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PRINCIPLES

OF THE

SOLUTION OF EQUATIONS OF THE HIGHER DEGREES, WITH APPLICATIONS.

BY GEORGE PAXTON YOUNG,

Toronto, Canada.

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$$\Delta_1^{\frac{1}{m}}\delta_{m-1}^{\frac{1}{m}},\quad \delta_2^{\frac{1}{m}}\delta_{m-2}^{\frac{1}{m}}, \ldots\ldots,\quad \delta_{m-1}^{\frac{1}{m}}\Delta_1^{\frac{1}{m}},$$

$$\Delta_1^{\frac{2}{m}}\delta_{m-2}^{\frac{1}{m}},\quad \delta_2^{\frac{2}{m}}\delta_{m-4}^{\frac{1}{m}}, \ldots\ldots,\quad \delta_{m-1}^{\frac{2}{m}}\delta_2^{\frac{1}{m}},$$

$$\Delta_1^{\frac{3}{m}}\delta_{m-3}^{\frac{1}{m}},\quad \delta_2^{\frac{3}{m}}\delta_{m-6}^{\frac{1}{m}}, \ldots\ldots,\quad \delta_{m-1}^{\frac{3}{m}}\delta_3^{\frac{1}{m}},$$

and so on, are the roots of a rational equation of the $(m-1)^{\text{th}}$ degree. The $\frac{m-1}{2}$ terms

$$\Delta_1^{\frac{1}{m}}\delta_{m-1}^{\frac{1}{m}},\quad \delta_2^{\frac{1}{m}}\delta_{m-2}^{\frac{1}{m}}, \ldots\ldots,\quad \delta_{\frac{m-1}{2}}^{\frac{1}{m}}\delta_{\frac{m+1}{2}}^{\frac{1}{m}},$$

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PRINCIPLES.

§1. It will be understood that the surds appearing in the present paper have *prime numbers* for the denominators of their indices, unless where the contrary is expressly stated. Thus, $2^{\frac{1}{15}}$ may be regarded as $h^{\frac{1}{5}}$, a surd with the index $\frac{1}{5}$, h being $2^{\frac{1}{3}}$. It will be understood also that no surd appears in the denominator of a fraction. For instance, instead of $\frac{2}{1 + \sqrt{-3}}$ we should write $\frac{1 - \sqrt{-3}}{2}$. When a surd is spoken of as occurring in an algebraical expression, it may be present in more than one of its powers, and need not be present in the first.

§2. In such an expression as $\sqrt{2} + (1 + \sqrt{2})^{\frac{1}{3}}$, $\sqrt{2}$ is *subordinate* to the *principal* surd $(1 + \sqrt{2})^{\frac{1}{3}}$, the latter being the only principal surd in the expression.

§3. A surd that has no other surd subordinate to it may be said to be *of the first rank;* and the surd $h^{\frac{1}{c}}$, where h involves a surd of the $(a - 1)^{\text{th}}$ rank, but none of a higher rank, may be said to be *of the a^{th} rank.* In estimating the rank of a surd, the denominators of the indices of the surds concerned are always supposed to be prime numbers. Thus, $3^{\frac{1}{6}}$ is a surd of the second rank.

§4. An algebraical expression in which $\Delta_1^{\frac{1}{m}}$ is a principal (see §2) surd may be arranged according to the powers of $\Delta_1^{\frac{1}{m}}$ lower than the m^{th}, thus,

$$\frac{1}{m}\left(g_1 + k_1 \Delta_1^{\frac{1}{m}} + a_1 \Delta_1^{\frac{2}{m}} + b_1 \Delta_1^{\frac{3}{m}} + \ldots + e_1 \Delta_1^{\frac{m-2}{m}} + h_1 \Delta_1^{\frac{m-1}{m}}\right) (1)$$

where g_1, k_1, a_1, etc., are clear of $\Delta_1^{\frac{1}{m}}$.

§5. If an algebrical expression r_1, arranged as in (1), be zero, while the coefficients g_1, k_1, etc., are not all zero, an equation

$$\omega \Delta_1^{\frac{1}{m}} = l_1 \qquad (2)$$

must subsist; where ω is an m^{th} root of unity; and l_1 is an expression involving only such surds exclusive of $\Delta_1^{\frac{1}{m}}$ as occur in r_1. For, let the first of the coefficients h_1, e_1, etc., proceeding in the order of the descending powers of $\Delta_1^{\frac{1}{m}}$, that is not zero, be n_1, the coefficient of $\Delta_1^{\frac{s}{m}}$. Then we may put

$$mr_1 = n_1\left\{f\left(\Delta_1^{\frac{1}{m}}\right)\right\} = n_1 \Delta_1^{\frac{s}{m}} + \text{etc.} = 0.$$

Because $\Delta_1^{\frac{1}{m}}$ is a root of each of the equations $f(x) = 0$ and $x^m - \Delta_1 = 0$, $f(x)$ and $x^m - \Delta_1$ have a common measure. Let their H. C. M., involving only such surds as occur in $f(x)$ and $x^m - \Delta_1$, be $\varphi(x)$. Then, because $\varphi(x)$ is a measure of $x^m - \Delta_1$, the roots of the equation

$$\varphi(x) = x^c + p_1x^{c-1} + p_2x^{c-2} + \text{etc.} = 0$$

are $\Delta_1^{\frac{1}{m}}$, $\omega_1\Delta_1^{\frac{1}{m}}$, $\omega_2\Delta_1^{\frac{1}{m}}$, ..., $\omega_{c-1}\Delta_1^{\frac{1}{m}}$; where ω_1, ω_2, etc., are distinct primitive m^{th} roots of unity. Therefore,

$$\Delta_1^{\frac{c}{m}}(\omega_1\,\omega_2..)(-1)^c = p_c.$$

Now c is a whole number less than m but not zero; and, by §1, m is prime. Therefore there are whole numbers n and h such that

$$\Delta_1^{\frac{cn}{m}}(\omega_1\,\omega_2..)^n(-1)^{cn} = \Delta_1^{\frac{1}{m}}\Delta_1^h(\omega_1\,\omega_2..)^n(-1)^{cn} = p_c^n.$$

Therefore, if $(\omega_1\,\omega_2..)^n = \omega$, and $l_1\Delta_1^h(-1)^{cn} = p_c^n$, $\omega\Delta_1^{\frac{1}{m}} = l_1$.

§6. Let r_1 be an algebraical expression in which no root of unity having a rational value occurs in the surd form $1^{\frac{1}{m}}$. Also let there be in r_1 no surd $\Delta_1^{\frac{1}{m}}$ not a root of unity, such that

$$\Delta_1^{\frac{1}{m}} = e_1 , \tag{3}$$

where e_1 is an expression involving no surds of so high a rank as $\Delta_1^{\frac{1}{m}}$ except such as either are roots of unity, or occur in r_1 being at the same time distinct from $\Delta_1^{\frac{1}{m}}$. The expression r_1 may then be said to have been *simplified* or to be *in a simple state.*

§7. Some illustrations of the definition in §6 may be given. The root $8^{\frac{1}{3}}$ cannot occur in a simplified expression r_1; for its value is 2ω, ω being a third root of unity; but the equation $8^{\frac{1}{3}} = 2\omega$ is of the inadmissible type (3). Again, the root $\sqrt{5}$ cannot occur in a simplified expression; for, ω_1 being a primitive fifth root of unity, $\sqrt{5} = 2(\omega_1 + \omega_1^4) + 1$; an equation of the type (3). Once more, a root of the cubic equation $x^3 - 3x - 4 = 0$, in the form $(2 + \sqrt{3})^{\frac{1}{3}} + (2 - \sqrt{3})^{\frac{1}{3}}$, is not in a simple state, because $(2 - \sqrt{3})^{\frac{1}{3}} = (2 - \sqrt{3})(2 + \sqrt{3})^{\frac{2}{3}}$.

§8. Let
$$p_1\Delta_1^{\frac{m-1}{m}} + p_2\Delta_1^{\frac{m-1}{m}} + \ldots + p_m = 0 ; \tag{4}$$

where $\Delta_1^{\frac{1}{m}}$ is a surd occurring in a simplified expression r_1; and p_1, p_2, etc., involve no surds of so high a rank as $\Delta_1^{\frac{1}{m}}$, except such as either are roots of unity, or occur in r_1 being at the same time distinct from $\Delta_1^{\frac{1}{m}}$. The coefficients p_1, p_2, etc., must be zero separately. For, by §5, if they were not, we should have $\omega\Delta_1^{\frac{1}{m}} = l_1$, ω being an m^{th} root of unity, and l_1 involving only surds in (4) distinct from $\Delta_1^{\frac{1}{m}}$; an equation of the inadmissible type (3).

§9. The expression r_1 being in a simple state, we may use R as a generic symbol to include the various particular expressions, say r_1, r_2, r_3, etc., obtained by assigning all their possible values to the surds involved in r_1, with the restriction that, where the base of a surd is unity, the rational value of the surd is not to be taken into account. These particular expressions, not necessarily all unequal, may be called *the particular cognate forms of R.* For instance, if $r_1 = 1^{\frac{1}{3}}$, R has two particular cognate forms, the rational value of the third root of unity not being counted. If $r_1 = (1 + \sqrt{2})^{\frac{1}{3}}$, R has six particular cognate forms all unequal. Should $r_1 = (2 + \sqrt{3})^{\frac{1}{3}} + (2 - \sqrt{3})(2 + \sqrt{3})^{\frac{2}{3}}$, R has six particular cognate forms, but only three unequal, each of the unequal forms occurring twice.

§10. Proposition I. An algebraical expression r_1 can always be brought to a simple state.

For r_1 may be cleared of all surds such as $1^{\frac{1}{m}}$ having a rational value. Suppose that r_1 then involves a surd $\Delta_1^{\frac{1}{m}}$, not a root of unity, by means of which an equation such as (3) can be formed. Substitute for $\Delta_1^{\frac{1}{m}}$ in r_1 its value e_1 as thus given. The result will be to eliminate $\Delta_1^{\frac{1}{m}}$ from r_1 without introducing into the expression any *new* surd as high in rank as $\Delta_1^{\frac{1}{m}}$, and at the same time not a root of unity. By continuing to make all the eliminations of this kind that are possible, we at last reach a point where no equation of the type (3) can any longer be formed. Then because, by the course that has been pursued, no roots of the form $1^{\frac{1}{m}}$ having a rational value have been left in r_1, r_1 is in a simple state.

§11. It is known that, if N be any whole number, the equation whose roots are the primitive N^{th} roots of unity is rational and irreducible.

§12. Let N be the continued product of the distinct prime numbers n, a, b, etc. Let ω_1 be a primitive n^{th} root of unity, θ_1 a primitive a^{th} root of unity, and so on. Let ω represent any one indifferently of the primitive n^{th} roots of unity, θ any one indifferently of the primitive a^{th} roots of unity, and so on. Let $f(\omega_1, \theta_1, \text{etc.},)$ be a rational function of ω_1, θ_1, etc. Then a corollary from §11 is, that if $f(\omega_1, \theta_1, \text{etc.}) = 0$, $f(\omega, \theta, \text{etc.}) = 0$. For t_1 being a primitive N^{th} root of unity, and t representing any one indifferently of the primitive N^{th} roots of unity, we may put

$$f(\omega_1, \theta_1, \text{etc.}) = a_1 t_1^{N-1} + a_2 t_1^{N-2} + \text{etc.} = 0,$$

$$\text{and } f(\omega, \theta, \text{etc.}) = a_1 t^{N-1} + a_2 t^{N-2} + \text{etc.};$$

where the coefficients a_1, a_2, etc., are rational. Should these coefficients be all zero, $f(\omega, \theta, \text{etc.}) = 0$. Should they not be all zero, let a_r be the first that is not zero. Then we may put

$$f(\omega_1, \theta_1, \text{etc.}) = a_r \left\{ \varphi\ (t_1) \right\} = a_r t_1^{N-r} + \text{etc.} = 0.$$

Therefore, t_1 is a root of the rational equation $\varphi(x) = 0$, being at the same time a root of the rational (see §11) equation $\psi(x) = 0$, whose roots are the primitive N^{th} roots of unity. Hence $\psi(x)$ and $\varphi(x)$ have a common measure. But by §11, $\psi(x)$ is irreducible. Therefore it is a measure of $\varphi(x)$; and the roots of the equation $\psi(x) = 0$ are roots of the equation $\varphi(x) = 0$. Therefore,

$$f(\omega, \theta, \text{etc.}) = a_r \left\{ \varphi(t) \right\} = 0.$$

§13. Another corollary is, that if

$$f(\omega_1, \theta_1, \text{etc.}) = h_1 \omega_1^{n-1} + h_2 \omega_1^{n-2} + \ldots + h_n = 0,$$

where h_1, h_2, etc., are clear of ω_1, the coefficients h_1, h_2, etc., are all equal to one another. For, by §12, because $f(\omega_1, \theta_1, \text{etc.}) = 0$, $f(\omega, \theta_1, \text{etc.}) = 0$. Therefore $\omega \left\{ f(\omega, \theta_1, \text{etc.}) \right\} = 0$. In $\omega \left\{ f(\omega, \theta_1, \text{etc.}) \right\}$ give ω successively its $n - 1$ different values. Then, in addition,

$nh_1 = h_1 + h_2 + \ldots + h_n$. Similarly, $nh_2 = h_1 + h_2 + \ldots + h_n \; \therefore \; h_1 = h_2$.
In like manner all the terms h_1, h_2, etc., are equal to one another.

§14. Proposition II. If the simplified expression r_1, one of the particular cognate forms of R, be a root of the rational equation $F(x) = 0$, all the particular cognate forms of R are roots of that equation.

For, let r_2 be a particular cognate form of R. By §12, the law to be established holds when there are no surds in r_1 that are not roots of unity. It will be kept in view that, according to §1, when roots of unity are spoken of, such roots are meant as $1^{\frac{1}{m}}$, m being a prime number. Assume the law to have been found good for all expressions that do not involve more than $n - 1$ distinct surds that are not roots of unity; then, making the hypothesis that r_1 involves not more than n distinct surds that are not roots of unity, the law can be shown still to hold; in which case it must hold universally. For, let $\varDelta_1^{\frac{1}{m}}$, not a root of unity, be a surd of the highest rank (see §3) in r_1. Then $F(r_1)$ may be taken to be the expression (1), and $F(r_2)$ to be the expression formed from (1) by selecting particular values of the surds involved under the restriction specified in §9. In passing from r_1 to r_2, let $\varDelta_1^{\frac{1}{m}}$, a_1, etc., become respectively $\varDelta_2^{\frac{1}{m}}$, a_2, etc. Then

$$m\left\{F(r_1)\right\} = h_1 \varDelta_1^{\frac{m-1}{m}} + e_1 \varDelta_1^{\frac{m-2}{m}} + \text{etc.} = 0.$$

$$\text{and } m\left\{F(r_1)\right\} = h_2 \varDelta_2^{\frac{m-1}{m}} + e_2 \varDelta_2^{\frac{m-2}{m}} + \text{etc.}$$

By §8, because r_1 is in a simple state, and $F(r_1) = 0$, the coefficients h_1, e_1, etc., are zero separately. But h_1 is clear of the surd $\Delta_1^{\frac{1}{m}}$. It therefore does not involve more than $n - 1$ distinct surds that are not roots of unity. Therefore, on the assumption on which we are proceeding, because $h_1 = 0$, $h_2 = 0$. In like manner, $e_2 = 0$, and so on. Therefore $F(r_2) = 0$.

§15. *Cor.* Let the simplified expression r_1 be the root of an equation $F(x) = 0$ whose coefficients involve certain surds $z_1^{\frac{1}{n}}, u_1^{\frac{1}{s}}$, etc., that have the same determinate values in r_1 as in $F(x)$. Then, if r_2 be a particular cognate form of R in which the surds $z_1^{\frac{1}{n}}, u_1^{\frac{1}{s}}$, etc., retain the determinate values belonging to them in r_1, r_2 is a root of the equation $F(x) = 0$. For $F(r_1) = 0$. Therefore, by the Proposition, $F(R) = 0$. Let R, restricted by the condition that the surds $z_1^{\frac{1}{n}}, u_1^{\frac{1}{s}}$, etc., retain the determinate values belonging to them in r_1, be R'. Then $F(R') = 0$. A particular case of this is $F(r_2) = 0$. The corollary established simply means that the surds $z_1^{\frac{1}{n}}, u_1^{\frac{1}{s}}$, etc., may be taken to be rational for the purpose in hand.

§16. The simplified expression r_1 being one of the particular cognate forms of R, let

$$r_1, r_a, \text{ etc.} \tag{5}$$

be the entire series of the particular cognate forms of R, not necessarily unequal to one another. Then, if the equation whose roots are the terms in (5) be $X = 0$, X is rational. In like manner, if those particular cognate forms of R, not necessarily unequal, that are obtained when certain surds $z_1^{\frac{1}{n}}, u_1^{\frac{1}{s}}$, etc., retain the determinate values belonging to them in r_1, be

$$r_1, r_r, \text{ etc.} \tag{6}$$

and if the equation whose roots are the terms in (6) be $X' = 0$, X' involves only surds found in the series $z_1^{\frac{1}{n}}, u_1^{\frac{1}{s}}$, etc. This is substantially proved by Legendre in his Théorie des Nombres, §487, third edition.

§17. PROPOSITION III. The unequal particular cognate forms of R, the generic expression under which the simplified expression r_1 falls, are the roots of a rational irreducible equation; and each of the unequal particular cognate forms occurs the same number of times in the series of the cognate forms.

As in §16, let the entire series of the particular cognate forms of R be the terms in (5), the equation that has these terms for its roots being $X = 0$. By §16, X is rational. Should X not be irreducible, it has a rational irreducible factor, say $F(x)$, such that r_1 is a root of the equation $F(x) = 0$. By Prop. II., because r_1 is in a simple state, all the terms in (5) are roots of the equation $F(x) = 0$, while at the same time, because $F(x)$ is a factor of X, all the roots of the equation are terms in (5). And the equation $F(x) = 0$, being irreducible, has no equal roots. Therefore its roots are the unequal terms in (5). Should $F(x)$ not be identical with X, put

$$X = \{F(x)\} \{\varphi(x)\}.$$

Because X and $F(x)$ are rational, $\varphi(x)$ is rational. Then, since $\varphi(x)$ is a measure of X, and the equation $F(x) = 0$ has for its its roots the unequal roots of the equation $X = 0$, the equations $F(x) = 0$ and $\varphi(x) = 0$ have a root in common. Consequently, since $F(x)$ is irreducible, it is a measure of $\varphi(x)$. Therefore $\{F(x)\}^2$ is a measure of X. Going on in this way we ultimately get $X = \{F(x)\}^N$; which means that each of the particular cognate forms of R has its value repeated N times in the series of the particular cognate forms.

§18. *Cor.* 1. The series (6) consisting of those particular cognate forms of R in which certain surds $z_1^{\frac{1}{n}}$, $u_1^{\frac{1}{s}}$, etc., retain the determinate values belonging to them in r_1, each of the unequal terms in (6) occurs the same number of times in (6); and the unequal terms in (6) are the roots of an irreducible equation whose coefficients involve only surds found in the series $z_1^{\frac{1}{n}}$, $u_1^{\frac{1}{s}}$, etc. Should X' not be irreducible, by which in such a case is meant incapable of being broken into lower factors involving only surds occurring in X', let it have the irreducible factor X''. That is to say, X'' involves only surds occurring in X', and has itself no lower factor involving only surds that occur in X''. We may take r_1 to be a root of the equation $X'' = 0$. Then, by Cor. Prop. II., all the terms in (6) are roots of that equation, all the roots of the equation being at the same time terms in (6). And the equation $X'' = 0$ being irreducible, has no equal roots. Therefore its roots are the unequal terms in (6). Put

$X' = (X'')(X''')$. Then, by the line of reasoning followed in the Proposition, X''' has a measure identical with X''. And so on. Ultimately $X' = (X'')^N$.

§19. *Cor.* 2. If r_2, one of the particular cognate forms of R, be zero, all the particular cognate forms of R are zero. For, by the proposition, the particular cognate forms of R are the roots of a rational irreducible equation $F(x) = 0$. And r_2, one of the roots of that equation, is zero, but the only rational irreducible equation that has zero for a root is $x = 0$. Therefore $F(x) = x = 0$. In fact, in the case supposed, the simplified expression r_1 is zero, and R has no particular cognate forms distinct from r_1.

§20. PROPOSITION IV. Let N be the continued product of the distinct prime numbers n, a, etc. Let ω_1 be a primitive n^{th} root of unity, θ_1 a primitive a^{th} root of unity, and so on. Then if the equation

$$F(x) = x^d + b_1x^{d-1} + b_2x^{d-2} + \text{etc.} = 0$$

be one in which the coefficients b_1, b_2, etc., are rational functions of ω_1, θ_1, etc., and if all the primitive n^{th} roots of unity, which, when substituted for ω_1 in $F(x)$, leave $F(x)$ unaltered, be

$$\omega_1, \omega_2, \ldots, \omega_s, \tag{7}$$

the series (7) either consists of a single term or it is made up of a cycle of primitive n^{th} roots of unity,

$$\omega_1, \omega_1^{\lambda}, \omega_1^{\lambda^2}, \ldots, \omega_1^{\lambda^{s-1}}; \tag{18}$$

that is to say, no term in (8) after the first is equal to the first, but $\omega_1^{\lambda^s} = \omega_1$. Also, if (let it be kept in view that n is prime) the cycle that contains all the primitive n^{th} roots of unity be

$$\omega_1, \omega_1^{\beta}, \omega_1^{\beta^2}, \ldots, \omega_1^{\beta^{n-2}}, \tag{9}$$

and if C_1 be the sum of the terms in the cycle (8), the form of $F(x)$ is

$$F(x) = x^d - (p_1C_1 + p_2C_2 + \ldots + p_mC_m)\,x^{d-1} + (q_1C_1 + q_2C_2 + \text{etc.})\,x^{d-2} + \text{etc.} \tag{10}$$

where each of the expressions in the series C_1, C_2, C_3, etc., is what the immediately preceding term becomes by changing ω_1 into ω_1^{β}, C_m through this change becoming C_1; and p_1, p_2, q_1, etc., are clear of ω_1.

For, assuming that there is a term ω_2 in (7) additional to ω_1, we may take ω_2 to be the first term in (9) after ω_1 that occurs in (7); and it may be considered to be $\omega_1^{\beta^m}$, which may be otherwise written ω_1^{λ}. Then, if $F(x)$ be written $\varphi(\omega_1)$, we have by hypothesis

$\varphi(\omega_1) = \varphi(\omega_1^{\lambda})$. Therefore, by §12, changing ω_1 into ω_1^{λ}, $\varphi(\omega_1^{\lambda}) = \varphi(\omega_1^{\lambda^2})$. Therefore $\varphi(\omega_1) = \varphi(\omega_1^{\lambda^2})$. And thus ultimately $\varphi(\omega_1) = \varphi(\omega_1^{\lambda^z})$, or $\varphi(\omega_1) = \varphi(\omega_1^{\beta^{mz}})$, z being any whole number positive or negative. But $\omega_1^{\lambda^z}$ includes all the terms in (8). Therefore each of these terms is a term in (7). Suppose if possible that there is a term in (7), say $\omega_1^{\beta^h}$, which does not occur in (8). Then, just as we deduced $\varphi(\omega_1) = \varphi(\omega_1^{\beta^{mz}})$ from the equation $\varphi(\omega_1) = \varphi(\omega_1^{\beta^m})$, we can, because still farther $\varphi(\omega_1) = \varphi(\omega_1^{\beta^h})$, deduce $\varphi(\omega_1) = \varphi(\omega_1^{\beta^{mz+hu}})$. Because $\omega_1^{\beta^h}$ lies outside the cycle (8), h is not a multiple of m. And it is not less than m, because $\omega_1^{\beta^m}$ is the first term in (9) after ω_1, which, when substituted for ω_1 in $\varphi(\omega_1)$, leaves $\varphi(\omega_1)$ unaltered. Therefore $h = qm + v$, where q and v are whole numbers, and v is less than m but not zero. Put

$$z = -(h+q), \text{ and } u = m+1 \therefore mz+hu = v \therefore \varphi(\omega_1) = \varphi(\omega_1^{\beta^v});$$

which, because v is less than m but not zero, and $\omega_1^{\beta^m}$ is the first term in (9) after ω_1 which, when substituted for ω_1 in $\varphi(\omega_1)$, leaves $\varphi(\omega_1)$ unaltered, is impossible. Hence, no term in (7) lies outside the cycle (8), while it has also been shown that all the terms in (8) are terms in (7). Therefore the terms in (7) are identical with those constituting the cycle (8). We have now to determine the form of $F(x)$. The expressions, C_1, C_2, etc., taken together, are the sum of the terms in (9). Therefore $\quad C_1 + C_2 + \ldots + C_m = -1. \quad$ (11)
Because (9) contains all the primitive n^{th} roots of unity, we may put

$$F(x) = x^d - \{p + (p+p_1)\,\omega_1 + (p+p_2)\,\omega_1^{\beta} + \text{etc.}\}\,x^{d-1} + \text{etc.}; \quad (12)$$

where p, p_1, etc., are clear of ω_1. But $F(x)$ remains unaltered when ω_1 is changed into $\omega_1^{\beta^m}$. Therefore

$$F(x) = x^d - \{p + (p+p_1)\,\omega_1^{\beta^m} + \text{etc.}\}\,x^{d-1} + \text{etc.} \quad (13)$$

Therefore, equating the coefficients of x^{d-1} in (12) and (13),

$$(p - p_1) + \ldots + (p_{m+1} - p_1)\,\omega_1^{\beta^m} + \text{etc.} = 0.$$

Here, by §13, the coefficients of the different powers of ω_1 have all the same value. And one of them, $p - p_1$, is zero. Therefore

$p_{m+1} = p_1$. That is to say, the coefficient of $\omega_1^{\beta^m}$ or ω_1^{λ} is the same as that of ω_1. In like manner the coefficients of all the terms in (8) are the same. Therefore one group of the terms that together make up the coefficient of x^{d-1} in (12) is properly represented by $-(p + p_1)C_1$. In the same way another group is properly represented by $-(p + p_2)C_2$, and so on. Hence

$$F(x) = x^d - \{p + (p + p_1)\,C_1 + (p + p_2)\,C_2 + \text{etc.}\}x^{d-1} + \text{etc.}$$

And by (11) this is equivalent to (10). The form of $F(x)$ has been deduced on the assumption that the series (7) contains more than one term; but, should the series (7) consist of a single term, the result obtained would still hold good, only in that case each of the expressions C_1, C_2, etc., would be a primitive n^{th} root of unity.

§21. A simplified expression will not cease to be in a simple state, if we suppose that any surd that can be eliminated from it, without the introduction of any new surd, has been eliminated.

§22. Proposition V. In the simplified expression r_1, one of the particular cognate forms of R, modified according to §21, let the surd $\Delta_1^{\frac{1}{m}}$ of the highest rank be not a root (see §1) of unity. Then, if the particular cognate forms of R obtained by changing $\Delta_1^{\frac{1}{m}}$ in r_1 successively into the different m^{th} roots of the determinate base Δ_1, be

$$r_1\,,\; r_2\,,\; \ldots\,,\; r_m, \qquad (14)$$

these terms are all unequal.

For the terms in (14) are all the particular cognate forms of R obtained when we allow all the surds in r_1 except $\Delta_1^{\frac{1}{m}}$ to retain the determinate values belonging to them in r_1. Therefore, by Cor. 1, Prop. III., each of the unequal terms in (14) has its value repeated the same number of times in that series. Let u be the number of the unequal terms in (14), and let each occur c times. Then $uc = m$. Suppose if possible that $u = 1$. This means that all the terms in (14) are equal. Therefore, r_1 being the expression (1),

$$mr_1 = r_1 + r_2 + \ldots + \text{etc.} = g_1.$$

Therefore the surd $\Delta_1^{\frac{1}{m}}$ can be eliminated from r_1 without the introduction of any new surd; which, by §21, is impossible. Therefore u is not unity. But, by §1, m is a prime number. And $m = uc$. Therefore $c = 1$ and $u = m$. This means that all the terms in (14) are unequal.

§23. *Cor.* 1. Let r_{a+1} be any one of the particular cognate forms of R; and let $\Delta_{a+1}^{\frac{1}{m}}$, h_{a+1}, etc., be respectively what $\Delta_1^{\frac{1}{m}}$, h_1, etc., become in passing from r_1 to r_{a+1}. Also let the m particular cognate forms of R, obtained by changing $\Delta_{a+1}^{\frac{1}{m}}$ in r_{a+1} successively into the different m^{th} roots of Δ_{a+1}, be

$$r_{a+1}, \; r_{a+2}, \; \ldots, \; r_{a+m}. \tag{15}$$

These terms are all unequal. For, because $\Delta_1^{\frac{1}{m}}$ is a principal surd in r_1, and r_2 is what r_1 becomes when $\Delta_1^{\frac{1}{m}}$ is changed into a surd whose value is $\omega_1\Delta_1^{\frac{1}{m}}$, ω_1 being a primitive m^{th} root of unity, the view may be taken that r_2 involves no surds additional to those found in r_1, except the primitive m^{th} root of unity ω_1. Therefore $r_1 - r_2$ involves no surds distinct from primitive m^{th} roots of unity that are not found in the simplified expression r_1. Therefore $r_1 - r_2$ is in a simple state. Let r_{a+2} be what r_{a+1} becomes by changing $\Delta_{a+1}^{\frac{1}{m}}$ into $\omega_1\Delta_{a+1}^{\frac{1}{m}}$. Then $r_{a+1} - r_{a+2}$ is a particular cognate form of the generic expression under which the simplified expression $r_1 - r_2$ falls. Therefore $r_{a+1} - r_{a+2}$ cannot be zero; for, if it were, $r_1 - r_2$ would, by Cor. 2, Prop. III., be zero; which, by the proposition, is impossible. Hence, the first two terms in (15) are unequal. In like manner all the terms in (15) are unequal.

§24. *Cor.* 2. Let $X_1 = 0$ be the equation whose roots are the terms in (14). When X_1 is modified according to §21, it is, by §16, clear of the surd $\Delta_1^{\frac{1}{m}}$. Should it involve any surds that are not roots of unity, take $z_1^{\frac{1}{c}}$ a surd of the highest rank not a root of unity in X_1; and, when $z_1^{\frac{1}{c}}$ is changed successively into the different c^{th} roots of the determinate base z_1, let

$$X_1, \; X_1', \; X_1'', \; \ldots, \; X_1^{(c-1)}, \tag{16}$$

be respectively what X_1 becomes. Any term in (16), as X_1', being selected, the m roots of the equation $X_1' = 0$ are unequal particular cognate forms of R. For, $z_2^{\frac{1}{c}}$ being a c^{th} root of z_1 distinct from $z_1^{\frac{1}{c}}$, let r_{a+1} be what r_1 becomes when $z_1^{\frac{1}{c}}$ becomes $z_2^{\frac{1}{c}}$; the expressions $\Delta_1^{\frac{1}{m}}$, h_1, etc., at the same time becoming $\Delta_{a+1}^{\frac{1}{m}}$, h_{a+1}, etc. Then we may put

$$X_1 = x^m + (bz_1^{\frac{c-1}{c}} + dz_1^{\frac{c-2}{c}} + \text{etc.})\, x^{m-1} + \text{etc.}; \qquad (17)$$

where b, d, etc., are clear of $z_1^{\frac{1}{c}}$. Therefore, because r_1 is a root of the equation $X_1 = 0$,

$$\left\{ \frac{1}{m} (h_1 \Delta_1^{\frac{m-1}{m}} + \text{etc.}) \right\}^m$$

$$+ (bz_1^{\frac{c-1}{c}} + dz_1^{\frac{c-2}{c}} + \text{etc.}) \left\{ \frac{1}{m} (h_1 \Delta_1^{\frac{m-1}{m}} + \text{etc.}) \right\}^{m-1} + \text{etc.} = 0.$$

All the surds in this equation occur in the simplified expression r_1. Therefore, by Prop. II.,

$$\left\{ \frac{1}{m} (h^{a+1} \Delta_{a+1}^{\frac{m-1}{m}} + \text{etc.}) \right\}^m$$

$$+ (bz_2^{\frac{c-1}{c}} + dz_2^{\frac{c-2}{c}} + \text{etc}) \left\{ \frac{1}{m} (h_{a+1} \Delta_{a+1}^{\frac{m-1}{m}} + \text{etc.}) \right\}^{m-1} + \text{etc.} = 0.$$

Therefore $\frac{1}{m} (h_{a+1} \Delta_{a+1}^{\frac{m-1}{m}} + \text{etc.})$ or r_{a+1} is a root of the equation

$$X_1' = x^m + (bz_2^{\frac{c-1}{c}} + \text{etc.})\, x^{m-1} + \text{etc.} = 0. \qquad (18)$$

Therefore also, by Cor. Prop. II., all the terms in (15) are roots of that equation. And, by Cor. 1, the terms in (15) are all unequal. Therefore the equation $X_1' = 0$ has m unequal particular cognate forms of R for its roots.

§25. *Cor.* 3. No two of the expressions in (16), as x_1 and X_1', are identical with one another. For, in order that X_1 and X_1' might be identical, the coefficients of the several powers of x in X_1 would need to be equal to those of the corresponding powers of x in X_1'; but, if one of the coefficients of X_1 be selected in which $z_1^{\frac{1}{c}}$ is present, this coefficient can be shown to be unequal to the corresponding coefficient in X_1' in the same way in which the terms in (15) were proved to be all unequal.

§26. *Cor.* 4. Any two of the terms in (16), as X_1 and X_1', being selected, the equations $X_1 = 0$ and $X_1' = 0$ have no root in common. For, suppose, if possible, that these equations have a root in common. Taking the forms of X_1 and X_1' in (17) and (18), since r_1 is a root of the equation $X_1' = 0$,

$$r_1^m + (bz_2^{\frac{c-1}{c}} + \text{etc.})\, r_1^{m-1} + \text{etc.} = 0. \tag{19}$$

All the surds in this equation except $z_2^{\frac{1}{c}}$ occur in r_1. It is impossible that $z_2^{\frac{1}{c}}$ can occur in r_1; for, $z_1^{\frac{1}{c}}$ occurs in r_1; and $z_2^{\frac{1}{c}} = \theta_1 z_1^{\frac{1}{c}}$, θ_1 being a primitive c^{th} root of unity; but this equation, if both $z_1^{\frac{1}{c}}$ and $z_2^{\frac{1}{c}}$ occurred in r_1, would be of the inadmissible type (3). Since $z_2^{\frac{1}{c}}$ does not occur in r_1, it is a principal (see §2) surd in (19). We may, therefore, keeping in view that r_1 is the expression (1) in which $\Delta_1^{\frac{1}{m}}$ is a principal surd, arrange (19) thus,

$$\varphi\,(\Delta_1^{\frac{1}{m}}) = \Delta_1^{\frac{m-1}{m}}\,(p_1 z_2^{\frac{c-1}{c}} + p_2 z_2^{\frac{c-2}{c}}, + \text{etc.})$$

$$+\, \Delta_1^{\frac{m-2}{m}}\,(q_1 z_2^{\frac{c-1}{c}} + q_2 z_2^{\frac{c-2}{c}} + \text{etc.}) + \text{etc.} = 0\,; \tag{20}$$

where p_1, q_1, etc., are clear of $z_2^{\frac{1}{c}}$. Then, ω_1 being a primitive m^{th} root of unity such that, by changing $\Delta_1^{\frac{1}{m}}$ into the m^{th} root of Δ_1 whose value is $\omega_1 \Delta_1^{\frac{1}{m}}$, r_1 becomes r_2,

$$\varphi\left(\omega_1 \Delta_1^{\frac{1}{m}}\right) = \omega_1^{m-1} \Delta_1^{\frac{m-1}{m}} \left(p_1 z_2^{\frac{c-1}{c}} + \text{etc.}\right)$$

$$+ \omega_1^{m-2} \Delta_1^{\frac{m-1}{m}} \left(q_1 z_2^{\frac{c-1}{c}} + \text{etc.}\right) + \text{etc.} \qquad (21)$$

The coefficients of the several powers of $\Delta_1^{\frac{1}{m}}$ in $\varphi\left(\Delta_1^{\frac{1}{m}}\right)$ cannot be all zero; for, if they were, we should have, from (21), $\varphi\left(\omega_1 \Delta_1^{\frac{1}{m}}\right) = 0$. This means that r_2 is a root of the equation $X_1' = 0$. But in like manner all the terms in (14) would be roots of that equation, and X_1' would be identical with X; which, by Cor. 3, is impossible. Since the coefficients of the different powers of $\Delta_1^{\frac{1}{m}}$ in $\varphi\left(\Delta_1^{\frac{1}{m}}\right)$ are not all zero, the equation (20) gives us, by §5, $\omega \Delta_1^{\frac{1}{m}} = l_1$, ω being an m^{th} root of unity, and l_1 involving only surds in $\varphi\left(\Delta_1^{\frac{1}{m}}\right)$ exclusive of $\Delta_1^{\frac{1}{m}}$. In l_1 we may conceive $z_2^{\frac{1}{c}}$ changed into $\theta_1 z_1^{\frac{1}{c}}$. Then l_1 involves only surds distinct from $\Delta_1^{\frac{1}{m}}$, all of them except the primitive c^{th} root of unity θ_1 being surds that occur in r_1. This makes the equation $\omega \Delta_1^{\frac{1}{m}} = l_1$ of the inadmissible type (3). Hence the equations $X_1 = 0$ and $X_1' = 0$ have no root in common.

§27. *Cor.* 5. Let X_2 be the continued product of the terms in (16). Then X_2, modified according to §21, is clear of $z_1^{\frac{1}{c}}$, in the same way in which X_1 is clear of $\Delta_1^{\frac{1}{m}}$. Also since, by Cor. 2, each of the equations $X_1 = 0$, $X_1' = 0$, etc., has m unequal particular cognate forms of R for its roots; and since, by Cor. 4, no two of these equations have a root in common, the mc roots of the equation $X_2 = 0$ are unequal particular cognate forms of R.

§28. PROPOSITION VI. Let the simplified expression r_1, modified according to §21, be a root of the rational irreducible equation of the N^{th} degree, $F(x) = 0$. Then if $\Delta_1^{\frac{1}{m}}$, not a root of unity, be a surd of the highest rank in r_1, N is a multiple of m. But if r_1 involve only surds that are roots of unity, one of them being the primitive n^{th} root of unity, N is a multiple of a measure of $n - 1$.

First, let $\Delta_1^{\frac{1}{m}}$, not a root of unity, be a surd of the highest rank in r_1. Taking the expression (1) to be r_1, let X_1 be formed as in §24, and let it be modified according to §21. It is clear of the surd $\Delta_1^{\frac{1}{m}}$. Should it involve a surd that is not a root of unity, let X_2 be formed as in §27. Setting out from r_1 we arrived by one step at X_1, an expression clear of $\Delta_1^{\frac{1}{m}}$, and such that the roots of the equation $X_1 = 0$ are unequal particular cognate forms of R. A second step brought us to X_2, an expression clear of the additional surd $z_1^{\frac{1}{c}}$, and such that the mc roots of the equation $X_2 = 0$ are unequal particular cognate forms of R. Thus we can go on till, in the series X_1, X_2, etc., we reach a term X_e into which no surds enter that are not roots of unity, the $mc \dots l$ roots of the equation $X_e = 0$ being unequal particular cognate forms of R. Should X_e modified according to §21, not be rational, its form, by Prop. IV., putting d for $mc \dots l$, is

$$X_e = x_d - (p_1C_1 + \dots + p_mC_m)x^{d-1} + (q_1C_1 + \dots + q_mC_m)x^{d-2} + \text{etc.};$$

where, one of the roots occurring in X_e being the primitive n^{th} root of unity ω_1, the coefficients p_1, q_1, etc., are clear of ω_1; and C_1 is the sum of the cycle of primitive n^{th} roots of unity (8) containing s or $\frac{n-1}{m}$ terms; and, the cycle (9) containing all the primitive n^{th} roots of unity, the change of ω_1 into ω_1^{β} causes C_1 to become C_2, and C_2 to become C_3, and so on, C_m becoming C_1. As was explained at the close of §20, the cycle (8) may be reduced to a single term, which is then identical with C_1. It will also not be forgotten that the roots of unity such as the n^{th} here spoken of are, according to §1, subject to the condition that the numbers such as n are prime. When C_1 in X_e is changed successively into C_1, C_2, etc., let X_e become

$$X_e,\ X_e',\ X_e'',\ \dots,\ X_e^{(m-1)} \qquad (22)$$

If X_{e+1} be the continued product of the terms in (22), the dm roots of the equation $X_{e+1} = 0$ can be shown to be unequal particular cognate forms of R. For, no two terms in (22) as X_e and X_e' are identical; because, if they were, X_e would remain unaltered by the change of ω_1 into ω_1^{β}; which, by Prop. IV., because ω_1^{β} is not a term in the cycle (8), is impossible. It follows that no two of the equations $X_e = 0$, $X_e' = 0$, etc., have a root in common. For, if the equations $X_e = 0$, and $X_e' = 0$ had a root in common, since X_e and X_e' are not identical, X_e would have a lower measure involving only surds found in X_e', because the surds in X_e are the same with those in X_e'. Let $\varphi(x)$ be this lower measure of X_e, and let r_1 be a root of the equation $\varphi(x) = 0$. Then, by Cor. Prop. II., all the d roots of the equation $X_e = 0$ are roots of the equation $\varphi(x) = 0$; which is impossible. In the same way it can be proved that no equation in the series $X_e = 0$, $X_e' = 0$, etc., has equal roots. Since no one of these equations has equal roots, and no two of them have a root in common, the dm roots of the equation $X_{e+1} = 0$ are unequal particular cognate forms of R. Also X_{e+1}, modified according to §21, is clear of the primitive n^{th} roots of unity. Should X_{e+1} not be rational, we can deal with it as we did with X_e. Going on in this way, we ultimately reach a *rational* expression X_z such that the $dm \ldots g$ roots of the equation $X_z = 0$ are unequal particular cognate forms of R. This equation must be identical with the equation $F(x) = 0$ of which r_1 is a root. For, by Prop. III., the equation $F(x) = 0$ has for its roots the unequal particular cognate forms of R. Therefore, because the roots of the equation $X_z = 0$ are all unequal and are at the same time particular cognate forms of R, X_z must be either a lower measure of $F(x)$ or identical with $F(x)$. But $F(x)$, being irreducible, has no lower measure. Therefore X_z is identical with $F(x)$. Therefore, the equation $F(x) = 0$ being the N^{th} degree, $N = mc \ldots lm \ldots g$. Hence N is a multiple of m. This is the result arrived at when r_1 involves a surd of the highest rank $\Delta_1^{\frac{1}{m}}$ not a root of unity. Should r_1 involve no surds except roots (see §1) of unity, we should then have set out from X_e regarded as identical with $x - r_1$. The result would have been $N = m \ldots g$. Therefore N is a multiple of m; and, because m is here the number of cycles of s terms each, that make up the series of the primitive n^{th} roots of unity, $ms = n - 1$. Therefore N is a multiple of a measure of $n - 1$.

§29. *Cor.* Let N be a prime number. Then, if r_1 involve a surd of the highest rank $\Delta_1^{\frac{1}{m}}$ not a root (see §1) of unity, $N = m$; for, the series of integers m, c, etc., of which N is the continued product, is reduced to its first term. If r_1 involve only surds that are roots of unity, $n - 1$ is a multiple of N; for $N = m \dots g$; therefore, because N is prime, it is equal to m; but $ms = n - 1$; therefore $n - 1 = sN$.

THE SOLVABLE IRREDUCIBLE EQUATION OF THE m^{th} DEGREE, m PRIME.

§30. The principles that have been established may be illustrated by an examination of the solvable irreducible rational equation of the m^{th} degree $F(x) = 0, m$ being prime. Two cases may be distinguished, though it will be found that the roots can in the two cases be brought under a common form; the one case being that in which the simplified root r_1 is, and the other that in which it is not, a rational function of roots of unity, that is, according to §1, of roots of unity having the denominators of their indices prime numbers. The equation $F(x) = 0$ may be said to be in the former case *of the first class*, and in the latter *of the second class*.

THE EQUATION $F(x) = 0$ OF THE FIRST CLASS.

§31. In this case, by Cor. Prop. VI., r_1 being modified according to §21, if one of the roots involved in r_1 be the primitive n^{th} root of unity ω_1, $n - 1$ is a multiple of m. Also the expression written X_e in Prop. VI. is reduced to $x - r_1$, so that

$$r_1 = p_1C_1 + p_2C_2 + \dots + p_mC_m .$$

The m roots of the equation $F(x) = 0$ being r_1, r_2, etc, we must have

$$\left.\begin{aligned} r_1 &= p_1 C_1 + p_2C_2 + \dots + p_mC_m , \\ r_2 &= p_mC_1 + p_1C_2 + \dots + p_{m-1}C_m , \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ r_m &= p_2 C_1 + p_3C_2 + \dots + p_1C_m . \end{aligned}\right\} \qquad (23)$$

For, by Prop. II., because r_1 is a root of the equation $F(x) = 0$, all the expressions on the right of the equations (23) are roots of that equation. And no two of these expressions are equal to one another. For, take the first two. If these were equal, we should have $(p_m - p_1) C_1 + (p_1 - p_2) C_2 +$ etc. $= 0$. Therefore, by §13, each of the terms $p_m - p_1$, $p_1 - p_2$, etc., is zero. This makes p_1, p_2, etc., all equal to one another. Therefore $r_1 = - p_1$; so that the primitive n^{th} root of unity is eliminated from r_1; which, by §21, is impossible. Hence the values of the m roots of the equation $F(x) = 0$ are those given in (23).

§32. Let r_1' be one of the particular cognate forms of the generic expression R under which the simplified expression r_1 falls. Then, because, by Prop. II., all the particular cognate forms of R are roots of the equation $F(x) = 0$, r_1' is equal to one of the m terms r_1, r_2, etc., say to r_z. I will now show that the changes of the surds involved that cause r_1 to become r_1', whose value is r_z, cause r_2 to receive the value r_{z+1}, and r_3 to receive the value r_{z+2}, and so on. This may appear obvious on the face of the equations (23); but, to prevent misunderstanding, the steps of the deduction are given. Any changes made in r_1 must transform C_1 into C_s, one of the m terms C_1, C_2, etc. In passing from r_1 to r_1', while C_1 becomes C_s, let r_2 become r_2', and p_1 become p_1', and p_2 become p_2', and so on. The change that causes C_1 to become C_s transforms C_2 into C_{s+1}, and C_3 into C_{s+2}, and so on. Therefore, it being understood that p_{m+1}, C_{m+1}, etc., are the same as p_1, C_1, etc., respectively,

$$r_1' = p_1'C_s + p_2'C_{s+1} + \text{etc.},$$

$$\text{and } r_2' = p_m'C_s + p_1'C_{s+1} + \text{etc.};$$

which may be otherwise written

$$\left.\begin{aligned} r_1' &= p_{m+2-s}' C_1 + p_{m+3-s}' C_2 + \text{etc.}, \\ r_2' &= p_{m+1-s}' C_1 + p_{m+2-s}' C_2 + \text{etc.} \end{aligned}\right\} \quad (24)$$

Therefore, form (24) and (23),

$$C_1(p_{m+2-z} - p_{m+2-s}') + C_2(p_{m+3-z} - p_{m+3-s}') + \text{etc.} = 0.$$

Therefore, by §13, $p_{m+2-s}' = p_{m+2-z}$, $p_{m+3-s}' = p_{m+3-z}$, etc.

Hence the second of the equations (24) becomes

$$r_2' = p_{m+1-z} C_1 + p_{m+2-z} C_2 + \text{etc.} = r_{z+1}.$$

Thus r_2 is transformed into r_{z+1}. In like manner r_3 receives the value r_{z+2}, and so on.

§33. By Cor. Prop. VI., the primitive n^{th} root of unity being one of those involved in r_1, $n - 1$ is a multiple of m. In like manner, if the primitive a^{th} root of unity be involved in r_1, $a - 1$ is a multiple of m, and so on. Therefore, if t_1 be the primitive m^{th} root of unity, t_1 is distinct from all the roots involved in r_1.

§34. From this it follows that, if the circle of roots r_1, r_2, ..., r_m, be arranged, beginning with r_c, in the order r_c, r_{c+1}, r_{c+2}, etc., and again, beginning with r_s, in the order r_s, r_{s+1}, r_{s+2}, etc., and if, t_1^a being one of the primitive m^{th} roots of unity,

$$r_c + r_{c+1} t_1 + r_{c+2} t_1^2 + \text{etc.} = r_s + r_{s+1} t_1^a + r_{s+2} t_1^{2a} + \text{etc.} \quad (25)$$

$r_c = r_s$. It is understood that in the series r_c, r_{c+1}, etc., when r_m is reached, the next in order is r_1, so that r_{m+1} is the same as r_1, and so on. In like manner r_{s+1} is the same as r_1, and so on. Since r_1, r_2, etc., do not involve the primitive m^{th} root of unity t_1, we can, by §12, substitute for t_1 in (25) successively the different primitive m^{th} roots of unity. Let this be done. Then, by addition,

$mr_c - (r_1 + r_2 + \text{etc.}) = mr_s - (r_1 + r_2 + \text{etc.})$. Therefore $r_c = r_s$.

§35. Proposition VII. Putting

$$\left.\begin{aligned} \Delta_1^{\frac{1}{m}} &= r_1 + t_1 r_2 + t_1^2 r_3 + \ldots + t_1^{m-1} r_m, \\ \Delta_2^{\frac{1}{m}} &= r_1 + t_1^2 r_2 + t_1^4 r_3 + \ldots + t_1^{2(m-1)} r_m, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Delta_{m-1}^{\frac{1}{m}} &= r_1 + t_1^{-1} r_2 + t_1^{-2} r_3 + \ldots + t_1 r_m, \end{aligned}\right\} \quad (26)$$

the terms, Δ_1, Δ_2, Δ_3, ..., Δ_{m-1}, (27)

are the roots of a rational irreducible equation of the $(m-1)^{\text{th}}$ degree $\varphi(x) = 0$, which may be said to be *auxiliary* to the equation $F(x) = 0$.

