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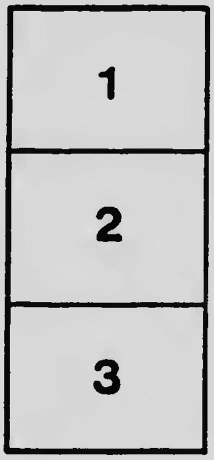
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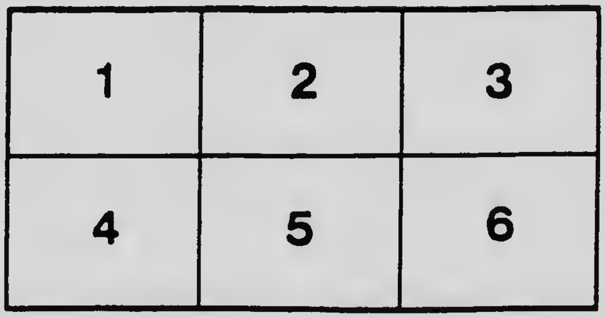
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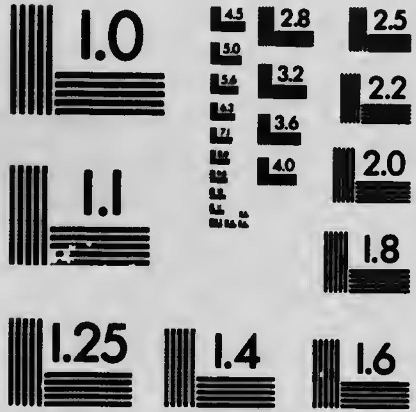


PLANE TRIGONOMETRY

A BRIEF TREATISE ON

PLANE TRIGONOMETRY

MOSTLY ON THE PRACTICAL SIDE
AND INTENDED FOR PRACTICAL
SCIENCE STUDENTS

BY

N. F. DUPUIS, M.A.
QUEEN'S UNIVERSITY

SECOND EDITION

KINGSTON
R. UGLOW & COMPANY
1906

PREFACE

The first edition of this little book was written as a help to students in practical Trigonometry in the School of Mining. The book proved to be so useful that, the first edition becoming exhausted, it was decided to issue this second edition.

While retaining the particular features of the first edition, considerable changes have been introduced in the way of greater expansion along certain lines which lead to a clearer understanding of the subject.

The book contains the essentials of Practical Trigonometry with a minimum of fancy work and of padding to increase the size of the volume. Exercises in transformations, which may be beautiful and interesting, but which are not of practical use, are not many.

The student is encouraged to work with natural functions, as in the experience and opinion of the writer these are more direct, more manageable, with small angles, and fully as expedient as logarithmic methods, and in many cases more so, to an expert arithmetician. Besides, it requires the use of simpler and less bulky tables. And in the case of very small angles it is well known that natural sines and tangents can be taken with very great accuracy, which is not the case with log-functions.

A perusal of some modern works on Trigonometry would naturally lead a student to the conclusion that logarithms form an essential and necessary part of Trigonometry, and that nothing practical in that subject can be done without them. The student should carefully avoid such a conclusion, as, however convenient logarithms may be in assisting to carry out long and peculiar arithmetical operations, they are not essential to any part of practical Trigonometry.

For these reasons logarithmic methods, instead of being placed at the very beginning as in some works, are relegated to the latter portions of the book, where they are given with all the fulness required.

In producing this second edition the author is much indebted to Prof. J. Matheson, for reading the proof sheets, and for other assistance.

N. F. D.

A BRIEF TREATISE ON TRIGONOMETRY

PART I.—PLANE TRIGONOMETRY

1. Of Decimal Approximation. In the practical measurements of length, weight, angles, etc., results are not usually expressible by integers or by definite fractions, but approximately by a series of decimal figures. Thus in measuring accurately the distance between two given points with a scale divided into tenths of an inch, we might estimate the hundredths and find the distance to be approximately 3.14 inches, say. If the scale were graduated to hundredths of an inch we might estimate to thousandths and find the distance to be 3.141 inches. But in any practical case of this kind our expression is probably only an approximation to the true distance, for even if every decimal figure be correct, the true distance might not be accurately expressible except by a larger number of decimals than it is convenient to employ.

It is obvious that, other things being the same, the more decimal places we include the closer is the approximation; but there is little or no purpose in employing more decimals than is necessary to express the degree of approximation required. Thus 3.1415926... expresses approximately the ratio of the circumference of a circle to its diameter. But this approximation is closer than is generally required in practice, and it may be both convenient and profitable to employ a number with fewer decimal places.

To two decimals the approximate expre[illegible] is 3.14; and to three places it is 3.142, for as we rejec[illegible]926 which is more than a half unit in the third decimal place, we should raise this place by a unit, and write 3.142 instead of 3.141. So also to four decimals the expression becomes 3.1416, because this latter expression is nearer 3.14159 than 3.1415 is.

Thus we have the rule—If the part rejected be less than one-half of the last unit retained, the last figure remains unchanged; but if the part rejected be greater than one-half of the last unit retained, one unit is to be added to the last figure.

Ex. The series of approximations to 1.8371296 is 1.8, 1.84, 1.837, 1.8371, 1.83713, 1.837130.

2. Degree of Approximation.—Take a number such as 1.4136 and suppose that it is correct in expression to four decimals. If the digit in the fifth decimal were less than 5 the number would undergo no change when rejecting this digit; but if this digit were greater than 5, the preceding figure would be increased by one unit. Hence we see that without knowing the value of the fifth digit, we are sure that the error in expression cannot be greater than 0.00005, or 5 units in the place following the last one written.

This is the maximum error, and the average error is not more than one-half of the maximum.

Ex. The distance from A to B is given as 2.423 ft., correct to the last unit.

Then, the greatest error of expression is <0.0005 ft., or <0.006 in. Or the expression is in error to a less quantity than 6 thousandths of an inch.

EXERCISE I.

1. A distance is given as 3.45926 miles. Find in inches the maximum error of expression.
2. In expressing an area in acres, how many decimals should be employed so that the error in expression may be less than 1 sq. in.?
3. In expressing a distance in miles, how many decimals is required to give it to the nearest inch?

4. The area of a field is given as 18.7415 ac. Give the greatest possible error of expression in square inches.

3. Contracted Multiplication and Division.—As we have to deal practically with numerical quantities extending to several places of decimals, and in which the integral parts are usually small, it will be convenient to establish some concise method of doing so.

If two decimal expressions each containing five decimals are to be multiplied together, there is no practical utility in retaining more than five decimals in the product; and even if we wished to retain 6 there is no advantage, but rather a disadvantage, in so multiplying as to have 10 decimal places, and then rejecting the last four. So that we seek a method which retains only the number of decimals required, which shall be correct in expression, and which avoids all unnecessary work.

A.—Multiplication.

I. Long method.

1.45612	
2.34168	
11	64896
87	3672
145	612
5824	48
43683	6
291224	
3.40976	70816

II. Contracted method.

1.45612
861432
291224
43684
5824
146
87
12
3.40977

The illustration is the multiplication of 1.45612 by 2.34168, retaining 5 decimal places in the product.

Explanation.—In I we have drawn a vertical line cutting off at the right all the decimals beyond the fifth in the result, and all the partial products from which these are obtained. We notice the saving in writing as none of these figures is written in II. We notice also that the result in II is correct to the nearest unit in the last figure, while rejecting the final 5 figures in the result of I requires that 6 be changed to 7.

Now as to the mode of operation.

In II, the unit's place (2) of the multiplier is written beneath the fifth decimal of the multiplicand, because we wish to retain 5 decimals in the product, and the remaining figures of the multiplier are then written in reversed order, that is 8614. 2 for 234168.

In obtaining the partial products we multiply as usual, but the figure on the extreme right, in any partial product, is got by multiplying the digit directly over the one by which we are multiplying, and adding whatever may be required.

Thus, the partial product for 4 is $4\times6=24$; we put down the 4, and carry the 2 forwards, etc.

But 4×2 and 4×1 are rejected, or are employed only to furnish a number to be carried forwards, if there be any.

The partial product by 6 is 7, that is $6\times5=30$, from which 3 is carried forwards, and $6\times4+3=27$, for which we write 7 and carry the 2 forwards.

In multiplying by the 8 we have to go back two places; thus 8×5 gives 4 to carry; then $8\times4+4=36$, from which we carry 4, as 36 is nearer 40 than 30; and finally $8\times1+4=12$.

To ensure accuracy in the result care must be taken to get the figures on the extreme right of the partial products correct. All the other steps are simple enough.

Ex. 2. To multiply 0.01745 by 0.478 and retain four decimal places in the product.

```
 0.01745          *=unit place of multiplier
   874*
 -------
     70....partial product by 4,
     12....partial product by 7,
      1....partial product by 8,
 -------
  .0083...complete product.
```

Ex. 3. Multiply 15.3718 by 0.43614 and retain 5 decimals in the product.

```
  153718
  41634*
 -------
  614872....partial product by 4,
   46115....partial product by 3,
    9223....partial product by 6,
     154....partial product by 1,
      62....partial product by 4,
 -------
 6.70426....complete product.
```

B.—DIVISION.

Ex. 1. Divide 1.34924 by 3.84125, retaining 5 decimals in the quotient.

```
       I. Long method.              II. Contracted method.
                                     Dividend    Divisor
3.84125)134924(35125                  134924    |384125
        115237|5                      115237    |-------
        ------------                  ------    |0.35125
         19686|50                      19687     Quotient
         19206|25                      19206
         ------------                  -----
           480|250                       481
           384|125                       384
           ------------                  ---
            96|1250                       97
            76|8250                       77
            ------------                  --
            19|30000                      20
            19|20625                      19
            ------------                  --
              |
```

We notice that all the figures to the right of the vertical line in I are omitted in II.

Rule of Procedure.—If the divisor and the dividend have not the same number of decimals, add a sufficient number of ciphers to the one having the deficit to make the number the same for both.

In the example given the number of decimals is the same. 384125 is not contained an integral number of times in 134924. So we place zero in the quotient, and instead of adding a cipher to the dividend we strike out the last figure of the divisor, 5. The partial quotient is now 3. Cancel the next last figure of the divisor, 2. The partial quotient is 5; but in multiplying by this 5 we must be careful to carry forward anything that may have come from the cancelled digits. Continue this operation as far as necessary. The number of figures cancelled is the number of decimals in the quotient.

Ex. 2. Divide 0.17462 by 3.22.

17462	322000
16100	0.05423
1362	
1288	
74	
64	
10	
10	

Ex. 3. Divide 1 by 0.01427 to 5 decimals in the quotient.

1000000	014270
9989	70.07708
1100	
999	
101	
100	
1	
1	

In this example, which is rather an unfavorable one, we add a cipher to the divisor so as to have 5 figures for cancellation; so that we work with 6 decimal places, and have only 5 in the quotient.

EXERCISE II.

1. Find to 6 decimals the circumference of a circle whose diameter is 1 mile.

2. Find to 5 decimals the diameter of a circle whose circumference is 25 feet.

3. Multiply 1.4142136 by 0.7071068, retaining 5 decimals in the product; 7 decimals in the product.

4. The sides of a rectangle being 13.3412 and 24.467 feet, find the area to 3 decimals

5. A measure 0.7312 in. long is taken 0.085 times. Find to 3 decimals the length measured.

6. A meter is 39.37 inches. Find the number of meters in 8.4712 feet, correct to 3 decimals.

7. Divide 1 by 2.30258509 to 7 decimals. (The quotient is the modulus of decimal logarithms.)

8. Divide 180 by 3.1415926 to 7 decimals. (The quotient is the number of degrees in 1 radian.)

9. Work out to 3 decimals the value of $(7920 \times 5280 \times 3.1415926)/1296000$. (The result is the number of feet in 1 second of latitude on the earth's surface.)

10. Work out the value of $\pi r \cos \varphi/180$ to 3 decimals, where $\pi = 3.14159$, $r = 3960$, $\cos \varphi = 0.7167$. (The result is the length in miles of 1 degree of longitude at latitude 44° 13′.)

11. A water wheel has a diameter of 12 feet, and its circumference moves at the rate of 6.347 feet per second. How many revolutions will it make in 1 hour? Also how long will it take to make 1000 revolutions?

12. A carriage wheel is 5 ft. 2 in. in diameter. How many revolutions will take it over 1 mile?

4. **Angles and their Measures.**—Angle is generated by the rotation of a variable line which passes through a fixed point not at infinity, and as the line may rotate in either one of two opposite directions, angles are either positive or negative.

The direction contrary to that of the hands of a clock over its dial is usually taken to be positive; but this is a convention and mostly a matter of convenience.

The angle between two given lines may be defined as that amount of rotation which is necessary to bring one line into the direction of the other.

For practical purposes this amount of rotation must be measured, that is, expressed in terms of some unit of its own kind. For just as a length must be expressed in terms of some unit of length as an inch, and weights in terms of some unit of weight as a pound or a gramme, so an angle must be expressed in terms of some unit of angle.

5. The Degree as Unit Angle.—The simplest unit of angle is the circumangle, or the amount of rotation which a line makes which turns through one complete revolution, and arrives again at its original direction. But this unit is too large for convenient use, and it is accordingly divided as follows:

1 circumangle = 4 right angles = 360 degrees. Hence 1 right angle = 90 degrees.

Then $1° = 60'$ and $1' = 60''$ where $°$ $'$ $''$ denote degrees, minutes and seconds.

The degree is then taken as the practical unit angle, and instruments for measuring angles are usually divided into degrees and parts of the degree.

This division of the angle, known as the sexagesimal or degree measure, is very ancient. It is also quite convenient, as it allows us to express $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{12}$, $\frac{1}{15}$, $\frac{1}{18}$, $\frac{1}{20}$, $\frac{1}{24}$, $\frac{1}{30}$, $\frac{1}{36}$, $\frac{1}{40}$, $\frac{1}{45}$, and $\frac{1}{60}$ of the circumangle in whole numbers of degrees.

The degree measure is the basis upon which rests the majority of trigonometric tables.

6. The Radian as Unit Angle—It is shown in geometry that in the same circle the lengths of arcs are proportional to, or vary as, the angles which they subtend at the centre. Hence if s be the length of an arc and θ be the angle which it subtends at the centre, we may write

$$s = m\theta,$$

where m is a constant but arbitrary multiplier.

The particular relation of s to θ will evidently depend upon what value we give to m. If we put $m = r$, we have

$$s = r\theta \quad \text{.............................(1)}$$

Then $\theta = \frac{s}{r}$, and the angle θ is said to be expressed in radian measure.

∴ And thus—*The radian measure of an angle is the ratio of the length of the subtended arc to the radius of the circle.*

Ex. If the length of the arc be 3.142 and the radius be 2, the measure of the angle is $\frac{3.142}{2}$, or 1.571 radians.

The relation $s = r\theta$ tells us that *the length of an arc of a circle is got by multiplying the radius by the radian measure of the angle which the arc subtends at the centre.*

Ex. If the radius of a circle be 3.425 and the radian measure of the angle subtended by a given arc be 1.5708, the length of the arc is $3.425 \times 1.5708 = 5.37999$ or 5.38 nearly.

If $s = r$, then $\theta = 1$; so that 1 *radian, or the unit of radian measure is the angle subtended at the centre by an arc equal in length to the radius.*

7. Connection Between the Degree and the Radian.—We have $\theta = \frac{s}{r}$. But if s is a semi-circumference of the circle we know that $s = \pi r$, where π stands for 3.1415926..., and putting πr for s gives—

$$\theta = \frac{\pi r}{r} = \pi;$$

and π is the radian measure of a straight angle or two right angles. But 180 is the degree measure of a straight angle. Whence, denoting radians by $^\wedge$,

$$180^\circ = \pi^\wedge.$$

$$\therefore 1^\circ = \frac{\pi}{180} = 0^\wedge.017453 \ldots\ldots\ldots\ldots (2)$$

and

$$1^\wedge = \frac{180}{\pi} = 57^\circ.29578 \ldots\ldots\ldots\ldots (3)$$

The first of these multipliers $\frac{\pi}{180}$ or 0.017453 changes degrees into radians; and the second multiplier $\frac{180}{\pi}$ or 57.29578 changes radians to degrees.

Ex. 1. $48° = 48 \times 0.01745 = 0^\wedge.8376$, and $1^\wedge.213 = 1.213 \times 57.29578 = 69°.5$.

These multipliers are very useful and should be committed to memory. They are reciprocals of one another.

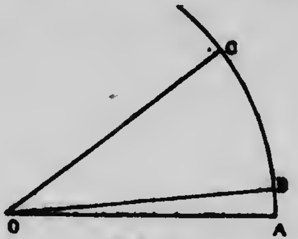


Fig. 1

Ex. 2. Let the angle *AOB* be 5°.264. Its radian measure is $5.264 \times .01745 = 0^\wedge.09186$. Also if the radius *OA* be 4 and the arc *AC* be 3.21, the radian measure of the angle *AOC* is 3.21/4. or $0^\wedge.8025$. And *AOC* in degrees is $0^\wedge.8025 \times 57.29578 = 45°.97986$, or nearly 46°.

8. If the angle *AOB*, Fig. 1, is very small, it is evident that the perpendicular from *B* to *OA* cannot differ much from the arc *BA*, so that for practical purposes we may take the perpendicular for the arc without any serious error. Thus for an angle of 1° on a circle of radius 1, the arc is 0.01745 and the perpendicular is 0.01745; for 2°, the arc is 0.03491 and the perpendicular is 0.03490. And thus for an angle as high as 2°, the arc and the perpendicular differ by only one unit in the 5th decimal place.

