in (112), zero included, which make $w^{g\sigma+1}$ a primitive n^{th} root of unity. Let two of these values of g be g_1 and g_2. Put

$$g_1\sigma + 1 = q_1 s + r_1$$

and

$$g_2\sigma + 1 = q_2 s + r_2,$$

q_1 and q_2 being whole numbers, and r_1 and r_2 whole numbers less than s. Suppose, if possible, that $r_1 = r_2$; then

$$(g_1 - g_2)\sigma = (q_1 - q_2)s,$$

which, as above, makes σ a multiple of s, and is therefore impossible. Consequently, the $s-1$ residues after multiples of s have been rejected from the $s-1$ different values of $g\sigma + 1$ are all different from one another.

§ 53. It can now be shown that equations

$$\left.\begin{aligned} (R_{mz}R_m^{-z})^{\frac{1}{4}} &= p_m \\ \text{and} \quad (R_{emz}R_{em}^{-z})^{\frac{1}{4}} &= p_{em} \end{aligned}\right\} \tag{114}$$

subsist for every integral value of z and every value of e that makes w^e a primitive n^{th} root of unity, p_m being a rational function of w^m, and p_{em} being what p_m becomes when w is changed into w^e. By (3) and (5), because R_1 is the fundamental element of the root of a pure uni-serial Abelian equation of the n^{th} degree,

$$(R_{mz}R_m^{-z})^{\frac{1}{n}} = k_1,$$

and

$$(R_{emz}R_{em}^{-z})^{\frac{1}{n}} = k_e,$$

k_1 being a rational function of w, and k_e being what k_1 becomes when w is changed into w^e. Therefore

$$\left.\begin{aligned} (R_{mz}R_m^{-z})^{\frac{1}{4}} &= k_1^m \\ \text{and} \quad (R_{emz}R_{em}^{-z})^{\frac{1}{4}} &= k_e^m \end{aligned}\right\} \tag{115}$$

In the second of these equations, give e a value, say c, falling under the form (112). Then

$$(R_{cmz}R_{cm}^{-z})^{\frac{1}{4}} = k_c^m. \tag{116}$$

Since σ is a multiple of 4, we may put $c = 4d + 1$. Therefore $cm = dn + m$. Therefore $w^{cm} = w^m$, and $w^{cmz} = w^{mz}$. Therefore (116) may be written

$$(R_{mz}R_m^{-z})^{\frac{1}{4}} = k_c^m.$$

This, compared with the first of equations (115), gives us

$$k_c^m = k_1^m. \tag{117}$$

Since k_1^m is a rational function of a primitive n^{th} root of unity, and the first of the cycles (101) contains all the primitive s^{th} roots of unity, we may put

$$k_1^m = a_0 + a_1 w^\sigma + a_2 w^{\sigma\lambda} + \ldots + a_{s-1} w^{\sigma\lambda^{s-2}}, \tag{118}$$