sqrt{bc}}\cos\theta,$$

and we have the two logarithmic forms:

(1) $l.\sin\theta=l.a-l.(b+c)$.

(2) $l.\cos\frac{A}{2}=l.(b+c)+l.\cos\theta-\frac{1}{2}(l.b+l.c)-l.2$.

(*a*), (*b*) and (*c*) all serve the same purpose, but (*a*) should not be used when the $\angle A$ is a small angle; and (*b*) should not be used when the $\angle A$ is nearly two right angles.

23. When two sides and the included angle are given we can, with natural functions, solve by the cosine formula under the form:

$$a^2=b^2+c^2-2bc\cos A,$$

which finds the third side.

(*a*) to adapt this case to logarithms we do as follows:

Since $$\frac{a}{b}=\frac{\sin A}{\sin B},\ \therefore\ \frac{a+b}{a-b}=\frac{\sin A+\sin B}{\sin A-\sin B}$$

$$=\frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}$$

$$=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)},$$

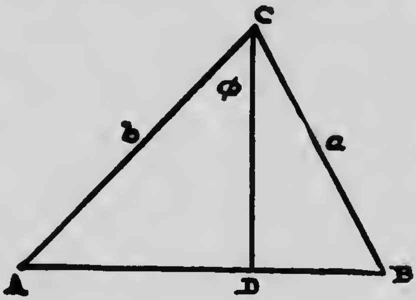
and $$\tan\frac{1}{2}(A+B)=\cot\frac{1}{2}C,$$

$\therefore$ finally, $$\tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\frac{1}{2}C\ldots\quad(22)$$

and $$l.\tan\frac{1}{2}(A-B)=l.(a-b)+l.\cot\frac{1}{2}C-l.(a+b)\ldots\quad(22l.)$$

This finds $\frac{1}{2}(A-B)$; and as $\frac{1}{2}(A+B)=90^\circ-\frac{1}{2}C$, we find A and B by addition and subtraction.

(*b*) A decidedly shorter solution, but one not adapted to logarithms, is the following:



Draw $CD \perp$ to AB, a, b and C being given.

Put $ACD = \phi$.

Then $BCD = C - \phi$.

$CD = b \cos \phi = a \cos (C - \phi)$

$= a \cos C \cos \phi + a \sin C \sin \phi$,

$\therefore\ b = a \cos C + a \sin C \tan \phi$,

and $$\cot A = \tan \phi = \frac{b - a \cos C}{a \sin C},$$

which makes A known, and thence B is known.

24. The expressions for the area and the circumradius are adapted to logarithms, their logarithmic forms being:

$$\left.\begin{aligned} l.\Delta &= \frac{1}{2}\{l.s + l.(s-a) + l.(s-b) + l.(s-c)\} \\ l.R &= l.a + l.b + l.c - l.4 - l.\Delta. \end{aligned}\right\} \ldots \quad (23)$$

EXERCISE XVI.

1. Find the angle (*a*) whose log-sine is 9.45062; (*b*) whose log-sine is 8.47165; (*c*) whose log-cosine is 9.99971; (*d*) whose log-tangent is 9.45674; (*e*) whose log-tangent is 12.41596.

2. Given log-cos θ, how can we find log-sec θ?

3. If $a = \dfrac{1 - \cos \theta}{1 + \cos \theta}$, show that $l.a = 2l. \tan \dfrac{\theta}{2}$.

4. Given $b = a \sin \theta - b \cos \theta$, show that $l.a = l.b + l. \cot \dfrac{\theta}{2}$.

5. In the triangle ABC, $a = 27.3$, $b = 34.1$, $c = 45.6$, to find the angle A.

($s = 53.5$, $s - a = 26.2$. Then

$$l. \cos \frac{A}{2} = \frac{1}{2}(l.53.5 + l.26.2 - l.34.1 - l.45.6)$$

$$= 9.97747. \quad \therefore\ \frac{A}{2} = 18^\circ\ 18', \text{ etc.}$$

INVERSE, OR CIRCULAR FUNCTIONS.