Ex. A light of glass in a window is known to be 24 in. high, and it is found to subtend an angle of 16′. Find the distance of the window.

We may take 24 for *s*, and $\frac{16}{60} \times 0.01745$ is the radian measure of the angle.

$\therefore s = r\theta$ becomes $24 = r\frac{16}{60} \times 0.01745$.

$\therefore r = 24 \times \frac{60}{16} \times 57.29578 = 5156.62$ in.

$= 429.7$ ft. nearly.

EXERCISE III.

1. Express 36° 14′ 20″ in radians.
2. Express 54″.35 in radians.
3. Express 0.456 radians in degrees, minutes and seconds.
4. With 1 mile radius find the arc which subtends 1″.
5. With 3960 miles as radius find the arc subtending 1″.
6. The earth's radius being 3960 miles, find the seconds in the angle subtended by 1 ft. on the earth's surface. By 101.33 feet on the earth's surface.
7. A house at the distance of 1 mile subtends an angle horizontally of 35′ 44″. Find the horizontal dimensions of the house.
8. A tree is known to be 76 ft. high. Find the angle in degrees which it subtends at a distance of 1 mile.
9. A cent is 1 in. in diameter. How far from the eye must the cent be placed to appear as large as the moon, *i.e.*, 32′ angular diameter?
10. How far from camp must a man 6 ft. tall go that he may subtend an angle of 50′?
11. A man 5 ft. 8 in. tall standing on the opposite bank of a river subtends an angle of 18′. Find the breadth of the river.
12. *A* and *B* are two places on the same meridian. *A's* latitude is 32° 14′ 12″ N., and *B's* is 27° 15′ 40″ N. Find the distance from *A* to *B* if the earth's radius be 3960 m.
13. If two places on the same meridian be 77 miles apart, and the difference in their latitudes be 1° 6′ 49″, find the earth's radius.

14. The earth's distance from the sun is 93000000 miles, and it makes its annual circuit in 365.2563 dys. Find the earth's velocity per second in its orbit.

TRIGONOMETRIC FUNCTIONS OR RATIOS.

9. Assume any triangle, *OPM*, right angled at *M*. Denote *OM* by x, *MP* by y. and *OP* by r. Then *OP* may be considered as the radius of a circle passing through *P* and meeting *OM* produced in *A*. Denote

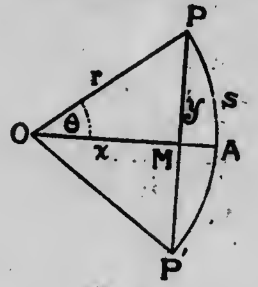


Fig. 2

the angle *POM* by θ. Then θ may be considered as being generated by the radius rotating from *OA* to *OP*. A downward rotation of *OA* as to *OP'* generates a negative angle, and if $AOP' = AOP$ in magnitude, then if $AOP = \theta$, $AOP' = -\theta$.

It is only in comparing angles that we need to consider negative angle.

Denote the arc *AP* by s.

From (1), $\theta^{\wedge} = \frac{s}{r}$, and $\theta^{\circ} = \frac{s}{r} \cdot \frac{180}{\pi}$ . . . . (4)

The sides of the right-angled triangle *OPM* give us six ratios in all, namely:

$$\frac{y}{r}, \frac{x}{r}, \frac{y}{x}, \frac{x}{y}, \frac{r}{x}, \frac{r}{y}.$$

These six ratios form the six trigonometric functions of θ, often but improperly called the trigonometric ratios of θ. The ratios are of the sides of the triangle taken every way in pairs, but they are not ratios into which θ enters, and are not therefore ratios of θ.

But as each ratio is expressible by an infinite series of ascending powers of θ, they are properly functions of θ; and to distinguish them from other functions they are called trigonometrical.

These ratios have distinctive names as follows:

$\frac{y}{r}$ is the sine of θ, contracted to sin θ,

$\frac{x}{r}$ is the cosine of θ, contracted to cos θ,

$\frac{y}{x}$ is the tangent of θ, contracted to tan θ,

$\frac{x}{y}$ is the cotangent of θ, contracted to cot θ,

$\frac{r}{y}$ is the cosecant of θ, contracted to cosec θ,

$\frac{r}{x}$ is the secant of θ, contracted to sec θ.

This assumed triangle, with its notation and the particular names of the ratios, should be mastered and remembered. The following statements may help:

With r as denominator the ratio is sine or cosine; sine, when the other side is opposite the angle, and cosine when adjacent to the angle.

With r as numerator the ratio is secant or cosecant; cosecant when the other side is opposite the angle, and secant when adjacent to the angle.

Without r the ratio is tangent or cotangent.

10. Derivation and Meaning of Co-functions.—It will be noticed from the list in Art. 9, that there are 3 direct functions, sine, tangent, and secant, and 3 co-functions, cosine, cotangent, and cosecant.

Now $\cos\theta = \frac{x}{r} = \sin OPM =$ sine of complement of θ

and this is contracted to complement-sine θ, and finally to cos θ.

Similarly, $\cot\theta$ = tangent of complement of θ,
and $\text{cosec}\,\theta$ = secant of complement of θ.

Ex. 1. As 90 and $\frac{\pi}{2}$ both denote a right angle, the first in degree measure and the other in radian measure, we have

$$\sin\theta = \cos(90^\circ - \theta^\circ) = \cos\left(\frac{\pi}{2} - \theta^c\right) \ldots\ldots\ldots\ldots (5)$$

$$\cot\theta = \tan(90^\circ - \theta^\circ) = \tan\left(\frac{\pi}{2} - \theta^c\right) \ldots\ldots\ldots\ldots (6)$$

etc., etc., etc.

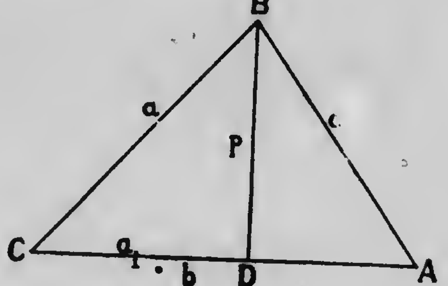


Fig. 3

Ex. 2. In the triangle ABC, let the angles be denoted by A, B, C, respectively, and the opposite sides by a, b, c.

Then $\frac{BD}{BC}$ is the equivalent of $\frac{y}{r}$ and is $\sin C$,

Whence $BD = p = a \sin C$.

Similarly $BD = p = c \sin A$.

whence $a \sin C = c \sin A$;

or $\frac{a}{\sin A} = \frac{c}{\sin C} = \frac{b}{\sin B}$ by symmetry.

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \ldots\ldots\ldots\ldots (7)$$

This relation between the sides of a triangle and the sines of the opposite angles is the *sine formula*, and is much employed.

Again $\frac{CD}{CB}$, equivalent to $\frac{x}{r}$, is $\cos C$.

$$\therefore\ CD = a \cos C.$$

Similarly, $AD = c \cos A$.

and by addition, $b = a \cos C + c \cos A$ .... (8)

Again $\frac{BD}{CD}$, equivalent to $\frac{y}{x}$, is $\tan C$.

Or $p = a_1 \tan C$.

and $a_1 = p \cot C$.

The student should aim to make himself so familiar with the ratios and their names as to be able, in any right-angled triangle, to write them down at sight; and he should remember that the letters A, B, C, a, b, c denoting the angles and sides are only symbolic, and capable of being replaced by any letter which may for the time be convenient.

For the purpose of practice the following exercise is introduced.

EXERCISE IV.

1. The triangle ABC is right-angled at B, and $\angle BAC = \theta$, and $AB = a$. BD is $\perp$ on AC, DE is $\perp$ on BC. EF is parallel to CA, and FG is parallel to BD.

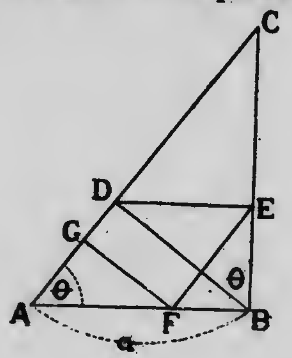


Fig. 4

Express the following line-segments about the figure in terms of a and the functions of θ.

(i) BC. It is readily seen that $BC = a \tan \theta$.
(ii) AC. (iii) BD (iv) BE
(v) DE. $DE = BD \sin \theta$, and $BD = a \sin \theta$. $\therefore DE = a \sin^2 \theta$.
(vi) EF. (vii) FB. (viii) FG.
(ix) AF. (x) AG. (xi) EC.

2. A beam 20 feet long has one end held up by an upright prop so as to make an angle of 30° with the horizontal. (i) What is the length of the prop? (ii) How far is it from the foot of the beam to the foot of the prop? (iii) What is the shortest distance from the beam to the foot of the prop?

3. By drawing a square and its diagonal prove that,

(i) $\tan 45° = 1$, $\cot 45° = 1$. (ii) $\sin 45° = \frac{1}{\sqrt{2}} = \cos 45°$. (iii) $\text{secant } 45° = \sqrt{2} = \text{cosec } 45°$.

4. By drawing a perpendicular from a vertex of an equilateral triangle to the opposite side, prove that—

(i) $\sin 30° = \frac{1}{2}$. (ii) $\cos 30° = \frac{\sqrt{3}}{2}$. (iii) $\tan 30° = \frac{1}{\sqrt{3}}$

(iv) $\sin 60° = \frac{\sqrt{3}}{2}$. (v) $\cos 60° = \frac{1}{2}$. (vi) $\tan 60° = \sqrt{3}$.

(vii) $\sec 60° = 2$.

11. Tables giving the values of the trigonometric functions for every degree and minute from 0° to 90°, or through a right angle, are called trigonometric tables, and in particular tables of the *natural functions*, to distinguish them from the logarithms of those quantities, which are tabulated under the name of logarithmic functions or log-functions.

The tabulated quantities are given to a certain number of decimals, usually not less than 4 and not greater than 7; and we thus have 4-place, 5-place, 6-place and 7-place tables.

Tables of the functions are generally accompanied by directions for their use, and these directions, on account of variations in the construction of the tables, are particular and cannot be given here. But the writer would tender this advice. Become expert at working with 4 and 5-place decimals, and employ natural functions rather than logarithmic ones. Natural functions are more direct and simple in their application, and offer less chances for errors in work. Besides, to a person skilled in working with decimals, operations are even more expeditious with natural functions than with log-functions.

12. Inter-relation of the Functions.—The functions of an angle are so related that when one of them is given, the others may be found. Thus only one function is absolutely necessary; but to have the others given also is a matter of great convenience.

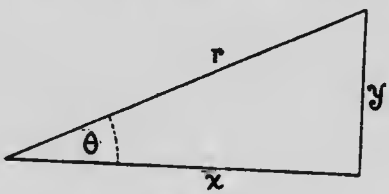


Fig. 5

Going to our reference triangle,

(1). $\left(\frac{y}{r}\right)^2+\left(\frac{x}{r}\right)^2=\frac{y^2+x^2}{r^2}=\frac{r^2}{r^2}=1.$

$\therefore \sin^2\theta+\cos^2\theta=1.$

(2). $\left(\frac{r}{x}\right)^2-\left(\frac{y}{x}\right)^2=\frac{r^2-y^2}{x^2}=\frac{x^2}{x^2}=1.$

$\therefore \sec^2\theta-\tan^2\theta=1$, and $1+\tan^2\theta=\sec^2\theta.$

(3). $\left(\frac{r}{y}\right)^2-\left(\frac{x}{y}\right)^2=\frac{r^2-x^2}{y^2}=\frac{y^2}{y^2}=1.$

$\therefore \operatorname{cosec}^2\theta-\cot^2\theta=1.$

(4). $\frac{y}{x}\cdot\frac{x}{y}=\frac{r}{x}\cdot\frac{x}{r}=\frac{r}{y}\cdot\frac{y}{r}=1.$

$\therefore$ tan θ. cot $\theta=$sec θ. cos $\theta=$cosec θ. sin $\theta=1$.

(5). $\frac{y}{x}=\frac{y}{r}\div\frac{x}{r}$ $\therefore$ tan $\theta=\frac{\sin\theta}{\cos\theta}$.

(6). $\frac{y}{r}=\frac{y}{x}\div\frac{r}{x}$ $\therefore$ sin $\theta=\frac{\tan\theta}{\sec\theta}$.

Collecting these results we may conveniently arrange them under three heads as follows:

$$\text{i.}\quad \left.\begin{cases}\sin^2\theta+\cos^2\theta=1\\ \sec^2\theta-\tan^2\theta=1\\ \text{cosec}^2\theta-\cot^2\theta=1\end{cases}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

$$\text{ii.}\quad \left.\begin{cases}\tan\theta.\cot\theta=1,\text{ or }\cot\theta=1/\tan\theta\\ \cos\theta.\sec\theta=1,\text{ or }\sec\theta=1/\cos\theta\\ \sin\theta.\text{cosec}\,\theta=1,\text{ or }\text{cosec}\,\theta=1/\sin\theta\end{cases}\right\}\ldots\ldots(10)$$

$$\text{iii.}\quad \left.\begin{cases}\tan\theta=\sin\theta/\cos\theta\\ \sin\theta=\tan\theta/\sec\theta\end{cases}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

The preceding relations are in constant use in the practice of trigonometry and should be thoroughly mastered.

Ex. 1. Given tan $\theta=\frac{b}{a}$, to find sin θ and cos θ.

From (iii), $\sin\theta=\frac{\tan\theta}{\sec\theta}=\frac{b}{a}/\sqrt{1+\tan^2\theta}=\frac{}{\sqrt{a^2+b^2}}$

$$\cos\theta=\frac{1}{\sec\theta}=\frac{1}{\sqrt{1+\tan^2\theta}}=\frac{a}{\sqrt{a^2+b^2}}$$

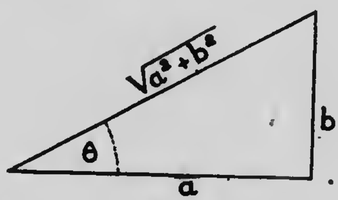


Fig. 6

Or as follows: In the triangle of Fig. 6 if a be the base and b the altitude, the hypothenuse is $\sqrt{(a^2+b^2.)}$

Then $\tan\theta=\frac{b}{a}$, and $\sin\theta=\frac{b}{\sqrt{a^2+b^2}}$, and $\cos\theta=\frac{a}{\sqrt{a^2+b^2}}$.

Ex. 2. If $\tan\theta=\frac{4}{5}$, $\sin\theta=\frac{4}{\sqrt{4^2+5^2}}=\frac{4}{\sqrt{41}}$; $\cos\theta=\frac{5}{\sqrt{41}}$.

Ex. 3. If $\sin\theta=\frac{m}{n}$, $\tan\theta=\frac{m}{n}\sqrt{1-\frac{m^2}{n^2}}=\frac{m}{\sqrt{n^2-m^2}}$.

The following table gives the leading functions, sine, cosine, tangent, and secant, each in terms of the others:

	sin $=x$	cos $=x$	tan $=x$	sec $=x$
sin	x	$\sqrt{1-x^2}$	$\frac{x}{\sqrt{1+x^2}}$	$\frac{\sqrt{x^2-1}}{x}$
cos	$\sqrt{1-x^2}$	x	$\frac{1}{\sqrt{1+x^2}}$	$\frac{1}{x}$
tan	$\frac{x}{\sqrt{1-x^2}}$	$\frac{\sqrt{1-x^2}}{x}$	x	$\sqrt{x^2-1}$
sec	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{x}$	$\sqrt{1+x^2}$	x

EXERCISE V.

1. If $\tan\theta=\frac{1}{2}$, find $\sin\theta$, $\cos\theta$, $\sec\theta$.
2. If $\sin\theta=\frac{3}{4}$, find $\tan\theta$, $\cos\theta$, $\sec\theta$.
3. Find $\tan\theta$ when $\sin\theta=0$, $=1$, $=-1$.
4. Find $\sec\theta$ when $\sin\theta=0$, $=\frac{1}{\sqrt{2}}$, $=1$.

5. Express $(\tan\varphi+\cot\varphi)\cos\varphi$ in terms of $\sin\varphi$.
6. If $\sec\theta=2$, find $\sin\theta$ and $\tan\theta$.
7. If $\tan\theta-\cot\theta=1$, find $\tan\theta$.

$\tan\theta-\dfrac{1}{\tan\theta}=1$. $\therefore \tan^2\theta-\tan\theta-1=0$, and solving this as a quadratic in $\tan\theta$, we get $\tan\theta=(1\pm\sqrt{5})/2$.

8. If $\sin\theta+\cos\theta=1$, find $\cos\theta$.

13. The accompanying diagram may be useful as giving a graphic representation of the several functions.

OA is perpendicular to *OB*, so that *AOB* is a right angle, and the arc *APB* is a quadrant of a circle.

If we take the radius *OA* or *OP* as our unit length, we may avoid writing it as a denominator.

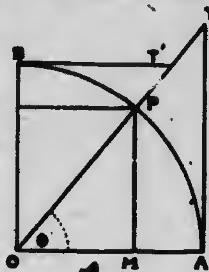


Fig. 7

Then *PM* represents $\sin\theta$, and *OM* represents $\cos\theta$. Also, *AT*, being a tangent line at *A*, represents $\tan\theta$, and *BT′* represents $\cot\theta$. Finally, *OT* represents $\sec\theta$, and *OT′*, $\operatorname{cosec}\theta$. The angle θ is represented by arc *AP*. If *P* moves towards *B*, θ increases, and *PM*, *AT* and *OT* all increase, while *OM*, *BT′* and *OT′* decrease.