For, let Δ be the generic expression of which Δ_1 is a particular cognate form; and let Δ' denote any one indifferently of the $m - 1$ particular cognate forms of Δ in (27). Because, by §33, the primitive m^{th} root of unity does not enter into r_1, r_2, etc., no changes made in r_1, r_2, etc., affect t_1. Also, by §32, if r_1 becomes r_z, r_2 becomes r_{z+1}, r_3 becomes r_{z+2}, and so on. Therefore the expression

$$(r_z + tr_{z+1} + t^2 r_{z+2} + \text{etc.})^m,$$

contains all the particular cognate forms of Δ; where z may be any number in the series 1, 2, ..., $m - 1$; and t denotes any one indifferently of the primitive m^{th} roots of unity. But this is equal to

$$\{t^{1-z}(r_1 + tr_2 + t^2 r_3 + \text{etc.})\}^m \text{ or } \Delta'.$$

The conclusion established means that all the differences of value that can present themselves in the particular cognate forms of Δ must arise from the different values of t that are taken in Δ', while the expressions r_1, r_2, etc., remain unaltered. And t has not more than $m - 1$ values. Hence there are not more than $m - 1$ unequal particular cognate forms of Δ. But the $m - 1$ forms obtained by taking the different values of t in Δ' are all unequal. For, selecting t_1 and t_1^a, two distinct values of t, suppose if possible that

$$(r_1 + t_1 r_2 + \text{etc.})^m = (r_1 + t_1^a r_2 + \text{etc.})^m$$

$$\therefore t_1^s (r_1 + t_1 r_2 + \text{etc.}) = r_1 + t_1^a r_2 + \text{etc.},$$

s being a whole number. This may be written

$$r_{m+1-s} + r_{m+2-s}\, t_1 + \text{etc.} = r_1 + t_1^a r_2 + \text{etc.} \qquad (28)$$

Therefore, by §34, $r_{m+1-s} = r_1$. This means, since all the m terms r_1, r_2, etc., are unequal, that $s = 0$. Hence (28) becomes

$$r_1 + r_2 t_1 + \text{etc.} = r_1 + r_2 t_1^a + \text{etc.}$$

Therefore

$$r_2 + r_3 t_1^a + \text{etc.} = r_2 t_1^{1-a} + r_3 t_1^{2-a} + \text{etc.}$$

$$= r_{a+1} + r_{a+2} t_1 + \text{etc.}$$

Therefore, by §35, $r_2 = r_{a+1}$. Therefore, because all the m terms r_1, r_2, etc., are unequal, $a = 1$; which, because t_1 and t_1^a were supposed to be distinct primitive m^{th} roots of unity, is impossible. Therefore no two of the terms in (27) are equal to one another. And it has been proved that there is no particular cognate form of Δ which is not equal to a term in (27). Therefore the terms in (27) are the unequal particular cognate forms of Δ Therefore, by Prop. III., they are the roots of a rational irreducible equation.

§36. Proposition VIII. The roots of the equation $\varphi(x) = 0$ auxiliary (see §35) to $F(x) = 0$ are rational functions of the primitive m^{th} root of unity.

For, let the value of Δ_1, obtained from (26), and modified according to §21, be

$$\Delta_1 = k_1 + k_2 t_1 + k_3 t_1^2 + \ldots + k_m t_1^{m-1},$$

where k_1, k_2, etc., are clear of t_1. Suppose if possible that k_1, k_2, etc., are not rational. We may take the primitive n^{th} root of unity ω_1 to be present in these coefficients. But ω_1 occurs in r_1, r_2, etc., and therefore also in Δ_1, only in the expressions C_1, C_2, etc. Therefore $\Delta_1 = d_1 C_1 + \ldots + d_m C_m$; where d_1, etc., are clear of ω_1. The coefficients d_1, d_2, etc., cannot all be equal; for this would make $\Delta_1 = -d_1$; which, by §21, is impossible. Hence m unequal values of the generic expression Δ are obtained by changing C_1 successively into C_1, C_2, etc., namely,

$$d_1 C_1 + d_2 C_2 + \dots + d_m C_m,$$
$$d_m C_1 + d_1 C_2 + \dots + d_{m-1} C_m,$$
$$\dots\dots\dots\dots\dots\dots$$
$$d_2 C_1 + d_3 C_2 + \dots + d_1 C_m.$$

To show that these expressions are all unequal, take the first two. If these were equal, we should have

$$(d_m - d_1)\, C_1 + (d_1 - d_2)\, C_2 + \text{etc.} = 0\,.$$

Therefore, by §13, $d_m - d_1 = 0$, $d_1 - d_2 = 0$, and so on; which, because d_1, d_2, etc., are not all equal to one another, is impossible. Since then Δ has at least m unequal particular cognate forms, Δ_1 is, by Prop. III., the root of a rational irreducible equation of a degree not lower than the m^{th}; which, by Prop. VII., is impossible. Therefore k_1, k_2, etc., are rational. Hence each of the expressions in (27) is a rational function of t_1.

§37. *Cor.* Any expression of the type $k_1 + k_2 t_1 + k_3 t_1^2 +$ etc., which is such that all the unequal particular cognate forms of the generic expression under which it falls are obtained by substituting for t_1 successively the different primitive m^{th} roots of unity, while k_1, k_2, etc., remain unaltered, is a rational function of t_1. For, in the Proposition, Δ_1 or $k_1 + k_2 t_1 +$ etc. was shown to be a rational function of t_1, the conclusion being based on the circumstance that Δ_1 satisfies the condition specified.

§38. Proposition IX. If g be the sum of the roots of the equation $F(x) = 0$,

$$r_2 = \frac{1}{m}\Big(g + \Delta_1^{\frac{1}{m}} + a_1 \Delta_1^{\frac{2}{m}} + b_1 \Delta_1^{\frac{3}{m}} + \dots$$
$$+ e_1 \Delta_1^{\frac{m-2}{m}} + h_1 \Delta_1^{\frac{m-1}{m}}\Big); \qquad (29)$$

For, z being one of the whole numbers, 1, 2, ..., $m - 1$, put

$$p_z = (r_1 + t_1^z r_2 + t_1^{2z} r_3 + \text{etc.})\,(r_1 + t_1 r_2 + t_1^2 r_3 + \text{etc.})^{-z}. \quad (30)$$

Multiply the first of its factors by t_1^{-z} and the second by t_1^z. Then

$$p_z = (r_2 + t_1^z r_3 + t_1^{2z} r_4 + \text{etc.})\,(r_2 + t_1 r_3 + t_1^2 r_4 + \text{etc.})^{-z}. \quad (31)$$

Hence p_z does not alter its value when we change r_1 into r_2, r_2 into r_3, and so on. In like manner it does not alter its value when we change r_1 into r_a, r_2 into r_{a+1}, and so on. Therefore, by §33, p_z is not changed by any alterations that may be made in r_1, r_2, etc., while t_1 remains unaltered. Consequently, if p_z be a particular cognate form of P, all the unequal particular cognate forms of P are obtained by substituting for t_1 successively in p_z the different primitive m^{th} roots of unity, while r_1, r_2, etc., remain unaltered. Therefore, by Cor., Prop. VIII., p_z is a rational function of t_1. When $z = 2$, let $p_z = a_1$; when $z = 3$, let $p_z = b_1$, and so on. Then, from (26) and (30), $\Delta_2^{\frac{1}{m}} = a_1 \Delta_1^{\frac{2}{m}}$, $\Delta_3^{\frac{1}{m}} = b_1 \Delta_1^{\frac{3}{m}}$ and so on. But, from (27), since g is the sum of the roots of the equation $F(x) = 0$,

$$r_1 = \frac{1}{m}\left(g + \Delta_1^{\frac{1}{m}} + \Delta_2^{\frac{1}{m}} + \ldots + \Delta_{m-1}^{\frac{1}{m}}\right).$$

By putting $a_1 \Delta_1^{\frac{2}{m}}$ for $\Delta_2^{\frac{1}{m}}$, $b_1 \Delta_1^{\frac{3}{m}}$ for $\Delta_3^{\frac{1}{m}}$ and so on, this becomes (29). Because a_1, b_1, etc., are rational functions of t_1, while Δ_1, the root of a rational irreducible equation of the $(m-1)^{\text{th}}$ degree, is also a rational function of t_1, the coefficients a_1, b_1, etc., involve no surd that is not subordinate to $\Delta_1^{\frac{1}{m}}$.

§39. Proposition X. If the prime number m be odd, the expressions

$$\Delta_1^{\frac{1}{m}} \Delta_{m-1}^{\frac{1}{m}},\ \Delta_2^{\frac{1}{m}} \Delta_{m-2}^{\frac{1}{m}},\ \ldots,\ \Delta_{\frac{m-1}{2}}^{\frac{1}{m}} \Delta_{\frac{m+2}{m}}^{\frac{1}{m}}, \tag{32}$$

are the roots of a rational equation of the $\left(\frac{m-1}{2}\right)^{\text{th}}$ degree.

By §32, when r_1, is charged into r_z, r_2 becomes r_{z+1}, r_3 becomes r_{z+2}, and so on. Hence the terms $r_1 r_2$, $r_2 r_3$, ... $r_m r_1$, form a cycle, the sum of the terms in which may be denoted by the symbol Σ_2^1. In like manner the sum of the terms in the cycle $r_1 r_3$, $r_2 r_4$, ..., $r_m r_2$, may be written Σ_3^1. And so on. In harmony with this notation, the sum of the m terms r_1^2, r_2^2, etc., may be written Σ_1^1. Now r_1 can only be changed into one of the terms r_1, r_2, etc.; and we have seen that, when it becomes r_z, r_2 becomes r_{z+1}, and so on. Such changes leave the cycle $r_1 r_2$, $r_2 r_3$, etc., as a whole unaltered.

Therefore, by Prop. III., Σ_2^1 is the root of a simple equation, or has a rational value. In like manner each of the expressions

$$\Sigma_1^1, \Sigma_2^1, \Sigma_3^1, \ldots, \Sigma_m^2, \tag{33}$$

has a rational value. From (26), by actual multiplication,

$$\Delta_1^{\frac{1}{m}} \Delta_{m-1}^{\frac{1}{m}} = \Sigma_1^1 + (\Sigma_2^1)\, t_1 + (\Sigma_3^1)\, t_1^2 + \text{etc.}$$

But Σ_2^1, Σ_3^1, etc., are respectively identical with Σ_m^1, Σ_{m-1}^1, etc. Therefore

$$\Delta_1^{\frac{1}{m}} \Delta_{m-1}^{\frac{1}{m}} = \Sigma_1^1 + (\Sigma_2^1)(t_1 + t_1^{-1}) + (\Sigma_3^1)(t_1^2 + t_1^{-2}) + \text{etc.} \tag{34}$$

Hence, since the terms in (33) are all rational, and since the terms in (32) are respectively what $\Delta_1^{\frac{1}{m}} \Delta_{m-1}^{\frac{1}{m}}$ becomes by changing t_1 successively into the $\frac{m-1}{2}$ terms t_1, t_1^2, etc., the terms in (32) are the roots of a rational equation of the $\left(\frac{m-1}{2}\right)^{\text{th}}$ degree.

§40. For the solution of the equation $x^n - 1 = 0$, n being a prime number such that m is a prime measure of $n - 1$, it is necessary to obtain the solution of the equation of the m^{th} degree which has for one of its roots the sum of the $\frac{n-1}{m}$ terms in a cycle of primitive n^{th} roots of unity. This latter equation will be referred to as the *reducing Gaussian equation* of the m^{th} degree to the equation

$$x^n - 1 = 0.$$

§41. Proposition XI. When the equation $F(x) = 0$ is the reducing Gaussian (see §40) of the m^{th} degree to the equation $x^n - 1 = 0$, each of the $\frac{m-1}{2}$ expressions in (32) is equal to n.

Let the sum of the primitive n^{th} roots of unity forming the cycle (8), which sum has in preceding sections been indicated by the symbol C_1, be the root r_1 of the equation $F(x) = 0$. This implies, since s is the number of the terms in (8), that $ms = n - 1$. Let us reason first on the assumption that the cycle (8) is made up of pairs of reciprocal roots ω_1 and ω_1^{-1}, and so on. Then, because the cycle consists of $\frac{s}{2}$ pairs of reciprocal roots, C_1^2 or r_1^2 is the sum of

s^2 terms, each an n^{th} root of unity. Among these unity occurs s times. Let ω_1 occur h_1 times; and let ω_1^λ the second term in (8), occur h' times. Since ω_1^λ may be made the first term in the cycle (8), it must, under the new arrangement, present itself in the value of r_1^2, precisely where ω_1 previously appeared. That is to say, $h' = h_1$. In like manner each of the terms in (8) occurs exactly h_1 times in the expression for r_1^2. The cycle (9) being that which contains all the primitive n^{th} roots of unity, let us, adhering to the notation of previous sections, suppose that, when ω_1 is changed into ω_1^β, C_1 or r_1 becomes C_2 or r_2, C_2 or r_2 becomes C_3 or r_3, and so on. On the same grounds on which every term in (8) occurs the same number of times in the value of r_1^2, each term in the cycle of terms whose sum is C_2 occurs the same number of times; and so on. Therefore

$$r_1^2 = s + h_1 C_1 + h_2 C_2 + \ldots + h_m C_m.$$

$$r_2^2 = s + h_m C_1 + h_1 C_2 + \ldots + h_{m-1} C_m,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$r_m^2 = s + h_2 C_1 + h_3 C_2 + \ldots + h_1 C_m.$$

Therefore, keeping in view (11), $\Sigma_1^1 = ms - (h_1 + h_2 + \ldots + h_m)$. But $s^2 - s$ is the number of the terms in the value of r_1^2 which are primitive n^{th} roots of unity. And this must be equal to

$$s\,(h_1 + \ldots + h_m).$$

Therefore

$$h_1 + h_2 + \ldots + h_m = s - 1 \;\therefore\; \Sigma_1^1 = ms + 1 - s = n - s.$$

Again, because r_1 is made up of pairs of reciprocal roots, and because therefore unity does not occur among the s^2 terms of which $r_1 r_2$ is the sum,

$$r_1 r_2 = k_1 C_1 + k_2 C_2 + \ldots + k_m C_m,$$

$$r_2 r_3 = k_m C_1 + k_1 C_2 + \ldots + k_{m-1} C_m,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$r_m r_1 = k_2 C_1 + k_3 C_2 + \ldots + k_1 C_m;$$

where k_1, k_2, etc., are whole numbers whose sum is s. Therefore $\Sigma_2^1 = -s$. In like manner each of the terms in (33) except the first is equal to $-s$. Therefore (34) becomes

$$\Delta_1^{\frac{1}{m}}\,\Delta_{m-1}^{\frac{1}{m}} = (n - s) - s\,(t_1 + t_1^2 + \text{etc.}) = n.$$

Let us reasnn now on the assumption that the cycle (8) is not made up of pairs of reciprocal roots. It contains in that case no reciprocal roots. By the same reasoning as above we get $\Sigma_1^1 = - s$. As regards the terms in (33) after the first, one of the terms C_1, C_2, etc., say C_z, must be such that the n^{th} roots of unity of which it is the sum are reciprocals of those of which C_1 is the sum. In passing from C_1 to C_z, we change r_1 into r_z. In fact, C_1 being r_1, C_z is r_z. This being kept in view, we get, by the same reasoning as above, $\Sigma_z^1 = n - s$. But, if any of the expressions C_1, C_2, etc., except C_z be selected, say C_a, none of the roots in (8) are reciprocals of any of those of which C_a is the sum. Therefore $\Sigma_a^1 = - s$. Therefore, from (34)

$$\Delta_1^{\frac{1}{m}} \; \Delta_{m-1}^{\frac{1}{m}} = - s + (n - s)\, t_1^{z-1}$$

$$- s \left\{ (t_1 + t_1^2 + \ldots + t_1^{m-1}) - t_1^{z-1} \right\} = n.$$

In like manner every one of the expressions in (34) can be shown to have the value n.

§42. Two numerical illustrations of the law established in the preceding section may be given. The reducing Gaussian equation of the third degree to the equation $x^{19} - 1 = 0$ is $x^3 - x^2 - 6x - 7 = 0$; which gives

$$r_1 = \tfrac{1}{3}(- 1 + \Delta_1^{\frac{1}{3}} + \Delta_2^{\frac{1}{3}}),$$
$$2\Delta_1 = 19\,(7 + 3\sqrt{3}),$$
$$2\Delta_2 = 19\,(7 - 3\sqrt{3}),$$
$$\Delta_1^{\frac{1}{3}}\,\Delta_2^{\frac{1}{3}} = 19.$$

The next example is taken from Lagrange's Theory of Algebraical Equations, Note XIV., §30. The Gaussian of the fifth degree to the equation $x^{11} - 1 = 0$ is $x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1 = 0$; which gives

$$r_1 = \tfrac{1}{5}(- 1 + \Delta_1^{\frac{1}{5}} + \Delta_2^{\frac{1}{5}} + \Delta_3^{\frac{1}{5}} + \Delta_4^{\frac{1}{5}});$$
$$4\,\Delta_1 = 11\,(- 89 - 25\sqrt{5} + 5p - 45q),$$
$$4\,\Delta_2 = 11\,(- 89 + 25\sqrt{5} - 45p - 5q),$$
$$4\,\Delta_4 = 11\,(- 89 - 25\sqrt{5} - 5p + 45q),$$
$$4\,\Delta_3 = 11\,(- 89 + 25\sqrt{5} + 45p + 5q),$$
$$p = \sqrt{(- 5 - 2\sqrt{5})},$$
$$q - \sqrt{(- 5 + 2\sqrt{5})},$$
$$pq = - \sqrt{5} \;\therefore\; \Delta_1\,\Delta_4 = 11^5.$$

§43. PROPOSITION XII. To solve the Gaussian.

The path we have been following leads directly, assuming the primitive m^{th} root of unity t_1 to be known, to the solution of the reducing Gaussian equation of the m^{th} degree to the equation $x^n - 1 = 0$. For, as in §41, the roots of the Gaussian are C_1, C_2, etc. Therefore g, the sum of the roots, is -1. Therefore

$$r_1 = \frac{1}{m}(-1 + \Delta_1^{\frac{1}{m}} + \Delta_2^{\frac{1}{m}} + \ldots + \Delta_{m-1}^{\frac{1}{m}}). \qquad (35)$$

By Prop. VIII., Δ_1, Δ_2, etc., are rational functions of t_1. Therefore

$$\left.\begin{aligned} \Delta_1 &= k_1 + k_2 t_1 + k_3 t_1^2 + \ldots + k_m t_1^{m-1} \\ \Delta_2 &= k_1 + k_2 t_1^2 + k_3 t_1^4 + \ldots + k_m t_1^{2(m-1)} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Delta_{m-1} &= k_1 + k_2 t_1^{-1} + k_3 t_1^{-2} + \ldots + k_m t_1 ; \end{aligned}\right\} \qquad (36)$$

where k_1, k_2, etc., are rational. From the first of equations (26), putting C_1 for r_1, C_2 for r_2, and so on,

$$\Delta_1 = (C_1 + t_1 C_2 + \text{etc.})^m.$$

By actual involution this gives us k_1, k_2, etc., as determinate functions of C_1, C_2, etc., and therefore as known rational quantities. For instance take k_1. Being a determinate function of C_1, C_2, etc., we have

$$k_1 = q_1 + q_2 C_1 + q_3 C_2 + \ldots + q_m C_{m-1} ;$$

where q_1, q_2, etc., are known rational quantities. But, by §13, the rational coefficients $q_1 - k_1$, q_2, etc., are all equal to one another. Therefore $k_1 = q_1 - q_2$. In like manner k_2, k_3, etc., are known. Therefore, from (36), Δ_1, Δ_2, etc., are known. Therefore, from (35), r_1 is known.

§44. PROPOSITION XIII. The law established in Prop. X falls under the following more general law. The $m - 1$ expressions in each of the groups

$$\left.\begin{aligned} &(\Delta_1^{\frac{1}{m}} \Delta_{m-1}^{\frac{1}{m}}, \ \Delta_2^{\frac{1}{m}} \Delta_{m-2}^{\frac{1}{m}}, \ldots, \Delta_{m-1}^{\frac{1}{m}} \Delta_1^{\frac{1}{m}},) \\ &(\Delta_1^{\frac{2}{m}} \Delta_{m-2}^{\frac{1}{m}}, \ \Delta_2^{\frac{2}{m}} \Delta_{m-4}^{\frac{1}{m}}, \ldots, \Delta_{m-1}^{\frac{2}{m}} \Delta_2^{\frac{1}{m}},) \\ &(\Delta_1^{\frac{3}{m}} \Delta_{m-3}^{\frac{1}{m}}, \ \Delta_2^{\frac{3}{m}} \Delta_{m-6}^{\frac{1}{m}}, \ldots, \Delta_{m-1}^{\frac{3}{m}} \Delta_3^{\frac{1}{m}} ;) \end{aligned}\right\} \qquad (37)$$

and so on, are the roots of a rational equation of the $(m - 1)^{\text{th}}$ degree.

The $m-1$ terms in the first of the groups (37) are the $\frac{m-1}{2}$ terms in (32) each taken twice. Therefore, by Prop. X., the law enunciated in the present Proposition is established so far as this groupe is concerned. The general proof is as follows. By (30) in §38, taken in connection with (26), $p_{m-z}\, \Delta_1^{\frac{m-z}{m}} = \Delta_{m-z}^{\frac{1}{m}}$. Therefore $\Delta_1^{\frac{z}{m}}\, \Delta_{m-z}^{\frac{1}{m}} = p_{m-z}\, \Delta_1$. But, by §38, p_{m-z} is a rational function of t_1; and, by Prop. VIII., Δ_1 is a rational function of t_1. Therefore $\Delta_1^{\frac{z}{m}}\, \Delta_{m-z}^{\frac{1}{m}}$ is a rational function of t_1. Also from the manner in which p_{m-z} is formed, when t_1 in $p_{m-2}\, \Delta_1$ is changed sucessively into $t_1\, t_1^2, \ldots, t_1^{m-1}$, the expression $\Delta_1^{\frac{z}{m}}\, \Delta_{m-z}^{\frac{1}{m}}$ is changed successively into the $m-1$ terms of that one of the groups (37) whose first term is $\Delta_1^{\frac{z}{m}}\, \Delta_{m-z}^{\frac{1}{m}}$. Therefore the terms in that group are the roots of a rational equation.

§45. *Cor.* The law established in the Proposition may be brought under a yet wider generalization. The expression

$$\Delta_1^{\frac{a}{m}}\, \Delta_2^{\frac{b}{m}}\, \Delta_3^{\frac{c}{m}} \ldots \Delta_{m-1}^{\frac{s}{m}} \qquad (38)$$

is the root of a rational equation of the $(m-1)^{\text{th}}$ degree, if

$$a + 2b + 3c + \ldots + (m-1)\,s = Wm,$$

W being a whole number. For, by (30) in connection with (26), $\Delta_2^{\frac{1}{m}} = p_2\, \Delta_1^{\frac{2}{m}}$, $\Delta_3^{\frac{1}{m}} = p_3\, \Delta_3^{\frac{1}{m}}$, and so on. Therefore (38) has the value

$$(p_2^b\, p_3^c \ldots)\, \Delta_1^{\frac{a+2b+3c+\ldots+(m-1)s}{m}}, \text{ or } (p_2^b\, p_3^c \ldots)\, \Delta_1^W.$$

This is a rational function of t_1, and therefore the root of a rational equation of the $(m-1)^{\text{th}}$ degree.

THE EQUATION $F(x) = 0$ OF THE SECOND CLASS.

§46. We now suppose that the simplified root r_1 of the rational irreducible equation $F(x) = 0$ of the m^{th} degree, m prime, involves, when modified according to §21, a principal surd not a root of unity. It must not be forgotten that, when we thus speak of roots of unity, we mean, according to §1, roots which have prime numbers for the denominators of their indices. In this case conclusions can be established similar to those reached in the case that has been considered. The root r_1 is still of the form (29). The equation $F(x) = 0$ has still an auxiliary of the $(m-1)^{\text{th}}$ degree, whose roots are the m^{th} powers of the expressions

$$\Delta_1^{\frac{1}{m}},\ a_1\,\Delta_1^{\frac{2}{m}},\ b_1\,\Delta_1^{\frac{3}{m}},\ \ldots\ldots,\ e_1\,\Delta_1^{\frac{m-2}{m}},\ h_1\,\Delta_1^{\frac{m-1}{m}}, \tag{39}$$

though the auxiliary here is not necessarily irreducible. Also, substituting the expressions in (39) for $\Delta_1^{\frac{1}{m}}$ $\Delta_2^{\frac{1}{m}}$, etc., in (37), the law of Proposition XIII. still holds, together with corollary in §45.

§47. By Cor. Prop. VI., the denominator of the index of a surd of the highest rank in r_1 is m. Let $\Delta_1^{\frac{1}{m}}$ be such a surd. By §21, the coefficients of the different powers of $\Delta_1^{\frac{1}{m}}$ in r_1 cannot be all zero. We may take the coefficient of the first power to be distinct from zero and to be $\frac{1}{m}$ for, if it were $\frac{k_1}{m}$, we might substitute $s^{\frac{1}{m}}$ for $k_1\Delta_1^{\frac{1}{m}}$, and so eliminate $\Delta_1^{\frac{1}{m}}$ from r_1, introducing in its room the new surd $s^{\frac{1}{m}}$ with $\frac{1}{m}$ for the coefficient of its first power. We may then put

$$r_1 = \frac{1}{m}\left(g + \Delta_1^{\frac{1}{m}} + a_1\,\Delta_1^{\frac{3}{m}} + \ldots\ldots + e_1\,\Delta_1^{\frac{m-2}{m}} + h_1\,\Delta_1^{\frac{m-1}{m}}\right); \tag{40}$$

where g, a_1, etc., are clear of $\Delta_1^{\frac{1}{m}}$. When $\Delta_1^{\frac{1}{m}}$ is changed successively into $\Delta_1^{\frac{1}{m}}$, $t_1^{-1}\,\Delta_1^{\frac{1}{m}}$, $t_1^{-2}\,\Delta_1^{\frac{1}{m}}$, etc., let

$$r_1,\ r_2,\ \ldots\ldots\ r_m, \tag{41}$$

be respectively what r_1 becomes, t_1 being a primitive m^{th} root of unity. By Prop. VI., the terms in (41) are the roots of the equation $F(x) = 0$. Taking r_n, any one of the particular cognate forms of R, let $\Delta_n^{\frac{1}{m}}$, a_n, etc., be respectively what $\Delta_1^{\frac{1}{m}}$, a_1, etc., become in passing from r_1 to r_n; and when $\Delta_n^{\frac{1}{m}}$ is changed successively into the different m^{th} roots of the determinate base Δ_n, let r_n become

$$r_n, \; r_n', \; r_n'', \; \ldots\ldots, \; r_n^{(m-1)}. \qquad (42)$$

By Prop. II., the terms in (42) are roots of the equation $F(x) = 0$; and, by §23, they are all unequal. Therefore they are identical, in some order, with the terms in (41). Also, the sum of the terms in (41) is g. Therefore g is rational.

§48. PROPOSITION XIV. In r_1, as expressed in (40), $\Delta_1^{\frac{1}{m}}$ is the only principal (see §2) surd.

Suppose, if possible, that there is in r_1 a principal surd $z_1^{\frac{1}{c}}$ distinct from $\Delta_1^{\frac{1}{m}}$. And first, let $z_1^{\frac{1}{c}}$ be not a root of unity. (It will be kept in view that when, in such a case, we speak of roots of unity, the denominators of their indices are understood, according §1, to be prime numbers.) When $z_1^{\frac{1}{c}}$ is changed into $z_2^{\frac{1}{c}}$, one of the other c^{th} roots of z_1, let r_1, a_1, etc., become respectively r_1', a_1', etc. Then

$$mr_1', = g + \Delta_1^{\frac{1}{m}} + a_1' \Delta_1^{\frac{2}{m}} + \text{etc} \qquad (43)$$

By Prop. II., r_1' is equal to a term in (41), say to r_n. And, by §48, putting t_{n-1} for t_1^{1-n},

$$mr_n = g + t_{n-1} \Delta_1^{\frac{1}{m}} + t_{n-1}^2 a_1 \Delta_1^{\frac{2}{m}} + \text{etc.} \qquad (44)$$

Therefore,

$$\Delta_1^{\frac{1}{m}} (1 - t_{n-1}) + \Delta_1^{\frac{2}{m}} (a'_1 - a_1 t_{n-1}^2) + \text{etc.} = 0. \qquad (45)$$

This equation involves no surds except those found in the simplified expression r_1, together with the primitive m^{th} root of unity. Therefore the expression on the left of (45) is in a simple state. Therefore, by §8, the coefficients of the different powers of $\Delta_1^{\frac{1}{m}}$ are separately zero. Therefore $t_{n-1} = 1$, $a_1' = a_1$, $b_1' = b_1$, and so on. But, as was shown in Prop. V., $z_1^{\frac{1}{c}}$ being a principal surd not a root of unity in the simplified expression a_1, a_1 cannot be equal to a_1' unless $z_1^{\frac{1}{c}}$ can be eliminated from a_1 without the introduction of any new surd. In like manner b_1 cannot be equal to b_1' unless $z_1^{\frac{1}{c}}$ can be eliminated from b_1. And so on. Therefore, because $a_1 = a_1'$, and $b_1 = b_1'$, and so on, $z_1^{\frac{1}{c}}$ admits of being eliminated from r_1 without the introduction of any new surd, which, by §21, is impossible. Next, let $z_1^{\frac{1}{c}}$ be a root (see §1) of unity, which may be otherwise written θ_1. Let the different primitive c^{th} roots of unity be θ_1, θ_2, etc.; and, when θ_1 is changed successively into θ_1, θ_2, etc., let r_1 become successively r_1, r_1', etc. Suppose if possible that the $c - 1$ terms r_1, r_1', etc., are all equal. Since $z_1^{\frac{1}{c}}$ is a principal surd in r_1, we may put $r_1 = h\theta_1^{c-1} + k\theta_1^{c-2} + \ldots + l$; where h, k, etc., are clear of θ_1. Therefore $(c - 1)\, r_1 = cl - (h + k + \text{etc.})$ Thus $z_1^{\frac{1}{c}}$ may be eliminated from r_1 without the introduction of any new surd; which by §21 is impossible. Since then the terms r_1, r_1', etc., are not all equal, let r_1 and r_1' be unequal. Then r_1' is equal to a term in (41) distinct from r_1, say to r_n. Expressing mr_1' and mr_n as in (43) and (44), we deduce (45); which, as above, is impossible.

§49. Proposition XV. Taking r_1, r_n, $\Delta_n^{\frac{1}{m}}$, etc., as in §47, an equation

$$t\, \Delta_n^{\frac{1}{m}} = p\, \Delta_1^{\frac{c}{m}} \tag{46}$$

can be formed; where t is an m^{th} root of unity, and c is a whole number less than m but not zero, and p involves only surds subordinate (see §3) to $\Delta_1^{\frac{1}{m}}$ or $\Delta_n^{\frac{1}{m}}$

By §47, one of the terms in (42) is equal to r_1. For our argument it is immaterial which be selected. Let $r_n = r_1$. Therefore

$$\begin{aligned}(h_n \Delta_n^{\frac{m-1}{m}} + e_n \Delta_n^{\frac{m-2}{m}} + \dots + \Delta_n^{\frac{1}{m}})\\ - (h_1 \Delta_1^{\frac{m-1}{m}} + e_1 \Delta_1^{\frac{m-2}{m}} + \dots + \Delta_1^{\frac{1}{m}}) = 0. \qquad (47)\end{aligned}$$

The coefficients of the different powers of $\Delta_n^{\frac{1}{m}}$ here are not all zero, for the coefficient of the first power is unity. Therefore by §5, an equation $t\Delta_n^{\frac{1}{m}} = l_1$ subsists, t being an m^{th} root of unity, and l_1 involving only surds exclusive of $\Delta_n^{\frac{1}{m}}$ that occur in (47). By Prop. XIV., $\Delta_1^{\frac{1}{m}}$ is a surd of a higher rank (see §3) than any surd in (47) except $\Delta_n^{\frac{1}{m}}$. Therefore we may put

$$l_1 = d + d_1 \Delta_1^{\frac{1}{m}} + d_2 \Delta_1^{\frac{2}{m}} + \dots + d_{m-1} \Delta_1^{\frac{m-1}{m}};$$

where d, d_1, etc., involve only surds lower in rank than $\Delta_1^{\frac{1}{m}}$. Then

$$\begin{aligned}\Delta_n = l_1^m &= (d + d_1 \Delta_1^{\frac{1}{m}} + \text{etc.})^m\\ &= d' + d_1' \Delta_1^{\frac{1}{m}} + d_2' \Delta_1^{\frac{2}{m}} + \text{etc.};\end{aligned}$$

where d', d_1', etc., involve only surds lower in rank than $\Delta_1^{\frac{1}{m}}$. By §8, since $\Delta_1^{\frac{1}{m}}$ is a surd in the simplified expressions r_1, the coefficients $d' - \Delta_n$, d_1', etc., in the equation

$$(d' - \Delta_n) + d_1' \Delta_1^{\frac{1}{m}} + d_2' \Delta_1^{\frac{1}{m}} + \text{etc.} = 0 \qquad (48)$$

are separately zero. Therefore $(d + d_1 \Delta_1^{\frac{1}{m}} + \text{etc.})^m = d'$. And, t_1 being a primitive m^{th} root of unity,

$$(d + d_1 t_1 \Delta_1^{\frac{1}{m}} + \text{etc.})^m = d' + d' t_1 \Delta_1^{\frac{1}{m}} + \text{etc.} = d'.$$

Therefore,

$$(d + d_1 t_1 \Delta_1^{\frac{1}{m}} + \text{etc.}) = t_1^a (d + d_1 \Delta_1^{\frac{1}{m}} + d_2 \Delta_1^{\frac{2}{m}} + \text{etc.}),$$

t_1^a being one of the m^{th} roots of unity. In the same way in which the coefficients of the different powers of $\Delta_1^{\frac{1}{m}}$ in (48) are separately zero, each of the expressions $d\,(1 - t_1^a)$, $d_1\,(t_1 - t_1^a)$, etc., must be zero. But not more than one of the $m - 1$ factors, $t_1 - t_1^a$, $t_1^2 - t_1^a$, etc., can be zero. Therefore not more than one of the $m - 1$ terms d_1, d_2, etc., is distinct from zero. Suppose if possible that all these terms are zero. Then $t\,\Delta_n^{\frac{1}{m}} = d$. Therefore the different powers of $\Delta_n^{\frac{1}{m}}$ can be expressed in terms of the surds involved in d and of the m^{th} root of unity. Substitute for $\Delta_n^{\frac{1}{m}}$, $\Delta_n^{\frac{2}{m}}$ etc., in (47), their values thus obtained. Then (47) becomes

$$Q - (h_1 \Delta_1^{\frac{m-1}{m}} + \ldots + \Delta_1^{\frac{1}{m}}) = 0; \qquad (49)$$

where Q involves no surds, distinct from the primitive m^{th} root of unity, that are not lower in rank than $\Delta_1^{\frac{1}{m}}$; which, because the coefficient of the first power of $\Delta_1^{\frac{1}{m}}$ in (49) is not zero, is, by §8, impossible. Hence there must be one, while at the same there can be only one of the $m - 1$ terms, d_1, d_2, etc., distinct from zero. Let

d_c be the term that is not zero. Then $t_1^c - t_1^a = 0$. Therefore $1 - t_1^a$ is not zero. Therefore $d = 0$. Therefore, putting p for d_c,

$$t\,\Delta_n^{\frac{1}{m}} = p\,\Delta_1^{\frac{c}{m}}.$$

§50. *Cor.* By the proposition, values of the different powers of $\Delta_n^{\frac{1}{m}}$ can be obtained as follows;

$$t\Delta_n^{\frac{1}{m}} = p\,\Delta_1^{\frac{c}{m}},\ t^2\,\Delta_n^{\frac{2}{m}} = q\,\Delta_1^{\frac{s}{m}},\ t^3\,\Delta_n^{\frac{3}{m}} = k\,\Delta_1^{\frac{z}{m}},\ \text{etc.}; \quad (50)$$

where p, q, etc., involve only surds that occur in Δ_1 or Δ_n; and c, s, z, etc., are whole numbers in the series $1, 2, \ldots, m-1$. No two of the numbers c, s, etc., can be the same; for they are the products, with multiples of the prime number m left out, of the terms in the series $1, 2, \ldots, m-1$, by the whole number c which is less than m. Therefore the series c, s, z, etc., is the series $1, 2, \ldots, m-1$, in a certain order.

§51. Proposition XVI. If r_n be one of the particular cognate forms of R, the expressions

$$t\Delta_n^{\frac{1}{m}},\ t^2 a_n\,\Delta_n^{\frac{2}{m}},\ \ldots,\ t^{m-2} e_n\,\Delta_n^{\frac{m-2}{m}},\ t^{m-1} h_n\,\Delta_n^{\frac{m-1}{m}}, \quad (51)$$

are severally equal, in some order, to those in (39), t being one of the m^{th} roots of unity.

By §47, one of the terms in (42) is equal to r_1. For our argument it is immaterial which be chosen. Let $r_n = r_1$. By Cor. Prop. XV., the equations (50) subsist. Substitute in (47) the values of the different powers of $\Delta_n^{\frac{1}{m}}$ so obtained. Then

$$\left(t^{-1} p\,\Delta_1^{\frac{c}{m}} + t^{-2} q a_n\,\Delta_1^{\frac{s}{m}} + \text{etc.}\right)$$
$$-\left(\Delta_1^{\frac{1}{m}} + a_1\,\Delta_1^{\frac{2}{m}} + \text{etc.}\right) = 0. \quad (52)$$

By Cor. Prop. XV., the series $\Delta_1^{\frac{c}{m}}$, $\Delta_1^{\frac{s}{m}}$, etc., is identical, in some order, with the series $\Delta_1^{\frac{1}{m}}$, $\Delta_1^{\frac{2}{m}}$, etc. Also, by §8, since $\Delta_1^{\frac{1}{m}}$ is a surd occurring in the simplified expression r_1, and since besides $\Delta_1^{\frac{1}{m}}$ there are in (52) no surds, distinct from the primitive m^{th} root of unity, that are not lower in rank than $\Delta_1^{\frac{1}{m}}$, if the equation (52) were arranged according to the powers of $\Delta_1^{\frac{1}{m}}$ lower than the m^{th}, the coefficients of the different powers of $\Delta_1^{\frac{1}{m}}$ would be separately zero. Hence $\Delta_1^{\frac{1}{m}}$ is equal to that one of the expressions,

$$t^{-1} p \Delta_1^{\frac{c}{m}}, \; t^{-2} q a_n \Delta_1^{\frac{s}{m}}, \text{ etc.} \tag{53}$$

in which $\Delta_1^{\frac{1}{m}}$ is a factor. In like manner $a_1 \Delta_1^{\frac{2}{m}}$ is equal to that one of the expressions (53) in which $\Delta_1^{\frac{2}{m}}$ is a factor. And so on. Therefore the terms $\Delta_1^{\frac{1}{m}}$, $a_1 \Delta_1^{\frac{2}{m}}$, etc., forming the series (39), are severally equal, in some order, to the terms in (53), which are those forming the series (51.)

§52. PROPOSITION XVII. The equation $F(x) = 0$ has a rational *auxiliary* (Compare Prop. VII.) equation $\varphi(x) = 0$, whose roots are the m^{th} powers of the terms in (39).

Let the unequal particular cognate forms of the generic expression Δ under which the simplified expression Δ_1 falls be

$$\Delta_1, \; \Delta_2, \; \ldots, \; \Delta_c. \tag{54}$$

By Prop. XVI., there is a value t of the m^{th} root of unity for which the expressions

$$t \Delta_2^{\frac{1}{m}}, \; t^2 a_2 \Delta_2^{\frac{2}{m}}, \; \ldots, \; t^{m-2} e_2 \Delta_2^{\frac{m-2}{m}}, \; t^{m-1} h_2 \Delta_2^{\frac{m-1}{m}} \tag{55}$$

are severally equal, in some order, to those in (39). Therefore Δ_2 is equal to one of the terms

$$\Delta_1, \; a_1^m \Delta_1^2, \ldots, e_1^m \Delta_1^{m-2}, \; h_1^m \Delta_1^{m-1}. \tag{56}$$

In like manner each of the terms in (54) is equal to a term in (56). And, because the terms in (54) are unequal, they are severally equal to different terms in (56). By Prop. III., the terms in (54) are the roots of a rational irreducible equation, say $\psi_1(x) = 0$. Rejecting from the series (56) the roots of the equation $\psi_1(x) = 0$, certain of the remaining terms must in the same way be the roots of a rational irreducible equation $\psi_2(x) = 0$. And so on. Ultimately, if $\varphi(x)$ be the continued product of the expressions $\psi_1(x)$, $\psi_2(x)$, etc., the terms in (56) are the roots of the rational equation $\varphi(x) = 0$.

§53. The equations $\psi_1(x) = 0$, $\psi_2(x) = 0$, etc., formed by means of the expressions $\psi_1(x)$, $\psi_2(x)$, etc., may be said to be *sub-auxiliary* to the equation $F(x) = 0$. It will be observed that the sub-auxiliaries are all irreducible.

§54. Proposition XVIII. In passing from r_1 to r_n, while Δ_1 becomes Δ_n, the expressions a_1, b_1, which, by Prop. XIV., involve only surds occurring in Δ_1, must severally receive determinate values, a_n, b_n, etc. In other words, a_1 being a particular cognate form of A, there cannot, for the same value of Δ_n, be two particular cognate forms of A, as a_n and a_N, unequal to one another. And so in the case of b_1, e_1, etc.

For, just as each of the terms in (42) is equal to a term in (41), there are primitive m^{th} roots of unity τ and T such that the expressions

$$\tau \Delta_n^{\frac{1}{m}} + \tau^2 a_n \Delta_n^{\frac{2}{m}} + \text{etc.}, \quad T \Delta_N^{\frac{1}{m}} + T^2 a_N \Delta_N^{\frac{1}{m}} + \text{etc.},$$

are equal to one another. Therefore, if $\Delta_N = \Delta_n$, in which case, by assigning suitable values to τ and T, $\Delta_N^{\frac{1}{m}}$ may be taken to be equal to $\Delta_n^{\frac{1}{m}}$,

$$\Delta_n^{\frac{1}{m}}(\tau - T) + \Delta_n^{\frac{2}{m}}(a_n \tau^2 - a_N T^2) + \text{etc.} = 0. \qquad (57)$$

Suppose if possible that the coefficients of the different powers of $\Delta_1^{\frac{1}{m}}$ in (57) are not all zero. Then, by §5, $t\,\Delta_n^{\frac{1}{m}} = l_1$; t being an m^{th} root of unity; and l_1 involving only surds of lower ranks than $\Delta_1^{\frac{1}{m}}$. Hence, by Prop. XV. and Cor. Prop. XV, $\Delta_1^{\frac{1}{m}}$ is a rational function of surds of lower ranks than $\Delta_1^{\frac{1}{m}}$ and of the primitive m^{th} root of unity; which, by the definition in §6, is impossible. Since then the coefficients of the different powers of $\Delta_n^{\frac{1}{m}}$ in (57) are separately zero, $\tau = T$, $a_n \tau^2 = a_N T^2$, therefore $a_n = a_N$.

§55. PROPOSITION XIX. Let the terms in (39) be written respectively

$$\Delta_1^{\frac{1}{m}}, \delta_2^{\frac{1}{m}}, \delta_3^{\frac{1}{m}}, \ldots, \delta_{m-1}^{\frac{1}{m}}. \tag{58}$$

The symbols Δ_1, δ_2, δ_3, etc., are employed instead of Δ_1, Δ_2, Δ_3, etc., because this latter notation might suggest, what is not necessarily true, that the terms in (56) are all of them particular cognate forms of the generic expression under which Δ_1 falls. Then (compare Prop. XIII.) the $m - 1$ expressions in each of the groups

$$\left.\begin{array}{l} (\Delta_1^{\frac{1}{m}} \delta_{m-1}^{\frac{1}{m}}, \delta_2^{\frac{1}{m}} \delta_{m-2}^{\frac{1}{m}}, \delta_3^{\frac{1}{m}} \delta_{m-3}^{\frac{1}{m}}, \ldots, \delta_{m-1}^{\frac{1}{m}} \Delta_1^{\frac{1}{m}},) \\ (\Delta_1^{\frac{2}{m}} \delta_{m-2}^{\frac{1}{m}}, \delta_2^{\frac{2}{m}} \delta_{m-4}^{\frac{1}{m}}, \delta_3^{\frac{2}{m}} \delta_{m-6}^{\frac{1}{m}}, \ldots, \delta_{m-1}^{\frac{2}{m}} \delta_2^{\frac{1}{m}},) \\ (\Delta_1^{\frac{3}{m}} \delta_{m-3}, \delta_2^{\frac{3}{m}} \delta_{m-6}^{\frac{1}{m}}, \delta_3^{\frac{3}{m}} \delta_{m-9}^{\frac{1}{m}}, \ldots, \delta_{m-1}^{\frac{3}{m}} \delta_3^{\frac{1}{m}},) \end{array}\right\} \tag{59}$$

and so on, are the roots of a rational equation of the $(m - 1)^{th}$ degree. Also (compare Prop. X.) the first $\frac{m-1}{2}$ terms in the first of the groups (59) are the roots of a rational equation of the $\left(\frac{m-1}{2}\right)^{th}$ degree.

In the enunciation of the proposition the remark is made that the series (54) is not necessarily identical with the series

$$\Delta_1, \delta_2, \delta_3, \ldots, \delta_{m-1}.$$

The former consists of the unequal particular cognate forms of Δ; the latter consists of the roots of the auxiliary equation $\varphi(x) = 0$. These two series are identical only when the auxiliary is irreducible. To prove the first part of the proposition, take the terms forming the second of the groups (59). Because $\delta_{m-2}^{\frac{1}{m}}$ represents $e_1 \Delta_1^{\frac{m-2}{m}}$,

$$e_1 \Delta_1 = \Delta_1^{\frac{2}{m}} \delta_{m-}^{\frac{1}{m}} .$$

Let E be the generic symbol under which the simplified expression e_1 falls. By Prop. XVIII., when Δ_1 is changed successively into the c terms in (54), e_1 receives successively the determinate values $e_1, e_2, \ldots, e_c$; and therefore $e_1 \Delta_1$ receives successively the determinate values

$$e_1 \Delta_1 , \ e_2 \Delta_2 , \ \ldots, \ e_c \Delta_c . \qquad (60)$$

There is therefore no particular cognate form of $E\Delta$ that is not equal to a term in (60). By Prop. XVI. there is a value of the m^{th} root of unity t for which the terms in (55) are severally equal, in some order, to those in (39). Let the term in (39) to which $t \Delta_2^{\frac{1}{m}}$ is equal be $q_1 \Delta_1^{\frac{n}{m}}$ Then, applying the principle of Cor. Prop. XV., as in Prop. XVI., it follows that the term in (39) to which $t^{m-2} e_2 \Delta_2^{\frac{m-2}{m}}$ in (55) is equal is $k_1 \Delta_1^{\frac{M-2n}{m}}$, M being a multiple of m, and $M - 2n$ being less than m. Therefore $e_2 \Delta_2$ is equal to $q_1^2 k_1 \Delta_1^{\frac{M}{m}}$, which is the product of two of the terms in (39) occuring respectively at equal distances from opposite extremities of the series. In other words, $e_2 \Delta_2$ is equal to an expression $\delta_m^{\frac{2}{m}} \delta_{m-2n}^{\frac{1}{m}}$ in the second of the groups (59). In like manner every term in (60) is equal to an expression in the second of the groups (59). Let the unequal terms in (60) be

$$e_1 \Delta_1 , \text{ etc.} \qquad (61)$$

Then, by Prop. III., the terms in (61) are the roots of a rational irreducible equation, say $f_1(x) = 0$. Rejecting these, which are distinct terms in the second of the groups (59), it can in like manner be shown that certain other terms in that group are the roots of a rational irreducible equation, say $f_2(x) = 0$. And so on. Ultimately, if $f(x)$ be the continued product of the expressions $f_1(x)$, $f_2(x)$, etc., the terms forming the second of the groups (59) are the roots of a rational equation of the $(m - 1)^{\text{th}}$ degree. The proof applies substantially to each of the other groups. To prove the second part, it is only necessary to observe that, in the first of the groups (59), the last term is identical with the first, the last but one with the second, and so on.

§56. *Cor.* 1. The reasoning in the proposition proceeds on the assumption that the prime number m is odd. Should m be even, the series Δ_1, δ_1, etc., is reduced to its first term. The law may be considered even then to hold in the following form. The product $\Delta_1^{\frac{1}{m}} \Delta_1^{\frac{1}{m}}$ is the root of a rational equation of the $(m-1)^{\text{th}}$ degree, or is rational. For this product is Δ_1, which, by Prop. XVII., is the root of an equation of the $(m-1)^{\text{th}}$ degree.

§56. *Cor.* 2. I merely notice, without farther proof, that the generalization in §45 in the case when the equation $F(x) = 0$ is of the first (see §30) class holds in the present case likewise.

Analysis of Solvable Equations of the Fifth Degree.

§58. Let the solvable irreducible equation of the m^{th} degree, which we have been considering, be of the fifth degree. Then, by Prop. IX. and §47, whether the equation belongs to the first or to the second of the two classes that have been distinguished, assuming the sum of the roots g to be zero,

$$r_1 = \tfrac{1}{5}\left(\Delta_1^{\frac{1}{5}} + a_1 \Delta_1^{\frac{2}{5}} + e_1 \Delta_1^{\frac{3}{5}} + h_1 \Delta_1^{\frac{4}{5}}\right), \qquad (62)$$

though, when the equation is of the first class, the root, as thus presented, is not in a simple state.

§59. Proposition XX. If the auxiliary biquadratic has a rational root Δ_1 not zero, all the roots of the auxiliary biquadratic are rational.

Because Δ_1 is rational, the auxiliary biquadratic $\varphi(x) = 0$ is not irreducible. Therefore, by Prop. VII., the equation $F(x) = 0$ is of the second (see §30) class. Therefore, by Prop. XIV., $\Delta_1^{\frac{1}{5}}$ is the only principal surd in r_1. Consequently, because Δ_1 is rational, a_1, e_1 and h_1 are rational. Therefore Δ_1, $a_1^5 \Delta_1^2, e_1^5 \Delta_1^3$, $h_1^5 \Delta_1^4$, which are the roots of the auxiliary biquadratic, are rational.

§60. Proposition XXI. If the auxiliary biquadratic has a quadratic sub-auxiliary $\psi_1(x) = 0$ with the roots Δ_1 and Δ_2, then $\Delta_2 = h_1^5 \Delta_1^4$, and $\Delta_1 = h_2^5 \Delta_2^4$; and $h_1 \Delta_1$ is rational.