25. When we have $\sin\theta = x$, we may write $\theta = \sin^{-1}x$, which we read "θ is the angle whose sine is x," and we call the symbol $\sin^{-1}x$ the inverse sine of x. The exponent, -1, here does not denote a reciprocal, as in algebra.

These functions, which equal in number the trigonometric functions, are known as circular functions, as $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, etc.

Circular or inverse functions are important in the subjects of the Differential and Integral Calculus. The important theorems in regard to them are, however, not numerous.

(*a*) To sum $\tan^{-1}x + \tan^{-1}y$, that is, to find an inverse tangent equal to the sum of these.

Let $\phi = \tan^{-1}x$ and $\theta = \tan^{-1}y$.

Then $\tan\phi = x$ and $\tan\theta = y$.

and
$$\tan(\phi + \theta) = \frac{\tan\phi + \tan\theta}{1 - \tan\phi\tan\theta}, \qquad (16)$$

or
$$\phi + \theta = \tan^{-1}\frac{x+y}{1-xy}.$$

Or, finally,
$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy} \dots \qquad (24)$$

and similarly,
$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy} \dots \qquad (25)$$

Ex.—
$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = \tan^{-1}1$$
$$= 45° \text{ or } \frac{\pi}{4}.$$

(*b*) To find the sum of $\sin^{-1}x + \sin^{-1}y$.

Let $\phi = \sin^{-1}x$ and $\theta = \sin^{-1}y$.

Then $\sin\phi = x$, $\quad \sin\theta = y$.

$\cos\phi = \sqrt{1-x^2}$, $\cos\theta = \sqrt{1-y^2}$,

and $$\sin(\phi+\theta) = \sin\phi\cos.\theta + \cos\phi\sin\theta.$$
$$= x\sqrt{1-y^2} + y\sqrt{1-x^2},$$
or $$\phi+\theta = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\},$$
and finally,
$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}.$$

EXERCISE XVII.

1. Prove that $\sin^{-1}x = \frac{\pi}{2} - \cos^{-1}x$.
2. Find $2\tan^{-1}\frac{1}{3}$ as an inverse tangent.
3. Show that $2\tan^{-1}\frac{1}{5} + 2\tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{7} = 45°$.
4. Prove that $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = 45°$.
5. Prove that $\sin^{-1}x + \sin^{-1}y = \cos^{-1}\{\sqrt{1-x^2}\sqrt{1-y^2} - xy\}$.
6. Find a circular function equal to $\cos^{-1}x + \cos^{-1}y$.
7. Show that $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = \frac{\pi}{2}$.
8. Show that $2\tan\frac{1}{3} + \tan^{-1}\frac{1}{7} = 45°$.

TRIGONOMETRICAL CONSTRUCTIONS.

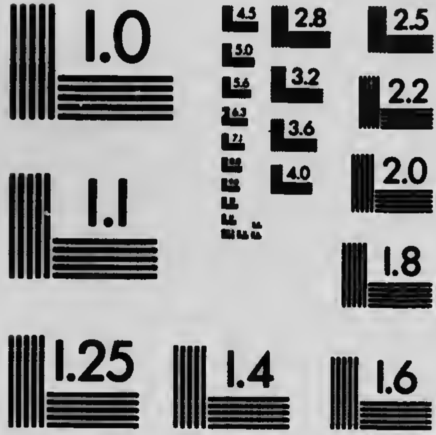
26. By trigonometrical construction we mean the finding, by graphical methods, of the values of such trigonometrical expressions as can be so found, and which have sufficient elements given to make them determinate.

On account of the great variety of such expressions, no very general principles of construction can be laid down, and even the construction for a given case may sometimes admit of a number of variations, of which some are more elegant than others.

A few examples will illustrate the subject.

