UNIVERSITY WORK.

MATHEMATICS.

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SOLUTIONS TO PROBLEMS IN MARCH NUMBER.

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1. Obtain an expression for the sum of the products of the first n natural numbers three and three together.

1. $(1+2+3+4\ldots\ldots n)^3 = 3(1+2+3+\ldots\ldots n)(1^2+2^2+\ldots\ldots n^2)2-(1^3+2^3+\ldots\ldots n^3)+6P$ (when P represents the sum of the products 3 at a time.)

$$\therefore \left\{\frac{n(n+1)}{2}\right\}^3 = 3\,\frac{n(n+1)}{2}\cdot\frac{n(n+1)(2n+1)}{6} - 2\left\{\frac{n(n+1)}{2}\right\}^2 + 6P$$

and $P = \dfrac{(n-1)(n-2)n^2(n+1)^2}{48}$.

2. Find the sum of the products of every three terms of an infinite geometrical progression.

2. $(a+ar+ar^2+\ldots\ldots ar^{n-1})^3 = 3(a+ar+\ldots\ldots ar^{n-1})(a^2+a^2r^2+\ldots\ldots a^2r^{2n-2}) - 2(a^3+a^3r^3,\text{ etc.},)+6P$.

$$\left\{\frac{a(1-r^n)}{1-r}\right\}^3 = 3\,\frac{a(1-r^n)}{1-r}\cdot\frac{a^2(1-r^{2n})}{1-r^2} - 2\,\frac{a^3(1-r^{3n})}{1-r^3}+6P.$$

From which P may be obtained.

3. Find the sum of the cubes of a series of quantities in $A.P.$

3. $a^3+(a+b)^3+(a+2b)^3+\ldots\ldots\{a+(n-1)b\}^3$
$=a^3+(a^3+3a^2b+3ab^2+b^3)+\ldots\ldots\{a^3+3a^2b(n-1)+3ab^2(n-1)^2+(n-1)^3b^3\}$
$=na^3+3a^2b\{1+2+\ldots\ldots(n-1)\}+3ab^2\{1^2+2^2+\ldots\ldots(n-1)^2\}+b^3\{1^3+2^3+\ldots+(n-1)^3\}$
$=na^3+3a^2b\,\dfrac{n(n-1)}{2}+3ab^2\,\dfrac{(n-1)n(2n-1)}{6}+b^3\left\{\dfrac{n(n-1)}{2}\right\}^2.$

4. Show that if $x^r+py^r+qz^r$ is exactly divisible by $x^2-(ay+bz)x+abyz$, then $\dfrac{p}{a^r}+\dfrac{q}{b^r}+1=0$.

4. $x^2-(ay+bz)x+abyz=(x-ay)(x-bz)$.
$\therefore$ $x^r+py^r+qz^r$ must be divisible by $x-ay$ and by $x-bz$.

$\therefore$ by substuting $\dfrac{x}{a}$ for y and $\dfrac{x}{b}$ for z,

$$x^r+p\cdot\frac{x^r}{a^r}+q\,\frac{x^r}{b^r}=0.$$

$$\therefore \frac{p}{a^r}+\frac{q}{b^r}+1=0.$$

5. In how many ways can $2n$ men be arranged in couples?

5. Two men can be taken from $2n$ in $\dfrac{\lfloor 2n}{\lfloor 2\lfloor 2n-2}$ ways. Two men can be taken from $2n-2$ in $\dfrac{\lfloor 2n-2}{\lfloor 2\lfloor 2n-4}$ ways, etc.,

$$\therefore \text{ total number ways } = \frac{\lfloor 2n}{\lfloor 2\lfloor 2n-2}\times\frac{\lfloor 2n-2}{\lfloor 2\lfloor 2n-4}\times\frac{\lfloor 2n-4}{\lfloor 2\lfloor 2n-6}\times\text{ etc.} = \frac{\lfloor 2n}{\{\lfloor 2\}^n}.$$

6. If $(by-cx)^2=(b^2-ac)(y^2-cz)$, prove that $(bx-ay)^2 = (b^2-ca)(x^2-az)$.

6. $(by-cx)^2=(b^2-ac)(y^2-cz)$
$\therefore b^2y^2-2bcxy+c^2x^2=b^2y^2-b^2cz-acy^2+ac^2z$
$\therefore -2bxy+cx^2=-b^2z-ay^2+acz$
$\therefore -2abxy+acx^2=-ab^2z-a^2y^2+a^2cz$
$\therefore b^2x^2-2abxy+a^2y^2=b^2x^2-ab^2z-acx^2+a^2cz$
i.e., $(bx-ay)^2=(b^2-ca)(x^2-az)$.