As in §52, t being a certain fifth root of unity, each term in (55) is equal to a term in (39). The first term in (55) cannot be equal to the first in (39), for this would make $\Delta_2 = \Delta_1$. Suppose if possible that the first in (55) is equal to the second in (39). Then, by equations (50), applied as in Prop. XVI.,

$$\left.\begin{array}{ll} t\,\Delta_2^{\frac{1}{5}} = a_1\,\Delta_1^{\frac{2}{5}}, & t^2\,a_2\,\Delta_2^{\frac{2}{5}} = h_1\,\Delta_1^{\frac{4}{5}}, \\ t^3\,e_2\,\Delta_2^{\frac{3}{5}} = \Delta_1^{\frac{1}{5}}, & t^4\,h_2\,\Delta_2^{\frac{4}{5}} = e_1\,\Delta_1^{\frac{3}{5}}, \\ \text{therefore } \Delta_2 = a_1^5\,\Delta_1^2, & a_2^5\,\Delta_2^2 = h_1^5\,\Delta_1^4, \\ e_2^5\,\Delta_2^3 = \Delta_1, & h_2^5\,\Delta_2^4 = e_1^5\,\Delta_1^3. \end{array}\right\} \qquad (63)$$

Now $a_1^5\,\Delta_1^2$, being equal to Δ_2, is a root of the equation $\phi_1(x) = 0$. And $a_1^5\,\Delta_1^2$, involving only surds that occur in r_1, is in a simple state. Therefore, by Prop. III., $a_2^5\,\Delta_2^2$ is a root of the equation $\phi_1(x) = 0$. Therefore $h_1^5\,\Delta_1^4$, and therefore also $h_2^5\,\Delta_2^4$ or $e_1^5\,\Delta_1^3$, are roots of that equation. Hence all the terms

$$\Delta_1\,,\ a_1^5\,\Delta_1^2,\ e_1^5\,\Delta_1^3,\ h_1^5\,\Delta_1^4, \qquad (64)$$

are roots of the equation $\phi_1(x) = 0$. But a_1, e_1, h_1, are all distinct from zero; for, by (63), if one of them was zero, all would be zero, and therefore $\Delta_1^{\frac{1}{5}}$ would be zero; which by §6, is impossible. From this it follows that no two terms in (64) are equal to one another; for taking $a_1^5\,\Delta_1^2$ and $e_1^5\,\Delta_1^3$, if these were equal, we should have $e_1\,t'\,\Delta_1^{\frac{1}{5}} = a_1$, t' being a fifth root of unity; which; which by §8, is impossible. This gives the equation $\phi_1(x) = 0$ four unequal roots; which, because it is of the second degree, is impossible. Therefore the first term in (55) is not equal to the second in (39). In the same way it can be shown that it is not equal to the third. Therefore it must be equal to the fourth. In like manner the first in (39) is equal to the fourth in (55). Because then $t\,\Delta_2^{\frac{1}{5}} = h_1\,\Delta_1^{\frac{4}{5}}$, and $\Delta_1^{\frac{1}{5}} = t^4\,h_2\,\Delta_2^{\frac{4}{5}}$, $h_2\,\Delta_2 = h_1\,\Delta_1$. But, just as it was proved in §56 that, the roots of the sub-auxiliary $\phi_1(x) = 0$ being the c terms Δ_1, Δ_2, etc., there is no particular cognate form of $E\Delta$ that is not a term in the series $e_1\,\Delta_1$, $e_2\,\Delta_2$, ..., $e_c\,\Delta_c$, it follows that, if h_1 be a particular cognate form of H, there is no particular cognate form of $H\Delta$ that is not equal to one of the terms $h_1\,\Delta_1$ and $h_2\,\Delta_2$. Hence, since $h_1\,\Delta_1 = h_2\,\Delta_2$, $H\Delta$ has no particular cognate form different in value from $h_1\,\Delta_1$. Therefore, by Prop. III., $h_1\,\Delta_1$ is rational.

§61. PROPOSITION XXII. The auxiliary biquadratic $\varphi(x) = 0$ either has all its roots rational, or has a sub-auxiliary (see §53) of the second degree, or is irreducible.

It will be kept in view that the sub-auxiliaries are, by the manner of their formation, irreducible. First, let the series (54), containing the roots of the sub-auxiliary $\psi_1(x) = 0$ consist of a single term Δ_1. Then, by Prop. III., Δ_1 is rational. Therefore, by Prop. XX., all the roots of the auxiliary are rational. Next, let the series (54) consist of the two terms Δ_1 and Δ_2. By this very hypothesis, the auxiliary biquadratic has a quadratic sub-auxiliary. Lastly, let the series (54) contain more than two terms. Then it has the three terms Δ_1, Δ_2, Δ_3. We have shown that these must be severally equal to terms in (64). Neither Δ_2 nor Δ_3 is equal to Δ_1. They cannot both be equal to $h_1^5 \Delta_1^4$. Therefore one of them is equal to one of the terms $a_1^5 \Delta_1^2$, $e_1^5 \Delta_1^3$. But in §60 it appeared that, if Δ_2 be equal either to $a_1^5 \Delta_1^2$ or to $e_1^5 \Delta_1^3$, all the terms in (64) are roots of the irreducible equation of which Δ_1 is a root. The same thing holds regarding Δ_3. Therefore, when the series (54) contains more than two terms, the irreducible equation which has Δ_1 for one of its roots has the four unequal terms in (64) for roots; that is to say, the auxiliary biquadratic is irreducible.

§62. Let $5u_1 = \Delta_1^{\frac{1}{5}}$, $5u_2 = a_1 \Delta_1^{\frac{2}{5}}$, $5u_3 = e_1 \Delta_1^{\frac{3}{5}}$, $5u_4 = h_1 \Delta_1^{\frac{4}{5}}$: and, n being any whole number, let S_n denote the sum of the n^{th} powers of the roots of the equation $F(x) = 0$. Then

$$S_1 = 0;\ S_2 = 10(u_1 u_4 + u_2 u_3);\ S_3 = 15\left\{\Sigma(u_1 u_2^2)\right\};$$

$$S_4 = 20\left\{\Sigma(u_1^3 u_2)\right\} + 30(u_1^2 u_4^2 + u_2^2 u_3^2) + 120\, u_1 u_2 u_3 u_4;$$

$$S_5 = 5\left\{\Sigma(u_1^5)\right\} + 100\left\{\Sigma(u_1^3 u_3 u_4)\right\} + 150\left\{\Sigma(u_1 u_3^2 u_4^2)\right\};$$

where such an expression as $\Sigma(u_1 u_2^2)$ means the sum of all such terms as $u_1 u_2^2$; it being understood that, as any one term in the circle u_1, u_2, u_4, u_3, passes into the next, that next passes into its next, u_3 passing into u_1.

THE ROOTS OF THE AUXILIARY BIQUADRATIC ALL RATIONAL.

§63. Any rational values that may be assigned to Δ_1, a_1, e_1, and h_1 in r_1, taken as in (62), make r_1 the root of a rational equation of the fifth degree, for they render the values of S_1, S_2, etc., in §62 rational. In fact, $S_1 = 0$, $25\, S_2 = 10\, \Delta_1 (h_1 + a_1 e_1)$, and so on

THE AUXILIARY BIQUADRATIC WITH A QUADRATIC SUB-AUXILIARY.

§64. PROPOSITION XXIII. In order that r_1, taken as in (62), may be the root of an irreducible equation $F(x) = 0$ of the fifth degree, whose auxiliary biquadratic has a quadratic sub-auxiliary, it must be of the form

$$r_1 = \tfrac{1}{5}\left\{(\Delta_1^{\frac{1}{5}} + \Delta_2^{\frac{1}{5}}) + (a_1\,\Delta_1^{\frac{2}{5}} + a_2\,\Delta_2^{\frac{2}{5}})\right\}; \qquad (65)$$

where Δ_1 and Δ_2 are the roots of the irreducible equation $\phi_1(x) = x^2 - 2\,px + q^5 = 0$; and $a_1 = b + d\,\surd\,(p^2 - q^5)$, $a_2 = b - d\,\surd\,(p^2 - q^5)$; p, b and d being rational; and the roots $\Delta_1^{\frac{1}{5}}$ and $\Delta_2^{\frac{1}{5}}$ being so related that $\Delta_1^{\frac{1}{5}}\,\Delta_2^{\frac{1}{5}} = q$.

By Prop. VII., when a quintic equation is of the first (see §30) class, the auxiliary biquadratic is irreducible. Hence, in the case we are considering, the quintic is of the second class. The quadratic sub-auxiliary may be assumed to be $\phi_1(x) = x^2 - 2\,px + k = 0$, p and k being rational. By Prop. XXI., the roots of the equation $\phi_1(x) = 0$ are Δ_1 and $h_1^5\,\Delta_1^4$. Therefore $k = (h_1\,\Delta_1)^5$; or, putting q for $h_1\,\Delta_1$, $k = q^5$. By the same proposition, $h_1\,\Delta_1$ is rational. Therefore q is rational. Hence $\phi_1(x)$ has the form specified in the enunciation of the proposition. Next, by Proposition XVI., there is a fifth root of unity t such that $t\,\Delta_2^{\frac{1}{5}} = h_1\,\Delta_1^{\frac{4}{5}}$. If we take t to be unity, which we may do by a suitable interpretation of the symbol $\Delta_2^{\frac{1}{5}}$, $\Delta_2^{\frac{1}{5}} = h_1\,\Delta_1^{\frac{4}{5}}$. This implies that $e_1\,\Delta_1^{\frac{3}{5}} = a_2\,\Delta_2^{\frac{2}{5}}$, a_2 being what a_1 becomes in passing from Δ_1 to Δ_2. Substituting these values of $e_1\,\Delta_1^{\frac{3}{5}}$ and $h_1\,\Delta_1^{\frac{4}{5}}$ in (62), we obtain the form of r_1 in (65), while at the same time $\Delta_1^{\frac{1}{5}}\,\Delta_2^{\frac{1}{5}} = h_1\,\Delta_1 = q$. The forms of a_1 and a_2 have to be more accurately determined. By Prop. XIV., $\Delta_1^{\frac{1}{5}}$ is the only principal surd that r_1, as presented in (62), contains. Therefore a_1 involves no surd that does not occur in Δ_1; that is to say, $\surd\,(p^2 - q^5)$ is the only surd in a_1. Hence we may put $a_1 = b + d\,\surd\,(p^2 - q^5)$; b and d being rational. But a_2 is what a_1 becomes in passing from Δ_1 to Δ_2. And Δ_2 differs from Δ_1 only in the sign of the root $\surd\,(p^2 - q^5)$. Therefore

$$a_2 = b - d\,\surd\,(p^2 - q^5).$$

§65. Any rational values that may be assigned to b, d, p and q in r_1, taken as in (65), make r_1 the root of a rational equation of the fifth degree; for they render the values of S_1, S_2, etc., in §62, rational. In fact, $S_1 = 0$, $25\,S_2 = 10\{q + q^2 b^2 - q^2 d^2 (p^2 - q^5)\}$, and so on.

The Auxiliary Biquadratic Irreducible.

§66. When the auxiliary biquadratic is irreducible, the unequal particular cognate forms of Δ are, by Prop. III., four in number, Δ_1, Δ_2, Δ_3, Δ_4. As explained in §55, because the equation $\varphi(x) = 0$ is irreducible, these terms are severaly identical with Δ_1, δ_2, δ_3, δ_4. Hence, putting $m = 5$, the first two terms in the first of the groups (59) may be written in the notation of (37),

$$\Delta_1^{\frac{1}{5}}\,\Delta_4^{\frac{1}{5}},\ \Delta_2^{\frac{1}{5}}\,\Delta_3^{\frac{1}{5}}\,; \tag{66}$$

and the second and third groups may be written

$$\left.\begin{array}{l} (\Delta_1^{\frac{2}{5}}\,\Delta_3^{\frac{1}{5}},\ \Delta_2^{\frac{2}{5}}\,\Delta_1^{\frac{1}{5}},\ \Delta_3^{\frac{2}{5}}\,\Delta_4^{\frac{1}{5}},\ \Delta_4^{\frac{2}{5}}\,\Delta_2^{\frac{1}{5}}) \\ (\Delta_1^{\frac{3}{5}}\,\Delta_2^{\frac{1}{5}},\ \Delta_2^{\frac{3}{5}}\,\Delta_4^{\frac{1}{5}},\ \Delta_3^{\frac{3}{5}}\,\Delta_1^{\frac{1}{5}},\ \Delta_4^{\frac{3}{5}}\,\Delta_3^{\frac{1}{5}}). \end{array}\right\} \tag{67}$$

§67. Proposition XXIV. The roots of the auxiliary biquadratic equation $\varphi(x) = 0$ are of the forms

$$\left.\begin{array}{l} \Delta_1 = m + n\sqrt{z} + \sqrt{s},\ \Delta_2 = m - n\sqrt{z} + \sqrt{s_1}, \\ \Delta_4 = m + n\sqrt{z} - \sqrt{s},\ \Delta_3 = m - n\sqrt{z} - \sqrt{s_1}\,; \end{array}\right\} \tag{68}$$

where $s = p + q\sqrt{z}$, and $s_1 = p - q\sqrt{z}$; m, n, z, p and q being rational; and the surd $\sqrt{s}$ being irreducible.

By Propositions XIII. and XIX., the terms in (66) are the roots of a quadratic. Therefore $\Delta_1\,\Delta_4$ and $\Delta_2\,\Delta_3$ are the roots of a quadratic. Suppose if possible that $\Delta_1\,\Delta_3$ is the root of a quadratic. By Propositions IX. and XIX., $\Delta_3^{\frac{1}{5}} = e_1\,\Delta_1^{\frac{3}{5}}$. Therefore $e_1^5\,\Delta_1^4$ is the root of a quadratic. From this it follows (Prop. III.) that there are not more than two unequal terms in the series,

$$e_1^5\,\Delta_1^4,\ e_2^5\,\Delta_2^4,\ e_3^5\,\Delta_3^4,\ e_4^5\,\Delta_4^4. \tag{69}$$

But suppose if possible that $e_1^5\,\Delta_1^4 = e_2^5\,\Delta_2^4$. Then, t being one of the fifth roots of unity, $te_1\,\Delta_1^{\frac{4}{5}} = e_2\,\Delta_2^{\frac{4}{5}}$ But, by Propositions IX. and XIX., $\Delta_2^{\frac{1}{5}} = h_1\,\Delta_1^{\frac{4}{5}}$. Therefore, $te_1\,\Delta_1^{\frac{4}{5}} = e_2\,h_1^4\,\Delta_1^3\,\Delta_1^{\frac{1}{5}}$. Therefore, by §8, $e_1 = 0$. Therefore one of the roots of the auxiliary biquadratic is zero; which because the auxiliary biquadratic is assumed to be irreducible, is impossible. Therefore $e_1^5 \Delta_1^4$ and $e_2^5 \Delta_2^4$ are unequal. In the same way all the terms in (69) can be shown to be unequal; which, because it has been proved that there are not more than two unequal terms in (69), is impossible. Therefore $\Delta_1 \Delta_3$ is not the root of a quadratic equation. Therefore the product of two of the roots, Δ_1 and Δ_4, of the auxiliary biquadratic is the root of a quadratic equation, while the product of a different pair, Δ_1 and Δ_3, is not the root of a quadratic. But the only forms which the roots of an irreducible biquadratic can assume consistently with these conditions are those given in (68).

§68. Proposition XXV. The surd $\surd s_1$ can have its value expressed in terms of $\surd s$ and $\surd z$.

By Propositions XIII. and XIX, the terms of the first of the groups (67) are the roots of a biquadratic equation. Therefore their fifth powers

$$\Delta_1^2 \Delta_3\,,\quad \Delta_2^2 \Delta_1\,,\quad \Delta_3^2 \Delta_4\,,\quad \Delta_4^2 \Delta_2\,, \tag{70}$$

are the roots of a biquadratic. From the values of Δ_1, Δ_2, Δ_3 and Δ_4 in (68), the values of the terms in (70) may be expressed as follows:

$$\left.\begin{aligned}
\Delta_1^2 \Delta_3 &= F + F_1 \surd z + (F_2 + F_3 \surd z) \surd s \\
&\quad + (F_4 + F_5 \surd z) \surd s_1 + (F_6 + F_7 \surd z) \surd s \surd s_1\,, \\
\Delta_2^2 \Delta_1 &= F - F_1 \surd z + (F_2 - F_3 \surd z) \surd s_1 \\
&\quad - (F_4 - F_5 \surd z) \surd s - (F_6 - F_7 \surd z) \surd s \surd s_1\,, \\
\Delta_4^2 \Delta_2 &= F - F_1 \surd z - (F_2 - F_3 \surd z) \surd s_1 \\
&\quad + (F_4 - F_5 \surd z) \surd s - (F_6 - F_7 \surd z) \surd s \surd s_1\,, \\
\Delta_3^2 \Delta_4 &= F + F_1 \surd z - (F_2 + F_3 \surd z) \surd s \\
&\quad - (F_4 + F_5 \surd z) \surd s_1 + (F_6 + F_7 \surd z) \surd s \surd s_1\,,
\end{aligned}\right\} \tag{71}$$

where F, F_1, etc., are rational. Let $\Sigma(\Delta_1^2 \Delta_3)$ be the sum of the four expressions in (70). Then, because these expressions are the roots of a biquadratic, $\Sigma(\Delta_1^2 \Delta_3)$ or $4F + 4F_7 \surd s \surd s_1$, must be rational. Suppose if possible that $\surd s_1$ cannot have its value expressed in terms of $\surd s$ and $\surd z$. Then, because $\surd s \surd s_1$ is not rational, $F_7 = 0$. By (68), this implies that $n = 0$. Let

$$\begin{aligned}
(\Delta_1^2 \Delta_3)^2 &= L + L_1 \surd z + (L_2 + L_3 \surd z) \surd s \\
&\quad + (L_4 + L_5 \surd z) \surd s_1 + (L_6 + L_7 \surd z) \surd s \surd s_1\,,
\end{aligned}$$

where L, L_1, etc., are rational. Then, as above, $L_7 = 0$. Keeping in view that $n = 0$, this means that $m^2 q = 0$. But q is not zero, for this would make $\surd s = \surd s_1$; which, because we are reasoning on the hypothesis that $\surd s_1$ cannot have its value expressed in terms of $\surd s$ and $\surd z$, is impossible. Therefore m is zero. And it was shown that n is zero. Therefore $\Delta_1 = \surd s$, and $\Delta_3 = - \surd s$. Therefore $\Delta_1 \Delta_3 = - \surd (p^2 - q^2 z)$; which, because it has been proved that $\Delta_1 \Delta_3$ is not the root of a quadratic equation, is impossible. Hence $\surd s_1$ cannot but be a rational function of $\surd s$ and $\surd z$.

§69. Proposition XXVI. The form of s is

$$h (1 + e^2) + h \surd(1 + e^2), \qquad (72)$$

h and e being rational, and $1 + e^2$ being the value of z.

By Prop. XXV., $\surd s_1 = v + c \surd s$, v and c being rational functions of $\surd z$. Therefore $s_1 = v^2 + c^2 s + 2vc \surd s$. By Prop. XXIV., $\surd s$ is irreducible. Therefore $vc = 0$. But c is not zero, for this would make $\surd s_1 = v$, and thus $\surd s_1$ would be the root of a quadratic equation. Therefore $v = 0$, and $\surd s_1 = c \surd s = (c_1 + c_2 \surd z) \surd s$, c_1 and c_2 being rational. Therefore

$$\surd (s s_1) = \surd (p^2 - q^2 z) = (c_1 + c_2 \surd z)(p + q \surd z)$$
$$= (c_1 p + c_2 q z) + \surd z (c_1 q + c_2 p) = P + Q \surd z.$$

Here, since $p^2 - q^2 z$ is rational, either $P = 0$ or $Q = 0$. As the latter of these alternatives would make $\surd (p^2 - q^2 z)$ rational, and therefore would make $\surd (p + q \surd z)$ or $\surd s$ reducible, it is inadmissible. Therefore $c_1 p + c_2 qz = 0$, and

$$\surd (p^2 - q^2 z) = (c_1 q + c_2 p) \surd z.$$

Now qz is not not zero, for this would make $\surd (s s_1) = \pm p$; which, because $\surd s$ is irreducible, is impossible. Therefore $c_2 = 0$. But, by hypothesis, $c_1 = 0$; therefore $\surd s_1$, which is equal to $(c_1 + c_2 \surd z) \surd s$, is zero; which is impossible. Hence c_1 cannot be zero. We may therefore put $ce = 1$, and $h (1 + e^2) = p$. Then $s = p + q \surd z = h (1 + e^2) + h \surd (1 + e^2)$. Having obtained this form, we may consider z to be identical with $1 + e^2$, q with h, and p with $h (1 + e^2)$.

§70. The reasoning in the preceding section holds good whether the equation $F(x) = 0$ be of the first (see §30) or of the second class. If we had had to deal simply with equations of the first class, the proof given would have been unnecessary, so far as the form of z is concerned; because, in that case, by Prop. VIII., Δ_1 is a rational function of the primitive fifth root of unity.

§71. PROPOSITION XXVII. Under the conditions that have been established, the root r_1 takes the form given without deduction in *Crelle* (Vol. V., p. 336) from the papers of Abel.

For, by *Cor.* Prop. XIII. (compare also *Cor.* 2, Prop. XIX.,) the expressions

$$\Delta_1^{\frac{1}{5}}\ \Delta_3^{\frac{2}{5}}\ \Delta_4^{\frac{3}{5}}\ \Delta_2^{\frac{3}{5}},\quad \Delta_2^{\frac{1}{5}}\ \Delta_1^{\frac{2}{5}}\ \Delta_3^{\frac{3}{5}}\ \Delta_4^{\frac{3}{5}},$$

$$\Delta_3^{\frac{1}{5}}\ \Delta_4^{\frac{2}{5}}\ \Delta_2^{\frac{3}{5}}\ \Delta_1^{\frac{3}{5}},\quad \Delta_4^{\frac{1}{5}}\ \Delta_2^{\frac{2}{5}}\ \Delta_1^{\frac{3}{5}}\ \Delta_3^{\frac{3}{5}}, \tag{73}$$

are the roots of a biquadratic equation. In the corollaries referred to, it is merely stated that each of the expressions in (73) is the root of a biquadratic; but the principles of the propositions to which the corollaries are attached show that the four expressions must be the roots of the same biquadratic. Let the terms in (73) be denoted respectively by

$$5A_1^{-1},\qquad 5A_2^{-1},\qquad 5A_3^{-1},\qquad 5A_4^{-1}.$$

Then $\Delta_1^{\frac{1}{5}}\ \Delta_3^{\frac{2}{5}}\ \Delta_4^{\frac{4}{5}}\ \Delta_2^{\frac{3}{5}} = \Delta_4^{\frac{1}{5}}\ (\Delta_1^{\frac{1}{5}}\ \Delta_3^{\frac{2}{5}}\ \Delta_4^{\frac{3}{5}}\ \Delta_2^{\frac{3}{5}})$ is an identity. Therefore

$$\tfrac{1}{5}\ \Delta_4^{\frac{1}{5}} = A_1\ (\Delta_1^{\frac{1}{5}}\ \Delta_3^{\frac{2}{5}}\ \Delta_4^{\frac{4}{5}}\ \Delta_2^{\frac{3}{5}}).\quad \text{Similarly,}$$

$$\tfrac{1}{5}\ \Delta_3^{\frac{1}{5}} = A_3\ (\Delta_3^{\frac{1}{5}}\ \Delta_4^{\frac{2}{5}}\ \Delta_2^{\frac{4}{5}}\ \Delta_1^{\frac{3}{5}})$$

$$\tfrac{1}{5}\ \Delta_2^{\frac{1}{5}} = A_2\ (\Delta_2^{\frac{1}{5}}\ \Delta_1^{\frac{2}{5}}\ \Delta_3^{\frac{4}{5}}\ \Delta_4^{\frac{3}{5}}),\ \text{and}$$

$$\tfrac{1}{5}\ \Delta_1^{\frac{1}{5}} = A_1\ (\Delta_4^{\frac{1}{5}}\ \Delta_2^{\frac{2}{5}}\ \Delta_1^{\frac{4}{5}}\ \Delta_3^{\frac{3}{5}}).$$

Substituting these values in (62), we get

$$r_1 = A_1\,(\Delta_1^{\frac{1}{5}}\ \Delta_3^{\frac{2}{5}}\ \Delta_4^{\frac{4}{5}}\ \Delta_2^{\frac{3}{5}}) + A_2\,(\Delta_2^{\frac{1}{5}}\ \Delta_1^{\frac{2}{5}}\ \Delta_3^{\frac{4}{5}}\ \Delta_4^{\frac{3}{5}})$$

$$+ A_3\,(\Delta_3^{\frac{1}{5}}\ \Delta_4^{\frac{2}{5}}\ \Delta_2^{\frac{4}{5}}\ \Delta_1^{\frac{3}{5}}) + A_4\,(\Delta_4^{\frac{1}{5}}\ \Delta_2^{\frac{2}{5}}\ \Delta_1^{\frac{4}{5}}\ \Delta_3^{\frac{3}{5}}). \tag{74}$$

This, with immaterial differences in the subscripts, is Abel's expression; only we need to determine A_1, A_2, A_3 and A_4 more exactly. These terms are the reciprocals of the terms in (73) severally divided by 5. Therefore they are the roots of a biquadratic. Also, no surds can appear in A_1 except those that are present in Δ_1, Δ_2, Δ_3 and Δ_4. That is to say, A_1 is a rational function of $\surd s$, $\surd s_1$ and $\surd z$. But it was shown that $\surd s_1 \surd s = he \surd z$. Therefore A_1 is a rational function of $\surd s$ and $\surd z$. We may therefore put

$$A_1 = K + K'\,\Delta_1 + K''\,\Delta_4 + K'''\,\Delta_1\,\Delta_4,$$

K, K', K'' and K''' being rational. But the terms A_1, A_2, A_4, A_3 circulate with Δ_1, Δ_2, Δ_4, Δ_3. Therefore

$$A_2 = K + K'\,\Delta_2 + K''\,\Delta_3 + K'''\,\Delta_2\,\Delta_3\,,$$
$$A_4 = K + K'\,\Delta_4 + K''\,\Delta_1 + K'''\,\Delta_1\,\Delta_4\,,$$
$$A_3 = K + K'\,\Delta_3 + K''\,\Delta_2 + K'''\,\Delta_2\,\Delta_3\,,$$

These are Abel's values.

§72. Keeping in view the values of Δ_1, Δ_2, etc., in (67), and also that $z = 1 + e^2$, and $s = hz + h\sqrt{z}$, any rational values that may be assigned to m, n, e, h, K, K', K'' and K''' make r_1, as presented in (74), the root of an equation of the fifth degree. For, any rational values of m, n, etc., make the values of S_1, S_2, etc., in §62, rational.

§73. It may be noted that, not only is the expression for r_1 in (71) the root of a quintic equation whose auxiliary biquadratic is irreducible, but on the understanding that the surds $\sqrt{s}$ and $\sqrt{z}$ in Δ_1 may be reducible, the expression for r_1 in (74) contains the roots both of all equations of the fifth degree whose auxiliary biquadratics have their roots rational, and of all that have quadratic subauxiliaries. It is unnecessary to offer proof of this.

§74. The equation $x^5 - 10x^3 + 5x^2 + 10x + 1 = 0$ is an example of a solvable quintic with its auxiliary biquadratic irreducible. One of its roots is

$$\omega^{\frac{1}{5}} + \omega\omega^{\frac{2}{5}} + \omega^3\omega^{\frac{3}{5}} + \omega^4\omega^{\frac{4}{5}}\,,$$

ω being a primitive fifth root of unity. It is obvious that this root satisfies all the conditions that have been pointed out in the preceding analysis as necessary. A root of an equation of the seventh degree of the same character is

$$\omega^{\frac{1}{7}} + \omega^4\omega^{\frac{2}{7}} + \omega^4\omega^{\frac{3}{7}} + \omega^2\omega^{\frac{4}{7}} + \omega^2\omega^{\frac{5}{7}} + \omega^6\omega^{\frac{6}{7}}\,,$$

ω being a primitive seventh root of unity. The general form under which these instances fall can readily be found. Take the cycle that contains all the primitive $(m^2)^{\text{th}}$ roots of unity,

$$\theta,\ \theta^{\beta},\ \theta^{\beta^2},\ \text{etc.} \qquad (75)$$

m being prime. The number of terms in the cycle is $(m-1)^2$. Let θ_1 be the $(m+1)^{\text{th}}$ term in the cycle (75), θ_2 the $(2m+1)^{\text{th}}$ term, and so on. Then the root of an equation of the m^{th} degree, including the instances above given, is

$$r_1 = (\theta + \theta^{-1}) + (\theta_1 + \theta_1^{-1}) + \ldots + \left(\theta_{\frac{m-3}{2}} + \theta_{\frac{m-3}{2}}^{-1}\right).$$

[*Read before the Canadian Institute, March 3rd,* 1883].

RESOLUTION

OF

SOLVABLE EQUATIONS OF THE FIFTH DEGREE,

BY GEORGE PAXTON YOUNG,

Toronto, Canada.

CONTENTS.

§1. By means of the laws established in the paper entitled "Principles of the Solution of Equations of the Higher Degrees," which is concluded in the present issue of the Journal of Mathematics, a criterion of the solvability of equations of the fifth degree may be found, and the roots of solvable quintics obtained in terms of given numerical coefficients. In certain classes of cases, the roots can be determined in terms of coefficients to which particular numerical values have not been assigned, but which are only assumed to be so related as to make the equations solvable.

SKETCH OF THE METHOD EMPLOYED.

§2. Let r_1, r_2, r_3, r_4, r_5, be the roots of the solvable irreducible equation of the fifth degree wanting the second term,

$$F(x) = x^5 + p_2 x^3 + p_3 x^2 + p_4 x + p_5 = 0. \tag{1}$$

It was proved in the "Principles" that

$$r_1 = \tfrac{1}{5}(\Delta_1^{\frac{1}{5}} + \Delta_2^{\frac{1}{5}} + \Delta_3^{\frac{1}{5}} + \Delta_4^{\frac{1}{5}}),$$

where Δ_1, Δ_2, Δ_3, Δ_4 are the roots of a biquadratic equation auxiliary to the equation $F(x) = 0$. It was also shown that the root can be expressed in the form

$$r_1 = \tfrac{1}{5}(\Delta_1^{\frac{1}{5}} + a_1 \Delta_1^{\frac{2}{5}} + e_1 \Delta_1^{\frac{3}{5}} + h_1 \Delta_1^{\frac{4}{5}}), \tag{2}$$

where a_1, e_1, h_1, involve only surds occurring in Δ_1; and no surds occur in Δ_1 except $\sqrt{(hz + h\sqrt{z})}$ and its subordinate $\sqrt{z}$; z being equal to $1 + e^2$, and h and e being rational. As in the "Principles,"

we may put $5u_1, = \Delta_1^{\frac{1}{5}}$, $5u_2 = \Delta_2^{\frac{1}{5}}$, $5u_3 = \Delta_3^{\frac{1}{5}}$, $5u_4 = \Delta_4^{\frac{1}{5}}$. Then

$$r_1 = u_1 + u_2 + u_3 + u_4. \tag{3}$$

Let S_1 be the sum of the roots of the equation $F(x) = 0$, S_2 the sum of their squares, and so on. Also let

$$\left.\begin{aligned}
\Sigma(u_1^2 u_3) &= u_1^2 u_3 + u_2^2 u_1 + u_3^2 u_4 + u_4^2 u_2,\\
\Sigma(u_1^3 u_2) &= u_1^3 u_2 + u_2^3 u_4 + u_3^3 u_1 + u_4^3 u_3,\\
\Sigma(u_1 u_3^2 u_4^2) &= u_1 u_3^2 u_4^2 + u_2 u_1^2 u_3^2 + u_3 u_4^2 u_2^2 + u_4 u_2^2 u_1^2;\\
\Sigma(u_1^5) &= u_1^5 + u_2^5 + u_3^5 + u_4^5;\\
\text{Then}\quad S_2 &= 10(u_1 u_4 + u_2 u_3),\ S_3 = 15\{\Sigma(u_1^2 u_3)\},\\
S_4 &= 20\{\Sigma(u_1^3 u_2)\} + \tfrac{3}{10}(S_2^2) + 60 u_1 u_2 u_3 u_4,\\
S_5 &= 5\{\Sigma(u_1^5)\} + \tfrac{5}{6}(S_2 S_3) + 50\{\Sigma(u_1 u_3^2 u_4^2)\}.
\end{aligned}\right\} \tag{4}$$

§3. It was proved in the "Principles" that $u_1 u_4$ and $u_2 u_3$ are the roots of a quadratic equation. But

$$25 u_1 u_4 = h_1 \Delta_1, \text{ and } 25 u_2 u_3 = a_1 e_1 \Delta_1.$$

Therefore, because a_1, e_1, h_1, involve no surds that are not subordinate to $\Delta_1^{\frac{1}{5}}$, $\sqrt{z}$ is the only surd that can appear in $u_1 u_4$ and $u_2 u_3$. Consequently we may put

$$u_1 u_4 = g + a \surd z, \text{ and } u_2 u_3 = g - a \surd z, \tag{5}$$

where g, a, are rational. It scarcely needs to be pointed out that these forms are valid whether the surd $\surd z$ is irreducible or not. Now $S_2 = 10 (u_1 u_4 + u_2 u_3) = -2 p_2$. Therefore

$$g = -\tfrac{1}{10}(p_2). \tag{6}$$

Again, it was shown in the "Principles" that the four expressions $u_1^2 u_3$, $u_2^2 u_1$, $u_3^2 u_4$, $u_4^2 u_2$; are the roots of a biquadratic equation. And, by the same reasoning as that employed in the case of $u_1 u_4$ and $u_2 u_3$, the only surds that can appear in these expressions are $\surd(hz + h\surd z)$, $\surd(hz - h\surd z)$, and $\surd z$. Let $hz + h\surd z = s$, and $hz - h\surd z = s_1$. Then

$$\surd s_1 = \left(\frac{\surd z - 1}{e}\right) \surd s, \text{ and } \surd s \surd s_1 = he \surd z.$$

Hence the expressions $u_1^2 u_3$, $u_2^2 u_1$, $u_3^2 u_4$, $u_4^2 u_2$, may have their values exhibited in terms of $\surd z$ and either of the surds $\surd s$, $\surd s_1$. Put

$$\left.\begin{aligned} u_1^2 u_3 &= k + c\surd z + (\theta + \varphi \surd z) \surd s, \\ u_4^2 u_2 &= k + c\surd z - (\theta + \varphi \surd z) \surd s, \\ u_2^2 u_1 &= k - c\surd z + (\theta - \varphi \surd z) \surd s_1, \\ u_3^2 u_4 &= k - c\surd z - (\theta - \varphi \surd z) \surd s_1; \end{aligned}\right\} \tag{8}$$

where k, c, θ, φ, are rational. These coefficients must bear a relation to g, a, in (5). In fact, because

$$(u_1^2 u_3)(u_4^2 u_2) = (u_1 u_4)^2 (u_2 u_3),$$

$$(g^2 - a^2 z)(g + a\surd z) = (k + c\surd z)^2 - (\theta + \varphi\surd z)^2 (hz + h\surd z).$$

Equating the rational parts to one another, and also the irrational parts,

$$\left.\begin{aligned} hz(\theta^2 + \varphi^2 z + 2\theta\varphi) &= k^2 + c^2 z - g(g^2 - a^2 z, \\ h(\theta^2 + \varphi^2 z + 2\theta\varphi z) &= 2kc - a(g^2 - a^2 z). \end{aligned}\right\} \tag{9}$$

Because $s_2 = 15 \{\Sigma u_1^2 u_3)\} = -3p_3$,

$$k = -\tfrac{1}{20}(p_3). \tag{10}$$

It will be convenient to retain the symbols g and k, whose values are given in (6) and (10). Again, because $u_1^3 u_2 = \dfrac{(u_1^2 u_3)(u_2^2 u_1)}{u_2 u_3}$, we have, from (5) and (8),

$$u_1^3 u_2 = \frac{g + a \surd z}{g^2 - a^2 z} \{k + c \surd z + (\theta + \varphi \surd z) \surd s\}$$
$$\{k - c \surd z + (\theta - \varphi \surd z) \surd s_1\}$$
$$= A + A' \surd z + (A'' + A''' \surd z) \surd s,$$

where A, A', A'', A''', are rational. The value of A is

$$A = \frac{1}{g_2 - a^2 z}\{g (k^2 - c^2 z) + ahez (\theta^2 - \varphi^2 z)\}. \tag{11}$$

Again, $u_1^5 = \dfrac{(u_1^2 u_3)^2 \; (u_2^2 u_1)}{(u_2 u_3)^2}$. That is, from (8) and (5) and (7).

$$u_1^5 = \frac{(g + a \surd z)^2}{(g^2 - a^2 z)^2} \} 2 (k + c \surd z)^2 - (g^2 - a^2 z) (g + a \surd z)$$
$$+ 2 (k + c \surd z) (\theta + \varphi \surd z) \surd s\}$$
$$\{k - c \surd z + (\theta - \varphi \surd z) \surd s_1 \}$$
$$= B + B' \surd z + (B'' + B''' \surd z) \surd s;$$

where B, B', B'', B''', are rational. Now, by (4),

$$S_4 = 20 \{\Sigma (u_1^3 u_2)\} + \tfrac{3}{10} (S_2^2) + 60 u_1 u_2 u_3 u_4 .$$

And $S_4 = 2p_2^2 - 4p_4$. Also $\Sigma (u_1^3 u_2) = 4A$; and, by (6), $10g = - p_2$; and, by (5), $u_1 u_2 u_3 u_4 = g_2 - a_2 z$. Therefore

$$p_4 = - 20A + 5g^2 + 15a^2 z. \tag{12}$$

Again, $S_5 = 5 \{\Sigma (u_1^5)\} + \frac{2}{3} (S_2 S_3) + 50 \{\Sigma (u_1 u_3^2 u_4^2)\}$.

And $\Sigma (u_1^5) = 4B$, $S_2 S_3 = 6 p_2 p_3 = 1200 gk$, and

$\Sigma (u_1 u_3^3 u_4^2) = u_1 u_4 (u_2^2 u_1 + u_3^2 u_4) + u_2 u_3 (u_1^2 u_3 + u_4^2 u_2)$.

Therefore $S_5 = 20B + 1000gk - 200acz$.

But $S_5 - 5p_2 p_3 + 5p_5 = S_5 - 1000gk + 5p_5 = 0$.

Therefore $$p_5 = - 4B + 40acz. \tag{13}$$

The values of p_4 and p_5 in (12) and (13) make the quintic

$$F(x) = x^5 + p_2 x^3 + p_3 x^2 + (5g^2 + 15a^2 z - 20A) x$$
$$+ (40acz - 4B) = 0. \tag{14}$$

§4. Assuming the coefficients p_2, p_3, etc., in (1), to be known, the coefficients in the equation $F(x) = 0$ as exhibited in (14) involve six unknown quantities, namely, a, c, θ, φ, e, h. The list does not include z, g, k; because $z = 1 + e^2$; and g and k are known by (6) and (10). To find the six unknown quantities we have six equations, which are here gathered together.

$$\left.\begin{aligned} p_4 &= -\,20A + 5g^2 + 15a^2 z, \\ p_5 &= -\,4B + 40acz, \\ B'' &= 1, \\ B''' &= 0, \\ hz\,(\theta^2 + \varphi^2 z + 2\theta\varphi) &= k^2 + c^2 z - g\,(g^2 - a^2 z) \\ h\,(\theta^2 + \varphi^2 z + 2\theta\varphi z) &= 2kc - a\,(g^2 - a^2 z). \end{aligned}\right\} \quad (15)$$

The first two of these equations are the equations (12) and (13). As to the third and fourth, it was proved in the "Principles" that the form of u_1^5 is $m + n\sqrt{z} + \sqrt{(hz + h\sqrt{z})}$, m and n being rational. This is saying in other words that $B'' = 1$ and $B''' = 0$. The last two of the equations (15) are the equations (9).

§5. The criterion of solvability of the equation $F(x) = 0$ may now be stated in a general way to be that the coefficients p_2, p_3, etc., must be so related that rational quantities, a, c, θ, φ, e, h, exist satisfying the equations (15). We also see what requires to be done in order to find the roots of the equation $F(x) = 0$ in terms of the given coefficients. By (3), r_1 is known when u_1, u_2, u_3, u_4 are known. But, B'' and B''' being respectively unity and zero,

$$u_1^5 = B + B'\sqrt{z} + \sqrt{s}, \qquad u_2^5 = B - B'\sqrt{z} + \sqrt{s_1},$$
$$u_4^5 = B + B'\sqrt{z} - \sqrt{s}, \qquad u_3^5 = B - B'\sqrt{z} - \sqrt{s_1}.$$

Therefore, to find r_1 we need to find B, B', z and h; which is equivalent to saying that we need to find the six unknown quantities a, c, θ, φ, e, h. Before pointing out how this may be done in the most general case, I will refer to some special forms of soluble quintics.

Case in which $u_1 u_4 = u_2 u_3$.

§6. A notable class of solvable quintics is that in which $u_1 u_4 = u_2 u_3$. It includes, as was proved in the "Principles," all the Gaussian equations of the fifth degree for the reduction of $x^n - 1 = 0$, n prime. It includes also other equations, of which examples will presently be given. Now, when $u_1 u_4 = u_2 u_3$, the root of the quintic can be found in terms of the coefficients p_2, p_3, etc., even while these coefficients retain their general symbolic forms; in other words, the root can be found in terms of p_2, p_3, etc., without definite numerical values being assigned to p_2, p_3, etc. This I proceed to show.

§7. By (5), because $u_1 u_4 = u_2 u_3$, $a = 0$. Thus, one of the six unknown quantities is determined, while we have still the six equations (15) to work with. It might be sufficient to say, that, from six equations five unknown rational quantities can be found. I will recur to this idea; but in the meantime the following line of reasoning may be pursued. From (11), $A = \frac{k^2 - c^2 z}{g}$. Therefore equation (12) becomes

$$gp_4 = -20(k^2 - c^2 z) + 5g^3. \qquad (16)$$

Also, because $a = 0$, equations (7) being kept in view,

$$u_1^5 = \frac{1}{g^2}$$

$$\{2(k^2 - c^2 z)(k + c\sqrt{z}) - g^3(k - c\sqrt{z}) + 2(k + c\sqrt{z})(\theta^2 - \varphi^2 z) he\sqrt{z}\}$$
$$+ B'' + B'''\sqrt{z})\sqrt{s}.$$

$$\therefore Bg^2 = k\{2(k^2 - c^2 z) - g^3\} + 2chez(\theta^2 - \varphi^2 z)$$

and $B'g^2 = c\{2(k^2 - c^2 z) + g^3\} + 2khe(\theta^2 - \varphi^2 z)$;

$$\therefore u_1^5 = \frac{1}{g^2}[k\{2(k^2 - c^2 z) - g^3\} + 2chez(\theta^2 - \varphi^2 z)]$$

$$+ \frac{\sqrt{z}}{g_2}[c\{2(k^2 - c^2 z) + g^3\} + 2khe(\theta^2 - \varphi^2 z)] + \sqrt{s}. \qquad (17)$$

Substitute in the second of equations (15) the value of B that has been obtained. Then

$$g^2 p_5 = -4k\{2(k^2 - c^2 z) - g^3\} - 8chez(\theta^2 - \varphi^2 z). \qquad (18)$$

The values of B'' and B''' are

$$\left.\begin{aligned} B'' eg^2 &= \theta\{M + 2e(k^2 - c^2 z)\} - \varphi z N = eg^2, \\ B''' eg^2 &= \theta N - \varphi\{M - 2e(k^2 c^2 z)\} = 0; \end{aligned}\right\} \qquad (19)$$

$$\left.\begin{aligned} &\text{where } M = -2(k^2 + c^2 z) + g^3 + 4kcz, \\ &\text{which may be written } M = 5kcz - P, \\ &\text{and } N = 2(k^2 + c^2 z) - g^3 - 4kc, \\ &\text{which may be written } N = P - 4kc. \end{aligned}\right\} \qquad (20)$$

The two equations (19) give us

$$\left.\begin{aligned} &\theta\{M^2 - zN^2 - 4e^2(k^2 - c^2 z)^2\} = eg^2\{M - 2e(k^2 - c^2 z)\}, \\ &\varphi\{M^2 - zN_2 - 4e^2(k^2 - c^2 z)^2\} = eg^2 N. \\ &\text{Therefore} \\ &\frac{\theta}{\varphi} = \frac{M - 2e(k^2 - c^2 z)}{N}. \end{aligned}\right\} \qquad (21)$$

Equating the value of $\frac{\theta^2 + \varphi^2 z + 2\theta\varphi}{\theta^2 + \varphi^2 z + 2\theta\varphi z}$ obtained from (21) with that derived from the last two of equations (15),

$$\frac{k^2 + c^2 z - g^3}{2kcz} = \frac{\{M - 2e(k^2 - c^2 z)\}^2 + N^2 z + 2N\{M - 2e(k^2 - c^2 z)\}}{\{M - 2e(k^2 - c^2 z)\}^2 + N^2 z + 2Nz\{M - 2e(k^2 - c^2 z)\}}. \quad (22)$$

The coefficients p_2, p_3, etc., in the equation $F(x) = 0$, being given, g and k are known by (6) and (10). Therefore, by (16), $c^2 z$ is known. Then (22) will be found to be a quadratic equation determinative of c. For, keeping in view the value of P in (20), (22) may be written

$$\frac{k^2 + c^2 z - g^3}{2kc^2 z} = \frac{\{4(k^2 + c^2 z)^2 + P^2\} - 8kPc - 16k(k^2 - c^2 z)(ce)}{\{4(k^2 - c^2 z)^2 - 16k^2 c^2 z - P^2\} c + 8kc^2 zP - 4(k^2 - c^2 z)P(ce)}$$

Because g, k, $c^2 z$ and P are known, this equation is of the form

$$H(ce) = Kc + L,$$

where H, K, L, are known. Therefore, since $c^2 e^2 = c^2 z - c^2$,

$$c^2(H^2 + K^2) + 2KLc + (L^2 - H^2 c^2 z) = 0;$$

from which c is known. Therefore, since $c^2 z$ is known, z is known. Therefore e is known. Therefore, by (21), θ and φ are known. Therefore, by (18) or either of the equations (9), h is known. Therefore, by (17), u_1^5 is known. In like manner, u_2^5, u_3^5, u_4^5, are known. Hence finally, by (3), r_1 is known.

§8. *Example First.* I will now give some numerical verifications of the theory. The Gaussian equation of the fifth degree for the reduction of $x^{11} - 1 = 0$, when deprived of its second term, is

$$x^5 - \frac{22}{5}x^3 - \frac{11}{25}x^2 + \frac{11 \times 42}{125}x + \frac{11 \times 89}{3125} = 0.$$

When a root of this equation is expressed as in (1), the value of r_1, as given by Lagrange, is

$$u_1^5 = \frac{11}{4(5)^5}\{-89 - 25\sqrt{5} + 5(19 - 9\sqrt{5})(-5 - 2\sqrt{5})\};$$

which, reduced to the form that we have adopted, is

$$u_1^5 = \frac{11}{4(5)^5}\left\{-89 + 25 \times \frac{89}{41}\sqrt{\left(\frac{5 \times 41^2}{89^2}\right)}\right\} + \sqrt{(hz + h\sqrt{z})};$$

where $h = -\dfrac{11^2 \times 89^2}{8 \times 41 \times (5)^8}, \sqrt{z} = -\dfrac{41}{89}\sqrt{5}$, and $e = -\dfrac{22}{89}$.

We have to show that this is the result to which the equations of the preceding section lead. The simplest way will be to find g, k and c^2z by means of (6), (10) and (16), and then to take the values of e and $\sqrt{z}$ given above, and to substitute them in equation (22). If the theory is sound, the equation ought in this way to be satisfied. When this equation has been satisfied, it will be unnecessary to pursue the verification farther. Because

$$p_2 = -\frac{22}{5}, \text{ and } p_3 = -\frac{11}{25}, \; g = \frac{11}{25} \text{ and } k = \frac{11}{20 \times 25}.$$

From (18), taken in connection with (21), che must be negative. Therefore

$$c = -\frac{11 \times 89}{4 \times 25 \times 41}, \; kc = -\frac{89}{80 \times 41}\left(\frac{11}{25}\right)^2,$$

$$kcz = -\frac{41}{16 \times 89}\left(\frac{11}{25}\right)^2, \; k^2 - c^2 z = -\frac{31}{100}\left(\frac{11}{25}\right)^2,$$

$$M = -\frac{2716}{89 \times 100}\left(\frac{11}{25}\right)^2, \; N = \frac{1224}{41 \times 100}\left(\frac{11}{25}\right)^2,$$

$$M - 2e(k^2 - c^2 z) = -\frac{4080}{89 \times 100}\left(\frac{11}{25}\right)^2.$$

These values reduce the equation (22) to the identity

$$\frac{89}{41} = \frac{89}{41}\left\{\frac{41(4080^2 + 5 \times 1224^2) - 89(2448 \times 4080)}{89(4080^2 + 5 \times 1224^2) - 205(2448 \times 4080)}\right\}.$$

§9. *Example Second.* The example that has been given is one in which the auxiliary biquadratic is irreducible, I will now take an example,

$$x^5 + 10x^3 - 80x^2 + 145x - 480 = 0, \qquad (23)$$

in which the auxiliary biquadratic has a sub-auxiliary quadratic. When the root of the equation (23) is put in the form (1),

$$u_1 = (1 + \sqrt{2})^{\frac{1}{5}}, \; u^4 = (1 - \sqrt{2})^{\frac{1}{5}},$$

$$u_2 = (1 + \sqrt{2})(1 + \sqrt{2})^{\frac{2}{5}},$$

$$u_3 = (1 - \sqrt{2})(1 - \sqrt{2})^{\frac{2}{5}},$$

the product of the roots $(1+\sqrt{2})^{\frac{1}{5}}$, $(1-\sqrt{2})^{\frac{1}{5}}$, being -1. Putting β for 28560, and λ for 28562,

$$g=-1,\ k=4,\ c\sqrt{z}=-3,\ z=\frac{\lambda^2}{\beta^2},\ c=\frac{3\beta}{\lambda},$$

$$k^2+c^2z=25,\ kc=\frac{12\beta}{\lambda},\ kcz=\frac{12\lambda}{\beta},$$

$$P=2(k^2+c^2z)-g^3=51,$$

$$M=\frac{48\lambda-51\beta}{\beta},\ N=\frac{51\lambda-48\beta}{\lambda},$$

$$M-2e(k^2-c^2z)=\frac{48\lambda-51\beta+14\times 338}{\beta}.$$

These values cause (22) to become

$$\frac{13}{12}=\frac{\lambda\{Q^2+(51\lambda-48\beta)^2\}+2\beta(51\lambda-48\beta)Q}{\beta\{Q^2+(51\lambda-48\beta)^2\}+2\lambda(51\lambda-48\beta)Q}$$

where $Q=48\lambda-51\beta+14\times 338$. This may be written

$$\frac{13}{12}=\frac{H\lambda+2K\beta}{H\beta+2K\lambda}.$$

In order that this equation may subsist, it is necessary that

$$H(13\beta-12\lambda)=2K(12\beta-13\lambda);$$

$$\text{or } \frac{H}{2}\left(\frac{\beta-24}{2}\right)=-\frac{K(\beta+26)}{2}.$$

But $H=(-80852)^2+(85782)^2=6537045904+7358551524$ $=13895597428$; $-K=(80852)(85782)=6935646264$; $\frac{\beta-24}{2}$ $=14268$; $\frac{\beta+26}{2}=14293$; and $6947798714\times 14268=6935646264$ $\times 14293=99131192051352$.