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



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1653 East Main Street
Rochester, New York 14609 USA
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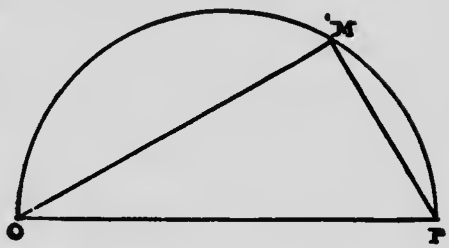
Ex. 1.—To construct an angle when its sine is given.

Take any line-segment, OP, as a radius, as the element of length must be involved, and on it describe the semi-circle OMP. In this semi-circle place the chord PM equal to $OP \times$ the given sine, and join OM. The $\angle POM$ is that required.

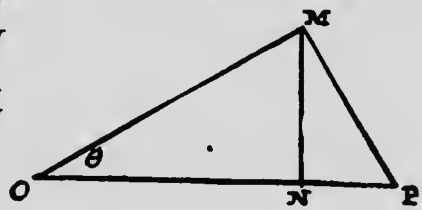


The proof is evident from the construction.

Ex. 2.—To construct $a \sin\theta \cos\theta$, where a and θ are given.

Draw a line $OP = a$, and OM making $\angle POM = \theta$. Draw $PM \perp OM$, and $MN \perp OP$. Then $OM = a \cos\theta$, and $MN = OM \sin\theta$.

$\therefore MN = a \sin\theta \cos\theta$.



Ex. 3.—To construct

$$a \cdot \frac{\sin\theta + \cos\theta}{\tan\theta}, \text{ or } a(\sin\theta + \cos\theta)\cot\theta.$$

Draw $AB = a$, and make $\angle BAD = \theta$, and draw $BC \perp AD$.

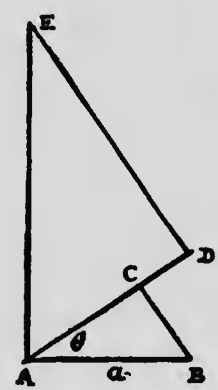
Take $CD = CB$, and draw $AE \perp AB$ and $DE \perp AD$, to meet in E.

Then, $AC = a \cos\theta$ and $CB = a \sin.\theta$, and

$\therefore AD = a(\sin\theta + \cos\theta)$.

But $\angle AED = \theta$, and $DE = AD \cot AED$,

$\therefore DE = a(\sin\theta + \cos\theta)\cot\theta$.



EXERCISE XVIII.

1. Construct an angle when (*a*) its tangent is given; (*b*) when its secant is given; (*c*) when its cotangent is given.

2. Construct x where $x = (\sin A - \sin B)\sqrt{a^2 - b^2}$, where A, B are given angles, and a, b are sides of a given rectangle.

3. Find by construction the rectangle $ab \sin\theta \cos\frac{\theta}{2}$, where a, b are given line segments and θ is a given angle.

4. Construct θ where $\theta = \sin^{-1}\frac{a}{b}$, a and b being given line segments.

5. Construct $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$.

6. Construct $a \sin\left(A + \tan^{-1}\frac{b}{a}\right)$, where A is a given angle and a and b are given line segments.

7. Construct the graph of $\sin\theta - \cos\theta$ from $\theta = 0$ to $\theta = 2\pi$.

8. Construct the graph of $\sin\theta + \cos\frac{\theta}{2}$ from 0 to 2π.

MISCELLANEOUS EXERCISES.

1. C is the centre of a circle with radius r, and P is a point without. Tangents from P touch the circle at T and T'. Find the tangent of TPT' in terms of r and PT.

2. In Ex. 1 find the length of chord TT'

3. The distance between graduation marks on the limb of a theodolite is 0.045 inch, and they represent 10′ of angle. Find the radius of the limb.

4. Prove the following relations:

(a) $\cos^4 A - \sin^4 A = 2\cos^2 A - 1$.

