8. (*a*) Given the three middle points of the sides of any triangle: construct it.

(*b*) Given the base, area, and the ratio of the sides of a triangle: construct it.

ALGEBRA—HONOURS.

Examiner—J. W. REID, B.A.

1. If $X = ax + cy + bz$, $Y = cx + by + az$, $Z = bx + ay + cz$, shew that $X^2 + Y^2 + Z^2 - YZ - ZX - XY$

$$(a^2 + b^2 + c^2 - bc - ca - ab)(x^2 + y^2 + z^2 - yz - zx - xy),$$

and also that

$$X^3 + Y^3 + Z^3 - 3XYZ = (a^3 + b^3 + c^3 - 3abc)(x^3 + y^3 + z^3 - 3xyz).$$

2. If $a + b\sqrt{-1} = x + y\sqrt{-1}$, find the values of x and y in terms of a and b.

Find the value of the expression

$$\frac{2a\sqrt{1+x^2}}{x + \sqrt{1+x^2}}$$

when we put $x = \frac{1}{2}\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)$

3. If $ax^2 + bx + c = 0$, and $a'x^2 + b'x + c' = 0$ have a common root, prove that

$$(ac' - a'c)^2 + (ab' - a'b)(cb' - bc') = 0.$$

If α and β are the roots of the quadratic $ax^2 + bx + c = 0$, from the quadratic whose roots are $(\alpha + \beta)$ and $(\alpha - \beta)^2$.

4. Solve the equations:

(1.) $(1 + x)^{\frac{1}{3}} + (1 - x)^{\frac{1}{3}} = 2^{\frac{1}{3}}$.

(2) $\begin{cases} \frac{x}{y} - \frac{y}{x} = \frac{x+y}{x^2+y^2} \\ \frac{x^2}{y^2} - \frac{y^2}{x^2} = \frac{x}{y^2} - \frac{y}{x^2} \end{cases}$

(3) $\begin{cases} x^2 - xy + y^2 = 37 \\ x^2 + xz + z^2 = 28 \\ y^2 + yz + z^2 = 19 \end{cases}$

5. If $A \propto B$ when C is invariable, and $A \propto C$ when B is invariable, then will $A \propto BC$ both B and C are invariable.

The value of diamonds varies as the square of their weights, and square of the value of rubies varies as the cube of their weights; a diamond of a carat is worth m times a ruby of b carats, and both together are worth £c; find the value of a diamond and ruby, each weighing x carats.

6 Insert n arithmetical means between two given terms a and b.

There are n arithmetical means between 1 and 31, such that the 7th mean: $(n-1)^{th}$ mean = 5:9; find n.

If a, b and c be the p^{th}, q^{th}, and r^{th} terms respectively of an arithmetic series; shew that

$$a(q-r) + b(r-p) + c(p-q) = 0.$$

7. Find the sum of a given number of quantities in Geometrical Progression, the first term, and the common ratio being supposed known. Find also the sum of the same series to infinity.

If P be the continued product of n quantities in Geometrical Progression, S their sum, and S_1 the sum of their reciprocals; shew that

$$P^2 = \left(\frac{S}{S_1}\right)^n.$$

8. Given M and N the m^{th} and n^{th} terms of a Harmonical Progression; find the $(m+n)^{th}$ term.

The term of 3 numbers in Harmonical Progression is 26, and the product of the extremes exceeds the square of the mean by the mean; find the numbers.

9. Find the number of permutations of n things taken r at time.

Given m things of one kind, and n things of another kind, find the number of permutations that can be formed containing r of the first and s of the second.

10. Assuming the Binomial Theorem for positive integral indices, prove it for fractional and negative indices.

Shew that

$$\left(\frac{1+2x}{1+x}\right)^n = 1 + n\left(\frac{x}{1+2x}\right) + \frac{n(n+1)}{1.2}\left(\frac{x}{1+2x}\right)^2$$

+ etc.

Find the greatest term in the expansion of

$$\left(1 + \frac{5}{6}\right)^{\frac{3}{2}}.$$

PROBLEMS—HONOURS.

Examiner—A. K. BLACKADAR, M.A.

1. A watch which is 10 minutes too fast at 12 o'clock noon on Monday loses 4 minutes and 12 seconds per day. What will be the true time on the following Saturday morning when the watch shews 8 o'clock?

2. If $a + b + c = 0$, prove that

$$\frac{a}{bc - a^2} + \frac{b}{ca - b^2} + \frac{c}{ab - c^2} = 0.$$

3. Having given for all values of n, the relation

$$a_1 . a_2 a_3 \ldots a_n = a_1 n^2,$$

find the sum to n terms of the series

$$a_1 + a_2 + a_3 + \ldots a_n.$$

4. If θ be an angle whose tangent is $\frac{1}{1 \cdot 3}$, and φ an angle whose tangent is $\frac{1}{1 \cdot 5}$, then will

$$\sin(\theta + \varphi) = \sin\frac{\pi}{3} \cdot \cos\frac{\pi}{5}.$$

5. Eliminate θ from the equations

$$(a + b)\tan(\theta - \varphi) = (a - b)\tan(\theta + \varphi),$$
$$a\cos 2\varphi + b\cos 2\theta = c.$$

6. Find x from the equation

$$\cot 2^{x-1}a - \cot 2^x a = \operatorname{cosec} 3a.$$

7. Solve the equations

$$\begin{cases} x + xy^2 = 18 \\ xy + xy^2 = 12 \end{cases}$$

8. Prove that

$$(1+x)^n + n(1+x)^{n-1}.x + \frac{n(n+1)}{1.2}(1+x)^{n-2}.x^2.$$
$$+ \ldots \text{to infinity} = (1+x)^{2n}.$$

9. n counters are marked with the numbers 1, 2, 3, 4, .. n respectively. Find the number of ways in which three may be drawn, so that the greatest and least together may be double of the mean.

10. In any plane triangle ABC, if cos A, cos B, cos C are in arithmetical progression and if $2s = a + b + c$, prove that $s - a$, $s - b$, $s - c$ are in harmonical progression.

11. If equilateral triangles be described on the sides of any triangle ABC (without the triangle), and the vertices be joined by the straight lines a, b, c; prove that

$$a^2 + b^2 + c^2 = \tfrac{1}{2}(AB^2 + BC^2 + CA^2) + 6\sqrt{3} \cdot$$
Area ABC.

12. If P, H, D, be the sides of a regular pentagon, hexagon, and decagon inscribed in a circle, prove that

$$P^2 = H^2 + D^2.$$

13. D is the middle point of the base BC of an isosceles triangle ABC; CF is perpendicular to AB; DE is perpendicular to CF; EG parallel to the base meets AD in G: prove that AG is to GA in the triplicate ratio of BD to DA.

14. A quadrilateral is circumscribed about a circle, prove that the line joining the middle points of the diagonals passes through the centre of the circle.

15. A circle with centre O and radius r is inscribed in a triangle ABC, and touches the sides in D, E, F. Circles are inscribed in the quadrilaterals $AEOF$, $BFOD$, $CDOE$. If r_1, r_2, r_3 be their radii, prove that

$$\frac{r_1}{r - r_1} + \frac{r_2}{r - r_2} + \frac{r_3}{r - r_3}$$

$$= \frac{r_1}{r - r_1} \cdot \frac{r_2}{r - r_2} \cdot \frac{r_3}{r - r_3}.$$