§ 55. In the very same way in which (83) was established, it can be proved that

$$R_{em}^{\frac{m}{n}}(R_{e\sigma}^{\lambda^{s-2}} R_{e\sigma\lambda}^{\lambda^{s-3}} \dots . R_{e\sigma\lambda^{s-2}})^{\frac{\sigma}{n}} \dots . (R_{e\beta}^{k^{b-2}} \dots .)^{\frac{\beta}{n}} = Q_e R_e^{\frac{\Delta}{n}}, \tag{121}$$

where Q_e is a rational function of w^e, and

$$\Delta = m^2 + \sigma^2 (s-1) \lambda^{s-2} + \tau^2 (t-1) h^{t-2} + \dots . + \beta^2 (b-1) k^{b-2}. \tag{122}$$

Because m is the continued product of the odd factors of n, m^2 is odd. But each of the expressions $s-1$, $t-1$, etc., is even. Therefore Δ is odd. Therefore Δ is prime to 4. Again, because m is the continued product of the odd factors of n, it is a multiple of b. And, because $s\sigma = b\beta$, σ is a multiple of b. In like manner τ is a multiple of b. In this way all the separate members of the expression for Δ in (122) except the last are multiples of b. And, by the same reasoning as was used in §44, $\beta^2 (b-1) k^{b-2}$ is not a multiple of b. Therefore Δ is prime to b. In like manner it is prime to s, t, etc. Therefore it is prime to n. Therefore there are whole numbers v and r such that

$$v\Delta = rn + 1.$$

Therefore, from (121),

$$R_{em}^{\frac{mv}{n}} (R_{e\sigma}^{\lambda^{s-2}} \dots .)^{\frac{\sigma v}{n}} (R_{e\tau}^{h^{t-2}} \dots .)^{\frac{\tau v}{n}} \dots . (R_{e\beta}^{k^{b-2}} \dots .)^{\frac{\beta v}{n}} = (Q_e^v R_e^r) R_e^{\frac{1}{n}}. \tag{123}$$

For any integral value of z, let $R_z^{\frac{v}{n}}$ be written $P_z^{\frac{1}{n}}$. Then, by (103), putting A_e^{-1} for $Q_e^v R_e^r$, (123) becomes

$$R_e^{\frac{1}{n}} = A_e (P_{em}^m \phi_{e\sigma}^\sigma \psi_{e\tau}^\tau \dots . F_{e\beta}^\beta)^{\frac{1}{n}}. \tag{124}$$

Therefore

$$R_1 = A_1^n (P_m^m \phi_\sigma^\sigma \psi_\tau^\tau \dots . F_\beta^\beta). \tag{125}$$

But P_m is the same as R_m^v. Therefore, by § 54, P_m is of the form of the fundamental element of the root of a pure uni-serial Abelian quartic. Therefore the expression for R_1 in (125) is identical with that in (104), and thus the form of the fundamental element in (104) is established. Also, it was necessary to take $R_0^{\frac{1}{n}}$ with its rational value, because, by § 5, $nR_0^{\frac{1}{n}}$ is the sum of the roots of the equation $f(x) = 0$. And equation (124) is identical with (109), which establishes the necessity of the forms assigned to all those expressions which are contained under $R_e^{\frac{1}{n}}$. It remains to prove that the expressions contained under $R_{ev}^{\frac{1}{n}}$, $\frac{n}{v}$ or y being a term in the series (107) distinct from n, have the forms assigned to them in (110). The details to be given here are very much a repetition of what is found in § 44; but, to prevent the confusion that might arise