(b) $\sqrt{1-\sin\theta} = (\sec\theta - \tan\theta)\sqrt{1+\sin\theta}$.

(c) $2\sec^2\theta = \dfrac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - 1} + \dfrac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta + 1}$.

(d) $\dfrac{\cos A}{1-\tan A} + \dfrac{\sin A}{1-\cot A} = \sin A + \cos A$.

(e) $4\cos^3 A = 3\cos A + \cos 3A$.

(f) $\sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A$.

5. Find any function of θ from the following:

(a) $2\sin\theta = 2 - \cos\theta$.

(b) $8\sin\theta = 4 + \cos\theta$.

(c) $\tan\theta + \sec\theta = 3/2$.

(d) $\sin\theta + 2\cos\theta = 1$.

(e) $\tan 2\theta + \cot\theta = 8\cos^2\theta$. (Express left-hand member in sines and cosines and reduce.)

6. In any circle prove that the chord of 108° is equal to the sum of the chords of 36° and 60°.

7. A person standing on a lighthouse notices that the angle of depression of a boat coming towards him is α, and that after m minutes it is β. How long after the first observation will it be before the boat reaches the lighthouse?

8. (*a*) From the cosine formula show that

$$c = (a+b)\sin\frac{C}{2}\sec\phi,$$

where
$$\tan\phi = \frac{a-b}{a+b}\cot\frac{C}{2}.$$

(*b*) Express the results of (*a*) in logarithmic form, and apply it to the case where $a = 25.33$, $b = 18.46$, and $C = 78^\circ\ 44'$.

9. (*a*) Prove that $a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos\left(\theta - \tan^{-1}\frac{b}{a}\right)$.

$$= \sqrt{a^2+b^2}\sin\left(\theta + \tan^{-1}\frac{a}{b}\right).$$

(*b*) Show that $a\cos\theta + b\sin\theta$ is a maximum when

$$\theta = \tan^{-1}\frac{b}{a}.$$

10. (*a*) Divide the angle A into two parts, such that the sum of the cosines of the parts is a given quantity, m.

(*b*) Obtain a geometric construction for this division.

11. Prove that $\tan^{-1}\frac{m}{n} - \tan^{-1}\frac{m-n}{m+n} = \frac{\pi}{4}$.

12. A side of a triangle is 4 and the opposite angle is 36°, and the altitude to another side is $\sqrt{5} - 1$. Find the other parts.

13. [illegible] th of the median to side a is m, and the parts into [illegible] divides its angle are α, β. Find the other parts of the tr[illegible]

14. Solve the triangle in which $a+b$, c and C are given. (Find $a-b$ by cosine formula.)

15. The altitude of a rock is 47°. After walking 1,000 feet towards it up a slope of 32°, the altitude is 77°. Find the vertical height of the rock above the first point of observation.

16. (*a*) A hill which rises 1 in 5 faces south. Show that a road on it which takes a N.-E. direction rises 1 in 7.

(*b*) What must be the direction of the road which going along the hill, rises 1 in 10?

17. A gable facing north has a vertical angle of 2γ. Show that when the sun is south at elevation α, the angle of the shadow of the gable on level ground is $2\tan^{-1}(\tan\alpha\tan\gamma)$.

18. A rod points towards the north pole of the heavens. Find the angle, θ, made by the shadow of the rod (on the level) with the meridian line, when the sun is h degrees west of the meridian, and the altitude of the pole is ϕ.

(This embodies the principles of construction of a horizontal sun-dial.)

19. (a) In Ex. 17 give a geometric construction for finding θ when h and ϕ are given.

(b) Lay off the hours of a dial for latitude 44° N.

20. (a) The quadrilateral $OPMQ$, in order, has $OP = OQ = r$, the $\angle\ POQ = 2\gamma$, and the angles OPN and OQM equal to α and β respectively. Find an expression for the value of PM.

(This embodies the principle of finding the distance of the moon from the earth.)

