$$R = b\sqrt{1 + \frac{T_1^2}{(2T_0)^2}} = b\,.\,T_1\sqrt{\frac{1}{T_1^2} + \frac{1}{(2T_0)^2}} =$$

$$b\,.\,\frac{T_1}{T} = -1.008\,.\,\frac{147}{137.5}$$

$R = -1.075$ feet, therefore

$$z = -4.8 - 1{,}075\,.\,e^{-\frac{t}{389}}\sin\left[69^\circ\,18' + \frac{t}{147}\right] + 1.008\cos\frac{t}{20}$$

$$v = +\,.00782\,e^{-\frac{t}{389}}\sin\frac{t}{147} - .0503\sin\frac{t}{20}$$

If at the time $t = \pi.\mathrm{T}$ the fluctuation of the outflow ceases it follows that

$$z_{\mathrm{T}} = -4.8 - 1.075\,e^{-\frac{20}{389}\pi}\sin\left(69^\circ\,18' + \frac{20}{147}\right) - 1.01 = 2.88\text{ ft.}$$

$$v_{\mathrm{T}} = .00782\,e^{-\frac{20}{389}\pi}\sin\frac{20}{147} = .00275 \text{ feet per second.}$$

For the movement that follows, we may use the equations (32) and (35)

$$z = -\epsilon h_1 + R_1\,e^{-\frac{t}{2T_0}}\sin\left(\beta_1 + \frac{t}{T_1}\right)$$

$$s = \frac{R_1}{T}e^{-\frac{t}{2T_0}}\sin\left(\gamma - \beta_1 - \frac{t}{T_1}\right) \quad \text{with } tg\,\gamma = \frac{2T_0}{T_1}$$

and R_1 and β_1 may be determined by the conditions $t = o$, $z = z_{\mathrm{T}}$, $s = s_{\mathrm{T}}$. We obtain

$R_1 \,.\, \sin\beta_1 = -9.6 + 2.88 + 4.8 = -1.92$ feet

$R_1 \,.\, \cos\beta_1 = .00275\,.\,147 - 1.92\,\frac{147}{389} = -.321$ feet

$R_1 = 1.95$ feet; β_1 lies in the third quadrant, i.e., negative.

$tg\,\beta_1 = 5.98975$ $\quad\beta_1 = -99^\circ\,30'$ $\quad$ Arc $\beta_1 = -1.737$

$\gamma = 68^\circ\,18'$ $\quad$ arc $\gamma = 1{,}210$

$$z = -4.8 + 1.95\,e^{-\frac{t}{389}}\,.\,\sin\left[-99^\circ\,30' + \frac{t}{147}\right]$$

$$s = -.0142\,e^{-\frac{t}{389}}\,.\,\sin\left[11^\circ\,12' + \frac{t}{147}\right]$$

Since at the time $t = o$ the velocity s is positive, a maximum first occurs. The time from the beginning of that maximum until the beginning of the second phase is $t_1 = 147\,(\gamma - \beta_1) = 433$ seconds.

$$z_1\min = -4.8 + 1.95\,e^{-\frac{433}{389}}\sin\gamma = -4.19\text{ feet}$$

The general course of the movement may be seen in Fig. 6.

In case of a continuous fluctuation of flow, expressed by the law $q = \epsilon Q_1\left(1 + f\sin\frac{t}{\mathrm{T}}\right)$ with $\epsilon = .5$ and $f = 1$ the forced oscillation becomes nearly a constant harmonic oscillation of the amplitude one foot measured from $z = -4.8$ feet and for a duration of period of 137.6 seconds.

The case of resonance might occur if $\mathrm{T} = T = 137.5$ seconds. This would be the case for an amplitude of the forced oscillation (measured from $z = -4.8$ feet) equal to

$$b = \pm\,\epsilon.f.h_1\sqrt{\frac{T_0^2}{\mathrm{T}^2} + 1\,.\,\frac{T_0}{T}} = \pm\,11.7\text{ feet.}$$

Herein the duration of period would be $T\,.\,2\pi = 137.5\,.\,2\,.\,\pi$ sec $= 863 = 14'\,23''$. It does not seem impossible for such periods to occur in regular operations as for railways, so that this investigation should be useful.

The preceding method, introducing a periodical function for the determination of a variable outflow in function of the time may be enlarged, introducing

$$q = \epsilon Q_1\left[f_0 + f_1\sin\left(\phi_1 + \frac{t}{\mathrm{T}}\right) + f_2\sin\left(\phi_2 + 2\,\frac{t}{\mathrm{T}}\right) + \ldots\right] \quad (80)$$

where we find the values for f and ϕ corresponding to a given variation of q by means of the Fourier series. The integration of the differential equation thus obtained involves no difficulty whatever. It is based upon the same method as that given in the last example. The values of z and s have the form

$$z = z_0 + R.e^{-\frac{t}{2T_0}}\sin\left(\beta + \frac{t}{T_1}\right) + \Sigma k_n\sin\left(\psi_n + n\frac{t}{\mathrm{T}}\right)$$

$$(81)$$

$$s = s_0 + \frac{R}{T}e^{-\frac{t}{2T_0}}\sin\left(\gamma - \beta + \frac{t}{T_1}\right) + \Sigma\frac{n.s_n}{\mathrm{T}}\cos\left(\psi_n + n\frac{t}{\mathrm{T}}\right)$$

Naturally the computation requires great care. The graphical demonstration is obtained by superposition of the projections obtained from the polar system with logarithmic spirals $r = R\,e^{-\frac{T_1}{2T_0}\,.\,\frac{t}{T_1}}$ and the circles with the radii k.

(To be continued.)