Suppose in figure 1, AB to be a circular curve with centre C. Now if we begin at B to reduce the curvature directly with distance, continuing this reduction until curvature is zero, maintaining the same central angle I. as in circular curve, the new curve will pass outside AB, having a tangent of FG, parallel to DE, the tangent of AB, and at a certain distance, KA, from it. As is customary A is called the P.C.; F the P.T.C., and B the P.C[1]. Again let θ be the angle which the curve at any point L, makes with the initial tangent FG, s the length of arc FL. Then since by hypothesis the curvature at F is zero and increases with the distance from P; therefore the curvature of any point L, distant s from P, is equal to a constant multiplied by s. For convenience to avoid fractions later, this curvature is expressed by $2ks$, k being the constant depending on the rate of change of curvature.

The radius of the curvature equals $\frac{ds}{d\theta}$, θ being any angle, and and curvature varies inversely as the radius of curvature, we have $\frac{d\theta}{ds} = 2ks$, or, after integrating, $\theta = ks^2$.

If y is the ordinate, we have $dy = ds \sin \theta = ds \sin ks^2$ (a). but $\sin a$ (a being any angle) in series is equal to

$$* \quad a - \frac{a^3}{3!} + \frac{a^5}{5!} - \frac{a^7}{7!} \; rc.$$

$$\text{therefore in } (a)\ dy = ds\,(ks^2 - \frac{k^3s^6}{3!} + \frac{k^5s^{10}}{5!} \ \cdots$$

$$\text{therefore } y = \frac{ks^3}{3} - \frac{k^3s^7}{7.3.2} + \frac{k^5s^{11}}{11.5.4.3.2} \ \cdots$$

$$= \frac{ks^3}{3} - \frac{k^3s^7}{42} + \frac{k^5s^{11}}{1320} \quad \text{(b)}$$

* $3! = 1.2.3$; $5! = 1.2.3.4.5$. and so on.

In the same way, for the abscissa x, we have $dx = ds \cos \theta = ds \cos ks^2$

$$\text{and } \cos a \text{ in series} = 1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \cdots$$

$$\text{therefore } dx = ds\left(1 - \frac{k^2s^4}{2!} + \frac{k^4s^8}{4!}\right) \ \cdots$$

$$\therefore \quad x = s - \frac{k^2s^5}{10} + \frac{k^4s^9}{216} - \cdots \text{(c)}$$

From these two equations (b) and (c) the length s of the arc FL is expressed in terms of the co-ordinates of the curve, but before using them the investigation must be carried further.

After passing this point two methods of procedure are open. First, reducing the curvature from the P. C.[1], in the same ratio as it was increased. Equation (b) will then give the radial ordinates for this part of the curve, besides the ordinates—from the tangent—for the first part. Also the curve and offset will bisect each other at P. If we should now place the second derivative for the first part at P equal to the second derivative of the second part at P, the K will be eliminated, leaving s to be ascertained for any given value of y or AK.

(The work of the above has been omitted here; it is simple application of the calculus, and, if interested, the reader may readily follow the steps through himself.)

Second—By continuing the curve to B, and placing the curve so that it shall be the circle of curvature, we will get for flat curves the same result as in the last case. When the offsets are larger, AK and the curve will not bisect each other, and the ordinates from the circle will differ slightly from the corresponding ones of the tangent. The author claims this to be theoretically the correct method, particularly if the curve is to be run with a transit. It also gives simpler formulæ.

Now from equation (1)

$$\text{(any arc) } S = \frac{ds}{d\theta} = \frac{1}{2ks} \text{ and } k = \frac{\theta}{s^2}$$

$$\therefore \quad = \frac{s}{2\theta}.$$

But at its limits S=R, and at the same time $2\,\theta = 2\,\mathrm{I}_{(\text{arc})}$; the s then being s'', the distance from the P. T. C. to the P. C.[1]

$$\text{Then } \mathrm{I}^\circ = \frac{s''}{2\ \mathrm{R} \sin 1^\circ} = 1.86\sqrt{\mathrm{FD}.}$$