Or, *as the angle increases from* 0° *to* 90° *the functions* $\sin\theta$, $\tan\theta$, *and* $\sec\theta$ *all increase, while the co-functions* $\cos\theta$, $\cot\theta$, *and* $\operatorname{cosec}\theta$ *all decrease.*

When *P* is at *A*, *MP* and *AT* are zero, and *OT* = the radius = 1. $\therefore \sin 0°=0$, $\tan 0°=0$, $\sec 0°=1$.

Also $\cos 0°=1$, $\cot 0°=\infty$, $\operatorname{cosec} 0°=\infty$.

When *P* is at *B*, $MP=1$; $TA=\infty$, $OT=\infty$, $OM=0$, $BT'=0$ and $OT'=1$. $\therefore \sin 90°=1$, $\tan 90°=\infty$, $\sec 90°=\infty$, $\cos 90°=0$, $\cot 90°=0$, $\operatorname{cosec} 90°=1$.

EXERCISE VI.

(*These exercises are intended to be worked by natural functions, and complete results are required*).

1. A plank 12 ft. long has one end raised from the floor until it describes an arc 3 ft. 4 in. long. Find (1) the angle that the plank makes with the floor; (2) the perpendicular height of the raised end.

2. The end of the plank of Ex. 1 is placed upon a box 22 inches high. What angle does the plank make with the floor?

3. A saw-cut through a board 20 inches wide, is 23.45 inches long. Find the angle which the cut makes with the edge of the board.

4. Across a board 12 inches wide a line is to be drawn so as to be 16.97 inches long. Find the angle that the line must make with the edge of the board.

5. A line is drawn from a vertex of a square to the mid-point of an opposite side. Find the angles which the line makes with the diagona's.

6. A hill has an incline of 36° 12′. In going 120 ft. up the hill how far does one go (*a*) on the level, (*b*) on the vertical?

7. A post 6 feet high stands upright on level ground. Find the length of its shadow when the sun's elevation is 18° 40′.

8. A vertical post 3 ft. high casts a shadow 5 ft. 8¼ in. long on level ground. Find the sun's elevation.

9. The legs of a pair of compasses are each 6 in. long. How far apart must the points be that the lines from joint to points may make an angle of 40°?

10. In the triangle ABC the angle $A = 64°\ 18'$, $B = 44°\ 13'$ and the altitude to side AB is 24. Find the remaining parts of the triangle.

11. Given two sides of a triangle 44 and 37, and altitude to side 44, 32, to find the angles of the triangle.

12. In the triangle ABC, $A = 58°\ 42'$, $B = 36°\ 20'$ and side $c = 20$. Find the other parts.

(Draw $CD \perp$ to AB. Put $AD = x$, $CD = y$. Then $y = x \tan A = (c - x) \tan B$. Whence x is found, etc.).

13. Given two sides of a triangle and the altitude to the third side to find the remaining parts.

COMPLEMENT AND SUPPLEMENT—NEGATIVE ANGLE.

14. Divide the circle into 4 quadrants by the two perpendicular diameters AC and BD. Then OP starting from OA and rotating in the positive direction describes first the quadrant or right angle AOB, then the quadrant BOC, then COD, and lastly DOA, thus completing the circumangle.

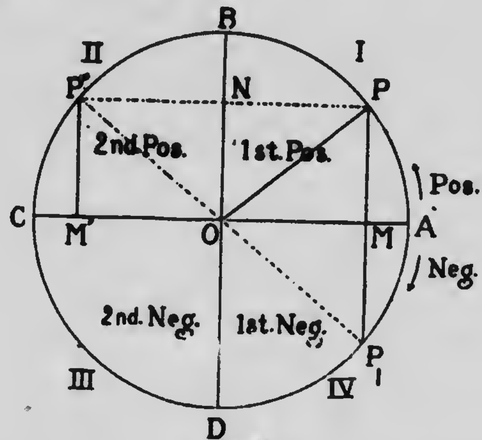


Fig. 8

In general trigonometry these are called the 1st, 2nd, 3rd, and 4th quadrants, in the order of rotation as given.

When OP lies in the 1st quadrant the angle AOP is said to be in the first quadrant, and the angle is then $> 0°$ and less than $\frac{\pi}{2}$ or $90°$. Similarly the angle AOP' is in the 2nd quadrant and is $> \frac{\pi}{2}$ or $90°$ and $< \pi$ or $180°$; and the angle AOP_1, described by the rotation of OA

through OB, OC and OD to OP_1, is said to be in the 4th quadrant, and is $>\frac{3\pi}{2}$ or 270° and $<2\pi$ or 360°.

In the triangle, however, no angle is so great as two right angles, so that all angles of the triangle lie within the first or the second quadrant.

Then angles of the third and fourth quadrants may be looked upon as negative angles formed by the rotation of OA downwards to OP_1, and so on to OD, and finally through 180° to OC.

The whole four quadrants will thus be covered by considering two quadrants to contain positive angles, from 0° to 180°, and two quadrants to contain negative angles, from 0° to −180°.

15. $\sin AOP = \frac{MP}{OP}$. Now OP, as the radius of the circle, is always positive, while MP is positive in both positive quadrants, and negative in both negative quadrants.

Hence *the sine of an angle is positive in the positive quadrants, and negative in the negative quadrants.* Or, *the sine of an angle has the same algebraic sign as the angle has.*

Again, $\cos AOP = \frac{OM}{OP}$, while $\cos AOP' = \frac{OM'}{OP}$. But from geometry, if OM is positive, OM' is negative as being measured in the opposite direction.

Hence the cosine of an angle is positive in the first quadrant and negative in the second; and it is readily seen that this is equally true when the quadrants are negative.

Therefore, *for both positive and negative angles, cosines are + in the first quadrant and − in the second.*

Ex. sin 65° is +, sin (−65°) is −, cos 72° is +, cos (− 72°) is +, sin 156° is +, sin (−156°) is −, cos 156° is −, cos (−156°) is −.

And $\sin(-\theta) = -\sin\theta$; $\cos\theta = \cos(-\theta)$.

16. Complement.—The complement of an angle is that angle which added to the given angle makes the sum 90° or a right angle. Or, when the sum of two angles is a right angle each angle is the complement of the other.

Ex. 1. Comp. $36° = 54°$; comp. $125° = -35°$; comp. $-40° = 130°$, etc.

Ex. 2. Comp. $\theta° = 90° - \theta°$, or comp. $\theta^{\wedge} = \frac{\pi}{2} - \theta^{\wedge}$.

AOP and POB are complementary angles, and $\sin AOP = \cos POB$; and $\cos AOP = \sin POB$.

Hence if $AOP = \alpha$, $POB = \frac{\pi}{2} - \alpha$

and α and $\frac{\pi}{2} - \alpha$ are complementary angles.

$$\therefore \left.\begin{aligned} \sin\left(\frac{\pi}{2} - \alpha\right) &= \cos\alpha \\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \sin\alpha \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (12)$$

Change the sign of α, then—

$$\left.\begin{aligned} \sin\left(\frac{\pi}{2} + \alpha\right) &= \cos(-\alpha) = \cos\alpha \\ \cos\left(\frac{\pi}{2} + \alpha\right) &= \sin(-\alpha) = -\sin\alpha \end{aligned}\right\} \quad \ldots\ldots\ldots (13)$$

Ex. 1. $\sin 72° = \cos 18°$ $\quad \sin 124° = \cos 34°$
$\cos 130° = -\sin 40°$ $\quad \cos(-50°) = \sin 40°$

Ex. 2. $\cos\left(\frac{3\pi}{2} - \theta\right) = \cos\left(-\frac{\pi}{2} - \theta\right) = \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta.$

$\cos\left(\frac{3\pi}{2} + \theta\right) = \cos\left(-\frac{\pi}{2} + \theta\right) = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta.$

17. Supplement.—Two angles are supplements of one another when their sum is two right angles. Thus AOP and POC are supplementary angles.

If the $\angle P'OC = \angle POA$, then $\angle AOP'$ is the supplement of $\angle AOP$. But AOP' and AOP have the same sine, since $MP = M'P'$.

Therefore *the sine of an angle is the same as the sine of its supplement.*

Again $OM' = -OM$, so that—

The cosine of an angle and the cosine of its supplement are equal in magnitude but opposite in sign.

Ex. 1. θ and $(\pi - \theta)$ are supplementary.

$\therefore \sin(\pi - \theta) = \sin\theta$; and $\cos(\pi - \theta) = -\cos\theta$.

Ex. 2. $\sin 142° = \sin 38°$; and $\cos 142° = -\cos 38°$.

Tables of trigonometric functions are given for angles lying between 0° and 90° only, and to find the function for any other angle it must be in some way reduced so as to lie between these limits.

EXERCISE VII.

1. Given that 0.6648 is the cosine of 48° 20′, find the two angles, less than 180°, of which it is the sine.

2. Given that 0.4731 is sin 28° 14′, find two angles of which it is the cosine.

3. Show two ways of taking from the tables (*a*) sin 145° 20′, (*b*) cos 157° 10′.

4. Show how to find from the table, (*a*) sin 128°, (*b*) cos 165°, tan 105°.

5. Prove that the tangent of an angle and the tangent of its supplement are equal in magnitude and opposite in sign.

The following table giving the signs and limits of the several functions in the different quadrants will be useful for reference.

	Q. I.		Q. II.		Q. III.		Q. IV.	
	1st Pos. Q.		2nd Pos. Q.		2nd Neg. Q.		1st Neg. Q.	
funct.	from	to	from	to	from	to	from	to
sine	+0	+1	+1	+0	−0	−1	−1	−0
cos	+1	+0	−0	−1	−1	−0	+0	+1
tan	+0	+∞	−∞	−0	+0	+∞	−∞	−0
sec	+1	+∞	−∞	−1	−1	−∞	∞	+1
cot	+∞	+0	−0	−∞	+∞	+0	−0	−∞
cosec	+∞	+1	+1	+∞	−∞	−1	−1	−∞

18. Graphs of Functions.—Taking any convenient unit length, draw a line *AB*, 3.14 units in length, and divide it into 18 equal parts, numbered 0°, 10°, 20°, ...180°. *AB* represents the length of the arc of a semicircle with 1 unit as radius, and *A*.10, *A*.20, etc., represent the angles 10°, 20°, etc.

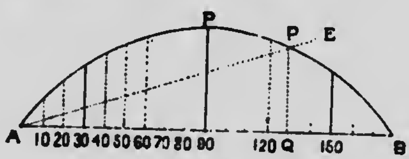


Fig. 9

At 10, 20, 30, etc., erect perpendiculars representing in length sin 10°, sin 20°, sin 30°, etc., with unit radius, that is the numbers 0.17, 0.34, 0.50, etc., and through the upper end-points of these perpendiculars draw the curve *APB*. This curve is the graph of the sine, and is called the *sinusoid*. The rise and fall of the curve pictures the variation in the value of the sine as the angle increases from 0° to 180°. From 180° to 360° the curve is similar in form but lies below the base line.

Graphs of the other functions are constructed in like manner.

Some difficult transcendental equations may be approximately solved by means of graphs.

Ex. Given $x=3 \sin x$, to find x. Here x is the radian measure of an angle, and we are asked to find the radian value of that angle which is three times its own sine.

Since $\frac{x}{3}=\sin x$, we may write—

$$\frac{x}{3}=y, \text{ and } y=\sin x.$$

But in the equation $y=\sin x$, y is ordinate to any point in the graph of the sine. And to get $y=\frac{1}{3}x$ we draw through A a line making with AB an angle whose tangent is $\frac{1}{3}$, as AE. This cuts the sine graph in P, and at this point the y of the sine graph and the y of the line are the same; or, $\frac{x}{3}=\sin x$. And by drawing the ordinate $PQ \perp$ to AB we find $AQ=x$ to be about 130°.

By graphing a small part around PQ upon a larger scale, a closer approximation may be obtained.

EXERCISE VIII.

1. Draw the graphs of (*a*) the cosine; (*b*) the tangent; (*c*) the secant, each through the two positive quadrants.
2. Solve the equation $x=\tan x$, where x is the radian measure of an angle.
3. Solve the equation $\tan x=3 \sin x$ by graphs. Also solve it without graphs.
4. Solve the equation $x=\sin x+\cos x$.
5. PM is $\perp$ to the diameter AB of a circle, P being on the curve. Determine the position of P when
   (*a*) The arc AP is twice PM.
   (*b*) The arc AP is equal to $OP+OM$, where O is the centre.

19. The Triangle.—The trigonometric solution of triangles consists in finding the remaining parts of a triangle when three parts, sufficient for the determination of a triangle, are given, and it is shown in geometry that a triangle is completely given when any three of its six parts are given, except the three angles, and two sides and the angle opposite the shorter side. In this latter case two triangles in general satisfy the conditions and the solution is said to be *ambiguous*.

The six parts of a triangle are the three angles, and the three opposite sides. These are usually denoted by A, B, C and a, b, c respectively, a being the side opposite $\angle A$, etc. But the student should accustom himself to employ any letters that may be convenient.

With the employment of natural functions the general solution of triangles is effected mostly by the application of two direct formulæ, which we proceed to develop.

20. **The Sine Formula.**—This has already been developed in article 10 (7). But the following method gives a more general result:

ABC is a triangle in its circumcircle. a, b, c are the sides and d is the circumdiameter.

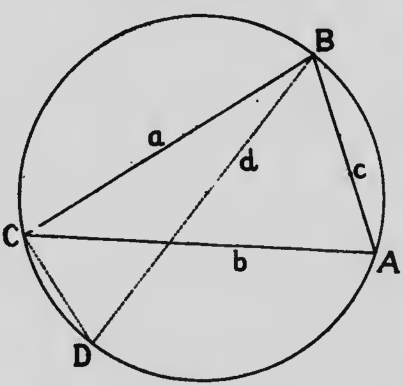


Fig. 10

Then $\angle D = \angle A$, as they stand on the same arc BC. But BCD is a right angle, being in a semicircle.

$$\therefore a = d \sin A\text{; or } d = \frac{a}{\sin A}$$

and as this must be true for each side and angle,

$$d=\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Hence, *in any triangle the sides are proportional to the sines of the angles opposite.*

We may also state the following theorem which is sometimes useful:

The sine of an angle of a triangle is the ratio of the opposite side to the circumdiameter of the triangle.

Ex. 1. In a triangle $a=20$, $b=45$, and $B=36°$, to find A.

$\because$ (14) $\frac{a}{\sin A}=\frac{b}{\sin B}$, $\therefore \sin A=\frac{a}{b}\sin B=\frac{20}{45}\sin 36°$.

and $\sin 36°=0.58779$. $\therefore \sin A=0.26124$.

And $A=$ either $15°\ 9'$ or $164°\ 51'$. Art. 17.

But it is shown in geometry that in any triangle the greater angle is opposite the greater side. And as $b>a$, so $B>A$. And as $B=36°$, A must be $15°\ 9'$.

Ex. 2. Given $a=14.14$, $b=20$, $A=30°$, to find B.

Here $\sin B=\frac{b}{a}\sin A=\frac{20}{14.14}\times 0.5=0.7072$.

And $B=45°$ or $135°$.

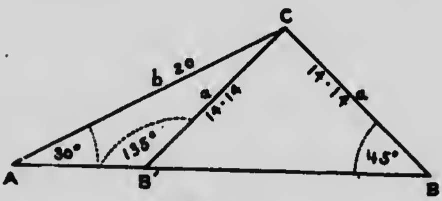


Fig. 11

And as $a<b$, so $A<B$, and both values of B are admissible. That is the solution is ambiguous, giving two different triangles, one with $B=45°$ and the other with $B=135°$.

The figure shows the nature of the ambiguity, as both of the triangles ABC and $AB'C$ have the given parts.

We notice then that there is an ambiguity whenever we seek to determine an angle from its sine alone, as there are two angles both less than 180° having the same sine, and both possible as an angle of a triangle, and unless we have some other means of knowing which of these two angles is to be taken, the solution is ambiguous.

EXERCISE IX.

1. In a triangle $a=20$, $b=26$, $A=35° 22'$ to find B.
2. In a triangle $a=35$, $b=48$, $A=62° 40'$ to find B.
3. In a triangle $A=51° 20'$, $B=16° 35'$, $c=17.45$ to find a, b, and C.
4. In a triangle $a=24.6$, $b=45.33$, $B=67° 15'$ to find all the other parts.
5. In a triangle $a=48$, $c=84$, $B=67° 20'$ to find the other parts.

(Draw $CD \perp$ to AB. Then $BD=a \cos B$, and $CD=a \sin B$. Thus BD and CD are known. Then $CD=AD \tan A=(c-a \cos B) \tan A$, and $\therefore \tan A=a \sin B/(c-a \cos B)$.)

6. A post 18 feet long leans to the north at an angle of 20° from the vertical. Find the length of its shadow on level ground when the sun is south at an elevation of 47° 50′.
7. Solve Ex. 6 on the condition that the post leans to the east at the same angle.
8. Solve Ex. 6 on the condition that the post leans to the south at the same angle.
9. A triangle right-angled at B has $a=4$, $c=10$, and the line BD meets AC in D and makes the angle $CBD=75°$. Find the length of BD.
10. In Ex. 9, $a=6$, and $c=13$. At what angle with AB must BD be drawn, meeting AC in D, so that BD may be 10?
11. The sides of a triangle being 13, 14, 15, its circum-diameter is 16.25. Prove that this is correct by showing that the sum of its three angles is 180°.
12. $ABCD$ is a quadrilateral right-angled at B and D. Show that $\sin A=\sin C=BD/AC$.
13. If α, β, γ be the altitudes of a triangle and a, b, c the sides, show that $\sin A . \sin B . \sin C=\alpha\beta\gamma/abc$.

