SOLVABLE IRREDUCIBLE EQUATIONS OF PRIME DEGREES.

§58. Let $f(x) = 0$ be a solvable irreducible equation of the prime degree n. Even if it be not a pure Abelian, the necessary and sufficient forms of its roots can, by means of the problems solved above, be determined *in all cases in which n is either the continued product of a number of distinct primes or four times the continued product of a number of distinct odd primes.*

§59. It is known that the root of the equation is of the form

$$k + R_1^{\frac{1}{n}} + R_2^{\frac{1}{n}} + \ldots . + R_{n-1}^{\frac{1}{n}}, \tag{138}$$

where k is rational; and

$$R_1, R_2, \ldots . , R_{n-1}, \tag{139}$$

are the roots of an equation of the n^{th} degree, that is, of an equation with rational coefficients. Let this equation be $\phi(x) = 0$. The root of the equation $f(x) = 0$ may also be expressed in the form

$$k + R_1^{\frac{1}{n}} + a_1 R_1^{\frac{2}{n}} + b_1 R_1^{\frac{3}{n}} + \ldots . + c_1 R_1^{\frac{n-1}{n}}, \tag{140}$$

where a_1, b_1, etc., are rational functions of R_1. The separate members of the expression (140) are severally equal to those of the expression (138); that is,

$$R_2^{\frac{1}{n}} = a_1 R_1^{\frac{2}{n}},\ R_3^{\frac{1}{n}} = b_1 R_1^{\frac{3}{n}}, \ldots . , R_{n-1}^{\frac{1}{n}} = c_1 R_1^{\frac{n-1}{n}}. \tag{141}$$

Therefore $R_2 = a_1^n R_1^2$. Hence, since a_1 is a rational function of R_1, R_2 is a rational function of R_1. The expression R_1 is thus the root of a pure Abelian equation, which, moreover, is known to be capable of having its roots arranged in a single circulating series, and therefore to be what we have called a pure uni-serial Abelian. A quotation from a remarkable memoir which was presented in 1853 by Herr Leopold Kronecker to the Academy of Berlin, and of which a translation is given in Serret's Cours d'Algèbre Supérieure (Vol. II, p. 654, 3d edition), will show how the case stands. In Kronecker's memoir μ indicates the degree of the equation, and is therefore our n, while A, B, C, etc., are quantities involved rationally in the coefficients of the equation $f(x) = 0$. Having given, after Abel, what are substantially the two forms (138) and (140), Kronecker adds: "Il est bien vrai que toute fonction algébrique, satisfaisant au problème proposé, doit pouvoir se mettre sous ces deux formes; mais ces formes sont encore trop générales, c'est-à-dire qu'elles renferment des fonctions algébriques qui ne répondent pas à la question. Je les ai donc étudiées de plus près, et j'ai trouvé d'abord que parmi les fonctions renfermées dans la forme (2)" [the