(b) If in (a) $\alpha = 145^\circ$, $\beta = 164^\circ\ 12'$, and $\gamma = 25^\circ$, show that $PM = 60r$ very nearly.

21. (a) $ABCD$, taken in order, determine a trapezoid with AB parallel to DC; and the diagonals meet in O. The angle $DAC = \alpha$, $ADB = 2p$, and the ratio $AC : OC = r$. Show that $p = (r-1)\frac{\alpha}{2}$, if the angles p and α be very small.

(b) Find p when $\alpha = 46''.8$ and $r = \left(\frac{365.26}{224.70}\right)^{\frac{2}{3}}$.

22. If r_1 be the radius of the circle escribed to side a of the triangle ABC, show that $r^1 = \dfrac{a}{\tan\frac{B}{2} + \tan\frac{C}{2}} = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$.

23. Prove the following:

(a) $r_1 r_2 r_3 = s\Delta = rs^2$.

(b) $r r_1 r_2 r_3 = \Delta^2$.

(c) $a = 2R\sin A$, where R is the circumradius.

(d) $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$, with symmetrical expressions for r_2 and r_3.

(e) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

24. Two wheels with radii r, r', have their centres d feet apart, and lie in the same plane. Find the length of the belt which goes around the wheels and (*a*) crosses between them, (*b*) does not cross between them.

25. In Ex. 24 (*a*), if $r + r'$ is constant, show that the length of belt is constant.

26. In Ex. 24, the wheels are 15 and 20 inches in diameter, and the axes are 120 inches apart. Find the length of (*a*) the open belt, (*b*) the crossed belt.

27. When a material body rests on an inclined plane, show that the ratio of the force tending down the plane to the pressure normal to the plane is the tangent of the angle of inclination of the plane.

When a body rests on a plane, and the plane is inclined until the body is just at the point of sliding down it, the tangent of the angle of inclination is called the *coefficient of friction*; and under reasonable conditions the coefficient of friction is constant for the same materials in the body and the plane.

Then the amount of friction, which acts as a force opposing motion, is the weight of the body multiplied by the coefficient of friction.

28. If the coefficient of friction of iron on iron be .16, find the inclination of an iron plane upon which an iron block is at the point of sliding.

29. If an iron plane have an inclination of 30°, find the force, acting along the plane, necessary (*a*) to slide a block of iron of 100 gms. up the plane, (*b*) down the plane.

30. The friction of a metal on oak is about 0.5. What force acting at 30° upwards will move 100 kgms. of iron along a level oak floor?

31. Three poles, each 20 feet in length, are joined at the top, and their feet rest at the vertices of an equilateral triangle with side 12 on level ground. (*a*) Find the angle of inclination of each pole. (*b*) Find the vertical height of the tops.

32. If, in Ex. 31, 100 lbs. be suspended from the tops of the

poles, find (*a*) the end pressure on a pole, (*b*) the horizontal thrust at the bottom of a pole; the weight of the pole being not considered.

33. ABC is a triangle of which AB and BC are rigid rods. C is fixed, and A is compelled to move in the line AC. If a force, p, be applied to A along AC, show that the force (*a*) acting perpendicularly to AC is $p \sin C / \sqrt{n^2 - \sin^2 C}$; (*b*) acting along BC is $p\{\cos C - \sin^2 C / \sqrt{n^2 - \sin^2 C}\}$; (*c*) acting perpendicularly to BC is $p\{\sin C + \sin C \cos C / \sqrt{n^2 - \sin^2 C}\}$, where n is the ratio $AB:BC$.

(This exercise embodies the principles of the cross-head and crank in the steam engine.)

34. From the corner of a cuboid a piece is cut off by a plane saw cut, which reaches to the distances a, b, c respectively on the three edges. Prove that the area of the section is $\frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$.