21. The Cosine Formula.—This formula may be developed in several different ways, but the following is one of the simplest:

In the triangle ABC, BD is the altitude from B.
Then $AD = c \cos A$
and $BD = c \sin A$.

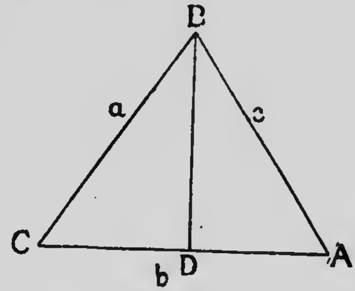


Fig. 12

But $BC^2 = BD^2 + CD^2$

$$= (c \sin A)^2 + (b - c \cos A)^2$$
$$= c^2(\sin^2 A + \cos^2 A) + b^2 - 2bc \cos A,$$

Or, $a^2 = b^2 + c^2 - 2bc \cos A$, since $\sin^2 A + \cos^2 A = 1$.

From this we may write two formulas in sets of three each. These are practically all the same, being variations obtained from symmetry, and any one of them may be quoted as *the cosine formula*.

$$(a) \quad \left\{\begin{array}{ll} a^2 = b^2 + c^2 - 2bc \cos A & \ldots\text{(i)} \\ b^2 = c^2 + a^2 - 2ca \cos B & \ldots\text{(ii)} \\ c^2 = a^2 + b^2 - 2ab \cos C & \ldots\text{(iii)} \end{array}\right\} \ldots\ldots\ldots\ldots (15)$$

$$(b) \quad \left\{\begin{array}{ll} \cos A = \dfrac{b^2 + c^2 - a^2}{2bc} & \ldots\text{(i)} \\ \cos B = \dfrac{c^2 + a^2 - b^2}{2ca} & \ldots\text{(ii)} \\ \cos C = \dfrac{a^2 + b^2 - c^2}{2ab} & \ldots\text{(iii)} \end{array}\right\} \ldots\ldots\ldots\ldots (16)$$

Set (*a*) makes known the third side of a triangle when two sides and the included angle are given; and set (*b*) makes known an angle when the three sides are given.

Notice that in (*a*) the side that appears on the left is opposite the angle which appears on the right; and in (*b*) the angle appearing on the left is opposite the side whose square is subtractive on the right, and that the denominator contains those sides whose squares are additive in the numerator.

Ex. 1. Given $a=20$, $b=28$, $C=44° \ 10'$ to find c.

From (15, iii) $c^2=a^2+b^2-2ab\cos C$,

$$\text{or } c^2=(20)^2+(28)^2-2\times 20\times 28\cos 44°\ 10'$$
$$=1184-1120\times 0.71732=380.6$$
$$\therefore c=19.51 \text{ nearly.}$$

Ex. 2. Given $a=50$, $b=42$, $c=80$, to find the three angles.

(1) For the angle opposite 50 we have—

$$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{42^2+80^2-50^2}{2\times 42\times 80}=0.84286.$$
$$\therefore A=32°\ 33'$$

(2) For the angle opposite 80 we have—

$$\cos C=\frac{50^2+42^2-80^2}{2\times 50\times 42}=-0.50857.$$

As $\cos C$ is negative, C lies in the 2nd quadrant and is greater than 90°.

But the angle whose cosine is 0.50857 is 59° 26′.

$\therefore C=180°-59°\ 26'=120°\ 34'$. Then $B=26°\ 53'$, as the sum of the angles is 180°.

EXERCISE X.

1. If $a=26$, $b=28$, $c=30$, find the angles.
2. If $a=24.3$, $b=17.75$, $A=27°\ 15'$, find c.
3. Given $a=15.71$, $b=18.37$, $B=162°\ 38'$ to find c.
4. Given $a=42.3$, $b=56.1$, $C=37°\ 44'$ to find A, B, c.
5. Starting from A I measure off 320 rods to B. I then change my direction through 42° 50′ and go 480 rods to C. How far is it from A to C in a straight line?
6. The road from A to B goes by way of C. From A to C is 23 miles direct north, and from C to B is 42

miles 27° east of north. How much will the road be shortened by going directly from A to B, and what will be the direction?

7. Starting from A I wish to measure a 10 mile straight line. At B, 4 miles from A, I find a large swamp. I turn to the right 50° and go 2.5 miles. I then turn to the left 97° from my previous course. How far must I go to strike my first line at C? And what is the distance from C to B?

8. The sides of a parallelogram are a and b and the angle between them is θ. Show that the two diagonals are given by $\sqrt{(a^2+b^2\pm 2ab\cos\theta)}$.

9. In any triangle $a=b\cos C+c\cos B$. Prove the cosine formula by transposing $b\cos C$ and squaring.

10. It is shown in geometry that in the triangle ABC, $BC^2=AB^2+AC^2-2AB.AD$ where D is the foot of the altitude from C. From this relation develop the cosine formula.

22. Area of the Triangle.—It is shown in geometry that the area of a triangle is equal to half that of the rectangle having the same base and altitude, and as the area of the rectangle is given by the product of the base and the altitude, that of the triangle is given by one-half this product.

Hence from figure of Art. 21,

$BD=c\sin A$, and the base AC is b.

$\therefore$ Denoting the area by $\triangle$ we get—

$$\triangle=\tfrac{1}{2}bc\sin A \qquad (17)$$

with two symmetrical expressions.

This expresses the area in terms of two sides and the included angle.

Ex. If $b=25$, $c=36$, $A=47°\ 18'$, $\triangle=\frac{1}{2}\times 25\times 36$ $\sin 47°\ 18'=450\times 0.73491=330.709\ldots.$

Again $\sin^2 A=1-\cos^2 A=1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2$, (16.i) and $1-\left[\frac{b^2+c^2-a^2}{2bc}\right]^2=\frac{(b+c)^2-a^2}{2bc}\cdot\frac{a^2-(b-c)^2}{2bc}$

$$=\frac{(a+b+c)\ (b+c-a)\ (c+a-b)\ (a+b-c)}{4b^2c^2}.$$

and putting $(a+b+c)=2s$, and hence $(a+b-c)=2(s-c)$, etc., we finally get

$$\sin A=\frac{2}{bc}\sqrt{\{s\ (s-a)\ (s-b)\ (s-c)\}}.$$

And from (17),

$$\Delta=\sqrt{\{s\ (s-a)\ (s-b)\ (s-c)\}}\ \ldots\ldots\ldots\ldots(18)$$

and as s is the half sum of the sides this gives the area in terms of the sides.

Various other expressions may be obtained for the area of the triangle, but those given are the most useful.

Ex. 1. Given $a=24$, $b=36$, $C=55°\ 20'$, to find the area.

$$\Delta=\frac{1}{2}\ ab\sin C=\frac{1}{2}\times 24\times 36\times\sin 55°\ 20'$$
$$=432\times .82248=355.31$$

Ex. 2. Given $a=13$, $b=14$, $c=15$, to find the area.

Here $s=21$, $s-a=8$, $s-b=7$, $s-c=6$.

$$\therefore\ \Delta=\sqrt{\{21\times 8\times 7\times 6\}}=84.$$

Ex. 3. Given $a=20$, $b=15$, $A=50°$, to find the area.

By the sine formula, $\sin B=\frac{15}{20}\sin 50°$

$=\frac{3}{4}\times 0.76604=0.57453$. And $B=35°\ 4'$, and the solution is not ambiguous.

Then $\Delta=\frac{1}{2}\times 20\times 15\times\sin 94°\ 56'=149.445$.

23. The Circumradius.—We have $\Delta=\frac{1}{2}\ bc\sin A$,

$$\therefore\ \frac{1}{\Delta}=\frac{2}{bc}\cdot\frac{1}{\sin A}=\frac{2}{bc}\cdot\frac{d}{a},\ (14).$$

$$\left.\begin{aligned}\text{Whence}\quad & d=\frac{abc}{2\Delta}\\ \text{and}\quad & R=\frac{abc}{4\Delta}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(18)$$

Other expressions for R are readily obtained.

Ex. 4. Given $a=26$, $b=28$, $c=30$, to find the point equidistant from the vertices of the triangle.

This point is evidently the circumcentre of the triangle, and R is the common distance.

Now (18) $\Delta=\sqrt{\{s\,(s-a)\,(s-b)\,(s-c)\}}=336.$

Whence $R=\dfrac{26\times28\times30}{4\times336}=16.25.$

EXERCISE XI.

1. Find the sines of the angles of the triangle whose sides are 39, 42, 45.
2. Find the area of the triangular field in which two sides are 14 and 23 rods, and the included angle is 76° 17′.
3. Find the area of the triangle whose sides are 52, 56 and 60.
4. Two sides of a triangle are respectively 14 and 23 feet, and the area is 125 sq. ft. Find the angle between the sides. Point out any ambiguity and explain it.
5. If in Ex. 4 the area is given as 172 sq. ft., explain the difficulty which arises.
6. What angle is between two given sides of a triangle when its area is a maximum?
7. A triangle is inscribed in a circle of 20 ft. diameter, and two sides are 8 ft. and 6 ft. Find the other parts of the triangle.
8. The diameter of a circle is 125 ft. Find the side of the equilateral triangle inscribed in it.
9. Find the circumradius of the triangle whose sides are 78, 91, and 100 ft. respectively.
10. The sides of a triangle are a, b, c. Find the length of the median to side b.

    (Let M be the foot of the median, and φ be the angle between the median and the base, b. Apply the cosine formula to the triangles BAM and BCM, and eliminate φ).
11. In Ex. 10, find any function of φ in terms of the sides of the triangle.

12. If θ be the angle of a parallelogram, and φ be the angle between the long diagonal and the base, show that $\tan\varphi = \dfrac{a \sin\theta}{b + a\cos\theta}$.

13. If a, b be adjacent sides of a parallelogram, θ be the angle between them, and φ be the angle between the diagonals, prove that $\tan\varphi = 2ab \sin\theta / (a^2 - b^2)$.

14. The area of a triangle is $2R^2 \sin A \sin B \sin C$.

15. The area of a triangle is $\frac{1}{4}\sqrt{\{2\Sigma a^2b^2 - \Sigma a^4\}}$.

16. The area of a triangle is $a^2. \sin B \sin C / 2 \sin A$.

17. To find the distance from A to P a base line AB, 1000 yds. long, is measured in any convenient direction. The angle $BAP = 14° \ 18'$, and $ABP = 114° \ 38'$. Find AP.

18. A tower 50 feet high stands on a mound. From a point on the level ground the angle of elevation of the top of the tower is 75°, and of the bottom 45°. Find the height of the mound.

19. AB is a base line 1000 feet long. AC is perpendicular to AB and the angle $ABC = 85° \ 10'$. AB is produced to D so that the angle $ACD = 84° \ 30'$. Find the distance AD.

24. Applications of Trigonometry to Forces, Etc.—In its applications to statics and dynamics Trigonometry plays an important part. And we here propose to give a number of elementary applications, partly as exercises in Trigonometric work, and partly so that when the student meets with them in other places they may not be altogether strange to him.

It is necessary for this purpose to assume certain fundamental principles which are fully established by experiment and in works upon statics. These are as follows:

(*a*) A force may be completely represented by a line-segment, the beginning of the segment being the point of application of the force, the direction of the segment giving the direction of the force, and the length of the segment representing the magnitude or greatness of the force. Thus OP represents a force which acts at

Fig. 13

O, along the line OP, and is proportional in magnitude to the length OP.

(*b*) If O were a bead free to move on a small wire OR, and OP were a thread exercising a pull or force

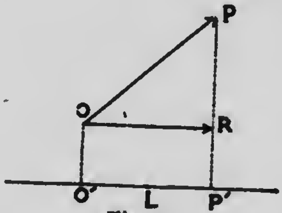


Fig. 14

along OP and represented in force by the length of OP, we know that O would move or tend to move along the wire OR.

The force thus acting along OR and exerting an influence to move the bead along OR, is a part of OP, and is called the resolved part of OP along OR.

Its magnitude is OR, found by drawing PR perpendicular to OR, and it is consequently expressed by $OP \cos POR$. And if L be any line parallel to OR, and OO', RP' be perpendicular to L, $O'P' = OR$ and is the projection of OP on L. Hence, *the resolved part of a force along any line is the force multiplied by the cosine of the angle between the direction of the force and that of the line.*

Cor. If OP is perpendicular to L the cosine of the angle is zero. Hence *a force exerts no effect, or has no resolved part perpendicular to its own direction.*

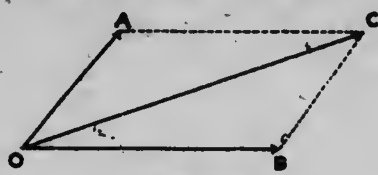


Fig. 15

(*c*) When two forces OA and OB, differing in direction, act on a point O, they are exactly represented in all their effects by the diagonal, OC, of the parallelogram OC, of which the sides represent the given forces. That is, the single force OC acting alone will produce exactly the same effect on O as the two forces OA and OB acting together. OC is then called the *Resultant* of OA and OB, and OA and OB themselves are *resolved components* of OC.

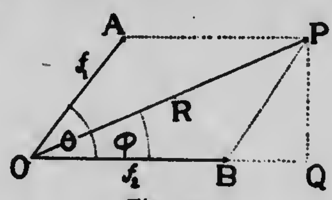


Fig. 16

Ex. Two forces f_1, f_2, (OA and OB), act on O at angle θ with one another, and R, (OP), is their resultant acting at angle φ with f_2.

Then $\quad OP^2 = OA^2 + OB^2 + 2OA \,.\, OB \cos\theta.$

$$\therefore\ R^2 = f_1^{\,2} + f_2^{\,2} + 2f_1 f_2 \cos\theta, \qquad (19)$$

which gives the magnitude of the resultant.

Again, if PQ be $\perp$ to OB,

$$\tan\varphi = \frac{PQ}{OQ} = \frac{PB\sin\theta}{OB + PB\cos\theta},$$

$$\text{or} \qquad \tan\varphi = \frac{f_1\sin\theta}{f_1\cos\theta + f_2}, \qquad (20)$$

which gives the direction of the resultant.

Cor. 1. If $\theta=0$ in (19), the forces are coincident in direction, $\cos\theta=1$ and $R^2=f_1^2+f_2^2+2f_1f_2=(f_1+f_2)^2$,

and $R=f_1+f_2$,

or the resultant is the sum of the forces.

Cor. 2. If $\theta=180°$ the forces act in opposite directions, and $\cos\theta=-1$, and $R^2=f_1^2+f_2^2-2f_1f_2$,

or $R=f_1-f_2$,

and the resultant is the difference of the two forces.

And if $f_1=f_2$, $R=0$. So that two forces equal in magnitude but acting in opposite directions upon the same point, neutralize each other and have their resultant zero.

Ex. Two forces of 6 and 8 pounds make an angle of 30°, to find their resultant and its direction.

$$R^2=6^2+8^2+2\times6\times8\cos30°=100+48\sqrt{3}.$$

and $\tan\varphi=6\sin30/(8+6\cos30)=0.2273.$

and $\varphi=12°\ 49'$.

EXERCISE XII.

1. Two forces of 12 and 20 act at an angle of 72°, to find their resultant and its direction.

2. A rope is thrown about a post and the ends are each pulled with a force of 100, and at an angle of 120° with each other. Find the pressure on the post.

3. Given two forces as 16 and 24 and their resultant as 32, to find their directions.

4. A string tied to a stone on a level floor pulls upwards at an angle of 26° 20′ with the floor. Find the force tending to move the stone along the floor. Also, the force tending to lift the stone.

5. In a canal a boat 20 feet from the tow-path is towed by a rope 55 feet long. What part of the pull on the rope is effective in moving the boat forward?

25. Let OX, OY be rectangular axes, and let f_1, f_2, etc., be forces acting at O and making angles α_1, α_2, etc., with OX.

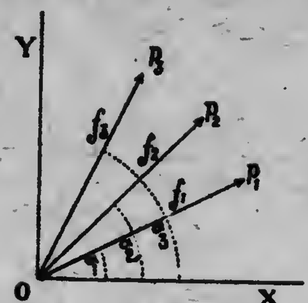


Fig. 17

Then if R be the resultant making angle φ with OX, the part of R resolved along OX must be equal to the sum of the resolved parts of all the forces along OX, etc.

$\therefore$ $R \cos \varphi = \Sigma f \cos \alpha$, resolving along OX;

and $R \sin \varphi = \Sigma f \sin \alpha$, resolving along OY.

$\therefore$ Dividing

$$\tan \varphi = \frac{\Sigma f \sin \alpha}{\Sigma f \cos \alpha} \dots\dots\dots\dots\dots\dots (21)$$

Squaring and adding

$$R^2 = (\Sigma f \sin \alpha)^2 + (\Sigma f \cos \alpha)^2 \dots\dots\dots\dots (22)$$

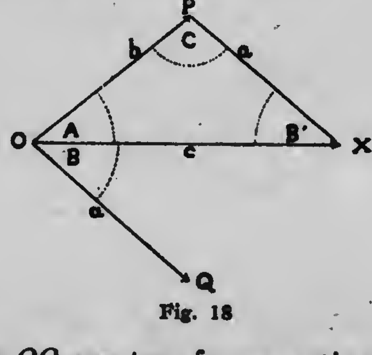


Fig. 18

Ex. OP and OQ are two forces acting at O, and PX is equal and parallel to OQ, so that OPX is a triangle, and PX represents in direction and magnitude the force OQ.

