3. (*a*). Solve $\left.\begin{array}{l} xy(x+y)=30 \\ x^3+y^3=35 \end{array}\right\}$

$(x+y)^3=x^3+y^3+3x^2y+3xy^2=35+3\cdot 30=125$. $\therefore x+y=5$, and hence $xy=6$. Then $x-y=1$; whence $x=3$, $y=2$.

(*b*). Solve $x^5-1=0$. This factors into $(x-1)(x^4+x^3+x^2+x+1)=0$. $\therefore x=1$ is one root.

The factor $x^4+x^3+x^2+x+1=0$ is a reciprocal equation. Divide through by x^2 and put $x+{}^1/_x=z$

Then $x^2+\frac{1}{x^2}+x+\frac{1}{x}+1=0$

But $z^2=x^2+\frac{1}{x^2}+2$; and the equation becomes, $z^2+z-1=0$

Whence $z=\frac{-1\pm\sqrt{5}}{2}$ and $x^2-xz+1=0$, or $x=\frac{z\pm\sqrt{z^2-4}}{2}$

Whence, by substitution for z, $x=\frac{1}{4}[\sqrt{5}-1+\sqrt{(2\sqrt{5}-10)}]$, which gives 4 values for x by varying signs of surds.

(*c*). Solve $\left.\begin{array}{l} x^3(y+3)+y^3(x+3)=183 \\ x+y=5. \end{array}\right\}$

The first equation gives $xy(x^2+y^2)+3(x^3+y^3)=183$.

But $x^2+y^2=(x+y)^2-2xy=25-2xy$.

And $x^3+y^3=(x+y)^3-3xy(x+y)=125-15xy$.

Whence $(xy)^2+10xy=96$; and $xy=6$ or -16.

Thence, having xy and $x+y$ we find

$x=3,\ 2,\ \frac{1}{2}(5+\sqrt{89}),\ \frac{1}{2}(5-\sqrt{89})$

$y=2,\ 3,\ \frac{1}{2}(5-\sqrt{89}),\ \frac{1}{2}(5+\sqrt{89})$;

the simultaneous values being arranged in pairs.

4. (*a*). Eliminate x and y from the equations $x+y=a$, $x^2+y^2=b^2$ $x^3+y^3=c^3$

Multiply together the first and second. Then $x^3+y^3+xy(x+y)=c^3+axy=ab^2$. $\therefore xy=\frac{ab^2-c^3}{a}$

But, squaring the first $x^2+y^2=a^2-2xy$. Whence $a^3+2c^3=3ab^2$; the relation required.

(*b*). If $z-a : z-b=z-c : z-d$, and $a^2+ad+d^2=b^2+bc+c^2$

Then $z=a+b+c+d$.

This is most easily shown by inverse working, i. e., by assuming the first and third relation and showing the necessity of the second. Thus the proportion becomes: $b+c+d : a+c+d=a+b+d : a+b+c$

Multiply extremes and means, equate and reduce, and the second relation results.

5 (*a*). If A varies directly as B when C is constant, and varies directly as C when B is constant, then A varies as BC when both B and C vary.

Let C be constant and let B take two values, B and B′ and let the corresponding values of A be A and A′. Then A : A′ = B : B′.

Now let B′ remain constant, while C varies to C′ and let A″ be the corresponding value of A. Then A : A″ = C : C′.

And compounding these proportions A : A″ = BC : B′C′, or A varies as BC.

(*b*). Illustrate the theorem in (*a*) by referring to the area, base and altitude of a triangle. If A, *b*, *a* be the area, base and altitude respectively: