By Professor Sylvester, F.R.S.

If $x^3+y^3+z^3=3\mu xyz$, and $x^6+y^6+z^6-x^3z^3-z^3y^3-y^3x^3=0$; show that (1) the variables will bear 18 distinct ratios to each other, if their order be taken into account; and (2) if their order be disregarded, these are reducible to the three following ratios:—

$$(1+q)^{\frac{1}{3}}:(1+q\rho)^{\frac{1}{3}}:(1+q\rho^2)^{\frac{1}{3}},\quad (1+q)^{\frac{1}{3}}:(1+q\rho)^{\frac{1}{3}}\rho:(1+q\rho^2)^{\frac{1}{3}}\rho^2,$$
$$(1+q)^{\frac{1}{3}}:(1+q\rho)^{\frac{1}{3}}\rho^2:(1+q\rho^2)^{\frac{1}{3}}\rho,$$

where $$q=\left(\frac{27}{\mu^3}-1\right)^{\frac{1}{3}} \text{ and } \rho^3=1.$$

[Professor Sylvester remarks that this Question contains virtually the complete analytical solution of the problem of finding the points in which a given cubic is osculable by other cubics in 9 consecutive points.]

Solution by W. J. C. Sharp, M.A.

If x^3, y^3, z^3 be the roots of $t^3-pt^2+q't-r=0$, we have, from the given relations $$p^3=27\mu^3r \text{ and } p^2=3q',$$

and the equation becomes $t^3-pt^2+\frac{p^2}{3}t-\frac{p^3}{27\mu^3}=0$,

or, if $$u=\frac{3t}{p},\ u^3-3u^2+3u-\frac{1}{\mu^3}=0,$$

$\therefore (u-1)^3=\mu^{-3}-1=q^3$ suppose, $\therefore u=1+q,\ 1+q\rho$, or $1+q\rho^2$, where ρ is an imaginary cube-root of unity, and x^3, y^3, z^3 are proportional to these quantities in some order, and, since the product of the three values of $u=q^3+1(=\mu^{-3})$, and only those values are to be chosen which make the cube root of this product μ^{-1}; the relations are as given in the Question.

If regard be paid to the order, each of these ratios will represent six different solutions, and therefore (1) is true.—*Educational Times.*

ORIGINAL PROBLEMS WITH SOLUTIONS.

By D. F. H. Wilkins, B.A., Bac. App. Sc., Mathematical Master, Mount Forest High School.

1. In Euclid II. 10 show that $AB.BD=EB.BG$.

2. Without using the Exponential Theorem prove the following: (See *Wolstenholme's Problems*, 1878, No. 296.)

If there be any n quantities whatever, $a, b, c, d, \ldots$, and if s_n represent their sum, s_{n-1} the sum of any $(n-1)$ of them, s_{n-2} the sum of any $(n-2)$ of them, etc.; and if

$$S^2_n\equiv(s_n)^t-\Sigma(s_{n-1})^n+\Sigma(s_{n-2})r-\&c.$$

then $S_n\equiv\lfloor\underline{n}\, abcd\ldots$;

$$S_{n+1}\equiv\lfloor\underline{n+1}\, abcd\ldots(a+b+c+d+\ldots)$$
$$S_{n+2}\equiv\lfloor\underline{n+2}\, abcd\ldots[2\Sigma(a^2)+3\Sigma(ab)]$$

3. Prove $\frac{xy}{4(x^2+y^2)}[(x+y)^2+2(x+y)^2(x^2+y^2)+(x^2-y^2)]$

$$\equiv(x^2+xy+y^2)^2-(x^2+xy+y^2)(x^2+y^2)+x^2y^2.$$

4. From the identity $\cos 2x\cos x\equiv\cos 3x+\cos x$, deduce

$$2^{2n-1}+\frac{(2n-2)(2n-3)}{\lfloor\underline{2}}2^{2n-3}+\frac{(2n-2)(2n-3)(2n-4)(2n-5)}{\lfloor\underline{4}}\cdot 2^{2n-5}+\ldots\equiv 3^{2n-2}+1.$$

5. ABC is an isosceles triangle, BC being the base. Forces act along the direction of the bisections of the exterior angles at B and C respectively, represented by these lines in magnitude. Show that their resultant passes through A.

Solutions.

1. $AD^2=AB^2+BD^2+2AB.BD$. (II. 4.)

$AD^2+DB^2=AB^2+2BD^2+2AB.BD$.

i.e. $AG^2=AB^2+2BD^2+2AB.BD$. (I. 47.)

i.e. $AE^2+EG^2=AB^2+2BD^2+2AB.BD$. (I. 47.)

i.e. $EB^2+EG^2=AB^2+2BD^2+2AB.BD$.

i.e. $EB^2+EG^2=4BC^2+2BD^2+2AB.BD$.

i.e. $2EB^2+BG^2+2EB.BG=4BC^2+2BD^2+2AB.BD$. (II. 4.)

$\therefore EB.BG=AB.BD$. (I. 47 and Ax-3.)

2 $S_n\equiv(s_n)^n-\Sigma(s_{n-1})^n+\Sigma(s_{n-2})^n$ &c.