Then, because $\angle B$ is negative, $R \sin \varphi = b \sin A - a \sin B =$ the resolved part $\perp$ to OX; and from the property of the triangle $b \sin A = a \sin B$, (14). $\therefore R \sin \varphi = 0$, and there is no resolved part of the resultant $\perp$ to OX.

Also $R \cos \varphi = b \cos A + a \cos B$, since $\cos(-B)$ is positive.

But in any triangle $b \cos A + a \cos B = c$, (8).

$\therefore c$, or OX is the resultant.

Hence, if two forces are represented in magnitude and direction by two sides (OP and PX) of a triangle taken in the same order, the resultant is represented in magnitude and direction by the third side (OX) taken in reversed order.

Ex. 1. A ball of weight w lies on a plane inclined at angle θ with the horizon. Find (*a*) the force required to keep the ball from rolling down the plane. (*b*) The pressure of the ball upon the plane.

AH is horizontal; AB the plane with the angle $BAH = \theta$; and W is the ball.

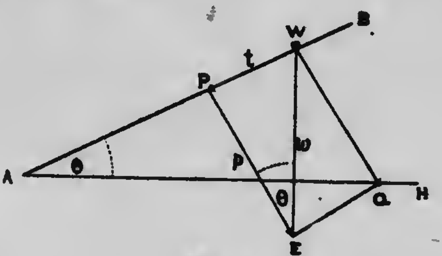


Fig. 19

Draw WE vertically downwards and of proper length to represent the weight w. This can be resolved into two forces, $WP = t$ acting down the plane, and WQ or $PE = p$ acting directly against the plane, and thus representing the pressure of the ball on the plane.

(*a*) $WP = t = w \sin \theta$ = the force acting down the plane, and therefore the force required, acting up the plane, to keep the ball at rest.

(*b*) $WQ = p = w \cos \theta$ is the pressure against the plane.

Cor. 1. If $\theta=0$ the plane is horizontal; $t=0$ and there is no tendency for the ball to move; and $p=w$, or the whole weight of the ball presses against the plane.

Cor. 2. If $\theta=\frac{\pi}{2}$, the plane is vertical; $t=w$ and the whole weight of the ball is the force along the plane; and $p=0$, or there is no pressure on the plane.

Ex. 2. A weight, w, is suspended by two strings, one of which make an angle α with the horizontal, and the other the angle β. To find the tension of the strings.

W is the body of weight w, and WA, WB are the strings, A and B being in the same horizontal line.

W is kept in equilibrium by w acting vertically

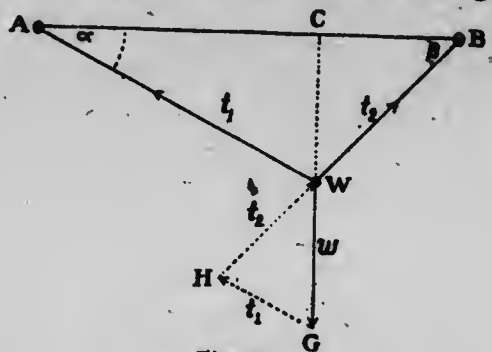


Fig. 20

downwards, and the two tensions t_1 and t_2 acting upwards along the strings.

Produce BW downwards to H, meeting GH drawn parallel to WA.

Then w, t_1 and t_2 form a triangle.

From the sine formula, $\frac{t_1}{w}=\frac{\sin HWG}{\sin WHG}=\frac{\sin CWB}{\sin AWB}$, since AWB is the supplement of WHG. So also $\alpha+\beta$ is the supplement of AWB.

$$\therefore t_1=\frac{\cos\beta}{\sin(\alpha+\beta)}w.$$

and similarly $$t_2=\frac{\cos\alpha}{\sin(\alpha+\beta)}w.$$

Cor. If the strings are equally inclined to the vertical, $\alpha=\beta$, and $t_1=t_2=\dfrac{\cos\alpha}{\sin 2\alpha}\,w=\dfrac{w}{2\sin\alpha}$.

EXERCISE XIII.

1. A ball of weight w rests on a plane inclined at angle θ. Find the force which acting horizontally will keep the ball at rest; and also the pressure on the plane.
   Discuss the case when θ becomes large.

2. Solve Ex. 1. When the force is applied at 45° to the horizontal.

3. A ball of 100 pounds hangs by a cord. A string fastened to the ball draws it 5° out of the vertical. Find the tension on the rope and on the string (*a*) when the pull is horizontal, (*b*) when the pull is ⊥ to the rope.

4. A rope 20 feet long is fastened to two supports in the same horizontal line and 15 feet apart, and a weight of 100 pounds is suspended from the centre. Find the tension of the rope and the horizontal thrust on the supports.

5. A boy's sled is attached to a sleigh by a rope making an angle θ with the road. What is the lifting force on the sled when the horizontal force is 10 pounds?

26. Orthogonal Projection.—AB is a line-segment and L is any line. AA' and BB' are perpendicular to L. Then $A'B'$ is the orthogonal projection of AB on L, and $B'A'$ is that of BA on L.

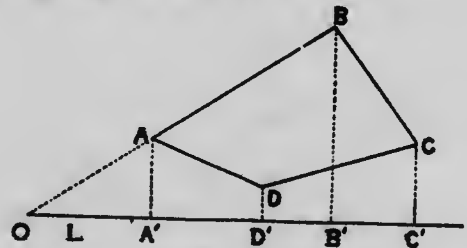


Fig. 21

Producing BA to meet L in O, we see that $A'B' = AB \cos AOA'$. Hence the projection of a line-segment upon any line is the line-segment multiplied by the cosine of the angle between it and the given line.

Cor. The projection of a segment upon a line $\perp$ to itself is zero.

$ABCD$ is a closed polygon and AB, BC, CD, DA are the sides in continuous order. The sum of the projections in the same order is $A'B'+B'C'+C'A'+A'D'$. But, as we start from A' and end at A', the sum is zero. Therefore—

The sum of the projections of the sides of a closed polygon taken in continuous order is zero, the projections being made on any line whatever.

In the practical application of this prolific principle care must be exercised that the terms in the projection are taken with their proper signs.

Ex. $OPQR$ is a concyclic quadrilateral having the angles at P and R right angles, and the angle at $O = \alpha$.

Pr. OP + Pr. PQ + Pr. QR + Pr. $RO = 0$, on any line. Projecting on OR, Pr. $OP = a \cos \alpha$, Pr. $PQ = x \sin \alpha$, Pr. $QR = 0$, and Pr. $RO = -b$. $\therefore a \cos \alpha + x \sin \alpha - b = 0$ finds x in terms of a, b and α.

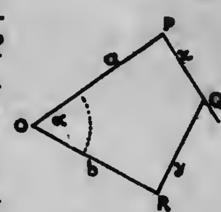


Fig. 22

Addition Theorems.

27. A theorem or formula which gives a function of the sum or of the difference of two angles in terms of functions of the single angles is called an *addition theorem*.

The principal addition theorems are for the sine, the cosine, and the tangent.

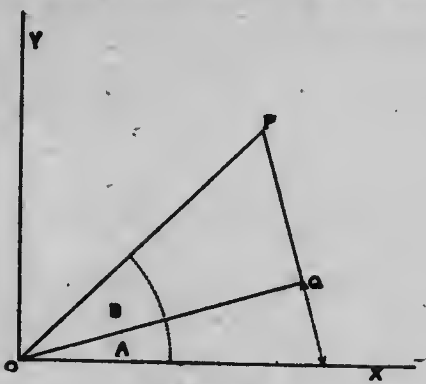


Fig. 23

OX, OY are two lines at right angles to one another, and OPQ is a triangle right-angled at Q.

Then,

$$\text{Pr. } OP + \text{Pr. } PQ + \text{Pr. } QO = 0$$

on any line.

But projecting on OX, Pr. $OP = OP \cos (A+B)$,
Pr. $PQ = PQ \sin A$,
Pr. $QO = -OQ \cos A$.

$\therefore OP \cos (A+B) + PQ . \sin A - OQ \cos A = 0$

and dividing throughout by OP, and transposing we get (*a*), $\cos (A+B) = \cos A \cos B - \sin A \sin B$.

Now, write $-B$ for B in (*a*). And because $\cos (-B) = \cos B$, and $\sin (-B) = -\sin B$, we get

(*b*), $\cos (A-B) = \cos A \cos B + \sin A \sin B$.

Again, writing $\frac{\pi}{2} - A$ for A in (*b*) gives—

$$\cos \left(\frac{\pi}{2} - \overline{A+B}\right) = \cos \left(\frac{\pi}{2} - A\right) \cos B + \sin \left(\frac{\pi}{2} - A\right) \sin B$$

or (*c*), $\sin (A+B) = \sin A \cos B + \cos A \sin B$.

And finally writing $-B$ for B in (*c*) gives

(*d*), $\sin (A-B) = \sin A \cos B - \cos A \sin B$.

These four formulas are in constant demand, and they are here collected for reference—

$$\left.\begin{aligned}\sin(A+B) &= \sin A\cos B+\cos A\sin B \ldots \text{i}\\ \sin(A-B) &= \sin A\cos B-\cos A\sin B \ldots \text{ii}\\ \cos(A+B) &= \cos A\cos B-\sin A\sin B \ldots \text{iii}\\ \cos(A-B) &= \cos A\cos B+\sin A\sin B \ldots \text{iv}\end{aligned}\right\}\ldots(23)$$

All the relations of the set may be obtained directly by projection. Thus by projecting the triangle OPQ on OY we obtain i, and by making $\angle POX=A$ and $\angle POQ=B$ we project on OX and on OY and obtain iv and ii.

These are left as exercises to reader.

28. Addition Theorem for the Tangent.—

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}.$$

Divide both numerator and denominator of the expression on the right by $\cos A \cos B$. This reduces the equation to

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}\ldots\ldots\ldots\ldots\ldots(24)$$

Writing $-B$ for B and remembering that $\tan(-B)=-\tan B$ we get—

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}\ldots\ldots\ldots\ldots\ldots(25)$$

29. Functions of Double Angles.—

Making $B=A$ in i (23) gives

$$\sin 2A=2\sin A\cos A.$$

Making $B=A$ in iii gives

$$\begin{aligned}\cos 2A &= \cos^2 A-\sin^2 A=\cos^2 A-(1-\cos^2 A)\\ &= 2\cos^2 A-1\\ &= 2(1-\sin^2 A)-1=1-2\sin^2 A\end{aligned}$$

$\therefore$ Collecting—

$$\left.\begin{aligned}\sin 2A &= 2\sin A\cos A, \ldots \text{i}\\ \cos 2A &= 2\cos^2 A-1, \ldots \text{ii}\\ &= 1-2\sin^2 A, \ldots \text{iii}\\ &= \cos^2 A-\sin^2 A, \ldots \text{iv}\end{aligned}\right\}\ldots\ldots(26)$$

Again, making $B=A$ in (24) gives—

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}\ldots\ldots\ldots\ldots\ldots(27)$$

Ex. 1. Prove that $\sin n\theta = 2 \sin \theta \cos (n-1)\theta + \sin (n-2)\theta$.

In the expansions of $\sin (\theta+\varphi)$ and of $\sin (\theta-\varphi)$ (i, ii of 23) write $(n-1)\theta$ for φ, and add the results. $\theta+\varphi=n\theta$, $\theta-\varphi=(2-n)\theta$, and $\sin (\theta+\varphi)+\sin (\theta-\varphi) = 2 \sin \theta \cos \varphi$. And substitution and transposition give—

$$\sin n\theta = 2 \sin \theta \cos (n-1)\theta + \sin (n-2)\theta \quad \ldots \ldots (28)$$

Cor. Making $n=3$ we get—

$$\sin 3\theta = 2 \sin \theta \cos 2\theta + \sin \theta = \sin \theta\,(2 \cos 2\theta + 1)$$
$$= \sin \theta\,(3-4 \sin^2\theta) = 3 \sin \theta - 4 \sin^3\theta \ldots\ldots (29)$$

EXERCISE XIV.

1. By putting $\varphi = (n-1)\theta$ in the expansions of $\cos (\varphi+\theta)$ and of $\cos (\varphi-\theta)$ and adding, deduce the formula, $\cos n\theta = 2 \cos \theta \cos (n-1)\theta - \cos (n-2)\theta$.
2. From Ex. 1, prove that $\cos 3\theta = 4 \cos^3\theta - 3 \cos \theta$.
3. Show that $\sin (A+60°) + \sin (A-60°) = \sin A$.
4. Show that $\cos (A+60°) + \cos (A-60°) = \cos A$.
5. Find surd expressions for sine and cosine of 15°, $(=45°-30°)$.
6. Find $\sin (\alpha+\beta)$ and $\cos (\alpha+\beta)$ when $\sin \alpha = \frac{1}{2}$ and $\sin \beta = \frac{1}{3}$.
7. Find sin 18° and cos 18° in surd expressions.

We have $2\times18°$ = the complement of $3\times18°$. Therefore $\sin 2\times18° = \cos 3\times18°$, or $2 \sin 18° \cos 18° = 4 \cos^3 18° - 3 \cos 18°$, Ex. 2. Thence $2 \sin 18° = 4 \cos^2 18° - 3$, which is a quadratic in sin 18°. Its solution gives $\sin 18° = \frac{\sqrt{5}-1}{4}$. Thence the cos 18°, etc.

8. Show that it is possible to find sin 3° in surd expressions. Compare Exs. 5 and 7.
9. To find the height of a hill a flag pole 20 ft. long is placed upon the hill. The elevation of the top of the pole, as seen from a certain point on the level, is 13° 20′ and the elevation of the bottom is 12°.

10. Prove that $\tan(45+\theta)=\dfrac{1+\tan\theta}{1-\tan\theta}$.

11. Find tan 15°, and also tan 75°.

12. Find an expression for $\tan 3\theta$ in terms of $\tan\theta$.

13. Find $\tan(A+B+C)$ in terms of tangents of the single angles A, B, and C.

14. From Ex. 13, show that if A, B, C are angles of a triangle $\tan A+\tan B+\tan C=\tan A.\tan B.\tan C$.

30. Functions of Half Angles.—

From the relations ii, iii (26) we get, by putting $2A=\theta$, and $\therefore A=\frac{1}{2}\theta$, the following relations—

$$\left.\begin{aligned}\cos^2\frac{\theta}{2}&=\frac{1}{2}(1+\cos\theta). &&\text{i}\\ \sin^2\frac{\theta}{2}&=\frac{1}{2}(1-\cos\theta). &&\text{ii}\\ \therefore\ \tan^2\frac{\theta}{2}&=\frac{1-\cos\theta}{1+\cos\theta}. &&\text{iii}\end{aligned}\right\}\ldots\ldots\ldots (30)$$

Now multiply numerator and denominator of the fraction on the right, in iii (30), by $1-\cos\theta$, and

$$\tan^2\frac{\theta}{2}=\frac{(1-\cos\theta)^2}{1-\cos^2\theta}=\left[\frac{1-\cos\theta}{\sin\theta}\right]^2.$$

$$\therefore \tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}\ldots\ldots\ldots\ldots\ldots (31)$$

and using $1+\cos\theta$ as multiplier,

$$\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}\ldots\ldots\ldots\ldots\ldots (32)$$

Ex. To prove that $\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\sec\theta+\tan\theta$,

$$\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}.\tan\frac{\theta}{2}}=\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}=\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}.$$

Multiply numerator and denominator by $\cos\frac{\theta}{2}+\sin\frac{\theta}{2}$, and we have—

$$\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}.$$

But $\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}=1$, $2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=\sin\theta$, $\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}=\cos\theta$.

$$\therefore \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\frac{1+\sin\theta}{\cos\theta}=\sec\theta+\tan\theta.$$

EXERCISE XV.

1. Given $\sin\theta=.1471$, find $\sin\frac{\theta}{2}$ and $\tan\frac{\theta}{2}$.

2. Show that $\tan\theta\tan\frac{\theta}{2}=\sqrt{(1+\tan^2\theta)}-1=\sec\theta-1$.

3. The centre of a circle is O, and AP is a tangent at the point A on the circle. $PQOR$ is a centre line cutting the circle in Q and R, and $\angle AOP=\theta$.
   (i). Find θ when $AP=AR$.
   (ii). Find the area of the triangle PAR.
   (iii). Show that $AR : RP=\cos\theta : \cos\frac{\theta}{2}$.
   (iv). When $\theta=60°$ prove that $RP^2=3AR^2$.

31. Formulas for Changing Sums and Differences of Functions into Products.—

Consider the four relations of set (23), art. 27, adding i and ii gives—

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B.$$

And subtracting ii from i,

$$\sin(A+B)-\sin(A-B)=2\cos A\sin B.$$

In each of these new equations write $\frac{1}{2}(\theta+\varphi)$ for A

and $\frac{1}{2}(\theta-\varphi)$ for B. Then $A+B=\theta$ and $A-B=\varphi$, and we get—

$$\sin\theta+\sin\varphi=2\sin\frac{1}{2}(\theta+\varphi)\cos\frac{1}{2}(\theta-\varphi)$$

and $$\sin\theta-\sin\varphi=2\cos\frac{1}{2}(\theta+\varphi)\sin\frac{1}{2}(\theta-\varphi).$$

Similarly, by adding and subtracting with regard to iii and iv we obtain in like manner—

$$\cos\theta+\cos\varphi=2\cos\frac{1}{2}(\theta+\varphi)\cos\frac{1}{2}(\theta-\varphi).$$

and $$\cos\varphi-\cos\theta=2\sin\frac{1}{2}(\theta+\varphi)\sin\frac{1}{2}(\theta-\varphi),$$

or $$\cos\theta-\cos\varphi=2\sin\frac{1}{2}(\varphi+\theta)\sin\frac{1}{2}(\varphi-\theta),$$

the change in the order of the letters in the latter case being due to the fact that if $\cos\theta-\cos\varphi$ is +, φ must be greater than θ, and therefore $\varphi-\theta$ is positive.

