For take ϕ, the general symbol under which are included all the particular terms in the series (1); and let the n^{th} power of ϕ, (n being a whole number), arranged so as to satisfy the conditions of Def. 8, be,

$$\phi^n = a + a_1 t_1 + a_2 t_2 + \&c. ; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

where the coefficients, a, a_1, &c., are rational; and each of the terms, t_1, t_2, &c., is either some power of an integral surd, or the continued product of several such powers. Suppose y_1^r to be one of the factors of t_1; the index of the surd y_1 being $\frac{1}{\lambda}$; and let the several λ^{th} roots of unity be, $1, z, z^2, \ldots\ldots, z^{\lambda-1}$. Then, from (3),

$$\phi_1^n = a + a_1 v_1 + a_2 v_2 + \&c.,$$
$$\phi_2^n = a + a_1 u_1 + a_2 u_2 + \&c.,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\phi_m^n = a + a_1 w_1 + a_2 w_2 + \&c. ;$$

where v_1, u_1, &c., are what t_1 becomes in passing from ϕ to ϕ_1, ϕ_2, &c.; and so of the other terms. Therefore,

$$\Sigma(\phi^n) = \phi_1^n + \phi_2^n + \ldots + \phi_m^n = \ldots + a_1(v_1 + u_1 + \ldots + w_1) + \&c.,$$
$$= \ldots\ldots + a_1 \Sigma(t_1) + \&c. ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

where, just as $\Sigma(\phi^n)$ represents the sum of the terms, ϕ_1^n, ϕ_2^n, ..., ϕ_m^n, so $\Sigma(t_1)$ represents the sum of the terms, $v_1, u_1, \ldots, w_1$. Now, in the series, v_1, u_1, &c., if any term v_1 be fixed upon, there are λ terms, including v_1, of the forms,

$$v_1, \; z v_1, \; z^2 v_1, \; \ldots\ldots, \; z^{\lambda-1} v_1.$$

The sum of these is zero. Strike these λ terms out of $\Sigma(t_1)$; and then, in the same manner, whatever term among those remaining in $\Sigma(t_1)$ be considered, it may be demonstrated to be one of a group whose sum is zero. And so on. Therefore $\Sigma(t_1)$ is zero. In like manner all the terms on the right hand side of equation (4), except the first, or $m\,a$, must vanish. Consequently, $\Sigma(\phi^n)$ is rational. If now we put

$$S_1 = \phi_1 + \phi_2 + \ldots\ldots + \phi_m,$$
$$S_2 = \phi_1^2 + \phi_2^2 + \ldots\ldots + \phi_m^2,$$
$$S_3 = \phi_1^3 + \phi_2^3 + \ldots\ldots + \phi_m^3,$$