This may be done in several ways, which we shall here illustrate :

1st. $4n^3 = (n+1)^4 - n^4 - 6n^2 - 4n - 1$

$$4(n-1)^3 = n^4 - (n-1)^4 - 6(n-1)^2 - 4(n-1) - 1$$
$$4(n-2)^3 = (n-1)^4 - (n-2)^4 - 6(n-2)^2 - 4(n-2) - 1$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$4 \cdot 3^3 = 4^4 - 3^4 - 6 \cdot 3^2 - 4 \cdot 3 - 1$$
$$4 \cdot 2^3 = 3^4 - 2^4 - 6 \cdot 2^2 - 4 \cdot 2 - 1$$
$$4 \cdot 1^3 = 2^4 - 1^4 - 6 \cdot 1^2 - 4 \cdot 1 - 1$$

$\therefore$ adding— $4\Sigma n^3 = (n+1)^4 - 1^4 - 6\Sigma n^2 - 4\Sigma n - n$

In which we have to substitute the values of Σn^2 and Σn, namely, $\frac{1}{6}(n)(n+1)(2n+1)$ and $\frac{1}{2}n(n+1)$, and reduce.

The objection to this method is that in order to find Σn^3 we must have Σn^2 and Σn.

2nd. In the series of cube numbers, the nth term is n^3. But if s_n denotes the sum of n terms, and s_{n-1} the sum of $n-1$ terms, the nth term is $s_n - s_{n-1}$, which is, therefore, n^3. Now, assume $s_n = an + bn^2 + cn^3 + dn^4$.

Then $s_{n-1} = a(n-1) + b(n-1)^2 + c(n-1)^3 + d(n-1)^4$.

$\therefore$ $s_n - s_{n-1} = n^3 = a - b + c - d + (2b - 3c + 4d)n + (3c - 6d)n^2 + 4dn^3$, and, equating co-efficients of like power of n, $4d = 1$, $3c - 6d = 0$, $2b - 3c + 4d = 0$, $a - b + c - d = 0$. Whence $d = \frac{1}{4}$, $c = \frac{1}{2}$, $b = \frac{1}{4}$, $a = 0$.

$$\therefore \quad s_n = \Sigma n^3 = \frac{n^2}{4}(1 + 2n + n^2) = \left(\frac{n(n+1)}{2}\right)^2 = (\Sigma n)^2$$

(For further explanation see Dupuis' Algebra, art 199.)

3rd. Taking $u_n = 1 + 8 + 27 + 64 + 125 + \ldots n^3$

First diff.	7	19	37	61...
Second "	12	18	24	
Third "	6	6		

$\therefore$ $u_0 = 1$, $\Delta = 7$, $\Delta^2 = 12$ $\Delta^3 = 6$. And substituting, in the formula, $\Sigma u_n = nu_0 + {}^nC_2\Delta + {}^nC_3\Delta^2 + {}^nC_4\Delta^3$, we obtain the same expression as before. (See Dupuis' Algebra, art 142.)

4th. Expressing n^3 as the sum of factorials we have $n^3 = n(n+1)(n+2) - 3n(n+1) + n$, and integrating between the limits o and n, $\Sigma n^3 = \frac{n(n+1)(n+2)(n+3)}{4} - \frac{3n(n+1)(n+2)}{3} + \frac{n(n+1)}{2}$, giving a result as before.

This last method requires some knowledge of the operations of finite differences.

(b). Deduce that the cube of any integer is the difference of two square integers. Find the two square integers whose difference is 512.

$$s_n = \Sigma n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 \qquad s_{n-1} = \Sigma(n-1)^3 = \left\{\frac{n(n-1)}{2}\right\}^2$$

$$\therefore s_n - s_{n-1} = n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 - \left\{\frac{n(n-1)}{2}\right\}^2$$

Also, if $n^3 = 512$, $n = 8$; and the numbers are $\left(\frac{8.9}{2}\right)^2$ and $\left(\frac{8.7}{2}\right)^2$, or $(36)^2$ and $(28)^2$