The Auxiliary Biquadratic with a Quadratic Sub-Auxiliary.

§64. Proposition XXIII. In order that r_1, taken as in (62), may be the root of an irreducible equation $F(x) = 0$ of the fifth degree, whose auxiliary biquadratic has a quadratic sub-auxiliary, it must be of the form

$$r_1 = \tfrac{1}{5}\left\{ (\Delta_1^{\frac{1}{5}} + \Delta_2^{\frac{1}{5}}) + (a_1 \Delta_1^{\frac{2}{5}} + a_2 \Delta_2^{\frac{2}{5}}) \right\}; \qquad (65)$$

where Δ_1 and Δ_2 are the roots of the irreducible equation $\phi_1(x) = x^2 - 2\,px + q^5 = 0$; and $a_1 = b + d \surd\,(p^2 - q^5)$, $a_2 = b - d \surd\,(p^2 - q^5)$; p, b and d being rational; and the roots $\Delta_1^{\frac{1}{5}}$ and $\Delta_2^{\frac{1}{5}}$ being so related that $\Delta_1^{\frac{1}{5}}\,\Delta_2^{\frac{1}{5}} = q$.

By Prop. VII., when a quintic equation is of the first (see §30) class, the auxiliary biquadratic is irreducible. Hence, in the case we are considering, the quintic is of the second class. The quadratic sub-auxiliary may be assumed to be $\phi_1(x) = x^2 - 2\,px + k = 0$, p and k being rational. By Prop. XXI., the roots of the equation $\phi_1(x) = 0$ are Δ_1 and $h_1^5\,\Delta_1^4$. Therefore $k = (h_1\,\Delta_1)^5$; or, putting q for $h_1\,\Delta_1$, $k = q^5$. By the same proposition, $h_1\,\Delta_1$ is rational. Therefore q is rational. Hence $\phi_1(x)$ has the form specified in the enunciation of the proposition. Next, by Proposition XVI., there is a fifth root of unity t such that $t\,\Delta_2^{\frac{1}{5}} = h_1\,\Delta_1^{\frac{4}{5}}$. If we take t to be unity, which we may do by a suitable interpretation of the symbol $\Delta_2^{\frac{1}{5}}$, $\Delta_2^{\frac{1}{5}} = h_1\,\Delta_1^{\frac{4}{5}}$. This implies that $e_1\,\Delta_1^{\frac{3}{5}} = a_2\,\Delta_2^{\frac{2}{5}}$, a_2 being what a_1 becomes in passing from Δ_1 to Δ_2. Substituting these values of $e_1\,\Delta_1^{\frac{3}{5}}$ and $h_1\,\Delta_1^{\frac{4}{5}}$ in (62), we obtain the form of r_1 in (65), while at the same time $\Delta_1^{\frac{1}{5}}\,\Delta_2^{\frac{1}{5}} = h_1\,\Delta_1 = q$. The forms of a_1 and a_2 have to be more accurately determined. By Prop. XIV., $\Delta_1^{\frac{1}{5}}$ is the only principal surd that r_1, as presented in (62), contains. Therefore a_1 involves no surd that does not occur in Δ_1; that is to say, $\surd\,(p^2 - q^5)$ is the only surd in a_1. Hence we may put $a_1 = b + d \surd\,(p^2 - q^5)$; b and d being rational. But a_2 is what a_1 becomes in passing from Δ_1 to Δ_2. And Δ_2 differs from Δ_1 only in the sign of the root $\surd\,(p^2 - q^5)$. Therefore

$$a_2 = b - d \surd\,(p^2 - q^5).$$

§65. Any rational values that may be assigned to b, d, p and q in r_1, taken as in (65), make r_1 the root of a rational equation of the