§ 9. A second property of the function $\phi_1$ is that an equation of the type

$$(\phi_z \phi_1^{-z})^{\frac{1}{n}} = F(w) \tag{10}$$

subsists for every integral value of $z$, $F(w)$ being a rational function of $w$. For, taking $z = \lambda$,

$$\phi_1^{\lambda} = P_1^{\lambda\theta} P_{\lambda}^{\lambda\epsilon} P_{\alpha}^{\lambda\delta} \ldots\ldots P_{\delta}^{\lambda\alpha} P_{\epsilon}^{\lambda^2} P_{\theta}^{\lambda}.$$

But $\lambda^2 = \alpha$, $\alpha\lambda = \beta$, ...., $\lambda g = \theta$. And $\lambda\theta = \lambda^{n-1}$. Since $\lambda^{n-1} - 1$ is a multiple of $n$, put $\lambda^{n-1} - 1 = cn$. Then

$$\phi_1^{\lambda} = P_1^{cn}(P_1 P_{\lambda}^{\theta} \ldots\ldots P_{\theta}^{\lambda}).$$

Comparing this with the second of equations (9),

$$\phi_{\lambda}\phi_1^{-\lambda} = P_1^{-cn}.$$

Therefore

$$(\phi_{\lambda}\phi_1^{-\lambda})^{\frac{1}{n}} = w' P_1^{-c}, \tag{11}$$

$w'$ being an $n^{\text{th}}$ root of unity. In like manner, from the second and third of equations (9),

$$\phi_{\alpha}\phi_{\lambda}^{-\lambda} = P_{\lambda}^{-cn}.$$

Substitute here the value of $\phi_{\lambda}$ in (11). Then $\phi_{\alpha}\phi_1^{-\alpha} = (P_{\lambda}^{-c} P_1^{-\lambda c})^n$. Therefore

$$(\phi_{\alpha}\phi_1^{-\alpha})^{\frac{1}{n}} = w''(P_{\lambda}^{-c} P_1^{-\lambda c}), \tag{12}$$

$w''$ being an $n^{\text{th}}$ root of unity. The equations (11) and (12) are of the type (10). Therefore an equation of the type (10) subsists when $z$ is equal either to $\lambda$ or to $\alpha$. In the same way we can go on to show that an equation of the type (10) subsists when $z$ is equal to any of the terms in (7). Should $z = 0$, $\phi_z\phi_1^{-z} = \phi_0$. Therefore, by § 8, $\phi_z\phi_1^{-z} = P_0^{mn}$. Hence in this case also $(\phi_z\phi_1^{-z})^{\frac{1}{n}}$ is a rational function of $w$. Therefore, whether $z$ be zero or one of the terms in the series (7), an equation of the type (10) subsists. This implies that an equation of the type (10) subsists for every integral value of $z$.

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## Criterion of Pure Uni-Serial Abelianism.

### *The Criterion Stated.*

§ 10. A Criterion of pure uni-serial Abelianism may now be given. Let $R_1$ be a rational function of the primitive $n^{\text{th}}$ root of unity $w$, and, $z$ being any integer, let $R_z$ be derived from $R_1$ by changing $w$ into $w^z$. Then, if $R_0^{\frac{1}{n}}$ is rational, and if the terms $R_1^{\frac{1}{n}}$, $R_2^{\frac{1}{n}}$, etc., are such that an equation of the type