35. At the vertices of an equilateral triangle line segments, a, b, c respectively, are drawn normal to the plane of the triangle. Show that the area of the triangle formed by connecting the outer points is $\frac{1}{4}\sqrt{\{3s^4 + 4s^2(\Sigma a^2 - \Sigma ab)\}}$, where s is the side of the equilateral triangle.

ANSWERS TO EXERCISES.

Exercise I.

1. 16587.608928. 2. 219.608928; 3.128928; 0.039072 ft.
3. 3.141593. 4. 4; 5. 5. 5; 7.
6. 2.178 sq. ft. 7. 1.0000000. 8. 1.00000.
9. 326 sq. ft. 60 sq. in. 10. 0.062.
11. 0.4342945. 12. 57.295780. 13. 101.881.

Exercise II.

1. $36^\circ.2389$. 2. $0^\circ.42093$. 3. $0^\circ.015097$.
4. $3^\circ\ 50'\ 49''.92$. 5. 0.30718 in. 6. $0''.009815$.
7. 60°. 8. $0^\wedge.8864$. 9. 55.178 ft.
10. $49'\ 8''$. 11. 214.9 ft. 12. 412.5 ft.
13. 1082.3 ft. 14. 15.079 ft. 15. 345.6 m.
16. 3977 m. 17. 18.52 m.
18. $0^\circ.549$; 2270.96 m.

Exercise IV.

3. $\frac{2}{3}\sqrt{2}$; $\frac{1}{4}\sqrt{2}$. 4. $\frac{5}{26}\sqrt{26}$; $\frac{1}{26}\sqrt{26}$.
7. $\frac{3}{5}$; $\frac{4}{5}$; $\frac{5}{4}$. 8. $(1-x^2)\sqrt{1-x^2}\ /\ x$.

Exercise V.

1. $15^\circ\ 54'\ 56''$; 39.52 in. 2. $8^\circ\ 47'$. 3. $58^\circ\ 32'$.
4. 45°. 5. $18^\circ\ 26'$; $71^\circ\ 34'$.
6. (a) 96.84 ft.; (b) 70.87 ft. 7. 17.76 ft. 8. $27^\circ\ 49'$.
9. $C=71^\circ\ 29'$; $a=34.41$; $b=26.63$. 10. 2.495 in.
11. $51^\circ\ 32'$; $59^\circ\ 52'$; $68^\circ\ 36'$.
13. $C=84^\circ\ 58'$; $a=17.156$; $b=11.896$.

Exercise VI.

1. 0.92988; 2.52786.
2. 0.259; 0.966.

Exercise VII.

1. $B = 48° 48'$; $C = 95° 50'$; $c = 34.37$.
3. $C = 112° 5'$; $a = 14.70$; $b = 5.37$.
4. $b = 42.42$; $A = 32° 21' 12''$; $C = 80° 23' 48''$.
5. 21.47 ft.
6. 16.5 ft.
7. 9.16 ft.
8. 6.2.
9. 8° 15'.

Exercise VIII.

1. $A = 53° 8'$; $B = 59° 32'$; $C = 67° 20'$.
2. 38.50.
3. 2.79.
4. $c = 34.38$; $A = 48° 52'$; $B = 93° 24'$.
5. 747 rods.
6. 1.64 m.; 17° 31' east of N.
7. 2.62 m.; 3.39 m.
15. 100° 10'; 92° 34'; 79° 50'; 87° 26'.

Exercise IX.

2. $\frac{4}{5}, \frac{56}{65}, \frac{12}{13}$.
3. 156.4 rods.
4. 1344.
5. 51° or 129°.
8. 26.25 ft.; 23° 35'; 17° 27'; 138° 58'.
9. 108 25 ft.
10. 52.5.
11. $\frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$.

Exercise X.

3. $(\sqrt{4m^2+1} - 1) / 2m$.
4. $\sqrt{\left(\frac{1-m}{1+m}\right)}$.
10. 2.801.

Exercise XII.