§10. *Example Third.* I will finally take an example,

$$x^5+20x^3+20x^2+30x+10=0, \tag{24}$$

in which the roots of the auxiliary biquadratic are all rational. By (6) and (10) and (16), $g=-2$, $k=-1$, $c^2z=0$. Therefore the denominator of the expression on the left of (22) is zero, while the numerator is not zero. Therefore the denominator of the expression on the right of (22) is zero. Or, $-g^6+4k^2g^3-8ek^4+4eg^3k^2=0$. Therefore $e=-\frac{12}{5}$. Therefore $z=\left(\frac{13}{5}\right)^2$, and $c=0$. Hence $M=-10$, $N=10$; and, if

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$$D = M^2 - zN^2 - 4e^2 (k^2 - c^2 z)^2 ,$$

$$D = - 104e^2 . \quad \text{Therefore, by (21), } \theta = - \frac{1}{12},$$

$$\varphi = \frac{25}{12 \times 13}, \; \theta^2 - \varphi^2 z = - \frac{1}{6} . \quad \text{Therefore by (9), } h = \frac{225}{26}.$$

Therefore using the symbols, B, B', as in §3,

$$B = - \frac{5}{2}, \; B' = - \frac{45}{26}, \; s = h (z + \sqrt{z}) = 81,$$

$$s_1 = h (z - \sqrt{z}) = 36.$$

Therefore $\quad u_1^5 = - 7 + 9 = 2, \; u_4^5 = - 7 - 9 = - 16,$

$$u_2^5 = 2 - 6 = - 4, \; u_3^5 = 2 + 6 = 8.$$

Hence, by (3),

$$r_1 = 2^{\frac{1}{5}} - 2^{\frac{2}{5}} + 2^{\frac{3}{5}} - 2^{\frac{4}{5}};$$

which is the solution of the equation (24).

§11. It was pointed out in §7 that, in the case we are considering, there are six equations and five unknown quantities. All the unknown quantities may be eliminated, and an equation $p' = 0$ obtained; where p' is a rational function of the coefficients p_2, p_3, etc. This elimination has been performed, under the direction of the author of the paper, by Mr. Warren Reid of Toronto, with the following result. Putting P, as in §7, for $2 (k^2 + c^2 z) - g^3$, let

$$A = - 2kc^2 zg^3 \{ 8 (k^2 + c^2 z) - 3g^3 \},$$

$$B = g^3 \{ 16k^2 c^2 z + 4 (k^2 + c^2 z)^2 - 5g^3 (k^2 + c^2 z) + g^6 \},$$

$$D = - 4 (k^2 - c^2 z) \{ - g^6 + 3g^3 (k^2 + c^2 z) - 2 (k^2 - c^2 z)^2 \},$$

$$A_1 = - 8kc^2 z [32kc^2 z (k^2 - c^2 z) - P \{ p_5 g^2 + 8k (k^2 - c^2 z) - 4kg^3 \}]$$

$$B_1 = \{ p_5 g^2 + 8k (k^2 - c^2 z) - 4kg^3 \} [-32k^2 c^2 z + g^3 \{ 4 (k^2 + c^2 z) - g^3 \}] + 64kc^2 z P (k^2 - c^2 z),$$

$$D_1 = - 16kc^2 zg^3 \{ 4 (k^2 + c^2 z) - g^3 \} + 4P (k^2 - c^2 z) \{ p_5 g^2 + 8k (k^2 - c^2 z) - 4kg^3 \}.$$

Then, since $10g = - p_2$, and $20k = - p_3$, and

$$20c^2 z = p_4 g - 5g^3 + 20k^2 ,$$

the quantities A, B, D, A_1, B_1, D_1, are known rational functions of p_2, p_3, etc. And

$$(B^2 + D^2)(A_1^2 - D_1^2 c^2 z) - (B_1^2 + D_1^2)(A^2 - D^2 c^2 z)$$
$$+ 4\{AB(B_1^2 + D_1^2) - A_1 B_1 (B^2 + D^2\}$$
$$\{AB(A_1^2 - D_1^2 c^2 z) - A_1 B_1 (A^2 - D^2 c^2 z)\} = 0. \qquad (25)$$

§12. To verify this result, the Gaussian equation in §8 may be used. Here

$$A = -\frac{11^6}{2^5 \times 5^{12}}\left(\frac{11^3 + 11^2 \times 19}{5^6}\right) = -\frac{11^8 \times 3}{2^4 \times 5^{17}}$$

$$B = \frac{11^3}{5^6}\left(\frac{11^4}{2^4 \times 5^9} + \frac{3^4 \times 7^2 \times 11^4}{2^4 \times 5^{12}} - \frac{9 \times 35 \times 11^5}{8 \times 5^{12}} + \frac{11^6}{5^{12}}\right)$$
$$= -\frac{9 \times 11^7}{4 \times 5^{16}}$$

$$D = \frac{11^2 \times 31}{5^{18}}\left(-11^6 + \frac{7 \times 27 \times 11^5}{8} - \frac{31^2 \times 11^4}{8}\right)$$
$$= \frac{3 \times 31 \times 11^6}{4 \times 5^{16}}.$$

$$A_1 = \frac{11^8}{2^6 \times 5^{18}}(19 + 31) = \frac{11^8}{2^5 \times 5^{16}}$$

$$B_1 = \frac{11^7}{2^4 \times 5^{18}}(-5^3 + 44 \times 41 - 19 \times 31) = \frac{11^7 \times 109}{8 \times 5^{17}}$$

$$D_1 = -\frac{11^6}{4 \times 5^{12}}\left(\frac{63 \times 11^2}{2 \times 5^6} - \frac{11^3}{5^6}\right) - \frac{11^7 \times 19 \times 31}{8 \times 5^{18}}$$
$$= -\frac{11^7 \times 26}{5^{17}}$$

Therefore

$$B^2 + D^2 = \frac{9 \times 11^{12} \times 41}{8 \times 5^{30}},\quad B_1^2 + D_1^2 = \frac{11^4 \times 11029}{2^6 \times 5^{33}},$$

$$A^2 - D^2 c^2 z = -\frac{9 \times 11^{14} \times 89}{2^6 \times 5^{35}},$$

$$A_1^2 - D_1^2 c^2 z = -\frac{11^{16} \times 40139}{2^{10} \times 5^{37}}.$$

By the substitution of these values, equation (25) becomes

$$\frac{11^{56} \times 3^4}{2^{26} \times 5^{136}} \{ 6265333^2 - 2886277 \times 13600357 \} =$$

$$\frac{11^{56} \times 3^4}{2^{26} \times 5^{136}} \{ 39254397600889 - 39254397600889 \} = 0.$$

§13. As an additional verification, the equation

$$x^5 + 10x^3 - 80x^2 + 145x - 480 = 0$$

may be taken. Here, by §9,

$$g = -1, k = 4, k^2 - c^2 z = 7, k^2 + c^2 z = 25.$$

Therefore

$$A = 2^3 \times 3^2 \times 7 \times 29, \; B = -2 \times 5 \times 17 \times 29,$$
$$D = 2^3 \times 3 \times 7 \times 29,$$
$$A_1 = -2^9 \times 3^4 \times 141, \; B_1 = 2^4 \times 3 \times 17 \times 2393,$$
$$D_1 = -2^7 \times 3^2 \times 13 \times 19.$$
$$B^2 + D^2 = 2^2 \times 29^2 \times 14281,$$
$$B_1^2 + D_1^2 = 2^8 \times 3^2 \times 5 \times 338016989,$$
$$A^2 - D^2 c^2 z = 0,$$
$$A_1^2 - D_1^2 c^2 z = 2^{14} \times 3^6 \times 5 \times 7 \times 17^2 \times 277.$$

By the substitution of these values, equation (25) becomes

$$2^{18} \times 3^6 \times 5 \times 7 \times 17^2 \times 29^4 \{ 277 \times 14281^2 + 5^7 \times 7 \times 338016989 - 2^8 \times 3 \times 141 \times 2393 \times 14281 \} = 0.$$

The Trinomial Quintic $x^5 + p_4 x + p_5 = 0$.

§14. In this case, by (6) and (10), $g = 0$, and $k = 0$. Therefore, by (11), $A = -\dfrac{he(\theta^2 - \varphi^2 z)}{a}$. Therefore, by (12),

$$p_4 = \frac{20he(\theta^2 - \varphi^2 z)}{a} + 15a^2 z. \qquad (26)$$

Also, by §3, $B = \dfrac{1}{a^2 z} \{ -a^3 z^2 c + 2hecz(\theta - \varphi^2) \}$. Therefore, by (13),

$$p_5 = -\frac{8hec}{a^2}(\theta^2 - \varphi^2 z) + 44acz. \qquad (27)$$

Hence the quintic becomes

$$F(x) = x^5 + \left\{ \frac{20he\,(\theta^2 - \varphi^2 z)}{a} + 15a^2 z \right\} x$$

$$+ \left\{ - \frac{8hec}{a^2} (\theta^2 - \varphi^2 z) + 44acz \right\} = 0. \qquad (28)$$

The criterion of solvability of a trinomial quintic of the kind under consideration is therefore that the coefficients p_4 and p_5 be related in the manner indicated in the form (28); while at the same time the last four of the equations (15), modified by putting $g = k = 0$, subsist between the rational quantities a, c, e, h, θ, φ. From these data, the three following equations may be deduced, v being put for $\frac{c^2}{a^3}$:

$$\left.\begin{aligned} &8ev^3 - 4zv^2 + z\,(3 - 4e)\,v - z^2 = 0, \\ &\frac{2p_4}{a^2} + \frac{5p_5}{ac} = 250z, \\ &4v\,(ze + 4zv - 8v^2) = \left(- 3z + \frac{p_4}{5a^2}\right)\{z + 4v\,(e - 1) + 8v^2\}. \end{aligned}\right\} \qquad (29)$$

The first of these equations is obtained from a comparison of the two equations (9), the second is obtained by putting p_4 and p_5 respectively equal to the values they have in (28); and the third is obtained by putting p_4 equal to the coefficient of the first power of x in (28).

§15. If any rational values of e and v can be found satisfying the first of equations (29), let such values be taken. Then, from the second and third of (29), a^2 and ac can be found. Therefore a and c are known. Therefore, by (21), θ and φ are known. Therefore, by (9), h is known. In this way all the elements for the solution of the quintic are obtained.

§16. For example, the three equations (29) are satisfied by the values.

$$e = \frac{1}{2},\ z = v = \frac{5}{4},\ c = \frac{25}{2},$$

$$a = 5, \therefore \theta = 0,\ \varphi = -\frac{4}{75},\ h = \frac{45 \times 25^3}{16}.$$

When these values are substituted in (28), the quintic becomes

$$x^5 + \frac{625x}{4} + 3750 = 0.$$

Then the values of u_1^5, u_2^5, u_3^5, u_4^5, obtained from the expression for u_1^5, in §3, are

$$u_1^5 = \frac{625}{4}\left\{-1-\sqrt{\left(\frac{5}{4}\right)}+\frac{3}{\sqrt{5}}\sqrt{\left(\frac{5}{4}+\sqrt{\frac{5}{4}}\right)}\right\},$$

$$u_4^5 = \frac{625}{4}\left\{-1-\sqrt{\left(\frac{5}{4}\right)}-\frac{3}{\sqrt{5}}\sqrt{\left(\frac{5}{4}+\sqrt{\frac{5}{4}}\right)}\right\},$$

$$u_2^5 = \frac{625}{4}\left\{-1+\sqrt{\left(\frac{5}{4}\right)}-\frac{3}{\sqrt{5}}\sqrt{\left(\frac{5}{4}-\sqrt{\frac{5}{4}}\right)}\right\},$$

$$u_3^5 = \frac{625}{4}\left\{-1+\sqrt{\left(\frac{5}{4}\right)}+\frac{3}{\sqrt{5}}\sqrt{\left(\frac{5}{4}-\sqrt{\frac{5}{4}}\right)}\right\}.$$

Hence, $r_1 = u_1 + u_2 + u_3 + u_4 = -1.52887 - 2.25035 + 2.48413 - 3.65639 = -4.95148.$

When any Relation is Assumed between the Six Unknown Quantities.

§17. In the case in which $u_1 u_4$ was taken equal to $u_2 u_3$ a relation was in fact assumed betwixt the six unknown quantites a, c, e, h, θ, φ; for, as we saw, to put $u_1 u_4 = u_2 u_3$ is tantamount to putting $a = 0$. Hence, as was noticed in §7, we had only five unknown quantities to be found from six equations. Now, when any relation whatever is assumed betwixt the six unknown quantities, the root of the quintic can be found in terms of the given coefficients p_2, p_3, etc., without any definite numerical values being assigned to the coefficients, because six rational quantities can always be found from seven equations.

The General Case.

§18. We have hitherto been dealing with solvable quintics, assumed to be subject to some condition additional to what is involved in their solvability. We have now to consider how the general case is to be dealt with. That is to say, we here make no supposition regarding the equation of the fifth degree $F(x) = 0$ except that it wants the second term and is solvable algebraically. In this case it is impossible to find the roots in terms of the coefficients p_2, p_3, etc., while these coefficients retain their general symbolic forms. But the equations in §3 enable us to find the roots when the coefficients receive any definite numerical values that render the equation solvable. For, we have the six equations (15) to determine the six unknown quantities a, c, e, h, θ, φ; and we can eliminate five of the unknown quantities, and obtain an equation involving only one unknown quantity. The unknown quantity appearing in this equation has a rational value; but there are known methods of finding the rational roots of any algebraical equation with definite numerical coefficients. Therefore the unknown quantity can be found. In this way all the six unknown quantities a, c, e, h, θ, φ, can be found. Hence the roots of the quintic can be found.

§19. *Note.*—From my friend, Mr. J. C. Glashan, of Ottawa, who read in manuscript the paper on the "*Principles of the Solution of Equations of the Higher Degrees,*" but did not see the present paper on the "*Resolution of Solvable Equations of the fifth Degree,*" I learn that, setting out from propositions demonstrated in the "Principles," he has arrived at important conclusions in the theory of Quintics, which will be made public without delay; but he has not communicated to me either his method or the results he has obtained.

PROCEEDINGS

OF

THE CANADIAN INSTITUTE,

SESSION 1884.

NINTH ORDINARY MEETING.

The Ninth Ordinary Meeting of the Session 1883-4 was held on Saturday, January the 12th, the President in the chair.

The minutes of last meeting were read and confirmed.

Mr. James Bain, jun. and Mr. John Notman, were appointed to represent the Institute on the Board of the Industrial Exhibition Association.

The Rev. William Clark of Trinity College was elected a member.

The following list of donations and exchanges were presented:

1. Transactions of the Royal Geological Society of Cornwall, Vol. X, Part 5.
2. Science, Vol. II, Nos. 46, 47 and 48.
3. The Ornithologist and Oölogist, for Jan., 1884.
4. Proceedings of the Royal Geographical Society for December, 1883.
5. Transactions of the Royal Scottish Society of Arts, Vol. XI, Part 1.
6. Science Record, Vol. II, No. 2, Dec. 15th, 1883.
7. Historical Collections of the Essex Institute, Vol. XX, Nos. 1 to 9, Jan. to Sept., 1883.
8. Journal of the Royal Microscopical Society, Vol. III, Part 6, Dec., 1883.
9. Journal of the Franklin Institute, for Jan., 1884.
10. Journal of the Transactions of the Victoria Institute, Vol. XVII, No. 67.
11. The Canadian Practitioner, Jan., 1884.
12. Micrometry, Reprinted from the Proceedings of the American Society of Microscopists, Chicago Meeting, 1883.
13. An Examination of some Controverted Points of the Physiology of Voice, by J. Wesley Mills, M. A., M. D. Read before the American Association for the Advancement of Science at Montreal, Aug., 1882.

14. Proceedings of the Philosophical Society of Glasgow, Vol. XIV.
15. The Canadian Entomologist for Nov., 1883.
16. Bulletin de la Société Géologique de France, Vol. XII, No. 1.
17. Mémoires des Travaux de la Société des Ingénieurs Civils for Oct., 1883.
18. Transactions of the Manchester Geological Society, Vol. XVII, Part 11.

Prof. R. Ramsay Wright, then presented the substance of a paper on the "Nervous System of the Cat-fish." Special attention was directed to the 'clavate' cells of the epidermis, to the branching of the fifth nerve, and to the relation existing between the air-bladder and the auditory organ. The paper is the first of a series on the cat-fish (*amiurus catus*) and will appear in a subsequent fasciculus of the Proceedings.

TENTH ORDINARY MEETING.

The Tenth Ordinary Meeting of the Session 1883-'84, was held on Saturday, the 19th of January, the President in the chair.

The minutes of last meeting were read and confirmed.

The following gentlemen were elected members of the Institute:—R. E. Kingsford, M.A.; Mr. D. O'Brooke, and Mr. J. Alfred Wilson.

The following list of donations and exchanges was read by the Hon. Secretary:

1. Science, Vol. III., No. 49, for January 11, 1884.
2. Journal of Speculative Philosophy, Vol. XVII., No. 4, October, 1883.
3. Proceedings of the Royal Geographical Society, Vol. VI., No. 1, Jan., 1884.
4. Trübner's American, European, and Oriental Literary Record, Vol. IV., Nos. 9 to 10, September to October, 1883.

Mr. W. Waugh Lauder then read a paper entitled, "The History of Musical Instruments." The instruments specially noticed were the Piano, Violin and Organ. In the discussion which followed, Mr. Notman, Mr. Wm. Anderson, Mr. Geo. E. Shaw, Mr. Paul Frind and Mr. Geo. Murray, took part.

ELEVENTH ORDINARY MEETING.

The Eleventh Ordinary Meeting of the Session 1883-'84, was held on Saturday, January the 26th, 1884, the Second Vice-President, Mr. Geo. Murray, in the chair.

The Minutes of last meeting were read and confirmed.

Mr. Stephen Nairn and Dr. John McConnell were elected members of the Institute.

The following list of donations and exchanges was read by the Hon. Secretary :

1. Science, Vol. III., No. 50, January 18th, 1884.
2. The Monthly Weather Review for December.
3. Map of Winnipeg and environs, by Alan Macdougall, Esq., C. E., F.R.S.E. Presented by the author.
4. Proceedings of the Boston Society of Natural History, Vol. XXII., Part 2, Nov., 1882, to Feb., 1883.

The President, Mr. J. M. Buchan, then read a paper entitled "Flora Hamiltonensis," a list of plants collected in the vicinity of Hamilton.

FLORA HAMILTONENSIS.

In preparing this list I have adopted the classification and nomenclature employed by Professor Gray, in the fifth edition of his Manual. Plants, the names of which are marked with an asterisk, are admitted on the authority of the late Judge Logie. All plants included occur within seventeen miles of Hamilton :—

RANUNCULACEÆ.

Clematis verticillaris, DC. Chedoke.
" Virginiana, L.
Anemone cylindrica, Gray.
" Virginiana, L.
" Pennsylvanica, L.
" nemorosa, L.
* " nemorosa, L., var. quinquefolia. Oaklands.
Hepatica triloba, Chaix.
" acutiloba, DC.
Thalictrum anemonoides, Mx.
" dioicum, L.
" Cornuti, L.
Ranunculus aquatilis, L., var. trichophyllus, Chaix.
* " multifidus, Pursh.
" abortivus, L.
" sceleratus, L.

RANUNCULACEÆ—*Continued.*

Ranunculus recurvatus, Poir.
" Pennsylvanicus, L. Also at Fullarton.
" fascicularis, Muhl. Also at Walkerton.
" repens, L.
" bulbosus, L.
" acris, L.
Caltha palustris, L.
Coptis trifolia, Salisb.
Aquilegia Canadensis, L.
Delphinium Consolida, L.
Actaea spicata, L. var. rubra, Mx.
" alba, Bigel.

MAGNOLIACEÆ.

Liriodendron tulipifera, L.

MENISPERMACEÆ.

Menispermum Canadense, L.

BERBERIDACEÆ.

* Berberis vulgaris, L.
Caulophyllum thalictroides, Mx.
Podophyllum peltatum, L.

NYMPHÆACEÆ.

Nymphæa tuberosa, Paine.
Nuphar advena, Ait.

SARRACENIACEÆ.

Sarracenia purpurea, L.

PAPAVERACEÆ.

Papaver argemone, L.
Chelidonium majus, L.
Sanguinaria Canadensis, L.

FUMARIACEÆ.

* Adlumia cirrhosa, Raf. Rare.
Dicentra cucullaria, DC.
" Canadensis, DC.
* Fumaria officinalis, L. Burlington Beach.

CRUCIFERÆ.

Nasturtium officinale, R. Br.
" silvestre, R. Br., Dundas.
" palustre, DC.
" " " var. hispidum.
Dentaria diphylla, Mx.
" heterophylla, Nutt.
" laciniata, Muhl.
Cardamine rhomboidea, DC.
* Cardamine rhomboidea, DC. var. purpurea, Torr.
* Cardamine patensis, L. Millgrove.
" hirsuta, L.
Arabis hirsuta, Scop.
" laevigata, DC.
" Canadensis, L.
Erysimum cheiranthoides, L.
Sisymbrium officinale, Scop.
" canescens, Nutt. Burlington Beach.
Brassica sinapistrum, Boissier.
" nigra, Gray.
Camelina sativa, Crantz. Also at Paris.
Capsella Bursa-pastoris, Moench.
Lepidium Virginicum, L.
" ruderale, L.
" campestre, L.
Cakile Americana, Nutt.

CAPPARIDACEÆ.

Polanisia graveolens, Raf.

VIOLACEÆ.

Viola blanda, Willd.
" cucullata, Ait.
" " " var. cordata.
* " sagittata, Ait. The Cemetery.

VIOLACEÆ—*Continued.*

Viola canina L. var. silvestris, Regel.
" rostrata, Pursh.
" Canadensis, L.
" pubescens, Ait.

CISTACEÆ.

Helianthemum Canadense, Mx.
Lechaea minor, Lam.

DROSERACEÆ.

* Drosera rotundifolia, L. Ancaster.

HYPERICACEÆ.

* Hypericum Kalmianum, L.
* " ellipticum, Hook. Freeman's Lot.
" perforatum, L.
" corymbosum, Muhl.
" mutilum, L.
Elodes Virginica, Nutt.

CARYOPHYLLACEÆ.

Saponaria officinalis, L.
* Silene inflata, Smith.
" antirrhina, L.
" noctiflora, L.
Lychnis Githago, Lam.
Arenaria serpyllifolia, L. Also at St. Thomas.
" stricta, Mx.
* " lateriflora, L. Burlington Beach.
Stellaria media, Smith.
" longiflora, Muhl.
" longipes, Goldie.
Cerastium vulgatum, L.
" viscosum, L.
* " oblongifolium, Torr. [Query.] Woods behind Captain Nichols's Farm.
" arvense, L.

PORTULACACEÆ.

Portulaca oleracea, L.
" grandiflora, Hook.
Claytonia Virginica, L.

MALVACEÆ.

Malva rotundifolia, L.
" moschata, L.
Abutilon Avicennæ, Gaertn. Dundas. Mouth of Stony Creek.

TILIACEÆ.

Tilia Americana, L.

LINACEÆ.

Linum Virginianum, L.

GERANIACEÆ.

Geranium maculatum, L.
" pusillum, L.
" Robertianum L.

GERANIACEÆ—*Continued.*

Erodium Cicutarium, L'Her.
Impatiens pallida, Nutt.
" fulva, Nutt.
Oxalis stricta, L.

RUTACEÆ.

Xanthium Americanum, Mill.

ANACARDIACEÆ.

Rhus typhina, L.
" toxicodendron, L.
" " " var. radicans. Mountain, above Reservoir.

VITACEÆ.

Vitis Labrusca, L. Mountain, East of Reservoir.
" cordifolia, Lam.
" cordifolia, Lam., var. riparia, Gray. This well-marked variety is very common, but I have never seen it in either flower or fruit.
Ampelopsis quinquefolia, Mx.

RHAMNACEÆ.

* Rhamnus alnifolia, L'Her. Millgrove.
Ceanothus Americanus, L.

CELASTRACEÆ.

Celastrus scandens, L.
Euonymus Americanus L., var. obovatus, Torr. and Gray.

SAPINDACEÆ.

Staphylea trifolia, L.
Acer spicatum, Lam.
" saccharinum, Wang.
" dasycarpum, Ehrhart.
" rubrum, L.

POLYGALACEÆ.

* Polygala Nuttallii, Torr. and Gray. [Query.] Prince's Island.
" verticillata, L.
" Senega, L.
" paucifolia, Willd.

LEGUMINOSÆ.

Trifolium arvense, L.
" pratense, L.
" repens, L.
Melilotus officinalis, Willd.
" alba, Lamm.
Medicago lupulina, L.
Robinia Pseudacacia, L.
Astragalus Canadensis, L.
" Cooperi, Gray.
Desmodium nudiflorum, DC.
" acuminatum, DC.

LEGUMINOSÆ—*Continued.*

Desmodium cuspidatum, Torr. & Gr.
" paniculatum, DC.
" Canadense DC.
* Lespedeza repens, Torr. and Gray. The Dell, Ancaster.
* " violacea, Pers. The Dell, Ancaster.
" hirta, Ell.
" capitata, Mx.
*Vicia hirsuta, Koch.
Lathyrus maritimus, Bigelow.
* " pratensis, L. Ancaster.
" ochroleucus, Hook.
" palustris, L.
" palustris, L., var. myrtifolius, Gray. Also at Toronto.
Apios tuberosa, Moench
Phaseolus diversifolius, Pers.
Amphicarpæa monoica, Nutt.
* Baptisia tinctoria, R. Br.

ROSACEÆ.

Prunus Americana, Marshall.
" Pennsylvanica, L.
" Virginiana, L.
" serotina, Ehrhart.
* Spiræa salicifolia, L. Millgrove.
Gillenia trifoliata, Moench.
Agrimonia Eupatoria, L.
Geum album, Gmelin.
" strictum, Ait.
" rivale, L.
Waldsteinia fragarioides, Tratt.
Potentilla Norvegica, L.
" paradoxa, Nutt.
" Canadensis, L.
" argentea, L. Also at Paris.
" anserina, L.
" palustris, Scop.
Fragaria Virginiana, Ehrhart.
" vesca, L.
* Dalibarda repens, L.
Rubus odoratus, L.
" triflorus, Richardson.
" strigosus. Mx.
" occidentalis, L.
" villosus, Ait.
" Canadensis, L.
Rosa Carolina. L.
" lucida, Ehrhart.
" blanda, Ait.
" rubiginosa, L.
* " micrantha, Smith.
Cratægus oxyacantha, L. Spontaneous on bluff overlooking Dundas Marsh.
" coccinea, L.

ROSACEÆ—*Continued.*

Cratægus tomentosa, L.
" tomentosa, L., var. pyrifolia, Gray.
" Crus-Galli, L.
Pyrus coronaria, L.
* " arbutifolia, L., var. melanocarpa. Millgrove Marsh.
" aucuparia, Gærtn. Apparently indigenous near Dundas Marsh.
Amelanchier Canadensis, Torr. and Gray, var. *Botryapium*, Gray.
" Canadensis, Torr. and Gray, var. rotundifolia, Gray.
" Canadensis, Torr. and Gray, var. oblongifolia, Gray.
" Canadensis, Torr. and Gray, var. with notched petals 2-4 feet high flowering a few days later than the preceding variety.

SAXIFRAGACEÆ.

Ribes cynosbati, L.
* " hirtellum, Mx.
" rotundifolium, Mx.
" lacustre, Poir.
" floridum, L.
* " rubrum, L. Millgrove.
Saxifraga Virginiensis, Mx.
*Parnassia Caroliniana, Mx. Ancaster.
Mitella diphylla, L.
" nuda, L.
Tiarella cordifolia, L.
* Chrysosplenium Americanum, Schwein. Ancaster.

CRASSULACEÆ.

Penthorum sedoides, L.
Sedum ternatum, Mx. The Mountain.
" Telephium, L.

HAMAMELACEÆ.

Hamamelis Virginica, L.

HALORAGEÆ.

Myriophyllum *spicatum*, L.
* " verticillatum, L.
* " heterophyllum, Mx. Waterdown Creek.

ONAGRACEÆ.

Circæa Lutetiana, L.
" Alpina, L.
Epilobium angustifolium, L.

ONAGRACEÆ—*Continued.*

Epilobium coloratum, Muhl.
Œnothera biennis, L., var. grandiflora.
" biennis, L., var. muricata.
* " pumila, L. Land's Inlet.

LYTHRACEÆ.

Nesaea verticillata, H. B. K.

CUCURBITACEÆ.

* Sicyos angulatus, L.
Echinocystis lobata, Torr. and Gray. In a thicket near Waterdown Creek, and apparently indigenous.

UMBELLIFERÆ.

* Hydrocotyle Americana, L. Ancaster.
Sanicula Canadensis, L.
" Marilandica, L.
Heracleum lanatum, Mx.
Pastinaca sativa, L.
Archangelica atropurpurea, Hoffm.
Conioselinum Canadense, Torr. and Gray.
* Thaspium aureum, Nutt. Prince's Island.
Zizia integerrima, DC.
Cicuta maculata, L.
" bulbifera, L.
Sium lineare, Mx
Cryptotænia Canadensis, DC.
Osmorrhiza longistylis, DC.
" brevistylis, DC.
Carum carui, L.

ARIALIACEÆ.

Aralia racemosa, L.
" nudicaulis, L.
" quinquefolia, Gray.
" trifolia, Gray.

CORNACEÆ.

Cornus Canadensis, L.
" florida, L.
" circinata, L'Her.
" stolonifera, Mx.
" paniculata, L'Her.
" alternifolia, L.

CAPRIFOLIACEÆ.

* Linnæa borealis, Gronov. Lake Medad.
Symphoricarpus racemosus, Mx.
* Symphoricarpus vulgaris, Mx.
* Lonicera flava, Sims.
" parviflora, Lam.
" hirsuta, Eaton.
" ciliata, Muhl.
Lonicera Tartarica, L. Mountain side, west of Queen Street.

CAPRIFOLIACEÆ—*Continued.*

Diervilla trifida, Moench.
Triosteum perfoliatum, L.
Sambucus Canadensis, L.
" pubens, Mx.
* Viburnum nudum, L., var. cassinoides. Millgrove.
" pubescens, Pursh.
" acerifolium, L.
" Opulus, L. Also at Fullarton.

RUBIACEÆ.

Galium Aparine, L.
" asprellum, Mx.
" trifidum, L.
" triflorum, Mx.
" boreale, L.
" verum, L
Cephalanthus occidentalis, L.
Mitchella repens, L.

DIPSACEÆ.

Dipsacus silvestris, Mill.

COMPOSITÆ.

Liatris cylindracea, Mx. Railway cutting in Burlington Heights. Also at the Humber.
Eupatorium purpureum, L.
" perfoliatum, L.
" ageratoides, L.
Aster corymbosus, Ait.
" macrophyllus, L.
" lævis, L., var. lævigatus, Willd.
" lævis, L., var. cyaneus, Hoffm.
" azureus, Lindley.
" undulatus, L.
" cordifolius, L.
" sagittifolius, Willd.
" multiflorus, Ait.
" Tradescanti, L.
" miser, L.
" simplex, Willd.
" tenuifolius L. Also at Port Rowan.
" carneus, Nees.
" puniceus, L.
" Novæ Angliæ.
" graminifolius, Pursh.
" ptarmicoides, Torr. and Gray.
Erigeron Canadense, L.
Erigeron bellidifolium, L.
" Philadelphicum, L.
" annuum, Pers.
" strigosum, Muhl.
Diplopappus umbellatus, Torr. and Gray.
Solidago squarrosa, Muhl.
" bicolor L.

COMPOSITÆ—*Continued.*

Solidago bicolor L. var. concolor.
" latifolia, L.
" caesia, L.
* " stricta, Ait.
" speciosa, Nutt.
" Virga-aurea, L., var. humilis.
" rigida, L.
" patula, Muhl.
" arguta, Ait., var. juncea.
" " " " scabrella.
" Muhlenbergii, Torr. and Gray.
" altissima, L.
" ulmifolia, Muhl.
" nemoralis, Ait.
" Canadensis, L.
" " " var. scabra.
" serotina, Ait.
" gigantea, Ait.
" lanceolata, L.
Inula Helenium, L
Polymnia Canadensis, L.
" " var. discoidea.
Ambrosia artemisiæfolia, L.
Xanthium strumarium, L., var. echinatum.
Xanthium spinosum, L.
Rudbeckia laciniata, L.
" hirta, L.
Helianthus strumosus, L.
" divaricatus, L.
* " divaricatus, L., var. with leaves whorled in threes. Prince's Island.
" decapetalus, L.
Bidens frondosa, L.
" connata, Muhl.
" " var. petiolata.
" cernua, L.
" chrysanthemoides, Mx.
" Beckii, Torr.
* Helenium autumnale, L.
Maruta cotula, DC.
Achillea millefolium, L.
Leucanthemum vulgare, Lam.
Tanacetum vulgare, L.
Artemisia Canadensis, Mx.
Gnaphalium decurrens, Ives. Also at Fullarton.
* Gnaphalium polycephalum, Mx.
" uliginosum, L.
Antennaria margaritacea, R. Brown.
" plantaginifolia, Hook.
Erechthites hieracifolia, Raf.
Senecio vulgaris, L.
* Senecio palustris, Hook. Roadside, Burlington.

COMPOSITÆ—*Continued.*

Cirsium lanceolatum, Scop.
" discolor, Spreng.
" arvense, Scop.
Onopordon Acanthium, L.
Lappa officinalis, Allioni.
* Lampsana communis, L.
Leontodon autumnale, L.
Hieracium Canadensis, Mx.
" scabrum, Mx.
* " venosum, L. Ancaster.
" paniculatum, L.
Nabalus albus, Hook.
" " var. serpentarius.
" altissimus, Hook.
* Nabalus Fraseri, DC., var. integrifolius. Prince's Island.
Taraxacum Dens-leonis, Desf.
Lactuca sativa, L. Apparently spontaneous below Mountain View Hotel.
" Canadensis, L.
Mulgedium leucophæum, DC.
Sonchus oleraceus, L.
" asper, Vill.
" arvensis, L.

LOBELIACEÆ.

* Lobelia cardinalis, L. Near Waterdown.
" syphilitica, L.
" inflata, L.
" spicata, Lam.

CAMPANULACEÆ.

Campanula rotundifolia, L.
" aparinoides, Pursh.
" Americana, L.
Specularia perfoliata, DC.

ERICACEÆ.

Gaylussacia resinosa, Torr. and Gray.
Vaccinium vacillans, Solander.
* Chiogenes hispidula, Torr. and Gray. Millgrove.
Gaultheria procumbens, L.
* Cassandra calyculata, Don. Millgrove.
Ledum latifolium, Ait. Lake Medad.
Pyrola rotundifolia, L., var. asarifolia.
" " " uliginosa.
" elliptica, Nutt.
" secunda, L.
* Moneses uniflora, L. Lake Medad.
Chimaphila umbellata, Nutt.
* Pterospora Andromedea, Nutt. Wood on Cline's Farm.
Monotropa uniflora, L.

AQUIFOLIACEÆ.

Ilex verticillata, Gray.
Nemopanthes Canadensis, DC. Millgrove Marsh.

PLANTAGINACEÆ.

Plantago major, L.
" Kamtschatica, Cham. Also at Toronto and London.
" lanceolata, L.

PRIMULACEÆ.

Trientalis Americana, Pursh.
Lysimachia thyrsiflora, L.
* " stricta, Ait. East Flamboro'.
" quadrifolia, Ait.
" ciliata, L.
* Anagallis arvensis, L.
Samolus Valerandi, L., var. Americanus, Gray.

LENTIBULACEÆ.

Utricularia vulgaris, L.
" intermedia, Hayne.

OROBANCHACEÆ.

Epiphegus Virginiana, Bart.
* Conopholis Americana, Wallroth. Wood behind Cline's Mill.
Aphyllon uniflorum, Torr. and Gray.

SCROPHULARIACEÆ.

Verbascum Thapsus, L.
" Blattaria, L.
Linaria vulgaris, Mill.
Scrophularia nodosa, L.
Chelone glabra, L.
Pentstemon pubescens, Solander.
Mimulus ringens, L.
* Gratiola Virginiana L. Hall's Corners.
Ilysanthes gratioloides, Benth.
Veronica Anagallis, L.
" Americana, Schweinitz.
* " scutellata, L. Millgrove.
" officinalis, L.
" serpyllifolia, L.
" peregrina, L.
" arvensis, L. Also at Fullarton.
* " triphyllos.
* Gerardia purpurea, L. Waterdown Creek.
* " tenuifolia, Vahl. Prince's Island.
" flava, L.
Gerardia quercifolia, Pursh.
" pedicularia, L.
Castilleia coccinea, Spreng.
Pedicularis Canadensis, L.
Melampyrum Americanum, Mx.

VERBENACEÆ.

Verbena hastata, L.
" urticifolia, L.
Phryma Leptostachya, L.

LABIATÆ.

Teucrium Canadense, L.
Mentha viridis, L.
" piperita, L.
" Canadensis, L.
Lycopus Virginicus, L.
" Europæus, L. var. sinuatus.
* Pycnanthemum incanum, Mx. Oaklands.
Calamintha Clinopodium, Benth. Red Creek.
Satureia hortensis, L. Burlington Heights.
Hedeoma pulegioides, Pers.
Collinsonia Canadensis, L.
Monarda didyma, L. West of Capt. Nichols's Farm.
" fistulosa, L.
Lophanthus nepetoides, Benth. Waterdown Creek.
Nepeta cataria, L.
Brunella vulgaris, L.
Scutellaria galericulata, L.
" lateriflora, L.
* Marrubium vulgare, L.
Galeopsis tetrahit, L.
Stachys palustris, L., var. aspera.
Leonurus cardiaca, L.
* Lamium amplexicaule, L. Wylde's Grounds.
" album, L.

BORRAGINACEÆ.

Echium vulgare, L.
* Symphytum officinale, L. Cumminsville.
Onosmodium Carolinianum, DC. Burlington Beach.
Lithospermum arvense, L.
* " longiflorum, Spreng. Burlington Heights.
Myosotis palustris, Withering.
" " var. laxa.
Echinospermum Lappula, Lehm.
Cynoglossum officinale, L.
" Morisoni, DC.

HYDROPHYLLACEÆ.

Hydrophyllum Virginicum, L.
" Canadensis, L. Chedoke.
" appendiculatum, Mx. Red Creek.

POLEMONIACEÆ.

Phlox divaricata, L.

CONVOLVULACEÆ.

Convolvulus arvensis, L.
Calystegia sepium, R. Br.
* " sepium, R. Br., var. repens Railway Track, East.
* " spithamœa, Pursh. Dundas.
Cuscuta Gronovii, Willd.

SOLANACEÆ.

Solanum dulcamara, L.
" nigrum, L.
Physalis viscosa, L.
Lycium vulgare, Dunal. Near Van Wagner's Farm. Saltfleet.
Hyoscyamus niger, L.
Datura Stramonium, L.
" tatula, L. Burlington Beach.
Nicotiana rustica, L. West Flamboro and Shore of Bay, near G. W. R. Station.

GENTIANACEÆ.

*Halenia deflexa, Grisebach. Ancaster.
Gentiana crinita, Froel.
* " alba, Muhl. Rare.
" Andrewsii, Griseb. Dundas Marsh.
* " acuta, Mx. [Query].
* Menyanthes trifoliata, L. Lake Medad.

APOCYNACEÆ.

Apocynum androsæmifolium, L.
" cannabinum, L.

ASCLEPIADACEÆ.

Asclepias Cornuti, Decaisne.
" phytolaccoides, Pursh.
" incarnata, L.
" tuberosa, L.

OLEACEÆ.

Fraxinus Americana, L.
" sambucifolia, Lam.

ARISTOLOCHIACEÆ.

Asarum Canadense, L.

PHYTOLACCACEÆ.

Phytolacca decandra, L. Stony Creek. Also at Port Rowan.

CHENOPODIACEÆ.

Chenopodium album, L.
" glaucum, L.
" hybridum, L.
" Botrys, L.
" ambrosioides, L.

CHENOPODIACEÆ—*Continued.*

Blitum capitatum, L.
" Bonus Henricus, Reichenbach.
Atriplex patula, L. var. littoralis.
" " " " hastata.

AMARANTACEÆ.

Amarantus hypochondriacus, L.
" paniculatus, L. Also at Guelph.
" retroflexus, L.
" albus, L.

POLYGONACEÆ.

Polygonum Pennsylvanicum, L.
" incarnatum, Ell.
" Persicaria, L
" Hydropiper, L.
" acre, H. B. K.
" hydropiperoides, Mx.
" amphibium, L, var. aquaticum.
" amphibium, L., var. terrestre.
" Virginianum, L. Red Creek.
" aviculare, L.
" " " var. erectum.
" arifolium, L.
" sagittatum, L.
" Convolvulus, L.
" dumetorum, L.
Fagopyrum esculentum, Moench.
Rumex orbiculatus, Gray.
" verticillatus, L.
" crispus, L.
" obtusifolius, L. East of City.
" acetosella, L.

CERATOPHYLLACEÆ

Ceratophyllum demersum, L.

LAURACEÆ.

Sassafras officinale, Nees.
* Lindera Benzoin, Meisner. The Dell Ancaster.

THYMELEACEÆ.

Dirca palustris, L. Carlisle.

ELÆAGNACEÆ.

Shepherdia Canadensis, Nutt.

SANTALACEÆ.

Comandra umbellata, Nutt.

EUPHORBIACEÆ.

Euphorbia polygonifolia, L.
" maculata, L.
" hypericifolia, L. Waterdown.

EUPHORBIACEÆ—*Continued.*

* Euphorbia platyphylla, L. The beach near Stony Creek.
* " obtusata, Pursh. [Query].
" Helioscopia, L.
" cyparissias, L.
" Peplus, L.
Acalypha Virginica, L.

URTICACEÆ.

Ulmus fulva, Mx.
" Americana, L.
Urtica gracilis, Ait.
Laportea Canadensis, Gaudichaud.
Pilea pumila.
Boehmeria cylindrica, Willd.
Cannabis sativa, L.
Humulus Lupulus, L. Red Creek.

PLATANACEÆ.

Platanus occidentalis, L.

JUGLANDACEÆ.

Juglans cinerea, L.
" nigra, L.
Carya alba, Nutt.
" porcina, Nutt.
" amara, Nutt.

CUPULIFERÆ.

Quercus alba, L.
" macrocarpa, Mx. E. Flamborough and Burlington Beach.
" Prinus, L., var. acuminata, Mx.
" coccinea, Wang., var. tinctoria, Gray.
" rubra, L.
Castanea vesca, L., var. Americana, Mx.
Fagus ferruginea, Ait.
Corylus rostrata, Ait.
Ostrya Virginica, Willd.
Carpinus Americana, Mx.

BETULACEÆ.

Betula lenta, L.
" lutea, Mx.
" papyracea, Ait.
Alnus incana, Willd.

SALICACEÆ.

* Salix tristis, Ait. Rocks near Ancaster.
" humilis, Marshall.
" discolor, Muhl.
" cordata, Muhl., var. myricoides, Gray.
" livida, Wahl., var. occidentalis, Gray.

SALICACEÆ—*Continued.*

Salix lucida, Muhl.
" nigra, Marsh.
" longifolia, Muhl. Burlington Beach.
Populus tremuloides, Mx.
" grandidentata, Mx.
" balsamifera, L.

CONIFERÆ.

Pinus strobus, L.
Abies nigra, Poir. Millgrove.
" alba, Mx. Lake Medad.
" Canadensis, Mx.
" balsamea, Marshall.
Larix Americana, Mx.
Thuja occidentalis, L.
Juniperus Virgiania, L.
" communis, L.
Taxus baccata, L., var. Canadensis, Gray.

ARACEÆ.

Arisæma triphyllum, Torr.
Calla palustris, L.
Symplocarpus fœtidus, Salisb.
Acorus Calamus, L.

LEMNACEÆ.

Lemna minor, L.
" polyrrhiza, L.
" trisulca, L.
Wolffia Columbiana, Karsten.
" Brasiliensis, Weddell.

TYPHACEÆ.

Typha latifolia, L.
Sparganium eurycarpum, Engelm.
" simplex, Hudson, var. angustifolium, Gray.

NAIADACEÆ.

Potamogeton natans, L.
" amplifolius, Tuckerm.
" lucens, L., var. minor.
" perfoliatus, L.
" compressus, L.
" pauciflorus, Pursh.
" pectinatus, L.

ALISMACEÆ.

Alisma plantago, L., var. Americanum, Gray.
Sagittaria variabilis, Engelm.

HYDROCHARIDACEÆ.

Anacharis Canadensis, Planchon.
Vallisneria spiralis, L.

ORCHIDACEÆ.

Orchis spectabilis, L.

ORCHIDACEÆ—*Continued.*

* Habenaria tridentata, Lindl. Millgrove.
* " virescens, Spreng. Prince's Island.
* " viridis, R. Br., var. bracteata, Reichenbach. Mountain at head of Queen Street.
* " hyperborea, R. Br. Sulphur Spring.
" Hookeri, Torr.
* " orbiculata, Torr.
* " leucophæa, Gray. Millgrove.
* " psychodes, Gray. Millgrove.
* " fimbriata, R. Br. Land's Farm.
Goodyera pubescens, R. Br.
* Spiranthes cernua, Richardson. The Dell, Ancaster.
* Pogonia ophioglossoides, Nutt. Millgrove.
* Calypso borealis, Salisb. Lake Medad.
* Corallorhiza innata, R. Br. Prince's Island.
" odontorhiza, Nutt.
" multiflora, Nutt.
Cypripedium parviflorum, Salisb.
" pubescens, Willd.
* " spectabile, Swartz. Lake Medad.
* " acaule, Ait. Millgrove.

AMARYLLIDACEÆ.

* Hypoxys erecta, L. Prince's Island.

IRIDACEÆ.

Iris versicolor, L.
Sisyrinchium Bermudiana, L., var. anceps, Gray.

DIORCOREACEÆ.

Dioscorea villosa, L. Near Dundas Marsh.

SMILACEÆ.

Smilax hispida, Muhl.
" herbacea, L.

LILIACEÆ.

Trillium grandiflorum, Salisb.
" erectum, L.
" erectum, L., var. album, Pursh.

LILIACEÆ—*Continued.*

Medeola Virginica, L.
Uvularia grandiflora, Smith.
Prosartes lanuginosa, Don.
Streptopus roseus, Mx.
Clintonia borealis, Raf.
Smilacina racemosa, Desf.
" stellata, Desf.
" trifolia, Desf.
" bifolia. Ker.
Polygonatum biflorum, Ell.
Lilium Philadelphicum, L.
* Lilium Canadense, L. Ancaster.
" superbum, L.
Erythronium Americanum, Smith.
Allium tricoccum, Ait.
Asparagus officinalis, L. Burlington Beach.

JUNCACEÆ.

Luzula pilosa, Willd.
" campestris, DC.
Juncus effusus, L.
" bufonius, L.
" tenuis, Willd.
" Alpinus, Villars, var. insignis, Fries.
" acuminatus, Mx.
" nodosus, L.
" nodosus, L., var. megacephalus, Torr.

PONTEDERIACEÆ.

Pontederia cordata L.
Schollera graminea, Willd.

CYPERACEÆ.

Cyperus diandrus, Torr.
" strigosus, L.
" filiculmis, Vahl.
Eleocharis obtusa, Schultes.
" palustris, R. Br.
" acicularis, R. Br.
* Scirpus pungens, Vahl.
" validus, Vahl.
" fluviatilis, Gray.
" atrovirens, Muhl.
" Eriophorum, Mx., var. cyperinus.
Eriophorum Virginicum, L. Millgrove.
* " polystachyon, L. The Dell, Ancaster.
Carex polytrichoides, Muhl.
" bromoides, Schk.
" teretiuscula, Good.
" vulpinoides, Mx.

CYPERACEÆ—*Continued.*

Carex stipata, Muhl.
" sparganioides, Muhl.
" cephalophora, Muhl.
" rosea, Schk.
" tenella, Schk.
" trisperma, Dew.
" stellulata, L., var. scirpoides.
" scoparia, Schk.
" lagopodioides, Schk.
" cristata, Schw.
" straminea, Schk., var. tenera, Dew.
" stricta, Lam.
" crinita, Lam.
" aurea, Nutt.
" gracillima, Schw.
" platyphylla, Carey.
" digitalis, Willd.
" retrocurva, Dew.
" laxiflora, Lam.
" " " var. blanda.
" " " " plantaginea, Booth.
" " " var. latifolia.
" pedunculata, Muhl.
" Novæ Angliæ, Schw.
" Emmonsii, Dew.
" Pennsylvanica, Lam.
" varia, Muhl.
" scabrata, Schw.
" riparia, Curtis.
" comosa, Booth.
" hystricina, Willd.
" tentaculata, Muhl.
" intumescens, Rudge.
" lupulina, Muhl.
" Schweinitzii, Dew.
" Tuckermani, Boott.
" retrorsa, Schw.

GRAMINEÆ.

Leersia Virginica, Willd.
" oryzoides, Swartz.
Zizania aquatica, L.
Alopecurus aristulatus, Mx.
Phleum pratense, L.
Vilfa aspera, Beauv. Burlington Beach.
Vilfa vaginæflora, Torr.
Sporobolus cryptandrus, Gray.
Agrostis scabra, Willd.
" perennans, Tuckerm.
" vulgaris, With.
" alba, L.
Muhlenbergia Mexicana, Trin.
* " diffusa, Schreber.

GRAMINEÆ—*Continued.*

Muhlenbergia glomerata, Trin.
" silvatica, Torr. & Gray.
Cinna arundinacea, L.
Brachyelytrum aristatum, Beauv.
Calamagrostis Canadensis, Beauv.
" confinis, Nutt.
Oryzopsis asperifolia, Mx.
" melanocarpa, Muhl.
* Eleusine Indica, Gærtn.
Dactylis glomerata, L.
Eatonia Pennsylvanica, Gray.
Glyceria Canadensis, Trin.
* " elongata, Trin. Binbrook.
" nervata, Trin.
" pallida, Trin.
" aquatica, Smith.
" fluitans, R. Br.
Poa annua, L.
" compressa, L.
" caesia, Smith.
" serotina, Ehrhart.
" pratensis, L.
" debilis, Torr.
Eragrostis poæoides, Beauv.
* Festuca tenella, Willd.
" ovina, L.
" elatior, L., var. pratensis, Gray.
" nutans, Willd.
Bromus secalinus, L.
" Kalmii, Gray.
" ciliatus, L.
Phragmites communis, Trin.
Lolium perenne, L.
Triticum repens, L.
" " " var. nemorale.
" caninum, L.
Elymus Virginicus, L.
" Canadensis, L.
" " " var. glaucifolius.
" striatus, Willd.
Gymnostichum Hystrix, Schreber.
Danthonia spicata, Beauv.
Avena striata, Mx. Lake Medad.
* Aira flexuosa, L.
Holcus lanatus, L.
* Anthoxanthum odoratum, L.
Phalaris arundinacea, L.
" canariensis, L.
Panicum glabrum, Gaudin.
" sanguinale, L.
" capillare, L.
" latifolium, L.
" xanthophysum, Gray.
" dichotomum, L.