Now the number of terms in S_n is

$$\frac{n(n+1)(n+2)\ldots(2n-1)}{\lfloor\underline{n}}-\frac{n}{\lfloor\underline{1}}\cdot\frac{(n-1)n(n-1)\ldots(2n-2)}{\lfloor\underline{n}}$$
$$+\frac{n(n-1)}{\lfloor\underline{2}}\cdot\frac{(n-2)(n-1)n\ldots(2n-3)}{\lfloor\underline{n}}-\&c.$$

But $\frac{n(n+1)(n+2)\ldots(2n-1)}{\lfloor\underline{n}}$ is the coeff. of x^n in $\left(\frac{1}{1-x}\right)^n$;

$\frac{(n-1)n(n+1)\ldots(2n-2)}{\lfloor\underline{n}}$ " " " $\left(\frac{1}{1-x}\right)^{n-1}$;

$\frac{(n-2)(n-1)n(n+1)\ldots(2n-3)}{\lfloor\underline{n}}$ " " $\left(\frac{1}{1-x}\right)^{n-2}$;

$$\therefore \text{coeff. of } S_n\equiv\text{coeff. of } x^n \text{ in }\left[\left(\frac{1}{1-x}\right)^n-n.\left(\frac{1}{1-x}\right)^{n-1}+\frac{n(n-1)}{\lfloor\underline{2}}\left(\frac{1}{1-x}\right)^{n-2}-\&c.\right]$$

$$\equiv\text{coeff. of } x^n \text{ in}\left[\frac{1}{1-x}-1\right]^n$$

$$\equiv\ \text{"}\ \ \text{"}\ \ \frac{x^n}{(1-x)^n}$$

$$\equiv\ \text{"}\ \ \text{"}\ \ x^n\left[1+nx+\frac{n(n+1)}{\lfloor\underline{2}}x^2+\&c.\right]$$

$$\equiv 1.$$

i.e. when the series in each bracket is expanded, all the terms but one cancel. The uncancelled term is $\lfloor\underline{n}\, abcd\ldots$

(*Vide Gross' Algebra, Example* 4, *p.* 73.)

Similarly the coeffs. of S_{n+1} and S_{n+2} are respectively the coeffs. of x^{n+1} and x^{n+2} in the identity $\left(\frac{1}{1-x}-1\right)^n\equiv\frac{x^n}{(1-x)^n}$; *i.e.* there rem n and $\frac{n(n+1)}{\lfloor\underline{2}}$ uncancelled terms containing respectively x^{n+1} & x^{n+2}.

In the first case these terms may be shown to be of the form

$\Sigma(a^2bcd\ldots)$ and the coeff. $\equiv\frac{\lfloor\underline{n+1}}{2}$.

$\therefore\ 2S_{n+1}\equiv\lfloor\underline{n+1}\, abcd2\ldots(\Sigma(a))$.

In the second case these terms are of the form $\Sigma 2a^3bcd\ldots+\Sigma 3a^2b^2cd\ldots$ and the coefficient may be seen to be $\frac{\lfloor\underline{n+2}}{3.4}$,

$\therefore\ 12S_{n+2}\equiv\lfloor\underline{n+2}\, abcd2\ldots[2\Sigma(a^2)+3\Sigma(ab)]$.

3. $$\frac{xy}{4(x^2+y^2)}[(x+y)^4+2(x+y)^2(x^2+y^2)+(x^2-y^2)^2]$$

$$\equiv\frac{xy}{4(x^2+y^2)}[(x+y)^2+(x+y)^2)][(x+y)^2+(x-y)^2]$$

$$\equiv\frac{2xy(x+y)^2}{4(x^2+y^2)}[2(x^2+y^2)]$$

$$\equiv xy(x+y)^2$$

$$\equiv x(x+y)y(x+y)$$

$$\equiv(x^2+xy)(xy+y^2)$$

$$\equiv[x^2+xy+y^2-y^2][x^2+xy+y^2-x^2]$$

$$\equiv[(x^2+xy+y^2)-y^2][(x^2+xy+y^2)-x^2]$$

$$\equiv(x^2+xy+y^2)^2-(x^2+y^2)(x^2+xy+y^2)+x^2y^2.$$

4. $$\cos x\equiv 1-\frac{x^2}{\lfloor\underline{2}}+\frac{x^4}{\lfloor\underline{4}}-\frac{x^6}{\lfloor\underline{6}}+\ldots+(-1)^{n-1}\frac{x^{2n}}{\lfloor\underline{2n}}+\&c.$$

$$\cos 2x\equiv 1-\frac{(2x)^2}{\lfloor\underline{2}}+\frac{(2x)^4}{\lfloor\underline{4}}-\frac{(2x)^6}{\lfloor\underline{6}}+\ldots+(-1)^{n-1}\frac{(2x)^{2n}}{\lfloor\underline{2n}}+\&c.$$

$\therefore$ Multiplying the above and picking out the coefficient of x^{2n-2}, we have

$$\frac{2^{2n-2}}{\lfloor\underline{2n-2}}+\frac{2^{2n-4}}{\lfloor\underline{2n-4}}+\frac{2^{2n-6}}{\lfloor\underline{2n-6}}+\&c.$$

i.e. $$\frac{1}{\lfloor\underline{2n-2}}\left[2^{2n-2}+\frac{(2n-2)(2n-3)}{\lfloor\underline{2}}.\ 2^{2n-4}+\frac{(2n-2)(2n-3)(2n-4)(2n-5)}{\lfloor\underline{4}}\ 2^{2n-6}+\&c\right.$$