1. 16.16 lbs.; 21° 49' with greater force.
2. 13.08 lbs.; 23° 25' with greater force.
3. 62.26 grms.; 156° 45' with greater force.
6. 51.96 lbs.; 30 lbs.

8. 3.804 miles per hour; 1.236 miles per hour.

9. (*a*) 100.38 lbs.; (*b*) 99.62 lbs., 8.72 lbs.

11. Tension $=\frac{5}{6}w$, horizontal force $=\frac{2}{3}w$, vertical force $=\frac{1}{2}w$.

12. $t_1=\frac{w\cos\beta}{\sin(\alpha+\beta)}$, $t_2=\frac{w\cos\alpha}{\sin(\alpha+\beta)}$, hor. force $=\frac{w\cos\alpha\cos\beta}{\sin(\alpha+\beta)}$.

13. 21.82 lbs.

14. Tension $=\frac{w\cos\alpha}{\sin(\alpha+\beta)}$, end thrust $=\frac{w\cos\beta}{\sin(\alpha+\beta)}$.

Exercise XIII.

5. $(\sqrt{3}-1)/2\sqrt{2}$, $(\sqrt{3}+1)/2\sqrt{2}$.

7. (*a*) 45°; (*b*) $\frac{1}{2}\cdot\frac{r^2\sin^3\theta}{\cos^2\theta}$; (*d*) 60°; (*e*) 60°; (*i*) 53° 8′.

Exercise XIV.

2. $\frac{\sqrt{3}-1}{\sqrt{3}+1}$, $\frac{\sqrt{3}+1}{\sqrt{3}-1}$. 3. $\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$. 8. $\frac{1}{4}(\sqrt{5}-1)$.

9. $\{(\sqrt{5}-1)(\sqrt{3}+1)-\sqrt{10+2\sqrt{5}}(\sqrt{3}-1)\}/8\sqrt{2}$.
$\{\sqrt{10+2\sqrt{5}}(\sqrt{3}+1)+(\sqrt{5}-1)(\sqrt{3}-1)\}/8\sqrt{2}$.

Exercise XV.

1. $\frac{1}{4}(\cos\theta-\cos 3\theta)$. 2. $\sin(A+B)/\cos A\cos B$.

Exercise XVI.

1. (*a*) 16° 23′ 40″; (*b*) 1° 41′ 51″; (*c*) 2° 6′; (*d*) 15° 58′ 25″; (*e*) 89° 46′ 48″.

Exercise XVII.

2. $\tan^{-1}\frac{3}{4}$. 6. $\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}$.

Miscellaneous Exercises.

1. $2r.PT \,/\, (PT^2 - r^2)$.
2. $2r.PT \,/\, \sqrt{PT^2 + r^2}$.
3. 15.47 in.
7. $m.\dfrac{\cos\alpha \sin\beta}{\sin(\beta - \alpha)}$.

10. (a) $2\theta = A + \cos^{-1}\dfrac{m}{2}\sec\dfrac{A}{2}$.

12. $(\sqrt{5} - 1)$ cosec 36°, and $(\sqrt{5} - 1) \cot 36° + \sqrt{10 + 2\sqrt{5}}$.

13. $2m \sin\alpha \,/\, \sin(\alpha + \beta)$, $2m \sin\beta \,/\, \sin(\alpha + \beta)$.

15. $1000\sqrt{2} \sin 47°$

16. (b) N.E. 15° S., or N.W. 15° S.

18. $\tan\theta = \sin\phi \tan h$.

24. (a) $\dfrac{l}{2} = (r + r')\pi - (r + r')\cos^{-1}\dfrac{r + r'}{d} + \sqrt{d^2 - (r + r')^2}$.

(b) $\dfrac{l}{2} = r\pi - (r - r')\cos^{-1}\dfrac{r - r'}{d} + \sqrt{d^2 - (r - r')^2}$.

30. 44.8 kgms.

31. (a) 69° 44′; (b) 18.76 ft.

32. (a) 35.5 lbs.; (b) 12.31 lbs.