GRAMINEÆ—*Continued.*

Panicum depauperatum, Muhl.
" Crus-Galli, L.
" " " var. hispidum, Gray.
Setaria glauca, Beauv.
" verticillata, Beauv.
" viridis, Beauv.
" Italica, Kunth.
Cenchrus tribuloides, L. G. W. Ry., about a mile east of Dundas.
Andropogon furcatus, Muhl.
" scoparius, Mx.
Sorghum nutans, Gray.

EQUISETACEÆ.

Equisetum arvense, L.
" pratense, Ehrhart.
" silvaticum, L.
" limosum, L.
* " palustre, L. Oaklands.
" hiemale, L.
" variegatum, Schleicher.
" scirpoides, Mx.

FILICES.

Polypodium vulgare, L.
Adiantum pedatum, L.
Pteris aquilina, L.
Pellæa atropurpurea, Link. Mountain below Chedoke.
Woodwardia Virginica, Smith.
* Asplenium Trichomanes, L. Lake Medad.
" thelypteroides, Mx.
" Filix-fœmina, Bernh.
Camptosorus rhizophyllus, Link.
Phegopteris hexagonoptera, Fée.
* " Dryopteris, Fée. Sulphur Spring.
Aspidium Thelypteris, Swartz.
" Noveboracense, Willd.
* " spinulosum, Swartz, var. dilatatum.
" spinulosum, Swartz, var. Boottii.
" spinulosum, Swartz, var. intermedium.
" spinulosum, Swartz, var. dumentorum. Ravine below Chedoke.
" cristatum, Swartz, var. Clintonianum.
" Goldianum, Hook.
" marginale, Swartz.
" acrostichoides, Swartz.

FILICES—*Continued*

Cystopteris bulbifera, Bernh.
" fragilis, Bernh.
Struthiopteris Germanica, Willd.
Onoclea sensibilis, L.
Dicksonia punctilobula, Kunze.
Osmunda regalis. L. Millgrove.
" Claytoniana, L.
" cinnamomea, L.

FILICES—*Continued.*

Botrychium Virginicum, Swartz.
" lunarioides, Swartz.

LYCOPODIACEÆ.

Lycopodium clavatum, L.

HYDROPTERIDES.

Azolla Caroliniana, Willd. Dundas Marsh and Burlington Beach.

NOTES.

Viola striata, Ait. In Professor Macoun's Catalogue of Canadian Plants, this is stated on the authority of the late Judge Logie to be common near Hamilton. I have never found it, and I am sure that it is not common.

Lathyrus venosus, Muhl., occurs at St. Thomas.

Cichorium Intybus, L., has naturalized itself at Port Rowan and Toronto.

Mimulus Jamesii, Torr., is abundant along the stream flowing into Grenadier Pond near the Humber.

Phlox subulata, L. I can confirm, from personal observation, the fact which Mr. Wilkins was, I believe, the first to discover, that this species is indigenous in the County of Norfolk.

Rumex sanguineus, L., occurs at London and Barrie.

Ulmus racemosa, Thomas, occurs at St. Thomas.

Juniperus Sabina, L., *var. procumbens*, Pursh., which I formerly reported as occurring, proves to be J. *Virginiana*, L.

In the discussion which followed, Mr. Geo. E. Shaw, Mr. T. Mackenzie, Mr. Henry Montgomery, Mr. James Bain, jun., and the reader of the paper took part.

Mr. Fred. Phillips read a paper on "The Antiquity of the Negro Race," the object of which was to show that the negro race made its appearance before the white races.

A discussion ensued, in which the President, Mr. John Notman, and Mr. Montgomery took part.

TWELFTH ORDINARY MEETING.

The Twelfth Ordinary Meeting of the Session 1883-'84 was held on Saturday, February 2nd, 1884, Dr. Geo. Kennedy, Third Vice-President, in the chair.

The minutes of last meeting were read and confirmed.

The following list of donations and exchanges received since last meeting was read :—

1. The Financial Reform Almanack for 1884 ; presented by the Cobden Club.

2. Museum of Comparative Zoölogy at Harvard College, Vol. XI., Nos. 5, 6, 7.
3. Proceedings of the American Academy of Arts and Sciences, Vol. XI., pp. 45—210.
4. Journal of the Franklin Institute for February, 1884.
5. Science Record, January 15, 1884.
6. Science, for January 25, 1884.
7. Proceedings of the Academy of Natural Sciences of Philadelphia, Part 2, June to October, 1883.
8. Nye Alcyonider Gorgonider, og Pennatulider, tilhørende Norges Fauna; from the Royal Museum of Bergen. (Norwegian Fauna.)

Prof. G. P. Young then read a paper entitled, "The Real Correspondents of Imaginary Points."

After the reading of the paper, remarks were made upon the subject by Prof. Galbraith and Mr. Alfred Baker.

THIRTEENTH ORDINARY MEETING.

The Thirteenth ''nary Meeting of the Session 1883-'84 was h l on Saturday, February 9th, 1884, the President in the Chair.

The minutes of last meeting were read and confirmed.

The following list of donations and exchanges received since last meeting was read :—

1. Transactions of the New York Academy of Sciences, Vol. II., Nos. 3 to 8. Contents and Title Page, Vol. I.
2. Annals of the New York Academy of Sciences, Nos. 12 and 13, Vol. II.
3. The Canadian Practitioner, for February, 1884.
4. Science, Vol. III., No. 52, for February, 1884.
5. Mémoires et Compte Rendu des Travaux de la Société des Ingénieurs Civils, November, 1883.
6. Bulletin of the Museum of Comparative Zoology at Harvard College, Vol. XI., No. 8.

It was moved and seconded "That the Council be a Committee, with power to add to their number, to arrange for the reception : nd entertainment of such members of the British Association as may visit Toronto during the month of September."—*Carried.*

Mr. W. H. VanderSmissen then read a paper by the Rev Prof. Campbell of Montreal, on

THE KHITAN LANGUAGES; THE AZTEC AND ITS RELATIONS.

My translation of the Hittite Inscriptions found at Hamath and Jerabis, in Syria, is the only one yet published with an explanation of the process by which it was accomplished. The Rev. Dunbar I Heath has sent me copies of his papers in which the Hamath inscriptions are translated as Chaldee orders for musical services, but no process is hinted at by the learned author. In the discussion which followed the reading of one of these papers, a well-known Semitic scholar remarked, "that so long as no principle was laid down and explained as to the system by which the characters had been transliterated, it would be impossible to express an opinion on the value of the proposed reading." Whatever may be the merits of my translation, it does not make default in this respect. The process is simple and evident. The phonetic values of the Aztec hieroglyphic system are transferred to corresponding hieroglyphic characters in the Hittite inscriptions. Common Hittite symbols are the arm, the leg, the shoe, the house, the eagle, the fish. These are also found as Mexican hieroglyphics. There is nothing to tell us what their phonetic values are in Hittite, because hardly any other remains of the Hittite language have survived. But in Aztec we know that these values are the first syllables of the words they represent. Thus an arm being called *neitl*, gives the phonetic value *ne* for the hieroglyphic representing an arm. A leg being called *meztli*, furnishes *me*. A shoe gives *ca* from *cactli*; a house, also, *ca* from *calli*; an eagle, *qua* from *quauhli*; and a fish, *mi* from *michin*. But the question has been raised, "What possible connection can there be between the Hittites or Khita of ancient Syria and the Aztecs of Mexico?" As well might we ask what connection can there be between Indian Brahmins and Englishmen; between European Osmanli and Siberian Yakuts. Geographical separation in such case, is simply the result of a movement that has been going on from early ages. Men are not plants nor mere animals to be restricted to floral and faunal centres. The student of history, who has followed the Hunnic and Mongolian hordes in their devastating course across two continents, will not be surprised to find that well-known Iroquois scholar, the Abbé Cuoq, suggesting the relationship of the Iroquois with the wandering and barbarous Alans and Huns. Still less surprise should be experienced when the more cultured Aztecs of Mexico are connected with an ancient Old World civilization. Aztec history does not begin till the 11th century of our era, and even that of the Toltecs, who preceded the Aztecs, and were of the same or of an allied race, goes no farther back than the 8th. The period of their connection with Old World history as a displaced Asiatic people is thus too early to be accounted for by the invasions of the Mongols, but coincides with the eastern movements of the Khitan, who, after centuries of warfare on the borders of Siberia, disappeared from the historian's view in 1123. It is certainly a coincidence that the Aztecs should claim to be of the noble race of the Citin, and that *citli*, the hare, or, in the plural, *citin*, should be the totem or heraldic device of their nation.

Since I wrote the article on the Khitan Languages, in which I traced the Chinese Khitan backwards to central Siberia about the sources of the Yenisei, where, according to Malte Brun, the Tartars called their mounds Lî Katei, or the tombs of the Cathayans, I have received from Mr. Vl. Youferoff, of the Imperial Society of Geography at St. Petersburg, copies of the chief inscriptions from that region. These triumphantly confirmed my supposition that the Katei and the Khita or Hittites were the same people, by presenting characters occupying a somewhat intermediate position in form between the Hittite hieroglyphics and the more cursive script of our Mound Builders. The rude representations of animals and other natural objects accompanying some of the inscriptions are precisely of the type furnished by the Davenport Stone. One inscription, which I deciphered and the translation of which is now before the Imperial Society of Geography, relates the victory of Sekata, a Khitan monarch, the Sheketang of the Chinese hostorians, over two revolted princes or chiefs dwelling at Uta or Utasa in Siberia. As in the case of the Syrian Hittite inscriptions, I have translated the Siberian one by means of the Japanese, using the Basque, the Aztec, and other languages of the Khitan family, for confirmation. Whatever foreign influences may have done to modify the physical features, the character, language, religion, and arts of the Japanese, and, in lesser measure, of the Coreans, there can be no doubt that these are at basis Hittite or Khitan. Already at the commencement of my Hittite studies I had noted the agreement of many characters in the Corean alphabet with those of Hamath and Jerabis on the one hand, and, on the other, with those on our mound tablets. The Rev. John Edwards of Atoka with great kindness procured for me, from a member of the Japanese Imperial Household at Tokio, a work on the ancient writing of the Japanese. One of the forms of writing exhibited in this work and occupying much space is very similar to the Corean, and is undeniably of the same origin. I have not yet had time to investigate the volumes thoroughly, but as they appear to contain samples of ancient alphabets with guesses at their signification rather than complete inscriptions, little progress may be anticipated by means of them. Nevertheless the existence in Japan of a syllabary of so Hittite a type as the Corean in ancient times is confirmatory of the Khitan origin of the Japanese. As for the relations of American civilizations, such as those of the Mexicans, Muyscas, and Peruvians, with that of Japan, I need only refer to the writings of so accurate and judicious an observer as Humboldt.

Returning to the Hittites of Syria, who figure so largely in the victorious annals of the Egyptian Pharaohs and Assyrian kings, and whose empire came to an end towards the close of the 8th century B.C., we find that, although apart from my own conclusions no definite opinion has been reached regarding their language beyond the mere fact that it was Turanian, guesses have been made by scholars whose hypotheses even are worthy of consideration. Professor Sayce believes the Hittite language to have been akin to that furnished by the ancient Vannic inscriptions of Armenia. The Vannic language, according to Lenormant, belongs to the Alarodian family, of which the best known living example is the Georgian of the Caucasus. Now it is the Caucasus that I have made the starting point of Hittite migration, which terminated at Biscay in the west, and in the east, reaching the utmost bounds of Northern Asia, overflowed into America. Not only the Georgians, I unhesitatingly assert, but most of the other Caucasian families, the Circassians, Lesghians, and Mizjeji at least, should be classed as Alarodians, or better still as Khitan. So far I have found no evidence from ancient Caucasian inscriptions, though such I believe have been discovered; but an evidence as conclusive is furnished by the languages of the Caucasian families I have named as compared with those which are presumably of Hittite origin in the Old World and in the New. In the remainder of this paper, I propose chiefly to set forth the relations of the Aztec language, by means of which I transliterated the Hittite inscriptions, with the Caucasian tongues, which of all Khitan forms of speech are in closest geographical propinquity to the ancient habitat of the Hittite nation. Before doing so I may set forth the principal members of the Khitan family at the present day.

THE KHITAN FAMILY.

1. Old World Division.

Basque.
Caucasian = Georgian, Lesghian, Circassian, Mizjeji.
Siberian = Yeniseian, Yukahirian, Koriak, Tchuktchi, Kamtchadale.
Japanese = Japanese, LooChoo, Aino, Corean.

2. American Division.

Dacotah.
Huron-Iroquois including Cherokee.
Choctaw-Muskogee including Natchez.
Pawnee including Ricaree and Caddo.
Paduca = Shoshonese, Comanche, Ute, &c.
Yuma = Yuma, Cuchan, Maricopa.
Pueblos = Zuni, Tequa, &c.
Sonora = Opata, Cora, Tarahumara, &c.
Aztec including Niquirian.
Lenca = Guajiquiro, Opatoro, Intibuca.
Chibcha or Muysca.
Peruvian = Quichua, Aymara, Cayubaba, Sapibocono, Atacameno, &c.
Chileno = Araucanian, Patagonian, Fuegian, &c.

The Nahuatl, or language of the Aztecs, as distinguished from other tribes of diverse speech inhabiting Mexico, has long been a subject of no little difficulty to philologists. It is not that its grammatical construction is peculiar, but because its vocabulary exhibits combinations of letters or sounds that have come to be regarded as its almost peculiar property. The most important of these is the sound represented by *tl*, whether it be initial, medial or final. The Aztecs of Nicaragua drop the *tl* altogether or reduce it to *t*; hence some writers have supposed theirs to be the true form of the language, and the literary tongue of Mexico a corruption. Upon this an argument has been founded for the southern origin of the Nahua race. But, as Dr. Buschmann and others have shewn, a mere casual survey of the languages of more northern peoples, the Sonora and Pueblo tribes, and the great Paduca family, reveals the fact that they contain a considerable proportion of Aztec words, and that in them, as in the Nahuatl of Nicaragua, the Aztec *tl* disappears or is converted into *t*, *d*, *k*, *s*, *r* or *l*. Here therefore it is claimed by others is an argument for the northern derivation of the Mexicans.

If we carry forward the work of comparison, having regard to certain laws of phonetic change, we shall find, as I profess to have done, that the vocabulary, and to a large extent the grammar, of the Aztecs are those of all the greater families in point of culture and warlike character of the Northern and Southern Continents. Nor do the Aztec and its related American languages form a family by themselves. They have their counterparts, as I have indicated, in many regions of the Old World. If my classification of these languages be just, there should, among a thousand other subjects of interest, be found some explanation of the great peculiarity of Aztec speech to which I have referred.

The Aztec combination *tl* appears, although to no very great extent, in the Koriak, Tchuktchi, and Kamtchatdale dialects. It has no place in Corean, Japanese, or Aino, and only isolated instances of its use are found in the Yukahirian and Yeniseian languages. Of the four Caucasian tongues which pertain to the Khitan family, two, the Georgian, and Mizjeji, are almost as destitute of such a sound as the Corean and Japanese; while the Circassian and Lesghian vocabularies, by their frequent employment of *tl*, reproduce in great measure the characteristic feature of the Nahuatl. It is altogether wanting in the Basque, and is a combination foreign to the genius of that language. Yet there is no simpler task in comparative philology than to show the radical unity of the Basque and Lesghian forms of speech. Such a comparison, as well as one of the Lesghian dialects among themselves and with the other Caucasian languages, will enable us to decide whether the *tl* of the Lesghian and Circassian forms part of an original phonetic system, or is an expedient, naturally adopted by speakers whose relaxed vocal organs made some other sound difficult or impossible, to stave off the process of phonetic decay by substituting for such sound the nearest equivalent of which they were capable.

In order first of all to exhibit the common origin of the Basque and the Lesghian, I submit the following comparison of forms, the relations of which are apparent to the most casual observer. The Lesghian vocabulary is that of Klaproth, contained in his Asia Polyglotta; the Basque is derived from the dictionaries of Van Eys and Lecluse. It will be observed that the Lesghian almost invariably differs from the Basque :—

1. In substituting *m* for initial *b*.

2. In dispensing with initial vowels; or, when they cannot be dispensed with, in prefixing to them *b* or *p*, *t* or *d*.

3. In generally rendering the Basque aspirate, together with *ch* and *g*, by the correspondingly harder forms *g*, *k* and *q*.

4. In occasionally adding final *l* or *r*.

(The last named letters *l* and *r* are interchangeable in the Khitan as they are in all other families of speech.)

COMPARISON OF BASQUE AND LESGHIAN.

Rule 1.	English.	Basque.	Lesghian.
	beard	bizar	mussur, muzul
	head	buru	mier, maar
	nail	behatz	maats
	back	bizkhar	machol, michal
	to-morrow	bihar	michar (Georgian)
Rule 2, *a*.	skin	achala	quli
	hand	ahurra	kuer
	river	uharre	chyare, uor
	thunder	ehurzuria, curciria	gurgur
	hair	ileak	ras
	cold	otzo	zoto
	no	ez	zu
	left hand	ezquerra, ezker	kuzal, kisil
	milk	eznea	sink
	star	izarra	suri
	day	eguna	kini
Rule 2. *b*.	deer	oreina	burni
	clothes	aldar	paltar
	child	aurra	durrha
	stone	arri, harri	tsheru, gul
Rule 3.	great	handi	kundi
	house	eche	akko
	hail	harri	goro
	smoke	gue	kui
	tooth	hortz	kertschi
	leaf	orri	kere
	finger	erhi	kilish
Rule 4.	rain	uria	kural
	son	seme	chimir
	great	zabala	chvallal

The following, though generally agreeing, present same exceptions to the above rules.

English.	Basque.	Lesghian.
heaven	ceru	ser
bird	chori	zur
red	gori, gorri	hiri
blue, green	urdin	crdjin
death	heriotze	haratz
old	agure, zar, zahar	herau, etshru
throat	cinzur	seker
white	churia, zuria	tchalasa
wood	zura	zul
leg	aztal	uttur
tree	zuhatsa	guet, hueta
fire	su	zo
high	gan	okanne
tongue	mia	mas

A comparison of the Basque with the other Caucasian languages, Georgian, Circassian, and Mizjeji, would display similar relations with some modification of the laws of phonetic change.

If now we ask what the Basque does with the Lesghian *tl*, we shall find that it represents that sound chiefly by the letters *r* and *l*. This equivalency of *tl*, and sometimes of *ntl*, to *r* and *l* also appears in comparing the Lesghian dialects among themselves or with other Caucasian languages.

COMPARISON OF LESGHIAN FORMS IN *tl* WITH OTHER CAUCASIAN AND BASQUE FORMS.

English.	Lesghian.	Other Forms.
hair	tlozi	ras, *Lesghian.*
bone	tlusa	rekka "
wood	thludi	redu-kazu "
tomorrow	shishatla	shile "
night	retlo	rahle "
sheep	betl	bura "
maize	zoroto-roodl	tzozal-lora "
goat	antle	arle
six	antlko	ureekul
nail	matl	mare, *Mizjeji*
low	tlukar	lochun "
eight	bitlno	bar, barl "
sun	mitli	malch
"	" beri, *Lesghian.*	marra, *Circassian.*
flesh	ytl	gll "
forehead	tlokva	illech "
easy	intlaugu	illesu "
"	"	errecha, *Basque.*
loins	tlono	errainac "
water	htli	ur "
butter	yetl	guri "
hair	tlozi	ileac "
earth	ratl	lurra, laur "

The following represent the exceptions to the rule both in form and in numerical proportion :—

ENGLISH.	LESGHIAN.	OTHER FORMS.
yellow	tlela	dula. *Lesghian.*
day	tlyal	thyal, tehzal "
horn	tlar	adar, *Basque.*
knee	tlon	belaun "

From the preceding examples it appears that the Lesghian sounds represented by *tl*, *thl*, *ntl*, are the equivalents of *r* and *l* generally, and sometimes of *d* or *t*. The latter exception probably finds its explanation in Basque, for in the dialects of that language an occasional permutation of *r* and *l* into *t* and *d* takes place. Thus *ideki* to take away, becomes *ireki*, and *iduzki* the sun, becomes *iruzki*, while *elur* snow, sometimes assumes the form *edur*, and *belar* grass, that of *bedar*. The last exception cited, that in which the Lesghian *tlon* is compared with the Basque *belaun*. is really no exception, for *elaun* is the true representation of *tlon*, the initial *b* being prosthetic to the root, as is frequently the case in Basque. Among many examples that might be given, I may simply cite *belar* the ear, as compared with the Mizjeji *lerk*.

Turning now to the Aztec, on the supposition that it is related to the Basque and Caucasian languages, we naturally expect to find on comparison a coincidence of roots and even of words following upon the recognition of *tl* and *ntl* as the equivalents of *r* and *l* in these forms of speech. The fact that the Aztec alphabet is deficient in the letter *r* favours such an expectation. But our comparison must be made with due caution. Any one who has examined a Mexican dictionary, such as that of Molina, must have been struck with the remarkable preponderance of words commencing with the letter *t* over those beginning with any other letter of the alphabet. These words comprise considerably more than one third of the whole lexicon. A certain explanation of this is found in the fact that the two particles *te* and *tla* possess, the former an indefinite personal, and the latter a substantive, signification, and thus enter largely into the structure of compound words. Whatever its grammatical value in Aztec, however, it appears, on comparing the Aztec vocabulary with its related forms of speech, that initial *t* or *te*, which leaving *tl* out of account still occupies one fifth of the lexicon, is frequently prosthetic to the root.

The following are some of the chief laws of phonetic change derived from a comparison of the Aztec and Lesghian languages. These may be found operating to almost as great an extent in the Lesghian dialects among themselves :—

1. The Aztec combinations *tl*, *ntl*, are either rendered in Lesghian by the same sounds, or by *r* or *l*. In some cases in which phonetic decay has set in, the Aztec *tl* is either omitted or represented by a dental. The Lesghian occasionally renders the Aztec *l* and *ll* by *tl*.
2. The interchange of *p* and *m*, which appeared in comparing the Basque and the Lesghian, for the Aztec is deficient in the sound of *b*, characterizes a comparison of the Aztec with the Caucasian languages.
3. A similar interchange of *n* and *l*, or the ordinary equivalents of l, such as marked the Iroquois in comparison with the Basque, occasionally characterizes the relations of the Aztec and Caucasian tongues.
4. The Lesghian, as already indicated, persists in the rejection of initial vowels, and the same is generally true of reduplications and medial aspirates.
5. As in many Aztec words initial *t* forms no part of the root, but is a prosthetic particle, it finds no place in such cases in the corresponding Lesghian term.
6. The Lesghian occasionally strengthens a word by the insertion of medial *r* before a guttural, for which of course there can be no provision in Aztec.

I have not thought it desirable to burden this paper with laws relating to other changes, as the relation of the compared words will be sufficiently apparent; but, for the purpose of illustration, I have added corresponding terms from other Khitan languages exemplifying the rules set forth.

COMPARISON OF AZTEC AND LESGHIAN FORMS.

ENGLISH.	AZTEC.	PHONETIC CHANGE.	LESGHIAN.	ILLUSTRATIONS.
water	atl	ar al	htli	ur, *Basque*
low	tlatzintli	latzili, latziri	tlukur	liuchtliu, *Koriak*
day	tlacatli	lacali, lacari	tlyal, djekul	allochal, teluchtat, *Koriak*
knee	tlanquaitl	lancail, lancair	tlon	zangar, *Basque* cconcor, *Quichua*
deer	mazatl	mazal, mazar	mitli	mool, *Yuma*
earth	tlalli	ralli, larri	ratl	lurra, *Basque*
night	tlalli	" "	retlo, rahle	neillhe, *Choctaw*
yesterday	yalhua	alhua	hutl	hooriz, *Dacotah*
ice	cetl	cel, cer	zer, zar	kori, *Japanese*
wind	ehecatl	ehecal, ehecar	churi	gygalkei, *Koriak*
sheep	ichcatl	ichcal, ichcar	kir	achuri, *Basque* ccaora, *Aymara*

English.	Atzec	Phonetic Change.	Lesghian.	Illustrations.
mud	zoquitl	zokil, zokir	zehur	chulu, *Corean*
stone	tetl	tel, ter	tsheru	tol "
dust	teuhtli	teuhli, teuhri	chur	turo, *Quichua*
grass	quilitl	kilil, kirir	cher, gulu	kyrau, *Yeniseian*
star	citlalli	cilalli, cilarri	suri	zirari, *Aino*
hair	tzontli	tzoli, tzori	tshara	thorok, *Corean*
skin	cuatl	cual, cuar	quli	ecara, *Quichua*
eye	ixtli	ishli, ishri	chuli	okahra, *Iroquois*
wood	quauitl	kauil, kauir	zul	kullu, *Quichua*
"	"	kauit	guet, hueta	zuhaitz, *Basque*
foot	icxitl	ieshil, ieshir	kash	ochsita, *Iroquois*
year	xiuitl	shiuil, shiuir	thahel	osera, "
god	teotl	teol, teor	saal, zalla	chail, koil, *Yukahiri*
clothes	tlatqtl	ratkl, latkr	paltar, retelkum	aldarri, aldagarri, *Basque*
cold	cecuiztli	cecuizli, cecuizri	chuatzala	hutseelo, xetchur, *Yuma*
mountain	tepetl	tepel, teper	dubura	neit-tippel, *Koriak*
moon	metztli	metzli, metzri	moots, bars	muarr, *Shoshonese*
leg	metztli	" "	maho	onitsa, *Iroquois*
hand	maitl	mail, mair	ku-mur	masseer, *Shoshonese*
honey	necutli	neculi, necuri	nutzi, nuzo	miski, *Quichua*
				mitzi, *Japanese*
bread	tlaxcalli	lashcalli, rashcalli	zulha	lagul, *Yukahiri*
				rajah, *Yeniseian*
copper	tepuztli	tepuzli, tepuzri	dupsi	tup, thep, *Yeniseian*
				tetiopulgun, *Kamtchatdale*
mouth	camatl	camal, camar	sumun, moli	simi, *Quichua*
				homal-galgen, *Koriak*
belly	xillantli	shillal, shillar	siarad	kolid, *Kamtchatdale*
feather	yhuitl	ywil, ywir	bel, pala	puru, *Quichua*
ram	quiahuitl	kiavil, kiavir	gvaral	kuhl-kishen, *Koriak*
woman	cihuatl	cival, civar	tshaba	sipi, *Corean*
				sungwal, *Shoshonese*
bird	to-totl	tol, tor	adjari, zur	tori, *Japanese*
name	to-caitl	cail, cair	zyer, zar	chareigtsh, *Kamtchatdale*
				teguala, *Sonora*
beard	te-nchalli	nchalli, ncharri	muzul, mussur	hannockquell, *Shoshonese*
river	at-oyatl	oyil, oyar	uor, chyare	hahuiri, *Aymara*
throat	t-uzquitl	uzkil, uzkir	seker	eztarri, *Basque*
back	to-puztli	puzli, puzri	machol	bizkhar, "
				kaptcher, *Koriak*
sun	to-natiuh	natiul	mitzi	mtchi, *Japanese*
				inti, *Quichua*
evening	te-otlac	olak, orak	sarrach, *Mizjeji*	sonrek, *Iroquois*
snow	cepayauitl	payauil, payauir	marchala	pukoelli, *Yukahiri*
				pagolka, *Koriak*
man	macehualli	maceualli	murgul	birkhjarjat, *Yeniseian*
				mailik, *Pujuni*
small	tlocoton, tzocoton	locoton, tzocoton	chitina	cikadang, *Dakotah*
				iskitmi, *Choctaw*
sand	xalli	shalli, sharri	keru	challa, *Aymara*
shoulders	acolli	acolli, acorri	hiro	callachi, "
son	tepil-tzin	tepil, tepir	timir, chimir	comerse, *Yuma*
				tiperie, *Sonora*
woman, wife	tenamic	tenamic	ganabi	kanafe, *Corean*
fish	michin	michin	mugul, besuro	mughat, pughutsi, *Shoshonese*
to-day	axcan	ashcan	djekul	hichmin, *Aymara*
				tachan, *Mizjeji*
give	maca	maca	beekish	eman, emak, *Basque*
stone	topecat	topecat	teb	tipi, *Shoshonese*
black	caputztic	caputztic	kaba	shuputkat, *Dacotah*
hard	tepitztic	tepitztic	debehase	kibichii, *Japanese*
old	veue	veue	vochor	vucha, *Araucanian*
				apacha, *Aymara*
green	quiltic	kiltic	sholdisa	shereeat, *Dacotah*
great	yzachi	izachi	zekko	lashka, "
"	yzachipul	izachipul	chvallal	zabal, *Basque*
dog	chichi	chichi	choi	cocochi, *Sonora*
no	amo	amo	anu	ama, *Quichua*
I	ne	ne	na	ni, *Basque*
				na, *Aymara*
than	te	te	duz	zu, *Basque*
				ta, *Aymara*
he	ye, yehua	he, heua	heich	hau, *Basque*
				uca, *Aymara*

The Georgian does not exhibit the Aztec *tl*, but, as it is regarded by Professor Sayce as the living language most likely to represent the speech of the ancient Hittites, a brief comparison of its forms with those of the Aztec may not be out of place. Like the Lesghian it is impatient of initial vowels, and it generally agrees with that language in the laws of phonetic change, adding, however, this peculiarity, the occasional insertion of *v* before *l*. The *v* seems generally to represent *u*, or some similar vowel sound, and is probably such a corruption of the original as appears in the Samivel of Pickwick compared with the orthodox Samuel.

COMPARISON OF AZTEC AND GEORGIAN FORMS.

English.	Aztec.	Phonetic Change.	Georgian.	Illustrations.
fowl	tototl	totol, totor	dedalı	totolin, *Sonora*
red	chichiltic	chichiltic	tziteli	tsatsal, *Kamtchatdale*
blood	eztli	ezli, ezri	sischli	odol, *Basque*
				chri, *Dacotah*
house	calli	calli	sachli	cari, caliki, *Sonora*
mountain	quautla	kaula, kaura	gora	kkollo, *Aymara*
horn	quaquauitl	kakaul, kakaur	akra	quajra, "
sheep	ichcatl	icheal, ichcar	tschchuri	ccaora, "
wind	checatl	checal, checar	kari	helcala, *Sonora*
heart	yullotl	yullol, yullor	gulu	gullugu, *Kamtchatdale*
girl	ocuel	ocuel	okurza, kali	okulosoha, *Choctaw*
dog	yzcuintli	izkili, izkiri	dzagli, djogori	schari, *Shoshonese*
nose	yacatl	hacal, hacar	zchviri	surra, *Basque*
				cher, *Pueblos*
hair	tzontli	tzoli, tzori	tzvere (beard)	tsheron, *Kamtchatdale*
moon	metztli	metzli, metzri	mtvare	muarr, *Shoshonese*
silver	teo-quitlatl	kilal, kilar	kvartshili	cilarra, *Basque*
shoulder	te-puztli	puzli, puzri	mchari	buhun, *Lesghian*
tomorrow	muztli	muzli, muzri	michar	mayyokal, *Yuma*
leg	metztli	metzli, metzri	muchli	ametche, "
to kill	*miclia*	*miclia*	*mokluli*	*wakerio, enkerio, Iroquois*
mother	nantli	nali, nari	nana	nourha, *Iroquois*
snow	cepayauti	cepayauil, cepayauir	tovli	repaliki, *Sonora*
snake	cohuatl	coval, covar	gveli	toeweroe, *Shoshonese*
boy	tepil-tzin	tepil	shvili	tiperic, *Sonora*
lightning.	tlapetlani	lapelani	elvai	illappa, *Quichua*
				willhyap, *Yuma*
leaf	iatla-pallo	iala-pallo, iala-parro	pur-zeli	bil-tlel, *Kamtchatdale*
small	tzocoton	tzocoton	katon	cikadang, *Dacotah*
man	oquichtli	okichli, okichri	ankodj	oonquich, *Iroquois*
			oiakotsh, *Koriak*	aycootch, *Yuma*
			guru, *Aino*	ccari, *Quichua*

The Circassian language abounds in labials, and thus finds its best American representatives among the Dacotah dialects. Nevertheless it presents many words which come under the same general laws in relation to the Aztec that have characterized the Lesghian and Georgian.

COMPARISON OF AZTEC AND CIRCASSIAN FORMS.

ENGLISH.	AZTEC.	PHONETIC CHANGE.	CIRCASSIAN.	ILLUSTRATIONS.
hand	mapipi	mapipi	meppe	nape, *Dacotah* mashpa, *Shoshonese*
black	caputztic	caputztic	kvatsha	shupitcat, *Dacotah* yupikha, *Shoshonese*
heavy	etic	etic	ondogh	tekay, tekash, *Dacotah*
sister	teicu	teicu	tsheeyakh	itaku, itakisa, " tshakyhetch, *Koriak*
"	tepi, teciuapo	tepi	tabcha, tsheebk	cuhuba, *Muysca*
shoulder	tepuztli	tepuzli	damasha	tapsut, *Aino* gepuca, *Muysca*
smoke	poctli	pocli, pocri	bacha	ibusu, *Japanese*
lip	tenxi-palli	tenxi-palli	uku-fari	kuchi-biru, *Japanese*
meat	nacatl	nacal, nacar	mikel	niku, *Japanese*
easy	velchiu-aliztli	velchiu	plese, illesu	raku, " errecha, *Basque* arrangya, *Yukahiri*
child	acatl	acal, acar	kaala	jacuel, *Yuma*
boy, son	tepil-tzin	tepil	tshvalye, chvalay	akwal-nesuta, *Natchez*
man	tlacatl	lacal	tle	kelgola, *Kamtchatdale*
blood	eztli	ezli, ezri	tleh, kleh	odel, *Basque* huila, *Aymara*
dog	chichi	chichi	chhah	kahi, *Corean*
no	quixmo	kishmo	ekesima	hetschen, *Lesghian*
summer	xupan	shupan	gapne (spring)	tofah, *Choctaw*

As things which are equal to the same thing are equal to one another, it follows that, by the application of the same law of phonetic change, the vocabulary of the Aztec must coincide with that of the Basque, in spite of the fact that these two tongues have maintained a separate existence for some 2500 or 3000 years. Nothing can more convincingly prove the indestructibility of human speech, not only in mere thought-forms but in the *ipsissima verba*, than a comparison of the two vocabularies.

COMPARISON OF AZTEC AND BASQUE FORMS.

ENGLISH.	AZTEC.	INTERMEDIATE FORMS.	BASQUE.
sheep	ichcatl	kir, *Lesghian*; caaora, *Aymara*	achuri
nose	yacatl	zehviri, *Georgian*; cher, sodornah, *Pueblos*	sur, sudur
rain	quiavitl	gvaral, *Lesghian*; furi, *Japanese*	euri
star	citlalli	zirari, *Aino*: suri, *Lesghian*	izar
water	atl	htli, *Lesghian*; ul, ur, *Yeniseian*	ur
worm	ocuiloa	kihigir, *Aino*; kuru, *Quichua*	chicharia
bad	aquallotica	whalich, *Yuma*; achali, *Koriak*	char, charto
mountain	quautla	gora, *Georgian*; kar, *Yeniseian*	zerra
stone	tetl	tol, *Corean*; kell, *Yukahiri*	harri
ice	cetl	zer, *Lesghian*; chilen, *Mizjeji*	karroin
fish	atlan	ennen, *Koriak*; olloga, *Yukahiri*	arrain
wood	zalli	zul, *Lesghian*: kullu, *Quichua*	zura
bird	tototl	adjari, zur, *Lesghian*; garioha, *Iroquois*	chori
dog	yzcuintli	aghwal, schari, *Shoshonese*; tkari, *Mizjeji*	zacur
throat	tuzquitl	seker, *Lesghian*; iakwal, *Araucan*	eztar
old	veue	vochor " hachooli, *Choctaw*	agure
evening	teotlac	sarrach, *Mizjeji*; sourek, *Iroquois*	arrax, arrats
axe	tlateconi	adaganu, *Koriak*; atacarte, *Yuma*	aizkor
bread	tlaxcalli	lagul, *Yukahiri*; tikaru, *Shoshonese*	hazkurri
bow	tlauitolli	ratla, *Koriak*; gahlotrahde, *Cherokee*	uztadarra
thunder	tlaquaqualaca	yekilkegie, urgirgerkin, *Koriak*	churzuri
river	atoyatl	uor, chyare, *Lesghian*; hahuiri, *Aymara*	uharre
earth	tlalli	delchel, *Koriak*; ratl, *Lesghian*	lur
child	acatl	jacuel, *Yuma*; jali, *Yeniseian*	aur
clothes	tlatqtl	retelkum, paltar, *Lesghian*	aldagarri, aldarri
knee	tlanquaitl	cconcor, *Quichua*; hizanosara, *Japanese*	zangar

English.	Aztec.	Intermediate Forms.	Basque.
easy	velchiu-aliztli	illesu, *Circas*: arrangya, *Yukahiri*	errecha
shoulder	cuitlapantli	telpilgin, tschilpit, *Koriak*	sorbalda
silver	teoquitlatl	colaque, *Aymara*; kvartschili, *Georgian*	cilarra
speak	tlatoa	raton, *Iroquois*; arusi, *Aymara*	erran, erraitea
"	notza	ni, *Quichua*; hanasu, *Japanese*	mintza
five	macuilli	millgin, *Koriak*: marqui, *Sonora*	bortz
ten	matlactli	mari, *Araucan*; peeraga, *Dacotah*	amar
seven	chicome	shahemo, shacopi, *Dacotah*	zazpi
beard	tenchalli	hannockquell, *Shoshonese*; musur, *Lesghian*	bizar
to-morrow	muztli	mayyokal, *Yuma*; michar, *Georgian*	bihar
back	topuztli	kaptcher, *Koriak*; machol, *Lesghian*	bizkhar
"	"	hapar, *Yeniseian*; sobira, *Japanese*	guibel
walk	malquica	pulanujaha, *Yeniseian*; puriy, *Quichua*	ibilcea
blood	eztli	tleh, kleh, *Circassian*; huila, *Aymara*	odol
breast	telchiquiuh	tar, *Mizjeji*; teyga, *Yeniseian*	thilia
skin	cuatl	tsholoh, *Lesghian*; tshal, *Yukahiri*	azal, achal
nail	yztetl	oocheelah, *Iroquois*; onzshil, *Yukahiri*	atzazal
frog	cueyatl	kayra, *Quichua*; kayeru, *Japanese*	iguela
come	vallauh	ela, *Choctaw*; or, *Corean*.	el, hel
great	yzachipul	oboloo, *Shoshonese*; chvallal, *Lesghian*	zabal
tree	quauitl	kotar, " guet, hueta, *Lesghian*	zuhaitz
to-day	axcan	wakum, *Araucan*; tachan, *Mizjeji*	egun
cold	yztic	izits, *Shoshonese*; echta, *Circassian*	ozt
"	cecuiztli	hutseelo, xetchur, *Yuma*	etsbero
child	tetel-puch	hailpit, *Yuma*; bikh-jal, *Yeniseian*	mut-il
small	tepiton	dahab, tkivisa, *Lesghian*	tipia
boy, son	tepil-tzin	tiperic, *Sonora*; timir, chimir, *Lesghian*	seme
lip	tenxipalli	kuchibiru, *Japanese*; uku-fai, *Circassian*	ez-pana
man	oquichtli	chojashin, *Koriak*; haasing, *Adahi*	gizon
mouse	vecacotl	achacollo, achaca, *Aymara*; dsugoh, *Circass.*	sagu
mouth	camatl	simi, *Quichua*; khaipi, *Atacama*	auba
name	tocatl	zar, *Lesghian*; chinna, *Iroquois*	izen, icen
sister	teciuapo	tsheebk, shupeh, *Circass.*; cuhuba, *Muysca*	aizpa
black	yapalli	millh, *Yuma*; shawagare, *Shoshonese*	beltz
wind	checatl	aeate, *Sonora*; ahekin, "	aicea
all	ixquich	hoaheasse, *Dacotah*; eezahk, *Circassian*	guci
enemy	teyaouh	toka, " tautyok, *Corean*	etsaya
give	maca	muy-scua, *Muysca*; beckish, *Lesghian*	eman, emak
sick	cocoxqui	ceotas, *Atacama*; joatsh, *Yukahiri*	gaicho, gaitz
I	ne	nah, *Pueblo*; na, *Aymara*; na, *Lesghian*	ni
thou	te	too, " ta, " de, *Dacotah*	zu
he	ye	ihih, " uca, " ecah, "	hau

Thanks to the survival of Lesghian forms in *tl*, the disguise of the Aztec has been penetrated, and we are thus enabled to assert, first of all, that the apparently widely divergent Peruvian dialects, the Quichua, Aymara, Atacameno, &c., are really its near relations. There is therefore every reason to believe that the Peruvians were the Toltecs, who preceded the Aztecs as rulers of Mexico, and who, under their king, Topiltzin Acxitl, withdrew to the south in 1062, and there founded the kingdom of the Sun. The Peruvian annals place the accession of their first historical monarch, Sinchi Rocca, in the same year. Passing over the intermediate kingdom of Bogota, the home of the Chibchas or Muyscas, which was distinctively Peruvian in character, and another Toltec remnant, the Lencas of Honduras, we come to the north of the Aztec country, where the Sonora, Pueblos, and Paduca tribes dwell, who have already been associated with the Aztecs by several writers. To these I would add the comparatively small but philologically important Yuma and Pujuni families. In all of these tribes we may recognize the barbarous Chichimecs through whom the Aztecs passed on their way to empire. But of the same race are the central stocks, the Dacotah and Pawnee; and to no other belong the eastern families of the Huron-Cherokees, and the Choctaw-Muskogees. The Algonquins of the north, like the Maya-Quichés of Central America, are of a totally distinct branch of the Great Turanian division. The samples of Mound Builder language furnished by the Davenport, the Grave Creek, and the Brush Creek Stones add their evidence to that of the written characters in favour of a connection of the Mound Builders with the Aztecs and related tribes. The Dacotah Mandans, the Choctaws, the Natchez, and the Aztecs, have been severally set forth as the Mound Builders. The true Mound Builders may have been none of these, but a distinct tribe of Allighewi or Alleghenies, for whom we must look elsewhere, still, however, to find them a portion of the same great family. Ancient traces of this tribe appear in the Hittite country of the Nairi in Mesopotamia, where Elisansu was situated; in the Alazonus river of Albania in the Caucasus; in the nation of the Halizoni of Pontus mentioned by Homer; in the Scythic Alazonians of Herodotus; and in Alzania, a mountain region of the Basques. It is not at all improbable that the ancient name survives in those of the Alasar and Allakaweah, sub-tribes of the Dacotahs, but this only tends to prove that a people of the same race as the Dacotahs, and not necessarily the Dacotahs themselves, were the Mound Builders.

There is abundant reason for believing the tradition of most of the American tribes I have mentioned to the effect that their ancestors passed over the sea or great river and traversed a region of intense cold before arriving at their destination in more hospitable climates. Kamtchatka must have been their point of departure from the Old World, whether they reached that point from the Siberian Desert or journeyed thitherward from Corea and Japan by the Kurile Islands. There they set foot on the Aleutian chain which carried them safely over to the coast of Alaska. In Kamtchatdale there are many Aztec traces, and some which exhibit an exaggeration of the peculiarity of Aztec speech with which this paper is mainly occupied. Such is the rendering of the Aztec verb *tlacotla*, to love, by the elongated but distinctly recognizable form *tallochtelasin*. And, with the Kamtchatdale, the Aztec connection, which has been illustrated by comparative vocabularies, embraces all the hitherto unclassified languages of Northern Asia and Europe. The same forms that prevail over a great part of the American continent, somewhat disguised yet easily recognizable, are found in Japan and in Siberia, in the Caucasus and in Biscay.

Some time ago I alluded to a passage in the Paschal Chronicle in which the Dardanians of the Troad are referred to as Hittites, and since then Professor Sayce has seen reason for connecting the whole Trojan family with that ancient and illustrious people. Strabo tells us that at Hamaxitus in the Troad the Teucri, near relations of the Dardani, consecrated a temple to Apollo Smintheus as a memorial of the destruction of their bow-strings and other leathern articles by an army of rats or mice. The same story is told by Herodotus of the Assyrian army, opposed by the Egyptian Sethos, whose name, being the equivalent of Sheth, is truly Hittite. This same story lives in America among the Utes of the Paduca or Shoshonese family, as related by Professor Powell, and among the Muskogees, as told by Dr. Brinton. Hamaxitus, the Trojan town where the legend was localized, was in all probability a transported Hittite Hamath, for in the form Hamaxia it occurs in the peculiarly Hittite country Cilicia, where Cetii dwelt in ancient times, and where Hittite kings held limited sway in the days of Rome's supremacy. The Scythic Hamaxoeci very probably bore no closer relation to the chariot or *Hamaxa* than the Muskogees do to *musk*. These words Hamaxitus, Hamaxia, and Hamaxoeci designated a tribe, sub-tribe or caste, which originally had its chief representatives in the Syrian Hamath. They were scribes, the most likely people to preserve and hand down traditions of the past, the Amoxoaquis of the Mexicans, and the Amautas of the Peruvians. Through them this legend, and many others which recall old world stories, have found a resting-place on the American continent. Many writers on comparative mythology have been led to connect American tribes with Aryans and Semites by failing to recognize what Accadian studies have fully established, that the Turanians were the instructors in mythology and in many other things of these more highly favoured divisions of the human race.

The decipherment of the Hittite and Siberian inscriptions by the Aztec is but the first step in the solution of problems relating to ancient Old World populations, which are supposed either to have been exterminated or to have lost their independent existence. And the superior purity of the Aztec language as preserved by a literary people, spite of its dialectic peculiarities, will enable the philologist to shed light on many points of etymology and construction in the languages of Europe and Asia to which it is related. Take, for instance, the world *totolh-tetl*, an egg. Its meaning is clear, for *totolh* is *tototl* a fowl, and *tetl* denotes a stone. By a simple postposition of the nominative, therefore, the Aztec word for egg means the stone of the bird. In Yukahirian the word used is *nonton-daul*. Now *nonda* means a bird in Yukahirian, a form doubtless of the Lesghian *onotsh*, and the Japanese *ondori*, a fowl; but *daul*, which is just the Aztec *tetl*, does not now designate a stone in that language. The form has undergone change and is now *kell*, but there can be no doubt that *daul* or *tol* was once the Yukahirian name for stone, as it now is the Mizjeji, Corean and Choctaw form. The Basque word, which I have not found any explanation of among the Basque etymologists, is *arrolchia* or *arroltz*. Here the order of the Aztec and the Yukahirian is inverted, for *arri* denotes a stone, and *ollo* or *oilo*, a fowl. The final *chi* or *zi* before the article *a*, is the mark of the genitive which is now *aco* or *eco*. Hence, literally translated, *arrolchia* is "stone fowl of the." The Iroquois has entirely lost the etymology of his word *onhonchia*, in which the Basque *r* and *l* have been replaced by *n*; and the same is the case with the Peruvian, who, by following his usual practice, like the Lesghian, of removing the initial vowel, and simply changing the *l* to *n*, makes the word *runto*. The Circassian *kutarr* is probably of the same composition, for *kut* should represent *kuttey*, fowl, and *arr*, though not now a Circassian word, was so at the time when Circassians and Basques were one people, and derived their respective tribal and local names, Chapsuch and Guipuzcoa, from the Hittite land of Khupuscai. It is interesting to note, as exhibiting the vicissitudes of language, that the Corean, who calls a stone *tol* or *tor*, retains *arr*, the primitive term, to denote an egg, just as the Aztecs frequently employed *tetl* to express the same without any prefix.

There is a Basque word, the derivation of which puzzles the lexicographers, although some have ventured to derive the only Basque term denoting a boy from the Latin. It is *mutil*, or with the article *mutilla*. In Lesghian, *motshi* is a boy, in Japanese, *musuko*, in Sonoro, *te-machi*; but, as a rule, the *m* of these languages is replaced in others of the Khitan family by an ordinary labial. A similar difficulty in Basque attends the connected word *illoba*, which may mean a nephew or niece, or a grandchild. I am disposed to see in these terms the same word as the Aztec *tetelpuch*, which appears to mean "the offspring of somebody," or "of a person," for *tetech*, which in composition becomes *tetel*, denotes personality. The Aztec *puch*, offspring, would thus be the same as the Basque *ba*, and *mut*. That the *mut* of *mutil* corresponds with the *mus* of the Japanese *musuko*, appears from the comparison of another Basque word of similar form, *mutchitu*, mouldy. This answers to the Japanese equivalent *museta*, as *mutil* does to *musuko*. The Aztec word for mouldy is *poxcauhqui*, and, although there can be no connection between mustiness and offspring, answers in form to *puch*, as *mutchitu* to *mutil* and *museta* to *musuko*. The *ba* of *illoba* is but an abbreviated form of *puch*, such as appears in the Aino *po*, the Yeniseian *puwo*, and the Circassian *ippa*. The Basque word for child is *nerabea*, *norhabe*, which connects with *nor*, *norbait*, somebody, just as the LooChoo *worrabi*, also meaning child, shows its relation to *waru*, the Japanese *aru*, likewise denoting "somebody." It appears therefore that "somebody's wean" is a thoroughly Khitan conception. In Georgian, *boshi* which may be taken as the root word, means "child," and in Lesghian *vashsho*. But the Aino *vas-asso* and *boy-otchi* seem to be compound terms, like the Choctaw *poos-koos* and the Dacotah *wah-cheesh* and *bak-katte*. Similar forms are the Iroquois *wocca-nanne*, and the inverted Muyscan *guasgua-fucha*. The abbreviation of *boshi* or *puch* to *ba*, *be* or *bi*, as in the Basque and LooChoo, finds its parallel in the Yeniseian *dul-bo*, a doubly apocopated *tetel-puch*. The Yuma *hail-pit* seems almost to reproduce the Basque form, which inverted would read *il-mut*. One of the Sonora dialects, as we have seen, gives *te-machi* for boy; one of the Iroquois, *ihiha-wog*; the Choctaw, *chop-pootche*; and the Shoshonese, *ah-puts*. In the Old World, the Corean furnishes *tung-poki*; the Kamtchatdale, *kamsanapatch*, a long form as in the Dacotah *menarkbetse*; and the Yeniseian, *pigge-dulb* and *bikh-jal*. But the Yeniseian and Kamtchatdale also designate a son by the simple word for offspring, *bit*, and *petsch* in the respective languages. In the Georgian, Circassian, and Peruvian Aymara, this simple form seems to be reserved for the girls, for daughter in these languages is *bozo*, *pchu*, and *ppucha*. The Aztec prefixes to the word offspring *puch*, one of its terms denoting woman, female, the whole being *teich-puch*. This is the *tshide-petch* of the Kamtchatdale, and, with inversion of parts, the *bai-taga* of the Yukahiri. Other corresponding Khitan forms for girl, daughter, are the Circassian *pus-pa*, the Yeniseian *bikh-jalja*. the Koriak *gna-fiku* and *goe-behkak*, the Kamtchatdale *uchtshi-petch*, the Corean *bao-zie*, and the Japanese *musu-me*: and, in America, the Paduca or Shoshonese *wya-pichi*, the Dacotah *weet-achnong*, and the Iroquois *kaunuh-wukh* and *echrojeha-wak*. The Basque word for girl, *ala-ba*, *ala-bichi*, is in harmony with *illoba*, *nerabea*, and the inverted *mut-illa*, and corresponds with the Yeniseian *bikh-jalja*. Besides these more conspicuous forms there are many others which exhibit a common formation. Among the Yuma words denoting boy, and the equivalents of *hail-pit* in other dialects, occur *her-mai* and *yle-moi*, in which the Basque *mut* and Japanese *musu* are abbreviated into *mai* and *moi*. Of the same structure are the Peruvian Quichua *huar-ma* and the Circassian *ar-ps*. Two other words for boy, the Japanese *bo-san*, and the Araucanian *bo-tum*, belong to the same category; and there are many other forms, such as the Adahi *talla-hache*, in which the labial of *boshi* or *puch* has been converted into an aspirate, to which I need refer no farther. The Aztec *tetel-puch* and *teich-puch* are the types of the many terms mentioned, which exhibit the singular agreement, with phonetic variations, of the Khitan languages in the formation of these compounds.