Collecting the four results we have—

$$\left.\begin{aligned}
\sin A+\sin B&=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)\ldots \text{i}\\
\sin A-\sin B&=2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)\ldots \text{ii}\\
\cos A+\cos B&=2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)\ldots \text{iii}\\
\cos A-\cos B&=2\sin\frac{1}{2}(B+A)\sin\frac{1}{2}(B-A)\ldots \text{iv}
\end{aligned}\right\}(35)$$

Ex. 1. $\sin 4\theta+\sin 2\theta=2\sin 3\theta\cos\theta$, and $\sin 4\theta-\sin 2\theta=2\cos 3\theta\sin\theta$.

Ex. 2. To express $\cos(n+1)\theta.\cos(n-1)\theta$ as a sum of functions. This, from its form, must be the sum of two cosines. But $\frac{1}{2}(\cos a+\cos b)=\cos\frac{1}{2}(a+b)\cdot\cos\frac{1}{2}(a-b)$. And comparing, $\frac{1}{2}(a+b)=(n+1)\theta$ and $\frac{1}{2}(a-b)=(n-1)\theta$.

$\therefore a = 2n\theta$, $b = 2\theta$, and

$$\cos(n+1)\theta . \cos(n-1)\theta = \frac{1}{2}(\cos 2n\theta + \cos 2\theta).$$

Ex. 3. To express $\cos\theta \sin^3\theta$ in multiples of θ and without powers or products of functions.

From (29) $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$.

$$\therefore \cos\theta \sin^3\theta = \frac{3}{4}\cos\theta\sin\theta - \frac{1}{4}\sin 3\theta\cos\theta,$$

and $\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$; and $\sin 3\theta\cos\theta = \frac{1}{2}(\sin 4\theta + \sin 2\theta)$.

$$\therefore \cos\theta\sin^3\theta = \frac{1}{4}\sin 2\theta - \frac{1}{8}\sin 4\theta.$$

32. To express $\tan A + \tan B$, and $\tan A - \tan B$ as products.

$$\tan A + \tan B = \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B}.$$

$$\left.\begin{aligned} \therefore \tan A + \tan B &= \frac{\sin(A+B)}{\cos A\cos B} \quad\ldots\ldots\text{i} \\ \text{Similarly } \tan A - \tan B &= \frac{\sin(A-B)}{\cos A\cos B} \quad\ldots\text{ii} \end{aligned}\right\} \ldots\ldots(34)$$

EXERCISE XVI.

1. Express $\cos\theta\sin^2\theta$ in multiples of θ.
2. Show that $\sin\theta + \cos\theta = 2\sin\frac{\pi}{4}\cos(\theta - \frac{\pi}{4})$.
3. Show that $\sin\theta + \cos\varphi = 2\sin\left[\frac{\pi}{4} + \frac{\theta-\varphi}{2}\right]\cos\left[\frac{\pi}{4} - \frac{\theta+\varphi}{2}\right]$.
4. If A, B, C are the angles of a triangle, prove—

i. $\cot B\cot C + \cot C\cot A + \cot A\cot B = 1$.

ii. $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

{$\because A+B+C=\pi$, $C=\pi-(A+B)$ and $\sin C=\sin (A+B)$

$\therefore \sin A+\sin B+\sin C=\sin A+\sin B+\sin (A+B)$

$=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)+2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A+B)$ etc.}

5. Show that $\sin(\theta+\varphi)\sin(\theta-\varphi)=(\sin\theta+\sin\varphi)(\sin\theta-\sin\varphi)=\sin^2\theta-\sin^2\varphi$.

6. Prove that $\cos(\theta+\varphi)\cos(\theta-\varphi)=\cos^2\theta-\sin^2\varphi=\cos^2\varphi-\sin^2\theta$.

7. Express as sums, or differences (i) $\sin 2\theta\cos 3\theta$; (ii) $\cos 4\theta\sin 3\theta$; (iii) $\cos 2\theta\cos 6\theta$, (iv) $\sin\theta\sin 5\theta$.

8. Express as products, (i), $\sin 4\theta+\sin\theta$; (ii) $\sin 6\theta-\sin 2\theta$; (iii), $\cos 4\theta+\cos\theta$; (iv), $\cos 3\theta-\cos 5\theta$.

32. Angles as Auxiliaries.—Owing to the inter-relation of the trigonometric functions, they may often be profitably employed in the solution of problems in algebra and geometry, from the final result of which the function has been eliminated. Angles whose functions are so employed may be called *auxiliary angles by elimination.*

Ex. 1. Let $\sin\theta=a$ and $\tan\theta=b$. Then because there is a fixed relation between $\sin\theta$ and $\tan\theta$, so there must be the same fixed relation between a and b, and as $\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$, so $b=\frac{a}{\sqrt{1-a^2}}$.

Ex. 2. To find the value of $\sqrt{a^2+b^2}$, where a and b are large numbers.

$\sqrt{a^2+b^2}=a\sqrt{1+\frac{b^2}{a^2}}$. Now, since the tangent of an angle may have any value whatever, we may get an angle θ such that $\tan\theta=\frac{b}{a}$. Then $\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\tan^2\theta}$ $=\sec\theta$, and $\sqrt{a^2+b^2}=a\sec\theta$.

In this example we make use of the relation between the $\tan\theta$ and $\sec\theta$ in order to avoid squaring and extracting the square root.

Ex. 3. $ABCD$ is a square with side a. A', B', C', D' are points upon the sides such that $AB' = BC' = CD' = DA' = m$. To find the area of the square $PQRS$.

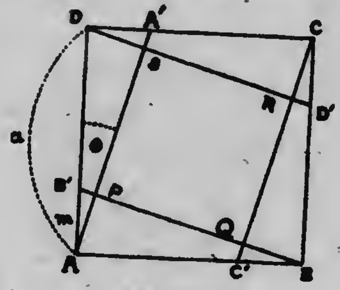


Fig. 24

Denote the angle DAA' by θ.

Then $AS = a \cos\theta$ and $AP = m \cos\theta$. $\therefore PS = (a-m)\cos\theta$; and $PS^2 = (a-m)^2 \cos^2\theta$. But $DA'/AD = \frac{m}{a} = \tan\theta$.

$\therefore \cos\theta = \frac{a}{\sqrt{a^2+m^2}}$, and $PS^2 = \frac{a^2(a-m)^2}{a^2+m^2}$; the required area.

Cor. If $m=0$, $PS^2 = a^2$; if $m=a$, $PS^2 = 0$; if $m = \frac{a}{2}$, $PS^2 = \frac{a^2}{5}$.

EXERCISE XVII.

$ABCD$ is a rectangle and BF and DE are perpendiculars upon the diagonal AC.

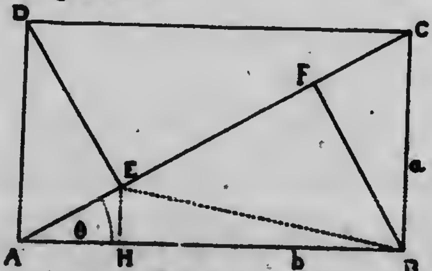


Fig. 25

Let the sides AB and BC be denoted by b and a respectively. Then every line about the figure, and the area of every triangle should be expressible in terms of a and b. Denote the angle CAB by θ. Then $\tan\theta = \frac{a}{b}$, and hence $\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$, and $\cos\theta = \frac{b}{\sqrt{a^2+b^2}}$.

1. To find the ratio $a:b$ when $AC = c.BF$, c being constant.

$$AC = b\sec\theta = \frac{b}{\cos\theta} = \sqrt{a^2+b^2};\ BF = a\cos\theta = \frac{ab}{\sqrt{a^2+b^2}},$$

and we must have $\sqrt{a^2+b^2} = c.\frac{ab}{\sqrt{a^2+b^2}}$.

whence $\left(\frac{a}{b}\right)^2 - c\left(\frac{a}{b}\right) + 1 = 0$, and $\frac{a}{b} = \frac{1}{2}(c \pm \sqrt{c^2-4})$.

2. Find the ratio $a:b$ when $DE = c.EF$.
3. Find the ratio $a:b$ when $EF = c.AC$.
4. To find BE,—

$BE^2 = AB^2 + AE^2 - 2AB.AE\ \cos\ \theta = b^2 + a^2\sin^2\theta - 2ab\ \sin\theta\cos\theta$

$$= b^2 + \frac{a^4}{a^2+b^2} - \frac{2a^2b^2}{a^2+b^2} = \frac{a^4 - a^2b^2 + b^4}{a^2+b^2} = \frac{a^6+b^6}{(a^2+b^2)^2}.$$

$\therefore BE = \sqrt{a^6+b^6}/(a^2+b^2)$.

5. Prove that $BE^2 = AC^2 - 3BF^2$.
6. Find the area of the parallelogram $BEDF$.
7. Find the area of the triangle BEC.
8. Find the area of the triangle BEH.
9. Show that $\cos EBF = ab/\sqrt{a^4 - a^2b^2 + b^4}$.
10. Show that $\tan ABE = \tan^3 BAE$.
11. Prove that the rectangle on AC and EF is equal to the difference between the squares on the sides AB and BC.

LOG-FUNCTIONS AND LOGARITHMIC FORMULAS.

33. Logarithms are used to simplify and extend arithmetical operations, and their proper relations are with arithmetic and algebra. And yet some works on Trigonometry are so constructed that one, who did not

know otherwise, would naturally infer that logarithms were essential in the practice of Trigonometry.

This however is not the case, as any, and every problem in practical Trigonometry may be accurately and readily solved without any reference to logarithms.

Logarithms are introduced in trigonometric operations and calculations as a matter of convenience, and because a large part of the practical work in Trigonometry consists of arithmetical operations, such as multiplying functions together, or dividing one function by another.

The trigonometric functions, being ratios, are numbers, and admit of having their logarithms taken after the manner of other numbers. These logarithms are called logarithmic functions, or log-functions.

Thus sin 50° is 0.76604. This number being a decimal, its logarithm is $\bar{1}.88425$ to five places. And as the sine of an angle can never exceed unity, the log-sines would in general have negative characteristics.

To avoid this the sines are multiplied by ten thousand millions or 10^{10}; or, in other words, 10 is added to the characteristic. And thus *l.* sin 50° is 9.88425. Similarly *l.* cos 50° is log (cos 50) + 10 or 9.80807. In like manner we get log-tangents, log-secants, etc., and these being tabulated in order form the tables of log-functions.

34. Tables differ so much in the details of their construction that it is not possible to give general rules for the "working" of all tables, but with any table of 5-place log-functions we may do as in the following examples:

Ex. 1. To find *l.* sin 37° 20′ 42″.

The table which usually registers to minutes of arc gives

$$l.\ \sin 37^\circ\ 20' = 9.78280.$$

$$l.\ \sin 37^\circ\ 21' = 9.78296.$$

Thus the *l.* sine increases by 16 units in going through 60″ or 1′, and hence it increases $\frac{42}{60} \times 16 = 11.2$ units in going over 42″. So adding 11 to 9.78280 gives 9.78291 as *l.* sin 37° 20′ 42″.

Ex. 2. To find $l.\cos 59° 47' 18''$. The table gives

$$l.\cos 59° 47' = 9.70180$$
$$l.\cos 59° 48' = 9.70158.$$

The log-cosine decreases 22 units for 60″, and hence it decreases $\frac{18}{60}\times 22 = 6.6$ units for 18″. Hence subtracting 7 from 9.70180 gives 9.70173 as $l.\cos 59° 47' 18''$.

In like manner proportional parts are to be taken for seconds in any log-function, remembering that for the sine, tangent, and secant the function and its logarithm increases as the angle increases, while the co-functions and their logarithms decrease with an increase of the angle.

Ex. 3. To find θ, when $l.\sin\theta = 9.81642$.

The table gives $9.81636 = l.\sin 40° 56'$
$9.81651 = l.\sin 40° 57'$
diff. 15 for 1′ or 60″.

But $9.81642 - 9.81636 = 6$. And $\frac{6}{15}$ of $60'' = 24''$

$$\therefore \theta = 40° 56' 24'.$$

The best tables have contrivances for finding these proportional parts with a minimum of labor.

35. The method of taking proportional parts, as illustrated in the preceding article, is quite sufficient for all ordinary cases which occur in practice. But for very small angles not rising above 20′ or so, the variations in $l.\sin$ and $l.\tan$ are so irregular as compared with equal increments of angles, as to render the method of proportional parts useless.

We may then do as follows: The equation $\sin n' = n \sin 1'$ is so nearly true that with five-place tables no error is appreciable when n is less than 20. with these restrictions we have then, $l.\sin n' = l.n + l.\sin 1'$

$$= l.n + 6.46373.$$

Ex. 1. To find $l.\sin 3' 14''.4$.

This has $n = 3'.24$, and $\log. 3.24 = 0.51054$.

Then $l.\sin 3' 14''.4 = 0.51054 + 6.46373 = 6.97427$, and this is correct to the last decimal place.

This principle can be extended as far as 60′ or 1°, if we subtract from the result, 1 when n lies between 20 and 50, and 2 when between 50 and 60.

A similar process applies to the finding of the $l.$ tangents of small angles, $l.\ \tan n' = l.\ n + l.\ \tan 1'$

$$= l.\ n + 6.46373,$$

except that instead of subtracting 1 and 2 between the limits given, we add to the final result.

Ex. 2. To find θ when $l.\ \sin \theta = 7.10485$.

This angle is evidently small, as the characteristic of the $l.$ sine is only 7. Therefore $7.10485 - 6.46373 = 0.64112$, which is the logarithm of 4.377.

Therefore $\theta = 4'.377 = 4'\ 22''.62$.

With the use of natural functions no such difficulties occur with the sine and tangent of small angles, as that is the very case in which proportional parts are most trustworthy.

36. The working rules of logarithms are included in the three statements—

(a) $\log a + \log b = \log (ab)$.

(b) $\log a - \log b = \log \dfrac{a}{b}$.

(c) $\log a^n = n \log a$.

Thus the addition of logarithms corresponds to the multiplication of numbers, and the subtraction of logarithms to the division of numbers; and there is no operation with logarithms corresponding to the addition or subtraction of numbers. So that given $\log a$ and $\log b$, we find $\log (ab)$ by addition, but we have no direct means of finding $\log (a+b)$ or $\log (a-b)$.

Hence formulas involving addition or subtraction of functions are not adapted to logarithms, and it becomes necessary to so transform such formulas as to replace additions and subtractions by multiplications and divisions.

Thus the sine formula is adapted to the use of logarithms, while the cosine formula is not.

Similarly, $\sin\phi + \sin\theta$ is not adapted to the use of logarithms, while its equivalent $2 \sin \frac{1}{2}(\varphi+\theta) \cos \frac{1}{2}(\varphi-\theta)$ is so adapted, as the addition and subtraction is between the angles and not between the functions.

Ex. Employ log-functions to find θ, when $\sin\theta = \sin A + \sin B$, where $A = 12°$, $B = 8°$,

$l. \sin\theta = \log(\sin 12° + \sin 8°) = \log(2 \sin 10° \cos 2°)$
$= \log 2 + l. \sin 10° + l. \cos 2° = 0.30103 + 9.23967 + 9.99973 = 9.54043 = l. \sin 20° 18' 32''$
$\therefore \theta = 20° 18' 32''$.

The student will find, by making the solution, that the process is very much more concise if natural functions are employed.

37. Transformation of Cosine Formula

$$a^2 = b^2 + c^2 - 2bc \cos A$$

so as to Find A When the Sides are Given.

Since $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, i. (16), and $\cos A = 2\cos^2\dfrac{A}{2} - 1$, i. (30).

$$\therefore 2\cos^2\frac{A}{2} = 1 + \frac{b^2+c^2-a^2}{2bc} = \frac{(b+c)^2-a^2}{2bc}$$
$$= \frac{(b+c+a)(b+c-a)}{2bc}$$

Now put $a+b+c=2s$. Then $b+c-a=2(s-a)$, etc., and this substitution gives—

$$\cos\frac{A}{2} = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}} \qquad \ldots\ldots(35)$$

Then $l. \cos\dfrac{A}{2} = \dfrac{1}{2}\{\log s + \log(s-a) - \log b - \log c\}$. (35.$l$.)

By this method we find $l.\cos\frac{A}{2}$, thence $\frac{A}{2}$ and finally A. A conjugate formula derived, in a similar manner, from the relation $\cos A=1-2\sin^2\frac{A}{2}$, is

$$\sin\frac{A}{2}=\sqrt{\left\{\frac{(s-b)\ (s-c)}{bc}\right\}}\ldots\ldots\ldots\ldots\ldots(36)$$

and $l.\sin\frac{A}{2}=\frac{1}{2}\{\log(s-b)+\log(s-c)-\log b-\log c\}$ (36.l.)

Of the two forms (35) and (36), (35) should not be used when the angle is known to be small, and (36) should not be used when the angle is near 180°.

Co-logarithm.—Define the co-logarithm of a number n by $\log\frac{1}{n}$. Then $c.\log n=\log\frac{1}{n}=\log 1-\log n=0-\log n=-\log n$.

Hence to add the co-logarithm of a number is equivalent to subtracting the logarithm of the number. Form (35.l.) may consequently be written—

$$l.\cos\frac{A}{2}=\frac{1}{2}\{\log s+\log(s-a)+c.\log b+c.\log c\}\ldots(37)$$

and we thus have only additions in the formula.

