TRIGONOMETRICAL THEOREMS.

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THE formulae for the sine and cosine of the sum or difference of two angles in terms of the sines and cosines of those angles are the most important in analytical trigonometry. They are to that branch of mathematics what Taylor's theorem is to the differential calculus or the parallelogram of forces to abstract dynamics, but yet they are rarely or never discussed with rigour or generality. Usually a geometric proof is given of the formula $\sin(A+B) = \sin A \cos B + \cos A \sin B$, when A and B are both acute; and then B is made to become $(-B)$ which gives the following modification:—

$$\text{Sin}\left\{A+(-B)\right\} = \sin A \cos(-B) + \cos A \sin(-B)$$
$$= \sin A \cos B - \cos A \sin B$$
$$= \sin(A-B)$$

Now it appears to me that an illegitimate assumption is made here, viz., that when the formula is proved to hold for a geometric sum, it is also true for an analytic sum. This is the more inexcusable, because the formula for the sine of the sum being admitted, that for the sine of the difference of two angles may be deduced, without either referring to the geometry or making any supposition to which we are not entitled from the definitions.

Thus $\cos(A-B) = \sin\left(\frac{p}{2} - \overline{A-B}\right)$ where p stands for 180°

$$= \sin\left(\overline{\frac{p}{2} - A} + B\right)$$
$$= \sin\left(\frac{p}{2} - A\right)\cos B + \cos\left(\frac{p}{2} - A\right)\sin B$$
$$= \cos A \cos B + \sin A \sin B.$$

Again—

$$\sin A = \sin(B + A - B)$$
$$= \sin B \cos(A-B) + \cos B \sin(A-B)$$
$$= \sin B(\cos A \cos B + \sin A \sin B) + \cos B \sin(A-B)$$
$$\therefore \sin(A-B) = \frac{\sin A - \cos A \cos B \sin B - \sin A \sin^2 B}{\cos B}$$
$$= \frac{\sin A(1 - \sin^2 B) - \cos A \cos B \sin B}{\cos B}$$
$$= \frac{\sin A \cos^2 B - \cos A \cos B \sin B}{\cos B}$$
$$= \sin A \cos B - \cos A \sin B. \qquad \text{Q. E. D.}$$

The greatest leap, however, is taken when the theorem being proved only for two acute angles it is assumed to hold universally. I have only seen one proof of it for angles of any magnitude, and that one is by Professor DeMorgan. His demonstration is made by means of the theory of projections, which, after all, is but a kind of reasoning in a circle, as the theory of projections depends upon trigonometry. At all events it seems less philosophical to advance one branch of science by means of another than to make use of the resources of the one itself when these are adequate to the occasion.

The following proof is given for the sine of the difference of two angles of any magnitude, and those for the other three formulæ are easily deducible therefrom. The formula for $\sin(A-B)$ when A and B are both acute being assumed, it is required to prove that $\sin(A'-B') = \sin A' \cos B' - \cos A' \sin B'$ when A' and B' have any magnitude.

In this case we may obviously put $A' = mp \pm A$; $B' = np \pm B$, where m and n are whole numbers, p stands for 180° and A and B are acute angles. We have then

$$\sin(A'-B') = \sin\left\{mp \pm A - (np \pm B)\right\}$$
$$= \sin\left\{(m-n)p \pm A \mp B\right\}$$
$$= \pm\sin(\pm A \mp B) \text{ according as } m-n \text{ is even or odd.}$$
$$= \pm\sin\left\{\pm A - (\pm B)\right\}$$

Let us first consider the case when $m-n$ is even, which can only be by m and n being either both odd or both even. We have then

$$\sin(A'-B') = \sin\left\{\pm A - (\pm B)\right\}$$
$$= \sin(\pm A)\cos(\pm B) - \cos(\pm A)\sin(\pm B)$$
$$= \sin(\pm A)\cos B - \cos A \sin(\pm B) \ldots\ldots\ldots \text{I.}$$

Now $\sin A' = \sin(mp \pm A) = \pm\sin(\pm A) \ldots\ldots\ldots$ II., according as m is even or odd.

$\cos B' = \cos(np \pm B) = \pm\cos B \ldots\ldots\ldots$ III., according as n is even or odd.

$\cos A' = \cos(mp \pm A) = \pm\cos A \ldots\ldots\ldots$ IV., according as m is even or odd.

$\sin B' = \sin(np \pm B) = \pm\sin \pm B \ldots\ldots\ldots$ V., according as m is even or odd.

We have then from

II. $\sin(\pm A) = \pm\sin A' \ldots\ldots\ldots$ VI.

III. $\cos B = \pm\cos B' \ldots\ldots\ldots$ VII.

IV. $\cos A = \pm\cos A' \ldots\ldots\ldots$ VIII

V. $\sin(\pm B) = \pm\sin B' \ldots\ldots\ldots$ IX.

Now, since we are considering the case when $m-n$ is even it is obvious from the previous remark that the same sign must be taken in VI. and VII. and likewise in VIII. and IX.; therefore the product of each pair of equations must be positive, that is,

$$\sin(\pm A)\cos B = \sin A' \cos B'$$
$$\cos A \sin(\pm B) = \cos A' \sin B'$$

Substituting these values in I. we get

$$\sin(A'-B') = \sin A' \cos B' - \cos A' \sin B' \ldots\ldots\ldots \text{X.}$$

Finally, when $m-n$ is odd if m be even n must be odd, and *vice versa:* hence

$$\sin(A'-B') = -\sin(\pm A)\cos B + \cos A \sin(\pm B)$$

Now, $\sin A' = \sin(mp \pm A) = \pm\sin(\pm A)$, according as m is even or odd.

$\cos B' = \cos(np \pm B) = \mp\cos B$ as n is odd or even.

Otherwise $\sin(\pm A) = \pm\sin A'$

$\cos B = \mp\cos B'$

$\therefore \sin(\pm A)\cos B = -\sin A' \cos B'.$

In the same way it may be shown that

$\cos A \sin(\pm B) = -\sin B' \cos A'$

Substituting these values in X. we get as before

$$\sin(A'-B') = \sin A' \cos B' - \cos A' \sin B'.$$

NUMBER.

A COURSE OF LESSONS PREPARATORY TO THE USE OF A TEXT-BOOK ON ARITHMETIC.

VI.

FOURTH STEP.—(*Concluded.*)

FRACTIONS.

Summary of the Exercises.

I.—Exercises on the names and values of Fractions.
II.—To convert whole numbers into Fractions.
III.—To convert Fractions into whole numbers.
IV.—Conversion of Fractions from one denomination to another.
V.—Addition of Fractional numbers.
VI.—To find how much must be added to a Fractional number, to produce a given number.

I.—*On the Names and Values of Fractions.*

Divide this apple into two equal pieces or parts. What is one of these parts called? What the other? And the two parts taken together? What, then, is a half?

A half is one of two equal parts of a whole.

How many half apples are there in one apple?

Then one whole apple is the same as ——.

How did I obtain the half of this apple?

By dividing it into two equal parts, and taking one of these parts.

If, instead of 2 parts, I divide the apple into three, what should I have? 3 equal parts. Say 3 thirds. And each of these parts? 1 third.

What, then, is the third part of a thing?

The two thirds?

To make one apple, how many thirds of an apple are necessary? Then one whole apple is the same as ——.

How did I obtain the third of the apple? How two thirds?

The idea of the *fourth* may be developed as the half and third, or as follows:—