CHAPTER VII

THE EXPONENTIAL AND LOGARITHMIC SERIES

1. The Exponential Series. It is proposed to make a brief study of the infinite series

$$1+\frac{x}{1}+\frac{x^2}{1.2}+\ldots\ldots+\frac{x^r}{1.2\ldots\ldots r}+\ldots\ldots \qquad (1)$$

a series which, on account of its simple form, might very easily have suggested itself for examination. This series has a finite limit for all finite values of x, a fact which will be assumed. Thus for $x=1$ the series is

$$1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\ldots\ldots+\frac{1}{1.2.3\ldots\ldots r}+\ldots\ldots \qquad (2)$$

This last series is seen to be less than

$$1+1+\frac{1}{2}+\frac{1}{2.2}+\frac{1}{2.2.2}+\ldots\ldots+\frac{1}{2^{r-1}}+\ldots\ldots$$

which after the first term is an infinite geometrical progression with common ratio $\frac{1}{2}$ so that its sum is

$$1+\frac{1}{1-\frac{1}{2}} \text{ or } 3.$$

Thus the series (2) has a finite limit between 2 and 3. This limit can, by taking a sufficient number of terms, be found to any degree of accuracy but it cannot be computed exactly. Its value is denoted by e so that

$$e=1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\ldots\ldots \textit{ in inf.} \qquad (I)$$

Approximately $e=2{\cdot}7182818$

Denote the series (1) by $F(x)$, thus indicating that it is a function of x. Then putting for x the values m and n we have