primes (§47–§57). Finally, from the relation between the solvable irreducible equation of prime degree n and the pure uni-serial Abelian equation of degree $n-1$, the necessary and sufficient forms of the roots of the irreducible solvable equation of prime degree n are shown to be determinable for all cases in which $n-1$ is either the continued product of a number of distinct primes, or four times the continued product of a number of distinct odd primes (§58–§64).

PRELIMINARY.

Corollary from a Law of Kronecker.

§4. It was proved by Kronecker that, n being any integer, the primitive n^{th} roots of unity are the roots of an irreducible equation, that is, of an irreducible equation with rational coefficients. We shall have occasion to make use of the following Corollary from this law: Let w and w' be two primitive n^{th} roots of unity, and let $F(w)$ be a rational function of w. Then, if $F(w)=0$, $F(w')=0$. For, by hypothesis,

$$F(w) = hw^s + h_1 w^{s-1} + \text{ etc.} = 0,$$

where h, h_1, etc., are rational. We assume s to be less than n, and h to be distinct from zero; therefore

$$h^{-1}\{F(w)\} = w^s + h^{-1}h_1 w^{s-1} + \text{ etc.} = 0.$$

Therefore w is a root of the equation $\phi(x) = x^s + h^{-1}h_1 x^{s-1} + \text{etc.} = 0$. If $\psi(x) = 0$ be the equation whose roots are the primitive n^{th} roots of unity, w is a root of the equation $\psi(x) = 0$. Therefore the equations $\phi(x) = 0$ and $\psi(x) = 0$ have a root in common. But, by Kronecker's law, the equation $\psi(x) = 0$ is irreducible. Therefore $\phi(x)$ is divisible by $\psi(x)$ without remainder. This implies that all the roots of the equation $\psi(x) = 0$ are roots of the equation $\phi(x) = 0$. Therefore $\phi(w') = 0$. Therefore $F(w') = 0$.

Principles established by Abel.

§5. Let $f(x) = 0$ be a uni-serial Abelian equation of the n^{th} degree, and let its roots, in the order in which they circulate, be the terms in (1). It is known (see Serret's Cours d'Algèbre supérieure, Vol. II, page 500, third edition) that

$$x_1 = R_0^{\frac{1}{n}} + R_1^{\frac{1}{n}} + R_2^{\frac{1}{n}} + \ldots + R_{n-1}^{\frac{1}{n}},$$