Form of the Root.

§60. The solutions of the problems investigated in the preceding part of the paper have furnished us with the necessary and sufficient form of the root of the pure uni-serial Abelian equation $\phi(x) = 0$ of degree $n - 1$. Let this be r_1. Let

$$w,\ w^{\lambda},\ w^{\lambda^2},\ \ldots\ldots,\ w^{\lambda^{n-2}} \tag{142}$$

be a cycle containing all the primitive n^{th} roots of unity. We may assume that λ is less than n. Let

$$1,\ \lambda,\ \alpha,\ \beta,\ \ldots\ldots,\ \delta,\ \varepsilon,\ \theta \tag{143}$$

be the indices of the powers of w in (143); that is, $\alpha = \lambda^2$, $\beta = \lambda^3$, and so on. The $n - 1$ roots of the equation $\phi(x) = 0$ can be arranged in a single circulating series. Let them, so arranged, be

$$r_1,\ r_\lambda,\ r_\alpha,\ \ldots\ldots,\ r_\epsilon,\ r_\theta. \tag{144}$$

It will be found that the terms $R_1^{\frac{1}{n}}$, $R_2^{\frac{1}{n}}$, etc., in (138), which are the same, in a certain order, as $R_1^{\frac{1}{n}}$, $R_\lambda^{\frac{1}{n}}$, $R_\alpha^{\frac{1}{n}}$, etc., with multiples of n rejected from the subscripts, are given by the equations

$$\left.\begin{array}{l} R_1^{\frac{1}{n}} = A_1 (r_1^\theta r_\lambda^\epsilon r_\alpha^\delta \ldots\ldots r_\delta^\alpha r_\epsilon^\lambda r_\theta)^{\frac{1}{n}} \\ R_\lambda^{\frac{1}{n}} = A_\lambda (r_1 r_\lambda^\theta r_\alpha^\epsilon \ldots\ldots \quad r_\epsilon^\alpha r_\theta^\lambda)^{\frac{1}{n}} \\ R_\alpha^{\frac{1}{n}} = A_\alpha (r_1^\lambda r_\lambda r_\alpha^\theta \ldots\ldots \quad r_\theta^\alpha)^{\frac{1}{n}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ R_\theta^{\frac{1}{n}} = A_\theta (r_1^\epsilon r_\lambda^\delta \quad \ldots\ldots \quad r_\theta^\theta)^{\frac{1}{n}} \end{array}\right\} \tag{145}$$

In (145) the subscripts of the factors of the expression for $R_1^{\frac{1}{n}} A_1^{-1}$ are the terms in (143), while the indices are the terms in (143) in reverse order. Because the series (144) circulates, R_λ is formed from R_1 by changing r_1 into r_λ, and, through the same change, R_λ becomes R_α, and so on.

Necessity of the above Forms.

§61. Here, assuming that the root of a solvable irreducible equation of degree n is expressible as in (138), we have to show that $R_1^{\frac{1}{n}}$, $R_2^{\frac{1}{n}}$, etc., have the forms (145).

§62. In (138) $R_1^{\frac{1}{n}}$ is an n^{th} root of R_1, one of the roots of a pure uni-serial Abelian equation $\phi(x) = 0$, the series of whose roots is contained in (139). But