A very common element in compound Aztec words is *palli*, which, besides denoting colour as in *ya-palli*, black, and *quil-palli*, green, appears to have the meaning of "contents, belonging to," just as the Japanese *iro* means colour, and *iru*, to hold or contain. So in Basque, *bal* is a root denoting colour in the abstract, and *bar*, a corresponding root signifying contents. In Aztec *tenxi-palli* means lip, but its derivation is only apparent in Japanese, in which language the word for lip is *kuchi-biru*. Now *kuchi* is the mouth, and *biru* is the original of *iru*, to hold, contain or enter. The Aztec *tenxi* does not appear in the dictionaries as a word for mouth, *camatl* being the term employed; but the related Shoshonese family furnishes *atongin*, *tungin*, and the Adahi, *tenanat*. The Circassian lip is *uku-fari*, plainly the same word as the Japanese and Aztec, although *uku* is not the present Circassian term for mouth. The Corean form is *ipsi-oor*, in which *ipsi* represents the Corean *ipkoo*, the mouth, and *oor*, the Japanese *iru* or *biru*. So also the Natchez adds *er* to *heche* the mouth, and calls the lip *ehec-er*. The Araucanian, from a primitive word *ia*, like the Dacotah *ea*, the Yuma *yu*, the Circassian *je*, *ja*, the Corean *ii* and the Basque *aho*, all meaning mouth, forms, with the equivalent of *palli*, *biru* and *fari*, *ia-pelk*, lip. The Circassian alone retains the sound of *itsha*, *utsha* for mouth, which appears in the inverted Lesghian *mur-tschi*, and Mizjeji *bar-dash*, their equivalenti for *uku-fari*. In Iroquois the lip is *osk-wenta*. By the conversion of *r* and *l* into *n*, which characterizes the Iroquois in comparison with most of the other Khitan languages, *wenta* represents an original *bar*, *pel*, *berta* or *palta*. The double meaning of this root which has appeared in the Aztec *palli*, the Japanese *iro* and *iru*, and the Basque *bel* and *bar*, holds good in the case of the Iroquois, for colour is *wensera*, in which *wen* is the radical, and *iowente* means "accompanying or belonging to." The form *wen* is by no means so common in Iroquois as to make this a chance coincidence. The first part of the word *osk-wenta* is an abbrevation of a common form denoting the mouth. In the Basque we are warranted in rejecting Van Eys's derivation of *ezpana*, the lip, from the root *es*, to shut, inasmuch as the same root in *eztarri*, the throat, would be manifestly out of place. In *ez* therefore we detect the ancient form for mouth which the Circassian gives as *itsha*, and the Natchez as *heche*. And in *pana*, when it is remembered that the change of *l* to *n* is not uncommon in the Basque dialects, there is no difficulty in seeing an archaic *pala*, even if the Iroquois *wen* did not justify the connection. The Aztec *tenxipalli* has derived its *enxi*, for the *t* is prosthetic, from such a strengthened form of the *ez*, *eche*, mouth, as is found in the Yukahiri *anga*, *angya*, and in the Lenca *ingh*. The following table will set more clearly before the eye these relations of the Khitan languages in the Old World and in the New :—

FORMS OF THE AZTEC *palli*.

	COLOUR.	CONTENTS, PERTAINING TO	LIP.
Aztec	*palli*	*palli*	*tenxi-palli*
Japanese	*iro biro*	*iru*, *biru*	*kuchi-biru*
Iroquois	*wensera*,	*iowente*	*osk-wenta*
Basque	*bel*	*bar*	*ez-pana*

A somewhat similar instance is afforded in the Aztec word for leaf, *iatla-pallo* or *quauhatla-palli*, of which the first part is the word denoting a tree. The same is the case with *eatcha* in the corresponding Yuma term *eatcha-berbetsen*. But the *tlel* of the inverted Kamtchatdale *bil-tlel*, the *djitsha* of the Yukahiri *pal-djitsha*, and the *zeli* of the Georgian *pur-zeli*, no longer mean tree in these tongues. The Kamtchatdale now uses *utha* and *uuda*, diminished forms of the

Lesghian *hueta* and the Basque *zuaitz*. The Yukahiri has conformed to the Lesghian *dzul* in *tshal*; and the Georgian, with its *che*, *tka*, and *tcheka*, more nearly approaches the Yuma and other American forms. Still *tlel*, *djitsha* and *zeli* are thoroughly Khitan in character, answering to the Circassian *zla*, the Basque *zuhatsa*, and the Lesghian *dzul* and Yukahiri *tshal*. Such examples suffice to show how difficult it must be to gain a thorough acquaintance with the structure of our American languages, without having reference to the stock from which they are derived, as well as the paramount value of these languages in all matters affecting the construction of the Basque and Caucasian, the Siberian and Japanese tongues.

Whether the Aztec *tl* was an original element in Hittite speech, or a corruption arising after the dispersion in 717 B.C., we shall not know definitely until the inscriptions of Syria and Asia Minor, of India, Siberia, and Japan, yield a vocabulary of sufficient extent to enable us to judge. It is very probable that it existed as a substitute for *r* in certain Khitan tribes from a very early period, since, in the land of the Nairi, the Assyrian inscriptions mention a town Citlalli, in which we recognize the Aztec word for star, the equivalents for which in Araucanian, Atacameno, Shoshonese, Aino, Lesghian and Basque are *schalela*, *halar*, *shul*, *zirari*, *suri*, and *izarra*. The land of the *Nairi* or *Nahri*, the *Naharina* of the Egyptian records, has been generally regarded as a form of the Semitic *Naharaim*, the rivers, whence the designation Mesopotamia. But the word is purely Turanian, and designates primarily a people, not a country. The Egyytian form is the most perfect, as it preserves the medial aspirate and retains the Hittite plural in *n*. It is just the Aztec national designation *Nahuatl*, *Nauatl*, or *Navatl*, which, by the application of the law of phonetic change, becomes *Nahuar*, *Nauar* or *Navar*. The Aztec word means "that which is well-sounding, or a fluent speaker," but most of the words derived from the same root have either the meaning of *law* or *measure* or of *interpretation*. The fluent speaker probably was looked upon as one who spoke with regard to the laws of language and in measured tones, and the interpreter as one who converted the idiom of barbarians into the well-regulated language of the Aztecs. The Japanese preserve the word in two forms, *nori*, meaning law or measure, and *naori*, translation. In Basque it is represented by *neurri*, measure, and this in all probability is the same word as Navarre, a Basque province. As Khupuscai and the land of the Nahri are united in the Assyrian inscriptions, so, in Basque geography, are Guipuzcoa and Navarre. The Scythic Neuri of Herodotus were probably members of the same family. The Niquirans, who are Aztecs, settled in Nicaragua, preserve the ancient name but have hardened the aspirate into a guttural.

More than thirty years ago that veteran ethnologist, Dr. Latham, wrote the following: "The Kamskadale, the Koriak, the Aino-Japanese, and the Korean, are the Asiatic languages most like those of America. (Afterwards he includes the Yukahiri and elsewhere connects that language with the Yeniseian.) Unhesitatingly as I make this assertion · an assertion for which I have numerous tabulated vocabularies as proof—I am by no means prepared to say that one-tenth part of the necessary work has been done for the parts in question; indeed it is my impression that it is easier to connect America with the Kurile Islands and Japan, &c., than it is to make Japan and the Kurile Islands, &c., Asiatic." Nothing can be truer than the above statement made by one whose name should carry the greatest weight with all his scientific utterances to the minds of scholars. It is therefore simply incomprehensible how a writer on philological subjects of such high standing as Mr. Horatio Hale could be led to say, "Philologists are well aware that there is nothing in the languages of the American Indians to favour the conjecture (for it is nothing else) which derives the race from Eastern Asia." I venture on the contrary to assert that there is no philologist worthy of the name who, having carefully studied the languages of the New World and the Old with which this paper deals, has come to any other conclusion than that reached by Dr. Latham and myself. And if Mr. Hale will simply follow up the relations of the Basque, which he wisely connects with our American aboriginal languages, he will soon find himself among those very peoples of Eastern Asia whom he so summarily dismisses. Dr. Latham's Peninsular Mongolidae, including the Yeniseians, and the Americans, are neither Mongolic, Tungusic, (with the exception of the Tinneh , Finno-Samoyedic, Dravidian, or Monosyllabic. They have relations in India among the aboriginal northern peoples, and the Kadun or red Kariens of Birmah belong to the same race. But, with these exceptions, the Khitan do not connect with the Asiatic populations. Not till we reach the confines of Europe and Asia in the Caucasus, where another unclassified group of languages makes its appearance, do we find the relatives of the colonizers of America, and through them effect, what Mr. Hale would do *per saltum* across the Atlantic, a union with the Basques.

From these general considerations I turn to the special work set forth in this paper, that namely which exhibits the relation of the Aztecs to the Khitan family in general, and in particular with those branches of it which are found in the neighborhood of the ancient Hittite civilization. The meagreness of my vocabularies of the Caucasian languages compelled me to illustrate their connection by the closely related Basque in the case of the Hittite inscriptions which I recently translated. Some examples of the relation of the Hittite language spoken in Syria and Mesopotamia in the 8th and preceding centuries B.C., may fitly close the argument in favour of the Hittite or Khitan origin of these and their related languages.

COMPARISON OF HITTITE FORMS FROM THE MONUMENTS.

English.	Hittite.	Basque.	Japanese.	Aztec.
dependence	kakala	katalo	kakari	cacalic, cetilia
incite	kasakaka	kitzikatu, kilikatu	keshikake	cocolquitia
oppose	kakeka	jauki	giyaku	ixquaqua
desirous	manene	min	mune	mayanani
beseech	neka	nastu	negau	notza
modest	simaka	zimiko	tsume	temociui
country	kane	gune	kuni	cana
cut	kara	zilhetze	kiru	xeloa
he	ra	hura, bau	are	ye
small	sasa	chiki	sasai	xocoa
put	tara	ezarri	ateru	tlalia
fight	tiketi	zehatu, etsaigo	tekitai	teyaotia
between	neke	nas, nahas	naka	netech
hastily	sakasakasa	takataka	sekaseka	iciuhcayotica
destroy	kasa	chikitu	*kachi*	*cacayaca*
lay waste	susane	zuzi	susami	xixinia
accord	kane	on-gune	kanai	cen
come	al	el, hel	iru. kuru	vallauh
house	taku	tegi	taku	techan
I	ne	ni	mi	ne
within	tata	ta, hetan	tate	titech
at	ka	gau	oku	co
in	ne	an, n	ni	
vex	nebala	————	naburi	navallachia
hear	kika	————	kiki	caqui
ruler	basa	————	bushi	pachoa
friend	tineba	————	tomobito	tenamic

From these examples it appears that the best living representative of ancient Hittite speech is the Japanese, which, with the Aztec down to the time of Spanish conquest, has never ceased to be a literary language. Standing midway between the long-forgotten Hittite civilization of Syria and the now extinct native civilization of Mexico, Japan affords the most satisfactory starting point for the investigation of problems of world-wide interest that find their centre in the Khitan name. In its name Yamato it shows a closer connection with Hamath than with the land of the Nahri in Mesopotamia. As the home, therefore, of the scribes, whom the Peruvians called Amautas and the Aztecs Amoxoaquis, literature naturally flourished in its islands; and the believer in Holy Writ will see in Japanese culture and prosperity the result of the blessing of Him who is governor among the nations upon the Kenite "scribes that came of Hamath, the father of Beth-Rehob," Hittites indeed, but nobler than their fellows.*

Mr. Buchan was of opinion that it was impossible to pronounce an opinion upon the paper without examining the lists of words carefully, but the conclusion that the American Indians reached this continent from north-eastern Asia seemed exceedingly reasonable. He must, however, differ from Prof. Campbell in regard to the relationship of the Hinos and Japanese. Recent accounts had confirmed him in the view that they were radically different in language as well as in physique. He might mention that it had been clearly established that the Hinos were, as according to a pet theory of his they ought to be, a white race, seeing that they inhabited a moist and cloudy region. The contradicting accounts of previous travellers as to their colour were due to the Hino abhorrence of water, at least when applied externally.

Mr. Notman, Mr. Shaw, Mr. Dunlop and Mr. Murray also took part in the discussion.

* Mr. VanderSmissen has kindly called my attention to the fact that Professor Schleicher, whom in my former paper on the Khitan Languages I inadvertently represented as constituting grammatical construction the sou of language, really gives great prominence to the phonetic element, especially to that portion of it which expresses relation. I am glad to acknowledge this correction of an extreme statement by so competent a disciple of the great German philologist.

Mr. VanderSmissen also read a paper by the Rev. Dr. MacNish, of Cornwall, entitled :—

THE GAELIC TOPOGRAPHY OF WALES AND THE ISLE OF MAN.

In a paper which I had the pleasure of sending to the Canadian Institute during last year, I endeavoured to prove, by the examination of topographical names in England and Scotland and Ireland, that Celts who spoke Gaelic must have preceded the Cymry in the occupation of the British Isles. On the strength of evidence which appeared to me satisfactory, I came to the conclusion that "the first powerful stream of immigration into Great Britain and Ireland was Gaelic; *that the Scottish Gaels are the representatives of those Celts* who were the first to enter Britain and to travel northwards from the South of England to Scotland; and that the remote ancestors of the Scottish Gaels and the Celts who were the first to people Ireland, were one and the same people and spoke the same language."

I propose in this paper to examine the Topography of the Isle of Man and of Wales, in the hope that corroborative evidence can thus be obtained in favour of the theory, that Celts who spoke Gaelic preceded the Cymry in the occupation of Great Britain; and that the arrival of the Cymry must have been much later than that of the Gaels whose language is still discernible, after the glide of many centuries, in the names of headlands and mountains, and lochs, and bays, and rivers. It is reasonable to conjecture that the earliest occupants of Britain wended their way westward, and that a Celtic population settled in the Isle of Man long before the Romans invaded Britain; and that from Man many Celts must have passed into Ireland and at different times into Scotland. The Topography of the Isle of Man; the names which still survive and which a succession of foreign masters was powerless to obliterate; the language which the Manksmen speak down to our own day; and the literature which they have, though it is not very extensive,—combine to prove that the Isle of Man and its inhabitants are normally Gaelic, and that Manx is closely allied to Irish and especially to Scottish Gaelic. Dr. Joyce in his interesting work, *Irish Names of Places*, (Vol. I, p. 163), has this reference to Manannan Beg Mac y Leirr, who, the Manksmen aver, was the founder, father and legislator of their country. "One of the most celebrated characters among the people, *i.e.*, the Tuath de Danaan, was Manannan Mac Leir, of whom we are told in Cormac's Glossary and other ancient authorities, that he was a famous merchant who resided in and gave name to Inis Manann, or the Isle of Man . . ." The conjecture has been advanced, that the term Mannin is compounded of *meadhon*, middle, and *in*, an island; and that accordingly, it is a purely Gaelic word, signifying "*the middle island*." A glance at the map will show, that the Isle of Man is situated in a very convenient position so far as England, Ireland and Scotland are concerned; and that in the days of irregular and unprincipled warfare, it could not fail to be involved in the continual struggles that were going on in those kingdoms. Three armed legs form the present armorial bearing of the Isle of Man. The motto, *Quocunque jeceris stabit*, inasmuch as no transposition of the words can alter the true meaning, may be regarded as an ingenious allusion to the three alternatives which Man in the days of its independence possessed, of leaning for support on one or more of its more powerful neighbours. That the Manksmen could and can speak their own Gaelic after being subject to their Welsh neighbours for 400 years, and to the Danes for 153 years, and to the Norwegians for 200 years, and after owning the sway of England and Scotland for 139 years before the Isle of Man became the property of the Stanleys with whom it remained for 330 years, when it passed into the possession of the Lords and Dukes of Atholl, who surrendered every claim to it in 1829,—goes very far to show how strong the life of a language can be, and how its vitality can continue and be vigorous even when unfriendly forces of a powerful kind are, it may be, intent on destroying it.

Taylor in his *Words and Places*, (pp. 260, 261), maintains that *Man* signifies a district. He goes on to state that "the map of the Isle of Man contains about four hundred names, of which about 20 per cent. are English, 21 per cent. Norwegian, and 59 per cent. Celtic. These Celtic names are all of the most characteristic Erse type. It would appear that not a single colonist from Wales ever reached the island, which, from the mountains of Carnarvon, is seen like a faint blue cloud upon the water. There are 96 names beginning with *Balla*, and the names of more than a dozen of the highest mountains have the prefix *slieu*, answering to the Irish *sliebh* or *sliabh*. The Isle of Man has the Curraghs, the Loughs, and the Allens of Ireland faithfully reproduced." Taylor was doubtless at pains to make an accurate examination of the topographical names of Man. It is in the highest degree surprising that, after all the changes which passed over the Isle of Man, and in spite of the numerous languages which were spoken by those who successively exercised authority over its inhabitants, 59 per cent. of the topographical names should still be Gaelic, commemorating thus the early and powerful presence of the Gaels in the Island long before, it may be, Cæsar invaded Britain, or the Cymry forced their way as later Celts into the Albion of Aristotle. In his introduction to his Irish Grammar, Dr. O'Donovan thus writes: "The Manx is much further removed from the Irish than the Gaelic of Scotland. Its words are principally obscured by being written as they are pronounced without preserving the radical letters as in Irish." The translation of the Holy Scriptures into Manx forms the most important part of Manx Literature. The translators went avowedly on the principle of spelling words phonetically, of disregarding etymological considerations, and of making as near an approximation as might be possible to the manner in which the language was spoken, in order that every Manksman could easily read and understand the Scriptures in his native tongue. It naturally happens that no small ingenuity is at times necessary to discover the exact value of certain sounds and words in the Manx language as it is written. The judicious remarks of Dr. Joyce, (Vol. I, pp. 1, 2, 3,) apply with peculiar strikingness to the topographical names of Man: "The interpretation of a name involves two processes, the discovery of the ancient orthography, and the determination of the meaning of this original form. . . . A vast number of our local names are perfectly intelligible as they stand in their present Anglicized orthography, to any person who has studied the phonetic laws by which they have been reduced from ancient to modern forms . . . In numerous other cases, where the original forms are so far disguised by their English dress as to be in any degree doubtful, they may be discovered by causing the names to be pronounced in Irish by the natives of the respective localities. When pronounced in this manner they become in general perfectly intelligible to an Irish scholar . . . The meaning of a name otherwise doubtful will often be explained by a knowledge of the locality."

Words beginning with *Baile* are very common in Scotland, and especially in Ireland. *Baile* signifies a farm, a village, or town. Indeed, a casual comparison of the names in Man, and Scotland, and

Ireland, that begin with *Baile*, will show that there is a great similarity if not an identity between them. It will be sufficient to adduce a few examples of the presence of *Baile* in the Topography of Man :—

Bailegawne, *baile'ghobhainn :* the smith's town.
Bailenahown, *baile na h-aimhne :* the town of the river.
Balladoole, *baile 'n tulaich :* the town of the knoll.
Ballaquane, *baile'chuain :* the town of the ocean.
Ballaquinney, *baile'chuinne :* the town of the corner.
Balnabarna, *baile* and *bearna*, a gap or fissure.
Ballamahow, *baile* and *magh*, *a field ;* Irish, *Mayo*.
Baldwin, *baile* and *aodhuinn*, a brow or face.
Ballamona, *baile* and *monadh*, *a moor*.
Ballawhane, *baile* and *uaine*, green.
Ballaharry, *Ballagharaidh*, *baile* and *garadh*, a den.
Balloun, *baile* and *amhainn*, a river.
Ballaglass, *baile* and *glas*, grey.
Balla Kilmorrey, *baile*, *cill*, a church or graveyard, and *Muire*, Mary.
Ballysallach, *baile* and *salach*, filthy.
Ballaugh, *bealach*, *Balloch :* an opening or defile.
Ballamonamoar, *baile* and *monadh mor*, the large moor.
Ballure, *baile* and *ur*, new.
Ballacowle, *baile* and *cuil*, a corner.
Ballacooiey, *baile* and *coille*, a wood.
Ballaliece, *baile* and *leac*, a flat stone.
Ballacreggan, *baile* and *creag*, a rock.
Ballamagher, *baile* and *machair*, a field.
Ballnakilley, *baile* and *cill*, a church-yard.
Ballaskyr, *baile* and *sgeir*, a rock.
Ballabooie, *baile* and *buidhe*, yellow.

Words identical with those which have now been cited, are of frequent occurrence in the Topography of Scotland and Ireland. I have given the Gaelic derivation or equivalent of the names which have been taken from the Topography of the Isle of Man. Their Gaelic origin is unmistakable; and hence the inference may be reasonably drawn, that the same people gave names in the Isle of Man, in Scotland, and in Ireland, to the places in which *Baile* is found as one of the constituent elements, and that the language which was then spoken in Man and Scotland and Ireland was one and the same.

The names of hills and glens in the Isle of Man are likewise Gaelic, *e.g.* :—

Slieu mayll, *sliabh*, hill : and *maol*, bare.

Cronk na h--eiric, *cnoc*, hill : *eirig*, a ransom.

Cronk na Kielan, *cnoc*, *hill* : and *ceolan*, faint music.

Slieuwhallin, *sliabh* and *aluinn*, lovely.

Cronk Keeillowan, *cnoc*, *cill*, and *Eoghann Hugh* : Ewan.

Knockaloe, *cnoc* and *loch*, a lake.

Cronk ny marroo, *cnoc na marbh*, dead.

Slieudhoo, *sliabh* and *dubh*, black.

Cronkbourne, *cnoc* and *burn*, water.

Cronkurleigh, *cnoc* and *iolaire*, an eagle.

Glentrammon, *gleann*, Manx *glione*, a valley, and *druman*, a ridge or boortree.

Glen Darragh, *gleann* and *darach*, oak.

Glen Moy, *gleann* and *magh*, a plain.

Glion Mooar, *gleann* and *mor*, large.

Gliongawne, *gleann* and *gobhainn*.

Glenfaba, *gleann*, *faigh*, pasture, and *ba*, cattle.

Glencutchery, *gleann* and *cruitearach*, the occupation of a harper.

Glendoo, *glenn* and *dubh*, black.

So apparent is the Gaelic origin of the names of hills and valleys in Man, that any one who has a knowledge of Gaelic can with great facility determine the meaning of the names in question.

Poolvash is compounded of poll, a pond or pool ; and *bas*, death, the pool of death.

Port ny-Hinshey, *port*, a harbour ; and *innis*, an island ; *port na h-innse*, the harbour of the island. Such was the original name of the harbour of Peel.

Maugherakew, *machair*, a plain ; and *ceo*, mist.

Bowmaken, *bogha*, a bow ; and *ceann*, head.

Rushen, *rudha*, a promontory ; and *sean*, old.

Rue, *rudha* : a point.

Rievalle, *righ*, a king, and *baile*.

Ayre, *airidh* : a shealing.

Shellach point, *seileach*, willow.

There are many words beginning with *ceann*, a head, whose Gaelic origin is quite evident, *e.g.* :—

Kentraugh, *ceann*, a head ; and *traighe*, shore.
Kiondroghad, *ceann* and *drochail*, bridge.
Kenmoy, *ceann* and *magh*, a plain.
Kinskae, *ceann* and *sgiath*, a wing.
Kionsleau, *ceann* and *sliabh*.
Kenna, *ceann* and *ath*, a ford.

Such words as these indicate at once that they are of Gaelic origin, and that the Celts who imposed such names on the prominent physical features of the Isle of Man spoke the language which has been perpetuated over many centuries in the Highlands of Scotland.

Lhergydoo, *learg*, a slope, and *dubh*, black.
Slegaby, *slige*, a shell ; and *buidhe*, yellow.
Keillvael, *cill* and *maol*, bare.
Douglas, *dubh* and *glas*, grey.
Sulby, *suil*, an eye ; and *buidhe*, yellow.
Lazayre, *lios*, a fort ; and *airidh*, a shealing.
Lhen moar, *lean*, a plain ; and *mor*, large.
Garff, *garbh* : rough.
Braddan : a salmon
Cas na h-owne : the foot of the water.
Strathallan, *srath*, a valley ; and *aluinn*, splendid.
Cloughbane, *clach*, a stone ; and *ban*, white.
Loughan a yeeah, *lochan* a little lake ; and *geadh*, a goose.
Cregnesh, *creag*, a rock ; and *innis*, an island.
Caoilban, *caol*, narrow ; and *ban*, white.

It would be very easy to adduce examples in abundance from the Topography of Scotland and Ireland in which such roots are present, as *baile*, *amhuinn*, *monadh*, *cill*, *magh*, *maol*, *creag*, *sgeir*, *cnoc*, *loch*, *gleann*, *port*, *innis*, *learg*, *ceann*, *rudha*, *clach* ; roots which are of constant occurrence in the Topography of the Isle of Man. It is reasonable to conclude, that the power of the Gaels in the Isle of Man was paramount at some time in the far-off past, seeing that the successive waves of conquest which passed over the Island have failed to obliterate the traces of the Gael, and to destroy the proofs that names of rivers and hills and valleys furnish, regarding the people whose time of predominant occupation was so long as to enable them to leave indelible footprints of themselves and of their language in the names which the Topography perpetuates for the information of posterity.

Thomas Stephens, the well-known author of the Literature of the Kymry, avers, "that the Welsh or Kymry are the last remnant of the Kimmerioi of Homer and of the Kymry, the Cimbri of Germany." It is possible to cite the authority of two learned Welshmen in favour of the theory that the Gaels preceded the Kymry in the occupation of Britain. Edward Llwyd, the famous author of the *Archæologica Britannica*, who expended five years in travelling among the portions of Great Britain and Ireland where the Celtic languages were spoken, and who is justly regarded as the Father of Welsh Philology, thus writes in his Welsh preface to his book: "Nor was it only North Britain that these Gwydhelians (Gaels) inhabited in the most ancient times, but also England and Wales . . . Our ancestors did from time to time drive them northward . . . From Kintyre, in Scotland, where there are but four leagues of sea, and from the County of Galloway and the Isle of Man, they passed over into Ireland, as they have returned backwards and forwards ever since . . . There are none of the Irish themselves, so far as I know, . . . who maintain that they had possession of England and Wales. And yet, whoever takes notice of a great many of the names of the rivers and mountains throughout the kingdom. will find no reason to doubt that the Irish must have been the inhabitants when those names were impressed upon them. *i.e.*, upon the rivers and mountains." In his *Celtic Britain*, (p. 4,) Professor Rhys, of Oxford, who is himself a Welshman, and a Celtic scholar of large attainments, asserts that the Goidels (or Gaels), were undoubtedly the first Celts to come to Britain, as their geographical position to the west and north of the others would indicate. In connection with the Ogam Inscriptions, which are found in Wales, he remarks in his *Celtic Britain*, (p. 213,) that the Goidels belonged to the first Celtic invasion of Britain, and that some of them passed over into Ireland and made Ireland also Celtic. Some time later there arrived another Celtic people with another Celtic language. "These later invaders," he writes, "called themselves Brittons, and seized on the best portions of Britain, driving the Goidelic Celts before them to the west and north of the Island; and it is the language of these retreating Goidels of Britain that we have in the old Inscriptions and not of Goidelic invaders from Ireland. Their Goidelic speech, which was driven out by the ever-encroaching dialects of the Brythons, was practically the same language as that of the Celts of Ireland, of Man, and of Scotland." As Lhuyd and Professor Rhys give such an unambiguous opinion respecting the earlier presence of the Gaels in Britain, it may fairly be expected that the Topography of Wales will lend strength to the conclusions of those Welsh scholars.

The word *Aber* is of frequent occurrence in the Topography of Wales. It is in all likelihood a compound of *ath*, a ford, and *bior*, water—waterford.

In my previous paper on the Gaelic Topography of Britain, I adverted to the theory which was first advanced by Chalmers and which has as its advocates Dr. MacLauchlan, and Mr. Taylor, the author of *Words and Places,*—that, as Dr. McLauchlan contends, "the Generic *Aber* is in Scottish Topography found uniformly associated with specific terms purely Kymric," and that, as Mr. Taylor contends, "the Cymry held the Lowlands of Scotland as far as the Perthshire hills. The names in the valleys of the Clyde and the Forth are Cymric and not Gaelic." I remarked that Robertson and Skene have successfully refuted the theory in question. It is certainly a singular fact that if the Topography of Strathclyde is Cymric and not Gaelic, there are no *Abers* in the counties of Selkirk, Peebles, Ayr, Renfrew, Lanark, Stirling, Dumbarton and Galloway. Robertson, after examining the theory of which mention has been made, is fully justified apparently in employing this language in his Gaelic Topography of Scotland, (p. 512): "All the great features of nature within Scotland attest to the truth of the Caledonians being the first race; the mountains and the valleys all speak to us in their language—the *Gaelic* and not in *Welsh*. The author has proved beyond all controversy that there is not a mountain to be found in Scotland which bears a Welsh name, not a lake or river." *Aber* is of common occurrence in the names of places that lie along the sea-coast of Wales, *e. g.* :—

Abergeley : the confluence of the white river, (*geal*).
Aberconwy, *aber* and *conabhuinn*: confluence of rivers.
Aberhonnddu, *aber* and *abhuinn*, river; and *dubh*, black.
Abermaw, *aber* and *baw*: W filthy.
Abermynwy, *aber*, and a root resembling *monadh*, moor.
Abertawy, *aber* and *tabh* : water or ocean.

Aberteifi, *aber* and *tiefi*, akin to *Taff*, *Taw*, *Tow*, a root which occurs in Tay, Tagus, Thames.

Aberavon, *aber* and *abhuinn*, river.

The question now arises as to the best and most plausible manner of accounting for the presence of *Aber* in the Topography of Wales and of the Highlands of Scotland; and for its almost entire absence from the Lowlands of Scotland where a Cymric kingdom once existed, as well as from Cornwall, which has many points of resemblance in language and race and tradition with Wales. In the face of the certainty that a large part of Scotland where no *Abers* are found intervenes between Cumberland, which in its very name perpetuates the fact that it was at one time inhabited by Cymry,—and between the Highlands where *Abers* and *Invers* are of constant occurrence, it will be vain to seek for any satisfactory explanation of the presence of so many *Abers* in Scotland in the predominance which the Cymry at one time possessed in the South of that country. Is not the conjecture more reasonable that, as Wales and the Highlands of Scotland resemble each other very closely in their mountainous character, in the ruggedness of their soil, and in the number and strength and rapidity of their streams; and as no other portion of Britain has such an uneven and rugged surface as Wales and the Highlands of Scotland, a similar term should be employed to designate the frequent confluences of streams,—a term which is not found elsewhere, and which, so far as Wales and Scotland are concerned, finds an easy explanation when the concession is made, that it was used by one and the same people in the far-off ages to describe these meetings of streams and rivers, which are common to both countries. The difficulty vanishes when it is granted that *Aber*, which is a Gaelic word, was employed by the observant Gaels of a remote age to represent these confluences which they found in Wales, and which they found in the Highlands of Scotland after they had passed over the comparatively level Lowlands. It is noteworthy that Latham is disposed to regard *Aber* as the *Abor* in the word *Aborigines*, "the locality to which it applied being either the confluence of the rivers Anio and Tiber, or the mouth of the Tiber."

Cær or *Cader*, which is the Gaelic *Cathair*, a city or fortified place, enters into many of the Topographical names of Wales, *e. g.:* Cader Idris, Cardigan, Cærnarvon, Cærmarthen, Cardiff, &c. Joyce, in his *Irish Names of Places*, (Vol. I., p. 284-5), states that "in modern nomenclature, the word (*Cathair*) usually takes one of the two forms, *Caher* and *Cahir*, and that there are more than 300 townlands and towns whose names begin with one or the other of these two words,—all in Munster and Connaught . . . *Caher* itself is the name of more than thirty townlands, in several of which the original structures are still standing." Cathair is unmistakably present in such names as these in the Topography of Scotland : Carden, Carriden, Carlin, Carmyle, Carluke, Carlaverock. Carnervon is the name of a place in Aberdeenshire—*Cathair-an-ear-abhuinn* : the city of the East river." The contention of the advocates of the theory, that the Topography of Scotland is largely Cymric, and that *Caer* which occurs in such names as have been already cited, is an illustration of the correctness of the theory,—is altogether untenable. The very fact that *Cathair* enters so largely into the Topography of Ireland and Scotland, clearly indicates that the word is not strictly Cymric, but that it dates from a remote age when Celts, whose language was Gaelic, imposed the names which have come down to our time on similar physical peculiarities in Wales and Scotland and Ireland.

The word *Llan* which means area, yard, church, is frequently found in the Topography of Wales, *e. g.* : Llandaff, Llandeilo, Llanelly, Lampeter, &c. Joyce thus writes, (Vol. I., p. 321) : "*Lann*, in old Irish *land*, means a house or church . . . *Lann* is found in our earliest MSS., among others in those of Zeuss : it occurs also in an ancient charter . . . in the sense of house." The word *lann* occurs also in Gaelic, and has the same meaning that it has in Welsh and Irish. I am disposed to believe that between *lann* and the Gaelic word *lean*, a meadow, a green plain, there is a strong resemblance, if not an identity. Joyce admits that, in its ecclesiastical application, *lann* was borrowed from the Welsh, but contends that "when it means simply *house*, it is no doubt purely Irish and not a loan word." It is clear, therefore, that *lann* is a Gaelic word, and that it does not belong exclusively to the Cymry and to the Topography of Wales.

Loch is the term which Scottish and Irish Gaels employ to designate a *lake* or an inland sea, or arms of the sea. The Anglicised form of Loch in Ireland is *Lough*. *Llyn* is the word which occurs in the Topography of Wales to designate a lake, *e. g.* : In Cardiganshire there are Llyn Teifi, Llyn Gynon, Llyn Eiddwen. In the County of Carnarvon there are, among others, Llyn Cwlyd, (*caoilead*, narrowness), Llyn Eigian, (*aigein*. deep), Llyn Llydan, (*leathan*, broad). Llyn is doubtless the purely *Gaelic* word, *linne*, which signifies a pool, lake, gulf. *Linn* enters into such words as Lincoln, Linn, Loch *Linne*, Roslin, *Dublin*. Though a difference obtains between the use which is made of *linne* in the Topography of Wales and the sense which it bears in the Topography of Ireland and Scotland, the word is unquestionably Gaelic, and as much entitled to that parentage as *loch*, or *cnoc*, or *amhuinn*.

The root *moin* or *moine*, a mountain, *moss*, a mossy-place, enters into *mynydd*, the Welsh word for mountain, and into the Gaelic word *monadh*. *Moin* or *monadh* enters largely into the Topography of Scotland and Ireland, *e.g.*, in the former country, Moncrieff, Monimail, Monivaird, &c., and in the latter country, Monalour, Ardmhoin.

Carn, the Gaelic word for a heap of stones, raised over the tombs of heroes,—a word which is of common occurrence in Scotland and Ireland, *e. g.*, Cairngorm, Cairndow, Carn, Carnglass, Carnlea, &c., is present in *Carned Llewelyn* and in *Carned Dafydd*, in the County of Carnarvon.

Maol, bare, *a precipitous promontory*, *Mull*, *Moyle*, which occurs in such names as the Mull of Kintyre, the Mull of Galloway, Malin Head, Rathmoyle, Lismoyle, Dunmoyle,—is present in *Moel Siabod*, *Moel Hebeg*, in the County of *Carn*arvon, and in *Mael Famman* and *Mael y-Gaer*, in Flintshire. *Drum*, the well-known Gaelic word for a *ridge* or *back*, is in Carnarvonshire. *Kinmel* (*ceann*, a head, and *meall*, a round hillock), is in Denbighshire. *Dun* appears to enter into the first syllable of Denbigh, Dinbych, *Dun*beag, the little dun or fort.

Arran, which occurs in *Arran Fowddy*, is the name of an island in Scotland and of several islands on the western coast of Ireland. *Craig y Llyn*, the rock of the pool or lake, is in Glamorganshire.

So far, therefore, as the names of mountains and ridges and hillocks in Wales are concerned, it is evident that Gaelic words are commonly to be found.

The names of various places in Wales disclose their Gaelic origin very readily. *I*, the Gaelic word for island, as in Iona, forms the last syllable of *Anglesey*.

Maeltraeth, in the same country, seems to be compounded of *Maol*, smooth or bare; *traighe*, a beach or shore.

Penmore is *ceann*, and *mor*, large.

In *Brecknock*: *breac*, spotted, and *cnoc*, mountain, seem to enter as constituent elements.

Brynmawr, (*bryn*, hill; Irish, *bri*; Gaelic, *bruthach*, and *mawr mor*, large), signifies a large hill.

Crickhowel seems to be compounded of *creag* and *suil*, an eye. *Bangor* (*Beannchar*, pointed hill or rocks), is also the name of a place in Down, Ireland. CARREG CENNIN, in Caermarthen, is doubtless *Carraig Cheannfhionn*, the white-headed rock. *Pembroke*, (Welsh, *Penfro*), is compounded of *ceann* and *bru*, a country.

The Topography of Wales discloses its Gaelic origin very distinctly in the names of its rivers, *e. g.*, *Taff*, *Tave*, *Taw*, *Towey*, *Tow*, *Teifi*: here, are different forms of the same root, which appears also in Tagus, Tay, Thames, and which has the strongest similarity to *Tabh*, an Irish and Gaelic word, signifying *water or ocean*.

Severn : *seimh*, still, and *burn*, water.

Dee, *da*, *abh* : double water.

Dovy, *dobhaibh* ; boisterous.

Cowin, *cumhann* : narrow.

Alyn, *aluinn*, splendid ; or *al*, a stone : *abhuinn*, river.

Dwyrid, *Dur*, water ; or *duiread*, stubbornness.

Ogmore, *uisge*, *oice*, water ; and *mor*, large.

Verniew, *fearna* : alder tree.

Wye : Welsh, *Guy*, water ; *Buidhe*, yellow.

Honddu, *amhainn dubh* : black or dark river.

Conway, Comh, *con-amhainn-aimhne* : coming together of the river.

Seoint, *sinte* : extended.

Gwili, *goil*, *goileach* : boiling, raging.

Cothi, *cuthaich* : frantic.

Llwchwr, *luachair* : rushes.

Aled, *aillead* : beauty (?)

The citations which have been made from the Topography of Wales will suffice, I trust, to show conclusively, that the names of the Abers, and rivers, and forts, and hills, and lakes of Wales are of Gaelic origin ; and that the same Celtic people gave, in the unrecorded ages of the past, the names which the prominent physical features of Wales and Ireland and Scotland have preserved over the centuries, and by which, though at times in the midst of obscurity, those natural features are still wont to be described.

In the preface to his *Grammatica Celtica,* Zeuss (than whom there is no better authority), asserts "that it can by no means be established that there was a community or an identity of language between the British and the Irish, (*inter Britannos et Hibernos*), in the eighth or ninth century, or even at a much earlier date; although it is abundantly manifest that both dialects or languages have begun from one fountain." That statement of Zeuss may be construed legitimately enough in such a manner as to increase the value of the argument which can be drawn from topographical names, in favour of the theory that the Gaels preceded the Cymry in their occupation of Wales as well as of the other portions of the British Isles. May not the argument be fairly advanced, that, as the substratum of the Topography of Wales is distinctly Gaelic, and as Zeuss, as the result of his exhaustive and masterly examination of the oldest forms of the Celtic languages or dialects contends, that long before the eighth or ninth century there was no identity of language between what may be regarded as the Cymry and the Gael,—to the Celts who spoke Gaelic the honour belongs of laying the foundation of the Topography of Wales; for, although the topographical structure has many stones that are of Cymric growth, the stones that form the foundation and on which the entire structure rests, are of purely Gaelic origin, and have an indefeasible kinship with the foundations of similar structures in Scotland and Ireland.

The rapid survey which I have been able to present of the Topography of the Isle of Man and of Wales will, I trust, serve to corroborate the conclusions at which learned philologists such as Llwyd and Rhys arrived from different channels of reasoning and observation, and to strengthen the theory, if not to establish it on honest and satisfactory grounds, that the first powerful stream of Celtic immigration into Britain was Gaelic, and that the same Celts who gave names to Fintry and Bannockburn in Scotland, gave names also to Bantry and Kinsale in Ireland, to Aberavon and Carnarvon in Wales, and to Slieu Mayll and Poolvash in the Isle of Man.

A short discussion followed, in which Mr. Notman, Mr. Shaw, and Mr. Murray took part.

FOURTEENTH ORDINARY MEETING.

The Fourteenth Ordinary Meeting of the Session 1883-84, was held on Saturday, February 16th, 1884. In the absence of the President and Vice-Presidents, the chair was taken by Dr. Jos. Workman, who, later in the evening, retired on the entrance of the President.

The minutes of last meeting were read and confirmed.

Mr. Henry E. Morphy, B.A., was elected a member.

The following list of donations and exchanges received since last meeting was read :—

1. Journal of the Transactions of the Victoria Institute, to complete a set. Vols. 2, 6, 7, 8, 9, 10, bound, and Part 3, Vol. 15 ; Part 3, Vol. 16.
2. Science, Vol. 3, No. 53, Feb. 8th, 1884.
3. Transactions of the Manchester Geological Society, Part 12, Vol. 17, Sess. 1883–84.
4. Proceedings of the American Antiquarian Society, N. S., Vol. 3, Part 1.
5. Atti della Società Toscana di Scienze Naturali, residente in Pisa, Processi Verbali, Vol. 4, title-pages to do., Vols. 1 and 3.
6. Catalogue of Canadian Plants, Part 1. Polypetalæ, by John Macoun, F.R.S.C.
7. Report of Progress for 1880-81-82, of the Geological and Natural Hist. Survey of Canada. Maps to accompany the above Report.
8. Records of the Geological Survey of India, Vol. 16, Part 4, 1883.

Prof. J. Playfair McMurrich then read a paper on "The Skeleton of the Catfish," which will appear in the next fasciculus of the Proceedings of the Institute.

FIFTEENTH ORDINARY MEETING.

The fifteenth Ordinary Meeting of the Session 1883-84 was held on Saturday, February 23rd, 1884, the President in the chair.

The minutes of last meeting were read and confirmed.

The following list of donations and exchanges was read :

1. Science, Vol. 3, No. 54, February 15, 1884.
2. Bulletin of the Museum of Comparative Zoölogy at Harvard College, Vol. 11, No. 9.
3. Bulletin of the Buffalo Society of Natural Sciences, Vol. 4, No. 4.

4. The Pennsylvania Magazine of History and Biography, Vol. 7, No. 4, December, 1883.
5. Correspondenz-Blatt der deutschen Gesellschaft für Anthropologie, Ethnologie und Urgeschichte, 13 Nos., January, 1883, to January, 1884.
6. The Monthly Weather Review for January, 1884.
7. Journal of the Royal Geological Society of Ireland, Vol. 2, Part 1; Vol. 3, Part 1, 2, 3; Vol. 4, Part 2, 3; Vol. 6, Part 2.
8. The Canadian Entomologist, Vol. 15, No. 12.
9. Journal of the Anthropological Institute of G. B. and Ireland, Vol. 13, No. 3.
10. Proceedings of the Royal Geographical Society, Vol. 6, No. 2, February, 1884.

Mr. J Gordon Mouat then read a paper entitled,

A FEW CANADIAN CLIMATES.

Of the water influences which affect the climate of Canada, that of the Pacific Ocean is by far the most extended and far-reaching. The atmospheric drift of the middle latitudes bears it across the ranges of the Rocky Mountain system and diffuses its ameliorating warmth over the vast plains of the Saskatchewan and Athabasca. The influence of the Atlantic is limited to the few hundred miles over which the eastern surface winds from the sea are drawn inland towards the cyclonic areas advancing from the west. The St. Lawrence valley shows this influence in the winter temperature, which is higher than in the central parts of the continent on similar latitudes, and in a heavier precipitation. The unequal influence of the two oceans tends to throw the meridian of greatest summer heat and winter cold—which, were these influences equal, would lie in the central part of the continent—towards the eastern coast. But here nature has provided a check in the existence of Hudson's Bay and the Great Lakes, which temper the heat of summer and mitigate the winter's cold. It is not, therefore, in the meridian of the Great Lakes that the greatest extremes are found, but westward in the valleys of the Mississippi and Red Rivers.

The influence of the Great Lakes is very marked. In the lake region of the Province of Ontario the mean of the three coldest months varies from nearly 30° Fahr. to a little less than 15°. At similar latitudes in the Mississippi valley, and at almost similar elevation above the sea, the mean temperature of these months varies from 24° to 4°. The winter isothermal of 20° skirts the north shore of Lake Huron on the 46th parallel, descending in the Western States nearly to latitude 41°. The winter mean of 25° has in Ontario an average latitude of 43°, while in the Mississippi valley it reaches as far south as North-Western Missouri in latitude 39°. When the occasional extremes of winter cold are considered, the influence of the Great Lakes is found to be even more marked than in regard to average temperature. The lowest temperature in the past twelve years in Toronto, (lat. 43° 39′) was only −18°·4, Fahr.: Hamilton, (lat. 43°.16′) records −20°.5, and Windsor, (lat. 42°·19′) −19°.5,—while portions of the Niagara and Lake Huron districts show no temperatures lower than 12° below zero. Within shorter periods, not exceeding in any one instance eight years, the following temperatures were recorded at meteorological stations in the Mississippi and Missouri valleys :—

Cairo, Ill., lat. 37° 0′	—16°
St. Louis, Mo., lat. 38° 37′	—21.5
West Leavenworth, Kansas, 39° 20′	—29.0
Indianapolis, Ind., 39° 47′	—25.0
Lafayette, Ark	—17.0

To instance minimum temperatures in the past eight or nine years at stations further up the Mississippi valley is superfluous. Temperatures 40° below zero have been recorded at places in this valley no further north than the Canadian stations cited. During the present winter temperatures as low as—32° have been recorded in the State of Missouri. The lowest in Toronto has barely exceeded —13°. In the winter of 1874–5, the coldest on record in Ontario, when in Toronto the minimum temperature was −16°, temperatures as low as —39° were reported in Northern Illinois.

In short, the lake region of Ontario has as mild a winter mean as the Mississippi valley two hundred and fifty miles farther south, and eastward of the Rocky Mountains it is only to the south and east of a line drawn from Lake Erie to North-Western Texas that the thermometer does not occasionally fall as low as the lowest ever reached in the milder parts of the Province of Ontario.

It is interesting to notice in connection with the influence of the Great Lakes in modifying the cold of winter that the shore of Lake Michigan, opposite Chicago, has a mean winter temperature nearly four degrees higher than that of the city mentioned, and that while the pear grows with difficulty at Chicago, the much more tender peach grows luxuriantly far northward along the eastern side of Lake Michigan, and over several thousand square miles in the Province of Ontario. The area over which the peach can be grown in this Province is nearly ten thousand square miles. It is even found to succeed on favorable soils and situations at Owen Sound, on the Georgian Bay.

If the winter cold of the Province of Ontario is mitigated by the Great Lakes, so also is the summer heat. The great central plains of the Mississippi and Missouri in summer become so heated that the mean temperature of July in Missouri and Kansas is little less than that of New Orleans in the same month. The influence of the solar rays on these great interior plains is so great that the trade winds of the Atlantic, drawn eastward into the Gulf of Mexico, are deflected northward and, affected by the prevailing eastward drift of the atmosphere, are finally carried, charged with moisture, north-eastward occasionally to the Ohio valley and the borders of the Great Lake region. Far northward, in summer, torrid influences prevail. Temperatures of 110° and upward are experienced in Dakota and Montana, and even further north across the international boundary of 49° in the Canadian valleys of the tributaries of the Missouri. But the Great Lakes interpose a buffer against the easterly drift of the interior heat. The isothermals which in winter trend southward after leaving the lake region, in summer trend north-westerly beyond Lake Michigan. The July isothermal of 74°, which is found in Ontario only in the very warmest localities of the Province, reaches a parallel two hundred miles further north in the great plains of the west. The mean temperature of 70° for the three midsummer months, which in Ontario is found rarely northward of the 43rd parallel, is reached very nearly as far north as the 49th parallel in the North-Western States and Territories. It is not until October that latitude for latitude and altitude for altitude the mean temperatures of Ontario and the Mississippi valley are equalized. The decline in temperature thenceforward till winter has set in is more rapid in the Mississippi valley than in the region of the Great Lakes which, warmed by the summer's heat, delay the advent of winter several weeks after that season is established in the central parts of the continent. The advent of spring in the lake region is also later than in the west, partly owing to the retarding effects of the lake water, which has been chilled by the winter's cold, and partly to the greater distance from the now rapidly heating plains of the Lower Mississippi. The effect of this delay of spring is not disadvantageous, for the occurrence of the last frost damaging to vegetation is very nearly alike in point of time in the lake region and in the central parts of the continent, and in the former districts, vegetation being less advanced when that frost occurs, suffers less from its effects. The general effect of the greater liability of the Mississippi valley to intense frosts in winter, sudden changes and late frosts, is such that north of Tennessee no peach districts are found which compare, in immunity from injury through low temperatures, with the peach belts east of Lake Michigan and in the neighborhood of Lakes Erie, Ontario and Huron.

What is true of the annual and seasonal extremes of the lake region and the Western States, has its parallel in regard to the daily range of temperature. It is only once in many years that Toronto, which is fairly representative in this respect of the lake borders of Ontario, knows a range of forty degrees in any one day. The late Prof. Loomis, discussing the results of two years' records of over one hundred stations scattered over the continent north of the 35th parallel and between the Rocky Mountains and the neighborhood of the Atlantic, states that only in the Province of Ontario had he found stations at which the mercury had not ranged occasionally forty degrees in a single day. At the stations in the Mississippi valley and westward to the Rockies, greater changes than forty degrees were recorded several times in each of the two years; at several stations twenty to sixty times. Even as far south as Northern Texas sudden changes of remarkable extent are recorded by the American Signal Service. In one instance a fall from 80° to 18° within a few hours is noted; and on the 7th of September, 1881, on the northern borders of Texas, a sudden lowering of temperature proved fatal to over 300 cattle. The facts given show that in equability of climate the Province of Ontario is one of the most favoured districts in the temperate latitudes of this continent.