And the co-logarithm may be taken from the table by finding the desired logarithm, and beginning on the left and subtracting every figure from 9, except the extreme right, which is taken from 10.

Ex. Given the sides of a triangle $a=13.45$, $b=17.20$, $c=21.87$, to find A.

Here $s=\frac{1}{2}(13.45+17.20+21.87)=26.26$; $s-a=12.81$

$$\begin{array}{ll} \log s & =1.41929 \\ \log(s-a) & =1.10755 \\ c.\log b & =8.76447 \\ c.\log c & =\underline{8.66015} \\ & 2\,|\,\underline{19.95146} \\ & \quad 9.97573 \end{array}$$

$\therefore\ l.\cos\frac{A}{2}=9.97573.$

$\therefore\ \frac{A}{2}=18°\ 59'.$

$\therefore\ A=37°\ 58'.$

38. Transformation of Cosine Formula so as to Find a When A is Given; or to Find the Third Side When Two Sides and the Included Angle are Given.

We have $a^2=b^2+c^2-2bc\cos A$; and $\cos A=2\cos^2\frac{A}{2}-1$.

Therefore
$$a^2=b^2+c^2-4bc\cos^2\frac{A}{2}+2bc$$
$$=(b+c)^2-4bc\cos^2\frac{A}{2}$$
$$=(b+c)^2\left\{1-\frac{4bc}{(b+c)^2}\cos^2\frac{A}{2}\right\}.$$

Now, $(b+c)^2>4bc$, so that $\frac{4bc}{(b+c)^2}<1$.

Hence putting $\frac{2\sqrt{bc}}{b+c}\cos\frac{A}{2}=\sin\theta$, we have
$$a=(b+c)\cos\theta.$$

And we thus obtain the solution by means of the auxiliary angle θ. Put into logarithmic form this is—

$$\left.\begin{aligned} l.\sin\theta &= \log 2+\frac{1}{2}(\log b+\log c)+l.\cos\frac{A}{2}-\log(b+c) \\ \log a &= \log(b+c)+l.\cos\theta. \end{aligned}\right\} \quad (38)$$

Ex. Given $b=30$, $c=42$, $A=64°$, to find a.

$$\log 2=0.30103$$
$$\frac{1}{2}(\log b+\log c)=1.55018$$
$$l.\cos\frac{A}{2}=9.92842$$
$$1.77963$$
$$-\log(b+c)=-1.85733$$
$$l.\sin\theta=9.92230$$

$$\theta=56°\ 44'$$
$$l.\cos\theta=9.73921$$
$$\log(b+c)=1.85733$$
$$\log a=1.59654$$
$$\therefore a=39.49$$

39. Transformation of Sine Formula so as to Find the Two Remaining Angles When Two Sides and the Included Angle are Given.—

Since $\frac{a}{b}=\frac{\sin A}{\sin B}$, $\therefore \frac{a+b}{a-b}=\frac{\sin A+\sin B}{\sin A-\sin B}$.

$$=\frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$$

But $\tan\frac{1}{2}(A+B)=\cot\frac{1}{2}C$, since $A+B+C=180°$,

$$\therefore \tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\frac{1}{2}C \quad\ldots\ldots\ldots(39)$$

This gives $\frac{1}{2}(A-B)$; and $\frac{1}{2}(A+B)=90°-\frac{1}{2}C$; and $\frac{1}{2}(A-B)+\frac{1}{2}(A+B)=A$; and $\frac{1}{2}(A+B)-\frac{1}{2}(A-B)=B$, which makes known A and B.

As logarithmic formulas we have—

$$\left.\begin{aligned} l.\tan\tfrac{1}{2}(A-B)&=\log(a-b)+l.\cot\tfrac{1}{2}C-\log(a+b)\\ \tfrac{1}{2}(A+B)&=90°-\tfrac{1}{2}C \end{aligned}\right\}(39.l.)$$

Ex. Given $a=64$, $b=48$, $C=77°\ 20'$ to find A and B.

$\log(a-b)$	$=1.20412$	$90°\ 0'$
$l.\cot\frac{C}{2}$.	$=0.09680$	$\frac{1}{2}C=38°\ 40'$,
$-\log(a+b)$	$=-2.04922$	$\frac{1}{2}(A+B)=51°\ 20'$
$l.\tan\frac{1}{2}(A-B)$	$=9.25170$	$\frac{1}{2}(A-B)=10°\ 7'$.

$\therefore \frac{1}{2}(A-B)=10°\ 7'$. $\therefore A=61°\ 27'$, $B=41°\ 13'$.

EXERCISE XVIII.

1. From the log-function tables find the following—
   (*a*) *l.* sin 12° 32′ 13″. 5. (*b*) *l.* cos 47° 18′ 10″.
   (*c*) *l.* tan 3′ 17″. 3. (*d*) *l.* sin 27″. 34.
   (*e*) The angle whose *l.* sin is 9.47121.
   (*f*) The angle whose *l.* tan is 2.34712.
   (*g*) The angle whose *l.* sin is 7.14125.

2. If $x=\sqrt{a^2+b^2}$ show that if $l.\tan\theta=\log b-\log a$, then $\log x=\log a+l.\sec\theta$.

3. Find C when $\tan C=\tan A+\tan B$, where $A=32°\ 42'$, $B=48°\ 17'$, and you have only a table of log-functions.

4. Given $l.\cos\theta$, show how to find $l.\sec\theta$.

5. If $a=\dfrac{1-\cos\theta}{1+\cos\theta}$, show that $\log a=2\ l.\tan\dfrac{\theta}{2}$.

6. If $b=a\sin\theta-b\cos\theta$, show that $\log a=\log b+l.\cot\dfrac{\theta}{2}$.

7. In the triangle ABC, $a=27.3$, $b=34.1$, $c=45.6$, to find A.

8. In the triangle ABC, given $a=121.3$, $b=98.4$ and $C=124°$, to find c.

9. In a triangle $a=321$, $b=245$, $C=78°\ 10'$, to find A and B.

10. Calculate the value of $\sqrt{\cos^2 42°-\sin^2 42°}$ by log-functions.

Inverse or Circular Functions.

40. When we have $\sin\theta=x$ we may invert the functional symbol "sin" and write $\theta=\sin^{-1}x$.

The exponent, -1, does not here denote a reciprocal of a power as in Algebra, as "sin" is not a symbol of quantity but of operation.

The expression $\sin^{-1}x$ is read "the angle whose sine is x," or "the arc-sine of x," or "the inversed sine of x." The first designation is longest, but most truthful and expressive.

The functions which are called inverse functions or circular functions are equally numerous with the trigonometric ones. Thus we have $\cos^{-1}x$, $\tan^{-1}x$, $\sec^{-1}x$, etc. The figure will serve to illustrate these.

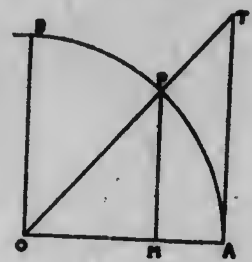


Fig. 26

Thus as $\frac{MP}{OP}=\sin\frac{AP}{OP}$,

so $\frac{AP}{OP}=\sin^{-1}\frac{PM}{OP}$.

Or, if we take OP as unit length in order to avoid fractional forms—

$AP=\sin^{-1}MP=\cos^{-1}OM=\tan^{-1}AT=\sec^{-1}OT$, etc.
Similarly, $PB=\cos^{-1}MP=\cot^{-1}AT=\operatorname{cosec}^{-1}OT$.

If $\sin\theta=p=\cos\left(\frac{\pi}{2}-\theta\right)$, $\theta=\sin^{-1}p$ and $\frac{\pi}{2}-\theta=\cos^{-1}p$.

$$\therefore\quad \sin^{-1}p+\cos^{-1}p=\frac{\pi}{2}$$

Similarly

$$\left.\begin{aligned}\tan^{-1}p+\cot^{-1}p&=\frac{\pi}{2}\\ \sec^{-1}p+\operatorname{cosec}^{-1}p&=\frac{\pi}{2}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ (40)$$

Circular functions are of very great importance in the operations of the differential, and especially, the integral calculus. The important theorems, however, are not numerous.

41. To sum $\tan^{-1}x+\tan^{-1}y$, that is to express this sum in the form of a single inverse tangent.

Let $\varphi = \tan^{-1}x$ and $\theta = \tan^{-1}y$.

Then $x = \tan\varphi$ and $y = \tan\theta$

and $\tan(\varphi+\theta) = \dfrac{\tan\varphi + \tan\theta}{1 - \tan\varphi\tan\theta} = \dfrac{x+y}{1-xy}$

$\therefore \varphi+\theta = \tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$.

and we may write

$$\left.\begin{aligned} \tan^{-1}x + \tan^{-1}y &= \tan^{-1}\frac{x+y}{1-xy} \\ \tan^{-1}x - \tan^{-1}y &= \tan^{-1}\frac{x-y}{1+xy} \end{aligned}\right\} \ldots\ldots\ldots (41)$$

Ex. 1. $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \tan^{-1}\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = \tan^{-1}1 = \dfrac{\pi}{4}$

Ex. 2. $2\tan^{-1}\dfrac{1}{5} = \tan^{-1}\dfrac{\frac{2}{5}}{1-\frac{1}{25}} = \tan^{-1}\dfrac{10}{24} = \tan^{-1}\dfrac{5}{12}$.

The meaning of example 1 is that if two angles be constructed such that the tangent of the one is $\frac{1}{2}$ and the tangent of the other is $\frac{1}{3}$, the two angles together will make up 45° or $\frac{1}{4}\pi$.

42. To sum $\sin^{-1}x + \sin^{-1}y$.

Let $\varphi = \sin^{-1}x$ and $\theta = \sin^{-1}y$

Then $x = \sin\varphi$, $y = \sin\theta$, $\sqrt{1-x^2} = \cos\varphi$, $\sqrt{1-y^2} = \cos\theta$.

But $\sin(\varphi+\theta) = \sin\varphi\cos\theta + \cos\varphi\sin\theta$
$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\therefore \varphi+\theta = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

$$\left.\begin{aligned} &\text{or } \sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\} \\ &\text{Similarly} \\ &\cos^{-1}x + \cos^{-1}y = \cos^{-1}\{xy - \sqrt{(1-x^2)(1-y^2)}\} \end{aligned}\right\} \ldots (42)$$

EXERCISE XIX.

1. Find $2 \tan^{-1} \frac{1}{3}$ as an inverse tangent.

2. Show that $2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \frac{\pi}{4}$.

3. Show that $2 \tan^{-1} \frac{1}{5} + 2 \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{7} = \frac{\pi}{4}$.

4. Prove that $4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} = \frac{\pi}{4}$.

5. Prove that $2 \sin^{-1} x = \sin^{-1} 2x\sqrt{1-x^2}$.

6. Show that $2 \cos^{-1} x = \cos^{-1}(2x^2-1)$.

7. If $x \sin\theta - \frac{1}{2}x^2 \sin 2\theta + \frac{1}{3}x^3 \sin 3\theta - + \ldots\ldots\ldots\ldots =$
$\tan^{-1} \dfrac{x \sin\theta}{1 + x\cos\theta}$,
show by changing the sign of x and subtracting, that
$x \sin\theta + \frac{1}{3}x^3 \sin 3\theta + \frac{1}{5}x^5 \sin 5\theta + \ldots = \frac{1}{2}\tan^{-1}\dfrac{2x\sin\theta}{1-x^2}$.

TRIGONOMETRIC CONSTRUCTIONS.

43. By trigonometric construction we mean the finding, by graphical methods, of the values of such trigonometric expressions as can be found, and which have sufficient elements given to make the construction determinate.

As a trigonometric function is a number it is necessary for construction that some line-segment enter in as a radius. But the length of this radius is quite immaterial.

On account of the endless variety of such trigonometric expressions as are capable of being constructed, no very general principles of operation can be laid down; and even the construction of a given case may admit of a number of variations of which some are more elegant than others. The following will serve to illustrate the subject:

Ex. 1. To construct an angle whose sine is given.

Take any line-segment OP as radius and on it describe

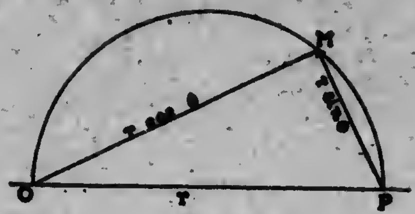


Fig. 27

the semicircle OMP. In this semicircle place the chord $PM = OP \times$ given sine, and join MO.

Then MOP is the required angle. This gives the smallest angle, but the supplement of MOP has the same sine.

A similar construction finds an angle whose cosine is given. For if the chord OM be made equal to r times the given cosine, MOP is the required angle.

Ex. 2. To construct an angle whose tangent is given.

Take an arbitrary segment, OA, as radius and draw

Fig. 28

$AT \perp$ to OA. Make $AT = r \times$ the given tangent, and join OT. TOA is the angle sought.

The same construction finds an angle whose secant is given. Except that AT is an indefinite line, and with O as centre and $OT = r \times$ the given secant, a circle is described cutting AT in T, giving TOA as the angle sought.

Ex. 3. Given $a \sin \theta + b \cos \theta = c$, to find the angle θ, where a, b and c are given line-segments.

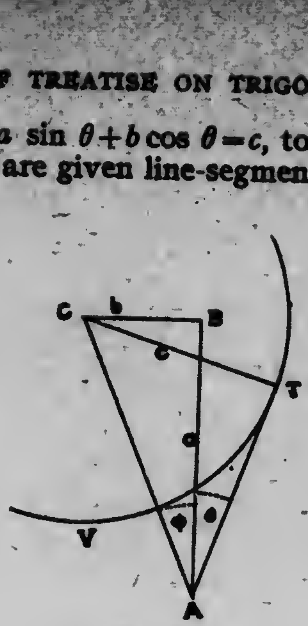


Fig. 29.

Let $a = r \cos \varphi$, $b = r \sin \varphi$. Then $r^2 = a^2 + b^2$, $\tan \varphi = \frac{b}{a}$, and $a \sin \theta + b \cos \theta = r \sin(\varphi + \theta) = c$. $\therefore$ In the right-angled triangle CBA, make $CB = b$. $BA = a$. Then $\angle CAB = \varphi$, and $CA = r$. With C as centre and c as radius, draw a circle TV, and AT a tangent to this circle. Then $c = r \sin(CAT)$. $\therefore CAT = \varphi + \theta$, and $BAT = \theta$.

Ex. 4. Construct the line-segment given by $a . \frac{\sin A + \cos A}{\tan A}$, where a and A are given.

$$a . \frac{\sin A + \cos A}{\tan A} = (a \sin A + a \cos A) \cot A.$$

Take $OB = a$ and draw OD making the $\angle DOB = A$. Draw $BC \perp$ to OC, and make $CD = BC$, and draw DP

Fig. 30

⊥ to OD to meet OP drawn ⊥ to OB. Then DP is the segment sought.

For $OC = a \cos A$, and $CD = CB = a \sin A$. And as POD is the complement of A, $OPD = A$, and $PD = OD \cot A = a(\cos A + \sin A) \cot A$.

EXERCISE XX.

1. Construct an angle whose cotangent is given.
2. Construct an angle whose cosecant is given.
3. Construct the rectangle $ab \sin\theta \cos\frac{\theta}{2}$, where a, b, θ are given.
4. Construct $x = (\sin A - \sin B)\sqrt{a^2 - b^2}$, where A, B are given angles and a, b given segments.
5. Find $\sin^{-1}\frac{a}{b}$ where a and b are given segments.
6. Construct $a \sin(A + \tan^{-1}\frac{b}{a})$ where A is a given angle and a and b given line-segments.
7. Construct $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$.

MISCELLANEOUS ARTICLES AND EXERCISES.

1. Prove the following relations:

(a) $\cos^4 A - \sin^4 A = 2\cos^2 A - 1$.

(b) $\sqrt{1-\sin\theta} = (\sec\theta - \tan\theta)\sqrt{1+\sin\theta}$.

(c) $2\sec^2\theta = \dfrac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - 1} + \dfrac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta + 1}$.

(d) $\dfrac{\cos A}{1-\tan A} + \dfrac{\sin A}{1-\cot A} = \sin A + \cos A$.

(e) $\sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A$.

(f) $\sin A + \sin B - \sin C = 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2}$ in any triangle.

2. Find any function of θ from the following:

(a) $2\sin\theta = 2 - \cos\theta$

(b) $8\sin\theta = 4 + \cos$

(c) $\tan\theta + \sec\theta = 3/$.

(d) $\sin\theta + 2\cos\theta = 1$.

(e) $\tan 2\theta + \cot\theta = 8\cos^2\theta$.

3. In any circle prove that the chord of 108° is equal to the sum of the chords of 36° and 60°.

4. A person on a light-house notices that the angle of depression of a boat coming towards him is α, and that after m minutes it is β. How long after the first observation will the boat reach the light-house?

5. (a) From the cosine formula show that

$$c = (a+b)\sin\frac{C}{2}\sec\varphi, \text{ if } \tan\varphi = \frac{a-b}{a+b}\cot\frac{C}{2}.$$

(b) Express the results of (a) in logarithmic form, and apply it to the case where $a = 25.33$, $b = 18.46$, $C = 78°\ 44'$.

6. Prove that $a\cos\theta + b\sin\theta$

$$= \sqrt{a^2+b^2}\cos\left[\theta - \tan^{-1}\frac{b}{a}\right].$$

7. Show from Ex. 6, that $a\cos\theta + b\sin\theta$ is a maximum when $\theta = \tan^{-1} b/a$.

8. (a) Divide analytically the angle A into two parts such that the sum of the cosines of the parts is a given quantity m.

(b) Obtain a geometric construction for this division.

