$$\therefore \sqrt{\left(\frac{1+x+2yz}{1-x}\right)} = \frac{y+z}{1-x}$$

and similarly for the other two.

2. Solve the equations

(1). $x^2 + 4xy + y^2 = 13$
$8xy - 7x^2 + y^2 = 13$

By subtraction we get

$$8x^2 - 4xy = 0$$

$\therefore x = 0$, which gives $y^2 = 13$
or $2x = y$ " " $x^2 = 1$
and $y = \pm 2 \pm \sqrt{17}$

(2). $(1+x)^{\frac{2}{n}} - (1-x)^{\frac{2}{n}}$

$$= (1-x^2)^{\frac{1}{n}}$$

Divide through by $(1-x)^{\frac{2}{n}}$; then

$$\left(\frac{1+x}{1-x}\right)^{\frac{2}{n}} - \left(\frac{1+x}{1-x}\right)^{\frac{1}{n}} = 1$$

$$\therefore \left(\frac{1+x}{1-x}\right)^{\frac{1}{n}} = \tfrac{1}{2}(1 \pm \sqrt{5})$$

$$\therefore \frac{1+x}{1-x} = \frac{(1 \pm \sqrt{5})^n}{2^n}$$

$$\therefore x = \frac{(1 \pm \sqrt{5})^n - 2^n}{(1 \pm \sqrt{5})^n + 2^n}$$

3. (1). If a be a root of the equation $f(x) = 0$, then $x-a$ is a factor of $f(x)$.

Since a is a root of the equation, the equation must be satisfied when a is substituted in it for x; that is $f(a)$ must $= 0$. But $f(a)$ is the remainder when $f(x)$ is divided by $x - a$, $\therefore$ $x-a$ divides $f(x)$ without remainder and is $\therefore$ a factor of it.

(2). The equation

$$4x^3 - 52x^2 + 49x - 12 = 0$$

has two equal roots; find all the roots.

Express the equation thus:

$$x^4 - 13x^2 + \tfrac{49}{4}x - 3 = 0$$

and let the roots be a, a, c.

Then the sum of these roots must be equal to 13, and the sum of their products, two at a time, must equal $\frac{49}{4}$.

$$\therefore 2a + c = 13$$
$$a^2 + 2ac = \tfrac{49}{4}$$

These two equations give

$$a = \tfrac{1}{2} \text{ and } c = 12\ ;$$

the roots of the given equations are $\therefore \frac{1}{2}, \frac{1}{2}, 12$.

(3.) The roots of the equation

$$x^4 - 10x^3 + 32x^2 - 38x + 15 = 0$$

are of the form $a+1$, $a-1$, $c+2$, $c-2$; find all the roots.

Since the sum of the roots must $= 10$ and the sum of their products two together $= 32$, we have

$$a + c = 5 \quad (1)$$

and $(a+1)(a-1) + (c+2)(c-2)$
$+ (a+1)(c+2) + (a+1)(c-2)$
$+ (a-1)(c+2) + (a-1)(c-2)$

(which reduces to

$$a^2 + c^2 + 4ac - 5) = 32$$
$$\therefore a^2 + c^2 + 4ac = 37 \quad (2)$$

From (1) and (2) we get

$$a = 2 \text{ or } 3,\ c = 3 \text{ or } 2$$

Now c cannot have the value 2, for then the root $c-2$ of the equation would be 0, but 0 is not a root of the equation.

$\therefore$ we must take $a = 2$, $c = 3$ and the required roots are 3, 1, 5, 1.

4. Sum the series

$$1^3 + 2^2 + 3^2 + 4^2 + \&c. + n^2.$$

See Todhunter's Algebra, § 460.

5. Show how to find the sum of an arithmetical progression, having given the first term, common difference and number of terms.

Sum to n term, the series whose first term is a, and the succession differences $b, 2b, 3b, \ldots. (n-1)b$.

This series is a, $a+b$, $a+b+2b$, &c., the last term being

$$a + b + 2b + \ldots (n-1)b,$$

$$\text{which} = a + \frac{n}{2}(n-1)b.$$

$\therefore$ the sum of the series $= na$

$$+ b\left\{1 + 3 + 6 + 10 + \ldots + \frac{n}{2}(n-1)\right\}$$

The series 1, 3, 6, 10,..., observing that