While the whole of the lake region of the Province of Ontario as far east as the Ottawa River experiences the modifying influence of the great lakes, the measure of that influence differs greatly according to elevation, and distance and direction from large bodies of lake water. In fact, the lake influence, while rendering the whole region more temperate than any part of the Mississippi Valley to the westward, increases the differences beyond those due to latitude, so that the part of the province south of the 46th parallel presents a much greater variety of climate than any other non-mountaincus district of equal area on the continent. Eastward from the Georgian Bay the effect of the great lakes in moderating heat and cold rapidly decreases, and continental conditions rather than semi-insular gradually come to prevail. Lake Ontario not lying in the direction from which the areas of low and high barometer advance on this region, has but a very limited influence. There being no large body of water to the north, such winter anti-cyclones as take a course to the Atlantic to the northward of the great lakes pour their refrigerating northern blasts down over this region.

At Ottawa the summers are hotter than at Toronto, Goderich and many other places a hundred miles or more further to the south, and *though the summers over the Ottawa district are shorter than in much* of the south-western part of the Province, the mean temperature of July is quite as hot as in most localities in the latter and the maximum temperature very frequently is higher than 95° in the shade; it occasionally exceeds 100° and usually is several degrees hotter than at Toronto, the eastern shore of Lake Huron, and even localities as far south as Lake Erie. The winters of Ottawa on the other hand average as low as 13° Fahr., and are much the same as at Moscow. The average minimum is about 30° below zero. Snow falls deep and the sleighing season is usually four months in length while in parts of south-western and southern Ontario, it is not as many weeks. Though the difference in latitude between Ottawa and Niagara is only about two degrees, the winters of the former place are at least as much colder than those of the latter as the winters of Niagara are colder than those of Memphis in Tennessee, eight degrees still farther south. Yet the sensible cold is not so great as this large excess might suggest; it is usually enjoyable, the atmosphere being dryer and there being more sunshine than in districts more within the influence of the lakes.

The district of Muskoka & Parry Sound, bordering on the Georgian Bay, experiences in greater measure the influence of the Georgian Bay and Lakes Huron and Superior in tempering the heat in summer and the cold in winter of winds from the western semi-circle This influence is necessarily much more marked in winter; though the elevation of much of the district makes the apparent amelioration less preceptible than it otherwise would be. The summers of Muskoka are cooler than those of any other part of Ontario south of the 47th parallel of latitude. But this tempering of the heat is due in large measure not so much to the influence of the Georgian Bay as to general elevation and the number of small lakes of great depth and coolness. Like the Ottawa Valley, though not to the same degree, the district is open to cold northerly winds in winter blowing outward from such centres of high pressure as move eastward to the Atlantic in high latitudes. Elevation adds to the cold of these north winds, which however are infrequent in some winters. At Huntsville (about lat. 45° 15′) in Eastern Muskoka, the temperature in January 1882 during the passage of almost the only severely cold anti-cyclone of the seaso n, fell under a north wind to a temperature 30° lower than was reached at Toronto, and actually 47° lower than at Windsor, less than three degrees further south and little more than 280 miles distant in a direct line. In severe winters, a large part of the Georgian Bay, encumbered with islands, freezes over and the tempering effect of the lake water is thus greatly diminished.

The winters of the large island of Manitoulin, which approaches the 46th parallel, are milder than those of Muskoka. Of the climate of the north shore of Lake Huron beyond the 46th parallel, the meteorological records are meagre. The district is protected against cold west winds in winter by Lake Superior, but is open to cold blasts from the north-west, north, and north-east. The winter isotherm of 20° skirts the coast; inland the winters are colder. The summers are said to be warmer than those of Muskoka, notwithstanding the higher latitude. Small lakes are less numerous, and are shallow and heat rapidly. Neighbourhood to the great breadth of land between Lake Huron and James' Bay—an area which sometimes becomes intensely heated in summer has probably also some effect on the summers of the district. The heat of the southerly winds is of course greatly tempered by the great length of Lake Huron stretching against them.

At a distance of from 12 to 20 miles north of the north shore of Lake Ontario extends from the Highlands of Grey in peninsular Ontario to the head of the Bay of Quinte, a ridge or watershed attaining at a few places an elevation of nearly one thousand feet above the sea, and doubtless having some effect on the climate of the basin of Lake Ontario. Eastward from the easterly termination of this ridge the land slopes back from the lake far inland to the central heights of the watershed between the Georgian Bay and the Ottawa River. The lake has an elevation of only 234 feet (264 according to American surveys between the Atlantic and Oswego) above the sea. This comparatively low level conduces to raise the temperature of the borders of the lake. The comparatively moderate temperature of winter induced by lake influence and low level, the presence of high land to the north and west, and distance from lake water to the west, render the snowfall of the district lighter than in any other part of the lake region, with the exception of the district immediately north of Lake Erie. Sweeping over these high lands the north-west and westerly winds of winter which in passing over Lake Huron absorb considerable moisture, precipitate most of that moisture, and on regaining the low level of the Ontario basin resume almost their normal dryness. Owing to the comparative narrowness of the lake, and the fact that the winds which blow across it are not common or prevalent winds, the north shore, especially in its westerly portion derives a comparatively small proportion of its rain and snowfall from the lake, and the average annual precipitation is less than in any other part of the lake region with the exception of a limited district immediately north of Lake Erie. Towards the east end of the lake the same influences which make the climate of Ottawa extreme begin more and more to prevail; and the duration of sleighing gradually increases, till at Kingston it is nearly three months in length.

The climate of Toronto fairly represents in kind the characteristics of the north shore. At a low level and protected by the lake against the warm southerly winds, and by Lake Huron and the Georgian Bay from the cold northerly and westerly winds of winter its seasonal and daily range is comparatively small. The summer is cooler than in almost any of the larger towns in Ontario; and few have winters as mild. The mean temperature of January—about 23° for the eight years, 1874–'81, is nearly nine degrees higher than in Montreal, and is higher than in the uplands to the south-west, or than near Chicago, a degree and three-quarters farther south. The average minimum of January is —3°.1, the average minimum of the year —11°.0; the absolutely lowest in the eight years cited, —16°.0; and in the past twenty-five years —18°.4. The latter temperature is not so low as has been recorded within the same period at Louisville, Kentucky, or St. Louis, Missouri. The average duration of sleighing appears to be between three and four weeks; in some winters there has been no sleighing whatever. An examination which I have made of the records of Toronto observatory for the past thirty Christmas days shows that only on four of these holidays, or little more than one in eight, has there been sufficient snow to permit the running of sledges, and on thirteen occasions the ground was bare. The interposition of the lake water against hot winds from southerly points of the compass greatly tends to prevent extremes of heat. The summer of Toronto is cooler than that of Montreal, the Ottawa Valley, and parts of the interior to the north, north-east and west of the city, and as cool as the eastern shore of Lake Huron. The mean temperature of July for the eight year period cited is 69°.01—which is little more than three degrees warmer than Paris, France, over five degrees farther north; and is less than two degrees for the same period warmer than Winnipeg, where though the latitude is higher by 6¼ degrees, full continental influences prevail. The freedom from warm extremes both winter and summer is more noticeable. The average maximum of January is only 46°.25. The absolute maximum (Dec. 31, 1875) of mid-winter in eight years was only 61°, while that of Galt, 56 miles westward and 520 feet higher, was 66°; that of Hamilton, 42 miles distant, but at the west end of the lake, 71°, and that of the Niagara district, 40 miles distant, nearly 80° in the shade. The average maximum of the year is only 91°.5; that of Hamilton is 96°.9, while over the Lake Erie district and over most of the inland parts of the Province as far as the Upper Ottawa, the average maximum is in most localities as high as 95°. The absolute maximum in twenty years past is only 95°.4. At Ottawa and even in Muskoka it has exceeded 100°, while at Hamilton it has reached 106°.3 in the shade. It is interesting to note in passing, that moderate as is the annual maximum at Toronto as compared with other localities in the Province, it is a little higher than at Charleston, South Carolina.

At Toronto, as, more or less, along the shores of the Great Lakes, a lake breeze by day and a land breeze by night, blow during hot, calm weather. These breezes usually do not affect the climate for more than a few miles from the shore. Inland, notwithstanding the increased elevation, the temperature is higher in the day time during the summer months than it is at Toronto.

Hamilton, only forty-two miles distant from Toronto, and only twenty-three minutes further south, has a much warmer climate, and illustrates in an interesting manner several of the peculiar differences due to situation. Like Toronto it is exposed to the northerly winds modified by the Georgian Bay a hundred miles to the northward, but it is in a measure protected from the north-easterly winds by the intervention of Lake Ontario. More important in its bearing on the climate is the fact that the southerly and south-westerly winds which in reaching Toronto, have part of their warmth abstracted by Lake Ontario, reach Hamilton after blowing over a considerable stretch of land. Hence the latter place attains much higher temperatures in all seasons of the year than are reached on the north shore: the mean temperature is also higher. In addition to these causes which tend to increase the daily and seasonal range, the situation of the city on a low plain with a steep escarpment on the south and a range of hills across the bay on the north, tends to the existence of great daily contrasts, for in certain conditions of weather, the heat appears to accumulate in the sheltered "ravine" while in other conditions the heavy cold night air of the upland pours over the "mountain" and displacing the warm air, settles beneath it.

A remarkable instance of the effects of situation in a ravine, cutting through an extended upland, is afforded by the records of Galt on the Grand River. In 1879 the writer had charge of the meteorological station in the valley of that town. On the edge of the plateau to the west, a little more than a mile distant from the ravine station and about 180 feet higher than the later, was a second station in charge of a careful observer, Mr. Alex. Barrie. The thermometers at both stations were protected by the fence and screens approved by the meteorological service and in use at Toronto Observatory, and great care had been exercised to make the conditions of exposure similar. Here while the average daily maximum temperature was about two degrees higher at the valley station than on the plateau, the relative temperatures were sometimes greatly reversed. On Oct. 10th 1879, the maximum at the plateau station was 90°.3, while at the valley station it was but 79°.3, eleven degrees lower. On another date in the same year the difference was still greater, the thermometer at the 9 p.m. reading on the plateau being 79°, when in the valley it was only 65°, or fourteen degrees lower. There being no station at Hamilton, other than in the valley, similar instances there of the inflow of cold air cannot be cited. But the effect of this occasional inflow is seen in the facts that while the mean temperature and monthly maxima at Hamilton are higher than at Toronto, the monthly minima, from July to October, are very nearly the same at both places. The following tables show the average monthly maxima and minima at Hamilton and Toronto over a period of eight years (1874-81):—

AVERAGE MONTHLY MAXIMA.

	JAN.	FEB.	MAR.	APR.	MAY.	JUNE	JULY.	AUG.	SEPT.	OCT.	NOV.	DEC.
Hamilton.	49·7	50·9	58·2	72·0	89·0	91·0	93·9	94·0	90·3	81·6	64·3	54·9
Toronto ..	46·2	44·5	50·8	66·0	83·8	86·2	89·6	87·4	84·3	71·3	57·4	48·7

AVERAGE MONTHLY MINIMA.

Hamilton.	—0·1	—1·6	6·3	18·9	31·1	42·2	49·9	47·4	37·5	26·9	11·2	2·5
Toronto ..	—3·1	—3·3	4·0	16·6	29·0	40·4	49·2	47·8	37·6	26·3	7·5	—1·7

The average yearly maximum at Hamilton is 96°.9, the average yearly minimum -7°.4; the lowest temperature recorded in the eight years (1874–81) from which these averages are obtained was -20°.5; the highest 100°5. The absolutely highest temperature on record was 106°.3 (July, 1868), a degree of heat which has not been reached at New Orleans, or at Naples or Calcutta, in a period of at least 18 years. The average annual maximum is quite as high as at New Orleans or cities to the eastward along the Gulf of Mexico.

The mean temperature of the different months at Toronto and Hamilton for the eight-year period mentioned is as follows:—

	JAN.	FEB.	MAR.	APR.	MAY.	JUNE.	JULY.	AUG.	SEPT.	OCT.	NOV.	DEC.
Toronto ..	22·7	22·2	28·7	40·2	54·2	62·6	69·0	67·8	60·3	47·6	35·1	26·4
Hamilton.	24·4	24·6	31·1	42·5	57·7	66·0	73·3	71·4	63·9	50·3	37·1	28·4

The mean of the year at Toronto is 44°74, and at Hamilton 47°47 or 2°73 higher. The daily range in Toronto is about 13° in January, and nearly 20° in July, while at Hamilton the figures for these months are respectively about 20° and 27°. The average daily maximum of July, at Hamilton, is above 84° in the shade, and not 79° in Toronto. In the warmest month ever recorded in Hamilton (July, 1868 , the mean temperature was 80° with an average daily maximum of 93° in the shade. In Toronto the mean of the same month was 75°.8, with a mean daily maximum of only 85°.4. These contrasts sufficiently illustrate the effect of the different situation of the two cities in regard to the water of Lake Ontario.

Along the south shore of Lake Ontario, eastward to Niagara, the general features of the climate of the belt of land referred to, resemble those of Hamilton, though the thermometer does not fall so low at night as in that city. The summer heats are intense, and temperatures above 70° have even been recorded in mid-winter. The season is, over much of the district, longer than at Hamilton, where the average period between the last fall of the temperature in spring to 32°, and the first descent in autumn to the freezing point, was for three years (1878-80), 186 days. The measure of protection afforded by Lake Ontario from the winds from northerly points of the compass increases, and the mean temperature of winter rises. Lake Erie also affords a measure of protection against the cold which in winters unusually severe in the Western States sometimes accompanies southwesterly winds. At Niagara the mean of winter is several degrees higher than at Hamilton, and nearly as high as at New York, and the average minimum of the year is little, if at all, below zero. The heat and duration of summer and the comparative mildness of winter make the district peculiarly well adapted to fruit growing. The peach-orchard area of the district is very large, and vineyards averaging four to five tons of grapes to the acre are numerous. The sweet potato and the peanut flourish in a degree unsurpassed in any other district in the province. The mulberry grows luxuriantly. The pseudo-papaw, and the tulip tree, *Liriodendron tulipfera*, grow wild in the woods and attain large proportions. At Niagara the writer has found fig-trees heavily laden with fruit, growing in the open air with but little winter protection ; and the soft-shelled almond, though of course but little cultivated, with slight winter protection, produces fruit equal to that of the common almond of commerce.

The north shore of Lake Erie, like the north shore of Lake Ontario, and for similar reasons, is marked by a tendency to the avoidance of great extremes of heat, though owing to latitude and the shallowness, and therefore greater warmth, of the water, the hot extremes of the summer months, and the mean temperature are higher than on the north shore of Ontario. In exceptionally severe winters, ice forms to a greater extent on the bays and indentations of Lake Erie than along the Lake Ontario coast, and though the mean temperature on the north shore of Erie is higher than on the same shore of Ontario. the winter maximum in such seasons is no greater than at Toronto. The snow of winter is light, and usually lies but a short time, even in winters when around Buffalo the depth is great and the sleighing of long duration.

The eastern shore of Lake Huron has a climate differing in several important particulars from the Canadian shores of Lakes Erie and Ontario, and illustrating more than these lakes the peculiar effect of a large body of water interposed against the prevailing westerly winds. The winters are nearly two degrees warmer than at Toronto, and are as mild as those of Hamilton, as free from cold extremes as at Niagara, and from warm extremes as at Toronto, yet the moisture of the lake winds makes the sensible cold appear greater than in the interior or in the Niagara District. Spring is retarded by the lake influence, and the mean of that season at Goderich is no higher than at Toronto; but on the other hand the autumn is several degrees warmer: summer is as cool as at Toronto, and comparatively free from very high temperatures. Goderich, lat. 43° 25′; altitude, 728 feet, has a mean temperature for the year a little higher than Toronto. Zero temperatures, and temperatures above 90° are rare; and the contrast in this respect with the Michigan shore opposite, is very marked. The climate is one of the most equable of the whole lake region, and surpasses in this respect almost every other district in the middle latitudes of the continent. The peach grows far north, and even on the Georgian Bay. Towards the southern part of the district, peach-growing is an important industry. Owing to the moisture of the lake winds, this shore is not so well adapted to the vine as the ordinary or low levels of peninsular Ontario. The rainfall and snowfall are both heavy, for to the rainfall brought by cyclonic areas, there is added the moisture gathered by westerly winds from the lake. The north-westerly winds, normally intensely dry, gather a large amount of moisture from the lake, and in winter when the land is chilled, this moisture is precipitated in snow flurries to a considerable depth. The interior of peninsular Ontario varies greatly in elevation, rising slowly and gradually from Lake Erie; more rapidly from Lake Huron and still more abruptly from the Georgian Bay, up to the

Highlands of Grey, where an elevation of 1,700 feet above the sea is attained. Consequently, considerable differences in climate exist in this interior. On the Highlands of Grey, and on the Lake Huron slope the snowfall is often excessively heavy, and the snow lies several feet in depth, when in some districts of the Province the ground is bare. Sleighing usually lasts for three months or more on the highest levels. Of the annual precipitation of this part of the interior, there are but few records, and these cover but a very short period. There is reason, however, to think that the annual precipitation in some localities, as in Muskoka, exceeds 50 inches, that is, amounts to nearly twice the precipitation of the dryest localities of the Province. The explanation of this heavy precipitation has already been sufficiently indicated.

The winter temperature of the central watershed, owing to great elevation, is cold, averaging in some localities below 20°. The extremes of cold, too, are great, though on these, as on the winter mean, the surrounding lakes exercise a moderating influence, and the temperature usually does not fall so low as at Ottawa or as in the Western States at even lower levels and much lower latitudes.

The difference in mean summer temperature between the lake shore and the highest land of the interior, is not great when the difference in altitude is considered. The mean of July, at the highest points, is about 65° and the maximum heat is about as high as on the Lake Huron shore. The degree of heat attained is due, in a large measure, to the extent of unbroken land to the south and southwest. At Owen Sound on the south shore of the Georgian Bay, so much does this large land area in the direction of the warm winds affect the climate, temperatures as high as 95° have been reported in the month of May. At elevations of 1,000 to 1,200 feet, the mean of summer is nearly as high as at Toronto, and the daily and yearly maxima are higher. The difference from the lake coasts and lower levels is chiefly in the existence of a greater daily and seasonal range on the high land and a shorter period of exemption from early and late frosts. On the long slope towards Lakes Erie and St. Clair, the mean temperature of all seasons gradually rises, and at some distance inland the mean temperature of summer exceeds that of the Erie coast by several degrees, and almost equals that of the very warmest localities of the Province. In extremes of warmth, both summer and winter, the temperatures are higher than in most localities near the lakes. At Galt, lat. 43° 20', altitude 870 feet, the mercury usually rises to 95°, and has exceeded 100°. London sometimes records a higher July mean than even Hamilton or Windsor. At Zurich, towards Lake Huron, 103° was reported in 1881. Perhaps as forcible an illustration of the tendency of the interior to develope extreme heat as can be given, is in the fact that while in 1881, at Brantford, lat. 43°10', altitude 720 feet, there were in May 7 days, in July 21 days, in August 16 days, and in September 7 days—51 in all—on which the mercury rose above 90° in the shade, and while the highest temperature was 99°, in Toronto there were but five days, in all, on which a temperature above 90° was reached, and the very highest was only 92°.7. Towards the south-western portion of this inland district, the absence of lake water to the south-west, between the foot of Lake Huron and the head of Lake Erie, fully admits the south-west wind, which is usually warm, and winter temperatures comparatively high are often recorded. An indication of the general climate of this Lake Erie slope is that the peach is grown, on suitable soils, to an elevation of about 1,000 feet above the sea.

In much of the interior of peninsular Ontario, thunder storms are numerous and more severe than on the north shore of Lake Ontario. Tornadoes also occur more frequently, though they are not so violent nor so frequent as in equal areas in Ohio, Indiana and the Central Western States. The snowfall of the Lake Erie slope rapidly diminishes as the distance from Lake Huron increases. North-west winds which near Lake Huron and in the highlands of Grey, bring several inches of snow in a single day are usually snowless over the southern half of the peninsula. At Galt the average duration of sleighing is not more than six weeks; southward and south-westward the period decreases to a few days. The advent of spring is one or two weeks earlier over much of the southern part of the district, than at Toronto, and winter-wheat harvest is almost as much earlier. Harvest usually commences in the beginning of July and has been known to begin in the end of June, as far northeast as Galt, and about the 15th of June a short distance north of Lake Erie.

The climate of Windsor on the Detroit River, lat. 42° 19', altitude 604 feet. is fairly representative of the climate of the extreme south western part of Ontario. Immediately to the north is Lake St. Clair, and not far beyond that lake, Lake Huron, affording protection from the cold north winds of anti-cyclones passing eastward north of the great lakes. To the south at no great distance is Lake Erie affording only a slight protection against the warmth of the south wind in winter. But against the cold in winter of westerly and north-westerly winds there is no shelter except such as the distant Lakes Michigan and Superior supply, and against the warmth generally, and in some winters the excessive cold, of the south-west wind there is little or no protection. Lake St. Clair is shallow, and in severe winters freezes over, and loses its protective influence, and both it and the very shallow westerly end of Lake Erie become in summer greatly heated, and not only lose the protective influence against extreme heat which lake-water generally exercises, but even at times, and especially in autumn, increase the heat. The extreme south-west has therefore a climate, on the average of the year warmer than almost any other part of the Province, but more variable also than most of peninsular Ontario.

The winter mean is the same as that of Hamilton, but with monthly extremes of heat and cold greater than in that city. The average yearly minimum is about the same as at Toronto. Owing to the great differences in the temperature of different winters in the Western and South-Western States, and the consequently great differences in the temperature of south-westerly winds in different winters, the temperature of the Windsor winters differs very much. In eight years (1874-81) the coldest January was 14°.7 which is lower than any January in the same period at Hamilton or Toronto, or eighty miles northward at Goderich. The warmest January on the other hand was 36°.2, or considerably higher than any at Toronto or Hamilton. December means varied from 18°7 to 38°9; March from 26°.6 to 41°.7; April from 37°.9 to 54°.2. Though the midsummer months show little difference in their mean temperature in different years, October means ranged from 46°.6 to 58°.9; May from 57°.2 to 65°.5, and September from 59°.0 to 72°.2; the last higher than any Toronto July in the same period.

The mean of the summer months is almost the same at Windsor as at Hamilton. In autumn, with the exception of the month of October, the two places are alike in mean temperature. It is the temperature of the spring and early summer that makes the mean of the year at Windsor (48°49) one degree warmer than the annual mean at Hamilton. April at Hamilton has a mean of 42°.5; at Windsor 45°.25; in May the figures are respectively 57°.7 and 60°.8; in

June Hamilton averages 66°.0 and Windsor 67°.85. The earlier springs of Windsor are due in part to latitude, in part to greater nearness to the rapidly advancing heat of the south-west, and in part to the fact that easterly winds which prevail in spring reach Hamilton from the deep, winter-chilled lake, and Windsor from the warmer land of Essex and Kent.

The following table shows the mean temperature of each month, the average monthly maxima and average monthly minima at Windsor, for the eight year period (1874-81.)

	JAN.	FEB.	MAR.	APR.	MAY.	JUNE.	JULY.	AUG.	SEPT.	OCT.	NOV.	DEC.
Mean	24·1	24·7	32·4	45·3	60·8	67·8	73·4	71·4	63·8	51·6	37·1	28·4
Mean max.	50·0	53·5	61·8	77·3	88·9	91·7	95·1	93·5	90·5	81·2	64·5	53·3
Mean min.	—3·0	—0·6	9·4	18·8	30·0	47·3	51·4	48·3	36·8	25·0	10·4	0·0

The mean of the year is 48°.49 ; the mean maximum 96°.25, (very nearly the same as at Hamilton) and the mean minimum,—10°.75 or 3°.4 lower than at Hamilton, and almost the same as at Toronto 2¼ degrees farther north. The absolutely highest temperature in the eight years referred to was 100°.6 (Sep., 1881) : the absolutely lowest —19°.5.

In the four coldest months the maxima were as follows:—Dec. 68°.3 ; Jan., 66°.9 ; Feb. 63°.4 ; March, 77°.4. The contrast with Toronto goes to show the effect of Lake Ontario in protecting against unseasonable temperatures. There the absolute maxima for those months were Dec., 61°.1 ; Jan., 57°.5 ; Feb., 51°.6 ; March, 58°4. Absence of lake-water to the west renders the precipitation small compared with the adjoining Huron district. The snowfall is light, and the general temperature of winter, and particularly the warm extremes, reduces the average period of sleighing to a few days. The fruits and flora generally are the same as in the Niagara district. The vineyards are very productive, averaging in good soils five tons of grapes, and nearly 700 gallons of wine (first drawing) to the acre—a yield probably unsurpassed either in California or in Europe.

The southernmost part of Ontario and of Canada, Pelee Island, a township of 17 square miles (lat. 41° 40′ to 41° 50′—further south than Rome), has a climate peculiarly interesting. The island lies almost midway between Sandusky, Ohio, 20 miles distant, and Leamington, Ont., and with Kelly's, an Ohioan island, six miles to the southward, and the peninsula of Point Pelee to the northward, marks the dividing line between the very shallow and island-dotted western extremity of Lake Erie, and the larger, deeper and unbroken area of the lake to the eastward. This peculiar position produces remarkable climatic effects. The water to the westward is generally not more than forty feet in depth, and under the hot summer sun becomes so heated that temperatures above 80° are sometimes registered at lake bottom in the habours along the neighbouring coasts. This high temperature not only tends to increase the average heat and length of summer, which here is almost as warm as at Cincinnati, but increases the warmth and length of autumn—which also is as warm and free from frosts as on the Ohio River—and reduces the difference between day and night temperatures to almost tropical smallness. Another effect, a physician on the island informs the writer, is that what corresponds with the nightly land breeze of the lake coasts in hot, calm weather, here blows not from the land, but from the deeper and cooler lake water to the eastward, into the heated western end of the lake. The effects in winter of the surrounding shallow water, vary with the severity of the seasons. In the milder winters the usual effects of water surroundings are experienced in a small daily and seasonal range. In severe winters the shallow archipelago of the western end of Lake Erie is encumbered with ice and sometimes freezes over, and Pelee partakes in greater measure of the continental character of the winter of the neighbouring mainland.

An examination of the records of the meteorological station on the island for a period of three and a half years bears out the deductions which otherwise could be made from the peculiar situation of Pelee.* The figures are interesting. The mean temperature, and mean monthly maxima and minima are as follows:

* The records, which through the courtesy of the Superintendent of the Meteorological Service, were furnished the writer, embrace the period between February 1st, 1879, and August 31st, 1882. The records for May, October and November, 1879, and April and November, 1880, are incomplete or wholly wanting. The mean temperature for these missing months has been approximated by the writer after careful examination of the records of Windsor and Sandusky, what is believed to be due allowance having been made for the peculiarities of the Pelee climate. The hours of observation were 7 a.m. and 2 and 9 p.m. The mean temperature is found by adding together the readings at the first two hours, and twice the 9 p.m. reading, and dividing the sum by 4. The maximum and minimum temperatures given are those of the

	Jan.	Feb.	Mar.	Apr.	May.	June.	July	Aug.	Sept.	Oct.	Nov.	Dec.
Mean	26·2	27·4	32·5	41·7	59·2	67·1	73·5	72·9	66·3	56·4	38·7	29·1
Mean max.	47·7	54·3	54·5	65·0	85·3	91·0	95·0	91·5	90·7	72·0	62·0	49·7
Mean min.	6·7	7·0	18·7	18·7	40·3	51·3	61·5	59·7	49·5	37·5	28·0	9·0

The mean temperature of the year is 49°·25: did the record extend over the eight years which have been used for the averages of Toronto, Hamilton and Windsor, it would probably appear a small fraction of a degree lower.

The coldest January averaged 16°.5, or 0°.7 higher than the same month at Windsor, while the warmest, (34°.8) was 1°.4 colder. The absolutely lowest temperature (—12°) occurred when the west end of the lake was covered with ice and was 5°.4 lower than at Windsor. The occurrence of lower temperatures than at Windsor during the same severe season suggests that the effect of a neighboring area of ice in extremely cold weather, is more favourable to the development of cold than is the vicinity of an unbroken land area, an explanation which may find additional illustration along the eastern side of the Georgian Bay. In mild winters the low extremes are higher at Pelee than at Windsor. In January 1880 the minimum at that town was 19°, while at Pelee it was only 25°. In the other months of the same winter the difference in favour of Pelee was from 4°.5 to 10°.5. The absolutely highest temperatures in the winter months were: Dec. 57°, Jan. 55°, Feb. 63°, March 60°. The extraordinary smallness of the mean daily range in winter is shown by a comparison between the averages of the 7 a.m. and the 2 p.m. readings. The average difference in Dec. is only 2°.2, in Jan. 3°.3, Feb. 6°.4, and March 5°.4. In December 1881 the average temperature was 34°.7, but the 2 p.m. reading was only 36°.1, and the 7 a. m. 34°.5, a total range of only 1°.6, between hours which represent, at this season

hours of observation only, but a careful consideration of the facts as to cloudiness, direction of wind, &c., at the times of their occurrence, and for some time before and after, leads to the conclusion that in many instances they represent within a fraction of a degree the true max. or min., as the case may be, and that in few instances can the highest or lowest temperatures have differed more than one or two degrees from these quantities as taken from hours of observation alone. Where the mean temperature of the month is not derived from the original records no attempt has been made at supplying maximum and minimum, or averages other than for mean temperature. The mean maximum and mean minimum of November is consequently derived from but one month, that of October from only two, those of January, February, April and May from three, and the remaining months of the year from four.

especially, very nearly the extremes of the day. The average daily range in January furnishes an interesting contrast with the range in the same month at Toronto and Hamilton.

April at Pelee is almost as cold as at Toronto, and is more than 3° colder than at Windsor, thirty miles further north. The effect of the cold lake water is shown in the fact that the highest maximum in this month was 82.°9, (April 1881) while in Pelee it was but 68°. Yet the last frost of the season is several weeks later at Windsor than in Pelee, where it occurs about the middle of April. In May, Pelee almost regains the normal temperature of the districts on the neighboring mainland: temperatures above 90° are recorded and frosts are known only in exceptional years.

The summers are hot and steady. In only one June in four years was a lower reading than 50° recorded. In July and August only once in the same period was there a lower reading than 60°. The daily range in summer is much greater than in winter but still not half so great as at most stations on the mainland of Ontario. The range between 7 a.m. and 2 p.m. for June is 8°.4, July 8°.6, Aug. 7°.5. The daily range above the mean temperature is in summer twice as high, as the range below the mean, the nights maintaining an almost even temperature of about 70° in July and August, while the day temperature rises in July to at least 80°. This daily maximum is not so high as that of some parts of the Ottawa Valley, and is much below the daily maximum of Hamilton and Windsor, where however the night temperatures fall considerably lower than at Pelee.

Intensely tropical weather frequently prevails for days together, when, though the mercury does not rise any higher than on the mainland, it does not fall at night below 80°. In the steaming atmosphere of this shallow lake such days must be very oppressive. The following are instances from the records:

	7 a.m.	2 p.m.	9 p.m.
July	86°	96°	83°
Aug	83°	95°	85°
Sept	82°	98°	84°

September, in regard to heat, is properly a summer month, its mean being higher than that of a Paris July, and little lower than that of a Toronto August. In 1881 the mean was 72°.9, with a minimum temperature of only 58°.

October averages 56°.4, nearly ten degrees warmer than at Toronto, and quite as warm as in the Ohio Valley. November prolongs the balmy, hazy weather which persists here for months, and it is not till about the 12th of the month that the first hoar frost of the season usually occurs on the warmer soils of the island.

In winter sleighing is rare. The rainfall in the warmer months is comparatively light, owing to the high temperature above the shallow surrounding waters checking condensation.

The mean period in which the mercury does not fall to 36°—the average point at which hoar frost here occurs—is nearly seven months in length, or quite as long as at Memphis, Tennessee, and much longer than throughout most of Ohio and Indiana. It extends from April 14th to Nov. 12th. The great length of the season, combined with the long steady heat admits of the full maturing of cotton, which at one Pelee farm visited by the writer, has been grown for many years without any special care either to secure protection or early maturity. Climatic conditions are more favorable to the cultivation of the Catawba grape on Pelee and adjacent islands than in any other part of America. Including the mainland on both sides of the lake, this district is the most famous wine district on the continent, with the exception of a small area in California, where however the yield per acre is not greater than here. On the islands alone, millions of gallons of wine are produced, and the area in vineyards can be greatly extended. The grape crop is never injured by frosts, and conditions in regard to moisture are more favourable to avoidance of loss through mildew than in the Ohio Valley, which formerly was the chief centre on this continent of the production of Catawba wine.

To find European parallels to the various climates of Ontario which have been described, would be no easy task Individual districts will find winter parallels in the Crimea, on the banks of the Danube, and at Berlin on the one hand, and on the other at St. Petersburg, Moscow, Astrachan and in Central Russia. The summers of parts of the Province are paralleled in those of Lisbon, Northern Spain and Italy, Southern France, the lower Danube and Constantinople, or in the cool summers of Paris and Berlin. The Ottawa Valley and the central and inland parts of the Province of Ontario have summers like those of Vienna. Toronto at any season of the year differs but little in temperature from Bucharest. The month of July at Hamilton and Windsor is almost as warm as at Oran, in Algiers, and but little cooler than at Jerusalem, in Syria. In general, it may be said that a line from the Danube through Bucharest to Moscow would furnish parallels to the climates along a line from Windsor north-easterly to Pembroke, on the Ottawa—though the summers of the latter place are warmer than those of Moscow.

By a British standard the summers of much of the Province may be considered long. May in south-western Ontario is warmer than July at Edinburgh. September is warmer than July at London, and warmer than September at Vienna. The vine, maize and sorghum fully mature in most parts of the Province south of the 46th parallel, and in not a few districts yield as abundantly as in any part of America or Europe. The limitations on the cultivation of the vegetals of similar latitudes in Europe is more in the intensity of the winter frosts than in the lack of a sufficiently long or warm summer.

NOTE.

The length and heat of Ontario summers contrasted with those of other places in Canada, and various places in Europe, may be seen by a glance over the following table. The means for Toronto, Hamilton, Windsor and Winnipeg are derived from the annual records of the Canadian Meteorological Service for eight years (1874-81); those for Montreal from same records for six years (1875-80); those for Pelee, from C. M. S. station reports for three and a half years. The averages for European Stations are quoted from Blodgett's "American Climatology," and are for periods, with few exceptions, longer than eight years.

MONTHLY MEANS OF CANADIAN SUMMERS.

	MAY.	JUNE.	JULY.	AUG.	SEPT.
Toronto	54°·2	62°·6	69°·0	67°·8	60°·3
Hamilton	57·6	66·0	73·4	71·3	63·9
Windsor	60·8	67·9	73·4	71·4	63·8
Pelee	59·2	67·1	73·5	72·9	66·3
Montreal, Que.	55·0	65·0	69·8	68·1	59·0
Winnipeg, Man.	52·9	61·8	67·3	64·1	51·9

MONTHLY MEANS OF EUROPEAN SUMMERS.

	MAY.	JUNE.	JULY.	AUG.	SEPT.
Edinburgh	50·3	56·0	58·7	56·8	53·4
Aberdeen	52·3	56·7	58·8	58·0	54·6
York	54·5	59·2	62·0	61·1	55·7
London	55·8	58·7	61·7	58·9	56·6
Dublin	54·4	60·2	61·5	61·4	56·5
Paris	58·1	62·7	65·6	65·3	60·1
Rochelle	59·4	67·5	69·0	66·5	62·4
Vevay	58·2	64·4	68·4	64·4	59·6
Munich	57·6	62·1	64·7	64·1	58·1
Berlin	56·5	63·3	65·8	64·4	58·4
Konigsberg	52·0	57·4	62·6	61·7	53·6
Vienna	62·1	67·5	70·7	70·0	61·9
Bucharest	56·3	62·5	68·1	65·2	58·3

The following members took part in the discussion which followed: The President, Dr. Barclay, Dr. Jos. Workman, Dr. O'Reilly, Mr. John Notman and Mr. David Boyle.

SIXTEENTH ORDINARY MEETING.

The Sixteenth Ordinary Meeting of the Session 1883-84 was held on Saturday, March 1st, 1884, the Third Vice-President, Dr. Geo. Kennedy, in the chair.

The minutes of last meeting were read and confirmed.

The following list of donations and exchanges was read:

1. Science, Vol. 3, No. 55, for Feb. 22nd, 1884.
2. Memoirs of the Boston Society of Natural History, Vol. 3, No. 8, on the development of Œcanthus Niveus and its Parasite, Teleas, by Howard Ayres.
3. Harvard University Bulletin, Vol. 3, No. 4, for January, 1884.
4. Historical Collections of the Essex Institute. Vol. 20, Nos. 10, 11, 12, for Oct., Nov. and Dec., 1883.

Dr. P. H. Bryce read a paper entitled, "Some Factors in the Malaria Problem."

The reader of the paper explained that he proposed reading extracts of a Report made upon malaria, prevalent in the lower district of the Grand River, under the direction of the Provincial Board of Health in September last.

After stating briefly the characters of the district regarding the nature of the soil and the underlying geological formations, in which he stated that coniferous strata are overlaid by Erie clays, and that the Saugeen clays overlie these, Dr. Bryce went on to explain how that, since the time of the building of the dams on the river for the purpose of supplying water by a feeder to the Welland Canal, malaria had been very prevalent up to the present. This he showed by the statements of old settlers and medical men, regarding that in past years, and by the tabulated reports of disease made by medical correspondents of the Board during the last year. After explaining the results upon the low-lying flats along the river of the damming back of the waters, the writer stated that there were three distinct elements of the problem, namely, the conditions of the soil, the ground-water and the air. Assuming that the various causes which it has been assumed cause malaria are all much less satisfactory than the germ theory, in which some bacterial organism, e. g., *Bacillus Malariæ*, is supposed to be the immediate cause, Dr. Bryce went into a discussion of how the local physical conditions might favour the free development of these germs, as it is well known that vegetable organic matter in a decaying state forms a favourite *nidus* for the development of bacteria of every kind. This material is largely present in some of the overflowed lands along the river, but free development of organic life in such depends upon the amount of water present in the soil. This necessarily varies with the dryness of the season and with the height of the river-water. This last point introduced the subject of its probable effects upon the ground-water of the low lands along the river. Through the denuded nature of the river-valley, the subsoil water of the neighboring higher lands naturally drain toward the valley along impervious beds of clay, and in some parts along the surface of the underlying rocks. This is seen in some parts in the presence of flowing wells. But, according to Miquel's experiments, it is not enough for the prevalence of germs in the air that they be developed in the soil. It is necessary that the upper layers of soil dry out sufficiently to allow the winds to carry these freely into the air. Further, their free development in the soil depends largely upon the amount of air in the soil, or oxygen. This it is clear must vary with the height of the ground-water, since as the water rises or falls the air must be less or more in the interstices of the soil. Hence, though ground-water conditions the amount of air in the soil, it is after all the oxygen of the air which determines the development of germs. But the next point in this connection is the fact that, as the temperature of the soil varies greatly from that of the contiguous atmosphere, especially during the warm summer weather, it follows that there is a regular circulation of ground air, new oxygen being constantly taken into the soil to supply the conditions of free zymotic development; and further, that this circulation probably serves to some extent as a vehicle for carrying the germs of the soil into the air. Upward currents of air during the day prevent an accumulation of atmospheric particles near the earth, and, on the other hand, the upper colder strata of air descending toward and after sundown, and especially in calm weather, cause accumulations near the earth of germs which have been carried up during the day. Hence, along with the increased humidity, is probably explained why night air is proportionately more malarious than day air. The influence of winds in greatly increasing the number of particles and germs in the air was also discussed and in this way, the writer explained, it was probable that the germs of malaria were laterally disseminated, and how they would tend so to increase each succeeding year as they found new centres of development. Hence it was apparent that forests, both mechanically, by breaking the force of winds, by keeping the air moist, by preventing extreme differences between day and night temperature, and by preventing undue drying out of the soil, would act favourably in preventing the wide-spread prevalence of malaria.

Discussing the matter of the influence of cold in causing malaria, the writer gave a number of selected experiments concerning the rapid decrease of body temperature under different physical surroundings, as temperature, wind and moisture.

He finally showed how drainage and the planting of forest trees would serve to lessen the conditions of soil favourable for the development of *Bacillus Malariæ*, the assumed immediate cause of the disease.

A discussion ensued in which Mr. W. Houston, Prof. J. P. McMurrich, Mr. Livingstone, Dr. Oldright, Mr. J. Notman, Dr. Bryce and the Chairman took part.

SEVENTEENTH ORDINARY MEETING.

The Seventeenth Ordinary Meeting of the Session 1883-84 was held on Saturday, March 8th, the President in the chair.

The minutes of last meeting were read and confirmed,

The following list of donations and exchanges was read :

1. Science Record, Vol. 2, No. 4. Feb. 15th, 1884.
2. Science, Vol. 3, No. 56, Feb. 29th, 1884.
3. The Canadian Practitioner for March, 1884.
4. Journal of the Franklin Institute, for March, 1884.
5. Journal of the Royal Microscopical Society, Series 2, Vol. 4, Part 1, for February, 1882.
6. The Canadian Entomologist, Vol. 16, No. 1, January, 1884.
7. Proceedings of the Royal Society of London, Vols. 31, 32, 33, 34, 35, and Part 1, Vol. 36, containing Nos. 206 to 228 inclusive, from March 24th, 1881, to Nov. 30, 1883.

A paper was then read by Mr. Wm. Houston on "Old English Spelling and Pronunciation."

In dealing with the subject, Mr. Houston dwelt for sometime on the changes which have taken place in the pronunciation of English words since Anglo-Saxon, in its various dialects, was the spoken language of the common people of England. The principal authority cited was Mr. A. J. Ellis, who has established by a wide induction from a variety of sources a considerable number of indisputable conclusions, though there are still many points left doubtful. As pronunciation changed, spelling should have changed also, and, as a matter of fact, it did so to some extent before the invention of printing, and to a less extent since; but the growing tendency of modern times is to allow the printers, to whom uniform spelling is a matter of great convenience, to fix the forms of words, not only absolutely but arbitrarily. The reader of the paper cited numerous instances of old spelling from Milton back to Chaucer to show (1) that spelling in Old English was more phonetic, and therefore better than now; (2) that spelling varied with pronunciation in the use of words by the same writer; and (3) that so far from adherence to a uniform system of spelling being regarded as a chief criterion of scholarship, old writers allowed themselves a great degree of latitude in their modes of spelling words. Spenser is an extreme instance of this free and easy view of orthography, for it is not uncommon to find him spelling the same word three or four different ways on the same page. In conclusion, Mr. Houston contended for greater freedom in orthography, not in the interest of diversity, but in the interest of simplicity of spelling.

The following gentleman took part in the discussion which followed: Dr. Workman, Dr. Bryce, Mr. J. Howard Hunter, Mr. D. Boyle, Mr. Murray, Mr. Shaw, Mr. Notman, Mr. Keys, Mr. Livingstone, Mr. Macdougall.

EIGHTEENTH ORDINARY MEETING.

The Eighteenth Ordinary Meeting of the Session 1883-84 was held on Saturday, March 15th, the Third Vice-President Dr. George Kennedy in the chair.

The minutes of the last meeting were read and confirmed.

The following gentlemen were elected members of the Institute:

T. C. L Armstrong, M. A., LL.B., Henry William Eddis, Esq., and Frank Arnoldi, Esq., Barrister.

The following list of donations and exchanges was read:—

1. Science, Vol. 3, No. 57, March 7, 1884.
2. The Canadian Record of Natural History and Geology, Vol. 1, No. 1, Montreal, 1884.
3. Transactions of the Ottawa Field Naturalists' Club, No. 4.
4. Annual Report of the Library Commissioners and Libarian of the Legislative Library of Nova Scotia, and the Librarian of the Nova Scotia Historical Society for the year 1883.
5. Transactions of the Royal Geological Society of Cornwall, Vol. 10, Part 6.
6. Annuaire de 1884, de la Société des Ingénieurs Civils, 37e. Année.
7. Appleton's Literary Bulletin, March, 1884.

PURCHASE.—38 Nos. of the Journal of the Franklin Institute of various years, to complete a set.

Mr. T. P. Hall, B. A., Fellow of University College, read a paper on "Photography and the Chemical Action of Light," illustrated by diagrams and apparatus.

After reviewing the history of photography, Mr. Hall showed the scientific value of this art, in leading to a more complete knowledge of the nature of radiant energy. The action of different parts of the spectrum upon various substances was explained in connection with wave-lengths and atomic vibrations, and the direction of future advances in photography indicated. The relation between transparency to certain rays and chemical composition, fluorescence, phosphorescence, colour-blindness, and other interesting subjects in this connection were discussed and illustrated. The following is an extract: "To make photographs which shall appear accurate to us we require a compound sensitive to the same rays and in the same relative degree as our eyes are. . . . Since, besides being deaf to an unknown variety of sounds, we are blind to nine-tenths of the light of the spectrum, it becomes a question of interest whether the lower animals are more or less blind than we. From his experiments on ants, Sir John Lubbock concludes that they are nearly or quite blind to red and yellow rays, and sensitive to green, blue, violet and ultra-violet rays. A photograph taken with silver chloride, which is very imperfect to us because the red and yellow rays are not represented, and violet and ultra-violet appear very bright, would therefore to the critical eye of an ant appear quite correct."

Dr. Bryce and Mr. VanderSmissen made remarks on the subject, after the reading of the paper.

NINETEENTH ORDINARY MEETING.

The Nineteenth Ordinary Meeting of the Session 1883-84 was held on Saturday, March 22nd, the President in the chair.

The minutes of last meeting were read and confirmed.

The following list of donations and exchanges was read :

1. Selected Papers of the Rensselaer Society of Engineers, Vol. 1, No. 1, January, 1884.
2. Transactions of the Manchester Geological Society, Vol. 17, Part 13.
3. Weather Review for February, 1884.
4. List of Fellows 1884, Royal Microscopical Society.
5. Science, Vol. 3, No. 58, March 14th, 1884.
6. Proceedings of the Royal Geographical Society, Vol. 6, No. 3, March, 1884.

Mr. W. J. Loudon, B. A., read a paper on the "Radiometer," illustrated by experiments.

The following members made observations on the subject: Mr. H. S. Howland, jun., Mr. Murray, Mr. Macdougall, Mr. McKenzie, Mr. Livingstone and Dr. Bryce.

TWENTIETH ORDINARY MEETING

The Twentieth Ordinary Meeting of the Session 1883-84 was held on Saturday, March 29th, the President in the Chair.

The minutes of last meeting were read and confirmed.

The following list of donations and exchanges received since last meeting, was read :

1. Science, Vol. 3, No. 59, March 21, 1884.
2. Life and Work of Darwin, by George Acheson, M. A.
3. Trübner's American, European and Oriental Literary Record, Vol. 4, Nos. 11, 12.
4. Transactions of the Oneida Historical Society at Utica, 1881.
5. Science Record, Vol. 2, No. 5, March 15, 1884.
6. Proceedings and Transactions of the Royal Society of Canada for the years 1882 and 1883. Vol. 1, Montreal, 1883.

Mr. Henry Brock then read a paper on

THE UPPER NIAGARA RIVER.

The Border land has long been a theme for novelist and poet, and pen-pictures of the stirring scenes amongst the Grampians, the plateaux and peaks of the Tyrol, and along the vine-clad banks of the Rhine have been depicted with the fervour of enthusiasm by the many lovers of what has tended towards forming the national character of their native land.

The border land of Canada has been the scene of many a heroic contest. And from Frontenac's struggles against the Mohawk and Iroquois, until the days of Earle's Hill and Limeridge, the Canadian, whether of French or English descent, has proved true to his native land. My intention this evening is to touch upon but a small portion of this interesting subject. Leaving out the country bordering on Memphremagog and Champlain, the noble St. Lawrence, and the clear flowing Detroit and St. Clair, I must content myself if I can bring before you a few reminiscences interesting to the antiquarian or historian of the Niagara Frontier, and particularly that portion of it which is commonly known as the "Upper Niagara River." Commencing at a point where 60 years ago a village hamlet stood with a few hundreds of a population, but where now 200,000 busy people are continuing the struggle for existence in the great city of Buffalo, and flowing in a north-westerly direction from Lake Erie, the Niagara River separates Canada from the United States. On the Canadian side of the river, just opposite Buffalo, are the remains of what was formerly Fort Erie. The fort is entirely dismantled, and is marked only by a few earthen ramparts fast settling down to the ordinary level. It was at this place that the Fenians of '66 crossed, although from the swiftness of the current of the river just below this point, which sweeps on at the rate of 10 miles an hour, and the width of the river, a few ordinary guns in their former embrasures at Fort Erie would prevent any such undisciplined raiders from again attempting to effect a crossing in safety. At present surrounded by the fast growing city, and forming now one of Buffalo's Parks, is the American Fort Porter. This fort is one of the military posts of the United States, forming with Forts Niagara and Detroit, links in that chain of forts, which in the United States extends from Maine to Oregon. It is garrisoned by a detachment of United States infantry, whose services were of great value to the city in the great railroad riots of 1877. Three miles from Black Rock the most northerly suburb of Buffalo is the Island called Grand Island, containing about 80 square miles of land, and forming a Township in Erie County State of New York. Flowing due north through the middle of the Island, is a small creek, called Burnt Ship Creek, emptying itself into the Basin separating Buckhorn Island from Grand. In this Basin the French, in 1759, anchored two small vessels containing the reinforcements which had been sent from Venago to raise the siege of Fort Niagara, if possible, which at that time was beleagured by the British under Sir William Johnson. After landing the men on Isle la Marine, now called Navy Island, they burnt and sunk these ships. Until a few years ago the charred timbers of these vessels were distinctly visible, but now, owing to the gradual filling up of this basin, they have completely disappeared. Some years ago, while fishing in the clear water in company with some American friends, we noticed what we first thought was a sunken log; but American inquisitiveness when once aroused cannot be pacified, save by complete and satisfactory investigation. A grappling hook was obtained and a long rope. By continued exertion we dragged the object on shore, and it was certainly a curiosity. It was evidently a wheel which was used as part of a primitive machine for dragging these small vessels over the portages. The wheel was about 8 feet in diameter, and was, although composed of probably over a hundred distinct parts, made entirely of wood, there not being a particle of iron in its composition. The wood was oak, and although it had been under water for nearly 120 years, was not in the least affected by any kind of rot or decay. Being too cumbrous to transport, it was left on the shore of the Island, and eventually the Philistine propensities of the agricultural natives destroyed this emblem of a departed age. I was fortunate enough the next year to be able to lay hold of some small pieces of the frame work, which I now have in my possession.

