§61. PROPOSITION XXII. The auxiliary biquadratic $\varphi(x) = 0$ either has all its roots rational, or has a sub-auxiliary (see §53) of the second degree, or is irreducible.

It will be kept in view that the sub-auxiliaries are, by the manner of their formation, irreducible. First, let the series (54), containing the roots of the sub-auxiliary $\varphi_1(x) = 0$ consist of a single term $⅃_1$. Then, by Prop. III., $⅃_1$ is rational. Therefore, by Prop. XX., all the roots of the auxiliary are rational. Next, let the series (54) consist of the two terms $⅃_1$ and $⅃_2$. By this very hypothesis, the auxiliary biquadratic has a quadratic sub-auxiliary. Lastly, let the series (54) contain more than two terms. Then it has the three terms $⅃_1$, $⅃_2$, $⅃_3$. We have shown that these must be severally equal to terms in (64). Neither $⅃_2$ nor $⅃_3$ is equal to $⅃_1$. They cannot both be equal to $h_1^5\,⅃_1^4$. Therefore one of them is equal to one of the terms $a_1^5\,⅃_1^2$, $e_1^5\,⅃_1^3$. But in §60 it appeared that, if $⅃_2$ be equal either to $a_1^5\,⅃_1^2$ or to $e_1^5\,⅃_1^3$, all the terms in (64) are roots of the irreducible equation of which $⅃_1$ is a root. The same thing holds regarding $⅃_3$. Therefore, when the series (54) contains more than two terms, the irreducible equation which has $⅃_1$ for one of its roots has the four unequal terms in (64) for roots; that is to say, the auxiliary biquadratic is irreducible.

§62. Let $5u_1 = ⅃_1^{\frac{1}{5}}$, $5u_2 = a_1\,⅃_1^{\frac{2}{5}}$, $5u_3 = e_1\,⅃_1^{\frac{3}{5}}$, $5u_4 = h_1\,⅃_1^{\frac{4}{5}}$; and, n being any whole number, let S_n denote the sum of the n^{th} powers of the roots of the equation $F(x) = 0$. Then

$$S_1 = 0;\ S_2 = 10\,(u_1 u_4 + u_2 u_3);\ S_3 = 15\,\{\Sigma(u_1 u_2^2)\};$$

$$S_4 = 20\,\{\Sigma(u_1^3 u_2)\} + 30\,(u_1^2 u_4^2 + u_2^2 u_3^2) - 120\,u_1 u_2 u_3 u_4;$$

$$S_5 = 5\,\{\Sigma(u_1^5)\} + 100\,\{\Sigma(u_1^3 u_3 u_4)\} + 150\,\{\Sigma(u_1 u_3^2 u_4^2)\};$$

where such an expression as $\Sigma(u_1 u_2^2)$ means the sum of all such terms as $u_1 u_2^2$; it being understood that, as any one term in the circle u_1, u_2, u_4, u_3, passes into the next, that next passes into its next, u_3 passing into u_1.

THE ROOTS OF THE AUXILIARY BIQUADRATIC ALL RATIONAL.

§63. Any rational values that may be assigned to $⅃_1$, a_1, e_1, and h_1 in r_1, taken as in (62), make r_1 the root of a rational equation of the fifth degree, for they render the values of S_1, S_2, etc., in §62, rational. In fact, $S_1 = 0$, $25\,S_2 = 10\,⅃_1\,(h_1 + a_1 e_1)$, and so on.