9. Prove that $\tan^{-1}\frac{m}{n}-\tan^{-1}\frac{m-n}{m+n}=\frac{\pi}{4}$.

10. A side of a triangle is 4 and the opposite angle 36°, and the altitude to another side is $\sqrt{5}-1$. Find the other parts of the triangle.

11. The length of the median to side a is m, and the parts into which it divides its angle A are α and β. Find the other parts of the triangle.

12. Solve the triangle in which $a+b$, c and A are given. (Find $a-b$ by cosine formula.)

13. The altitude of a rock is 47°. After walking 1000 feet towards it up a slope of 32°, the altitude is 77°. Find the vertical height of the rock above the first point of observation.

14. A hill rising 1 in 5 faces south. Show that a road which takes a north-east course rises 1 in 7.

15. Find the direction in Ex. 14 that the road may rise 1 in 10.

16. A gable facing north has a vertical angle of 2γ. Show that, when the sun is south at elevation α, the angle of the shadow of the gable is $2\tan^{-1}(\tan\alpha\tan\gamma)$.

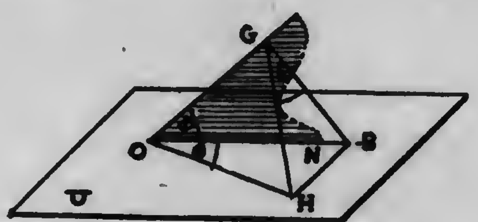


Fig. 31

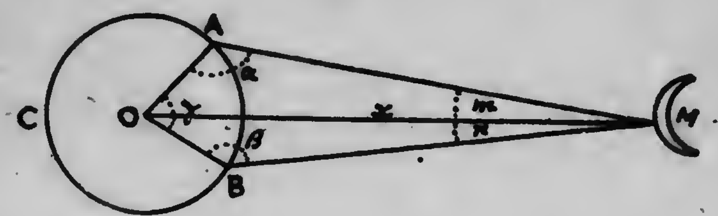
U is the plane of a dial and GON is the gnomon with the edge OG pointing to the north pole of the heavens, and $\angle\ GOB$ is the latitude φ. By the sun's apparent daily motion the angle BGH, or hour angle, h, between the gnomon and its shadow, is measured out uniformly. OH, making the angle $BOH=\theta$, is the edge of the shadow upon the dial. Then, $BH=GB \tan h=OB \tan \theta$. And $GB/OB=\sin \varphi$, whence $\tan \theta=\sin \varphi \tan h$.

Hence setting off $h=$ I hr. II hrs. III hrs., etc., we can lay off the corresponding values of θ, and thus form the hour lines on the dial.

17. Given φ, construct θ for various values of h.

18. Lay off the hour lines of a dial for lat. 44° N.

19. ABC represents the earth and M the moon. The moon is observed from A and from B. $\angle AOB=\gamma$, $OAM=\alpha$, $OBM=\beta$, to find OM.



The whole angle $m+n$ does not exceed 30′ or 40′, and $m-n$ is not more than 10′ or so. Hence $\cos \frac{1}{2}(m-n)$ is practically unity. On this supposition prove that

$$x=OM=r \sin \frac{1}{2}(\alpha+\beta) \cos \frac{1}{2}(\alpha-\beta) / \sin \frac{1}{2}(\alpha+\beta+\gamma).$$

20. If in Ex. 19, $\alpha=145°$, $\beta=164° 12'$ and $\gamma=50°$. Show that $x=60.58\, r$.

This means that the moon's distance from the centre of the earth is a little more than 60 radii of the earth.

21. S represents the sun, E the earth, and V the planet Venus. bb' is the path of Venus as seen from B and aa' as seen from A. The $\angle DBC=\alpha$ is measured by observation. Denote BDA by $2p$, and DVC by θ,

these angles being all less than 1′. Show that $p=\frac{1}{2}\alpha\left(\frac{AD}{VD}-1\right)$.

22. In Ex. 21, if $\left(\frac{AD}{VD}\right)^2=\left(\frac{365}{224}\right)^2$, find p when $\alpha=46''.8$.

(p is the sun's horizontal parallax, *i.e.*, the angle subtended at the sun by the earth's radius).

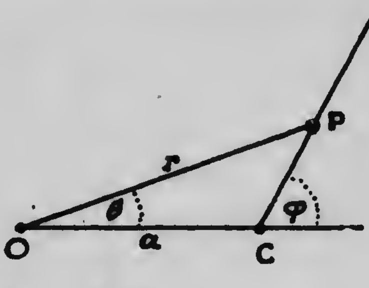
23. $OC=a$ and $OP=r$ are fixed lengths, and θ, φ are variable angles, θ being generated uniformly. Show

(i) $\cot\varphi=\cot\theta-\frac{a}{r}\operatorname{cosec}\theta$.

(ii) If $a>r$, φ can never be a right angle.

(iii) If $a=r$, $\varphi=\frac{\pi}{2}+\frac{\theta}{2}$.

These enter into the theory of eccentric motion.



24. Prove the following:

(*a*) $r_1 r_2 r_3=s\,\Delta=rs^2$.

(*b*) $r\,r_1 r_2 r_3=\Delta^2$.

(*c*) $a=2R\sin A$, where R is the circumradius.

(*d*) $r_1 = 4R \sin\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$, with symmetrical expressions for r_2 and r_3.

(*e*) $r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$.

25. Two wheels with radii r, r', have their centres d feet apart, and lie in the same plane. Find the length of the belt which goes around the wheels and (*a*) crosses between them, (*b*) does not cross between them.

26. In Ex. 25 (*a*), if $r+r'$ is constant, show that the length of belt is constant.

27. In Ex. 25, the wheels are 15 and 20 inches in diameter, and the axes are 120 inches apart. Find the length of (*a*) the open belt, (*b*) the crossed belt.

28. When a material body rests on an inclined plane, show that the ratio of the force tending down the plane to the pressure normal to the plane is the tangent of the angle of inclination of the plane.

When a body rests on a plane, and the plane is inclined until the body is just at the point of sliding down it, the tangent of the angle of inclination is called the *coefficient of friction*; and under reasonable conditions the coefficient of friction is constant for the same materials in the body and the plane.

Then the amount of friction, which acts as a force opposing motion, is the weight of the body multiplied by the coefficient of friction.

29. If the coefficient of friction of iron on iron be 0.16, find the inclination of an iron plane upon which an iron block is at the point of sliding.

30. If an iron plane have an inclination of 3°, find the force, acting along the plane, necessary to slide a block of iron of 100 gms. (*a*) up the plane, (*b*) down the plane.

31. The friction of a metal on oak is about 0.5. What force acting at 30° upwards will move 100 kgms. of iron along a level oak floor?

32. Three poles, each 20 feet in length, are joined at the top, and their feet rest at the vertices of an equilateral triangle with side 12 on level ground. (*a*) Find the angle of inclination of each pole. (*b*) Find the vertical height of the tops.

33. If, in Ex. 32, 100 lbs. be suspended from the tops of the poles, find (*a*) the end pressure on a pole, (*b*) the horizontal thrust at the bottom of a pole; the weight of the pole being not considered.

34. ABC is a triangle of which AB and BC are rigid rods. C is fixed, and A is compelled to move in the line AC. If a force, p, be applied to A along AC, show that the force (*a*) acting perpendicularly to AC is $p \sin C / \sqrt{n^2 - \sin^2 C}$; (*b*) acting along BC is $p\,\{\cos C - \sin^2 C / \sqrt{n^2 - \sin^2 C}\}$; (*c*) acting perpendicularly to BC is $p\{\sin C + \sin C \cos C / \sqrt{n^2 - \sin^2 C}\}$, where n is the ratio $AB : BC$.

(This exercise embodies the principles of the cross-head and crank in the steam engine.)

35. From the corner of a cuboid a piece is cut off by a plane saw cut, which reaches to the distances a, b, c respectively on the three edges. Prove that the area of the section is $\frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$.

36. At the vertices of an equilateral triangle line segments, a, b, c respectively, are drawn normal to the plane of the triangle. Show that the area of the triangle formed by connecting the outer points is $\frac{1}{4}\sqrt{\{3s^4 + 4s^2(\Sigma a^2 - \Sigma ab)\}}$, where s is the side of the equilateral triangle.

ANSWERS TO EXERCISES

EXERCISE I.

1. 0.3168. 2. 7. 3. 5. 4. 313.632.

EXERCISE II.

1. 3.141593 m.
2. 7.95775 ft.
3. 1.00000; 1.0000000.
4. 326.419.
5. 0.062 in.
6. 2.582.
7. 0.4342945.
8. 57.2957805.
9. 101.369.
10. 49.535.
11. 606.094; 1 hr. 38 min. 59.7 sec.
12. 325.292.

EXERCISE III.

1. 0.632477.
2. 0.0002635.
3. 26° 7′ 36″.8.
4. 0.30717 in.
5. 101 ft. 4.4 in.
6. 0.009865; 0.99962.
7. 54.88 ft.
8. 0°.8247 or 49′ 29″ nearly.
9. 8 ft. 11.43 in.
10. 412.5 ft.
11. 1082.3 ft.
12. 343.88 m.
13. 3961.7 m.
14. 18.52 m.

EXERCISE IV.

1. (ii) $a \sec\theta$. (iii) $a \sin\theta$. (iv) $a \sin\theta\cos\theta$.
   (vi) $a \cos\theta$. (vii) $a \cos^2\theta$. (viii) $a \sin^3\theta$.
   (ix) $a \sin^2\theta$. (x) $a \sin^2\theta\cos\theta$. (xi) $a \sin^2\theta\tan\theta$.
2. (i) 10 ft. (ii) 17.32 ft. (iii) 8.66 ft.

EXERCISE V.

1. 0.4472; 0.8944; 1.1180.
2. 1.1339; 0.6614; 1.5119.
3. 0; ∞; ∞.
4. 1; $\sqrt{2}$; ∞.
5. $1 / \sin\varphi$.
6. 0.8660; 1.7321.
8. 0 or 1.

EXERCISE VI.

1. 15° 54′ 56″; 39.48 in. 2. 8° 47′. 3. 58° 32′.
4. 45°. 5. 18° 26′; 71° 34′.
6. 96.84 ft.; 70.87 ft. 7. 17.76 ft.
8. 27° 49′. 9. 4.104 in.
10. $a=34.41$, $b=26.63$, $c=36.22$, $C=71°\ 29'$.
11. 59° 52′, 51° 32′, 68° 36′.
12. $C=84°\ 58'$, $a=17.157$, $b=11.896$.

EXERCISE VII

1. 41° 40′ and 138° 20′. 2. 61° 46′ and −61° 46′.

EXERCISE VIII.

2. 4.4933 3. 70° 32′ or −70° 32′. 4. 1.2587.
5. (*a*) $\angle AOP=108°\ 36'$. (*b*) $\angle AOP=73°\ 32'$.

EXERCISE IX.

1. 48° 48′ or 131° 12′. 2. Triangle imaginary.
3. $C=112°\ 5'$, $a=14.70$, $b=5.37$.
4. $A=30°\ 2'$, $C=82°\ 43'$, $c=48.76$.
5. $A=34°\ 4'$, $C=78°\ 36'$, $b=79.07$.
6. 21.48 ft. 7. 16.5 ft. 8. 9.16 ft.
9. 6.2. 10. 8° 14′.

EXERCISE X.

1. $A=53°\ 8'$, $B=59°\ 29'$, $C=67°\ 23'$.
2. 38.68. 3. 2.78.
4. $A=48°\ 49'$, $B=93°\ 27'$, $c=34.39$.
5. 747 rods. 6. 1.64 m.; 17° 31′ east of N.
7. 2.62 m.; 3.39 m.

EXERCISE XI.

1. $\frac{4}{5}, \frac{56}{65}, \frac{12}{13}$. 2. 156.4 rods. 3. 1344.
4. 50° 56′ or 129° 4′. 6. 90°.

7. 13.13, 17° 27′, 23° 35′, 138° 58′.

8. 108.25. 9. 52.5. 10. $\frac{1}{2}\sqrt{2c^2+2a^2-b^2}$.

11. $\sin\varphi = 4\Delta / b\sqrt{2c^2+2a^2-b^2}$; etc.

17. 1168.5 yds. 18. 18.3 ft. 19. 23 m. 1379.8 ft.

EXERCISE XII.

1. 26.31; 25° 42′ with the greater force.
2. 100. 3. $\theta = 75°\ 31'$; $\varphi = 28°\ 57'$.
4. 0.8962 f; 0.4436 f, where f is force along the string.
5. 0.9316.

EXERCISE XIII.

1. $w \tan\theta$; w sec. θ. 2. $\frac{w\sin\theta}{\sin(45°-\theta)}$; $\frac{w\sin 45°}{\sin(45°-\theta)}$
3. (a) 100.38 lbs.; 8.75 lbs. (b) 99.62 lbs.; 8.72 lbs.
4. 75.59 lbs.; 56.69 lbs. 5. $10\tan\theta$.

EXERCISE XIV.

5. $\frac{\sqrt{3}-1}{2\sqrt{2}}$, $\frac{\sqrt{3}+1}{2\sqrt{2}}$. 6. $\frac{1}{6}(2\sqrt{2}+\sqrt{3})$, $\frac{1}{6}(2\sqrt{6}-1)$.

9. 174.1 ft. 11. $\frac{\sqrt{3}-1}{\sqrt{3}+1}$, $\frac{\sqrt{3}+1}{\sqrt{3}-1}$.

12. $\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$.

13. $\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$.

EXERCISE XV.

1. 0.0738; 0.0740.

3. (i) 60°. (ii) $\frac{1}{2}r^2\tan\theta\,(1+\cos\theta)$.

EXERCISE XVI.

1. $\frac{1}{4}(\cos\theta - \cos 3\theta)$.

7. (i) $\frac{1}{2}(\sin 5\theta - \sin\theta)$; (ii) $\frac{1}{2}(\sin 7\theta - \sin\theta)$;

(iii) $\frac{1}{2}(\cos 8\theta + \cos 4\theta)$; (iv) $\frac{1}{2}(\cos 4\theta - \cos 6\theta)$.

8. (i) $2\sin\frac{5}{2}\theta.\cos\frac{3}{2}\theta$; (ii) $2\cos 4\theta.\sin 2\theta$;

(iii) $2\cos\frac{5}{2}\theta.\cos\frac{3}{2}\theta$; (iv) $2\sin 4\theta.\sin\theta$.

EXERCISE XVII.

2. $(-1\pm\sqrt{1+4c^2})/2c$. 3. $\pm\sqrt{\frac{1-c}{1+c}}$.

6. $ab.\frac{b^2-a^2}{b^2+a^2}$. 7. $\frac{1}{2}.\frac{ab^2}{a^2+b^2}$. 8. $\frac{1}{2}.\frac{a^2b^2}{(a^2+b^2)^2}$.

EXERCISE XVIII.

1. (*a*) 9.33660. (*b*) 9.83131. (*c*) 6.98070.
(*d*) 6.12237. (*e*) 17° 12′ 51″. (*f*) 0″.004587.
(*g*) 4′ 45″. 54.

3. 60° 27′. 7. 36° 36′. 8. 194.28.

9. $A = 60°\ 18'$, $B = 41°\ 32'$. 10. 0.3233.

EXERCISE XIX.

1. $\tan^{-1}\frac{3}{4}$.

MISCELLANEOUS EXERCISES.

2. (*a*) $\sin\theta = \frac{3}{5}$ or 1. (*b*) $\sin\theta = \frac{5}{13}$ or $\frac{3}{5}$.

(*c*) $\tan\theta = \frac{5}{12}$. (*d*) $\sin\theta = 1$ or $-\frac{3}{5}$.

(*e*) $\theta = 90°$ or $7°\ 30'$.

4. $m. \dfrac{\cos\alpha \sin\beta}{\sin(\beta-\alpha)}$. **5.** (*b*) 28.28.

8. (*a*) $\dfrac{A}{2} - \cos^{-1}\left(\dfrac{m}{2}\sec\dfrac{A}{2}\right)$ and $\dfrac{A}{2} + \cos^{-1}\left(\dfrac{m}{2}\sec\dfrac{A}{2}\right)$.

10. 18°, 126°, 2.1028, 5.5066.

11. Sides b, $c = 2\ m \sin\alpha\ /\ \sin(\alpha+\beta)$, $2\ m \sin\beta\ /\ \sin(\alpha+\beta)$.

12. $a = \dfrac{1}{2}(a+b) + \dfrac{1}{2}\cdot\dfrac{c^2 - c\ (a+b)\ \cos A}{(a+b) - c\ \cos A}$; $b = \dfrac{1}{2}(a+b) - \dfrac{1}{2}\cdot\dfrac{c^2 - c\ (a+b)\cos A}{(a+b) - c\cos A}$; etc

13. 1034.3 ft.

15. N. E. 15° 30′ S., or N. W. 15° 30′ S.

22. 9″.

25. (*a*) $\dfrac{l}{2} = (r+r')\pi - (r+r')\cos^{-1}\dfrac{r+r'}{d} + \sqrt{d^2-(r+r')^2}$.

(*b*) $\dfrac{l}{2} = r\pi - (r-r')\cos^{-1}\dfrac{r-r'}{d} + \sqrt{d^2-(r-r')^2}$.

27. (*a*) 295.03 in.; (*b*) 297.54.

29. 9°5′.

30. (*a*) 21.208 gms.; (*b*) 10.748 gms.

31. 44.8 kgms. **32.** (*a*) 69° 44′; (*b*) 18.76 ft.

33. (*a*) 35.5 lbs.; (*b*) 12.31 lbs.