where R_1 is a rational function of the primitive n^{th} root of unity w and of the known quantities involved in the coefficients of θ; and, z being any integer, R_z is derived from R_1 by changing w into w^z. Putting

$$x_{s+1} = R_0^{\frac{1}{n}} + w^s R_1^{\frac{1}{n}} + w^{2s} R_2^{\frac{1}{n}} + \dots + w^{(n-1)s} R_{n-1}^{\frac{1}{n}}, \tag{2}$$

the n roots of the equation $f(x) = 0$ are obtained by giving s in x_{s+1} successively the values $0, 1, 2, \dots, n-1$. Therefore $nR_0^{\frac{1}{n}}$ is the sum of the roots of the equation; consequently, $R_0^{\frac{1}{n}}$ is rational. An equation of the type

$$(R_z R_1^{-z})^{\frac{1}{n}} = F(w) \tag{3}$$

subsists for every integral value of z, $F(w)$ being a rational function of w and of the known quantities involved in the coefficients of θ. As w may be any one of the primitive n^{th} roots of unity, if the general primitive n^{th} root of unity be w^e, we may suppose w in R_1 to be changed into w^e. The n roots of the equation $f(x) = 0$ will then be obtained by giving t, in the expression

$$R_0^{\frac{1}{n}} + w^t R_e^{\frac{1}{n}} + w^{2t} R_{2e}^{\frac{1}{n}} + \text{etc.} \tag{4}$$

successively the values $0, 1, 2, \dots, n-1$. Abel's investigation shows that the form of the function $F(w)$ in (3) is independent of the particular primitive n^{th} root of unity denoted by w. Hence the change of w into w^e causes equation (3) to become

$$(R_{ez} R_e^{-z})^{\frac{1}{n}} = F(w^e), \tag{5}$$

the symbol F having the same meaning for every value of e.

Fundamental Element of the Root

§6. Because R_0, R_2, etc., are derived from R_1 by changing w into w^0, w^2, etc., the root x_1 can be constructed when R_1 is given. We may therefore call R_1 the fundamental element of the root. Examples of the way in which the root is constructed from its fundamental element will present themselves in the course of the paper.

A Certain Rational Function of the Primitive n^{th} Root of Unity, n being an Odd Prime Number.

§7. Taking n an odd prime number, there is a certain rational function of the primitive n^{th} root of unity w, of which we shall have occasion to make