right angle, and thence it follows that the triangle whose angle is right is the greatest.

Cor. 1. Of all parallelograms on the same base and with same perimeter, the rectangle is the greatest. For the other sides are equal, and the triangle which is half the rectangle is greater than a triangle which is half of any of the other parallelograms.

Cor. 2. A square is greater than any other parallelogram of equal perimeter. For, if the parallelogram be not a rhombus, construct a rhombus equiangular with it and of same perimeter. (IV. Cor. 3.) The rhombus will be greater than the parallelogram by III. But by IV., Cor. 1, the rhombus is less than the square. Therefore the square is still greater than the parallelogram.

Cor. 3. The space which can be enclosed by an indefinite straight line and a straight line of given length, which is divided into two segments, is greatest when the segments are equal and are perpendicular to each other.

V. In a given indefinite straight line to find a point, from which if two straight lines be drawn to two given points on the same side the given line, their sum should be a minimum.

Let A, B be the two given points, and KL the given straight line. Draw AC at right angles to KL and produce it to D, so that CD is equal to AC. Join BD cutting KL in E. Then AE, EB, drawn to E are together less than AF, FB drawn to any other point F. For $AE = ED$, and $AF = FD$. Therefore $AF + FB = DF + FB$, and $AE + EB = DE + EB = DB$. But $DF + FB$ is greater than DB; therefore $AF + FB$ is greater than $AE + EB$.

VI. The perimeter of an isosceles triangle is less than that of any other equal triangle standing on the same base.

Let ABC be an isosceles triangle, and DBC any other equal triangle upon the same base, so that AD is parallel to BC. Produce BA to E, making AE equal to AB; join ED. Then it may be shown that DE is equal to DC. Therefore $BD + DC = BD + DE$, which is greater than BE, i. e., than $BA + AC$; or the perimeter of ABC is less than that of the equal triangle DBC on the same base.

Cor. 1. If any polygon be not equilateral, another equal polygon may be found of the same number of sides and with a less perimeter.

Let $ABCDE$ be a polygon, and let AB be not equal to BC. On AC as base construct an isosceles triangle AHC equal to ABC. Then AH and HB are together less than AB, BC, and therefore the perimeter of the polygon $AHCDE$ is less than that of the equal polygon $ABCDE$,

Cor. 2. An isosceles triangle is greater than any scalene triangle of equal perimeter and on the same base.

For an isosceles triangle equal to the scalene triangle and on the same base will have a less perimeter, and therefore less than the perimeter of the first isosceles triangle, than which it will therefore be less in area; i.e., the given isosceles triangle is greater in area than the scalene triangle.

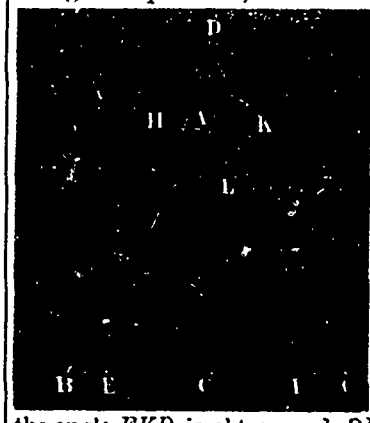
Cor. 3. Hence if any polygon be not equilateral, a greater polygon may be found of the same number of sides and of equal perimeter.

Let $ABCDE$ be a polygon; and let the sides AB, BC be unequal. On AC as base construct an isosceles triangle AHC such that $AH + HC = AB + BC$. Then the isosceles triangle AHC is greater than the triangle ABC of equal perimeter, and therefore the polygon $AHCDE$ is greater than the polygon $ABCDE$ of equal perimeter.

It will readily be seen that we are not justified in inferring *from the preceding* that, the perimeter being given, the area of a polygon is a maximum when it is equilateral, for, in addition to its being doubtful, so far as the preceding goes, whether there is a maximum at all or not, we may make a polygon whose sides are equal assume an infinite number of different *shapes*, and of course these are not all maxima.

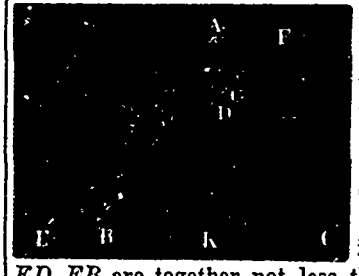
VII. If the base of an isosceles triangle be less than the base of an equal equilateral triangle, its side shall be greater than the side of the equilateral triangle.

Let ABC be an equilateral triangle and DEF an isosceles triangle of equal area, the base BC being greater than the base EF. Then DE shall be greater than AB. Though A draw HAK parallel to BC; join DB, AE, EK. Then because the triangles DEA, BEA are equal, DB is parallel to AE; and therefore HA is equal to BE; but HA is equal to AK; therefore BE is equal to AK, and they are parallel; therefore EK is equal to AB. Now the angle CKL is less than the angle CLE which is 60°, being equal to the angle BAC. Hence the angle EKD is obtuse, and DE is greater than EK. But EK is equal to AB. Therefore DE is greater than AB.



VIII. The perimeter of an equilateral triangle is less than that of any other equal triangle.

Let ABC be the equilateral triangle. If the other triangle be not isosceles, construct an isosceles triangle equal to it, whose perimeter will therefore be less than its perimeter. Let DEK be half of this isosceles triangle.



Then, as before, BD is parallel to AE. Produce BD to meet in F a line through A parallel to BC.

First let KE be not less than KA. Then the angle KAE $(=ADF)$ is not less than KEA $(=AFD)$; therefore AF, i.e., EB is not less than AD. Therefore ED, EB are together not less than ED, AD. But ED, DA are greater than AE, which is greater than AB. Therefore DE, EB are greater than AB, i.e., the semi-perimeter DE, EK is greater than AB, BK, and therefore the perimeter of the original triangle is greater than that of ABC.

Next let KE be less than KA. Make the angle EAG equal to the angle AEB. Then because KE is less than KA, the angle KAE is less than the angle KEA, and therefore AG falls above AD. Again, because CE is greater than CA, the angle CAE is greater than the angle CEA, and therefore AG falls below AC. Angle $AGF = GAE = AEB = AFG$; therefore $AF = AG$. Also angle $ADG = BDK > BAK$ or $> DAC > DAG$; $\therefore$ AG or AF or EB is greater than DG. Hence DE, EB are greater than ED, DG, i. e., than EG. But because $AG = EB$, angle $GAE = BEA$, and EA is common, therefore EG is equal to AB; and therefore DE, EB are greater than AB, and again the perimeter of the isosceles triangle and therefore of the original triangle is greater than that of the equilateral triangle.

Lastly, if the base of the isosceles triangle be less than that of the equilateral, its side is greater by the previous theorem (VII). If on a side of this isosceles triangle another isosceles triangle be constructed equal to it, this will have a less perimeter, but its perimeter will, by the previous cases, be greater than that of the equilateral triangle. Hence in this case also the equilateral triangle has the least perimeter.