frequent use. It will be convenient to describe it here, and to point out two of its properties. Let

$$w,\ w^{\lambda},\ w^{\lambda^2},\ \ldots\ldots,\ w^{\lambda^{n-2}}, \tag{6}$$

be a cycle containing all the primitive n^{th} roots of unity. The number λ may be assumed to be less than n. With a view to convenience in printing, the indices of the powers of w in (6) may be written

$$1,\ \lambda,\ \alpha,\ \beta,\ \ldots\ldots,\ \delta,\ \varepsilon,\ \theta; \tag{7}$$

that is to say, $\alpha = \lambda^2$, $\beta = \lambda^3$, and so on. Take P_1 a rational function of w, and, z being any integer, let P_z be what P_1 becomes when w is changed into w^z. Then the function to which we desire to call attention is

$$P_1^{\theta} P_{\lambda}^{\varepsilon} P_{\alpha}^{\delta} \ldots\ldots P_{\delta}^{\alpha} P_{\varepsilon}^{\lambda} P_{\theta}. \tag{8}$$

The *subscripts* of the factors of the expression (8) are the terms in (7), while the *indices* are the terms in (7) in reverse order. The expression (8) may be denoted by the symbol ϕ_1. From ϕ_1, as expressed in (8), derive ϕ_z by changing w into w^z, z being any integer. Then

$$\left.\begin{aligned} \phi_1 &= P_1^{\theta} P_{\lambda}^{\varepsilon} P_{\alpha}^{\delta} \ldots\ldots P_{\delta}^{\alpha} P_{\varepsilon}^{\lambda} P_{\theta} \\ \phi_{\lambda} &= P_1 P_{\lambda}^{\theta} P_{\alpha}^{\varepsilon} \ldots\ldots \quad P_{\varepsilon}^{\alpha} P_{\theta}^{\lambda} \\ \phi_{\alpha} &= P_1^{\lambda} P_{\lambda} P_{\alpha}^{\theta} \ldots\ldots \qquad P_{\theta}^{\alpha} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \phi_{\theta} &= P_1^{\varepsilon} P_{\lambda}^{\delta} \quad \ldots\ldots P_{\delta}^{\lambda} P_{\varepsilon} P_{\theta}^{\theta} \end{aligned}\right\} \tag{9}$$

The second of these equations is derived from the first by changing w into w^{λ}. This, since $\alpha = \lambda^2$ and $\beta = \lambda^3$, and so on, causes w^{λ} to become w^{α}, and w^{α} to become w^{β}, and so on. Hence it causes P_{λ} to become P_{α}, P_{α} to become P_{β}, and so on. Thus the second of equations (9) is obtained. The rest are obtained in a similar manner.

§8. One property which the function ϕ_1 possesses is that $\phi_0^{\frac{1}{n}}$ has a rational value. For

$$\phi_0 = P_0^{\theta} P_0^{\varepsilon} \ldots\ldots P_0^{\lambda} P_0 = P_0^{t},$$

where

$$t = 1 + \lambda + \lambda^2 + \ldots\ldots + \lambda^{n-2} = \frac{\lambda^{n-1} - 1}{\lambda - 1}.$$

Because (6) is a cycle of primitive n^{th} roots of unity, $\lambda^{n-1} - 1$ is a multiple of n. And, since λ is less than n, $\lambda - 1$ is not a multiple of n; therefore t is a multiple of n. Put $t = mn$; then

$$\phi_0 = (P_0^{m})^{n};$$

consequently, one of the values of $\phi_0^{\frac{1}{n}}$ is the rational quantity P_0^{m}.