Going down the American channel past Tonawanda, with its miles of timber wharves, and directly east of Navy and Grand Islands, we come to Cayuga Island and creek, a mile from whose mouth is the village of La Salle. The name of René Cavelier dit La Salle occupies a place in early Canadian annals second only, if second, to that of Champlain himself. At the mouth of this creek, six miles above the falls, was built the first European craft that ever navigated the waters of the upper lakes, the ill-fated Griffin, whose fate must, like that of many a noble vessel in modern days, be a matter of conjecture, since, after carrying La Salle on his way to the Mississippi, it was never afterwards seen or heard of. The water of this creek, like that of all the streams flowing into the Niagara, is of a dark brown colour, in striking contrast with the clear blue of the river itself. Three miles below Cayuga Creek is Schlosser's Island and landing, pronounced by the degenerate inhabitant of the river Slusher's. Here was one end of the portage round the Falls of which the other end was nine miles below at Lewiston. Here the canoes of the Indian and voyageur once again entered the stream on their voyage from Fort Frontenac to the fur depots at Machilimackinac. The current is very mild along this shore of the river, and until the lower end of Grand Island is reached, when it becomes very rapid, the voyageurs could propel themselves as easily and rapidly as along a placid inland lake. In 1750 the French constructed a stockade and fort at this point which they appropriately called Fort La Portage. It was burnt in 1759 by Chabert Joncaire who was in command of it when the British commenced the glorious campaign against the French, which gave us the "brightest jewel in the British crown." A short time after this the fort was rebuilt by Captain Joseph Schlosser, a German, who had served in the British army throughout the campaign. A few inequalities in the surface of the ground now mark the site of the guardian of the Portage, but some twenty or thirty years ago the outlines and ditches were still quite distinct. A monument of antiquity still stands some yards below the remains of the Fort in the shape of a stone chimney, which was the centre point of the French barracks and storehouses previous to 1759. Several houses have been at different times attached to it and have been burnt or destroyed, but still the chimney remains, solitary, moss-grown and grey, and will remain no doubt until the advancement of civilization and the necessities of commerce will cause its replacement by something more modern. It was at the wharf at Schlosser that the ill-fated steamer Caroline was fastened that night, in '37, when she was cut out by the loyalists from the Canadian shore. The Canadian militia, under Col. Allan McNab's command, at that time investing Navy Island, were in a complete state of ignorance concerning the river. The Falls were a source of great terror to the storming party, and a circuitous route was taken to reach Fort Schlosser that delayed them many hours. At present the hardy inhabitant of either shore safely crosses the river in a small boat or canoe within half a mile of the rapids, and adventurous youths land with impunity even on Goat Island, but in '37 the cutters out of the Caroline were esteemed greater heroes than even those who faced the bullets of the enemy; such is the power of nature compared with even the life-destroying gunpowder. The affair of the Caroline caused much international ill-feeling and was made the subject of much conjecturing and studying of international law. Evidently the same principles and arguments were quoted and cited, but by the opposite parties, when the Alabama claims came before the board of arbitrators at Geneva. Lying to the north-west of Grand Island, and west of Schlosser, is the small Island, formerly Isle la Marine, now Navy Island. The French, in 1759, built some small vessels on this Island, hence its name was literally translated when it came into possession of the English. Although hardly over three miles in circumference it was probably better known and more thought about, at one time of our national existence, than even Toronto itself. Here, in December, 1837, Wm. Lyon Mackenzie established his headquarters and issued his proclamations to the patriots, as the unfortunate rebels called themselves. In fact to this day, on the American side of the river, the trouble of '37 is referred to as the "patriot war." There was great uncertainty as to the number of the insurgents, who certainly had plenty of arms and ammunition. To this day may be seen in the upper rooms of the several farm houses on the Canadian shore the marks of the bullets, while every plowing turns up on Navy Island many a rusty cannon ball. There is still standing and in good repair three miles from the Village of Chippawa, and directly opposite the head of Navy Island, the house in which Captain Usher was shot. Upon the door of this house is painted in white letters "No. 8. 20 men." evidently its billeting capacity. There are yet on the island two log huts or cottages which were occupied by Mr. and Mrs. Mackenzie. Although degraded to agricultural purposes they still seem destined to out-last several more modern structures built near them. There could not have been much peace of mind for any of the Reformers there; Sir Allan McNab, while not exposing his men to too much personal danger, continued to ply the rebels with shot and shell. While Mrs. Mackenzie was attending to some culinary operations one day a shell, plunging through the roof, fell into a barrel of beans which formed part of the stock of provisions, and burst, scattering a week's provender, but fortunately the inmates all escaped. For the greater security of his followers Mackenzie caused an open space to be cut out of the forest in the centre of the island. This is still known by the name of Mackenzie's Field, and is now used as a pasture for cattle. The proximity of the island to the United States, its great capabilities for defence and its commanding the entrance to the Welland River (which river is one of the entrances to the Welland Canal), combine in making it an outpost of great military value in time of war. On Navy Island may be seen many trees and flowers growing wild which cannot be found in any other place nearer than the Southern States; amongst others are the magnolia, sassafras, and several varieties of wild grapes. The apricot and nectarine are also grown and attain great perfection. A mile and a half from the head of Navy Island, on the Canadian side at the mouth of the Welland River, is the Village of Chippawa. It was at one time, before the building of the Welland Canal, a prosperous place. It was the head of navigation and a tramway ran from it to Queenston, the port at the other end of the Portage. But the canal and railway came and Chippawa suffered the common lot and decreased in trade and population in proportion as the larger towns grew. It is one of the oldest settled portions of Canada, John Cummings a U. E. Loyalist having settled there in 1782. It was the scene of several battles in the war of 1812 between the British and Americans. Several buildings are yet standing which were built previously to 1812, and in one of them may be seen a room at that time used as a prison; the rings and staples for securing the prisoners are still there. At the mouth of the Welland River may be seen the outlines of a stockade and fort first constructed in 1812, and afterwards used in 1837. For the purposes of navigation and the security of the harbour, a canal about one hundred yards in length was cut from the southern shore of the Welland River through to the Niagara. The refuse earth was thrown to one side and has several times been mistaken for the ramparts of the old Fort. On making a personal investigation with several of the "oldest inhabitants" last year, we discovered distinct traces of the old Fort, only, however, a few yards from the mistaken ramparts. Chippawa, like Queenston, has fallen into decay, and has been completely outshadowed by the greater attractions at the Falls two miles away.

From Buffalo to the head of Navy Island the river is comparatively deep, averaging from twenty to thirty feet from shore to shore. Across the head of Navy Island the width is about two and a half miles. Opposite Chippewa it commences to narrow, and so on till the Falls are reached; the main, or "Canadian" current, as it is called, does not follow the middle of the river, but pursues a course of its own, running from the foot of Grand Island towards the American shore, past Schlosser's Island in a north-easterly direction, then, instead of following the straight course towards the head of Goat Island, it makes a sweep round the head of Grass Island towards the Canadian shore, almost due west, and skirting the banks just below Chippawa, flows precipitously over the Horse Shoe Fall. In the centre of the river, stretching from about half a mile above the rapids to within half a mile of Navy Island, there is a reef about two miles long and three-quarters of a mile broad. In no place on this reef is the water more than three feet deep, and at times during low water the heads of the larger stones peep above the surface. The water rushes over this reef at a great rate, and the bottom being composed entirely of rock, and the current not allowing any sediment to settle, the reef, on some windy days, to a stranger, looks very much like the commencement of the Rapids. On the American side of the river opposite to Chippawa a canal has been cut for water power; the opening of this canal forms a small harbour called Port Day. Several steam yachts are kept here, and as the channel does not extend along the American shore, these vessels have to strike across towards the Canadian shore before ascending the river. As this is only a few hundred yards above the rapids the sensations of nervous passengers are not to be envied. The country back from the Canadian shore was formerly settled by U. E. Loyalists. At the present time, however, the farms are every day going into the hands of persons of German and American descent, the original settlers flocking to the cities. These new inhabitants of the river front have no sentimental regard for historical remains, and ruthlessly plow up and tear down anything that is not in strict conformity with agricultural economy. In a very few years all that remains of Forts Erie, Schlosser and Porter will be swept away in "improvements." The relics of 1812 and 1837 will be sought for in vain by the archæologist, but the memory of the deeds that were done, and the devotion of the people who accomplished them, will live forever.

The following members took part in the discussion which followed:—The President, Mr. Murray, Prof. McMurrich, Mr. Livingstone and Dr. Workman.

TWENTY-FIRST ORDINARY MEETING.

The Twenty-first Ordinary Meeting of the Session 1883-84 was held on Saturday, April 5th, the Third Vice-President, Dr. George Kennedy, in the chair.

The minutes of last meeting were read and confirmed.

Mr. Chas. Levey, Mechanical Engineer, was elected a member of the Institute.

The following list of donations and exchanges received since last meeting was read :

1. Science, Vol. 3, No. 60, March 28, 1864.
2. Journal of the Franklin Institute for April, 1884.
3. Annual Report of Proceedings of the Belfast Naturalists' Field Club for 1882-83, Series 2, Vol. 2, Part 3.
4. Proceedings of the Academy of Natural Sciences of Philadelphia, Part 1, January to May, 1883, and Part 3, November and December, 1883.
5. Correspondenz-Blatt der Deutschen Gesellschaft für Anthropologie, Ethnologie, and Urgeschichte, 15 Jahrgang, Nos. 2 und 3 Februar und März, 1884.
6. The Canadian Practitioner for April, 1884.
7. Le Courrier de Europe, Semaine Française, for 1884, presented by Mr. Geo. E. Shaw.

Dr. E. A. Meredith and Prof. Galbraith were appointed Auditors of the accounts of the Institute for the year ending March 31st, 1884.

Prof. McMurrich, presented by title, a paper "On the Myology of the Catfish."

Mr. T. McKenzie, B.A., then read an abstract of a paper by A. B. Macallum, M.A., on "The Alimentary System of the Catfish," after which Mr. McKenzie read a paper by himself, on the "Vascular System and Glands of the Catfish."

These papers will appear, together with others on the same general subject, in the concluding fasciculus of the present volume.

TWENTY-SECOND ORDINARY MEETING.

The Twenty-second Ordinary Meeting of the Session 1883-84 was held on Saturday, April 12th, the President in the chair.

The minutes of last meeting were read and confirmed.

The following list of donations and exchanges was read:

1. Transactions of the Manchester Geological Society, Vol. 17, part 14, Session 1883-84.
2. Science, Vol. 3, No. 61, for April 4, 1884.
3. (a) Records of the Geological Survey of India, Vol. 15, Part 4; Vol. 16, parts 1—3. Vol. 17, part 1.
   (b) Memoirs of the Geological Survey of India, Vol. 19, parts 2, 3 and 4.
   (c) " of Palaeontologia Indica, Series X., Vol. 2, part 4.
   " " " " X., " 2, " 6.
   " " " " XII., " 4, " 1.
   " " " " XIII., " 1, " 4, Fasciculi 1, 2.
   " " " " XIV., " 1, " 4.
4. Publications of the Oneida Historical Society at Utica, No. 5, January 13, 1880.
   (a) Second Annual Address before the Society, by William Tracy, of New York.
   (b) Historical Fallacies regarding Colonial New York.
5. Bulletin of the Philosophical Society of Washington, Vol. 6, 1884.
6. (a) Fifteenth Annual Report of the American Museum of Natural History (Central Park, New York), March, 1884.
   (b) Bulletin of the American Museum of Natural History (Central Park, N. Y.), Vol. 1, No. 5, February 13, 1884.

7. Appleton's Literary Bulletin, No. 4, April, 1884.
8. California State Mining Bureau : Third Annual Report of the State Mineralogist, for the year ending June 1st, 1883.
9. (*a*) Bulletin of the Essex Institute, Vol. 14, January to December, 1882, Nos. 1—12,
   (*b*) Pocket Guide to Salem, Mass., 1883.
   (*c*) Plummer Hall : Its Libraries, its Collections, its Historical Associations.
   (*d*) The North Shore, Massachusetts Bay, 6th Ed., 1883.
10. Anales del Museo Nacional de México, Tomo III., Entrega 5*a*.
11 (*a*) Proceedings of the Literary and Philosophical Society of Liverpool during the 59th Session, 1869-70, No. 24.
   (*b*) Proceedings of the same Society during the 62nd Session, 1872-73, No. 27.
12 (*a*) Transactions of the Cambridge Philosophical Society, Vol. 11, parts 1 and 2 ; Vol. 13, part 3.
   (*b*) Proceedings of the Cambridge Philosophical Society, Vol. 1, 1843-1865, 16 Nos. complete ; Vol. 2, 1866 to 1876, parts 1—17, complete ; Vol. 4, part 6, for 1883.
13. Sitzungsberichte und Abhandlungen der Naturwissenschaftlichen Gesellschaft, "Isis," in Dresden, 1883, Juli bis December.
14. L'Académie Royale de Copenhague, Bulletin pour 1883, No. 2, (Mars-Mai.)
15. (*a*) Annuaire de L'Académie Royale des Sciences, des Lettres, et des Beaux Arts de Belgique, for 1881, 1882 and 1883 ; 47th to 49th year, Bruxelles, (3 Vols.)
   (*b*) Bulletin de L'Académie Royale de Belgique, 49*me* Année, 2*me* Série 1880, Tome 1 ; 50*me* Année, 3*me* Série, 1881, Tome 1, 2 ; 51*me* Année, 3*me* Série, 1882, Tome 3*me*, et 4*me* ; 52*me* Année, 3*me* Série, 1883, Tome 5*me*, (6 Vols.)

Dr. E. A. Meredith then read a paper entitled :—

"COMPULSORY EDUCATION IN CRIME."

The reader of the paper contended that so far as regards the suppression of vice and crime, our Common or County jails were little better than the abominable dens which Howard visited and denounced more than a century ago. Philanthropists, social reformers and Prison Congresses had worked earnestly during the last thirty or forty years, and their labours had in other departments produced good results. In institutions for saving children, such as Homes, Refuges and Industrial Schools, extraordinary progress had been made and in convict prisons for adults an extraordinary revolution had been carried out with equal success, especially in those conducted under the so-called "Crofter" or "Irish" system. The Common Jails alone lagged behind the age, and the reason probably is that they were the only institutions managed by *municipal bodies*, the others being either under the control of the State or of private individuals or societies. The jails have been improved materially, but not morally. The giant evil of Howard's time, the indiscriminate association of the prisoners is still permitted in the great majority of jails on the continent, whether in the United States or Canada. The jails had ceased to be in any sense either *deterrent* or *reformatory;* they are, on the contrary, *attractive* to criminals and to the last degree demoralizing to the inmates. They are nurseries of crime, hotbeds of vice, where criminals are manufactured at the cost of the country. The only remedy for this disgraceful state of things is the introduction of the " Separate System," a system which had been approved by all authorities on the subject, and carried out with marvellous success in many jails in England and on the continent of Europe, and in some few in the United States. The great merit of the separate system is that it stops the corruption and contamination which indiscriminate association of prisoners necessarily produces. In other words it puts an end to the " Compulsory Education in Crime" now going on in all our jails; and more than this, it represses crime, both by its deterrent and reforming influences. Dr. Meredith recommended that the separate system be made obligatory in all jails so soon as these are fitted for it.

In answer to a question by the President, Dr. Meredith explained that " Solitary Confinement" was stricter than " Separate Confinement," which latter meant merely separation from injurious influences, and not from visits of those who may benefit the prisoners. These would be brought together only under supervision.

Mr. B. B. Hughes spoke in high terms of the manner in which the Reformatory at Penetanguishene was conducted.

Mr. Douglas hoped that the Legislature would do something to remedy the evils mentioned in Dr. Meredith's paper.

Mr. Geo. Murray thought that young children should not be sent to the same prison as adults. He would endeavour to sift out the worse juvenile criminals, and he believed a large majority would remain with which we could deal in the way of reformation.

TWENTY-THIRD ORDINARY MEETING.

The Twenty-third Ordinary Meeting of the Session 1883-84 was held on Saturday, April 19th, the President in the chair.

The minutes of last meeting were read and confirmed.

The Rev. Hugh Johnston, M. A., B. D., was elected a member of the Institute.

The following list of donations and exchanges received since last meeting was read:

1. Memoirs of the Boston Society of Natural History, Vol. 3, No. 9, March, 1884.
2. Science, Vol. 3, No. 62, for April 11th, 1884.
3. The Manitoba Gazette, March 31st, and April 5th.
4. Journal of the Asiatic Society of Bengal, Vol. 52, Part 2, Nos. 2, 3, 4, 1883. Proceedings of the Asiatic Society of Bengal, No. 9, November, 1883.
5. The Canadian Entomologist, Vol. 16, No. 2, February, 1884.
6. Proceedings of the Royal Geographical Society, N. S., Vol. 6, No. 4, April, 1884.
7. Atti della Società Toscana di Scienze Naturali, Processi Verbali, Vol. 4, pp. 29-52. Processi Verbali, Indice del, Vol. 1, pp. 133 to 138.
8. Meteorological Service, Dominion of Canada, Monthly Weather Review, March, 1884.
9. Oneida Historical Society, 1879, Men of Early Rome, by D. E. Wager.

Capt. Gamble Geddes, A. D. C., then read a paper entitled,

AN ENTOMOLOGICAL TRIP IN THE ROCKIES.

MR. PRESIDENT AND GENTLEMEN,

It is with great pleasure that I take advantage of your kind invitation to read a brief paper upon a trip made by me to the Rocky Mountains last summer. As my object in undertaking this long journey was purely "Entomological," I had intended to prepare and read to you a paper upon the genera "Coliadæ" and "Argynnidæ," of our Diurnal Lepidoptera, (two of my favourite families,) but at the request of some of my friends, I am going to give you a rough outline of the entire trip, trusting that I may be enabled to make it of more interest to you, by exhibiting a few of the specimens and relics picked up by the way.

With this short preface and with your kind indulgence, I will begin:

The different points of interest between here and Winnipeg have been so thoroughly discussed by tourists of late years that it is needless for me to refer to my trip until a start is made from Winnipeg on June 9th, 1883.

The main object of this journey, was to make a collection of insects and especially of Rhopaloceres or the day butterflies, the first of the two great divisions into which the Lepidoptera have been divided. The different species of this division all fly by day: they have the antennæ terminated by a knob or club and comprise the Papilionidæ, Pieridœ, Lycœnidæ, Erycinidæ, Litytheadæ, Satyridæ, Hesperidæ, and so on.

The Heteroœres, the greater portion of which fly by night, embrace the Sphingidæ, Bombycidæ, Noctuæ, Geometridæ, &c., &c. In this division the most noticeable feature of distinction is the antennæ, which are of a feather-like appearance and taper to a point at the ends instead of the knob or club that the majority of the diurnals have.

I do not intend to enter into detail with regard to the species which I captured *en route*, but more to give a brief sketch of the trip and the beauty of the country through which I passed, as well as the barren parts.

After leaving Brandon on the 11th June, the next point of interest was Moosomin where we lay over a day, to visit Fort Ellis and Binscarth. The former is an old fort under the charge of Mr. W. J. McLean, a faithful officer of the H. B. C. The party with whom I was travelling were going on to Binscarth, the "stock farm," and property of the Scottish Ontario Co.

As Mr. McLean offered me the hospitality of the Fort, I decided to remain over and make such additions to my collections, as were to be taken in the neighbourhood of Fort Ellis.

I was well rewarded for my pains, as I succeeded in making some rare captures. Fort Ellis is situated 30 miles from the C. P. R. track and about 20 miles from Binscarth.

Here the Blackfeet and the Sioux Indians, (principally the former) were congregated in large numbers around the Fort. Their "tepees" or lodges were thick in every direction, and I had the opportunity of seeing how the Government agents distributed the pork, flour and blankets to those Indians who deserve them.

I was advised to go out with some of the squaws of the Blackfeet tribe, and get some wild turnips, though *why* they are so called, I never could and never will imagine. The root of the plant resembles the bulb of a small tulip, and when the outer skin is removed, the heart of the bulb tastes something like the kernel of an almond and quite as dry. The leaf of the plant reminded me of the lupin (perenniel), but it was too early in the season to take the flower, as a specimen.

The women of the Blackfeet and Sioux seem particularly partial to these roots. Armed with genuine "crow-bars" of *iron*, about four feet in length and from one and a-half inches to 2 inches in diameter, we sallied forth. It was a matter of amazement to me, to see the manner in which the squaws handled these iron bars. On the side of a steep hill they would let themselves down and holding on to a shrub, or the end of a rock with one hand they would with the other hand wield the bar (always pointed at the end) and soon the roots, which were generally about four inches to six inches down in the soil, would be dislodged. I tried to handle one of these bars myself and as I had to use two hands and the combined strength of my two arms to boot, I appeared to cause much merriment to my redskin friends, who looked upon me as a very poor specimen of the human race. I had a chance of purchasing a few samples of the bead-work of the Sioux women, a few articles of which I have brought with me, also one or two of their favourite pipes.

On the 15th June we reached Medicine Hat and the end of the track of the C. P. R., which had just then crossed the Saskatchewan River on a temporary trellis-work bridge. Here we had an opportunity of witnessing the wonderful rapidity with which this road was constructed; the contractors at that time were building from three to five miles a day.

On June 25th we started for Calgarry. On the 27th we reached the Bow River. The mosquitos were terrific. During the night our camp was set on fire by Indians, who hoped to make a stampede with our horses. Luckily, we discovered the grass on fire in two places, in time to put it out with wet blankets, and so saved our property.

The flora between Calgarry and Edmonton (my next halting-place) was simply lovely. The orange and cardinal lilies, or, as the Cree Indians call them, "Wappiconnaisa," the yellow ladies' slippers, anemones, wild rhubarb blossom, 4 feet high, and the plumed head of that lovely flower, Geum Triflorum, made a charming contrast to the innumerable shades of green of the foliage. As one looked into the differenc "coolies" and "hollows" in the prairie in passing, it appeared to be like a rich carpet of most exqu'site workmanship and colouring, but far, far more beautiful.

Upon the 7th July we reached Edmonton, and it became apparent that we were getting much farther to the north, as the days were so long and the nights so very short. I was surprised to hear from the proprietor of the hotel at Edmonton that upon the Sunday previous to our arrival he had put green peas from his own garden upon the table for his guests at dinner. I was not so much surprised however to see what he showed me the same evening we arrived, and that was half a field of beautiful potatoes cut off by the summer frosts and looking as black as ink alongside the other half, which had escaped. It was just as if some one had taken a ruler and drawn a line from one corner of the field to the other, and then painted one half black and the other green. I was very much disgusted with the cold and windy weather that we had at this point. It was impossible to collect butterflies and moths, and I was not at all sorry to start off again for the south.

On July 16th we reached Calgarry on the return trip. Here I met some old friends, and on the 19th July started off for Fort Macleod. This was a very interesting part of the trip, as we stopped at several ranches, amongst them "Oxley Ranche," the property of Mr. Staveley Hill and other English gentlemen, and the ranche of Mr. Stinson at High River. I must not forget to mention that at all the ranches I stopped at on my long journey and at all the posts of the Hudson Bay Co., I received the greatest hospitality, likewise from the N. W. Mounted Police. Whilst amongst the ranches I learned that the cow-boy's whip was called a "Quoit;" the rope for catching horses in "corral" is called a "Mecarte," a "Lariet" being a grass rope for the same purpose. "Chaps" or "chaparellos," are the leather breeches or leggings used for riding, and so on.

On July 24th I spent a rare day of collecting at Pincher Creek, being then the guest of Lieut.-Col. Macleod; also three or four days following I did good work, taking Argynnis Clio, Argynnis Artonis, as well as many rare Coliadæ.

Aug. 1—At the Garnett Ranche, a lovely Ranche in the foot-hills of the Rockies, where the peaks of the mountains tower above one over three sides of the ranche. Here I took many rare insects.

I met Dr. George Dawson, of the Geological Survey of Canada, and his party, at the Garnett Ranche. He had just returned from the Crow's Nest Pass, with specimens of natural history *generally*, but with notes upon the geological formations of the country *in particular*. I took here a new Polyomatus or Chrysophanus called Florus by Mr. W. H. Edwards, of Coalburgh, West Virginia).

Whilst a guest at the Garnett Ranche, I went out with one of the proprietors to get some trout. When I was catching butterflies, he was catching trout, averaging about 1½ lbs. each. He took 17 fine fish in a very short time. Upon the 4th August we reached our camping ground at the Crow's Nest Pass, and a lovely spot it was. Through the kindness of Col. Macleod, I was enabled to take along with me a folding boat made of canvas, with which we explored the lakes near the summit of the mountains known as the "Big Fish Lakes," and judging from the size of the fish taken, the name was very appropriate. I did some rare collecting through this new country, taking the ♀ Hermodur, a species described by Mr. Henry Edwards as a var. of Parnassius Smintheus, also Arg. Chariclea and Arg. Boisduvallii, Chrysophanus Mariposa and Thecla Edwardsii, one solitary specimen. We met large bands of the Stoney Indians throughout this Pass, who were trapping, shooting, and fishing. The Indians supplied us with meat from the mountain sheep or big horn (Ovis Montana), which made a delicious steak when broiled, reminding one both of mutton and venison. The band of Indians who were camping close to us were trapping beaver, and hunting bear and sheep, principally.

Whilst at Big Fish Lake, I saw three fine trout caught (more than once) at one cast, by Mr. Arthur Garnett, one of the most experienced fishermen I have ever met. I may say that our living here was really luxurious, after feeding on fat pork and porridge for a long time, the variety in our fare was most welcome, I can assure you. Let me advise any of you, gentlemen, who ever go for a trip to the Mountains to be well provided with fishing tackle, and lots of it, besides a good rifle, and shot gun as well. These articles are infinitesimally small in comparison with the "prog" you would have to pack your horses with, and with a bag of flour and some bacon you can live well. It is not absolutely necessary to take canned meat and vegetables along with you, as many explorers do, for in this lovely country you are independent, so to speak, with the quantity of fish and game that is always on hand in the neighbourhood.

After reaching the summit of the Mountains here, and returning to my headquarters in camp, we started back to the District of Old Man's River. Upon August 15th, I found myself at the Belly River District, from which place I started for the Koutanai Lakes. All through this beautiful grazing country, I was perfectly delighted with everything I saw.

The ranchers were all busy taking in hay for winter emergencies, although it is seldom required, for the snow is seldom too deep for the cattle to scratch it up to feed on the long grass underneath. The Chenook wind which blows through the mountains from the Pacific Ocean, melts the snow nearly as soon as it makes its appearance, and wheeled vehicles, principally heavy carts and buck boards supply the place of sleighs. From the Muirhead Ranche, I started out for the so-called Koutanai Lakes, where I was successful in capturing many fine butterflies, amongst them *Argynnis Leto*, ♂ and ♀.

The guide who took me up to the Lakes killed two grisly bears whilst I was in this part of the mountains, and I brought the skins back with me as a memento of the trip.

The name of the hills that one meets with on the prairie is " bute " and " cooley " or " lie " is applied to all hollow spots or valleys.

I must warn all who may be disposed to make a summer excursion through the mountains to the British Columbia side, to be well provided with a musquito-net ; I mean by this not only the small nets to wear over one's head and neck whilst riding or driving, but a strong net, capable of being fastened to the tent inside, and covering one's entire body at night time.

It may not be out of place here to relate one or two anecdotes about the extraordinary numbers of musquitos that infest the entire district through which I passed—not forgetting to mention the black-flies, sand-flies, horse-flies or " bull-dogs " as the old settlers call them, and the greatest *torment* of all the flying ants.

I remember one evening after we had pitched our tent for the night, and just about dusk, I set off with one of my fellow-travellers to inspect a curious rock, which was standing upright in the midst of a vast plain, with no other sign of stones or gravel of any kind anywhere near it. Our fire in the camp had driven the musquitos away from the immediate neighbourhood, and for the time I quite forgot the existence of these pests. My friend was wearing a dark blue pea-jacket and walked before me. Fortunately he was provided with a net to cover his face, but I had foolishly left mine behind. As soon as we stirred up the long grass with our feet, the musquitos arose in myriads, and after fighting them off for a short time, I looked ahead at my companion, and I declare I could not tell what colour his coat was, so thickly was his back covered with the insects, I confess that this was too much for me, and I turned and fled to the camp as fast as my legs would carry me in a most ignominious fashion.

In case you have not seen a "smudge" or read of one, I will describe it. A "smudge" is a refuge for horses and cattle that are attacked by flies and mosquitos. A "square" of logs dove-tailed at the four corners, is constructed just high enough to allow a horse standing up to put his head over the topmost log. Inside this square and on the ground you set fire to leaves and grass, and pile on to this wet foliage of plants, and make a heavy suffocating smoke. The horses will run madly towards this smoke from wherever they may be and hold their heads where the smoke is thickest. It is absolutely necessary to build a barricade of logs round these fires, as the horses will burn themselves in the fire often rather than suffer the torment of the flies.

Whilst driving one day to the Koutanai Lakes we had to pass through a cloud of black flying ants. My guide and I were both well covered up, but he had on a light-coloured felt hat which seemed to have some peculiar attraction, for they attacked him vigorously; there was a small opening at the back of his neck between his hat and the top of his coat-collar, and the ants fairly gnawed away that portion of his neck which was exposed. We came across a very intelligent man who acts as guide to exploring parties in the Koutanai District. He lives in a most lonely situation, quite near the mouth of the Koutanai Pass. He is familiarly known as Koutanai Brown, and I would recommend any one going to that solitary neighborhood to patronize this guide. He is a dead shot with a rifle and an excellent fisherman. He makes his living by trading to some extent with the Indians and shooting sheep and bears, himself. We had plenty of bear's meat while with Koutanai Brown. But as it had been dried in the sun, (and not smoked) it was decidedly "odoriferous," and I preferred watching the others enjoy it and partaking of salt pork instead.

I would like to call your attention on the map for a moment to the stretch of country lying between the Red Deer River and Fort Edmonton. Here the shrubs begin to appear as trees, and the trees increase in size as one proceeds north. Very fine timber is to be had in and around Edmonton and all along the banks of the Saskatchewan. Amongst all the farmers we met between Calgarry and Edmonton (with one exception), the opinion expressed as to the quality of the land and the nature of the climate, was unanimous. All agreed in saying that although the winters were severe, yet they could grow such fine crops and so rapidly, that the brief summer was amply long to mature the grain and get it harvested.

In conclusion, I have with me a list of the diurnals taken by me during this tour, and for the benefit of those entomologists who are present, I have looked it over and will call their attention to several of the species which are rare and which I will be glad to point out to them in my cases. I regret that I could not bring my cabinets down here to show my collection to the members of the Institute, but I will only say to those who are interested in this fascinating study, that it will afford me the greatest possible pleasure to look over my cabinets at Government House with them at any time that I may be honoured by a visit.

Thanking you gentlemen, for your kind attention, and trusting that I have not been encroaching too much upon your time, I beg to conclude.

LIST OF DIURNAL LEPIDOPTERA COLLECTED IN THE NORTH-WEST TERRITORY AND THE ROCKY MOUNTAINS.

1. Papilio Asterias, F. Edmonton.
2. " Troilus, L. Fort Macleod.
3. " Turnus, L. "
4. " Glaucus, L. "
5. " Eurymedon, Bd. Seen but not taken.
6. Parnassius Smintheus, Doubd. Crow's Nest Pass.
7. Dark var. Hermodur, H. Edw. Summit Pass.
8. Pieris Oleracea, Boisd. Koutanai.
9. " Occidentalis, Reak. Pincher Creek.
10. " Protodice, Boisd. Belly River.
11. " Rapæ, L. N. W. T.

12. Anthocaris Olympia, Edw. (v. rare). Summit.
13. " Ausonides, Boisd. Calgarry.
14. Colias Christina, Edw. Red Deer River.
15. " Occidentalis, Scud. (rare). Edmonton.
16. " Edwardsii, Behr. (rare). Edmonton.
17. " Astrea, Edw. (♀ new). Red Deer River.
18. " Alexandra, Edw. (rare) 5,000 ft. elevation Rocky Mountains.
19. " Eurytheme, Boisd. (rare). None taken W. of Moose Jaw.
20. " Hagenii, Edw. Fort Macleod.
21. " " " (diminutive form). Fort Macleod.
22. " Scudderii, Reak. Koutanai.
23. Argynnis Lais, N. S., Edw. Fort Edmonton.
24. " Cybele, F. " "
25. " Baucis, Edw. (not proved new yet). Fort Edmonton.
26. " Coronis, Behr. Belly River.
27. " " (dark varieties). Crow's Nest.
28. " Chariclea, Schneid. Crow's Nest.
29. " Boisduvalii, "
30. " Atlantis, Edw. "
31. " Eurynome, Edw. Belly River.
32. " V. Erinna. Red Deer River.
33. " V. Arge (?), Streck. Calgarry.
34. " Clio, Edw. (v. rare). Crow's Nest.
35. " Monticola, Behr. (v. rare). Summit.
36. " Edwardsii, Reak. (v. rare). Blackfoot Reserve.
37. " Artonis, Edw. (v. rare). Koutanai.
38. " Myrina, Cram. Edmonton.
39. " Aphrodite, F. "
40. Melitæa Nubigena, Behr. Crow's Nest.
41. " Palla (?), Boisd. "
42. " Chalcedon (?), Boisd. Garnett Ranche.
43. " Leanira. " "
44. Limenitis Disippus, Godt. Crow's Nest.
45. " Lorquini, Boisd. "
46. " Arthemis, Drury. N. W. T.
47. Vanessa Milberti, Godt, N. W. T.
48. " Antiopa, L. N. W. T.
49. Pyrameis Atalanta, L. N. W. T.
50. Grapta Satyrus. Edw. Crow's Nest.
51. " Progne, Cram. Fort Macleod.
52. Danais Archippus, F. Common.
53. Chionobas Chryxus, West (v. rare). Summit.
54. " Varuna, Edw. Calgarry.
55. " Uhleri (?), Reak. "
56. Erebia Epipsodea. Butl. Fort Ellis.

57. Satyrus Charon, Edw. Garnett Ranche.
58. " Silvestris, Edw. "
59. " Nephele, Kirby. Rocky Mountains.
60. " V. Boopis, Behr. "
61. " V. Ariana, Boisd. "
62. " V. Olympus, Edw. "
63. Cœnonympha Inornata, Edw. Calgarry and Edmonton.
64. " Ochracea, Edw. " "
65. Phyciodes Carlota, Reak. Brandon.
66. " Tharos, Drury. Edmonton.
67-68. Several varieties from North of Edmonton. Not determined.
69. Thecla Titus, F. Old Man's River.
70. " Edwardsii, Saund. (rare). Summit.
71. Chrysophanus Mariposa, Reak. (v. rare). Summit.
72. " Florus, Edw., Nov. Spec. (v. rare). Garnett's Ranche.
73. " Helloides, Boisd. Oxley Ranche.
74. " Americana, D'Urban. "
75. " Sirius, Edw. (v. rare). Fort Macleod.
76. Pyrgus Tessellata, Scud. Medicine Hat.
77. Amblyscirtes Vialis, Edw. (v. rare). Fort Ellis.
78. Thymelicus Garita, Reak. Fort Ellis.
79. Thanaos Brizo, Boisd. Fort Ellis.
80. Eudamus Pylades, Scud. "
81. Lycæna Anna, Edw. Belly River.
82. " Amyntula, Boisd. Calgarry.
83. " Sæpiolus, Boisd. Crow's Nest.
84. " Rustica. Fort Qu'Appelle.
85. " Pembina, Edw. Crow's Nest.
86. " Afra, Edw. Nov. Spec. Saskatchewan.
87. " Unknown Spec. sent for identification. Garnett Ranche.
88. Pamphila Zabulon, Bd. Lec. Calgarry.
89. " Manataaqua, Scud. (v. rare). Fort Macleod.
90. " Manitoba. Belly River.
91. " Uncas, Edw. "
92. " Cernes, Bd. Lec. Crow's Nest.
93. Argynnis Leto ♀, Behr. Fort Macleod.
94. " Bellona F. Fort Ellis.
95. Lycæna Fulla, Ew.
96. " Melissa Edw. Oxley Ranche.
97. " Neglecta, Edw. Fort Ellis.
98. " Lygdamas, Doubld. Fort Ellis.
99. " Icaroides, Bd. Red Deer River.
100. Pamphila Nevada, Edw. (?) Fort Macleod.
101. " Colorado, Scud. Medicine Hat.
102. " Idaho, Edw. Moose Jaw.

103. Pyciodes Camillus, Edw. Edmonton.
104. " Marcia, Edw. Edmonton.
105. " Nycteis, Doubld. Edmonton.
106. Argynnis Nevadennis, Edw. Calgarry.

The President, Dr. Bryce, Mr. Chas. Levey and Mr. B. B. Hughes took part in the discussion which followed.

TWENTY-FOURTH ORDINARY MEETING.

The Twenty-fourth Ordinary Meeting of the Session 1883-84 was held on Saturday, April 26th, 1884, the President in the chair.

The minutes of last meeting were read and confirmed.

The nomination of Office-bearers and Members of Council was made.

A communication was read from Mr. W. Thompson, President elect of Section A of the British Association for the Advancement of Science, giving notice of special discussions in the Section of Mathematical and Physical Science.

The following list of donations and exchanges was read :

1. Journal of the Transactions of the Victoria Institute, Vol. 17, No. 68.
2. List of Members, Council, &c., of the Royal Society of Edinburgh, November, 1883.
3. Science, Vol. 3, No. 63, April 18, 1884.
4. Ninth Annual Report of the Ontario Agricultural College and Experimental Farm for 1883.
5. Report of the Entomological Society of Ontario for 1883.
6. Science Record, Vol. 2, No. 6, April 15, 1884.
7. Report of Speeches at the Annual Dinner of the Institution of Civil Engineers, March 26, 1884.

Mr. Henry S. Howland, jun., then read a paper entitled,

"THE ART OF ETCHING."

Mr. Howland opened his subject with the following words :

"Very often we, who are engaged in mercantile life, seem to lose sight of the great value of having some interests, some tastes and some pursuits independent of, and in many ways directly opposite in their influence, to our regular business—something to engage our leisure moments, to keep us from becoming too much absorbed in the mere material and hardening act of money-getting, and at the same time by directing our thoughts into a different course, to be a wholesome recreation to our minds and a means of ennobling our hearts.

"Our long and wearisome days of business are usually spent in work without much change, our whole attention directed to practical things the poetical instincts of our natures receiving no culture, and so lacking development, unless quickened into life and activity by some powerful influence. * * *

"Now, while not at all asserting that we should not give to our business the care and attention which it may need, for indeed to make a true success of it, it must be uppermost in our thoughts, but just because of that very thing, because man, by the very constitution of his nature, needs variety and change, or he will develop into a mere machine, or, perchance, his health may fail, he must become interested in something else. And while giving to science and philosophy the tribute of respect and admiration which is their due, I insist that poetry, that painting, that architecture, that music, the fine arts in fact, will appeal to something in man's nature, which science, philosophy, the professions, or branches of mercantile industry, cannot reach. There is a part of man's nature which responds to beauty as to an electric thrill."

Mr. Howland then gave a brief history of the "Art of Etching" as first practiced by D rer about 1518, with its bright and its dark days, to its decline and comparative obscurity at the commencement of the present century, with its revival about 1860, and gradual growth in popularity to the present day.

The practical part was then carefully described, Mr. Howland illustrating the processes and modes of treatment, with plates and implements used. "Etching really means drawing upon a plate, generally of copper, which has previously been coated with a varnish-like substance called *ground*, with a point which removes the varnish wherever it touches, and then subjecting these exposed parts to the biting of an acid, so as to leave actual hollows in the metal."

Mr. Howland mentioned the names of Haden, Hamerton, Palmer, Whistler, Chattock, Law, Lelanne, Méryon, Jacquemart as being the leading etchers in Europe.

In America Stephen Parish, of Philadelphia, probably stands the highest, and we owe a great deal to such men as Henry Farrer, Thomas Moran, J. T. Bentley, F. S. Church, R. S. Gifford, Wm.

Sartain, J. C. Nicoll, Jas. D. Smillie, K. Van Elten, Walter Shirlaw, J. F. Sabin, F. Dielman, J. F. Cole, E. H. Miller, P. Moran, M. N. Moran, Samuel Coleman, for the work they have given us.

Mr. Howland expressed himself especially indebted to Mr. Stephen Parish, who was very generous in lending him a plate on which he had etched a picture called "An Old Acadian Inn-yard." Mr. R. J. Kimball, of New York, was very kind in sending a plate by Mr. Henry Farrer, President of the "New York Etching Club."

Mr. J. F. Bentley, a Canadian, now living in New York, kindly sent a large artist's proof of his picture called the "South Porch of St. Ouen."

Thanks are due to Mr. Jardine, Secretary of the "Ontario Society of Artists," for his kindness in lending a large collection of etchings. Not only did he volunteer the pictures, but he spent the greater part of an afternoon in hanging them.

Mr. Howland ended his paper with a short description of the beauties of etching, illustrated by a large number of etchings from the time of the invention of the art to the present day, and hoped that, the appreciation and support of this attractive art would go on increasing.

"When the artist by his skill awakens in those who view his pictures feelings or emotions similar to the promptings he had in the conception of his work, he is much nearer true art than when, by careful and minute detail, he gives the conscious feeling of reality. Hence in this particular, etching really seems well adapted for expressing the highest art. Something is given to awaken thought, rather than a passing pleasure only."

After the reading of the paper, the members present were invited to inspect the etchings which Mr. Howland had collected to illustrate his subject.

THIRTY-FIFTH ANNUAL MEETING.

The Thirty-fifth Annual Meeting was held on Saturday, May 3rd, the Second Vice-President, Mr. George Murray, in the chair.

The Minutes of last Annual Meeting were read and confirmed.

The following list of donations and exchanges received during the preceding week was read:

1. Science, Vol. 3, No. 64, April 25, 1884.
2. Verhandlungen der Berliner Gesellschaft für Anthropologie, Ethnologie und Urgeschichte, Sitzung vom 20 Jan., 10 Feb., 17 Feb., 17 März, 21 April, 19 Mai, 16 Juni, 21 Juli, 20 October, Nov. 17 and 24, Dec. 15, 1883. 12 Numbers.
3. Report of the Canadian Observations of the Transit of Venus, 6 December, 1882.
4. (1) Bulletin of the Natural History Society of New Brunswick, No. 3, 1884.
   (2) Annual Report of New Brunswick Natural History Society. Memorial Sketch of Prof. Ch. Fred. Hartt, by George U. Hay.
5. Bulletin of the Museum of Comparative Zoölogy at Harvard College, Cambridge.
   Vol. 1, Complete.
   " 2, Nos. 2-5.
   " 3, " 3, 6-16.
   " 5, " 1, 7-12, 14-16.
   " 6, complete.
   " 7, Nos. 2-10.
6. Journal of the Franklin Institute for May, 1884.
7. Journal of the Microscopical Society for April, 1884.
8. The Canadian Practitioner for May, 1884.
9. Proceedings of the Worcester Society of Antiquity for 1883, No. 20.

The following gentlemen were elected Honorary Members of the Canadian Institute:

Daniel Wilson, LL.D., Rev. John McCaul, D.D., Prof. Balfour Stewart, (Owen College, Manchester,) and the Abbé Provencher, Cap Rouge, Quebec.

The Hon. Secretary read the Annual Report of the Council as follows:

ANNUAL REPORT, SESSION 1883-84.

The Council of the Canadian Institute have the honour to lay before the members their Thirty-fifth Annual Report.

The attendance at the weekly meetings has been satisfactory, and a large number of papers have been read. These will compare favorably in average merit with those of any preceding Session. In addition to the regular work of the Institute, a course of three popular public lectures on sanitary subjects was arranged for and delivered in the Library under the joint auspices of the Institute and the Provincial Board of Health. The lecturers were Dr. Oldright, Dr. Cassidy and Dr. Bryce.

The number of members has increased from 225 to 236, and a larger number than heretofore have made use of the reading-room and library. As will be seen by reference to one of the appendices to this report, the number of books and periodicals taken out by members has nearly doubled. The number of Societies with which we exchange publications is now 140. The number of donations and exchanges received has been 800, as against 280 during the preceding year. One hundred and twenty volumes have been bound, and eighty volumes and numbers purchased to complete sets. It is much to be desired that funds should be forthcoming to bind the whole of the 700 volumes that are now awaiting the binder.

A change has been made in the method of publishing the Proceedings, which, it is believed, will have the effect of rendering our transactions more acceptable to our members without rendering them less valuable to other Societies.

The Council having devoted so much attention to the Library, Reading-room, Journal and Exchanges, has not been able to put the collections in the museum in order or increase them. This department, however, has not been altogether neglected. A few valuable skins have been stuffed, and the very handsome offer made by Mr. Brodie to furnish a collection of insects, provided the Institute supplied cases, has been accepted, and a number of cases have been placed at his disposal.

Herewith are submitted appendices, showing (1) the membership, (2) the financial condition of the Institute, which will be found very satisfactory, (3) the number and sources of the donations and exchanges, (4) the number of books and periodicals issued to members, (5) the list of periodicals subscribed for, and (6) the list of periodicals presented to the Institute, with the names of the donors.

All of which is respectfully submitted.

J. M. BUCHAN,
PRESIDENT.

APPENDIX I.

MEMBERSHIP.

Number of Members, March 31st, 1883	225
Withdrawals and Deaths during the past year	25
	200
Elected during the Session 1883-84	36
Total number of Members, April 1st, 1884	236
Composed of:	
Corresponding Member	1
Honorary Member	1
Life Members	17
Ordinary Members	217
Total	236

APPENDIX II.

TREASURER IN ACCOUNT WITH THE CANADIAN INSTITUTE, SESSION OF 1883-4.

To Summary	$ cts.
" Balance on hand	689 04
" Annual Subscriptions	588 00
" Rents	179 50
" Journals Sold	17 25
" Interest on Deposits	17 10
" Freight	1 20
	$1,492 09

By Summary	$ cts.
" Salaries	286 47
" Periodicals	244 34
" Interest on Mortgage	238 78
" Printing	222 79
" Fuel	142 23
" Postage	78 07
" Express	34 82
" Gas	42 19
" Furniture	32 80
" Stationery	25 92
" Repairs	24 39
" Water	24 00
" Contingencies	14 30
" Taxes	9 49
" Cash in Bank	71 50
	$1,492 09

Assets.

Building	$11,000 00
Warehouse	720 00
Ground	2,500 00
Library	5,500 00
Specimens	1,200 00
Personal Property	400 00
	$21,320 00

Liabilities.

Mortgage	$3,411 00
Balance in favour of Institute	17,909 00
	$21,320 00

Examined, compared with vouchers and found correct.

E. A. MEREDITH, } *Auditors.*
J. GALBRAITH, }

14th April, 1884.

APPENDIX III.

DONATIONS AND EXCHANGES.

Books and Pamphlets received from—

April 1, 1882, to April 1, 1883.		April 1, 1883, to April 1, 1884.	
Canadian	30	Canadian	90
United States	60	United States	300
Great Britain and Ireland	100	Great Britain and Ireland	200
India, and other British Colonies, exclusive of Canada	20	India, and other British Colonies, exclusive of Canada	40
Foreign	70	Foreign	170
Total	280	Total	800

The number of Societies with which the Institute exchanges is.. 140

The following are the principal Institutions that have supplied back numbers of their publications to completed sets.

Smithsonian Institution.
Essex Institute.
New York Academy of Sciences.
Academy of Natural Sciences, Philadelphia.
Worcester Society of Antiquity.
Harvard University Library.
Museum of Comparative Zoölogy at Harvard College.
Connecticut Academy of Arts and Sciences.
Historical Society of Pennsylvania.
Peabody Institute, Baltimore.
Entomological Society of Ontario.
Royal Scottish Society of Arts.
Anthropological Institute of Great Britain and Ireland.
Cambridge Philosophical Society.
Leeds Philosophical Society.
Royal Geological Society of Ireland.
Royal Dublin Society.
Royal Colonial Institute.
Royal Geographical Society.
Institution of Civil Engineers, G. B.
The Victoria Institute.
The Linnean Society.
New Zealand Institute.
Naturwissenschaftliche Gesellschaft "Isis," Dresden.
The Literary and Philosophical Society, of Liverpool.

NOTE.—The donations presented by the above, and some others have already been given in detail.

APPENDIX IV.

The number of books and periodicals issued to members :—

(1) From April 1, 1882, to April 1, 1883 450
(2) " " 1, 1883, " 1, 1884 860

APPENDIX V.

List of periodicals subscribed for :—

American Journal of the Medical Sciences.
Athenæum.
Atlantic Monthly.
Blackwood's Magazine.
Brain.
British Quarterly Review.
Builder.
Century Magazine.
Contemporary Review.
·Critic.
Edinburgh Review.
English Mechanic.
Fortnightly Review.
·Graphic.
Lancet.
London Quarterly Review.
Longman's Magazine.
Macmillan's Magazine.
Mind.
Nature.
Nineteenth Century.
North American Review.
Popular Science Monthly.
Princeton Review.
Punch.
Scientific American.
Scientific American Supplement.
Times, Weekly.
Westminster Review.

To the above have been added for the current year :—

Illustrated London News.
Saturday Review.
English Illustrated Magazine.
Harper's Monthly Magazine.
The Week.

The following were discontinued at the end of 1883 :—

The Builder.
St. James's Gazette.
Critic.
The Medical News.

APPENDIX VI.

Periodicals presented to the Institute, and the names of the donors :—

Das Echo—W. H. VanderSmissen, Esq., M.A.
Le Temps, Paris—Dr. C. W. Covernton.
Spectator—Prof. Hutton.
Le Figaro, for 1883. } Geo. E. Shaw, Esq., B.A.
Le Courrier de l'Europe, for 1884. }

On motion of Mr. J. C. Dunlop, seconded by Mr. Alan Macdougall, the Report was adopted.

The following Officers and Members of Council nominated at last meeting were elected for the ensuing year :

President, W. H. Ellis, Esq., M. A., M. B.
First Vice-President, George Murray, Esq.

Second Vice-President, George Kennedy, Esq., M. A., LL.D.
Third Vice-President, E. A. Meredith, Esq., LL.D.
Treasurer, John Notman, Esq.
Recording Secretary, James Bain, jun., Esq.
Corresponding Secretary, W. H. VanderSmissen, Esq., M. A.
Librarian, George E. Shaw, Esq., B. A.
Curator, David Boyle, Esq.

MEMBERS OF COUNCIL.

James Loudon, Esq., M. A., F. R. S. C.
J. M. Buchan, Esq., M. A.
Alan Macdougall, Esq., C. C., F. R. S. C.
P. H. Bryce, Esq., M. A., M. D.
Daniel Wilson, Esq., LL.D., F. R. S. E., F. R. S. C.
Alexander Marling, Esq., LL.B.

On motion of W. H. VanderSmissen, M. A., it was resolved: "That in Section III, Par. 6, of the Regulations, the words "an Editor" * be inserted after the word "Librarian."

It was moved by Mr. Alan Macdougall, and seconded by Mr. B. B. Hughes: "That the thanks of the Institute be presented to Mr. J. M. Buchan, the retiring President, in recognition of his valuable services rendered during the past year." Carried.

It was moved by Mr. Macdougall seconded by Dr. Cassidy: "That the thanks of the Institute be tendered to the retiring members of the Council in recognition of their valuable services during their term of office." Carried.

* The Rev. Henry Scadding, D.D., was elected Editor at a meeting of Council held on May 31st, 1884.