fore, by §8, $e_1 = 0$. Therefore one of the roots of the auxiliary biquadratic is zero; which because the auxiliary biquadratic is assumed to be irreducible, is impossible. Therefore $e_1^5 ⅃_1^4$ and $e_2^5 ⅃_2^4$ are unequal. In the same way all the terms in (69) can be shown to be unequal; which, because it has been proved that there are not more than two unequal terms in (69), is impossible. Therefore $⅃_1 ⅃_3$ is not the root of a quadratic equation. Therefore the product of two of the roots, $⅃_1$ and $⅃_4$, of the auxiliary biquadratic is the root of a quadratic equation, while the product of a different pair, $⅃_1$ and $⅃_3$, is not the root of a quadratic. But the only forms which the roots of an irreducible biquadratic can assume consistently with these conditions are those given in (68).

§68. Proposition XXV. The surd $\surd s_1$ can have its value expressed in terms of $\surd s$ and $\surd z$.

By Propositions XIII. and XIX, the terms of the first of the groups (67) are the roots of a biquadratic equation. Therefore their fifth powers

$$⅃_1^2 ⅃_3, \quad ⅃_2^2 ⅃_1, \quad ⅃_3^2 ⅃_4, \quad ⅃_4^2 ⅃_2, \tag{70}$$

are the roots of a biquadratic. From the values of $⅃_1$, $⅃_2$, $⅃_3$ and $⅃_4$ in (68), the values of the terms in (70) may be expressed as follows:

$$\left.\begin{aligned}
⅃_1^2 ⅃_3 &= F + F_1 \surd z + (F_2 + F_3 \surd z) \surd s \\
&\quad + (F_4 + F_5 \surd z) \surd s_1 + (F_6 + F_7 \surd z) \surd s \surd s_1, \\
⅃_2^2 ⅃_1 &= F - F_1 \surd z + (F_2 - F_3 \surd z) \surd s_1 \\
&\quad - (F_4 - F_5 \surd z) \surd s - (F_6 - F_7 \surd z) \surd s \surd s_1, \\
⅃_4^2 ⅃_2 &= F - F_1 \surd z - (F_2 - F_3 \surd z) \surd s_1 \\
&\quad + (F_4 - F_5 \surd z) \surd s - (F_6 - F_7 \surd z) \surd s \surd s_1, \\
⅃_3^2 ⅃_4 &= F + F_1 \surd z - (F_2 + F_3 \surd z) \surd s \\
&\quad - (F_4 + F_5 \surd z) \surd s_1 + (F_6 + F_7 \surd z) \surd s \surd s_1,
\end{aligned}\right\} \tag{71}$$

where F, F_1, etc., are rational. Let $\Sigma(⅃_1^2 ⅃_3)$ be the sum of the four expressions in (70). Then, because these expressions are the roots of a biquadratic, $\Sigma(⅃_1^2 ⅃_3)$ or $4F + 4F_7 \surd s \surd s_1$, must be rational. Suppose if possible that $\surd s_1$ cannot have its value expressed in terms of $\surd s$ and $\surd z$. Then, because $\surd s \surd s_1$ is not rational, $= 0$. By (68), this implies that $n = 0$. Let

$$(⅃_1^2 ⅃_3)^2 = L + L_1 \surd z + (L_2 + L_3 \surd z) \surd s$$
$$+ (L_4 + L_5 \surd z) \surd s_1 + (L_6 + L_7 \surd z) \surd s \surd s_1,$$